Twenty-eight

Ⅰ Finding potential functions

Ⅱ Parametrized curves

Ⅲ Parametrizing curves

Ⅳ Curl

⑪ Finding potential functions

Central Theorem

A vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is the gradient of a function exactly when

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

for all i and j.

Examples:

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$,

$\vec{F}(x,y) = (2xy, x^2)$.

$\vec{F} = \nabla f$ since

$$\frac{\partial F_1}{\partial x_2} = 2x = \frac{\partial F_2}{\partial x_1}$$

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$,

$\vec{F}(x,y,z) = (z, 2yz, x+y^2)$.

$\vec{F} = \nabla f$ since

$$\frac{\partial F_1}{\partial x_2} = 0 = \frac{\partial F_2}{\partial x_1}$$

$$\frac{\partial F_2}{\partial x_3} = 2y = \frac{\partial F_3}{\partial x_2}$$

$$\frac{\partial F_1}{\partial x_3} = 1 = \frac{\partial F_3}{\partial x_1}$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

① Take antiderivative of each F_i with respect to x_i.

② Write summands from ① for $i=1$, and cross out repeats for $i \geq 2$.

③ Write remaining summands from ① for $i=2$, and cross out repeats for $i \geq 3$.

④ Continue as in ② and ③ for remaining i.

⑤ Add a generic constant C.

⑥ Check that $\nabla f = \vec{F}$.

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x

antider. w.r.t. y

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → x^2y

antider. w.r.t. y

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → x^2y

antider. w.r.t. y → x^2y

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → x^2y

antider. w.r.t. y → x^2y

$$f(x,y) =$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → x^2y

antider. w.r.t. y → x^2y

$$f(x,y) = x^2y$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → x^2y

antider. w.r.t. y → x^2y

$$f(x,y) = x^2y$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → ~~x^2y~~

antider. w.r.t. y → ~~x^2y~$

$$f(x,y) = x^2y + C$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → ~~x^2y~~

antider. w.r.t. y → ~~x^2y~~

$$f(x,y) = x^2y + C$$

Check: $\nabla f(x,y) =$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → ~~x^2y~~

antider. w.r.t. y → ~~x^2y~~

$$f(x,y) = x^2y + C$$

Check: $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → ~~x^2y~~

antider. w.r.t. y → ~~x^2y~~

$$f(x,y) = x^2y + C$$

<u>Check</u>: $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2xy, x^2)$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y) = (2xy, x^2)$$

{passed conservative test}

antider. w.r.t. x → ~~x^2y~~

antider. w.r.t. y → ~~x^2y~~

$$f(x,y) = x^2y + C$$

<u>Check</u>: $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2xy, x^2) = \vec{F}(x,y)$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z, 2yz, x+y^2)$$

{passed conservative test}

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x

antider. w.r.t. y

antider. w.r.t. z.

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x: xz

antider. w.r.t. y: y^2z

antider. w.r.t. z: $xz+y^2z$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x → xz

antider. w.r.t. y → y^2z

antider. w.r.t. z. → $xz+y^2z$

$$f(x,y,z)$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x → xz

antider. w.r.t. y → y^2z

antider. w.r.t. z → $xz+y^2z$

$$f(x,y,z) = xz$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x → ~~xz~~

antider. w.r.t. y → y^2z

antider. w.r.t. z → ~~xz~~ $+ y^2z$

$$f(x,y,z) = xz$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x → ~~xz~~

antider. w.r.t. y → y^2z

antider. w.r.t. z → ~~xz~~ $+ y^2z$

$$f(x,y,z) = xz + y^2z$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x → ~~xz~~

antider. w.r.t. y → ~~y^2z~~

antider. w.r.t. z. → ~~$xz + y^2z$~~

$$f(x,y,z) = xz + y^2z$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x → ~~xz~~

antider. w.r.t. y → ~~y^2z~~

antider. w.r.t. z. → ~~$xz + y^2z$~~

$$f(x,y,z) = xz + y^2z + C$$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x → ~~xz~~

antider. w.r.t. y → ~~y^2z~~

antider. w.r.t. z. → ~~$xz + y^2z$~~

$$f(x,y) = xz + y^2z + C$$

<u>Check:</u> $\nabla f(x,y,z) =$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x: ~~xz~~

antider. w.r.t. y: ~~y^2z~~

antider. w.r.t. z: ~~$xz + y^2z$~~

$$f(x,y) = xz + y^2z + C$$

<u>Check:</u> $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z,\ 2yz,\ x+y^2)$$

{passed conservative test}

antider. w.r.t. x: ~~xz~~

antider. w.r.t. y: ~~y^2z~~

antider. w.r.t. z: ~~$xz+y^2z$~~

$$f(x,y) = xz + y^2z + C$$

<u>Check:</u> $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (z,\ 2yz,\ x+y^2)$

How to solve for f if $\vec{F} = \nabla f$ and $\vec{F}$ is known

$$\vec{F}(x,y,z) = (z, 2yz, x+y^2)$$

{passed conservative test}

antider. w.r.t. x: ~~xz~~

antider. w.r.t. y: ~~y^2z~~

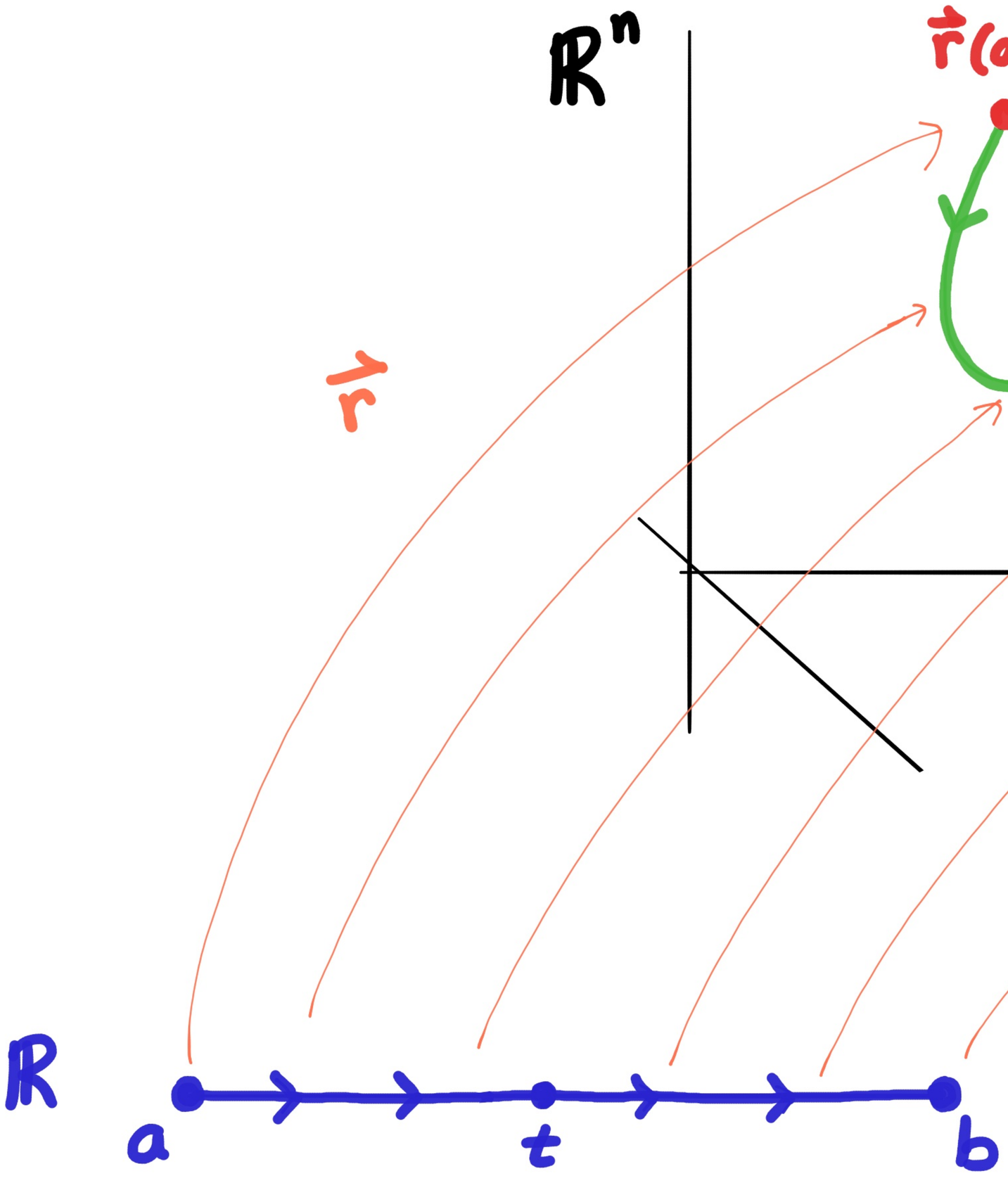
antider. w.r.t. z: ~~$xz + y^2z$~~

$$f(x,y) = xz + y^2z + C$$

<u>Check:</u> $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (z, 2yz, x+y^2) = \vec{F}$

Ⅱ Parametrized curves

A **smooth parametrized curve** C in $\mathbb{R}^n$ is a collection of points in $\mathbb{R}^n$ that is the range of some $\vec{r}:[a,b]\to\mathbb{R}^n$ where $\vec{r}(t)=(r_1(t),r_2(t),\ldots,r_n(t))$, each $r_i'(t)$ is continuous, and not all $r_i'(t)$ equal zero simultaneously.



$$\vec{r}:[a,b]\longrightarrow\mathbb{R}^n$$

$$a\le t\le b$$

Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$\vec{r}(t) = (r_1(t), r_2(t)) \quad , \quad a \le t \le b$$

Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$\vec{r}(t) = (r_1(t), r_2(t)) \quad , \quad a \le t \le b$$

$$= (x(t), y(t))$$

Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$\vec{r}(t) = \cancel{(r_1(t), r_2(t))}, \quad a \le t \le b$$

$$= (x(t), y(t))$$

Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$\vec{r}(t) = \cancel{(r_1(t), r_2(t))}, \quad a \le t \le b$$

$$= (x(t), y(t))$$

$$x = x(t), \quad y = y(t)$$

Convention

A curve C in $\mathbb{R}^2$ is parametrized by

~~$\vec{r}(t) = (r_1(t), r_2(t))$~~ , $a \le t \le b$

~~$= (x(t), y(t))$~~

$x = x(t)$, $y = y(t)$

Convention

A curve C in $\mathbb{R}^2$ is parametrized by

~~$\vec{r}(t) = (r_1(t), r_2(t))$~~

~~$= (x(t), y(t))$~~

$x = x(t)$, $y = y(t)$, $a \le t \le b$

Convention

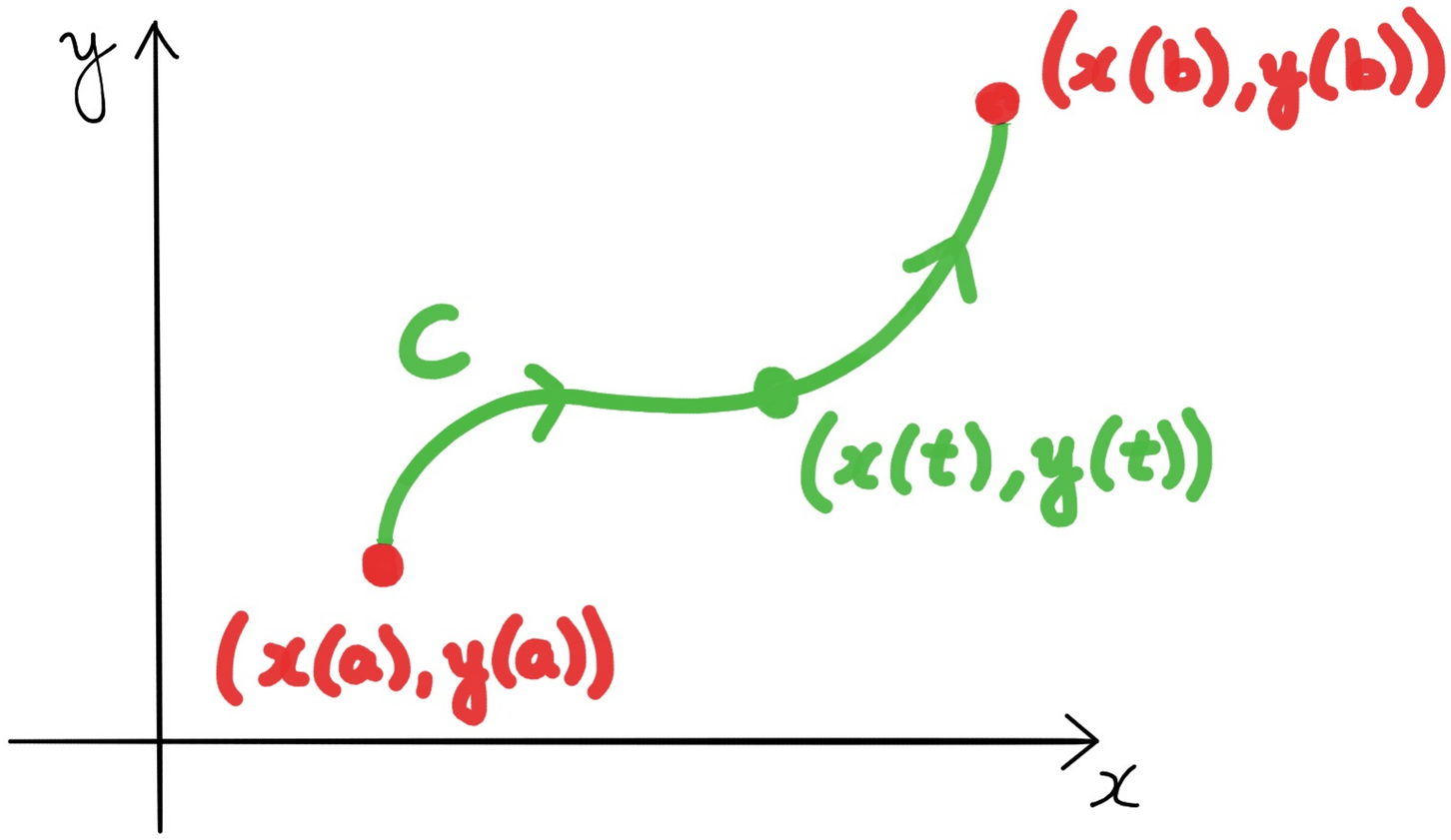
A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t) \ , \ y = y(t) \ , \quad a \le t \le b$$

