

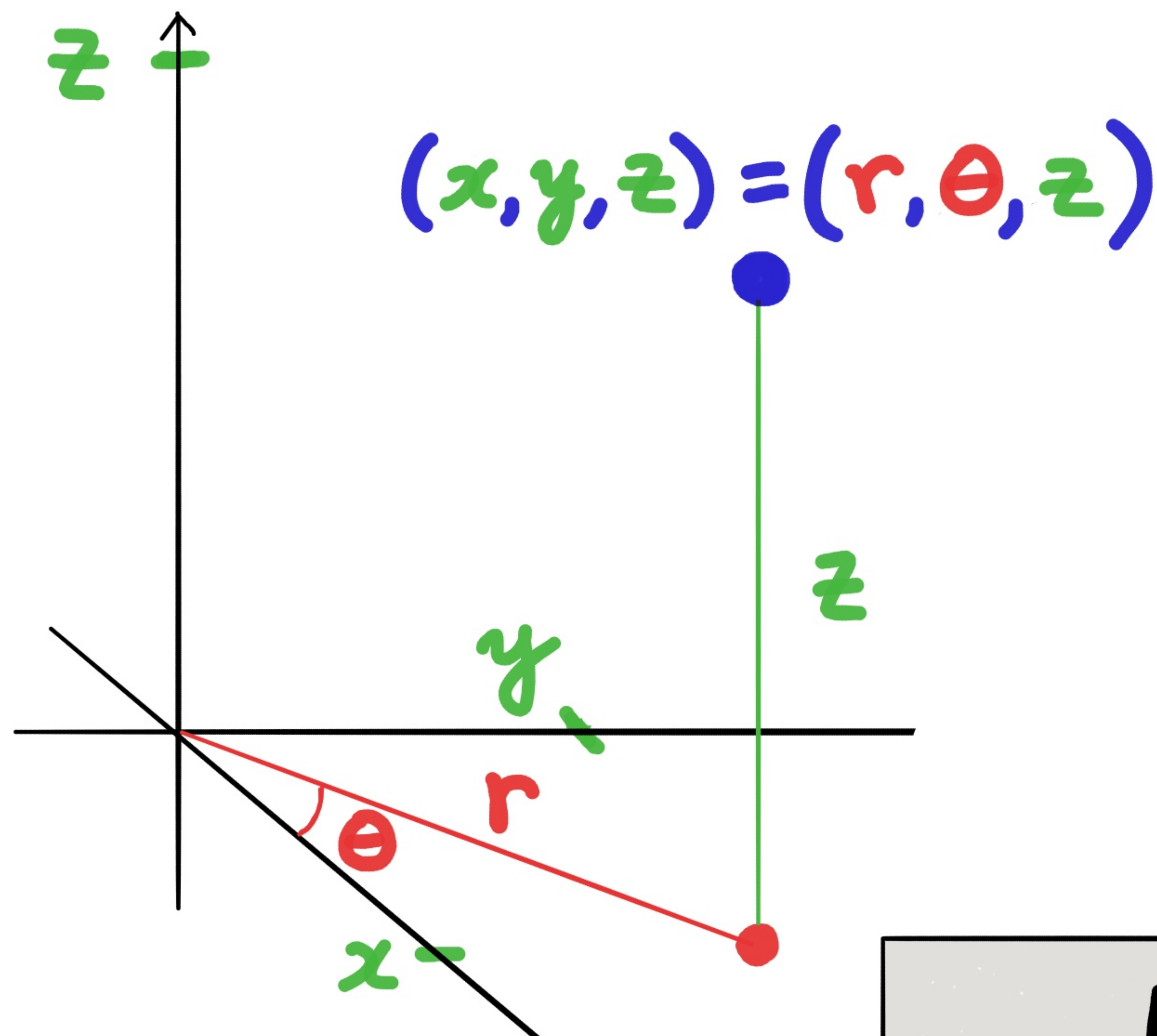
Twenty-four

① Cylindrical and spherical
coordinates for $\mathbb{R}^3$

② Triple integrals

① Cylindrical and
spherical coordinates
for $\mathbb{R}^3$

Cylindrical coordinates



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$r = \sqrt{x^2 + y^2}$$

$$0 \leq r$$

