Twenty-five

Ⅰ Determinants

Ⅱ Jacobians

Ⅲ One-to-one transformations

Ⅳ Change of variables

Ⅰ Determinants

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Example:

$$\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 3 \cdot 2 - 0 \cdot 0 = 6$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

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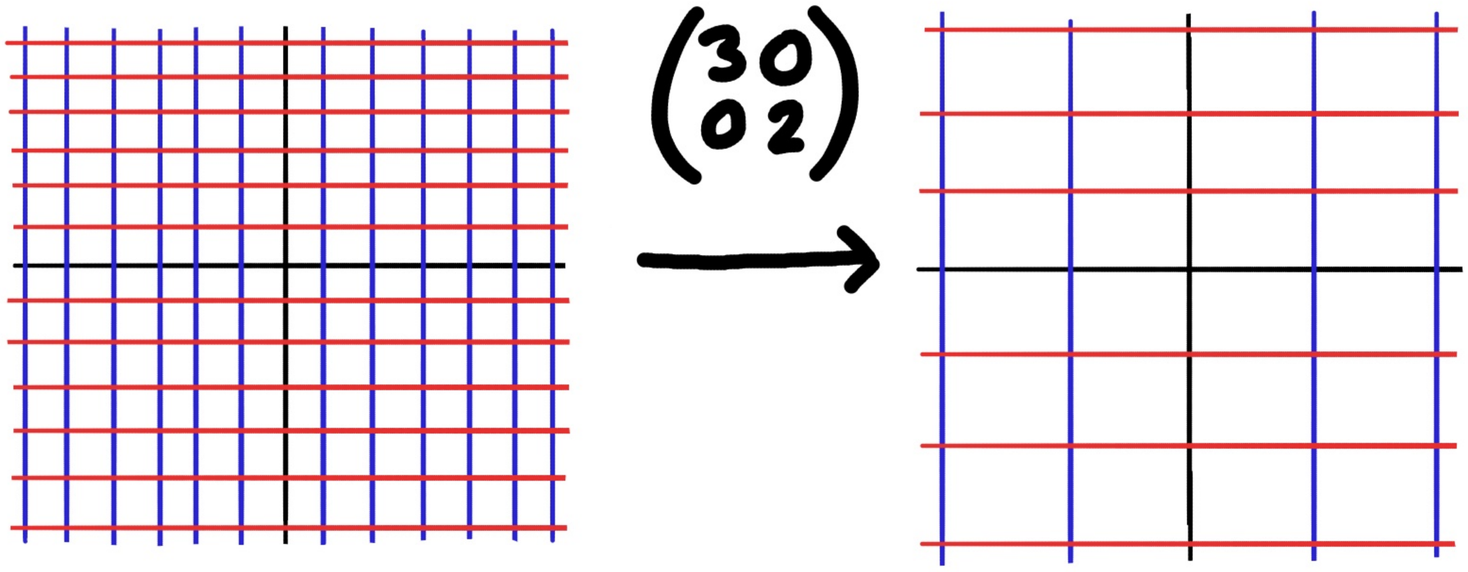
$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 2y \end{pmatrix}$$



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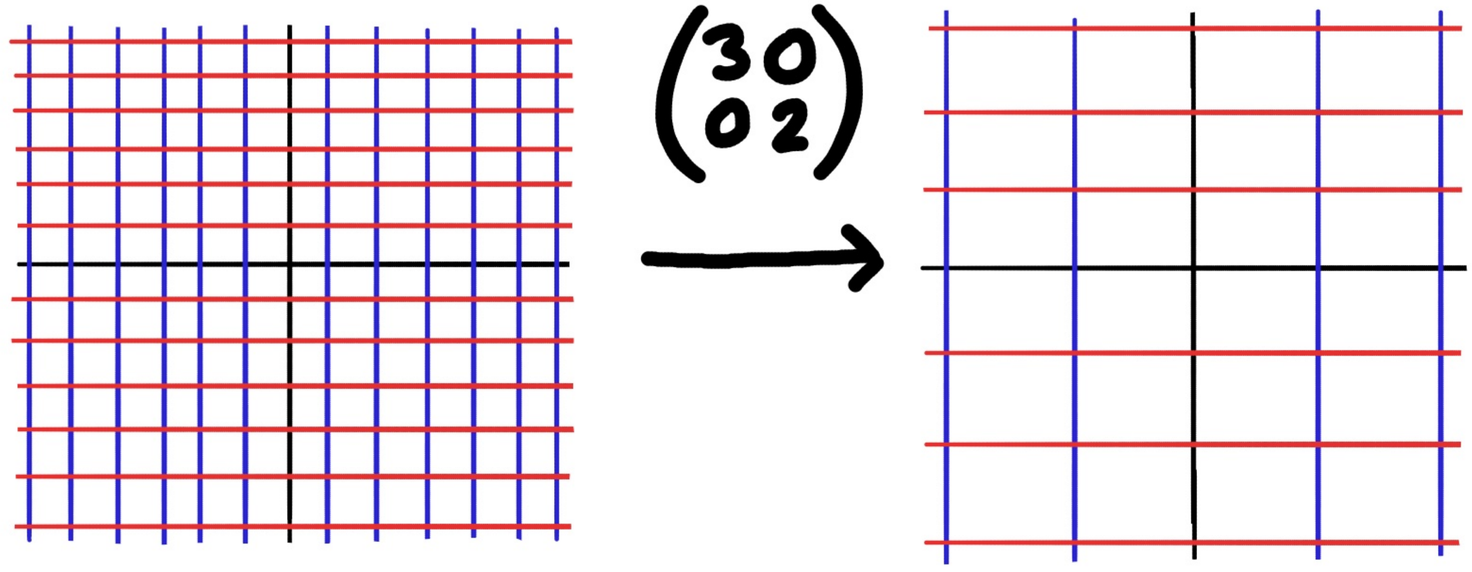

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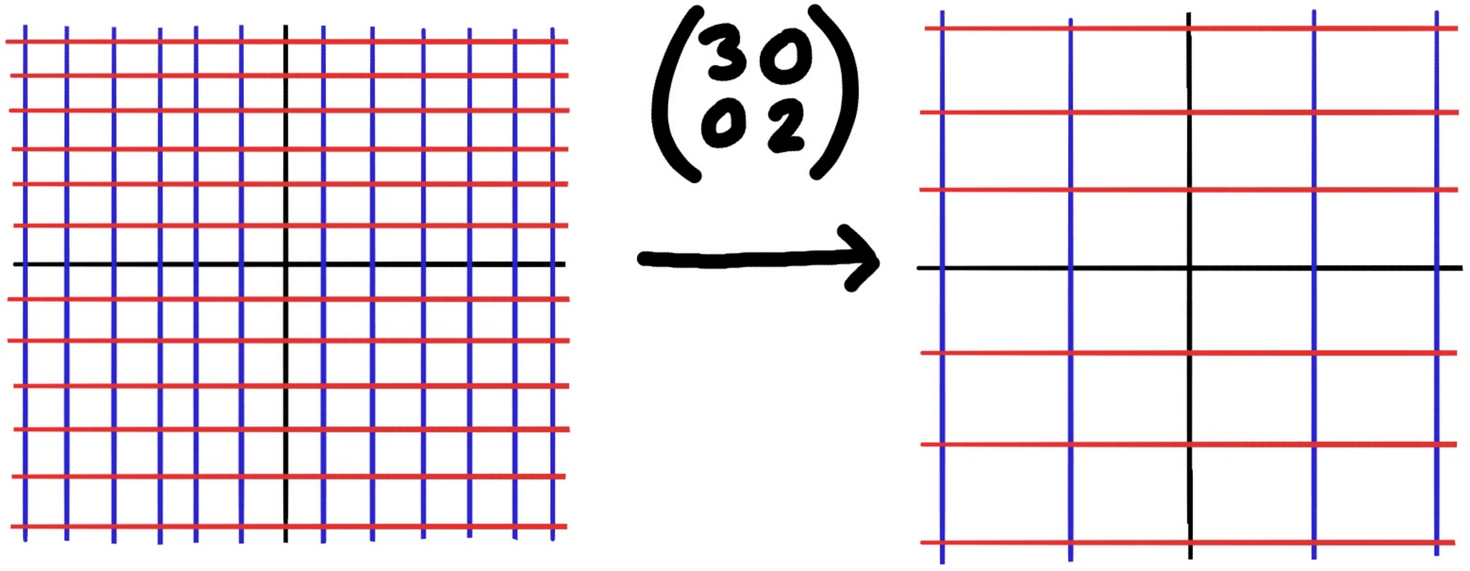

- $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ transforms $\varepsilon \times \varepsilon$ squares into $3\varepsilon \times 2\varepsilon$ rectangles.

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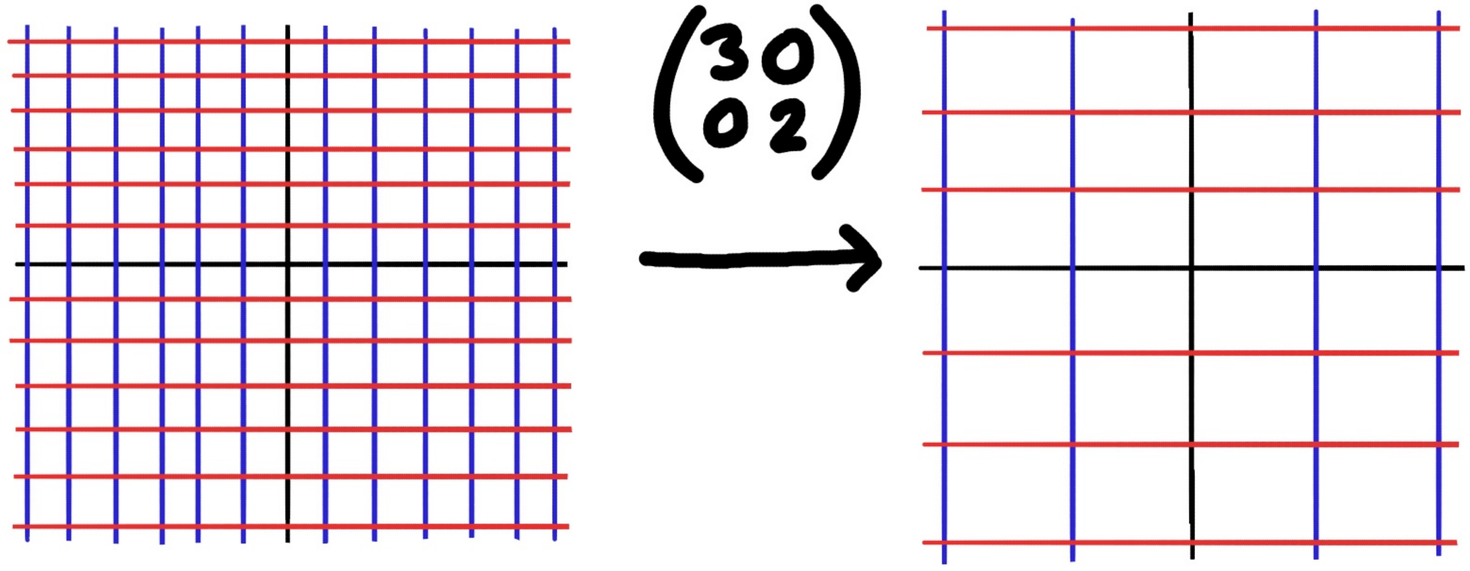

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- $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ transforms squares of area ε^2 into rectangles of area $6\varepsilon^2$

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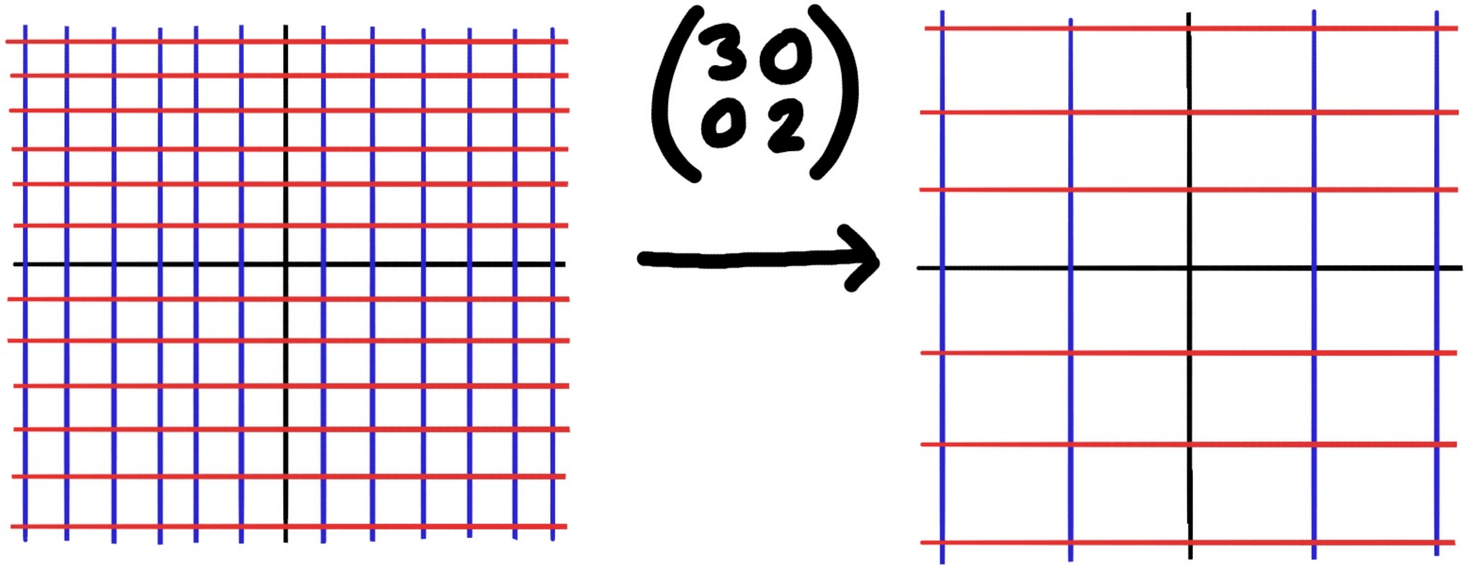


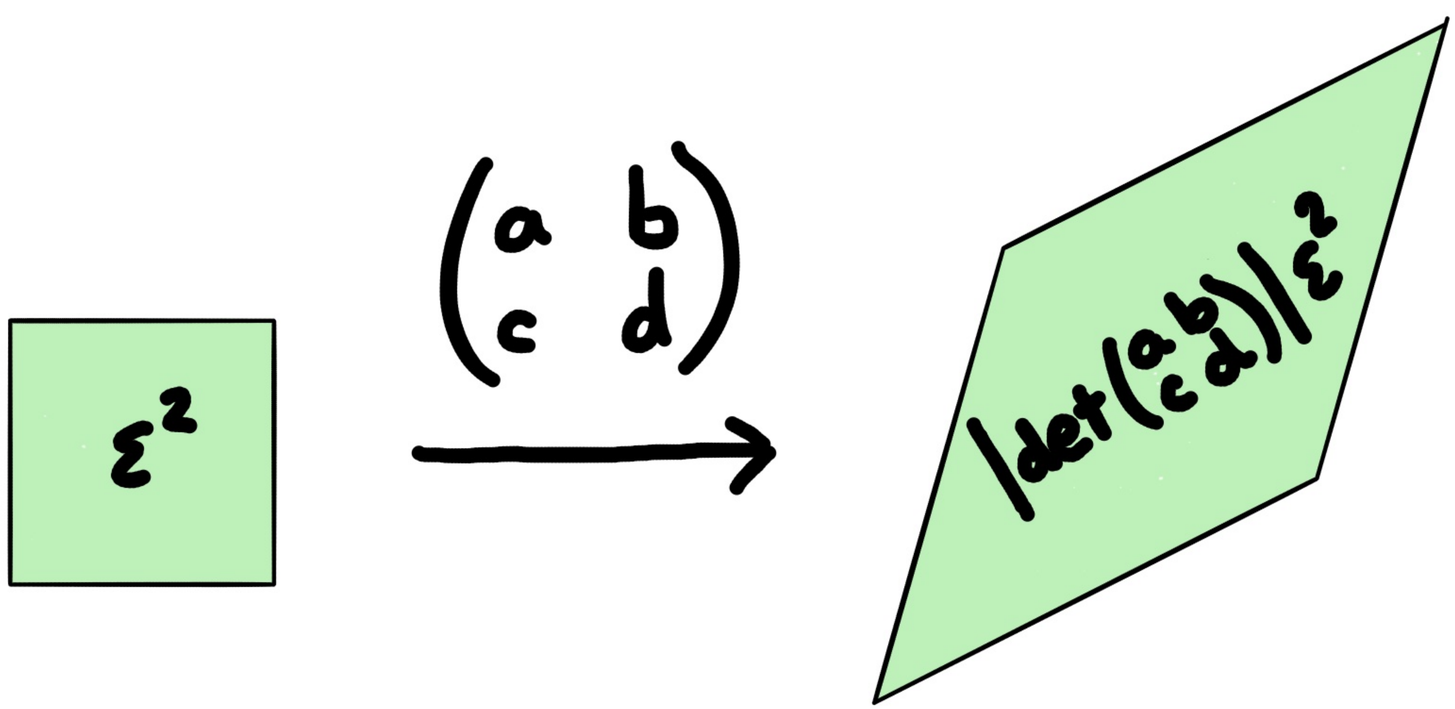
- $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ transforms $\varepsilon \times \varepsilon$ squares into $3\varepsilon \times 2\varepsilon$ rectangles.

- $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ transforms squares of area ε^2 into rectangles of area $6\varepsilon^2 = \left|\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right|\varepsilon^2$.

