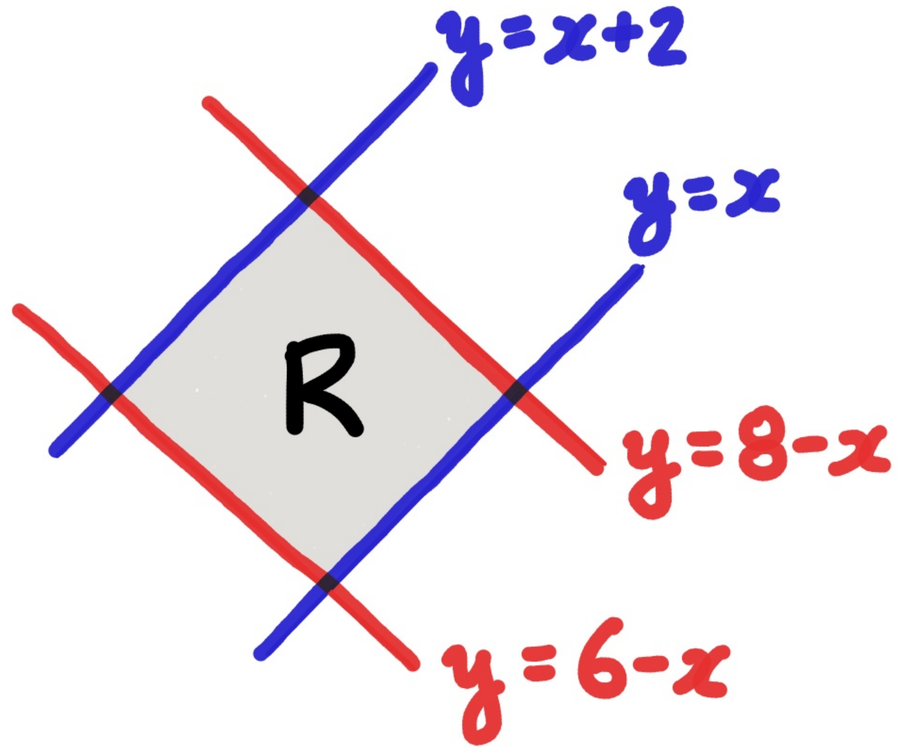
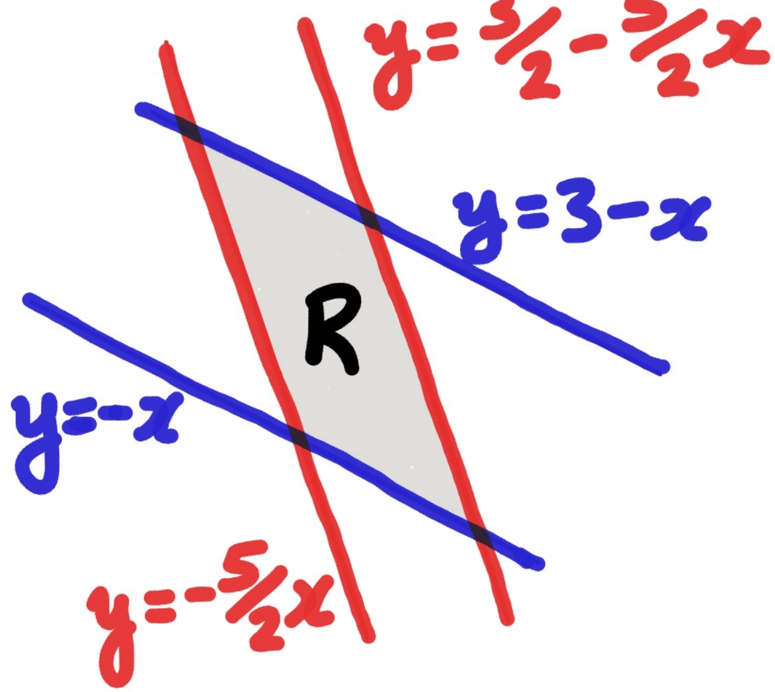
Twenty-six