$$0 \leq \theta \leq 2\pi$$

Spherical coordinates

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

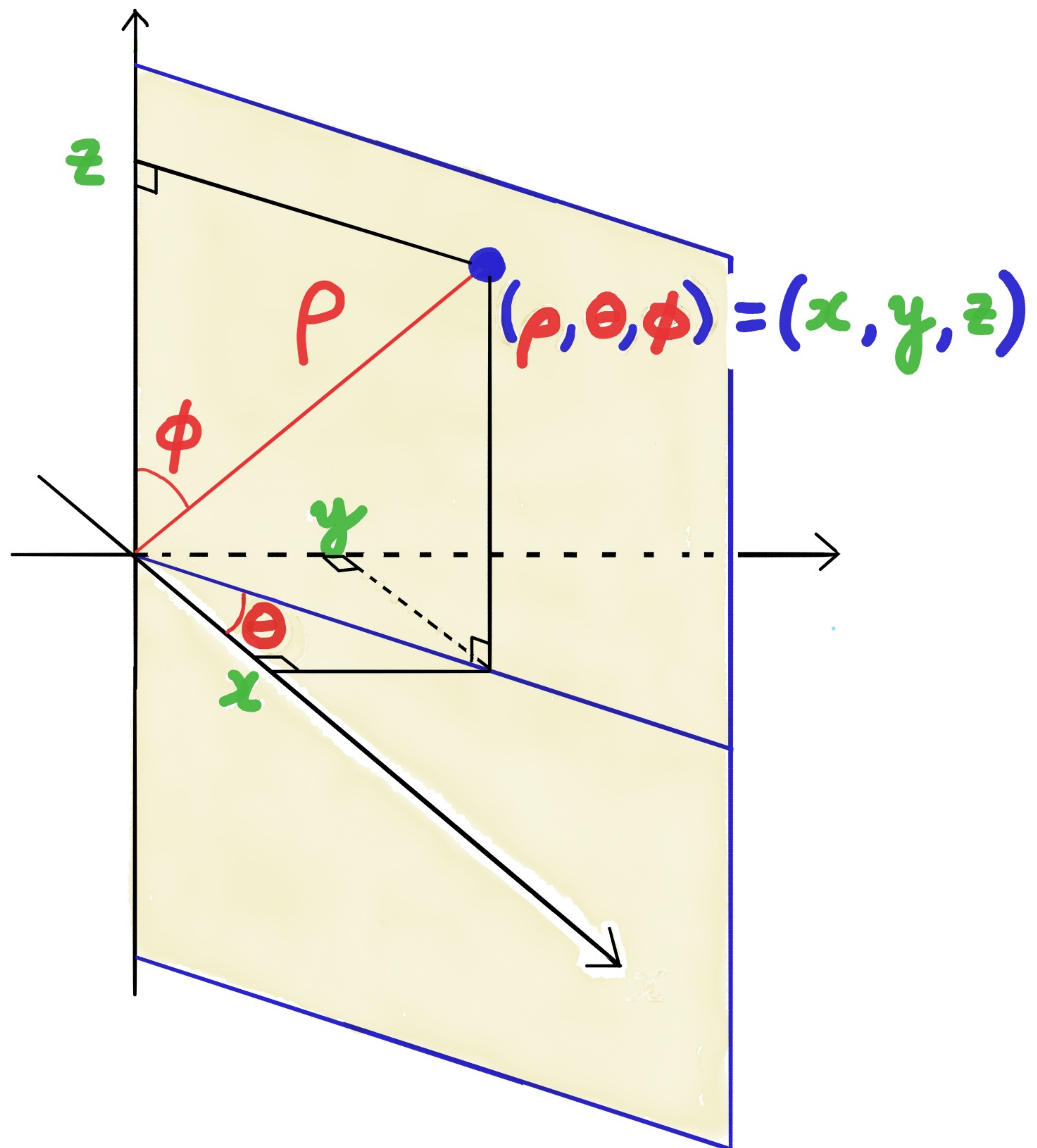
$$z = \rho \cos \phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

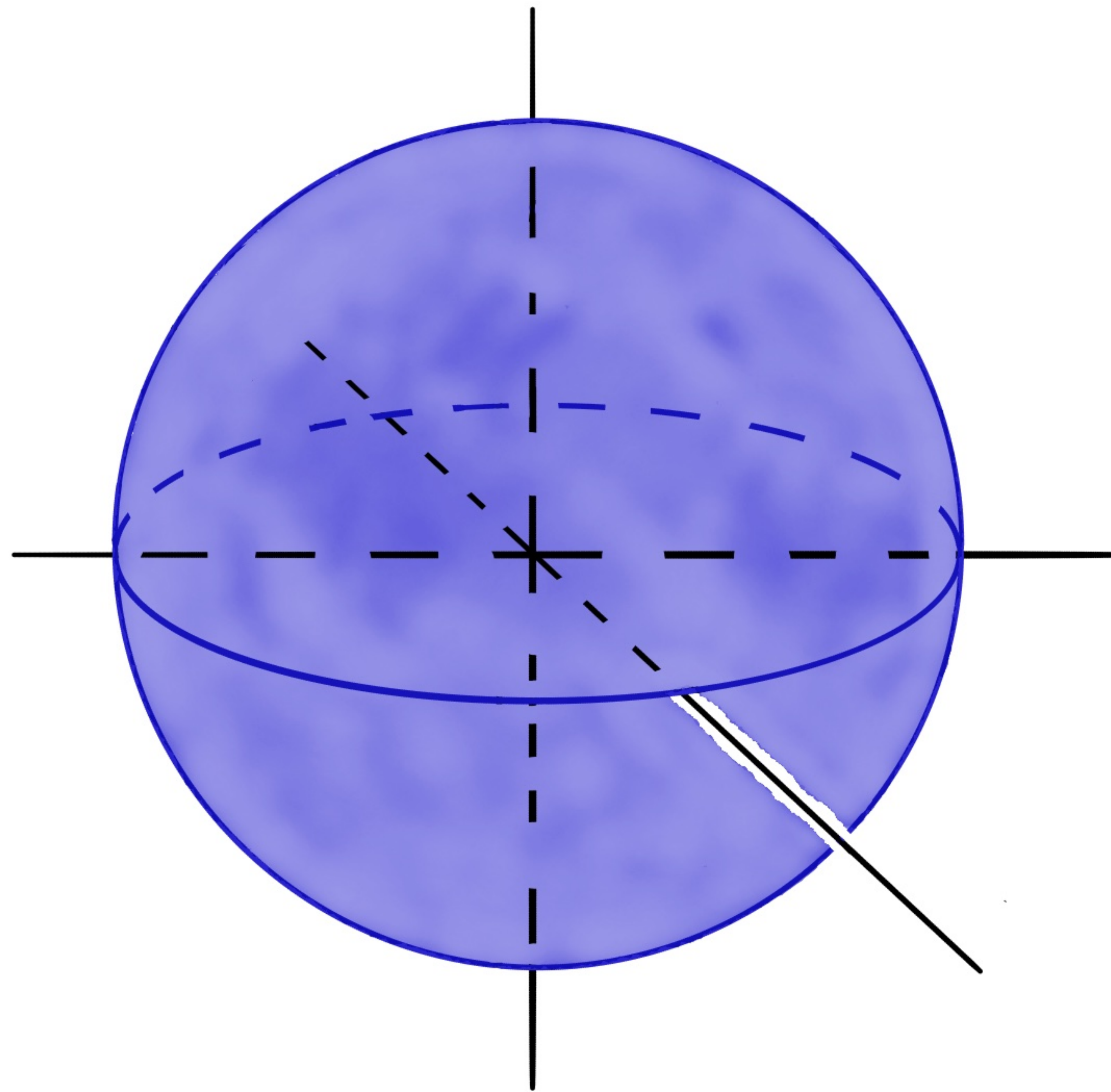
$$0 \leq \rho$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