General Formula:

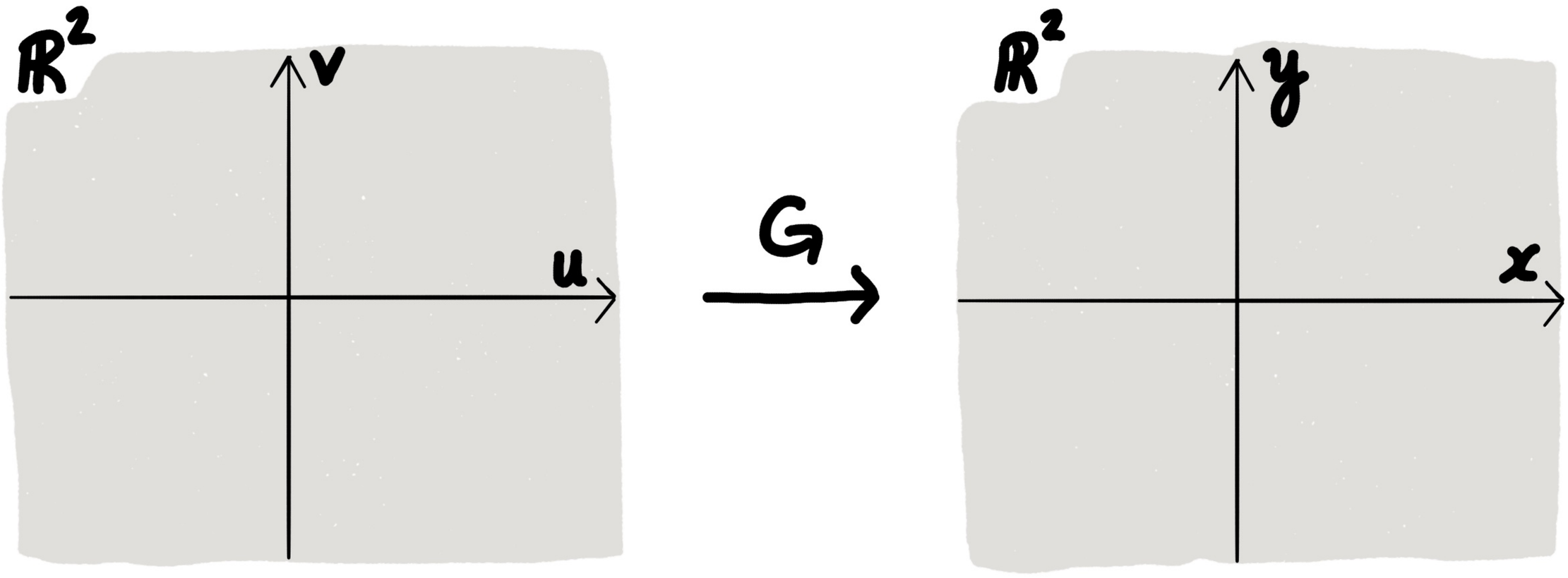
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

transforms squares of area ε^2 into parallelograms of area $\left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| \varepsilon^2$.



Ⅱ Jacobians

$$G: \underset{u,v}{\mathbb{R}^2} \longrightarrow \underset{x,y}{\mathbb{R}^2}, \qquad D_{(u,v)}G: \mathbb{R}^2 \to \mathbb{R}^2$$



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The Jacobian of G at $(u,v)\in\mathbb{R}^2$ is $\det(D_{(u,v)}G)$.

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Example:

- $G(u,v)=(3u,2v)$
- $D_{(u,v)}G=\begin{matrix}3u\\2v\end{matrix}\begin{pmatrix}\\\\\end{pmatrix}$

u v

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The Jacobian of G at $(u,v) \in \mathbb{R}^2$ is $\det(D_{(u,v)}G)$.

Example:

- $G(u,v) = (3u, 2v)$
- $$D_{(u,v)}G = \begin{matrix} 3u \\ 2v \end{matrix} \begin{pmatrix} \frac{\partial 3u}{\partial u} & \frac{\partial 3u}{\partial v} \\ \frac{\partial 2v}{\partial u} & \frac{\partial 2v}{\partial v} \end{pmatrix} \\ \qquad\qquad u \qquad\quad v$$

$$G:\underset{u,v}{\mathbb{R}^2}\to\underset{x,y}{\mathbb{R}^2},\qquad D_{(u,v)}G:\mathbb{R}^2\to\mathbb{R}^2$$

The **Jacobian** of G at $(u,v)\in\mathbb{R}^2$ is $\det(D_{(u,v)}G)$.

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  u v
- $\det(D_{(u,v)}G)=6$

Example:

- $G(u,v) = (u, u^2+v^2)$

- $D_{(u,v)}G = \begin{pmatrix} u & \\ u^2+v^2 & \end{pmatrix}$

u v

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- $$D_{(u,v)}G = \begin{matrix} u \\ u^2+v^2 \end{matrix} \begin{pmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial (u^2+v^2)}{\partial u} & \frac{\partial (u^2+v^2)}{\partial v} \end{pmatrix} \begin{matrix} \\ \\ u & v \end{matrix}$$

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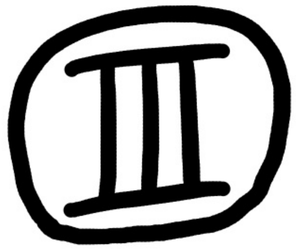
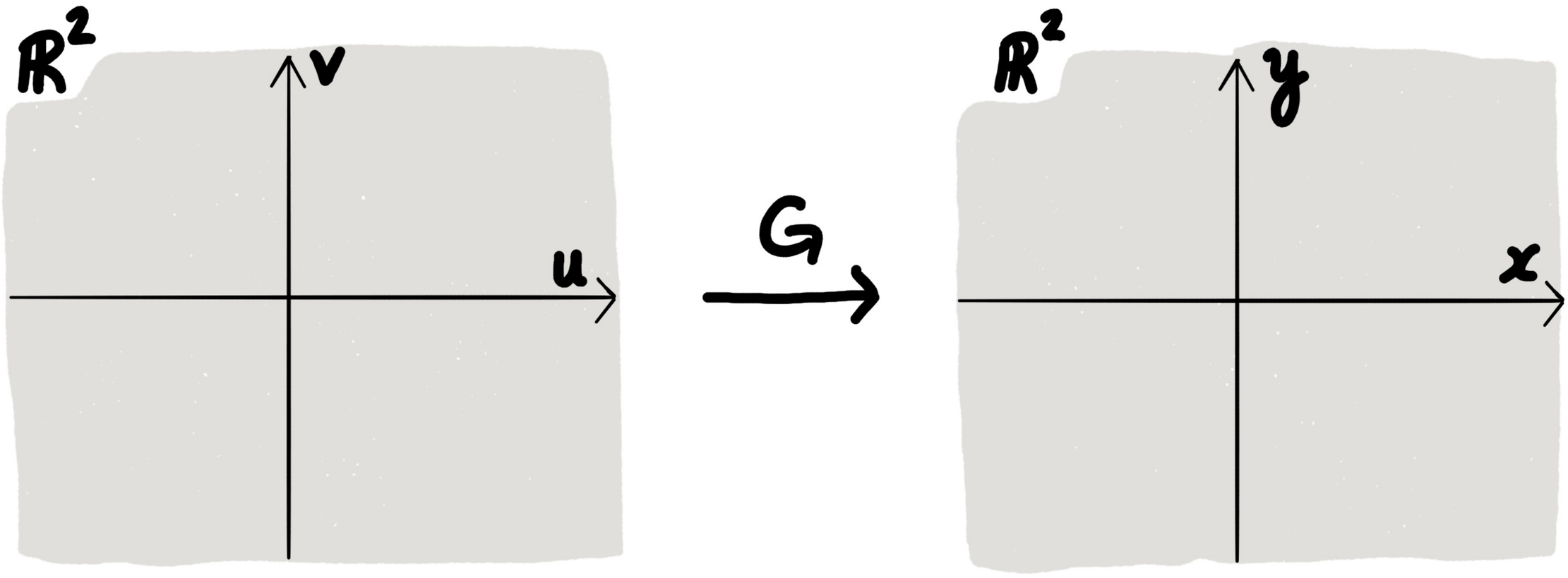
- $\det(D_{(u,v)}G) = 1 \cdot 2v - 0 \cdot 2u = 2v$



One-to-one transformations

- Let $G : \underset{u,v}{\mathbb{R}^2} \to \underset{x,y}{\mathbb{R}^2}$ be one-to-one

  (meaning if $p \neq q$, then $G(p) \neq G(q)$.)



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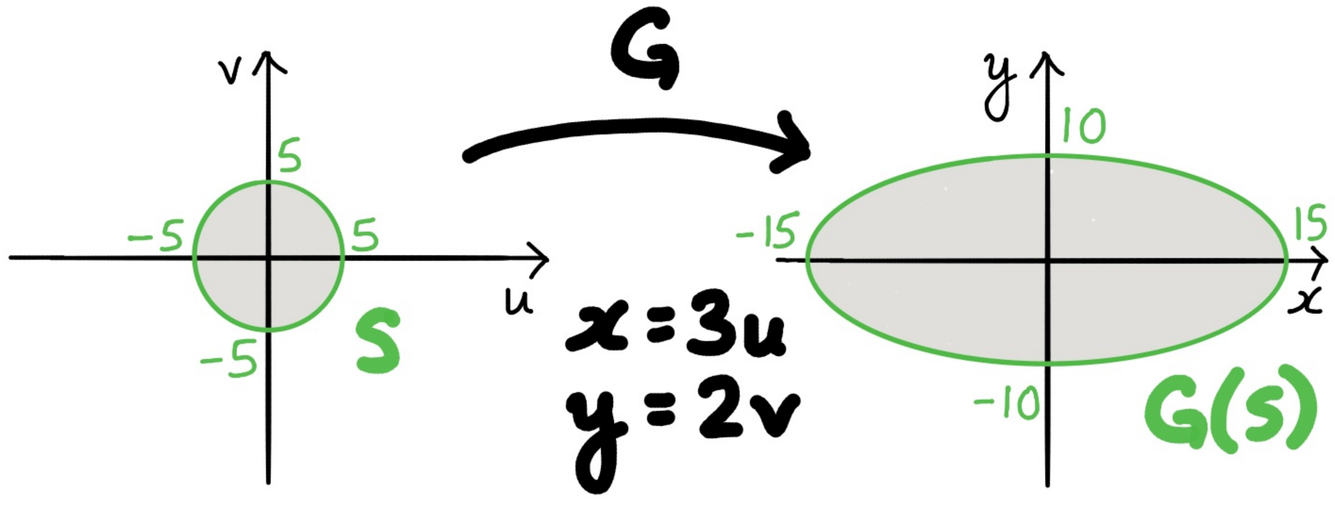
Example:

- $G(u,v) = (3u, 2v)$.
- S circle of radius 5.
- $G(S)$ ellipse.

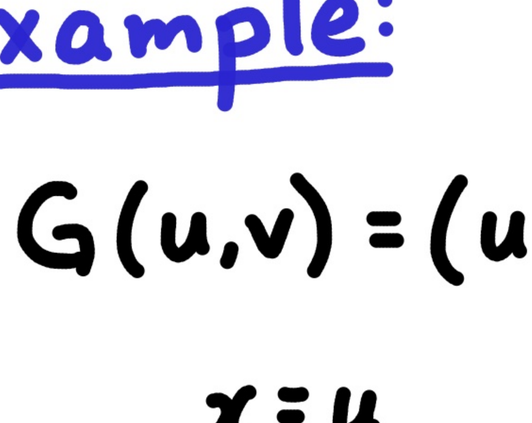


Example:

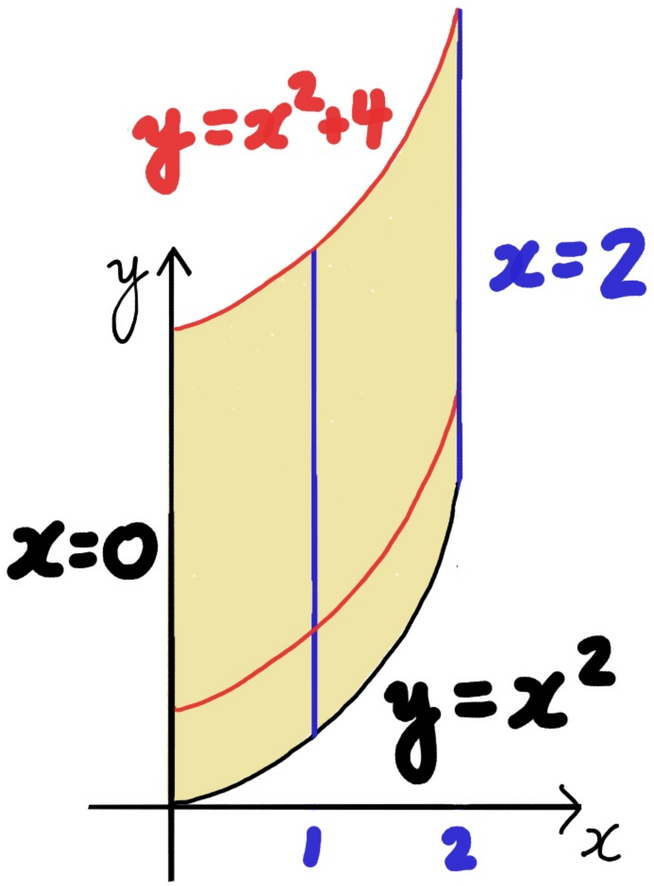
$$G(u,v) = (u, u^2+v^2)$$

$$x = u$$

$$y = u^2+v^2$$



G



$$S = \{(u,v) : 0 \le u \le 2, 0 \le v \le 2\}$$

$$G(S) = \{(x,y) : 0 \le x \le 2,\ x^2 \le y \le x^2+4\}$$

Ⅳ Change of variables

Change of variables

If,

- $f: \underset{x,y}{\mathbb{R}^2} \to \mathbb{R}$, C^0
- $G: \underset{u,v}{\mathbb{R}^2} \to \underset{x,y}{\mathbb{R}^2}$, C^1 and one-to-one
- $G(S) = R$

Then,

$$\iint_R f(x,y)\,dxdy = \iint_S f \circ G(u,v)\,\left|\det\left(D_{(u,v)}G\right)\right|\,dudv$$

proof:

- $D_{(u,v)}G$ transforms squares of area ε^2 into parallelograms of area $|\det(D_{(u,v)}G)|\varepsilon^2$.

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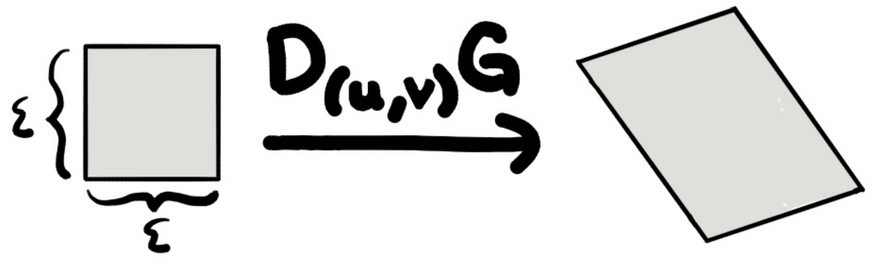

Area: $|\det(D_{(u,v)}G)|\varepsilon^2$

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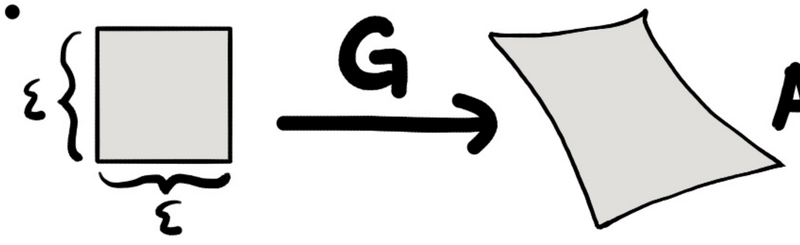
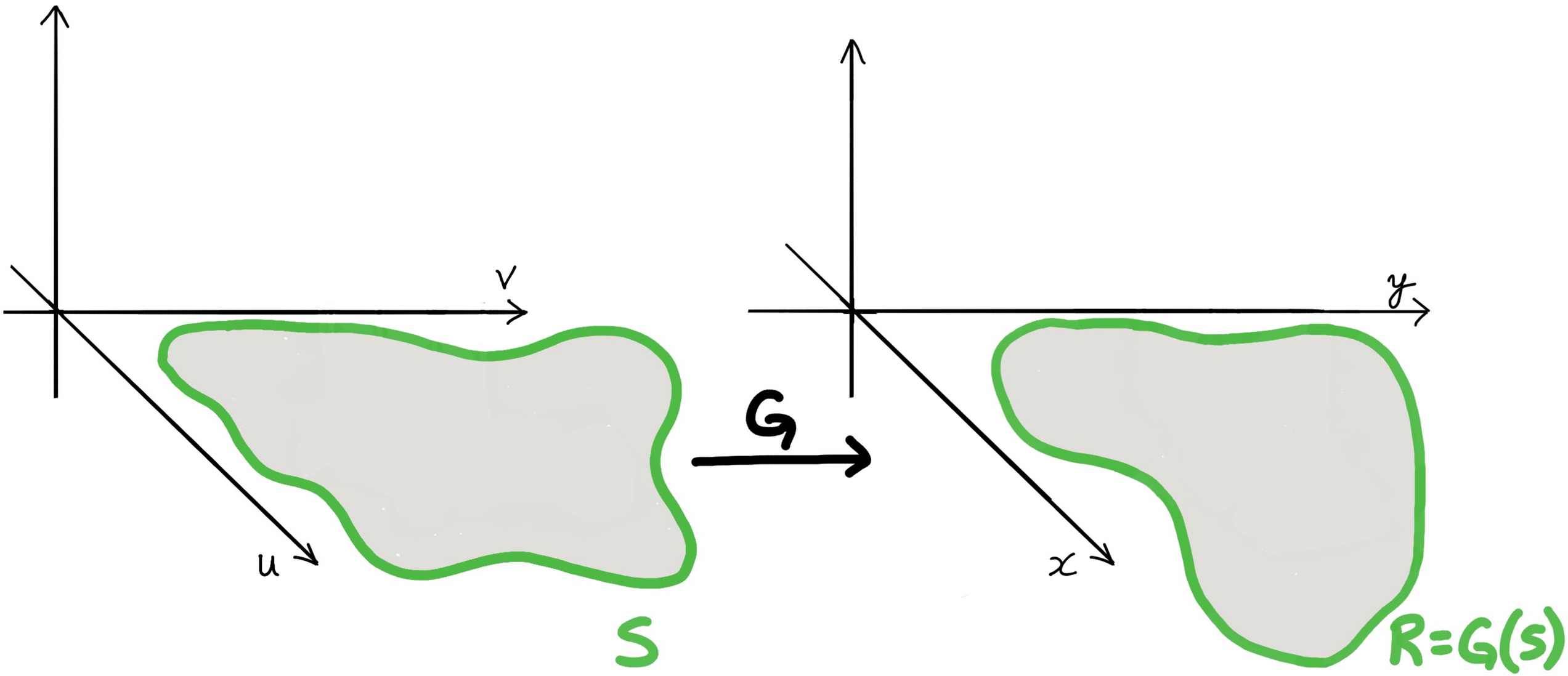
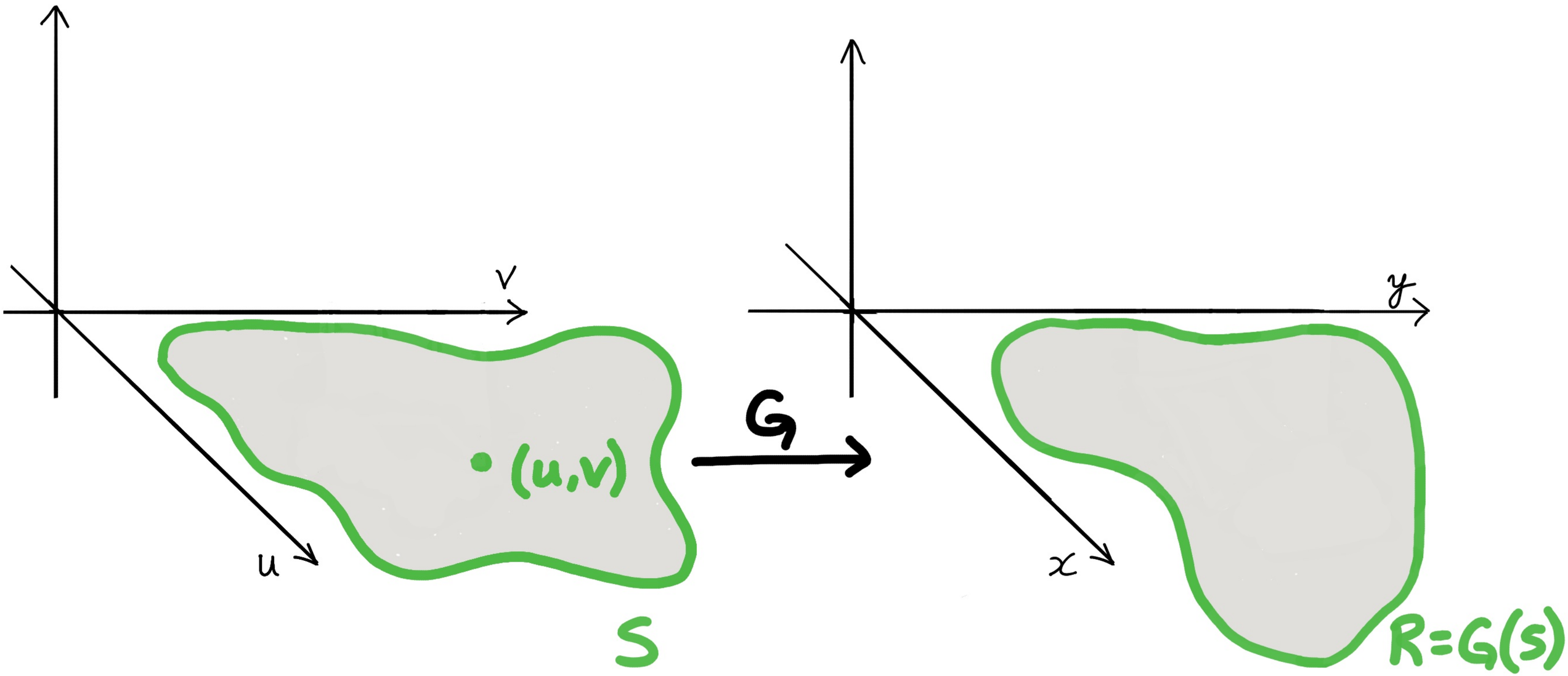