Convention

A curve C in $\mathbb{R}^2$ is parametrized by

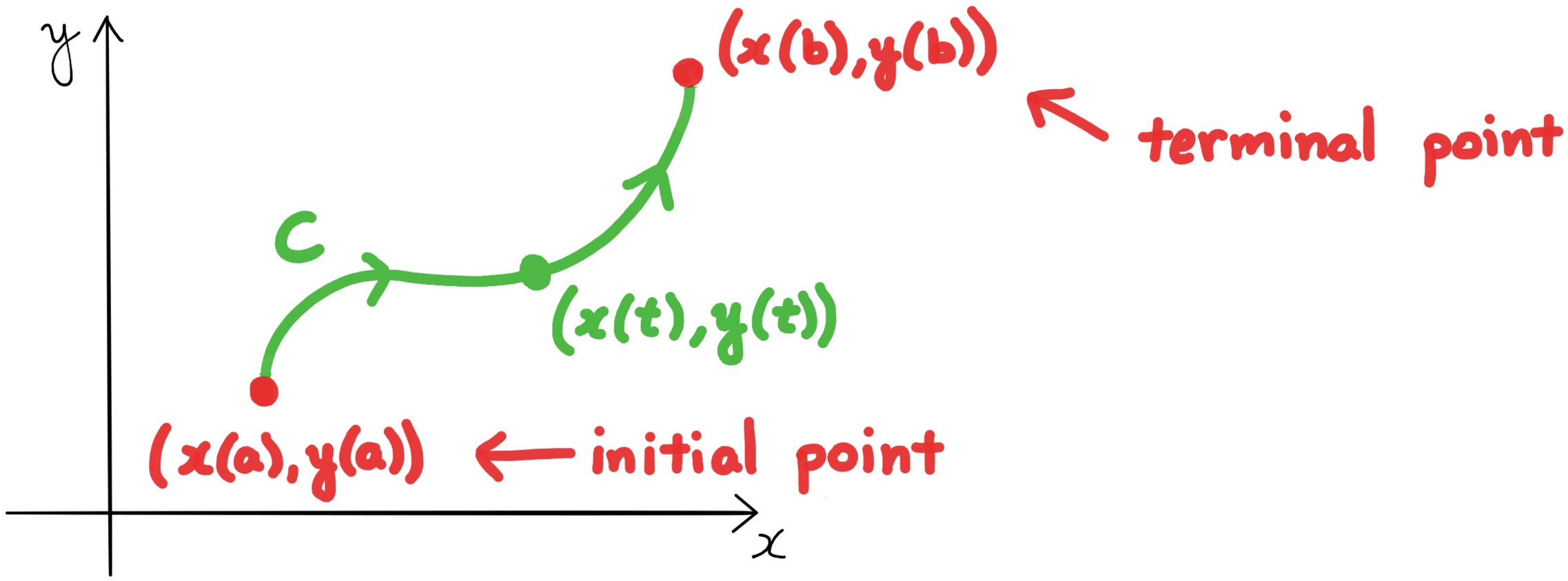
$$x = x(t) \ , \ y = y(t) \ , \quad a \le t \le b$$



Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t) \ , \ y = y(t) \ , \quad a \le t \le b$$



Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

Example:

C param. by

$x = \cos t,$

$y = \sin t,$

$0 \le t \le 2\pi.$

Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

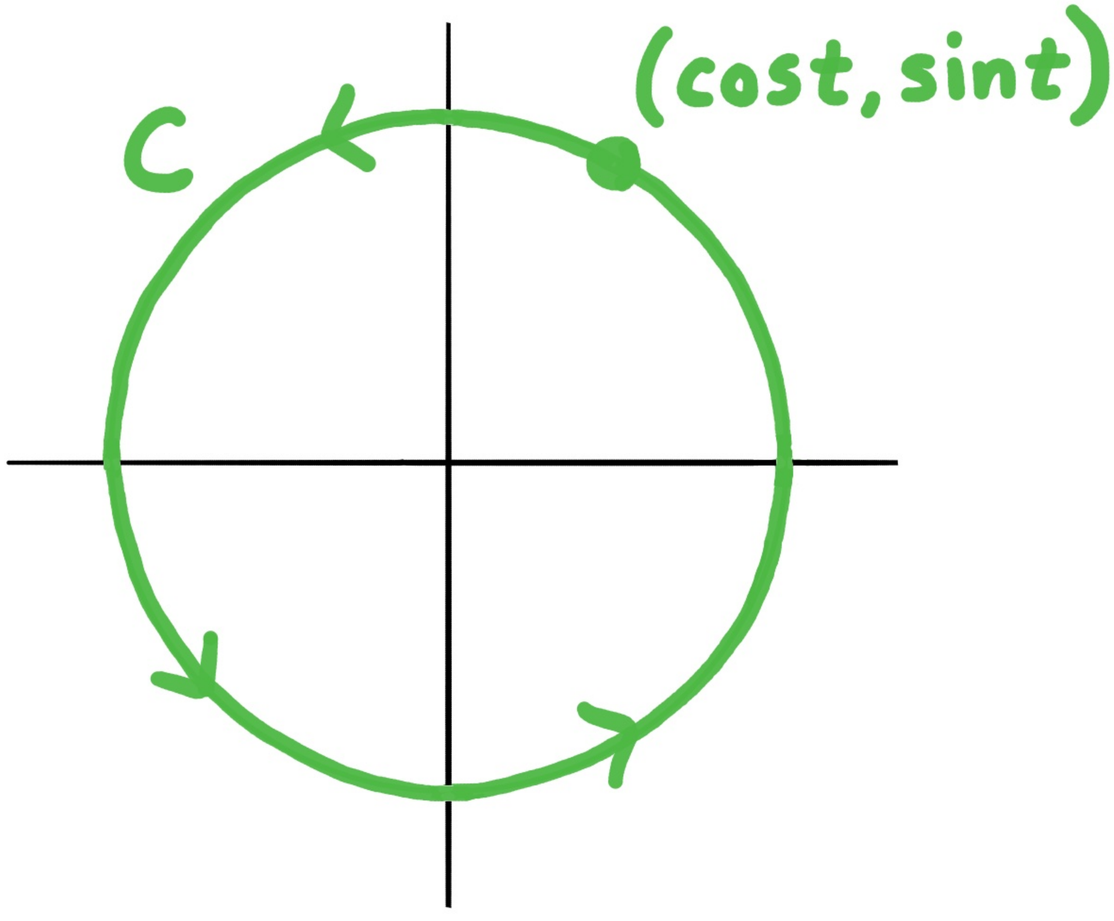
Example:

C param. by

$x = \cos t$,

$y = \sin t$,

$0 \le t \le 2\pi$.



Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

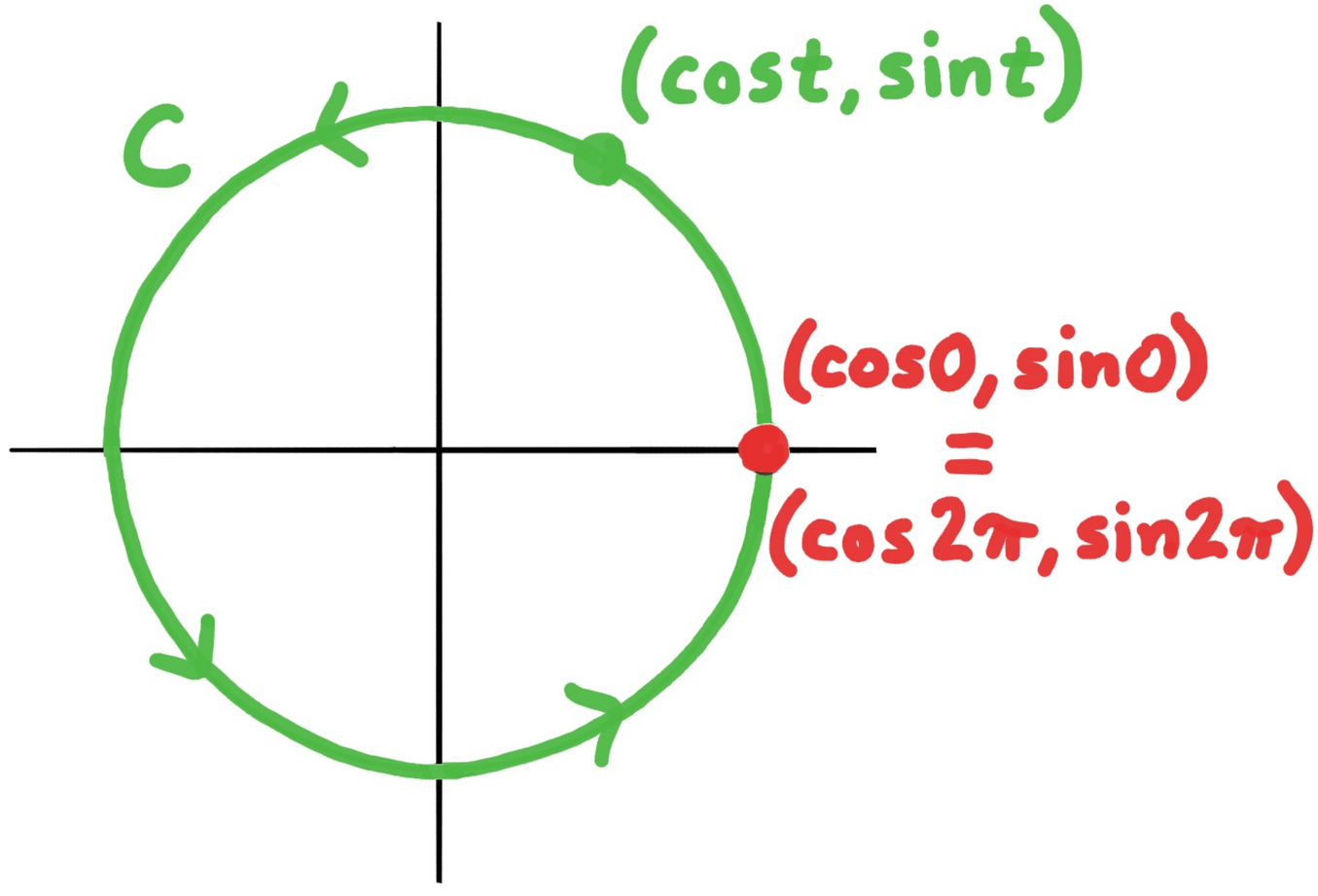
Example:

C param. by

$x = \cos t$,

$y = \sin t$,

$0 \le t \le 2\pi$.



Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

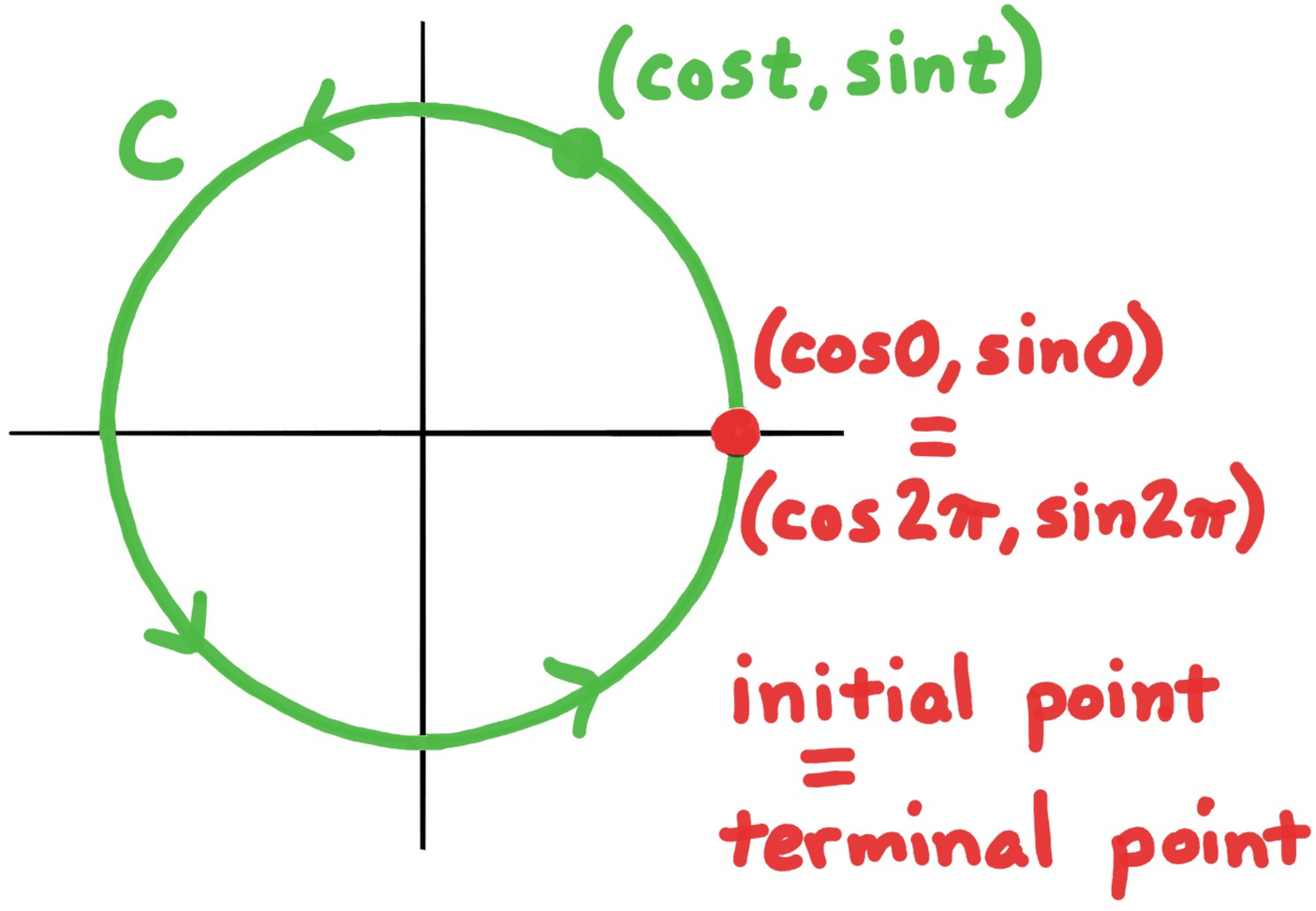
Example:

C param. by

$x = \cos t$,

$y = \sin t$,

$0 \le t \le 2\pi$.



Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

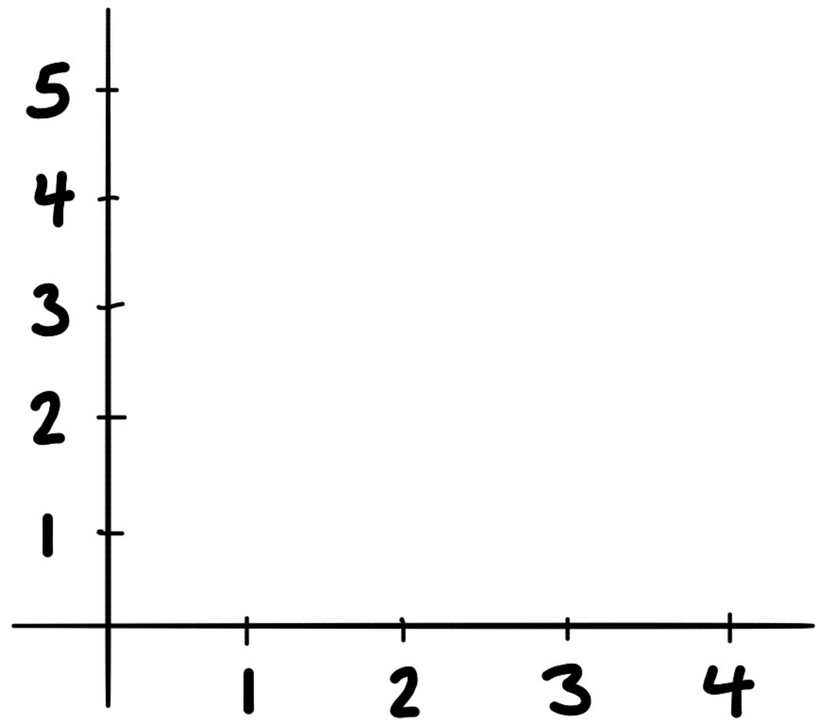
Example:

C param. by

$x = 1 + 3t$

$y = 5 - 4t$

$0 \le t \le 1.$



Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

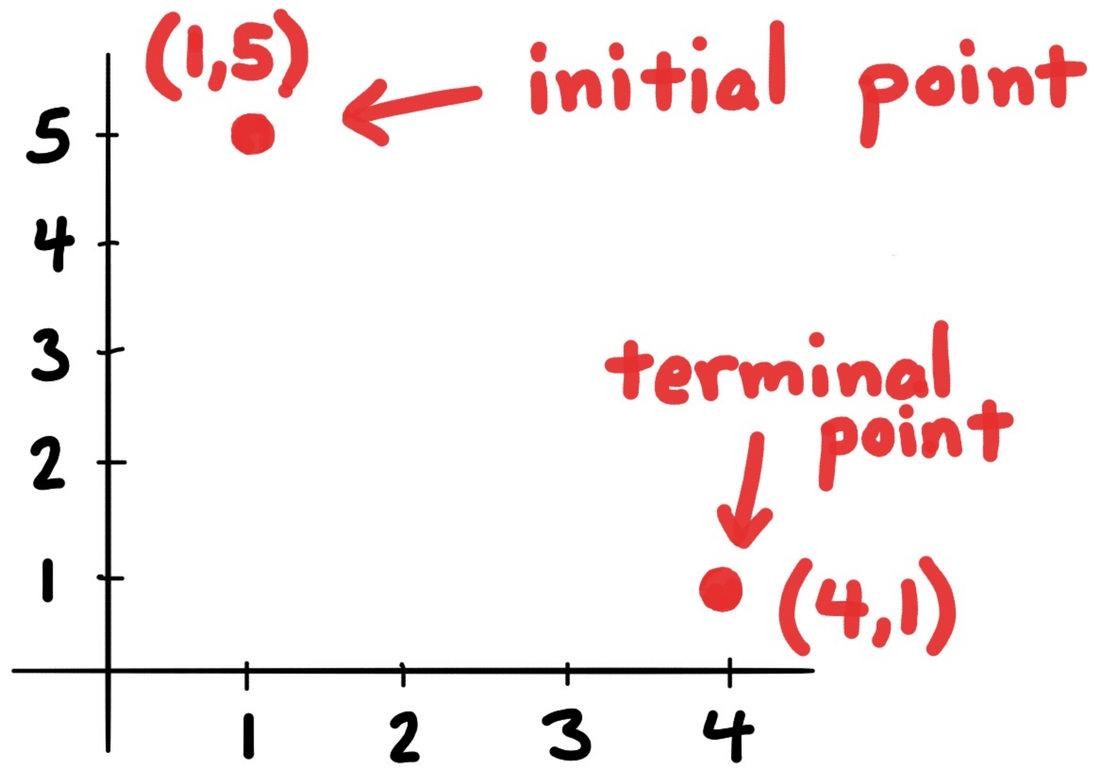
Example:

C param. by

$x = 1 + 3t$

$y = 5 - 4t$

$0 \le t \le 1.$



Convention

A curve C in $\mathbb{R}^2$ is parametrized by

$$x = x(t), \quad y = y(t), \quad a \le t \le b$$

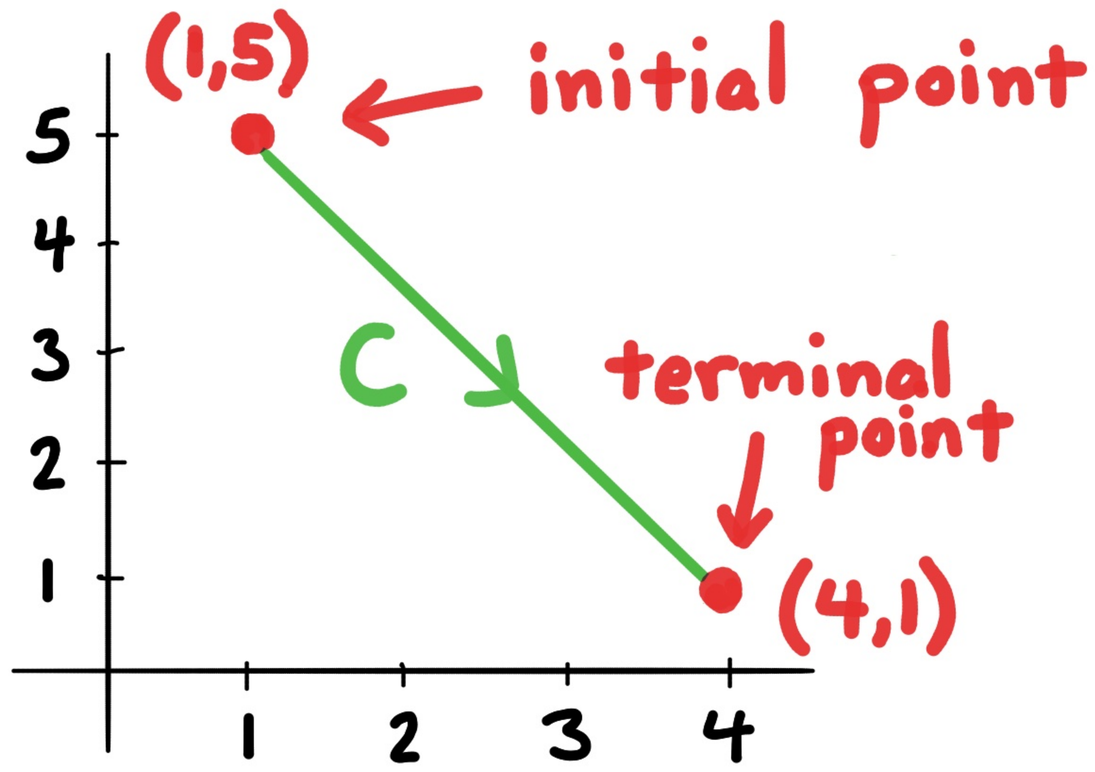
Example:

C param. by

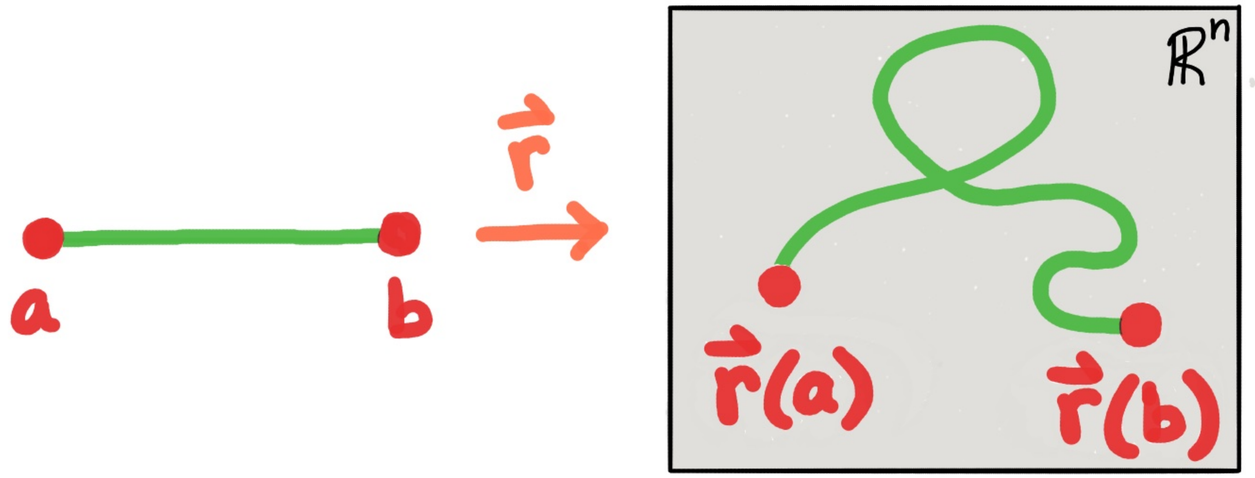
$x = 1 + 3t$

$y = 5 - 4t$

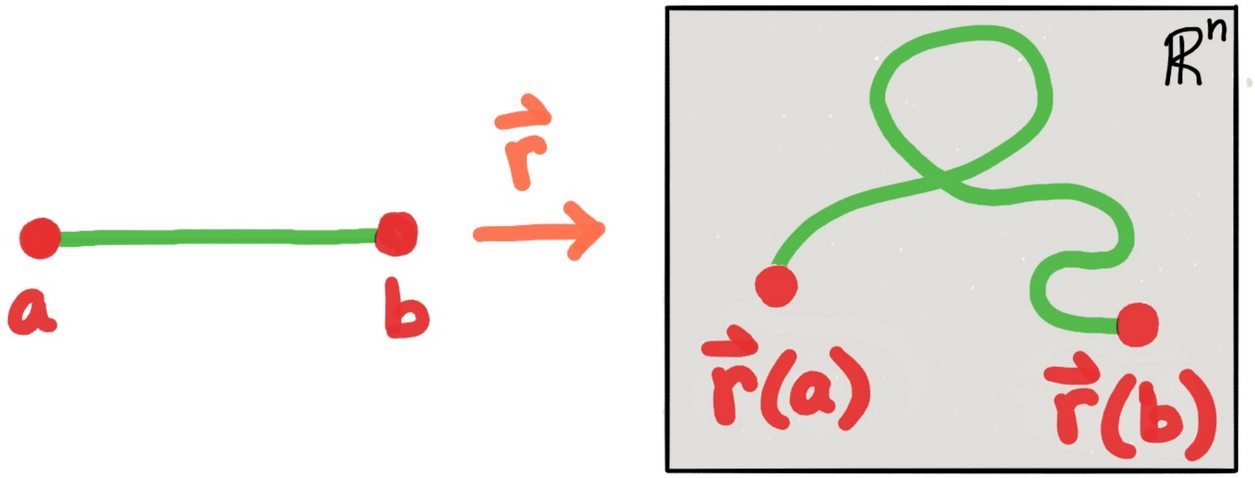
$0 \le t \le 1.$



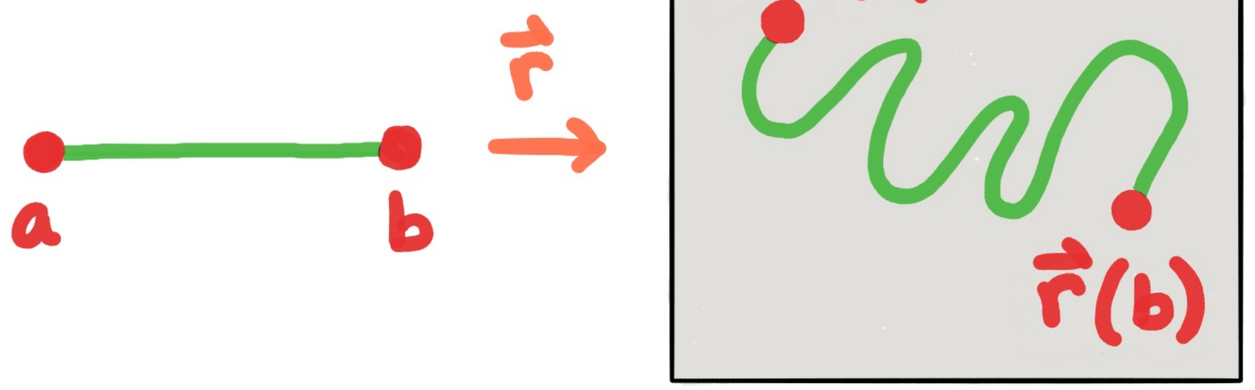
A **curve** in $\mathbb{R}^n$ is a continuous $\vec{r} : [a,b] \longrightarrow \mathbb{R}^n$.



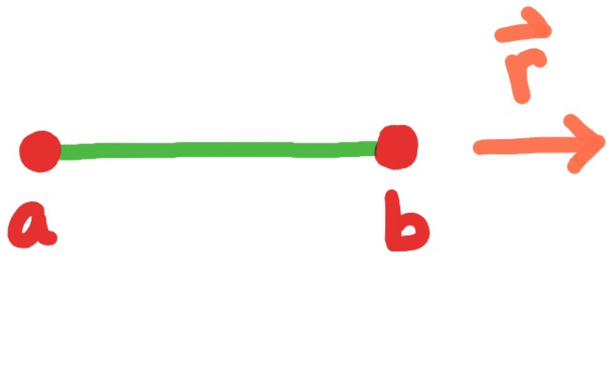
A **curve** in $\mathbb{R}^n$ is a continuous $\vec{r} : [a,b] \longrightarrow \mathbb{R}^n$.



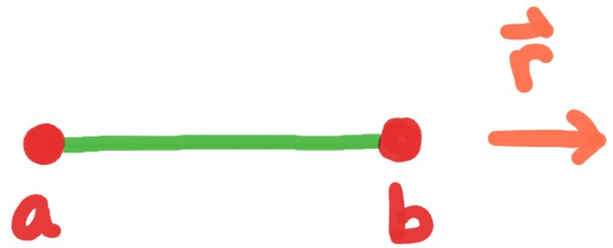
A curve is **simple** if it doesn't intersect itself.



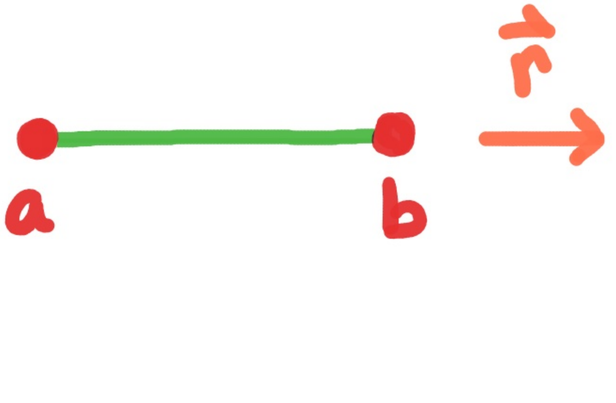
A curve in $\mathbb{R}^n$ is a continuous $\vec{r} : [a,b] \longrightarrow \mathbb{R}^n$.



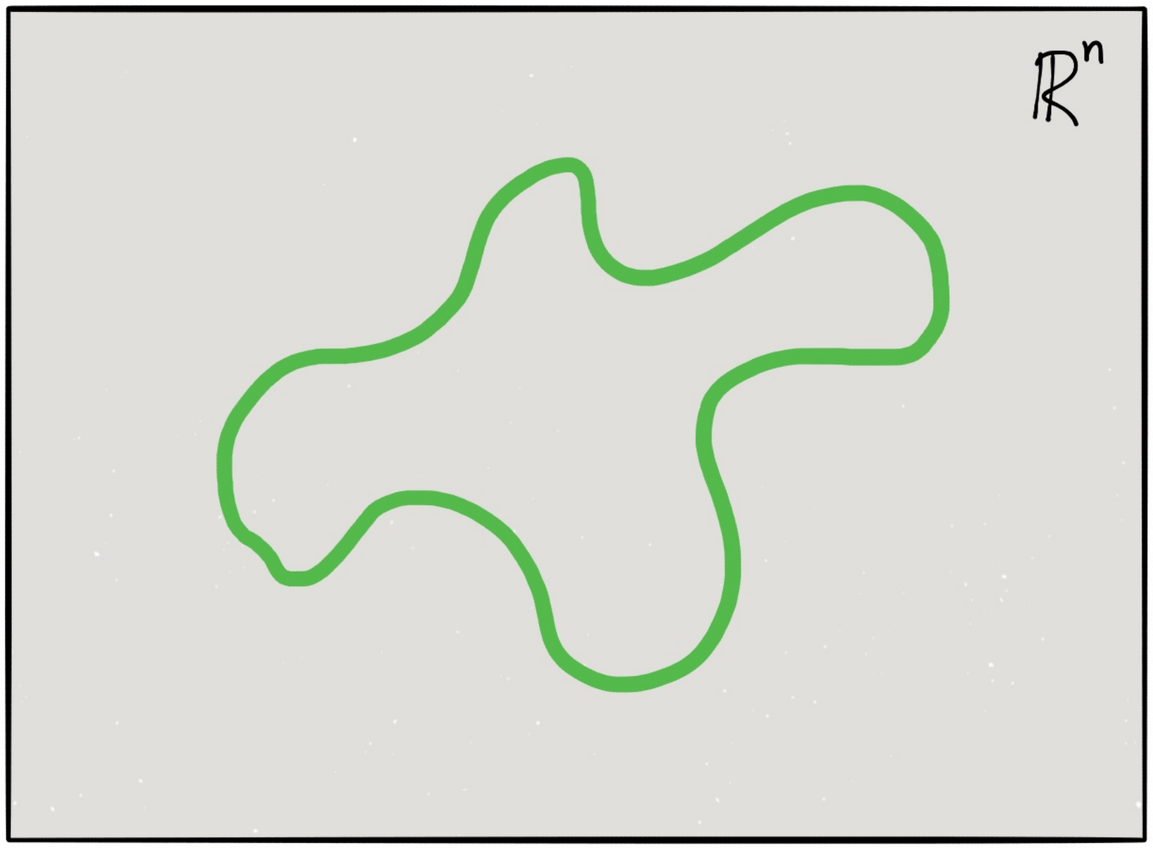
A curve is simple if it doesn't intersect itself.