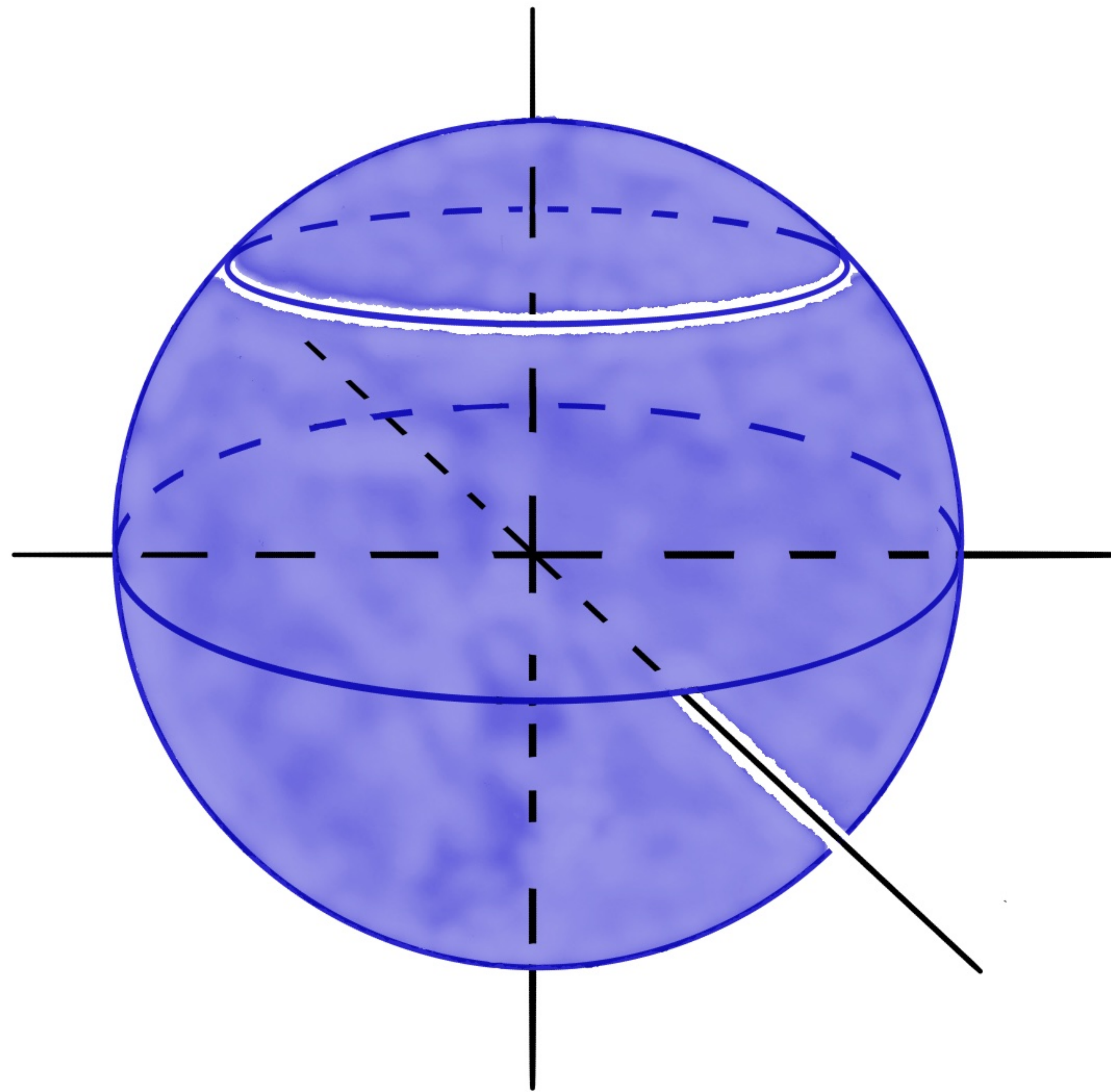


Spherical: $\rho = \rho_0, \phi = \phi_0$
is circle of radius $\rho_0 \sin \phi_0$