Area $\approx |\det(D_{(u,v)}G)|\varepsilon^2$


v
u
S
G
y
x
R=G(s)


v
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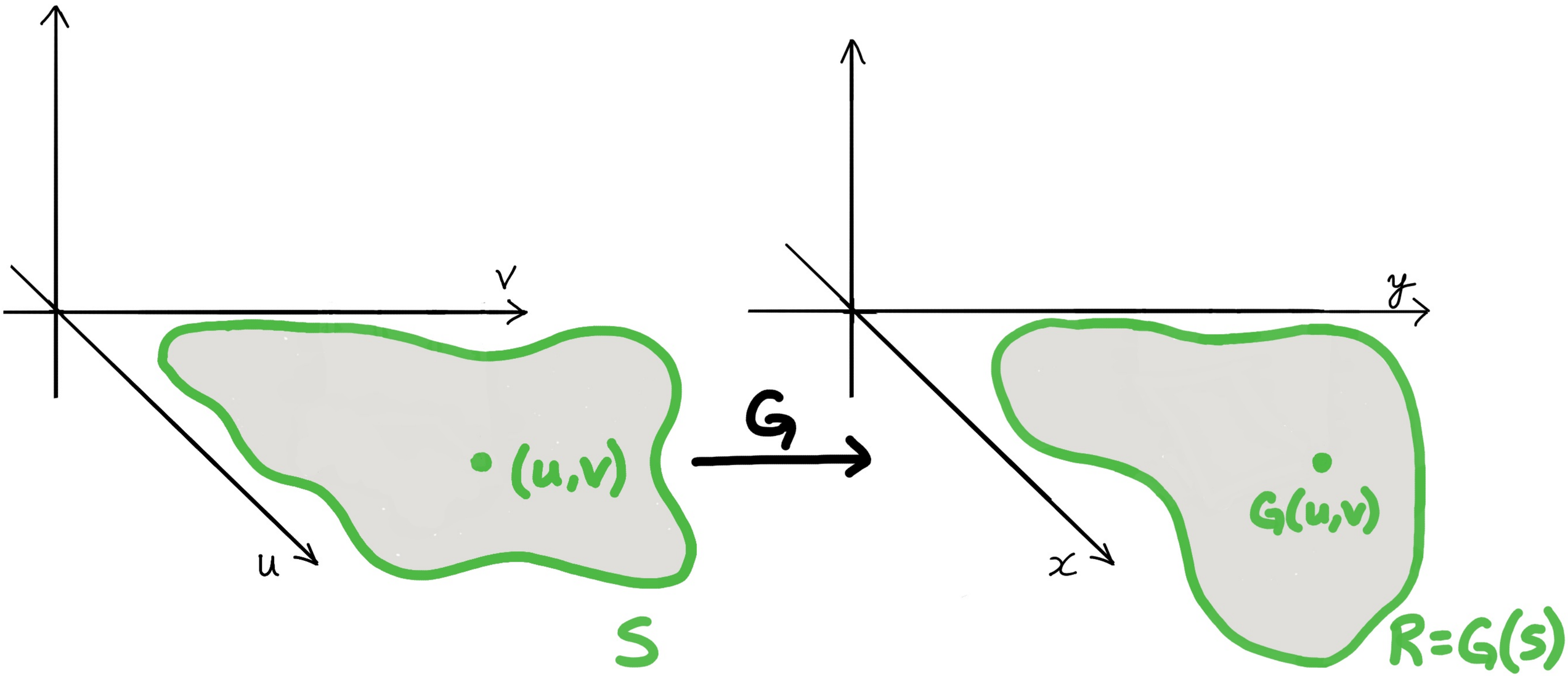
R=G(s)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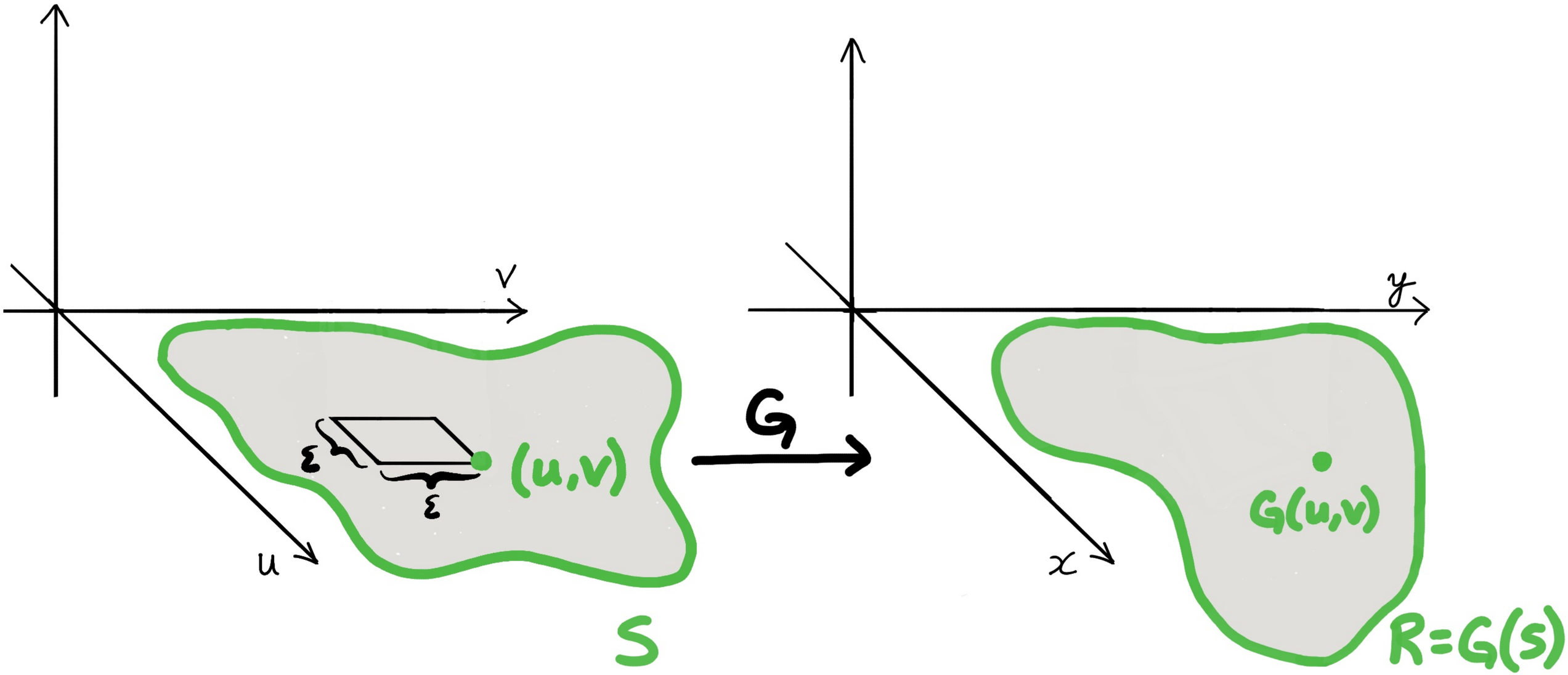
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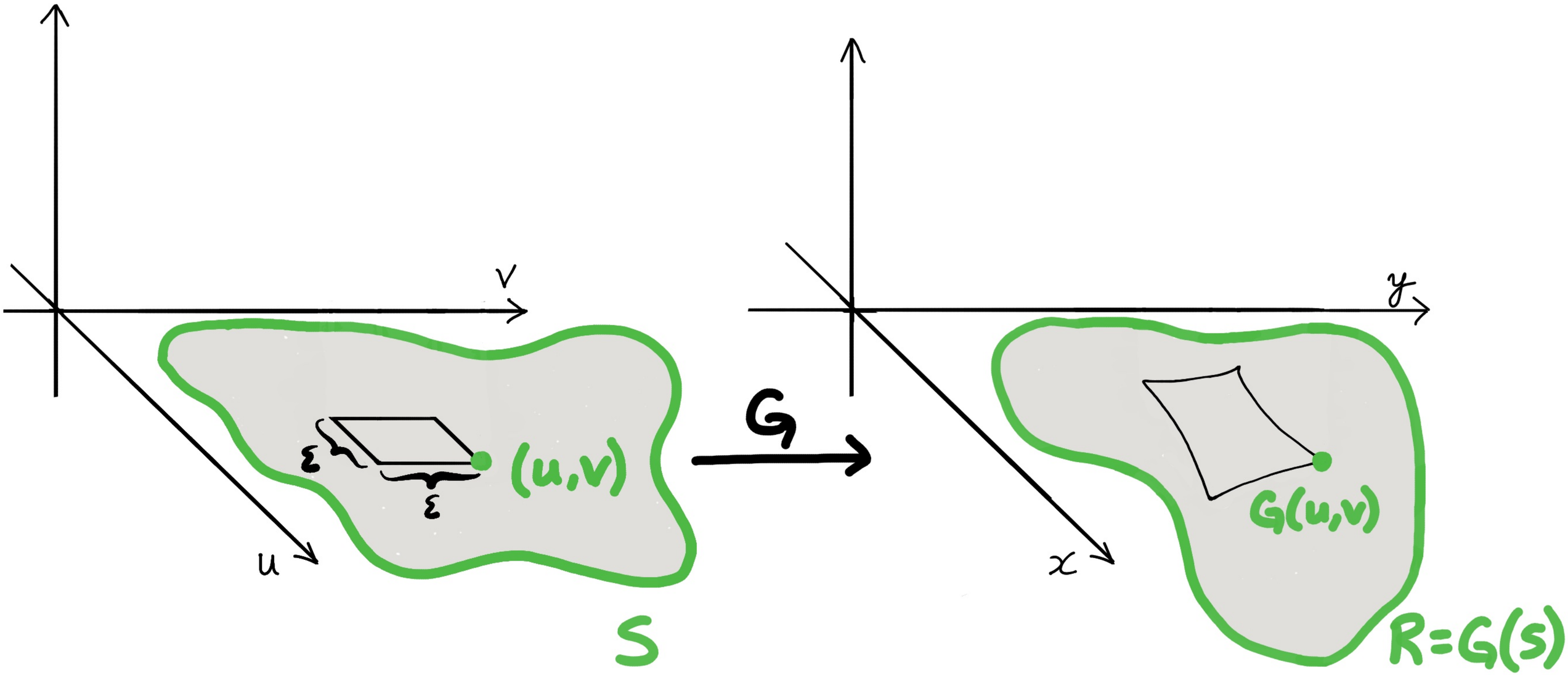
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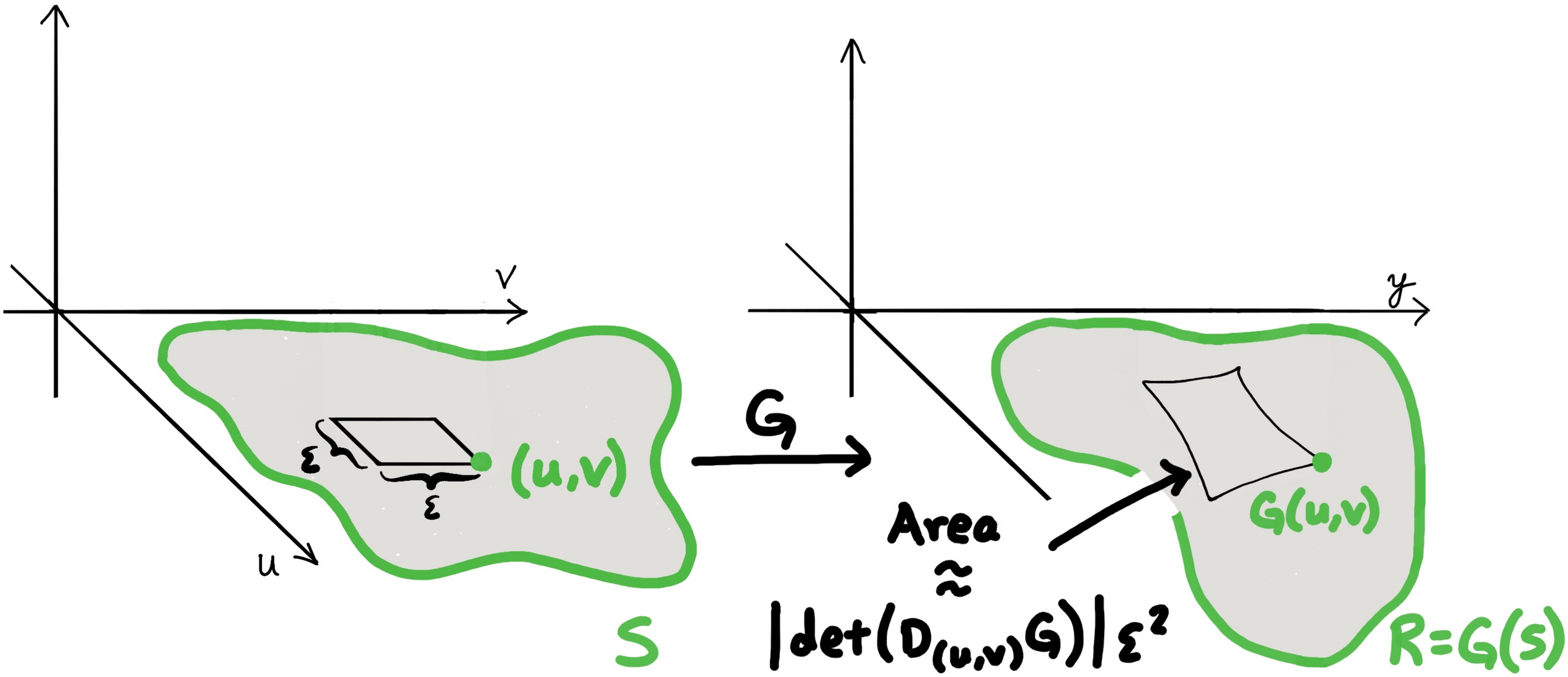
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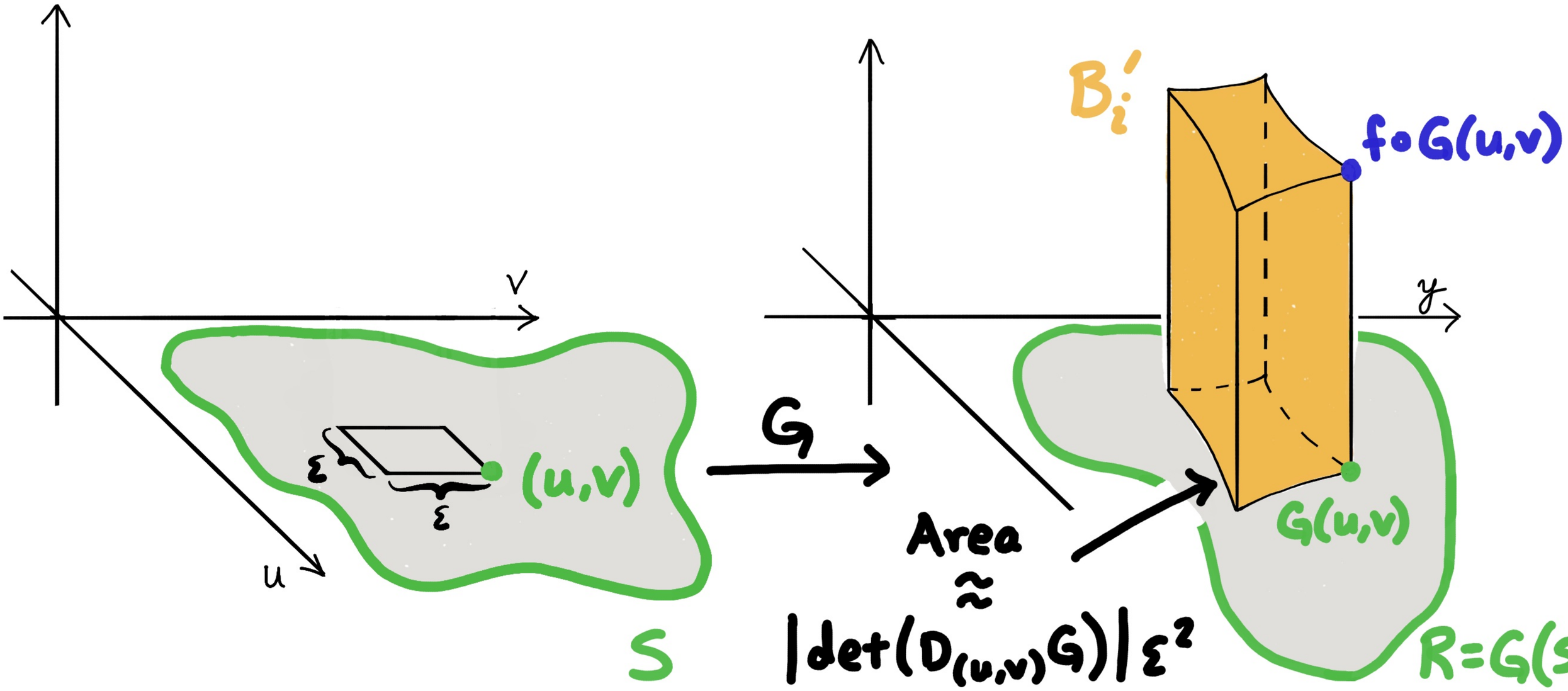
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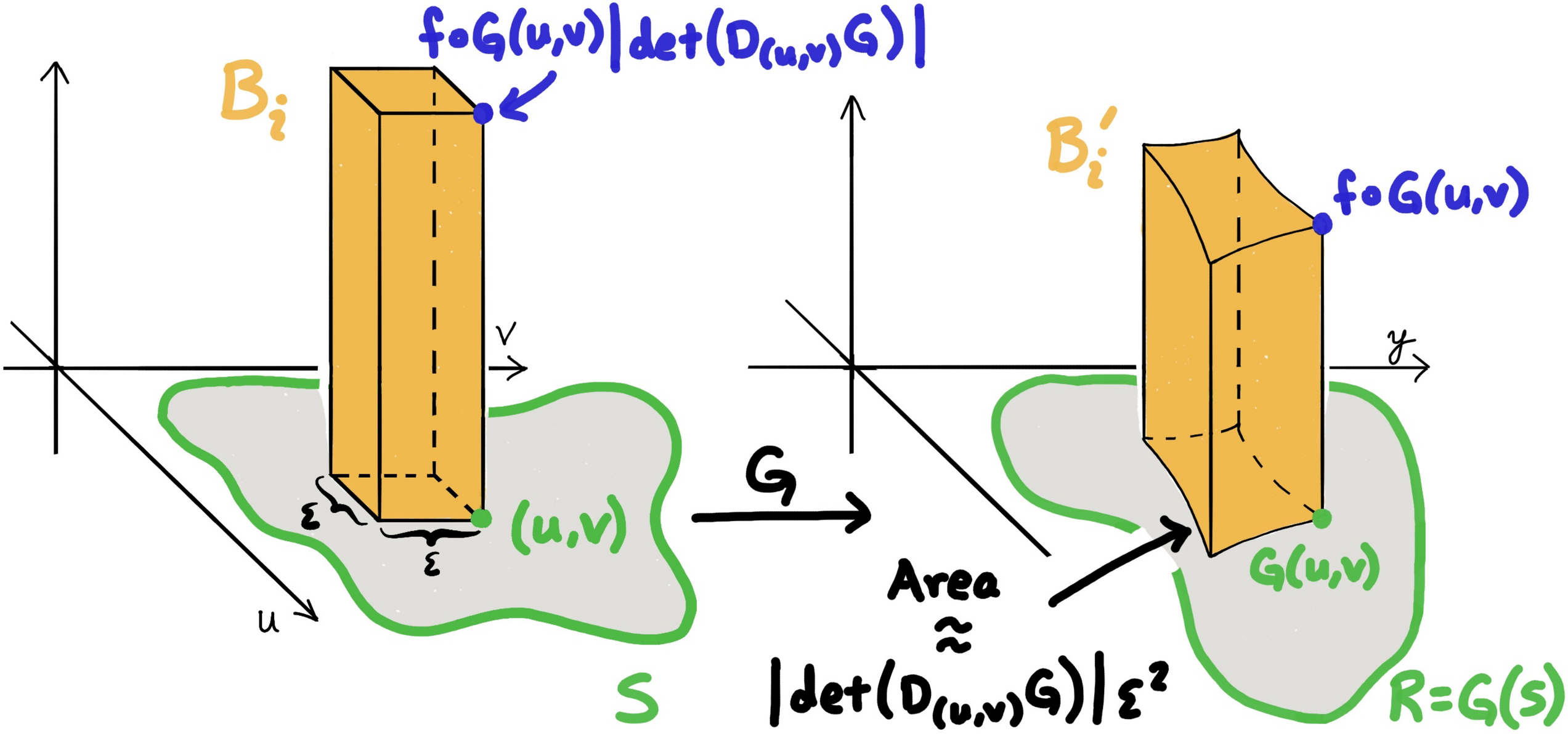
f∘G(u,v)|det(D(u,v)G)|
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S
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Area
≈
|det(D(u,v)G)|ε²
B_i'
f∘G(u,v)
y
G(u,v)
R=G(S)