Examples of Change of variables

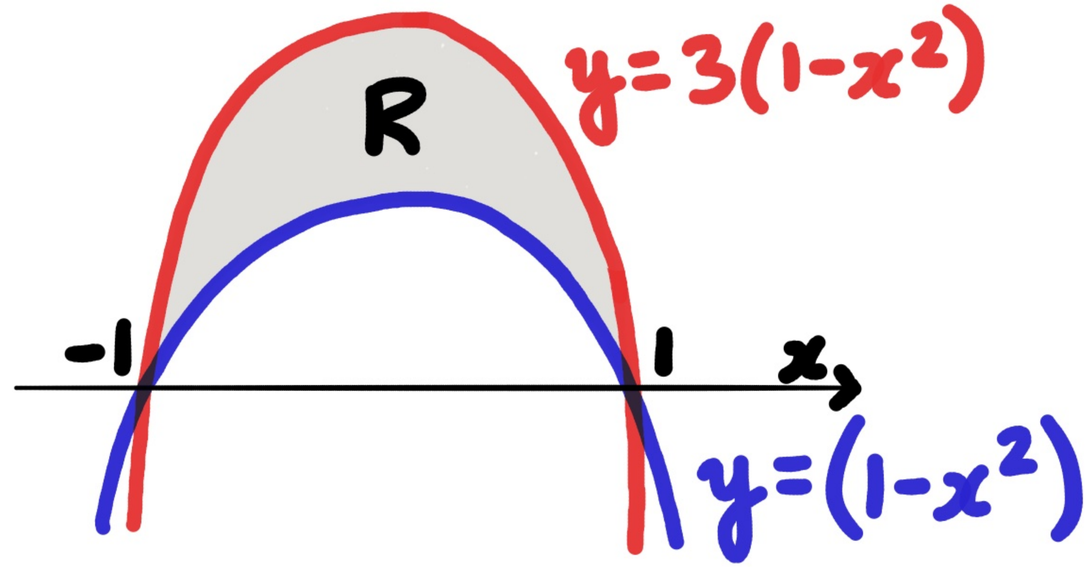
① Find $\iint_R \frac{x-y}{x+y}\,dxdy$ using change of variables with $G(u,v)=(u-v,u+v)$.



② Find $\iint_R \sin(x+y)\,dx\,dy$ using change of variables with $G(u,v) = (v-2u, 5u-v)$.

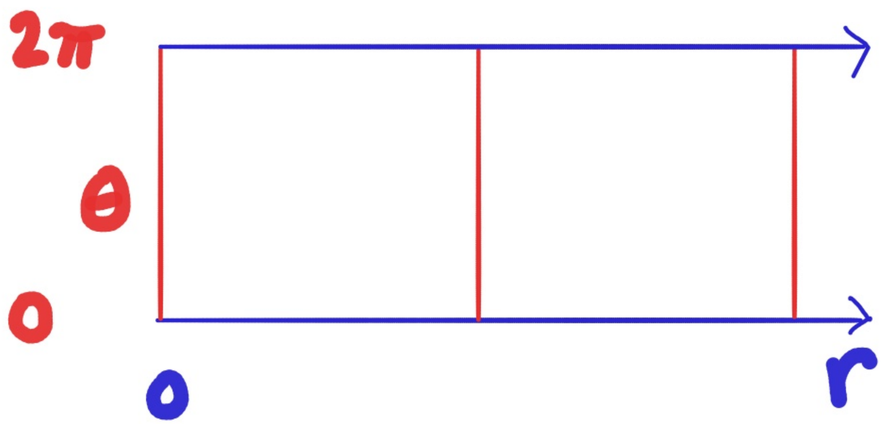


③ Find $\iint_R \frac{x^2}{y}\,dxdy$ using change of variables with $G(u,v)=(v,u(1-v^2))$.