$$\rho = \rho_0$$

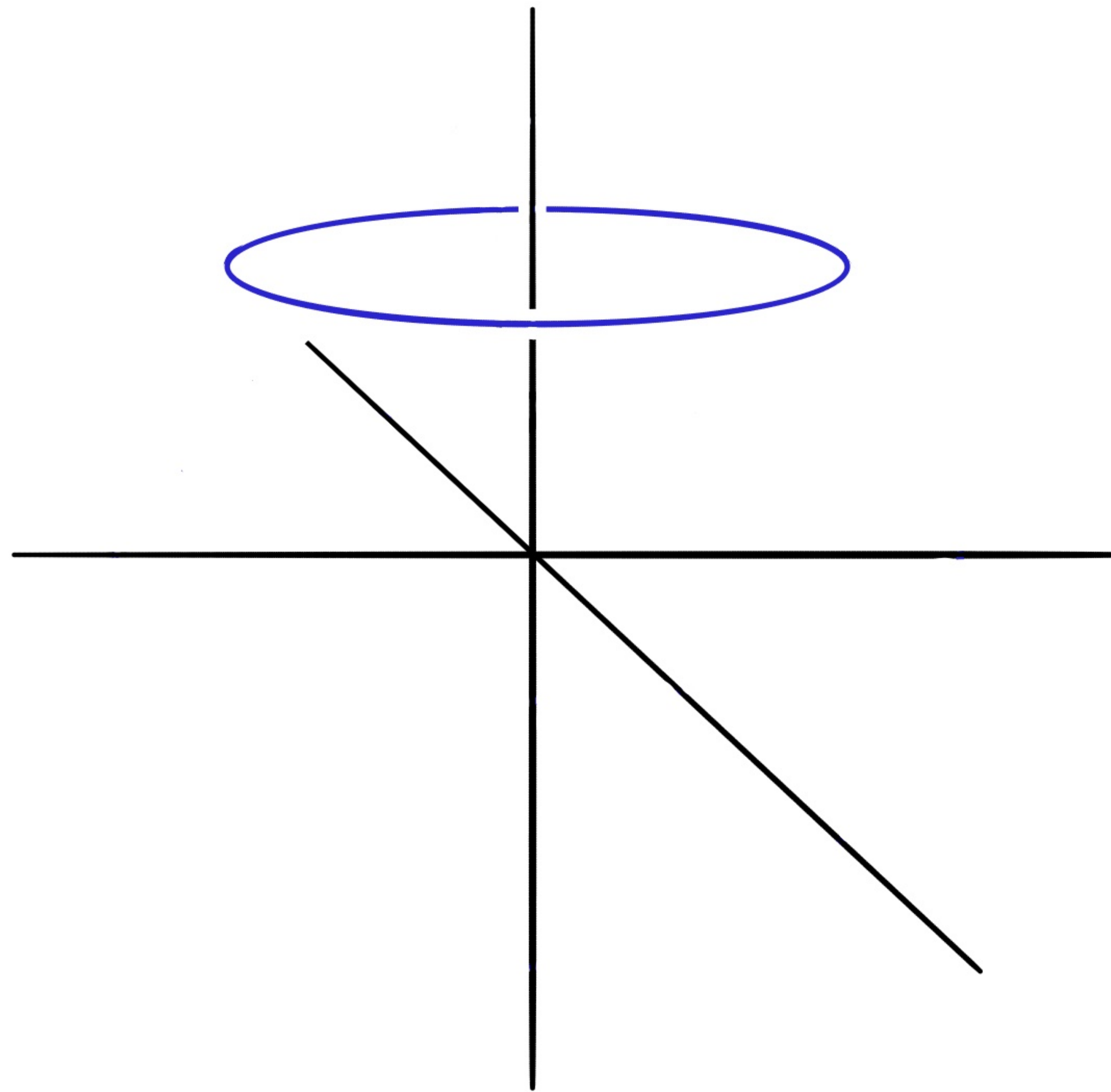
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$$\rho = \rho_0$$