$f \circ G(u,v) \left| \det(D_{(u,v)}G) \right|$

B_i

B_i'

$f \circ G(u,v)$

v

u

y

ε

ε

(u,v)

G

Area $\approx$ $\left| \det(D_{(u,v)}G) \right| \varepsilon^2$

$G(u,v)$

S

$R = G(S)$

$$\text{Volume}(B_i) = f \circ G(u,v) \left| \det(D_{(u,v)}G) \right| \varepsilon^2$$

height

area of base

$f \circ G(u,v) \left| \det(D_{(u,v)}G) \right|$

B_i

B_i'

$f \circ G(u,v)$

v

u

y

ε

ε

(u,v)

G

Area $\approx \left| \det(D_{(u,v)}G) \right| \varepsilon^2$

$G(u,v)$

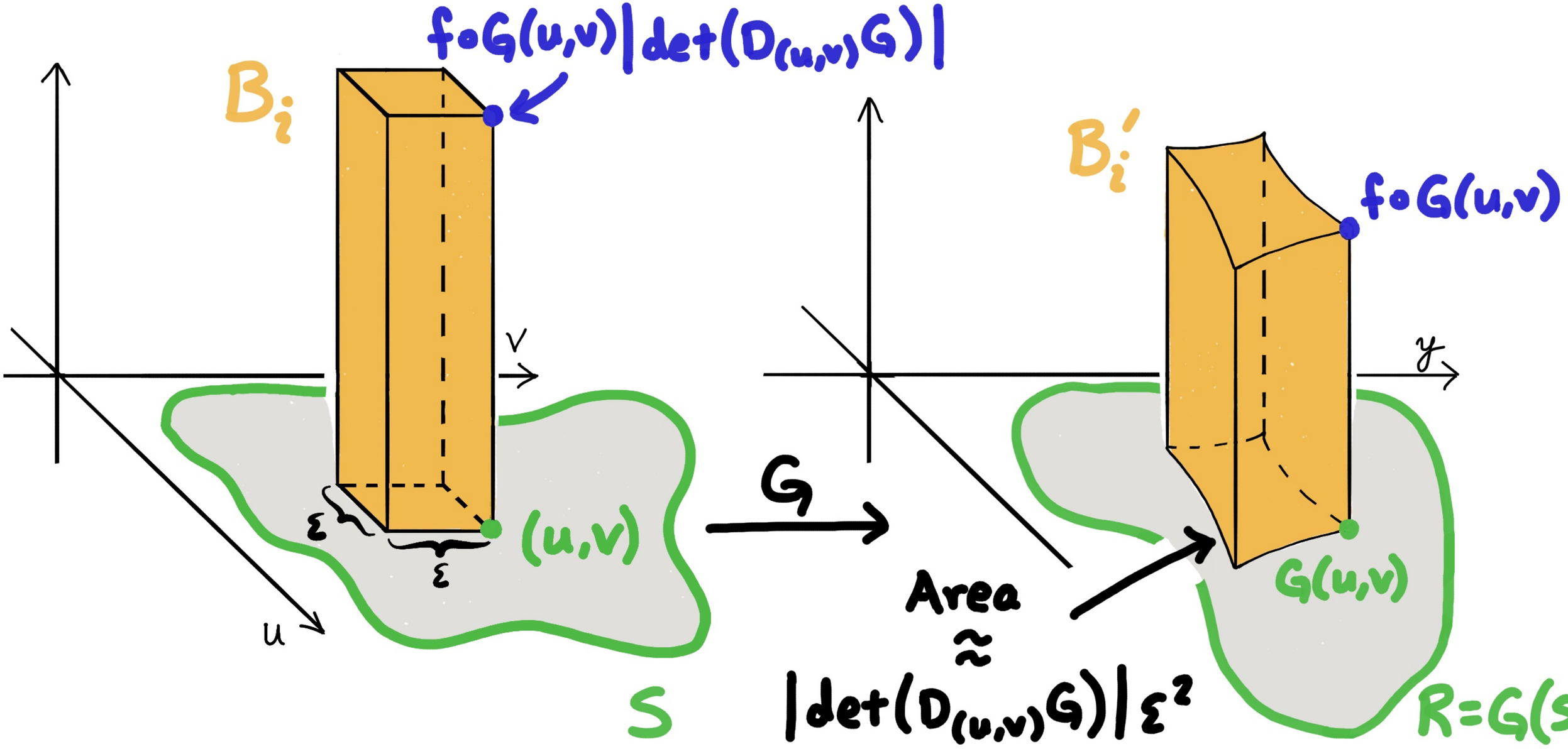
S

$R = G(S)$

$$\text{Volume}(B_i) = f \circ G(u,v) \left| \det(D_{(u,v)}G) \right| \varepsilon^2$$

height

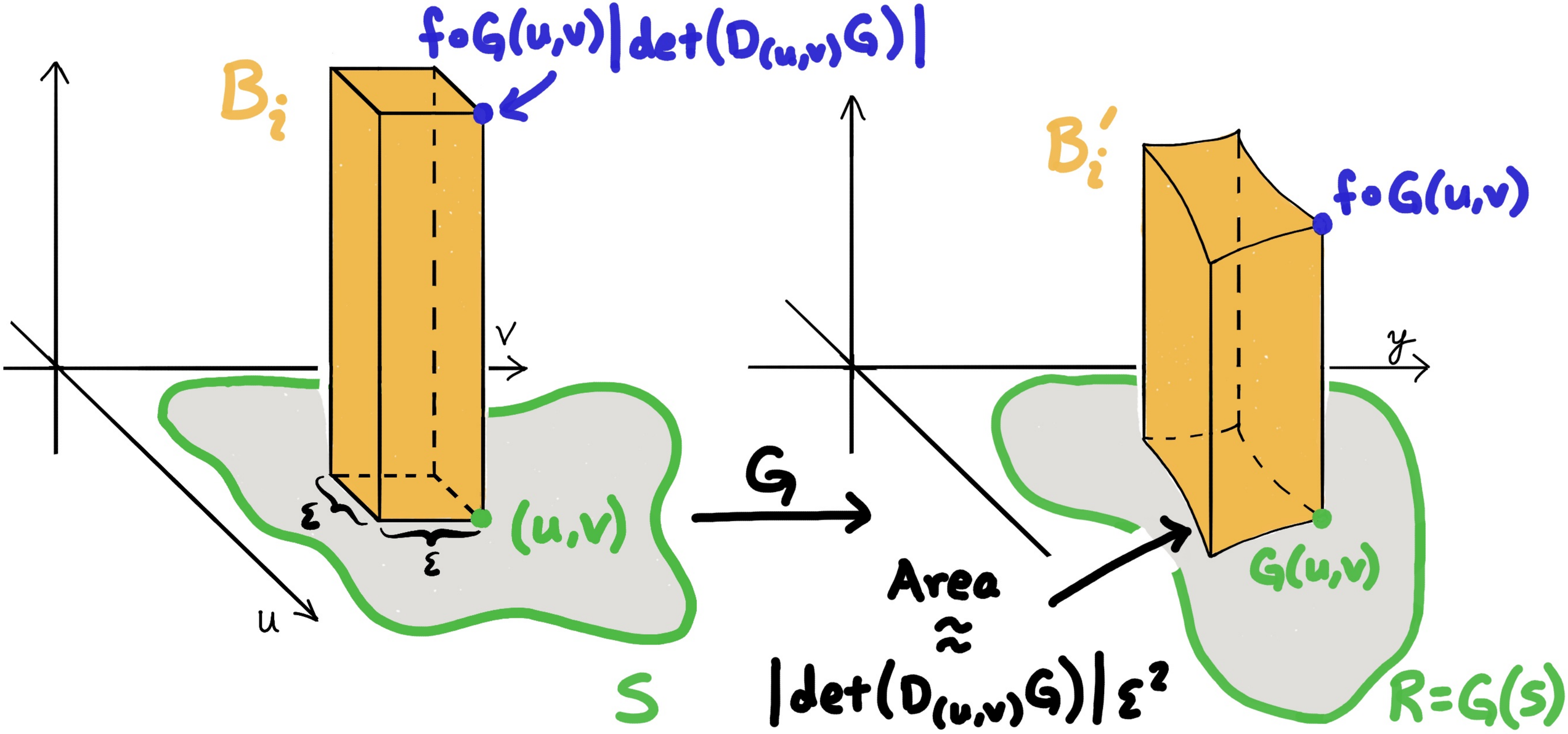
area of base



$$\text{Volume}(B_i) = f\circ G(u,v)\left|\det(D_{(u,v)}G)\right|\varepsilon^2 \approx \text{Volume}(B_i')$$

height

area of base



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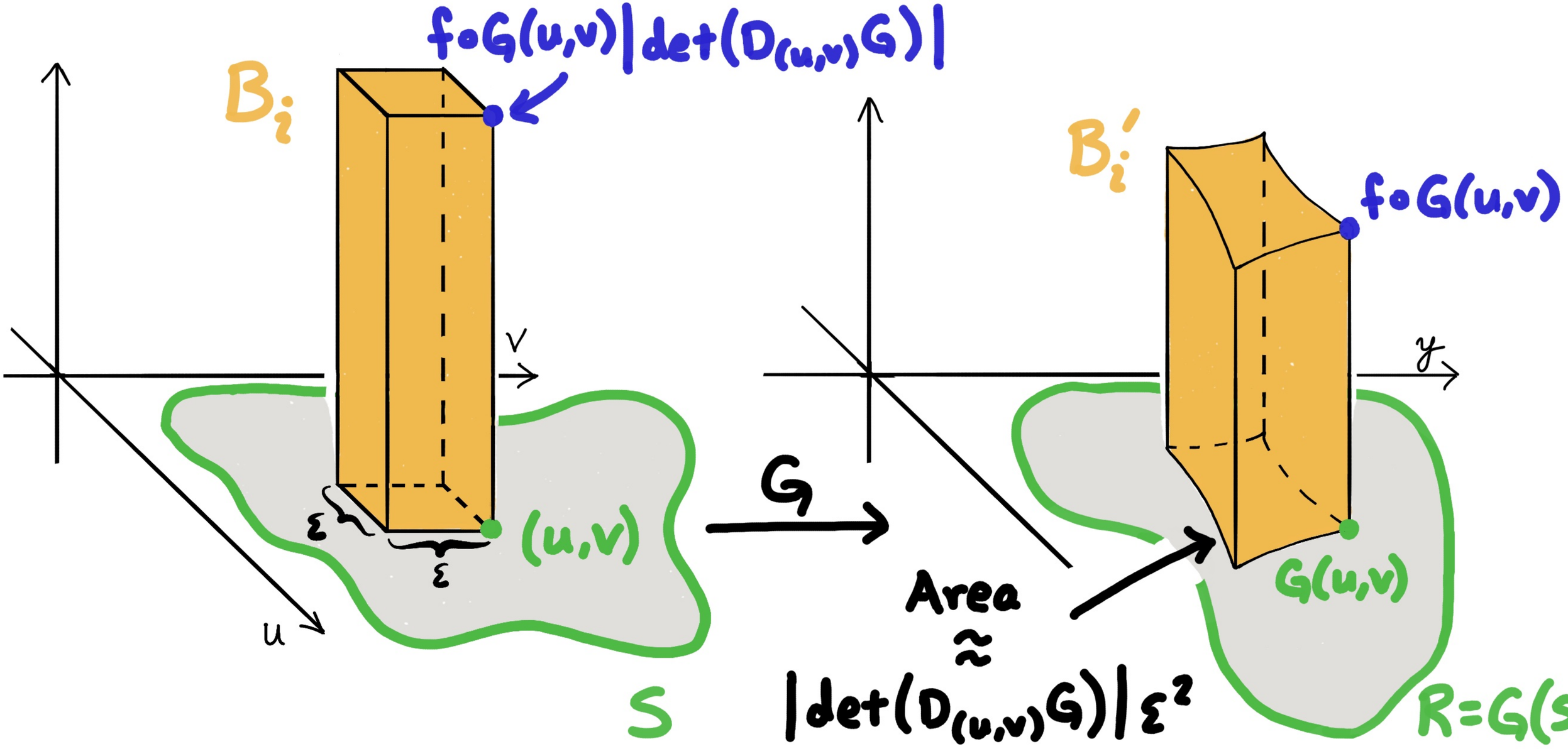
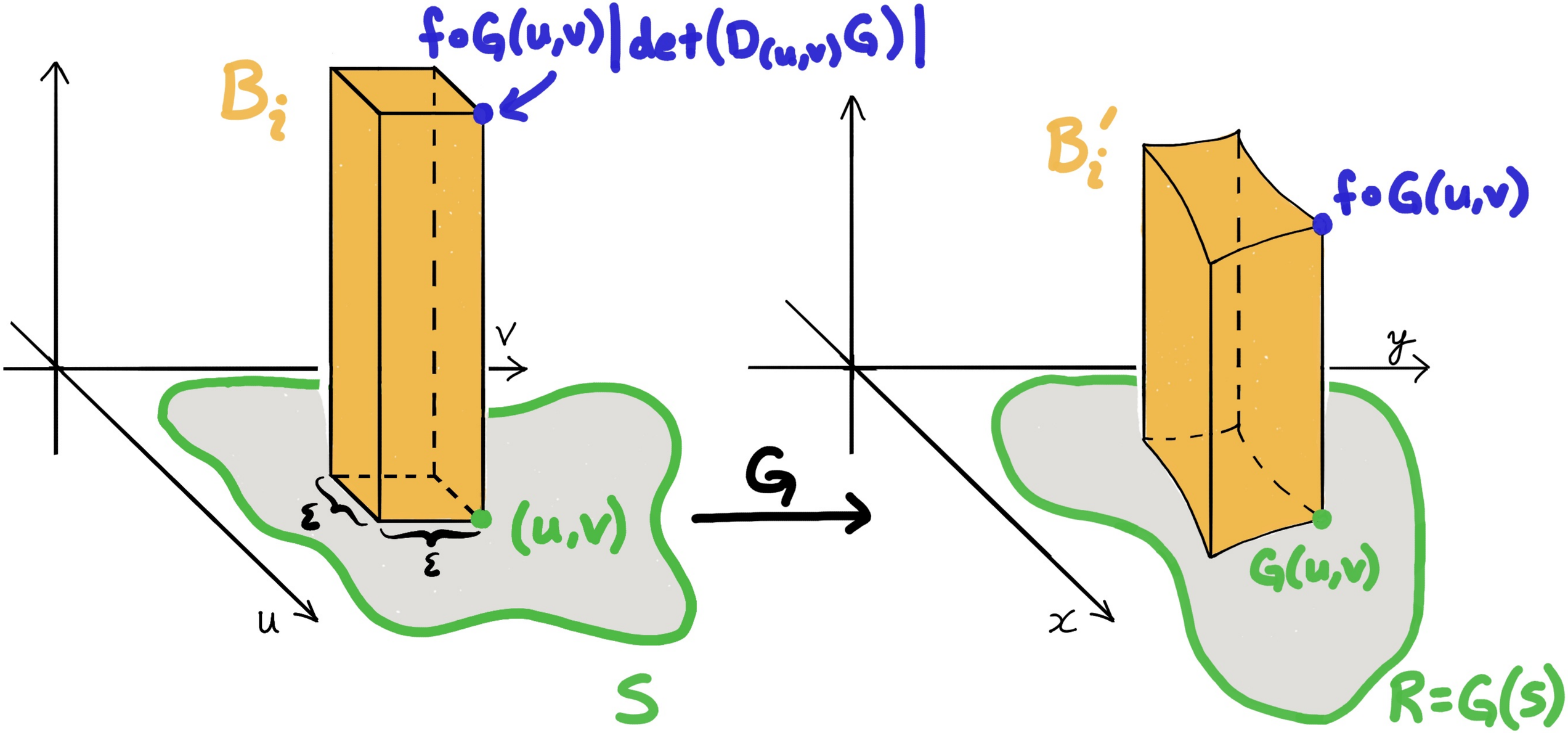