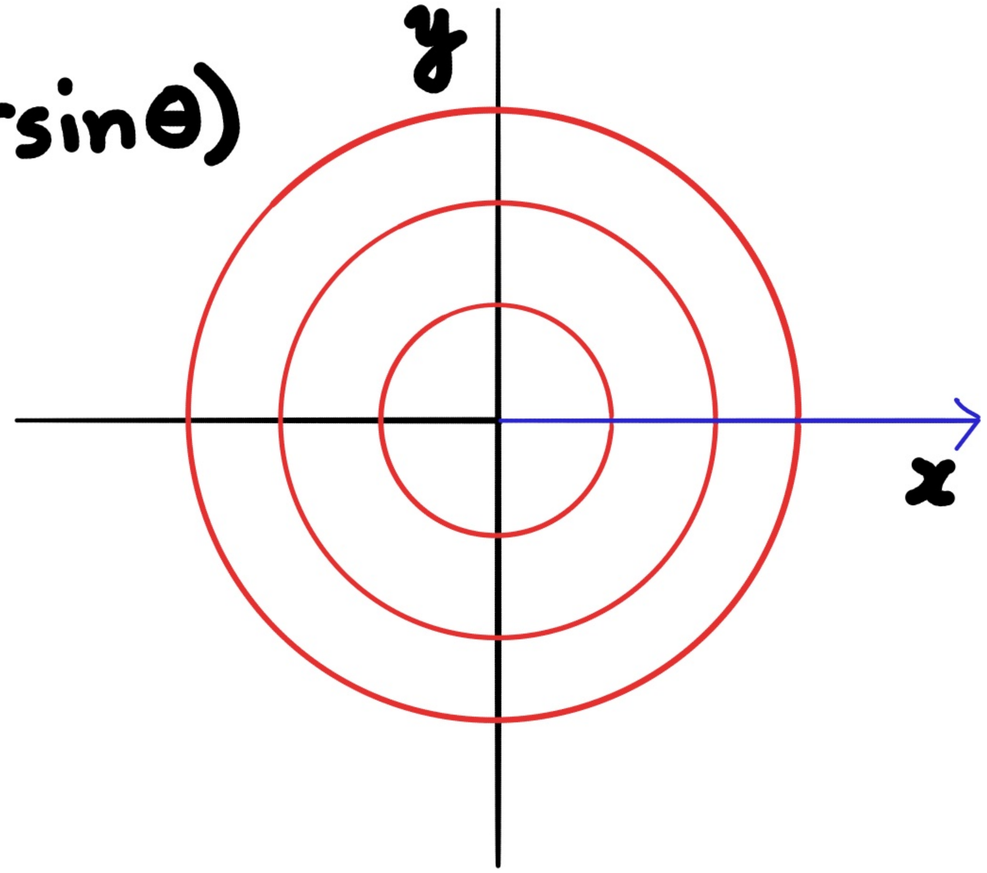


④ Polar COV

$$G(r,\theta)=(r\cos\theta, r\sin\theta)$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$



$$D_{(r,\theta)}G = \begin{pmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\left|\det D_{(r,\theta)}G\right| = \left| r\cos^2\theta + r\sin^2\theta \right| = |r| = r$$

⑤ Cylindrical COV

$$G(r,\theta,z)=(\underbrace{r\cos\theta}_{x},\underbrace{r\sin\theta}_{y},\underbrace{z}_{z})$$

$$D_{(r,\theta,z)}G = \begin{pmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} & \frac{\partial r\cos\theta}{\partial z} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial \theta} & \frac{\partial r\sin\theta}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\left|\det D_{(r,\theta,z)}G\right| = r$$

⑥ Spherical COV

$$G(\rho,\theta,\phi) = (\underbrace{\rho\sin\phi\cos\theta}_{x}, \underbrace{\rho\sin\phi\sin\theta}_{y}, \underbrace{\rho\cos\phi}_{z})$$

$$\left|\det D_{(\rho,\theta,\phi)} G\right| = \rho^2 \sin\phi$$

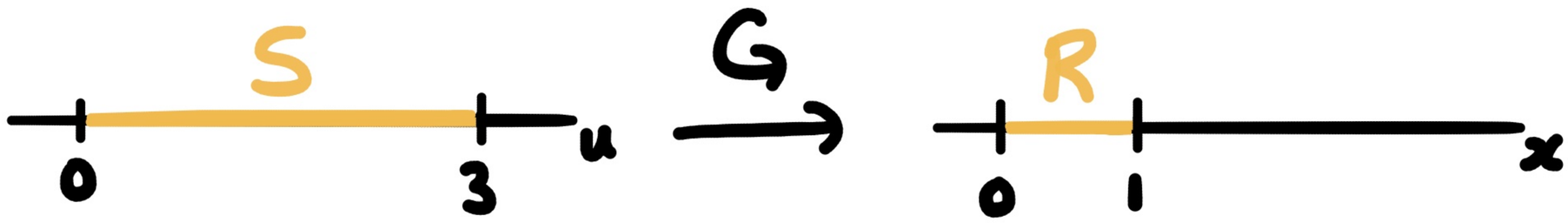
⑦ $\int_0^1 \sin(3x)\,dx = \int_0^3 \sin(u)\,\tfrac{1}{3}\,du$

$u = 3x$

⑦ $\int_0^1 \sin(3x)\,dx = \int_0^3 \sin(u)\,\tfrac{1}{3}\,du$

$u = 3x$

$G(u) = \underbrace{\tfrac{1}{3}u}_{x}$



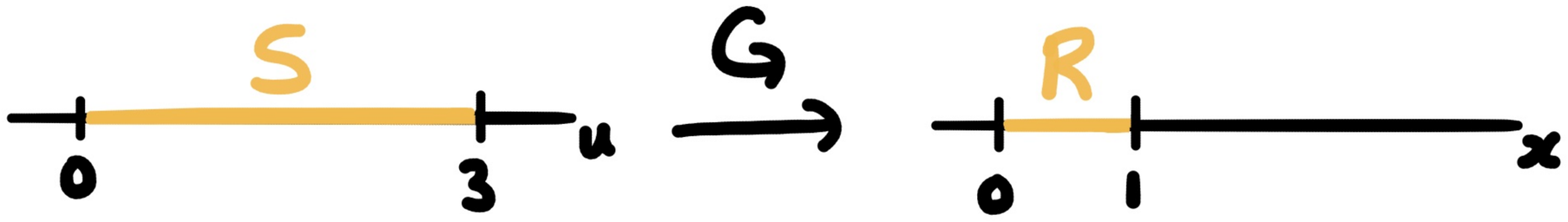
$D_u G = G'(u) = \tfrac{1}{3}$, so $|D_u G| = \tfrac{1}{3}$

⑦

$$\int_0^1 \sin(3x)\,dx = \int_0^3 \sin(u)\,\tfrac{1}{3}\,du$$

$3x \mapsto 3(\frac{1}{3}u)$

$G(u) = \underbrace{\frac{1}{3}u}_{x}$



$D_u G = G'(u) = \frac{1}{3}$, so $|D_u G| = \frac{1}{3}$