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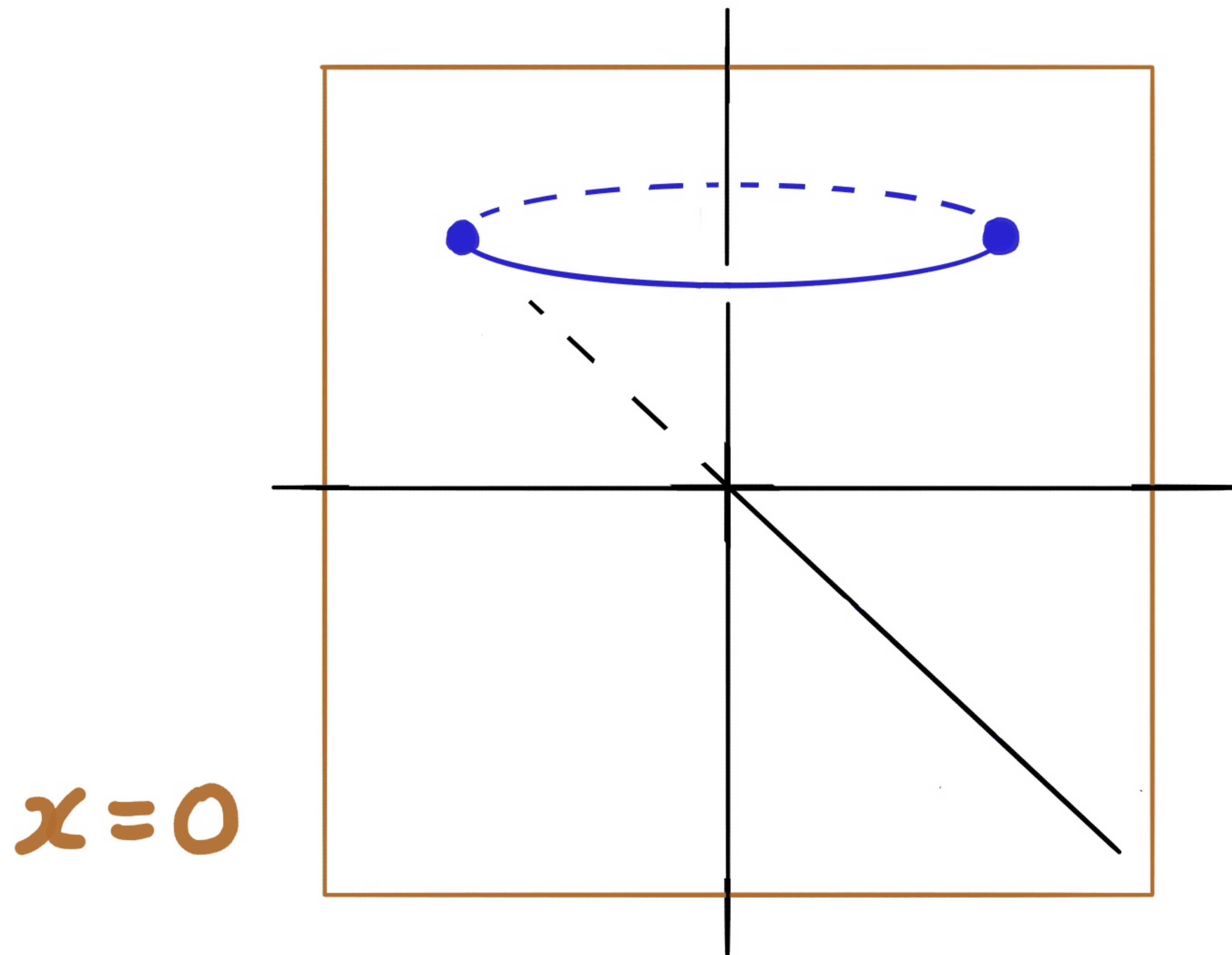
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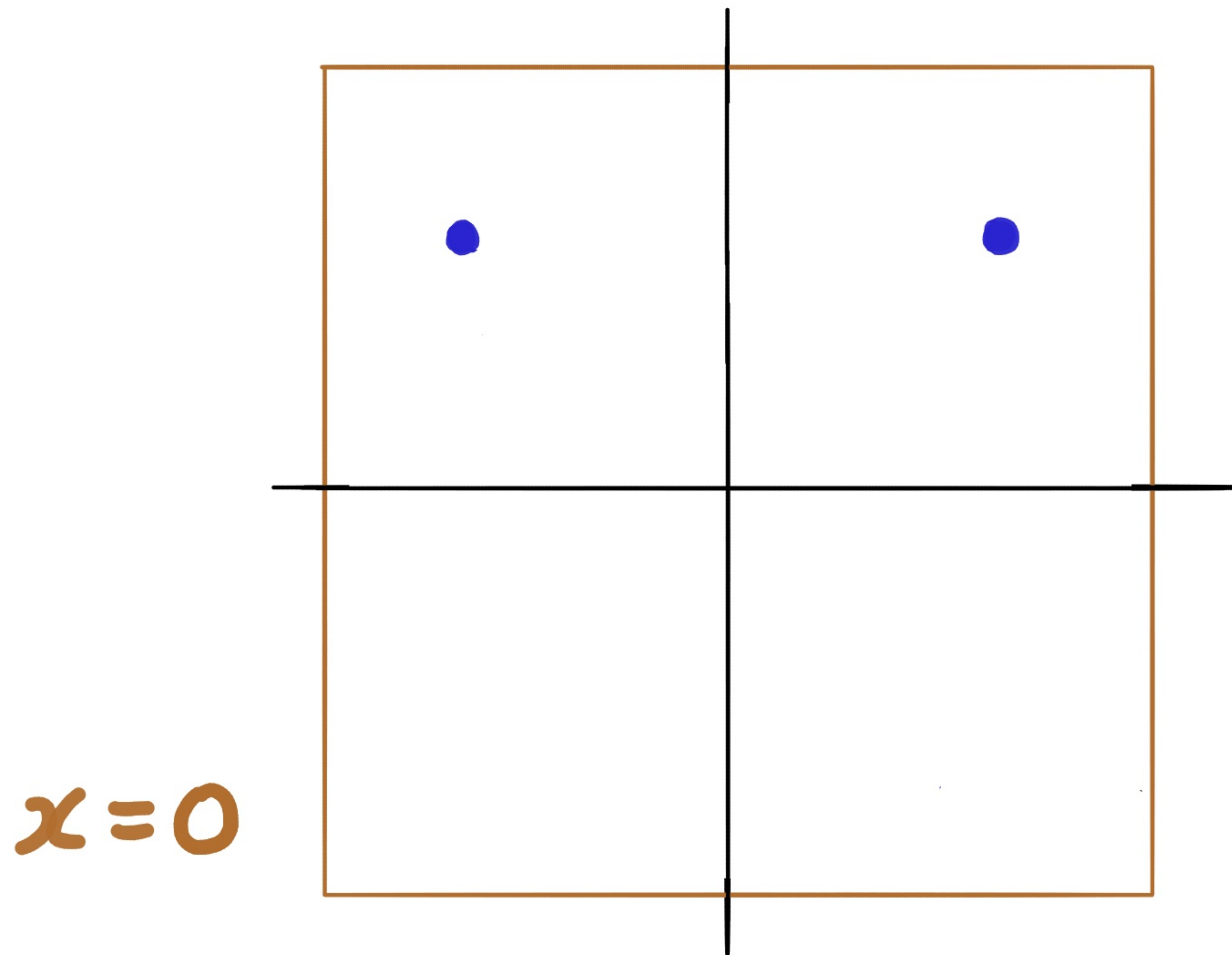