$$\text{Volume}\,(B_i) \approx \text{Volume}(B_i')$$



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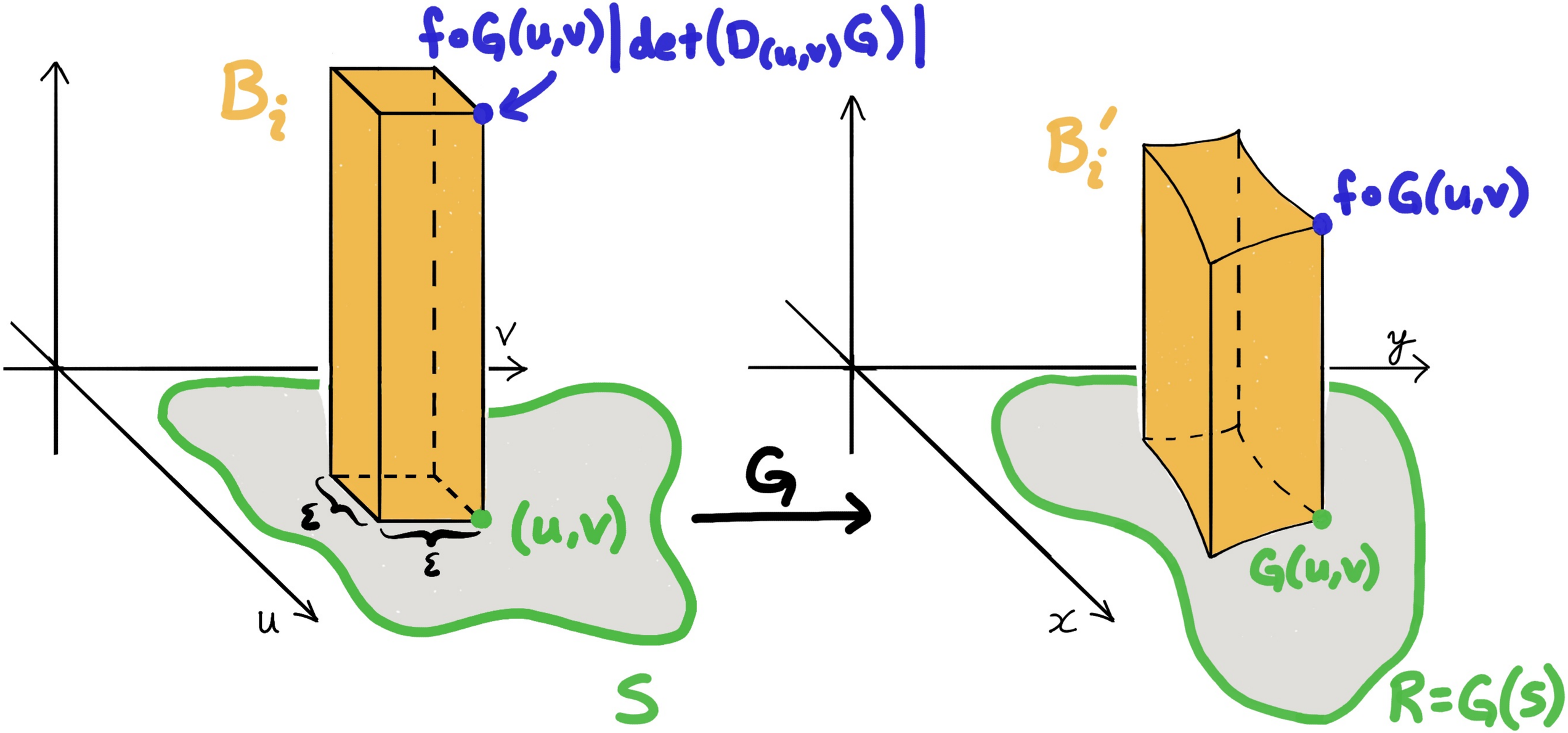

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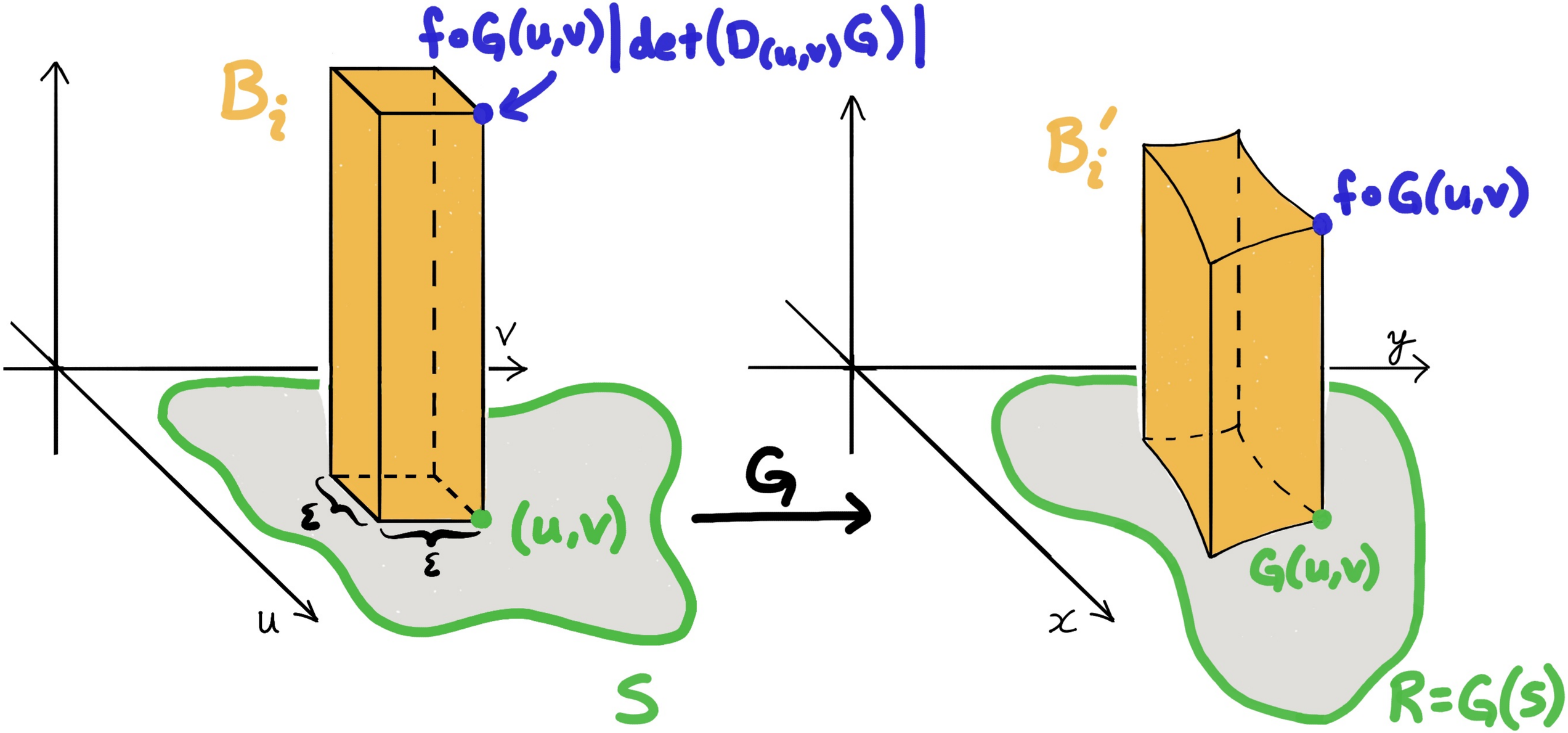

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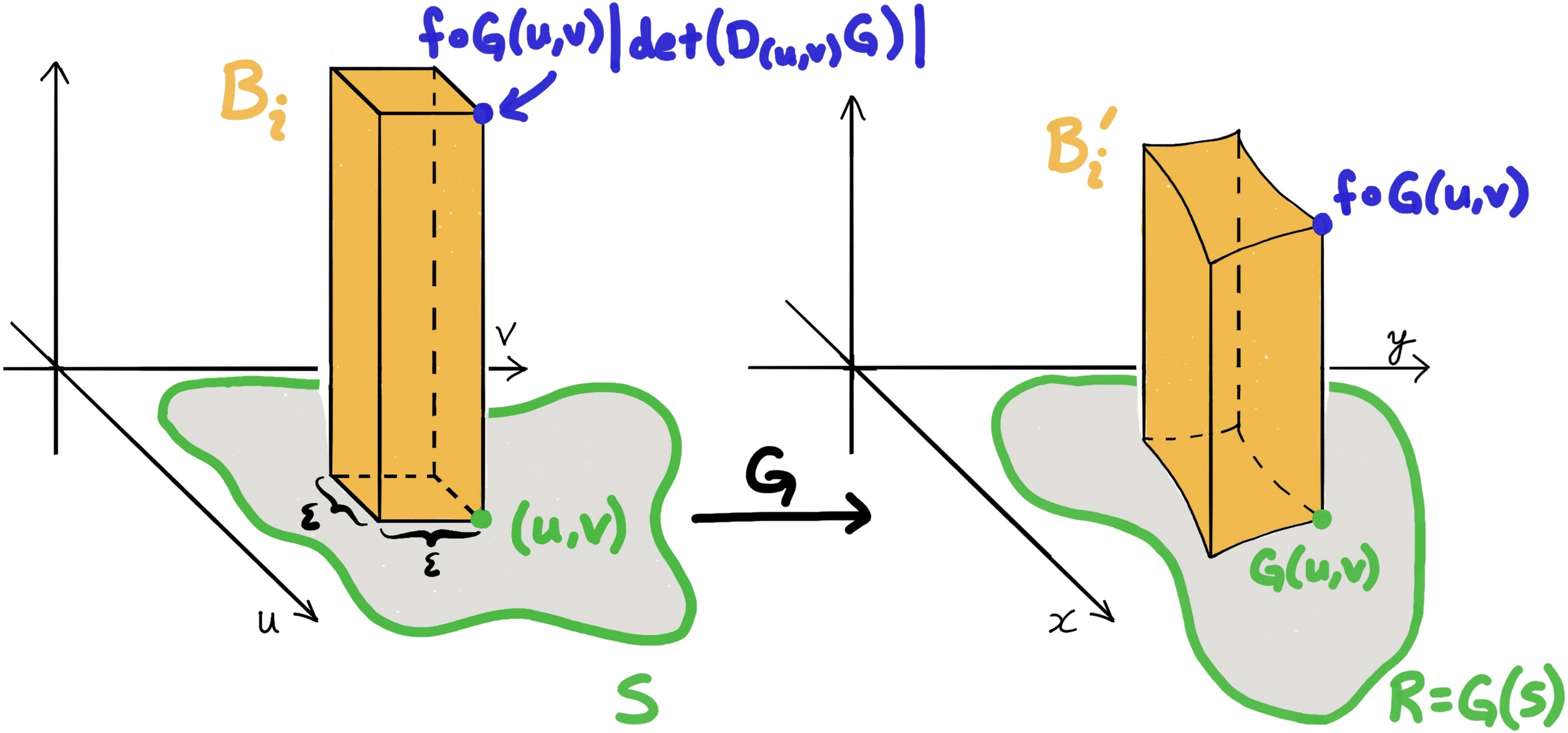

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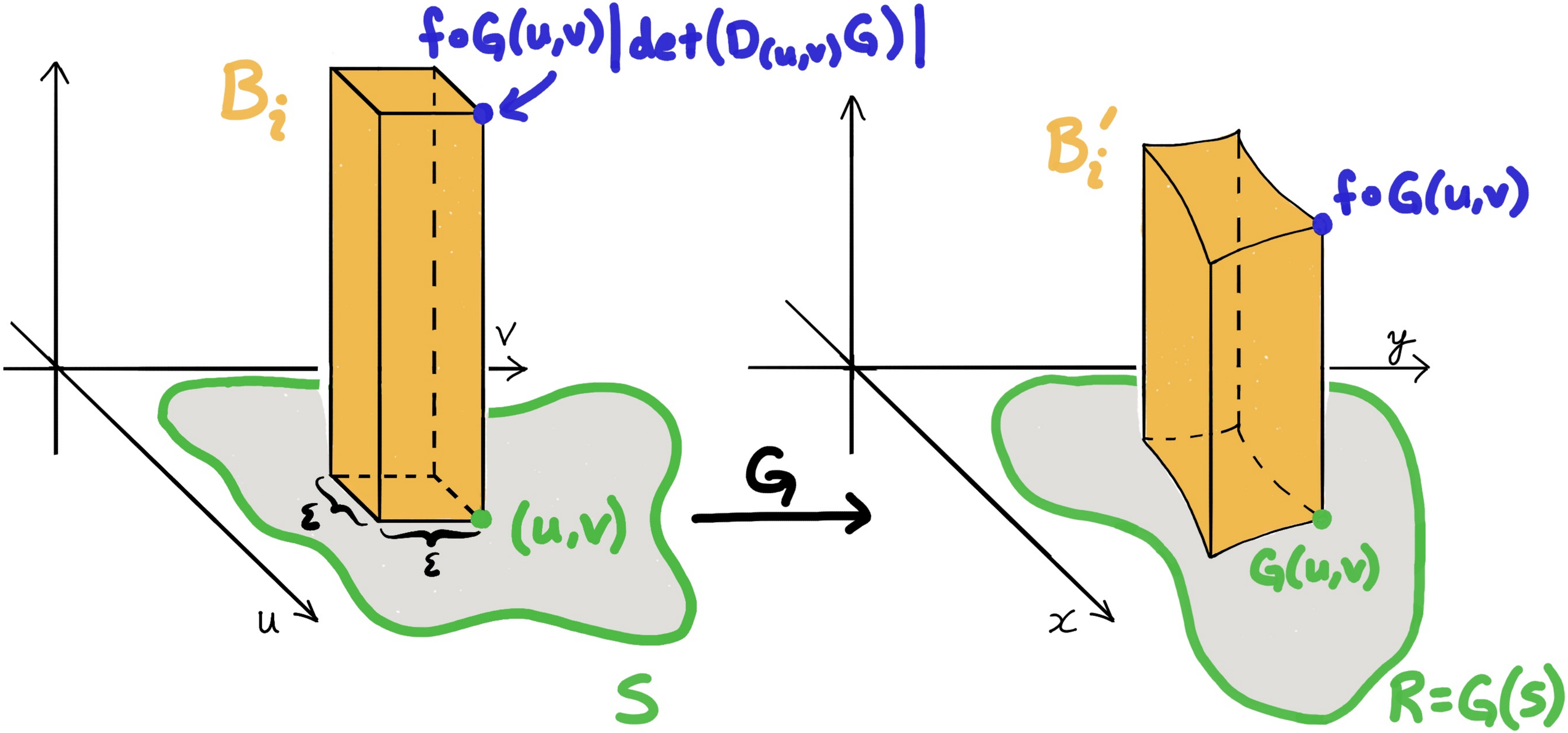