A curve is closed if its endpoints are the same.



simple closed curve



"floppy circle"

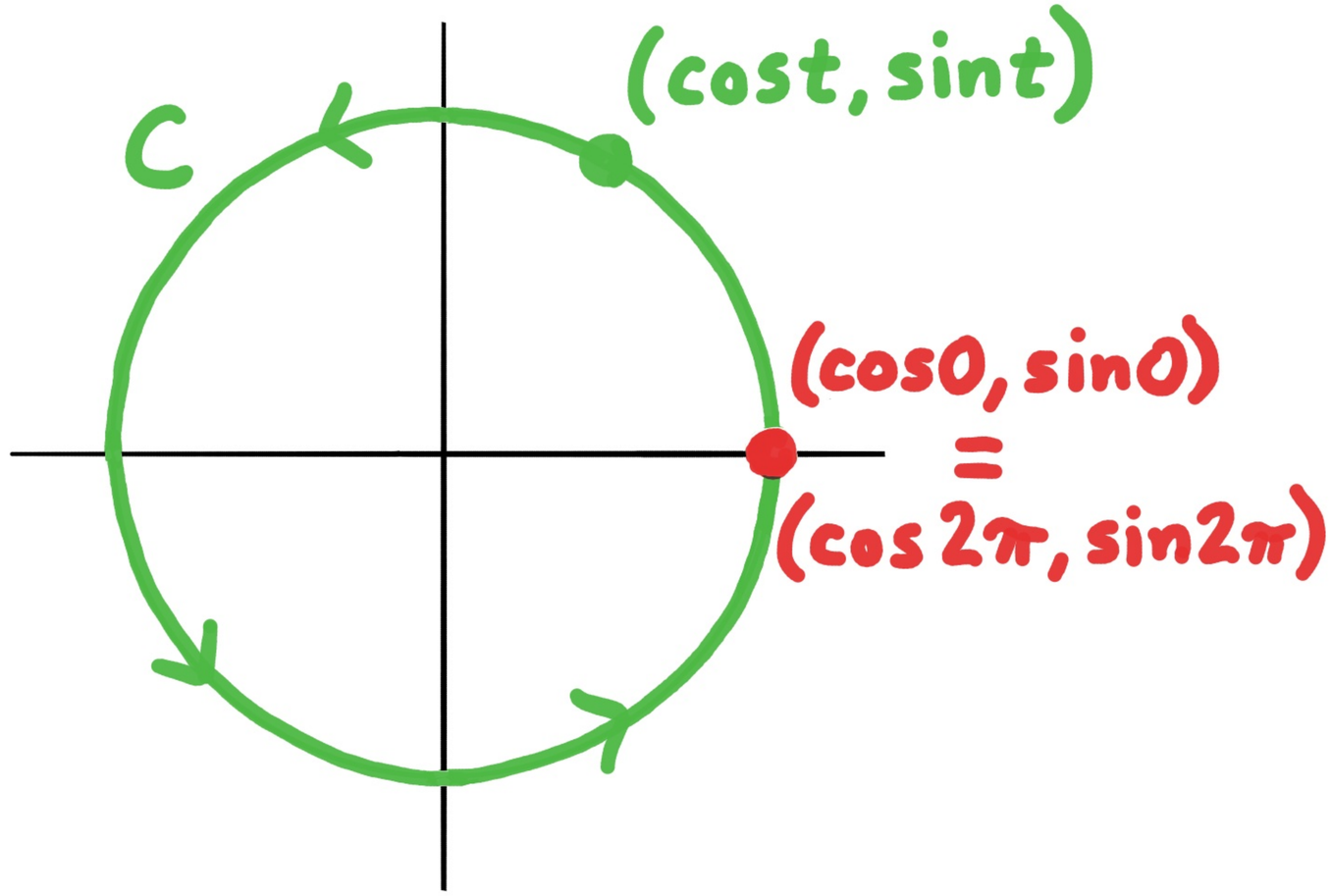
Example:

C param. by

$x = \cos t$,

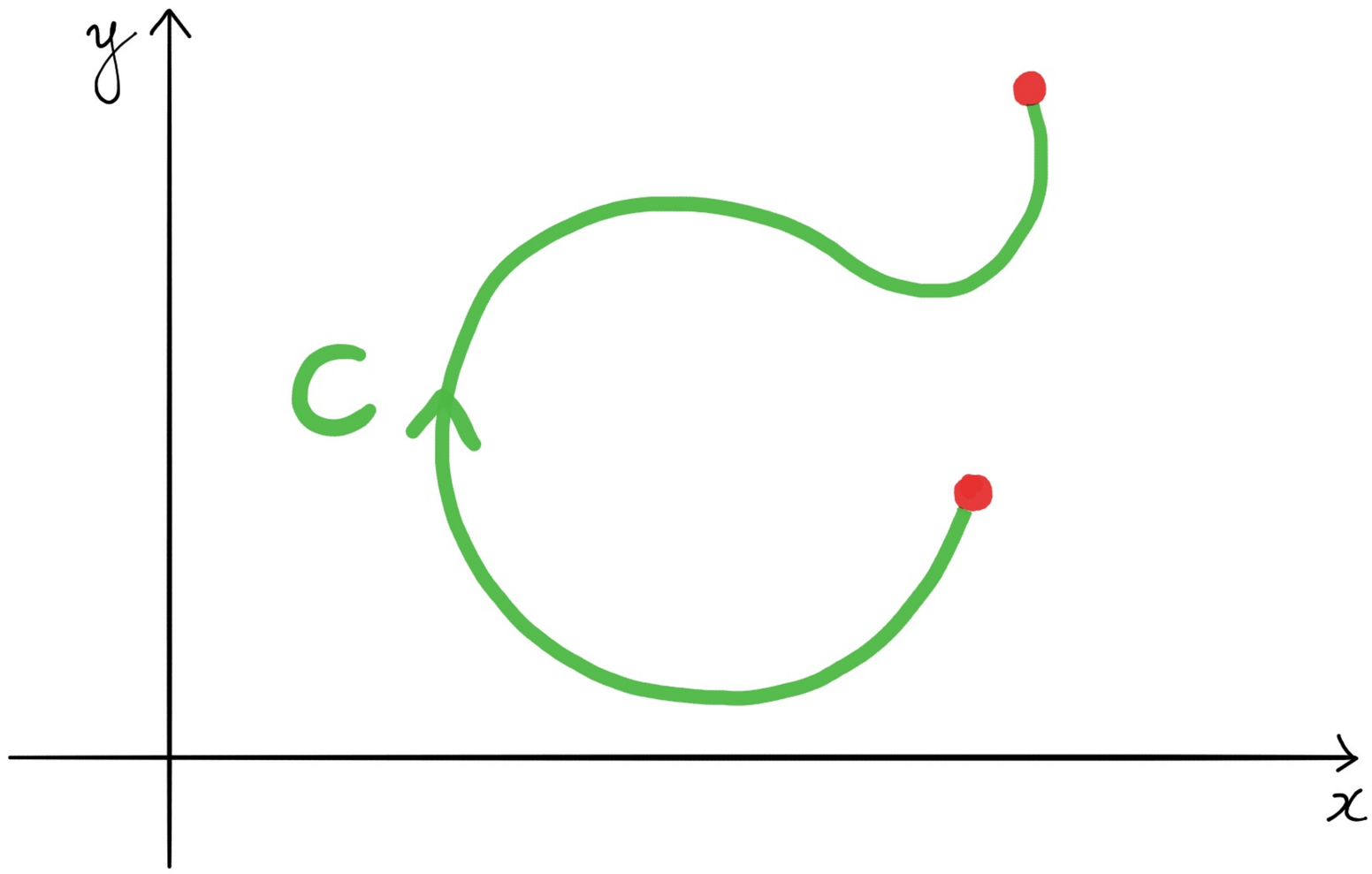
$y = \sin t$,

$0 \le t \le 2\pi$.

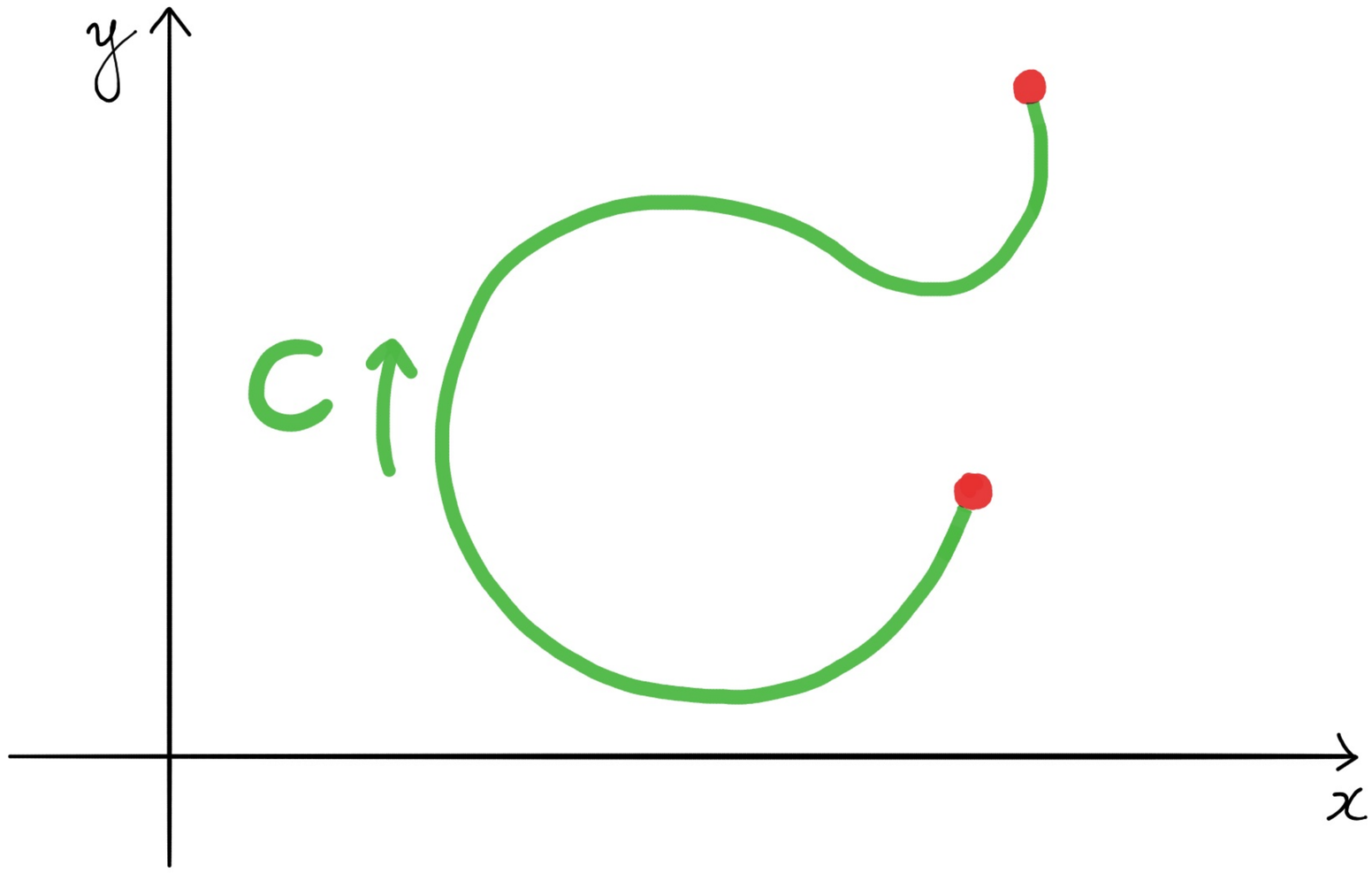


C is a simple closed curve.