$$x=0$$

$$\rho = \rho_0$$

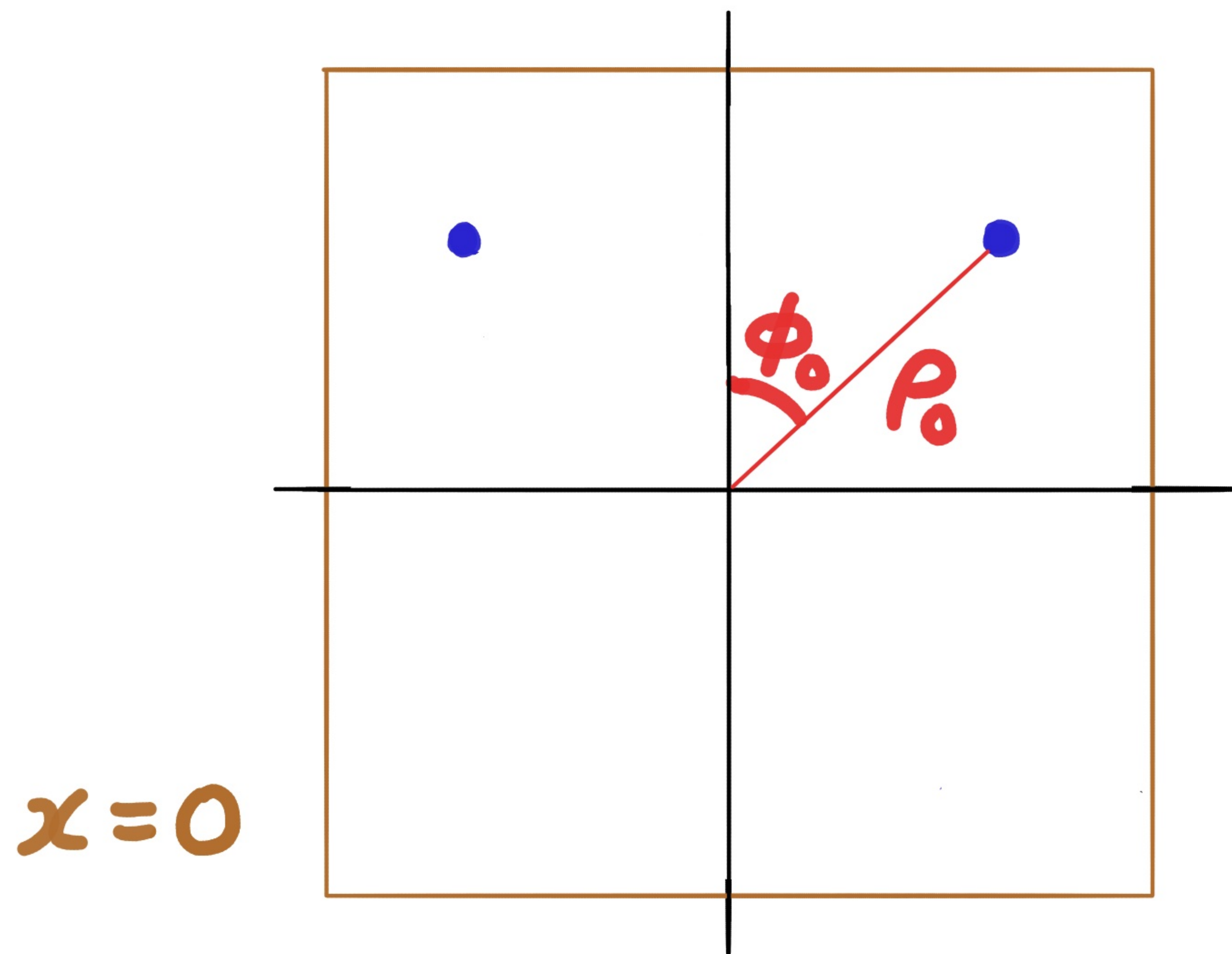
$$\phi = \phi_0$$

Spherical: $\rho = \rho_0, \phi = \phi_0$
is circle of radius $\rho_0 \sin \phi_0$