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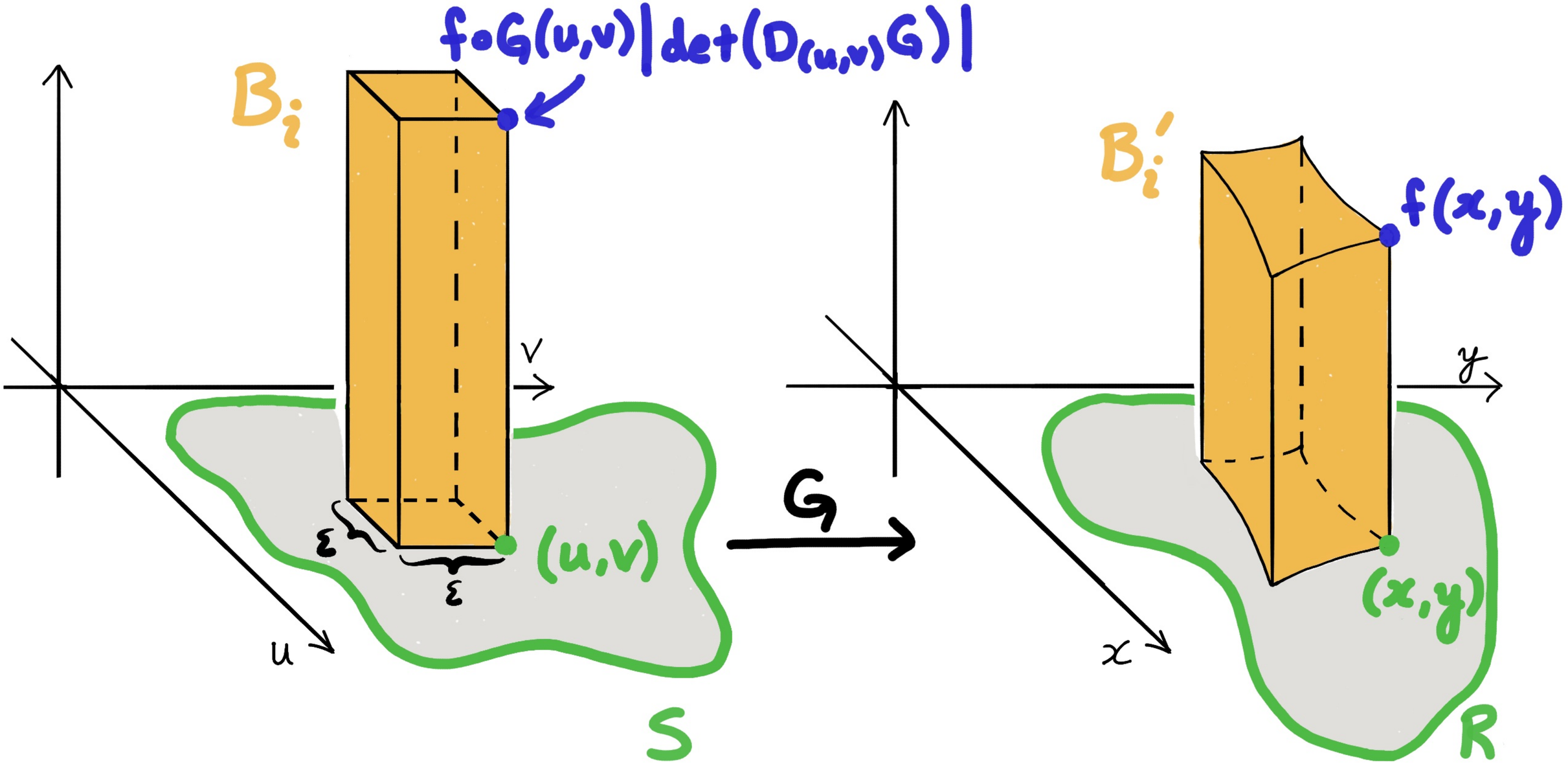

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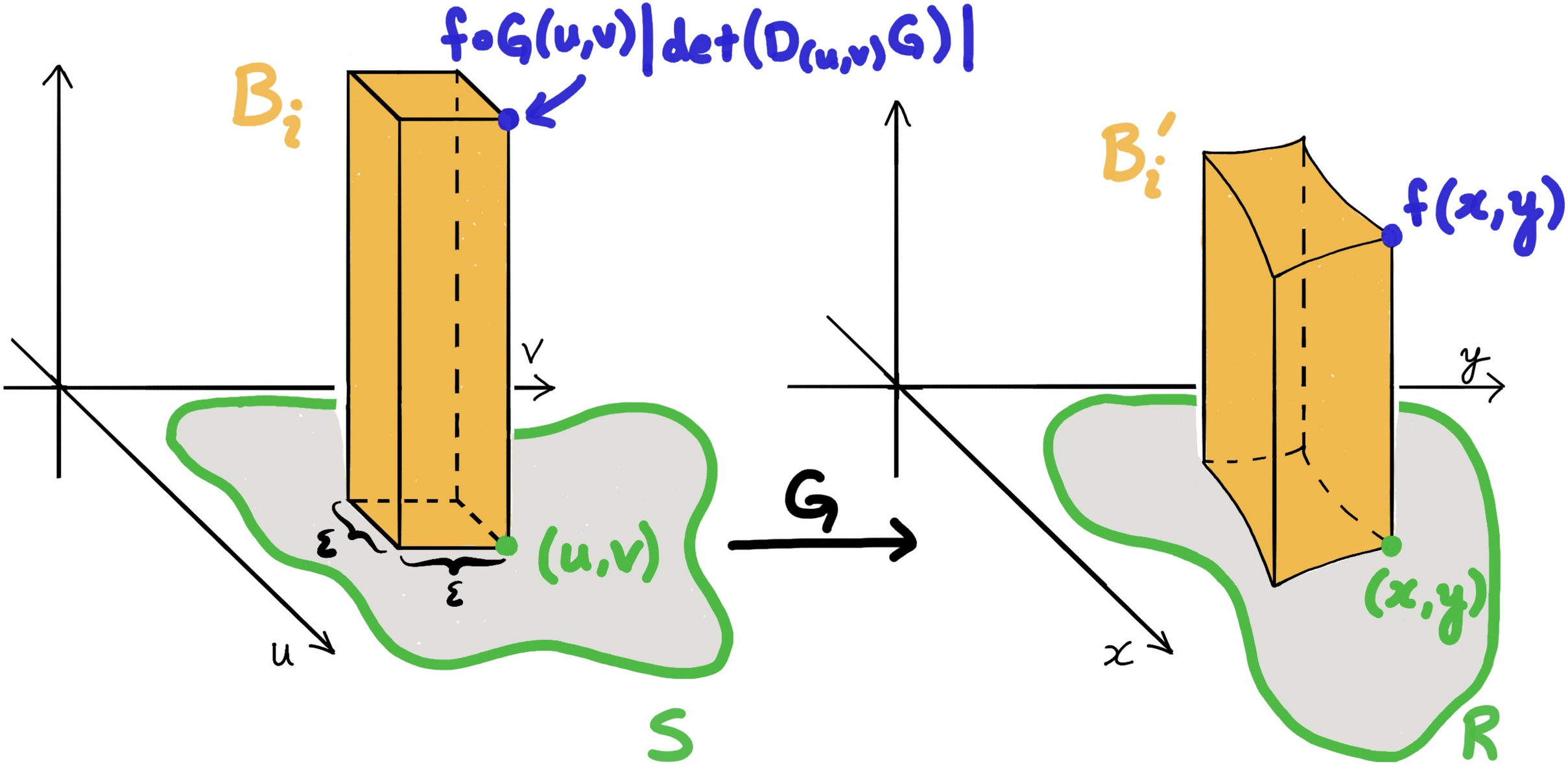

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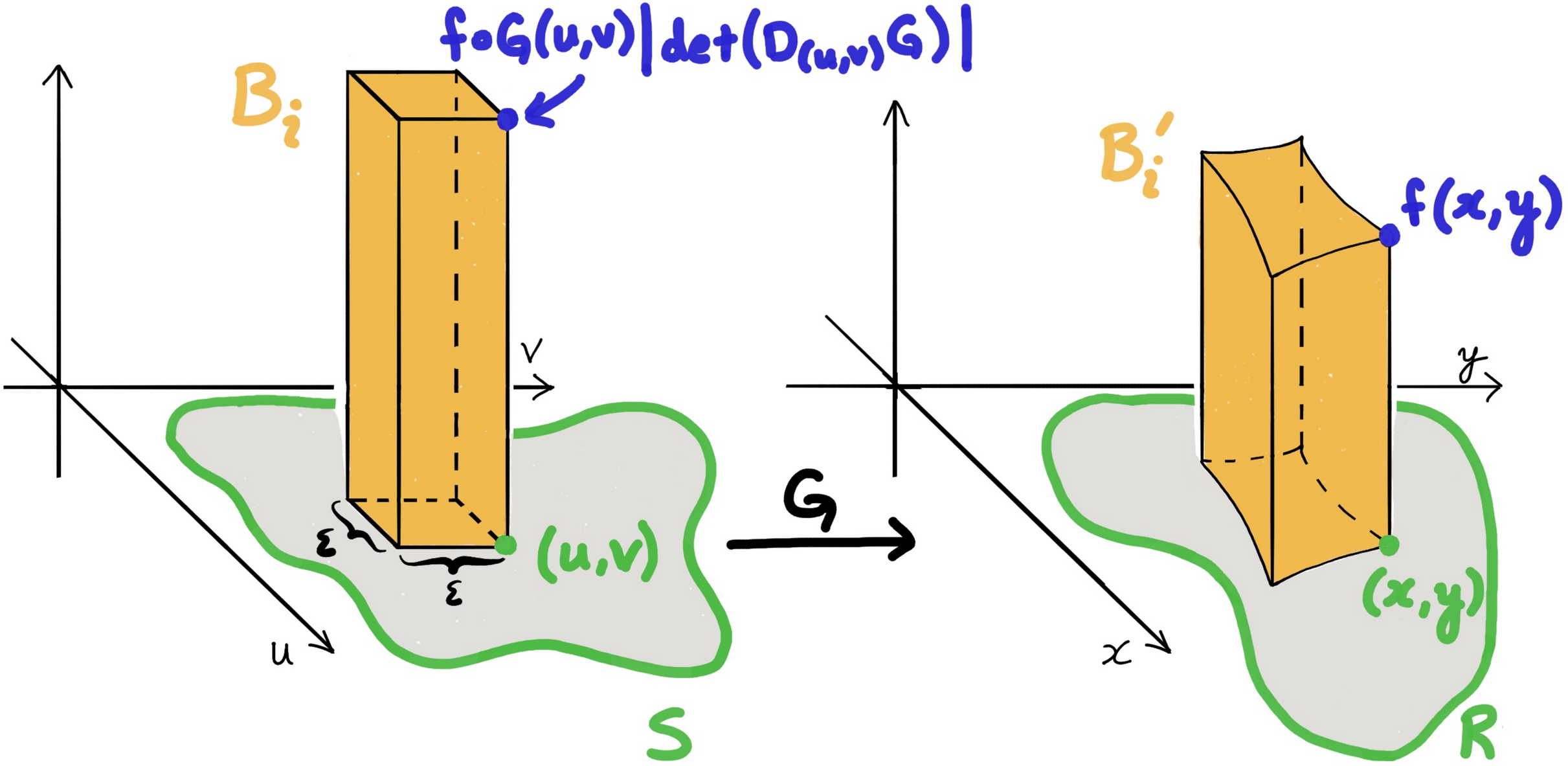

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END

Change of variables

If,

- $f: \underset{x,y}{\mathbb{R}^2} \to \mathbb{R}$, C^0
- $G: \underset{u,v}{\mathbb{R}^2} \to \underset{x,y}{\mathbb{R}^2}$, C^1 and one-to-one
- $G(S) = R$

Then,

$$\iint_R f(x,y)\,dx\,dy = \iint_S f \circ G(u,v)\,\left|\det\left(D_{(u,v)}G\right)\right|\,du\,dv$$

Change of variables

If,

- $G(u,v) = (x(u,v), y(u,v))$
- $G : S \longrightarrow R$

Then,

$$\iint_R f(x,y)\,dxdy = \iint_S f \circ G(u,v)\,\left|\det\left(D_{(u,v)}G\right)\right|\,dudv$$

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Then,

superior shape

regrettable region

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$R \mapsto S$ $\qquad$ $x \mapsto x(u,v)$

$dx\,dy \mapsto du\,dv$ $\qquad$ $y \mapsto y(u,v)$

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$R \mapsto S$

$dx\,dy \mapsto du\,dv$

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$y \mapsto y(u,v)$

Throw in $\left|\det(D_{(u,v)}G)\right|$.

Example:

① $\int_0^2 \int_{x^2}^{x^2+4} xy \, dy \, dx$

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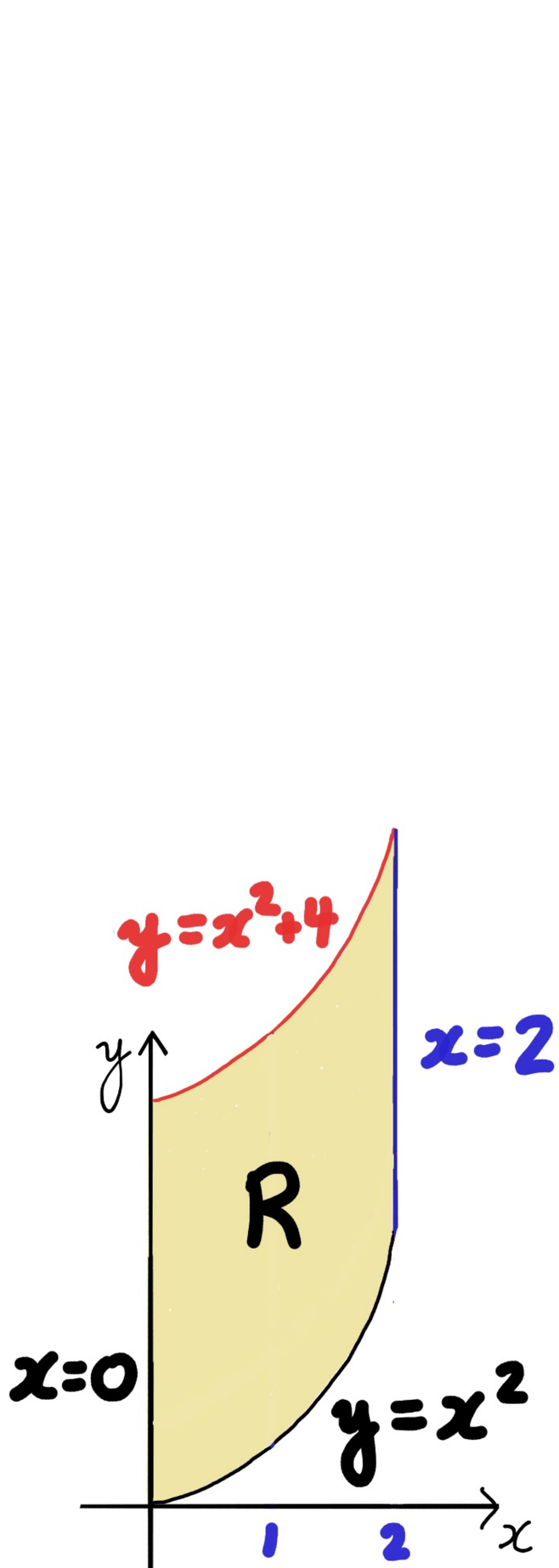

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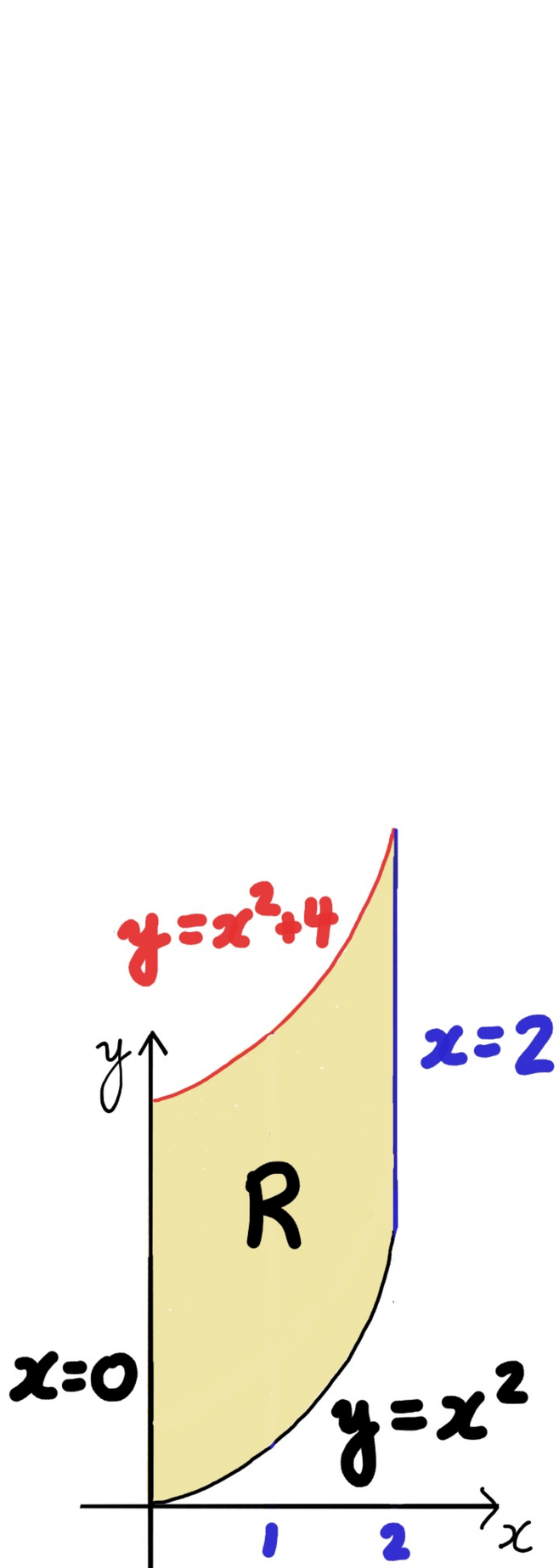