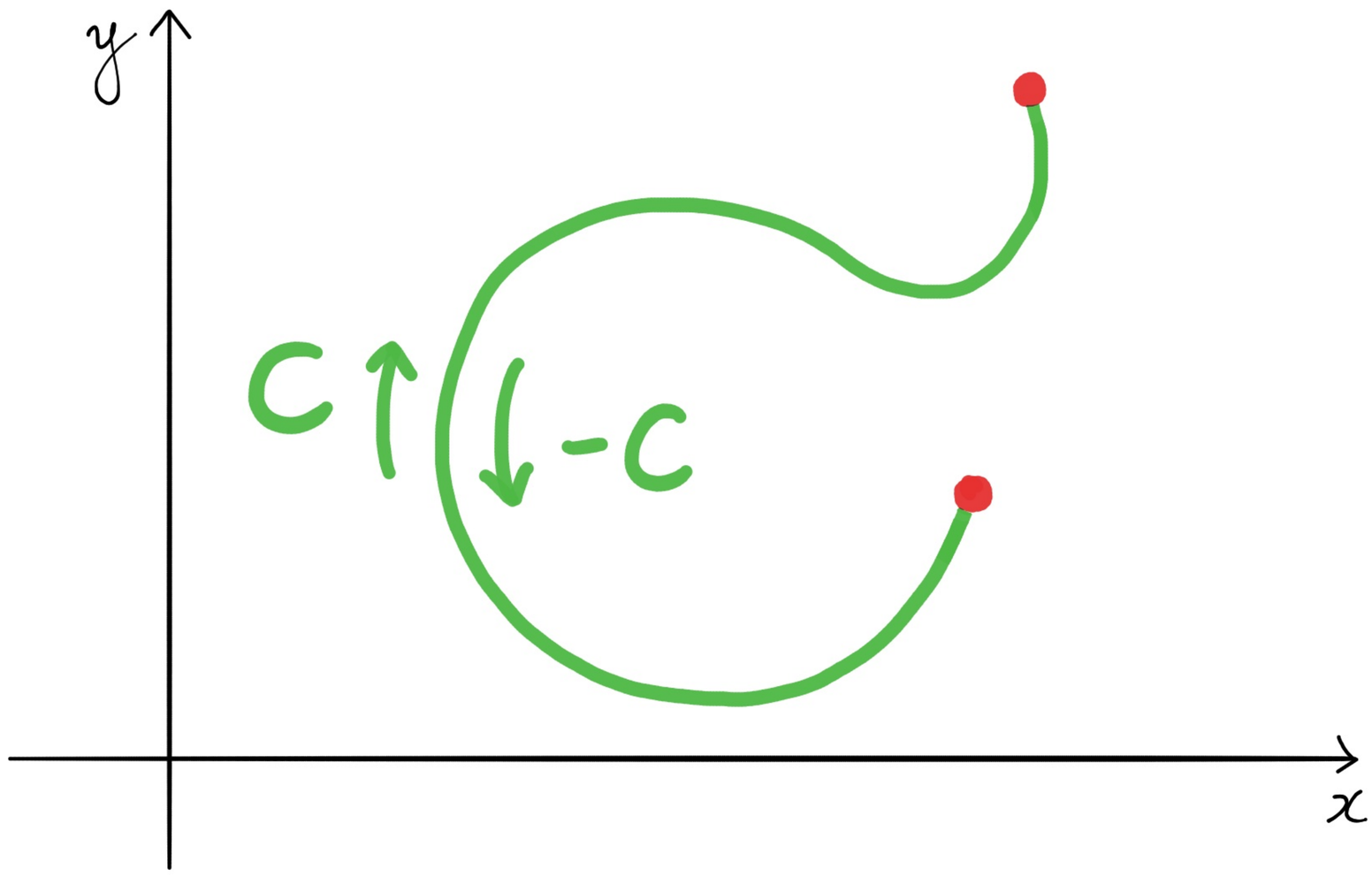
Negatives of curves



Negatives of curves

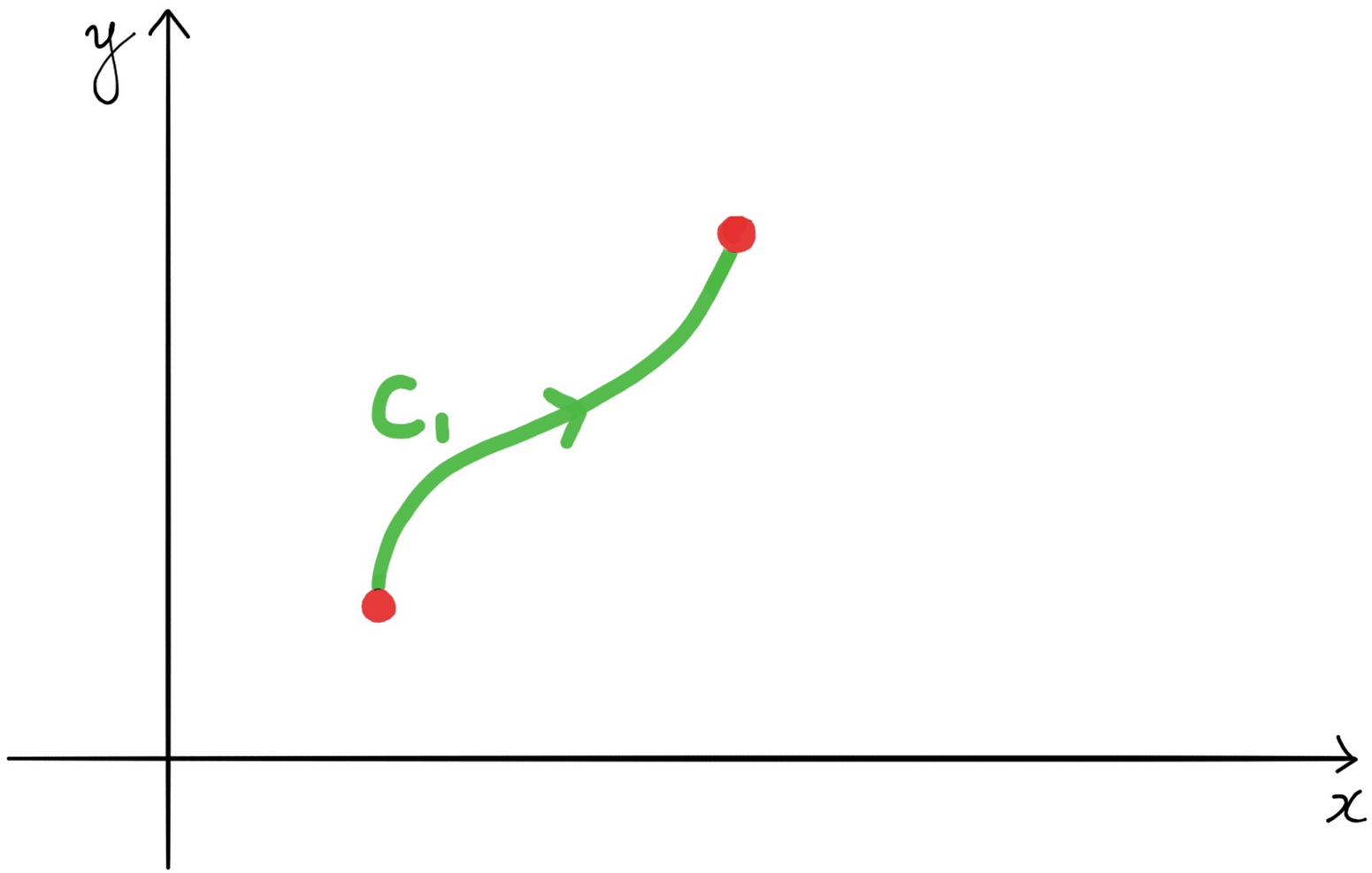


Negatives of curves

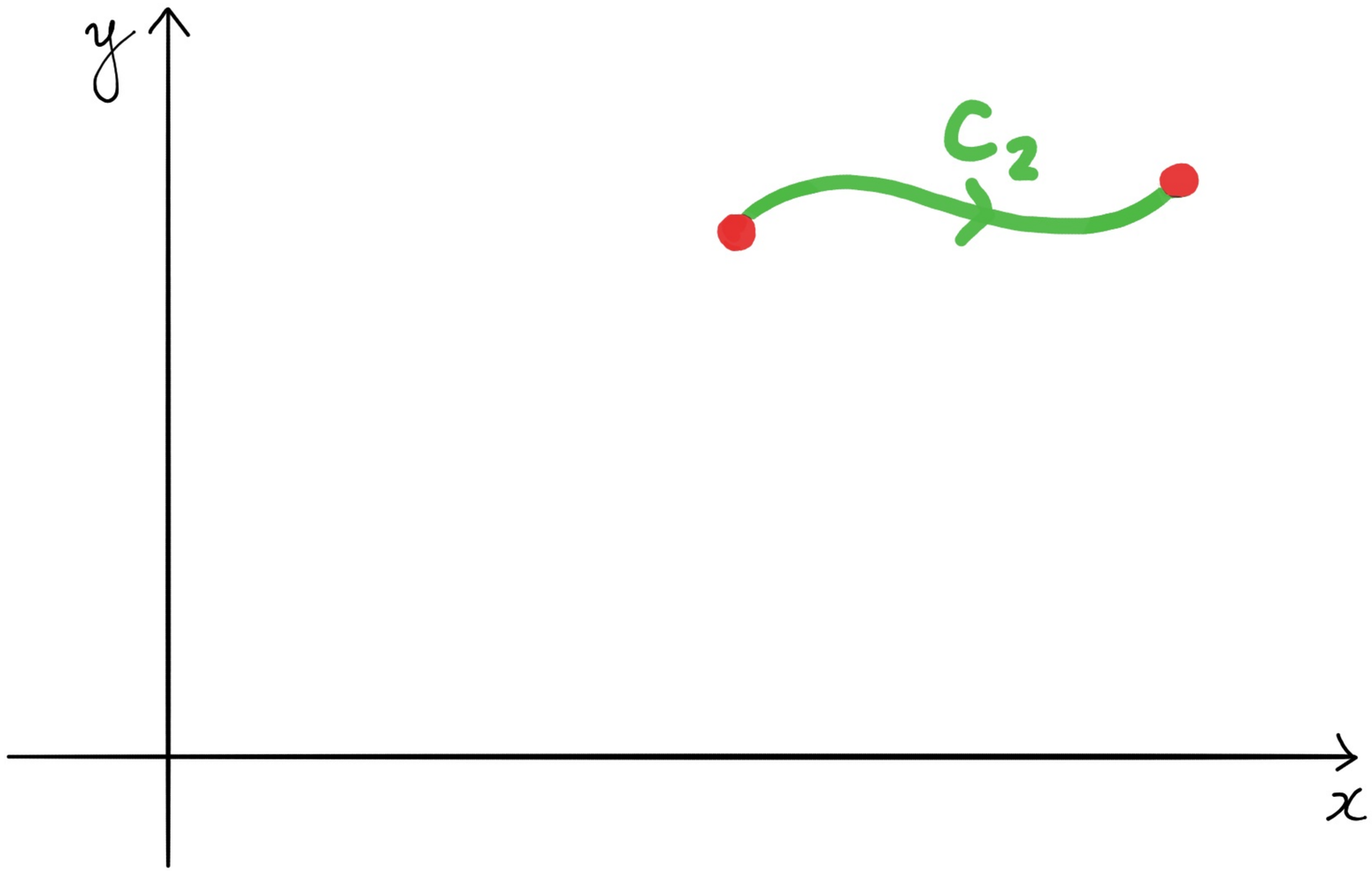


-C is C traversed backwards.

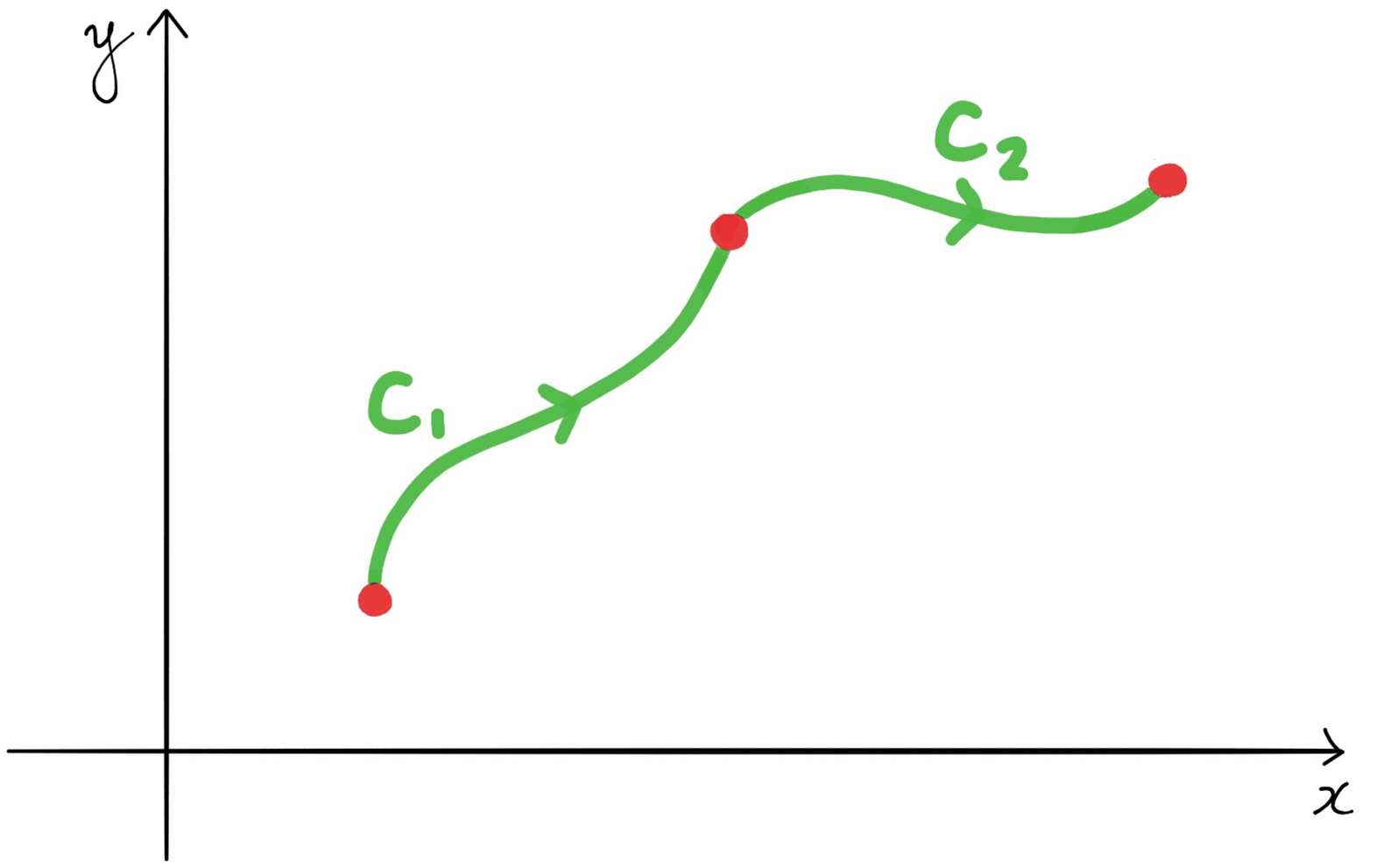
Unions of curves



Unions of curves

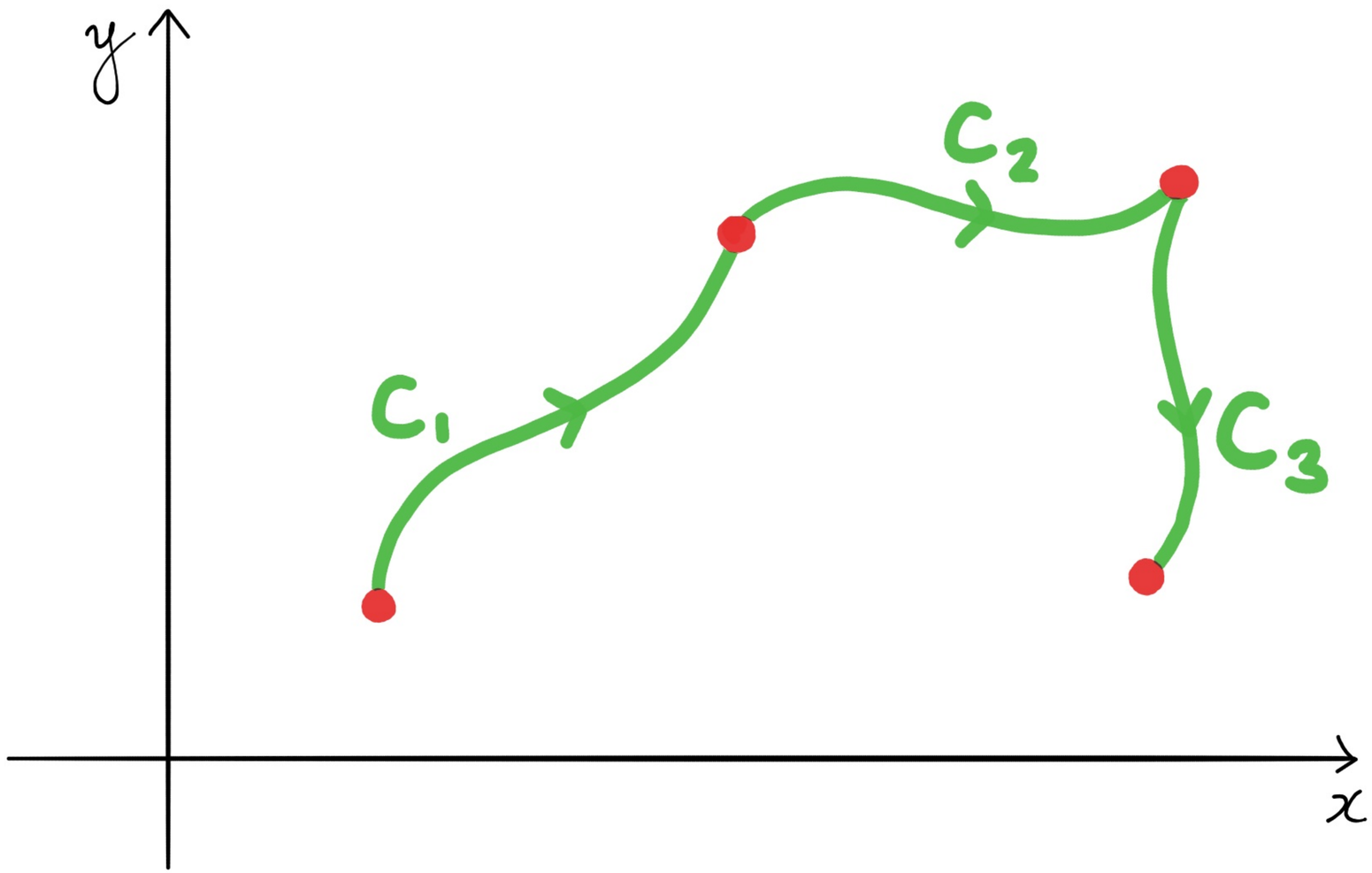


Unions of curves



$C_1 \cup C_2$ means the curve obtained by linking C_1 and C_2.