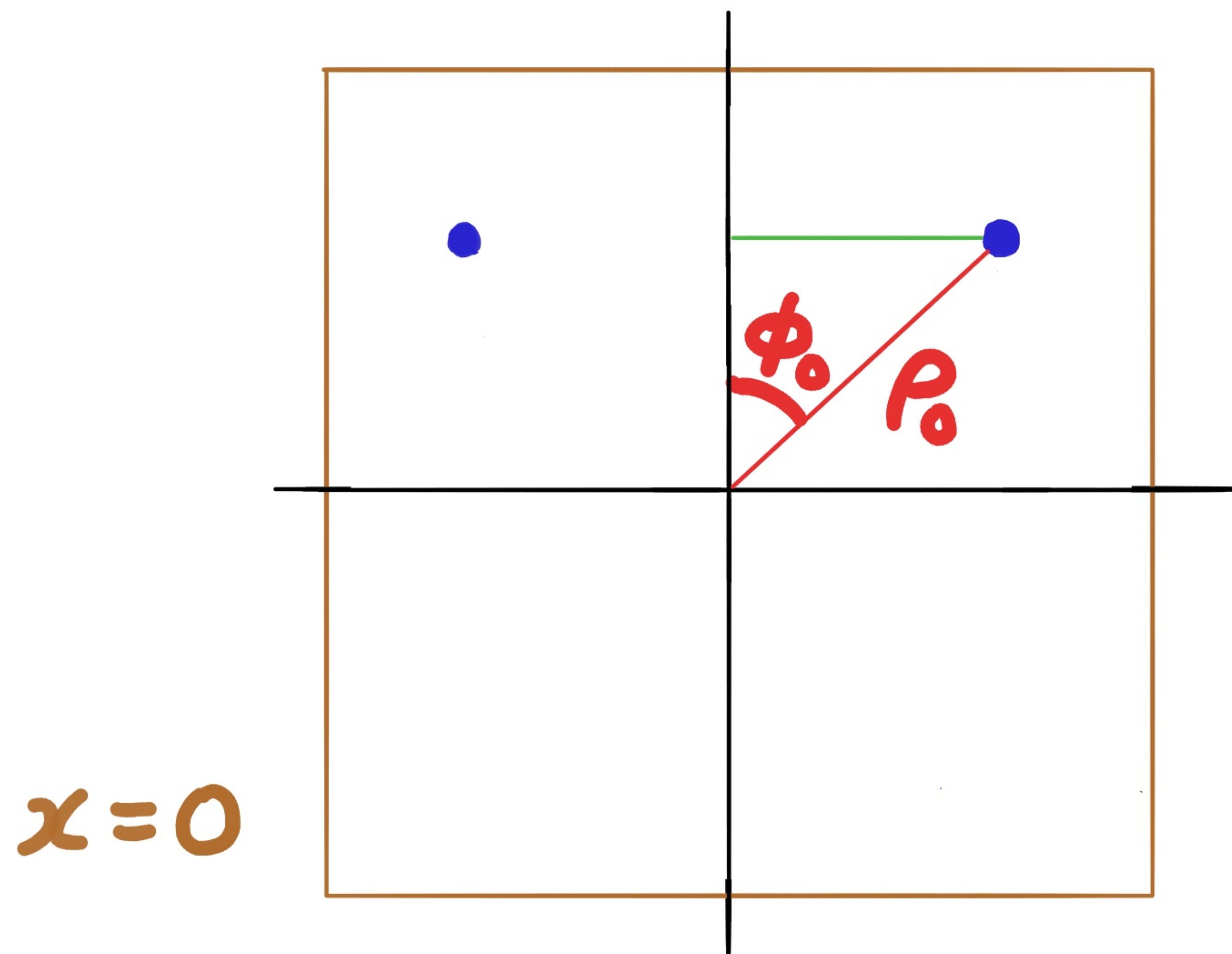
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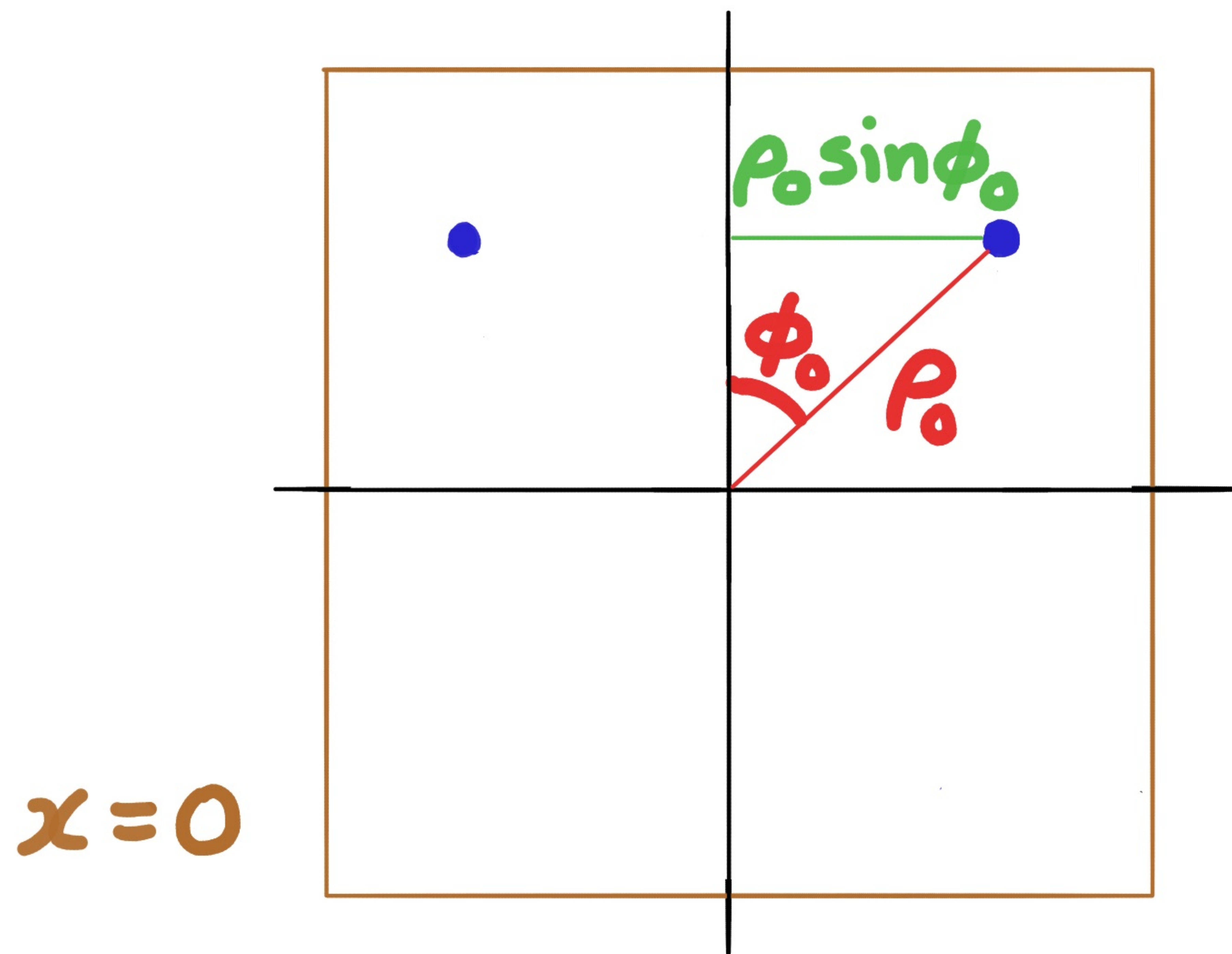