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$$G(u,v) = (u, u^2+v^2)$$

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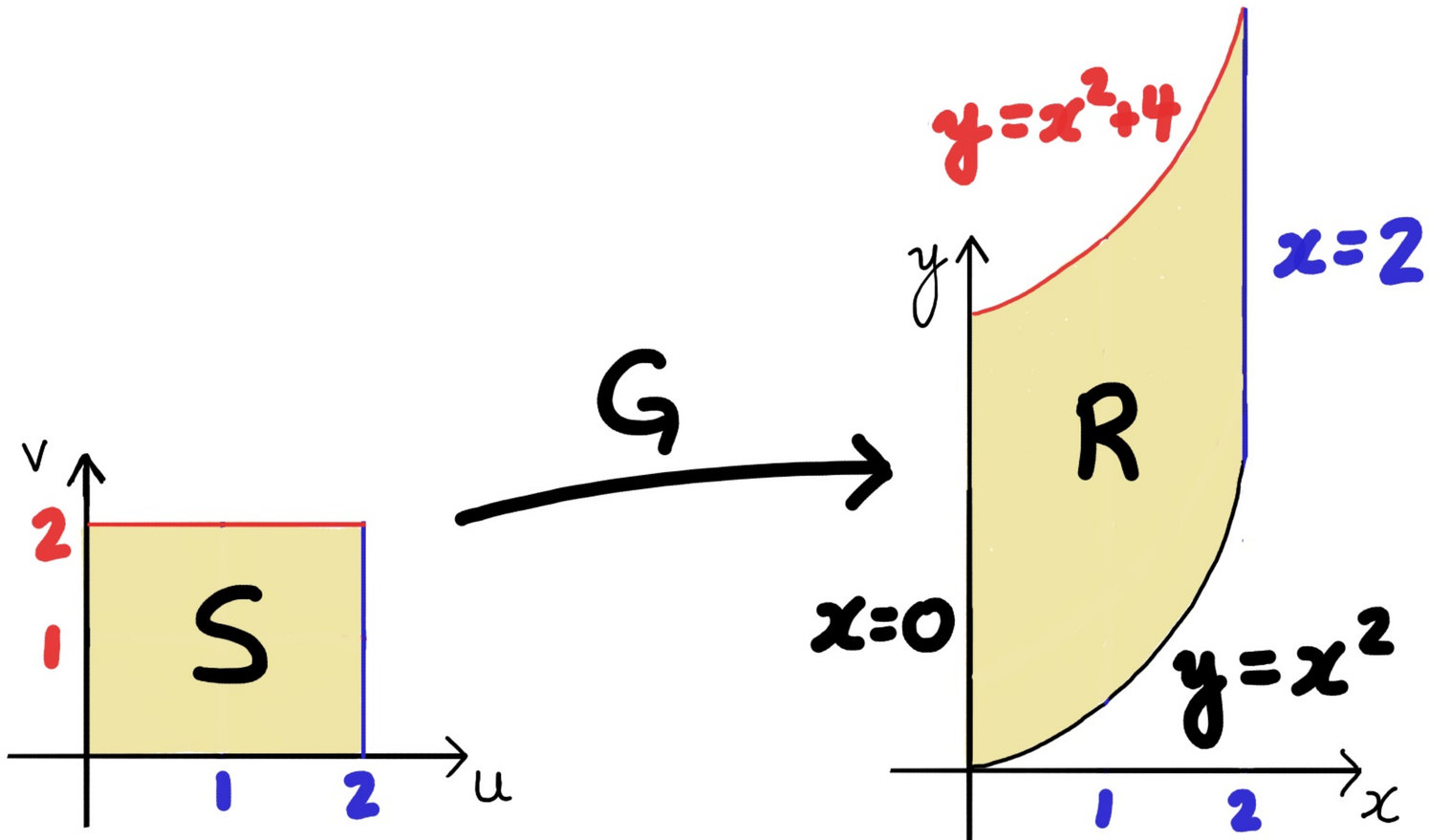

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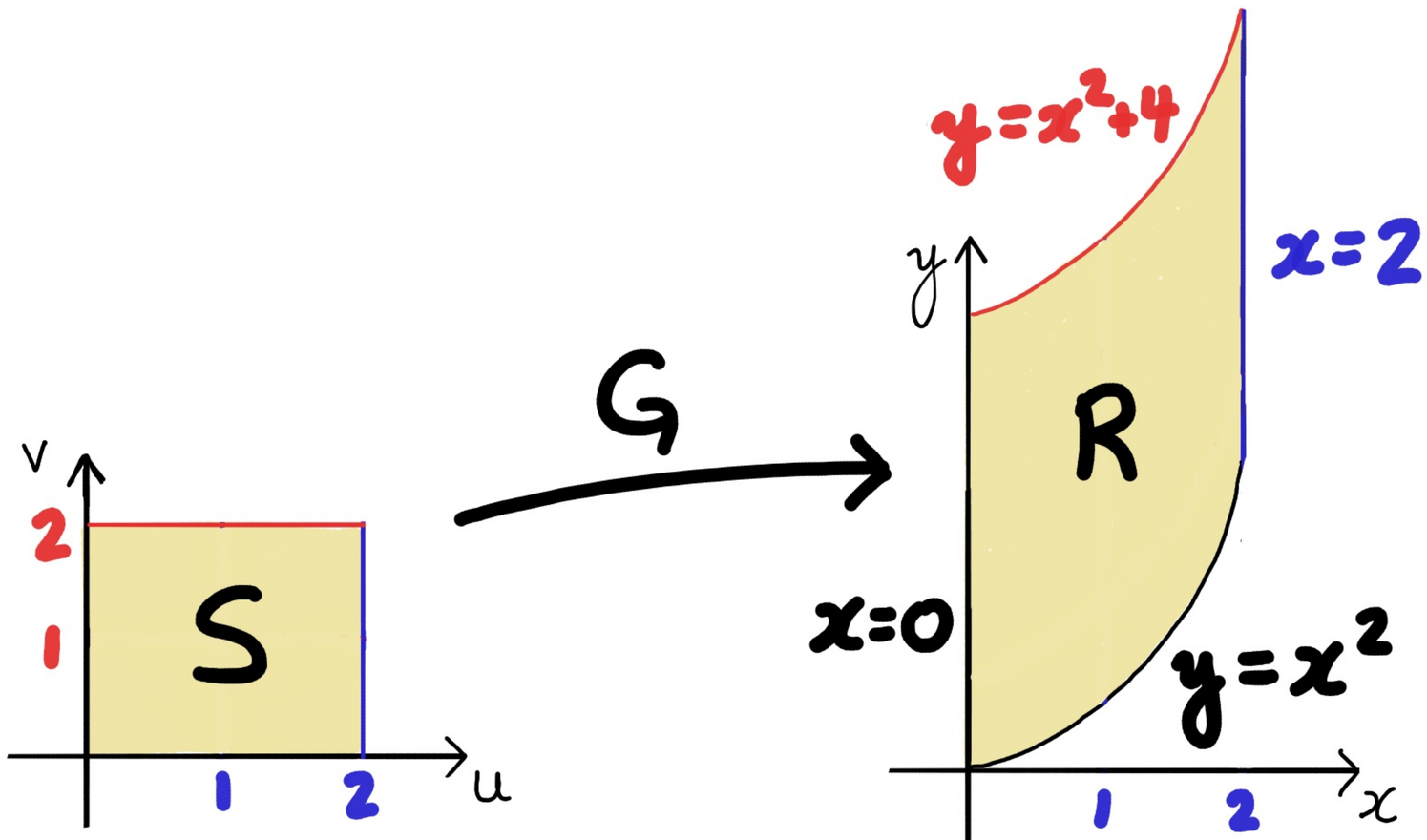

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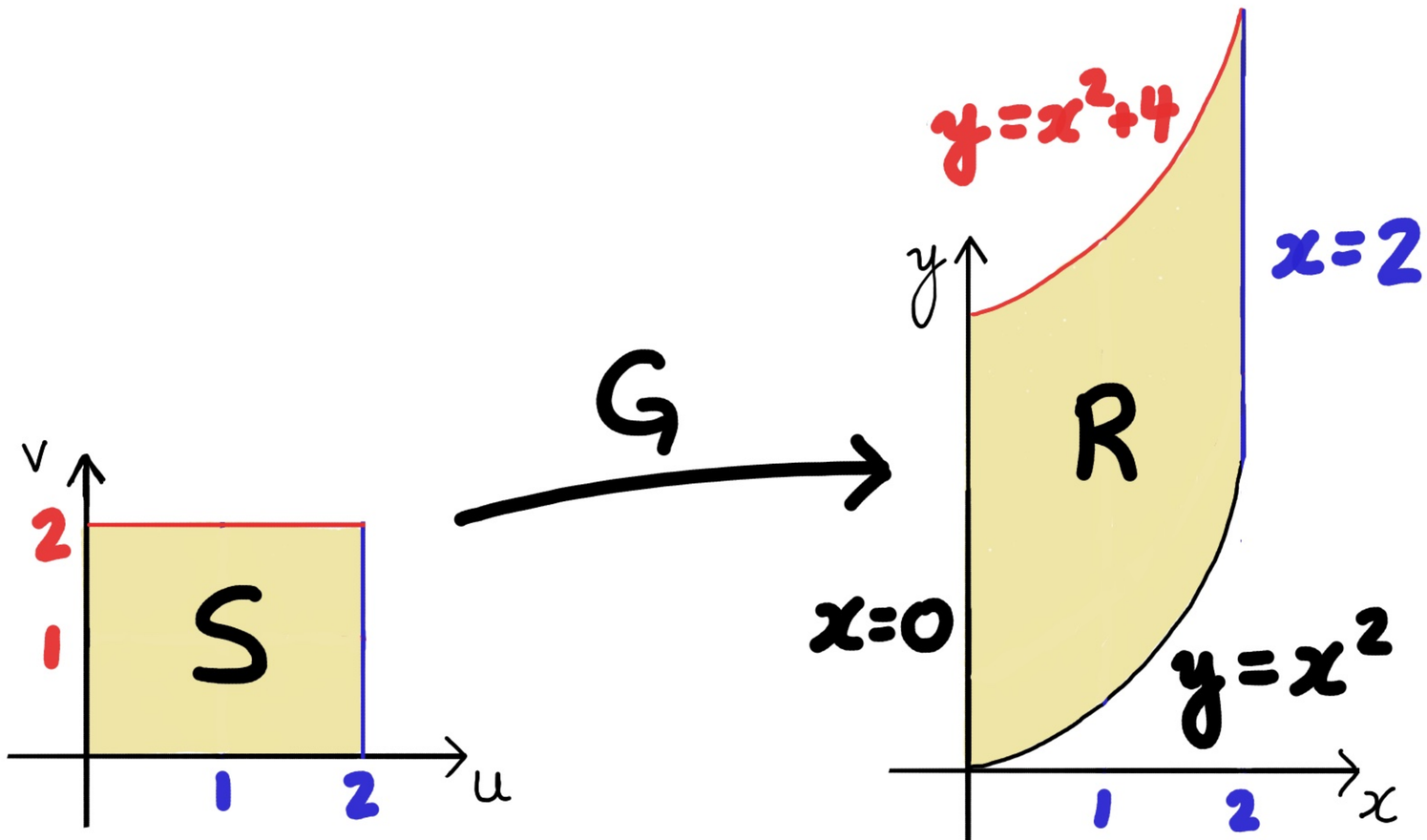

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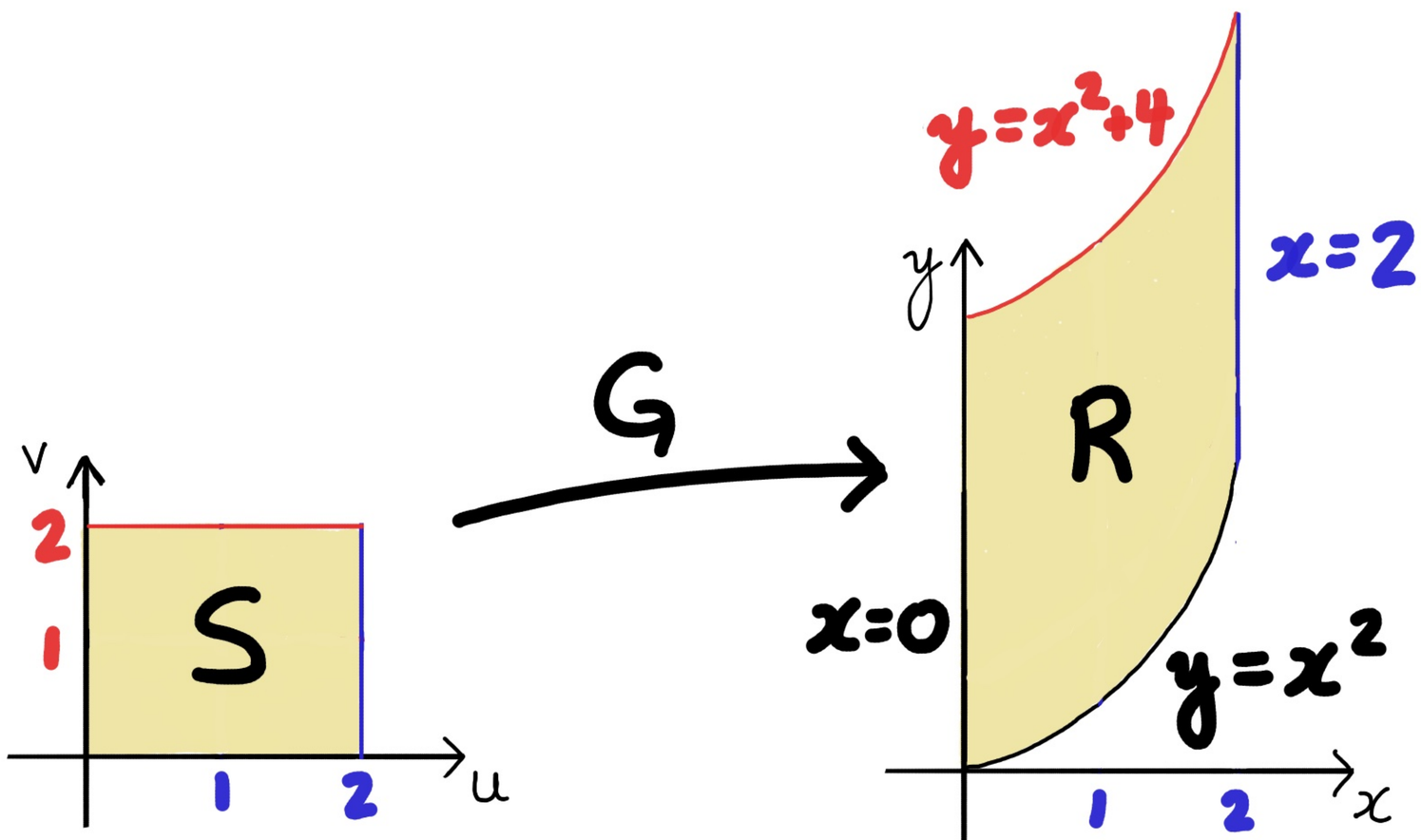

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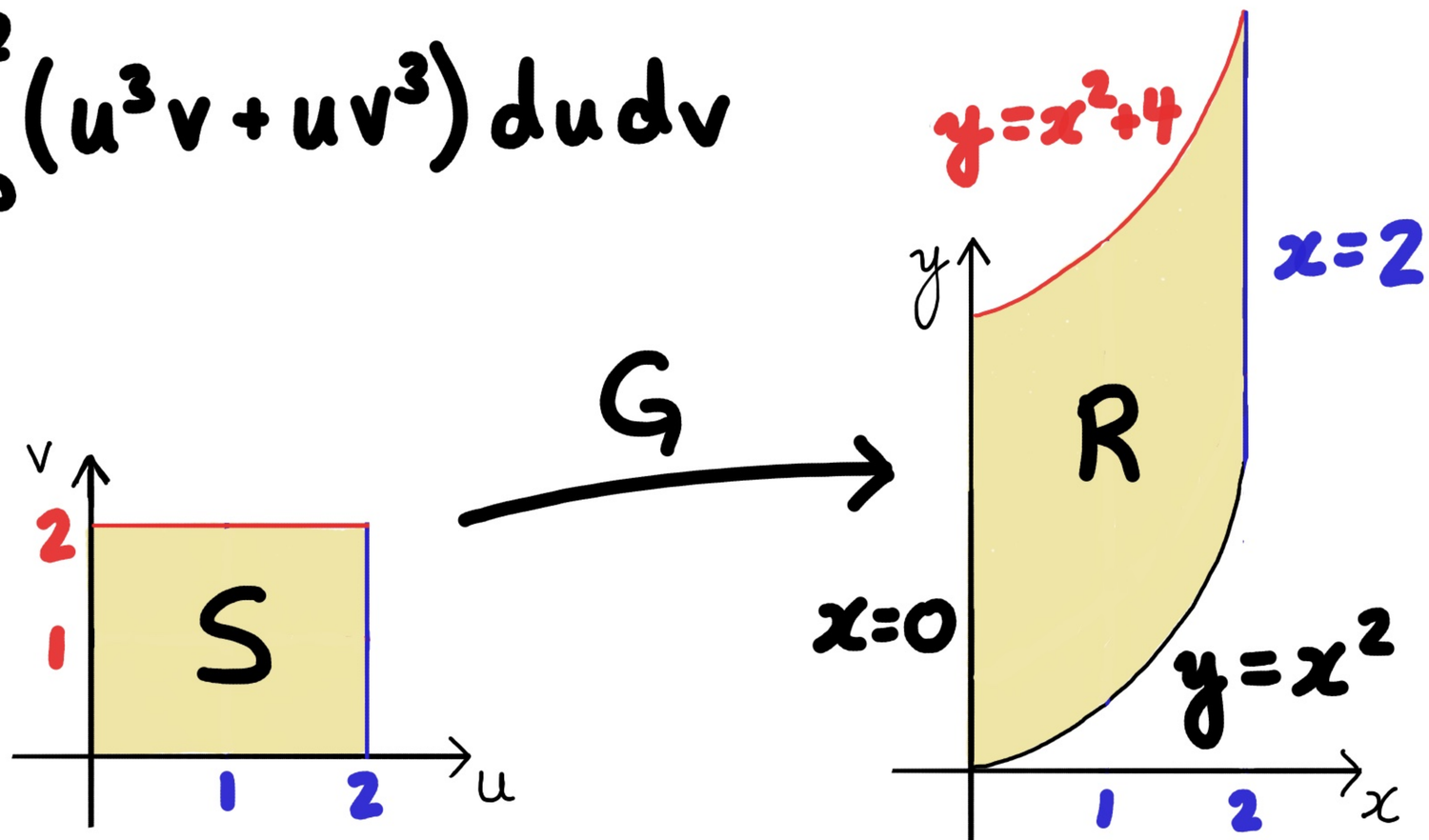