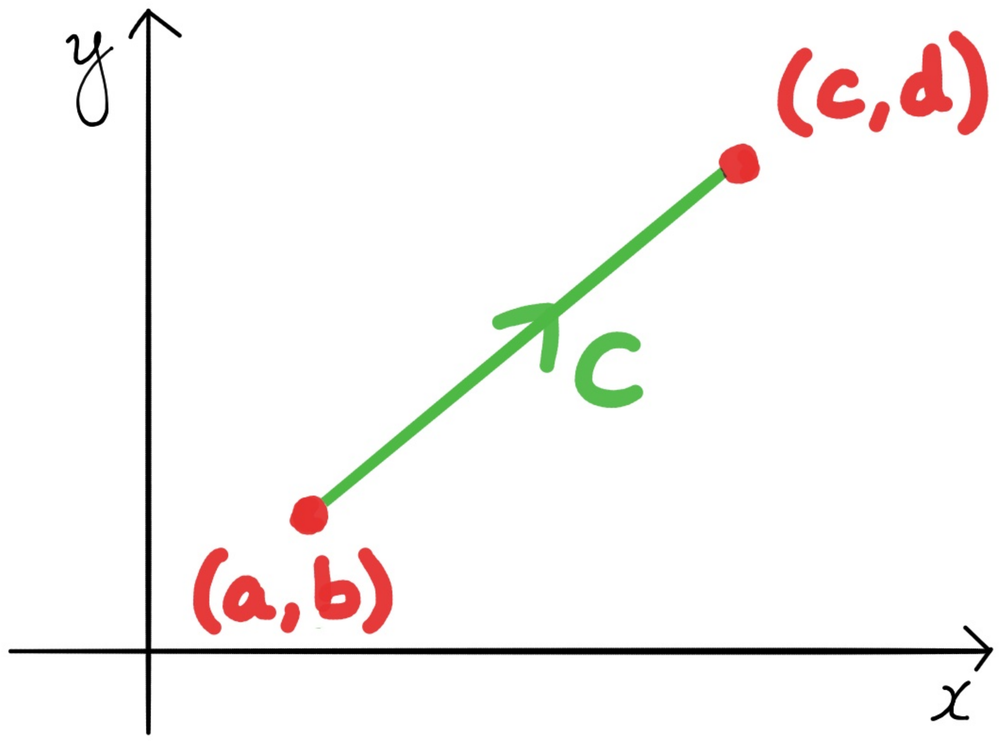
Unions of curves



$$C_1 \cup C_2 \cup C_3$$

Ⅲ Parametrizing curves

Parametrizing straight lines

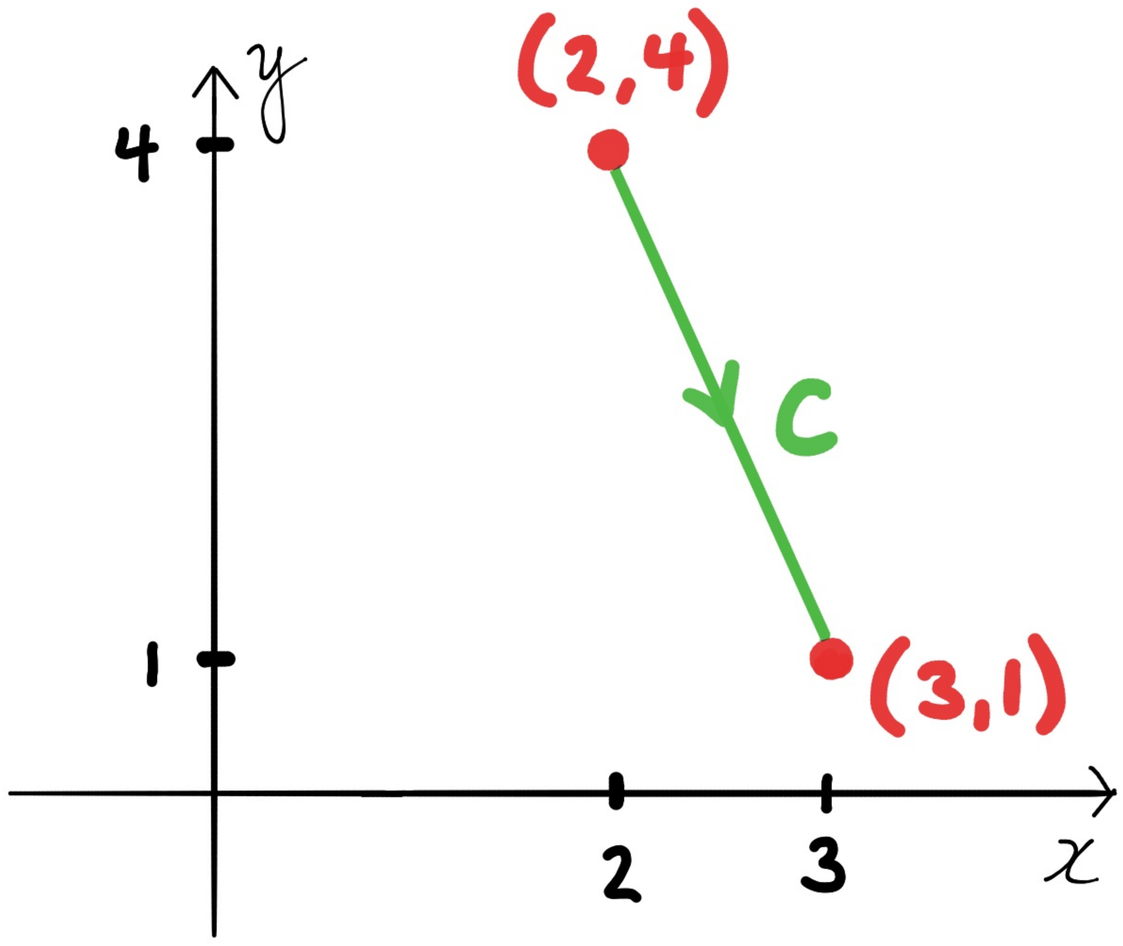


C: $0 \le t \le 1$

$x = a + (c-a)t$

$y = b + (d-b)t$

Example:

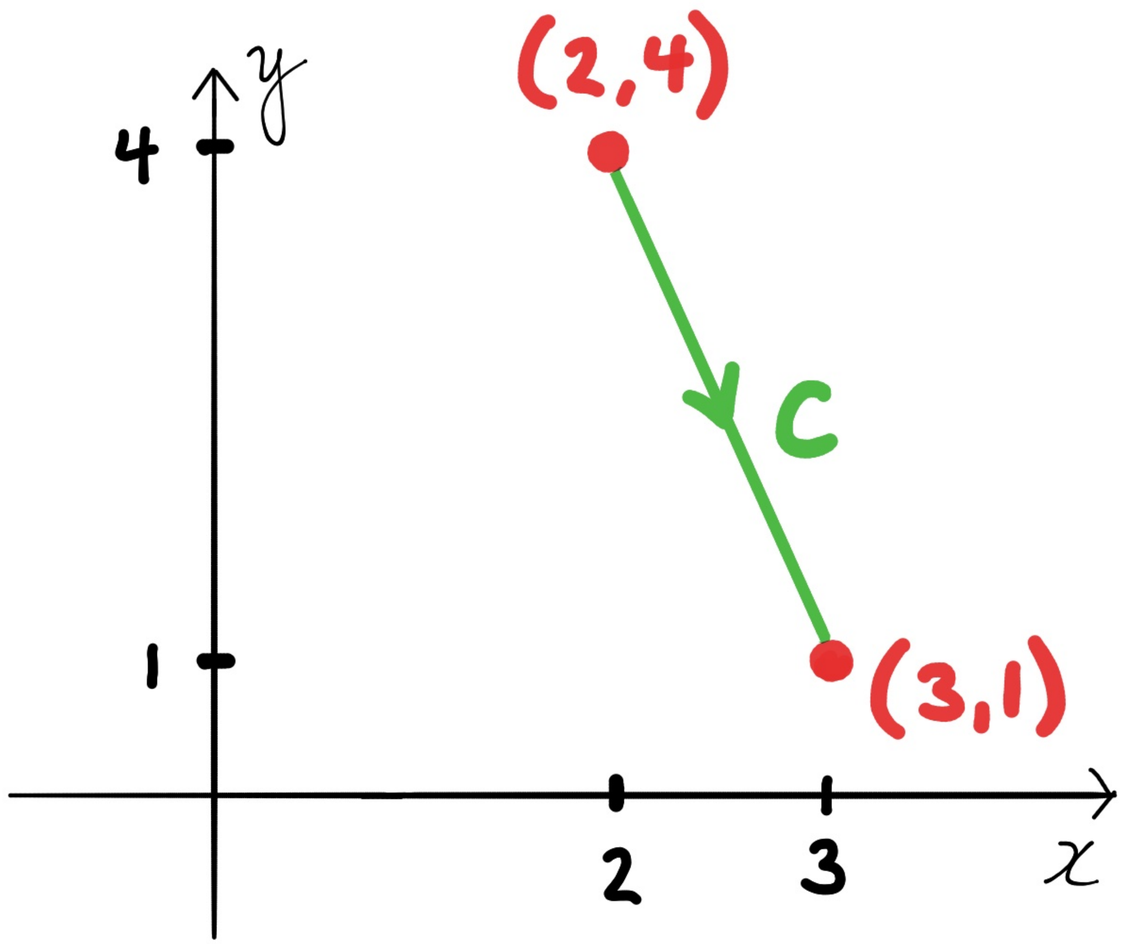


C: $0 \le t \le 1$

$x = 2 + (3-2)t$

$y = 4 + (1-4)t$

Example:

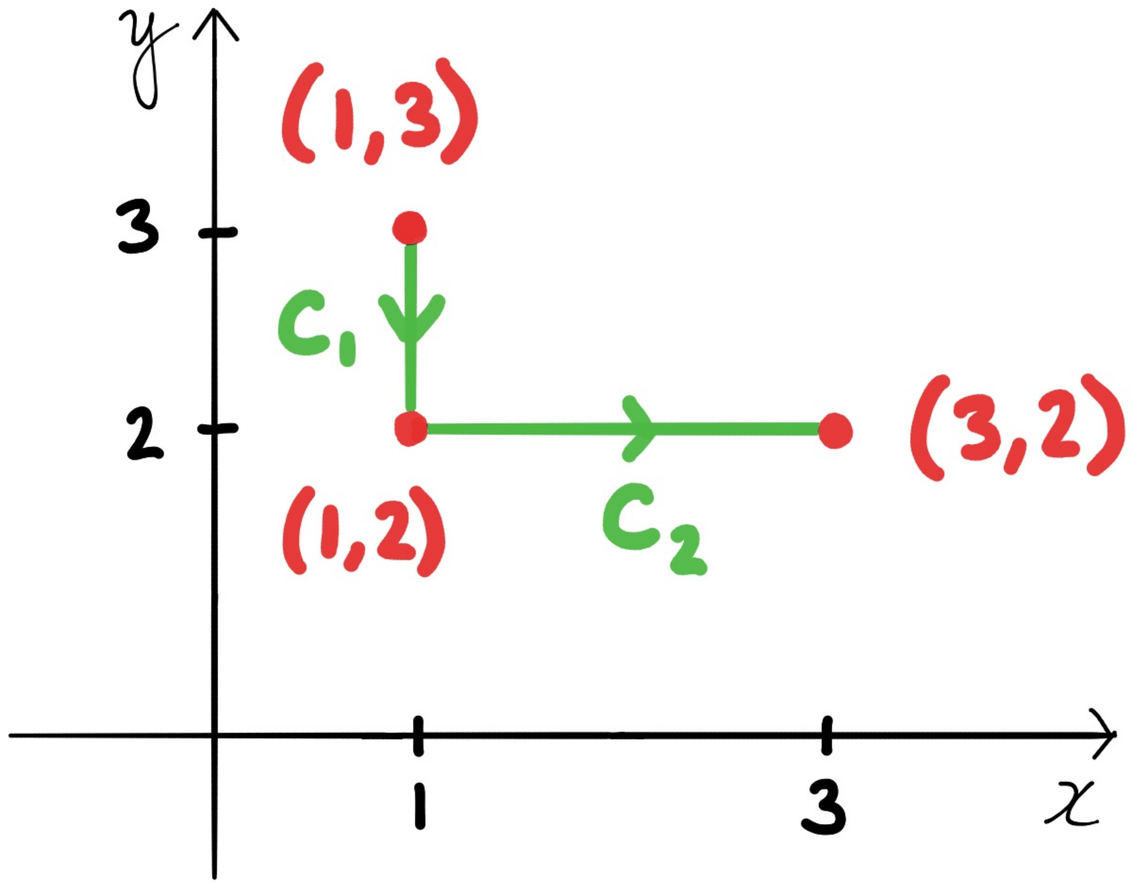


C: $0 \le t \le 1$

$x = 2 + t$

$y = 4 - 3t$

Example:



$$C = C_1 \cup C_2$$

C_1: $0 \le t \le 1$

$x = 1$

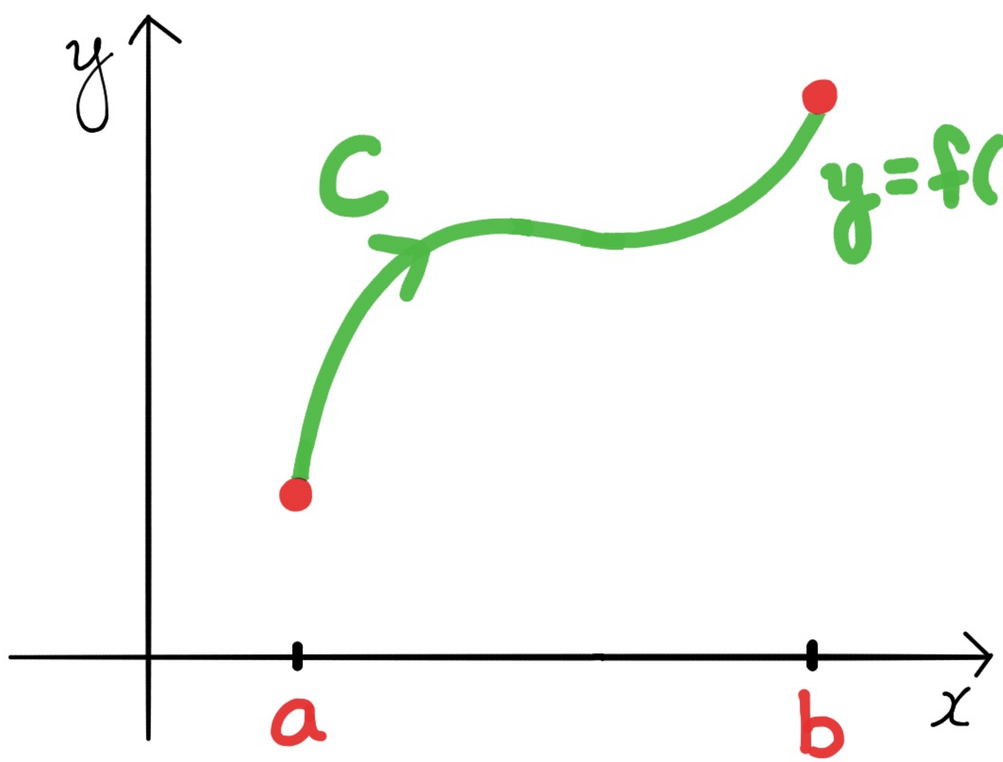
$y = 3 - t$

C_2: $0 \le t \le 1$

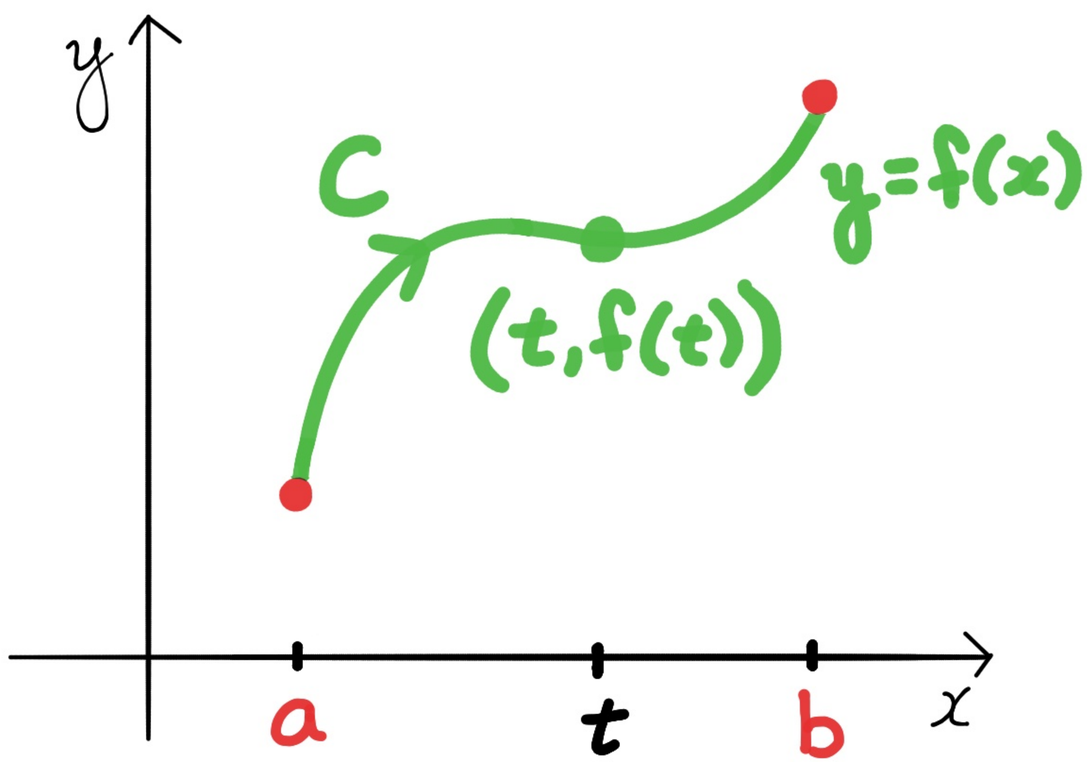
$x = 1 + 2t$

$y = 2$

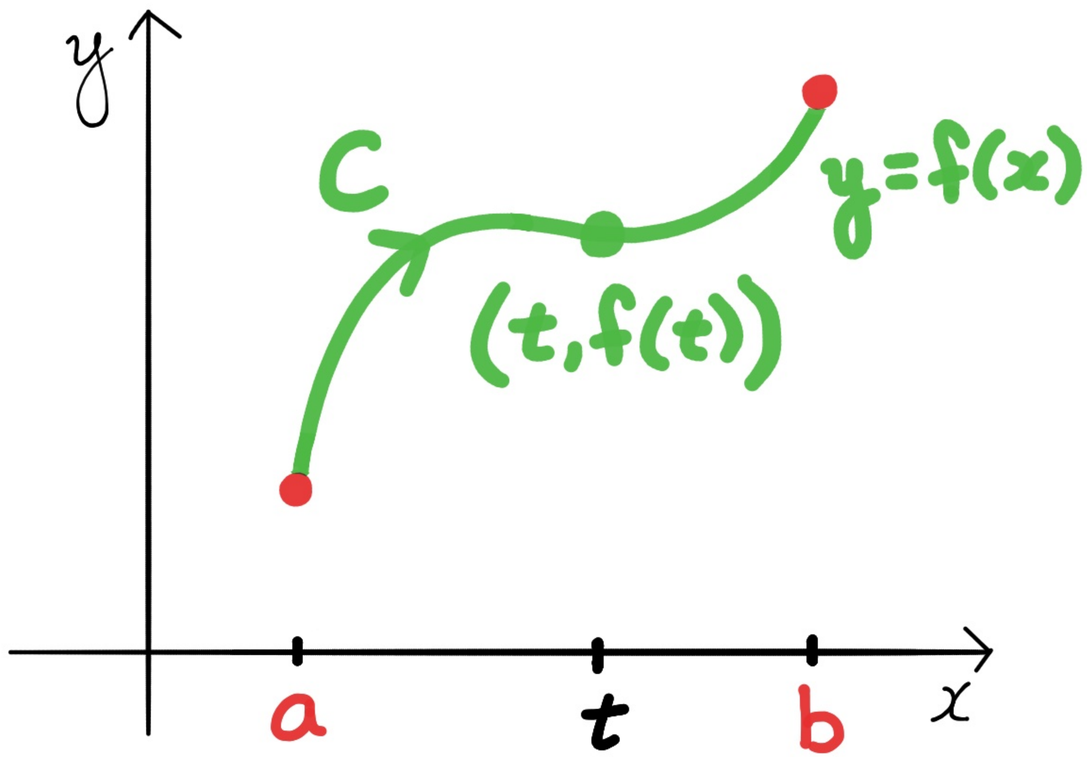
Parametrizing $y=f(x)$ graphs



Parametrizing $y=f(x)$ graphs



Parametrizing $y=f(x)$ graphs

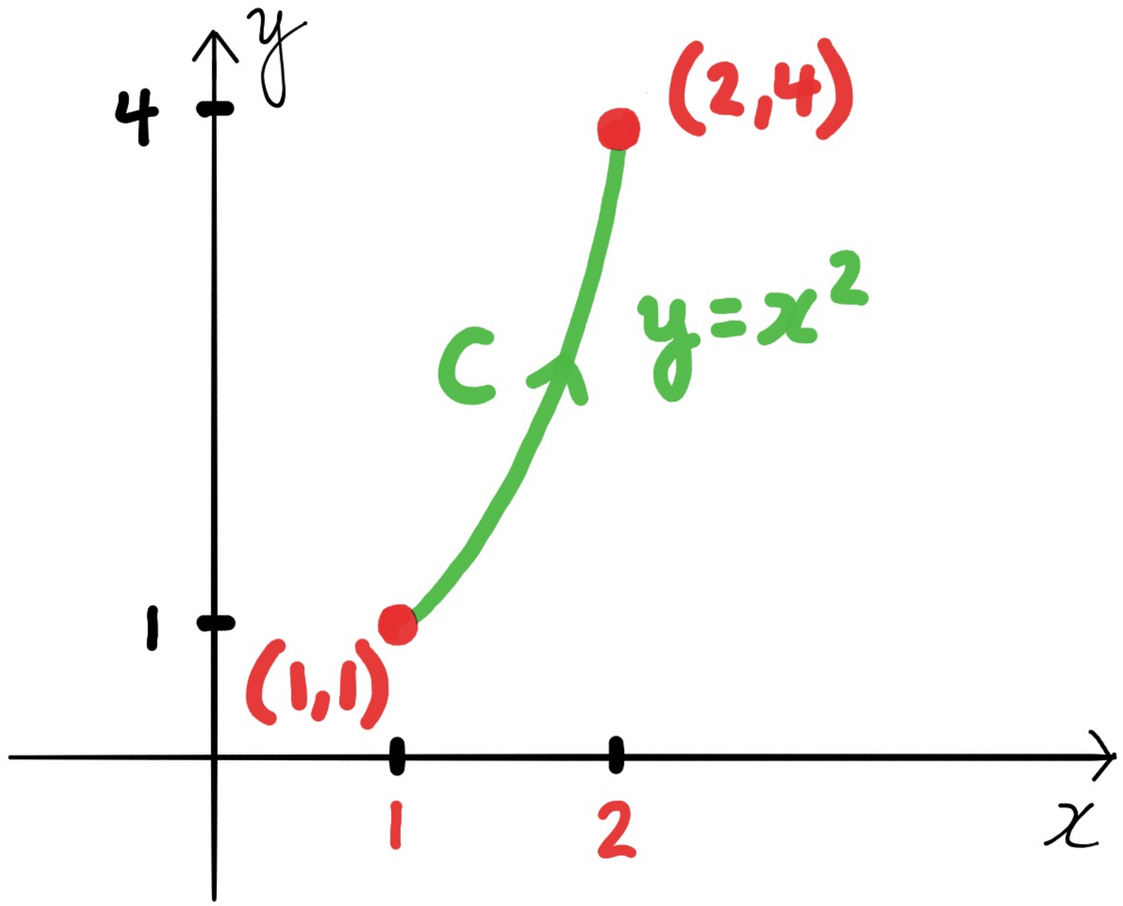


C: $a \le t \le b$

$x = t$

$y = f(t)$

Example:

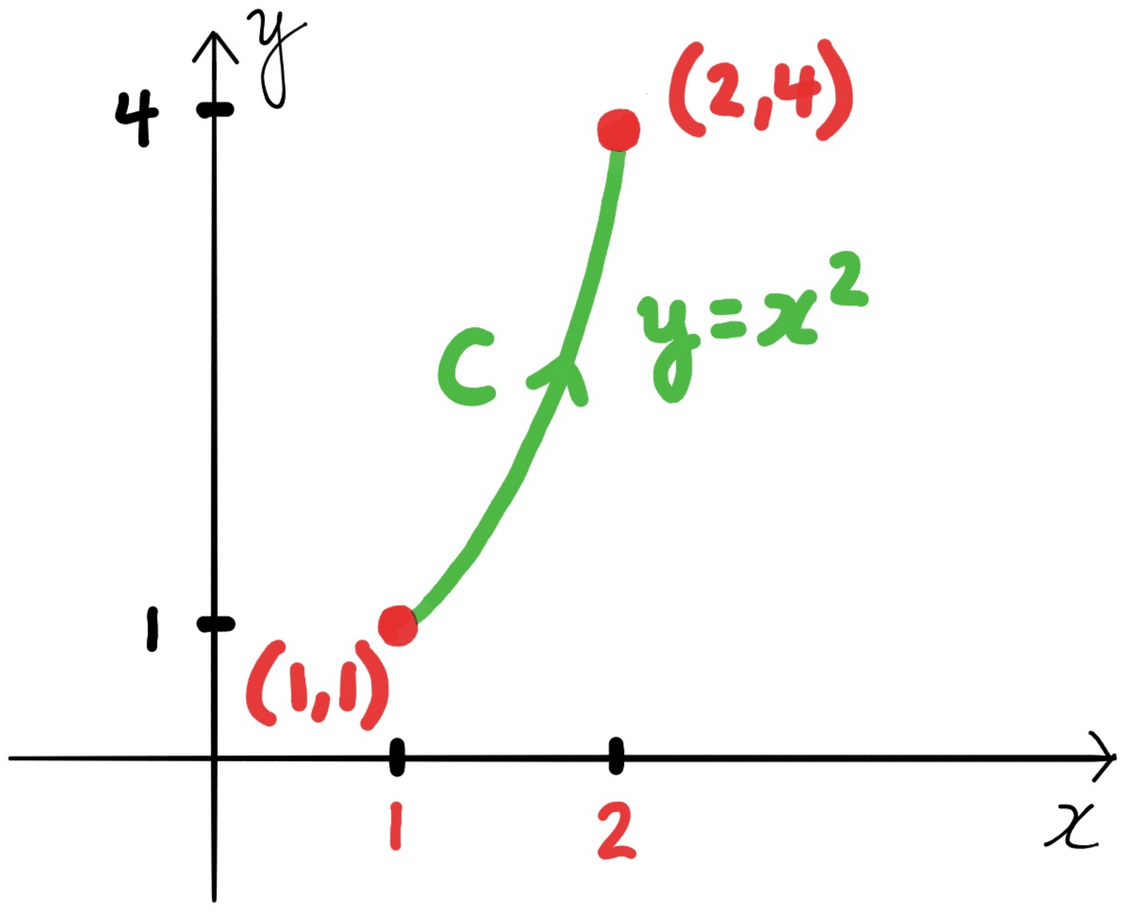


C: $1 \le t \le 2$

$x = t$

$y = f(t)$

Example:

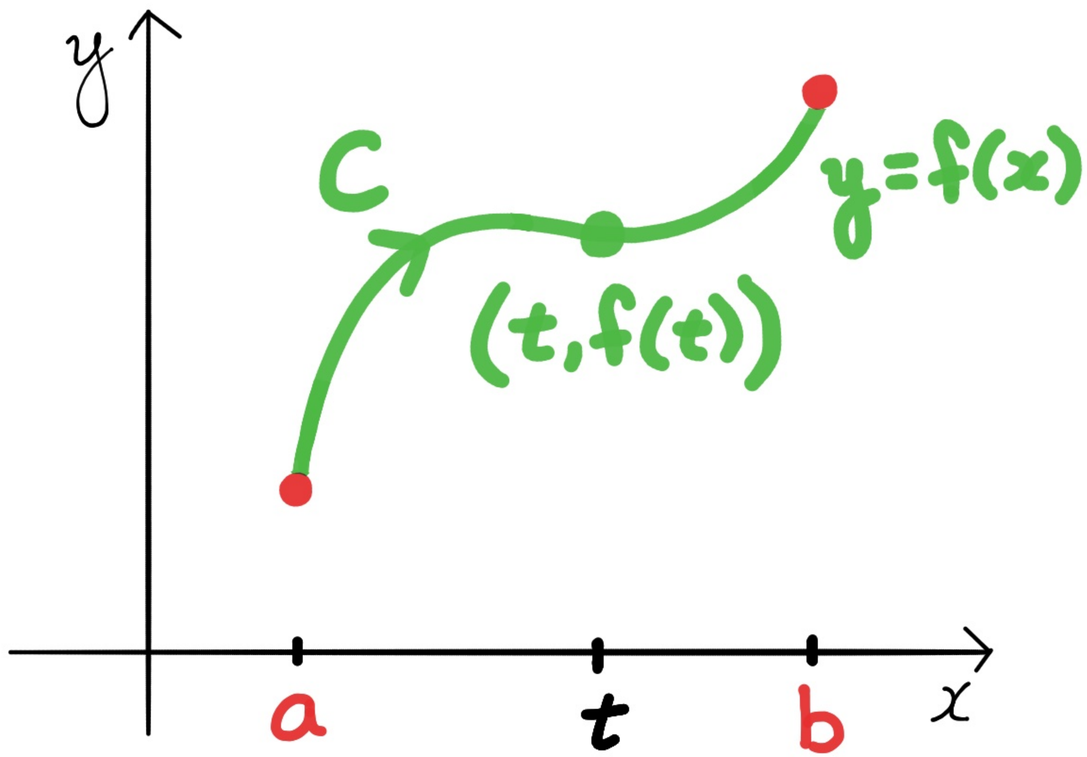


C: $1 \le t \le 2$

$x = t$

$y = t^2$

Parametrizing $y=f(x)$ graphs

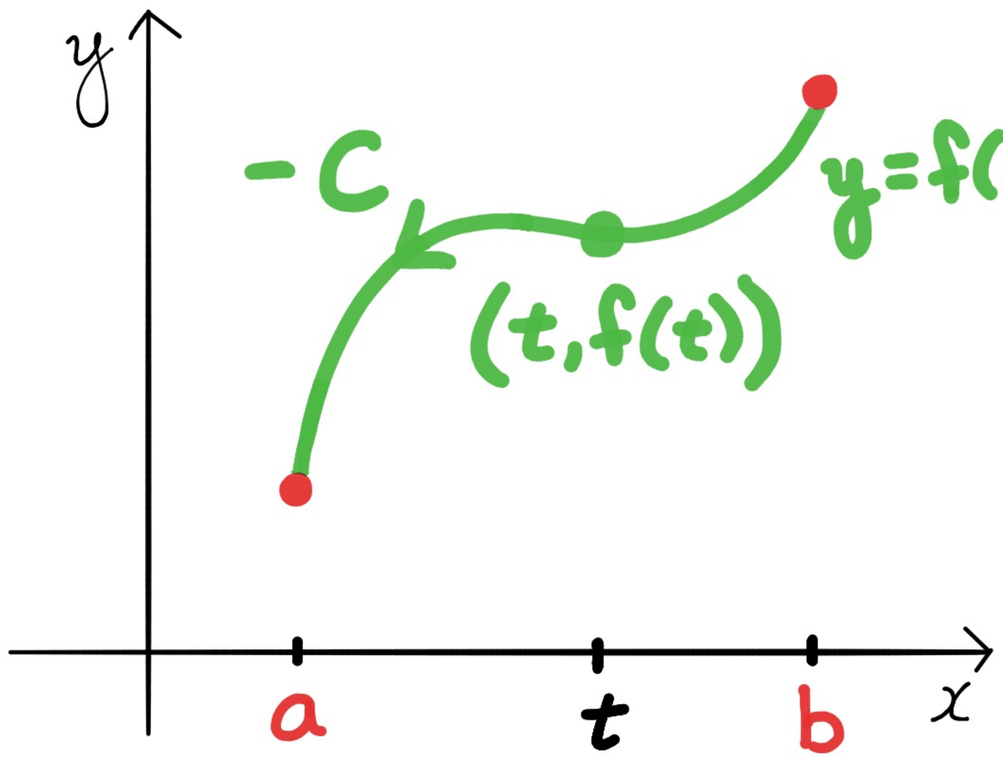


C: $a \le t \le b$

$x = t$

$y = f(t)$

Parametrizing $y=f(x)$ graphs

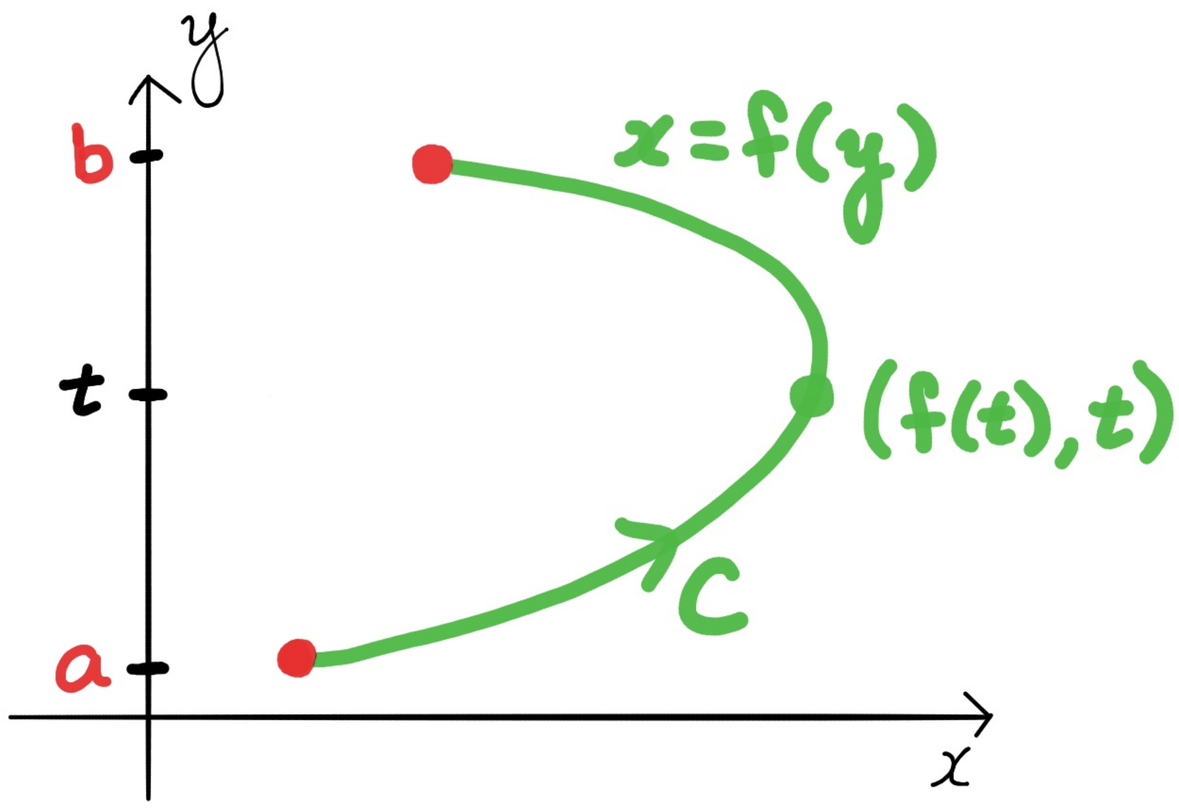


C: $a \le t \le b$

$x = t$

$y = f(t)$

Parametrizing $x=f(y)$ graphs

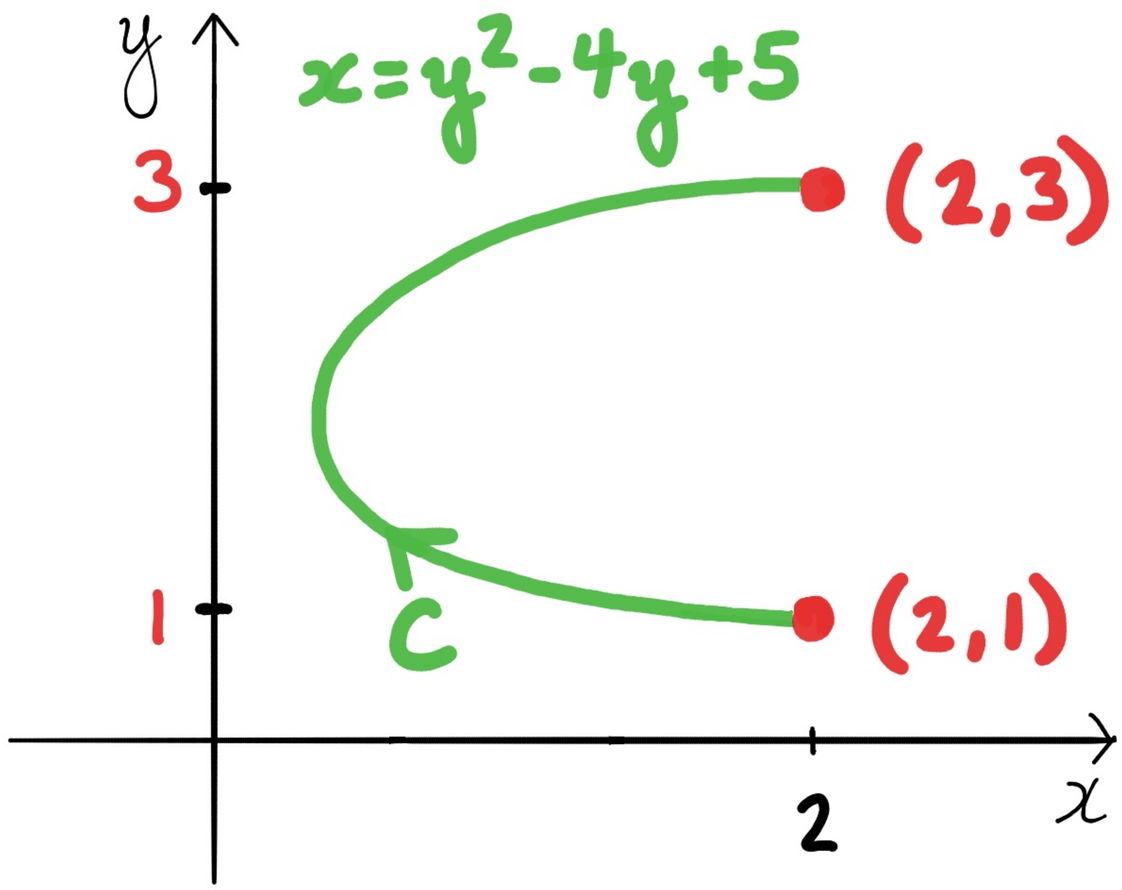


C: $a \le t \le b$

$x = f(t)$

$y = t$

Example:

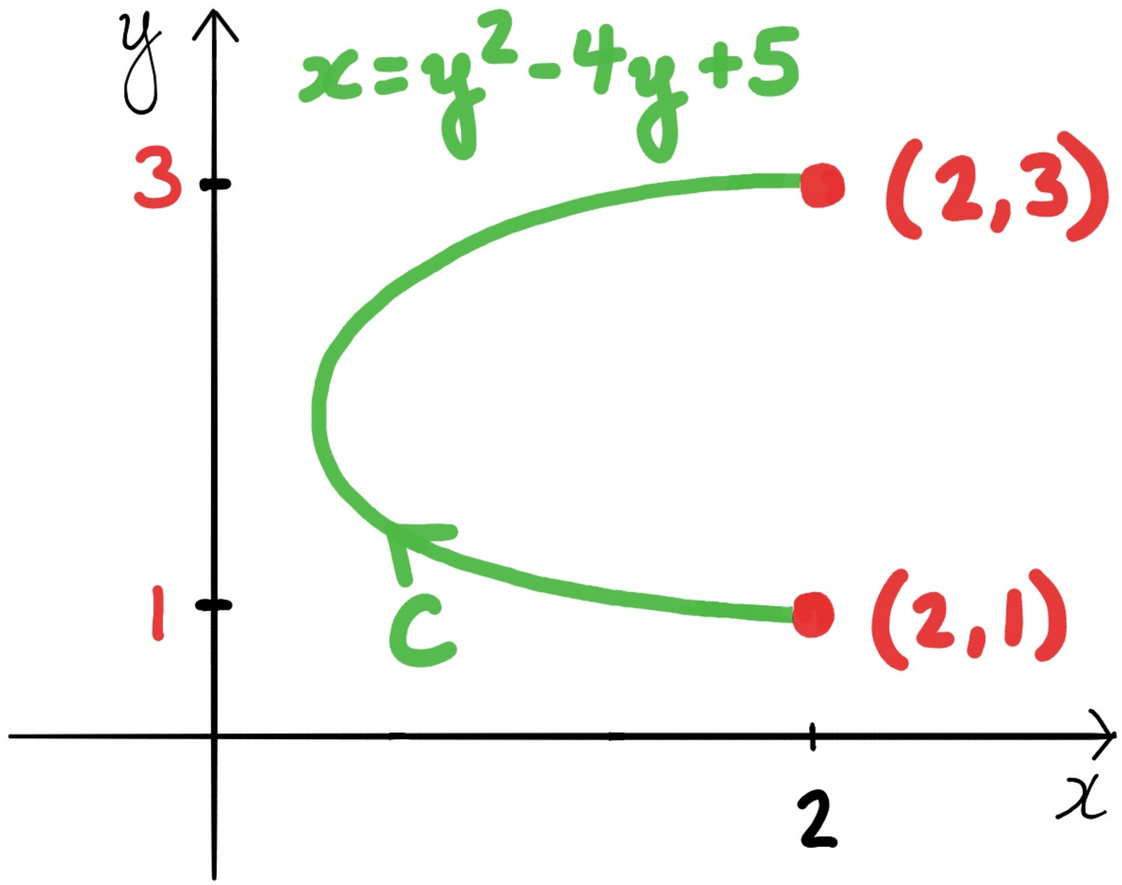


C: $1 \le t \le 3$

$x = f(t)$

$y = t$

Example:



C: $1 \le t \le 3$

$x = t^2 - 4t + 5$

$y = t$

Ⅳ Curl

Curl of a vector field

Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F} = (F_1, F_2, F_3)$

curl $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$\operatorname{curl} \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Curl of a vector field

Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F} = (F_1, F_2, F_3)$

curl $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$\text{curl } \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Example: $\vec{F}(x,y,z) = (-y, x, xyz)$

Curl of a vector field

Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F} = (F_1, F_2, F_3)$

curl $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$\operatorname{curl} \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Example: $\vec{F}(x,y,z) = (-y, x, xyz)$

$$\operatorname{curl} \vec{F}(x,y,z) = (xz - 0, 0 - yz, 1 + 1)$$

Curl of a vector field

Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F} = (F_1, F_2, F_3)$

curl $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$\operatorname{curl} \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Example: $\vec{F}(x,y,z) = (-y, x, xyz)$

$$\operatorname{curl} \vec{F}(x,y,z) = (xz, -yz, 2)$$

Lemma: For $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl} \vec{F} = \vec{0}$ exactly when $\vec{F}$ is conservative.

Lemma: For $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl} \vec{F} = \vec{0}$ exactly when $\vec{F}$ is conservative.

proof:

Lemma: For $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl} \vec{F} = \vec{0}$ exactly when $\vec{F}$ is conservative.

proof:

$\vec{F}$ is conservative when

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \text{ and } \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Lemma: For $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl}\vec{F} = \vec{0}$ exactly when $\vec{F}$ is conservative.

proof:

$\vec{F}$ is conservative when

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \text{ and } \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Hence,

$$\operatorname{curl}\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Lemma: For $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl} \vec{F} = \vec{0}$ exactly when $\vec{F}$ is conservative.

proof:

$\vec{F}$ is conservative when

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \text{and} \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Hence,

$$\operatorname{curl} \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Lemma: For $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl}\vec{F} = \vec{0}$ exactly when $\vec{F}$ is conservative.

proof:

$\vec{F}$ is conservative when

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \text{and} \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Hence,

$$\operatorname{curl}\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = (0,0,0)$$

Lemma: For $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl} \vec{F} = \vec{0}$ exactly when $\vec{F}$ is conservative.

proof:

$\vec{F}$ is conservative when

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \text{and} \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Hence,

$$\operatorname{curl} \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = (0,0,0) = \vec{0}.$$

Lemma: For $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\operatorname{curl}\vec{F} = \vec{0}$ exactly when $\vec{F}$ is conservative.

proof:

$\vec{F}$ is conservative when

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \text{and} \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

Hence,

$$\operatorname{curl}\vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = (0,0,0) = \vec{0}.$$

END