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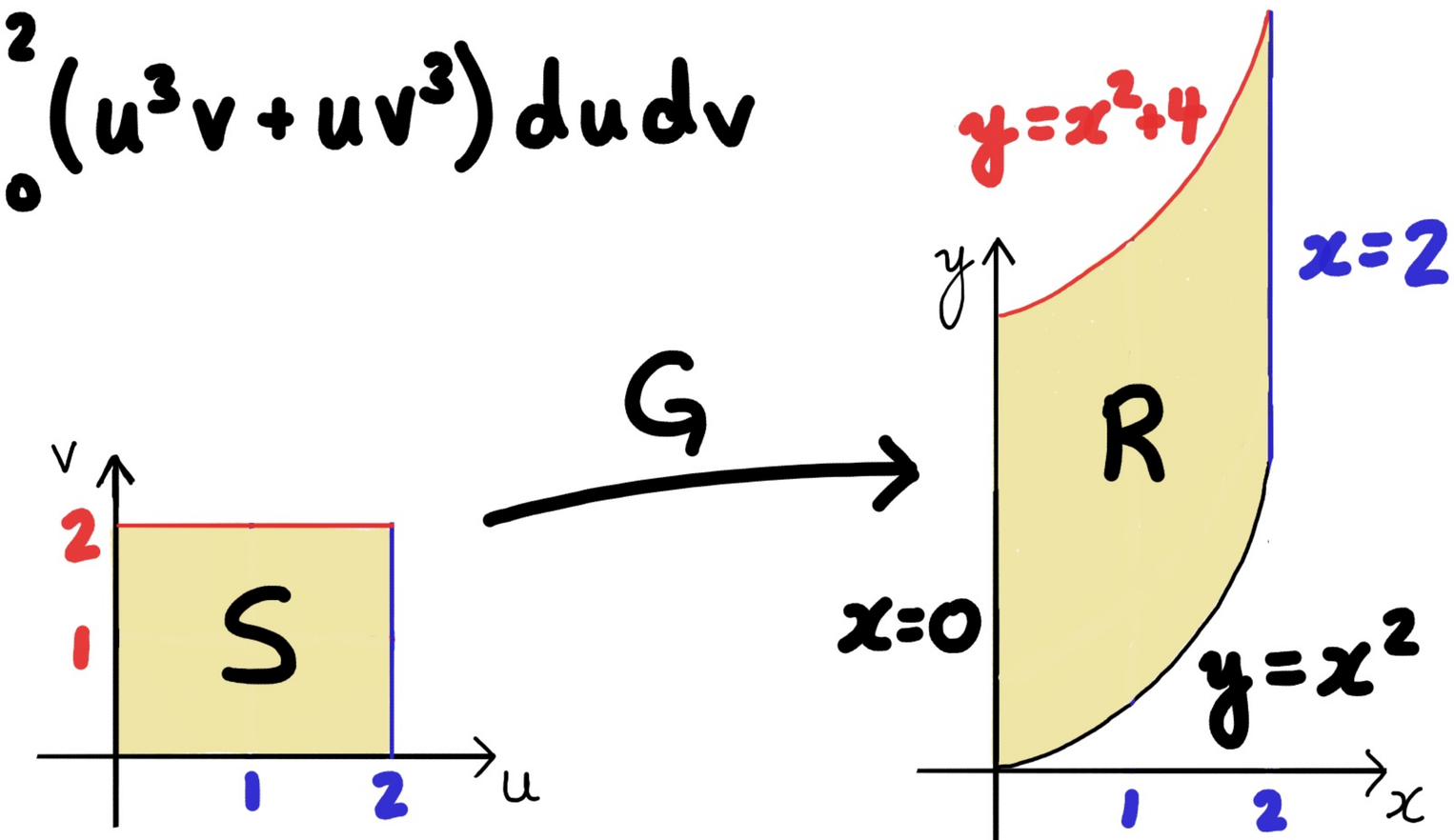
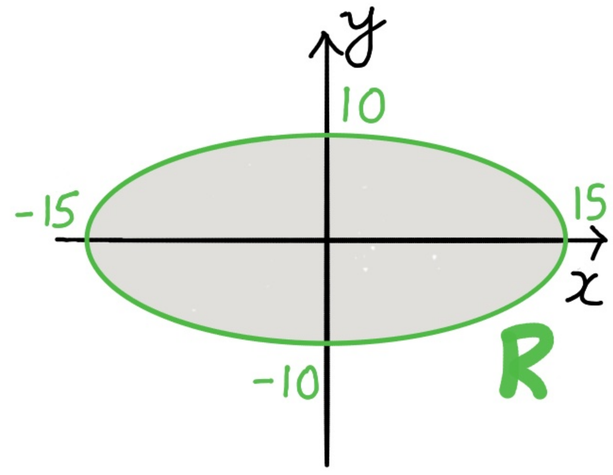


Example:

② Find $\iint_R e^{x^2/9 + y^2/4}\,dx\,dy$ where R is the region of $\mathbb{R}^2$ bounded by the ellipse $\frac{x^2}{15^2} + \frac{y^2}{10^2} = 1$.



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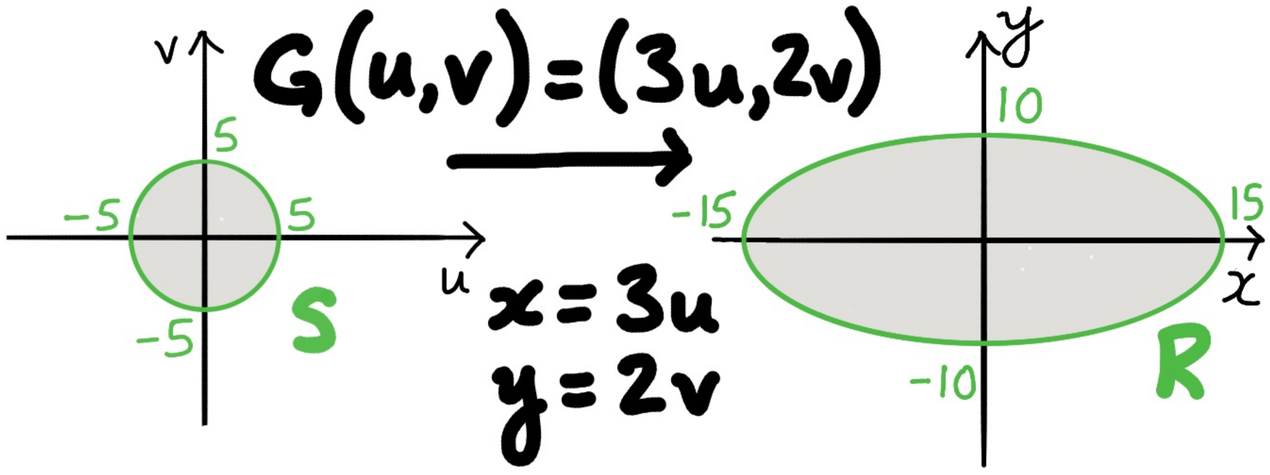

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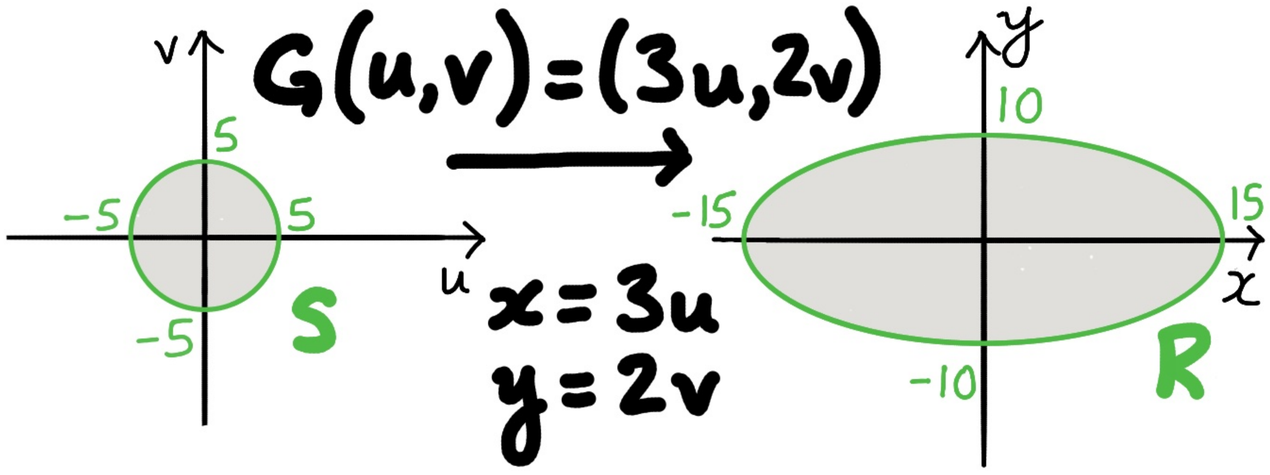

Solution:

$$\iint_R e^{x^2/9 + y^2/4}\,dx\,dy = \iint_S e^{(3u)^2/9 + (2v)^2/4} \left|\det\left(D_{(u,v)}G\right)\right| du\,dv$$

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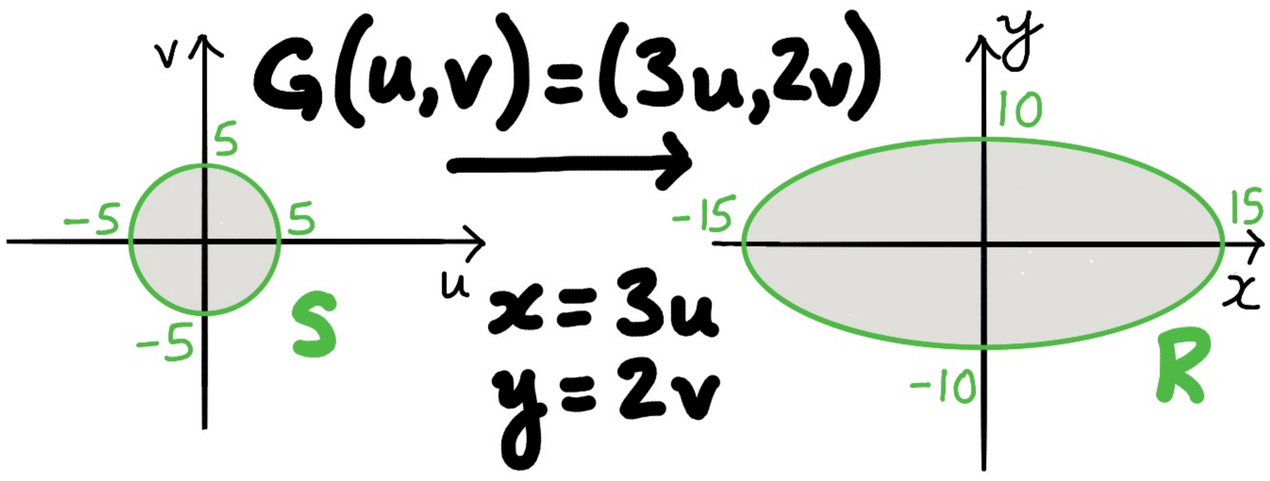

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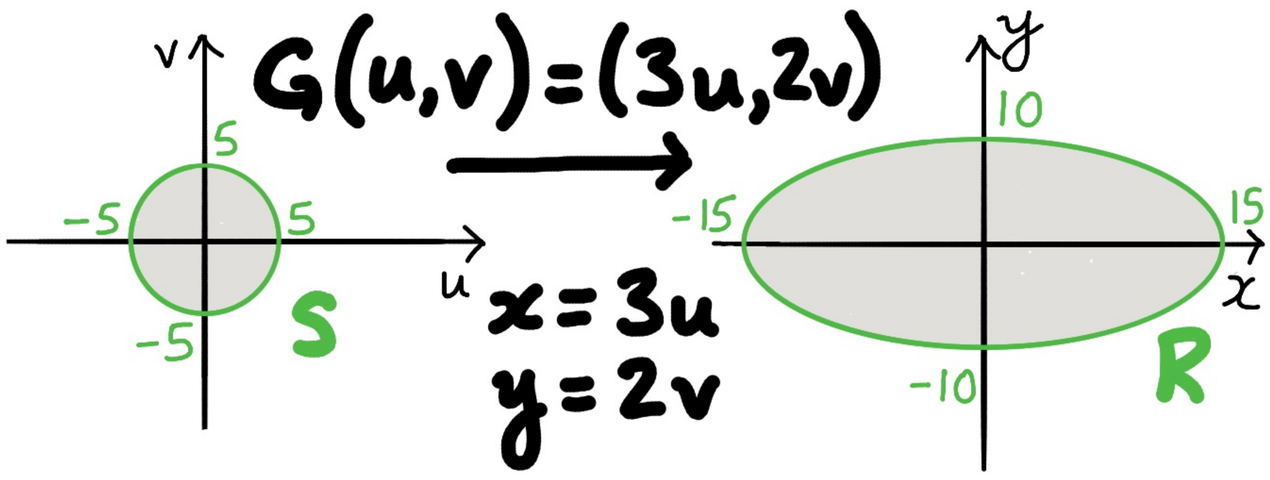

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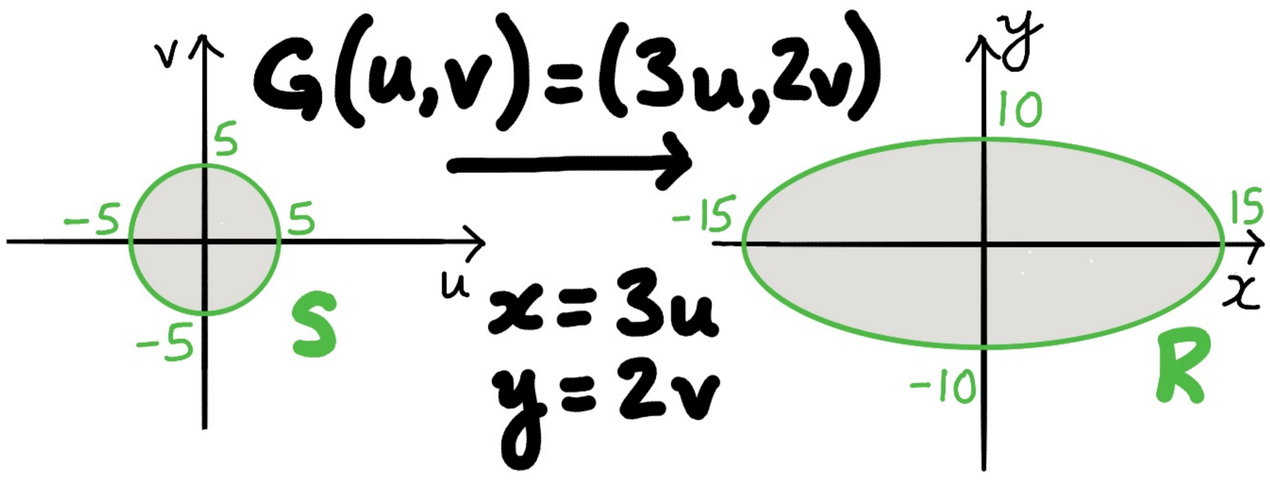

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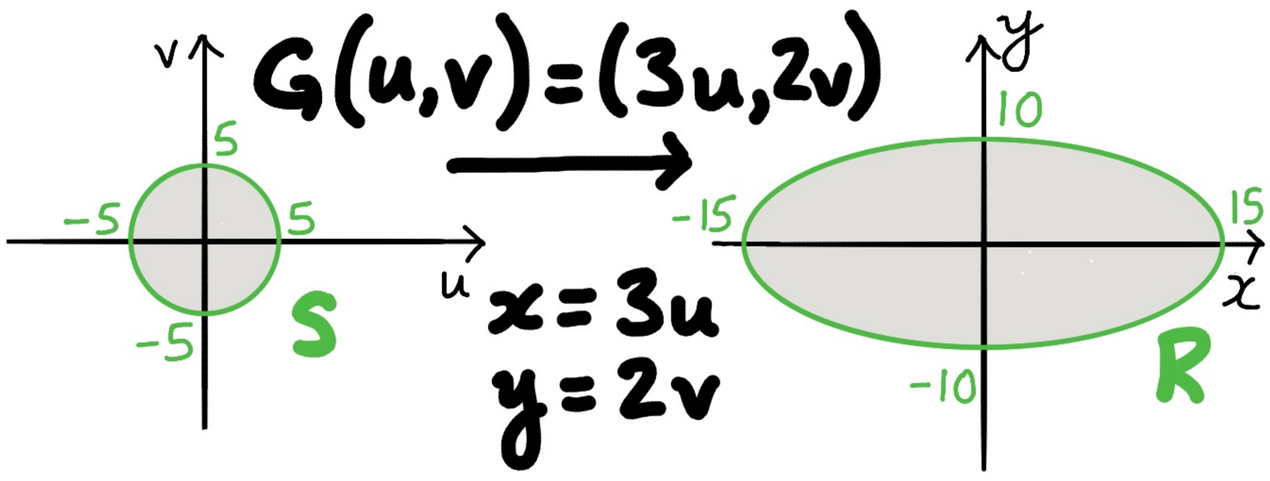

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Change of variables

If,

- $f: \underset{x,y,z}{\mathbb{R}^3} \to \mathbb{R}$
- $G: \underset{u,v,w}{\mathbb{R}^3} \to \underset{x,y,z}{\mathbb{R}^3}$; $G(u,v,w) = \big(x(u,v,w), y(u,v,w), z(u,v,w)\big)$
- $G(S) = R$

Then,

$$\iiint_R f(x,y,z)\,dx\,dy\,dz$$

$$=$$

$$\iiint_S f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)\,\big|\det\big(D_{(u,v,w)}G\big)\big|\,du\,dv\,dw$$

Change of variables

If,

- $f: \mathbb{R} \to \mathbb{R}$
- $g: \underset{u}{\mathbb{R}} \to \underset{x}{\mathbb{R}}$; $g(u) = x(u)$
- $g(S) = R$

Then,

$$\int_R f(x)\,dx = \int_S f(x(u))\,\left|\det(D_u g)\right|\,du$$

Change of variables

If,

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Then,

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