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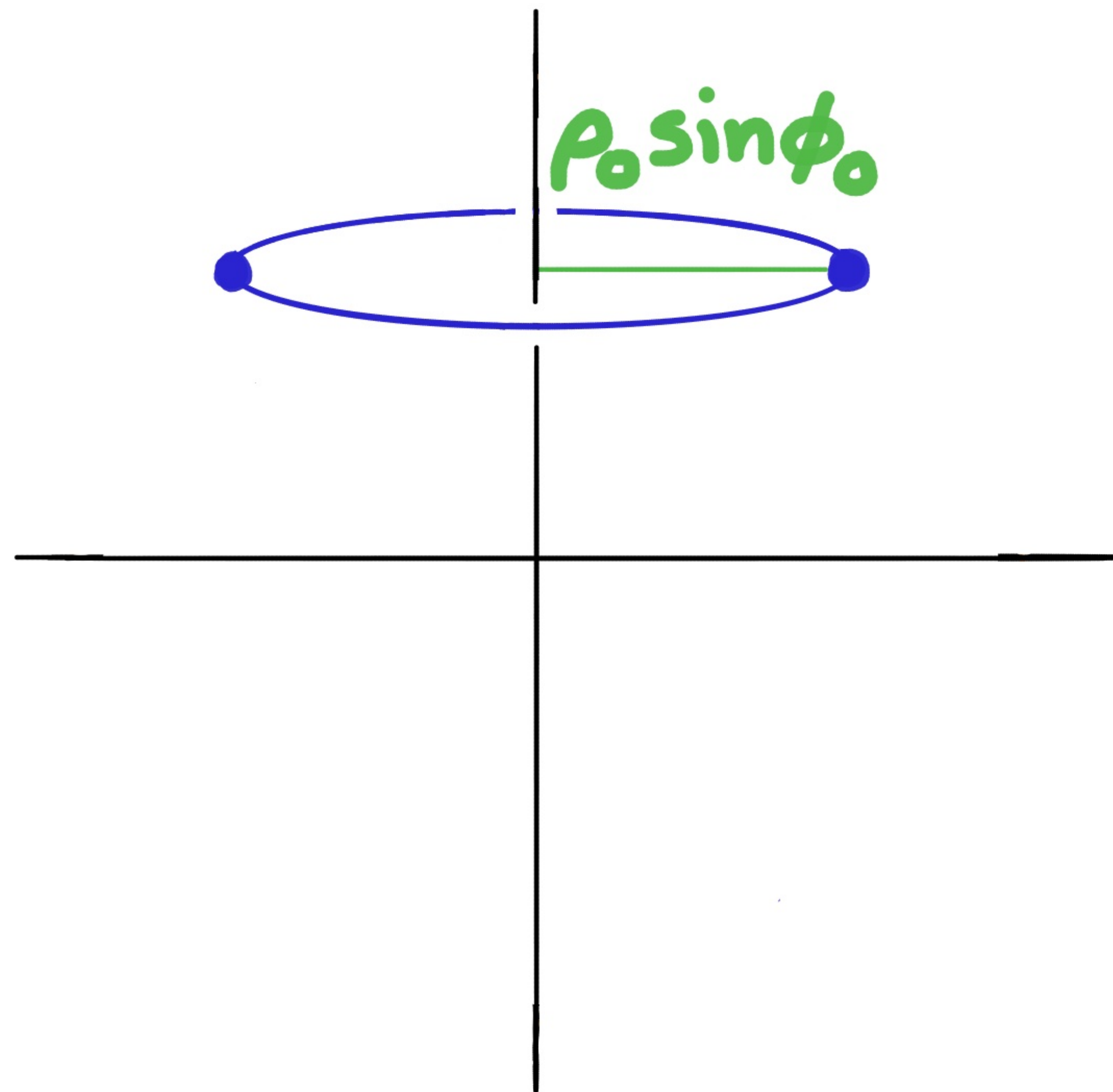
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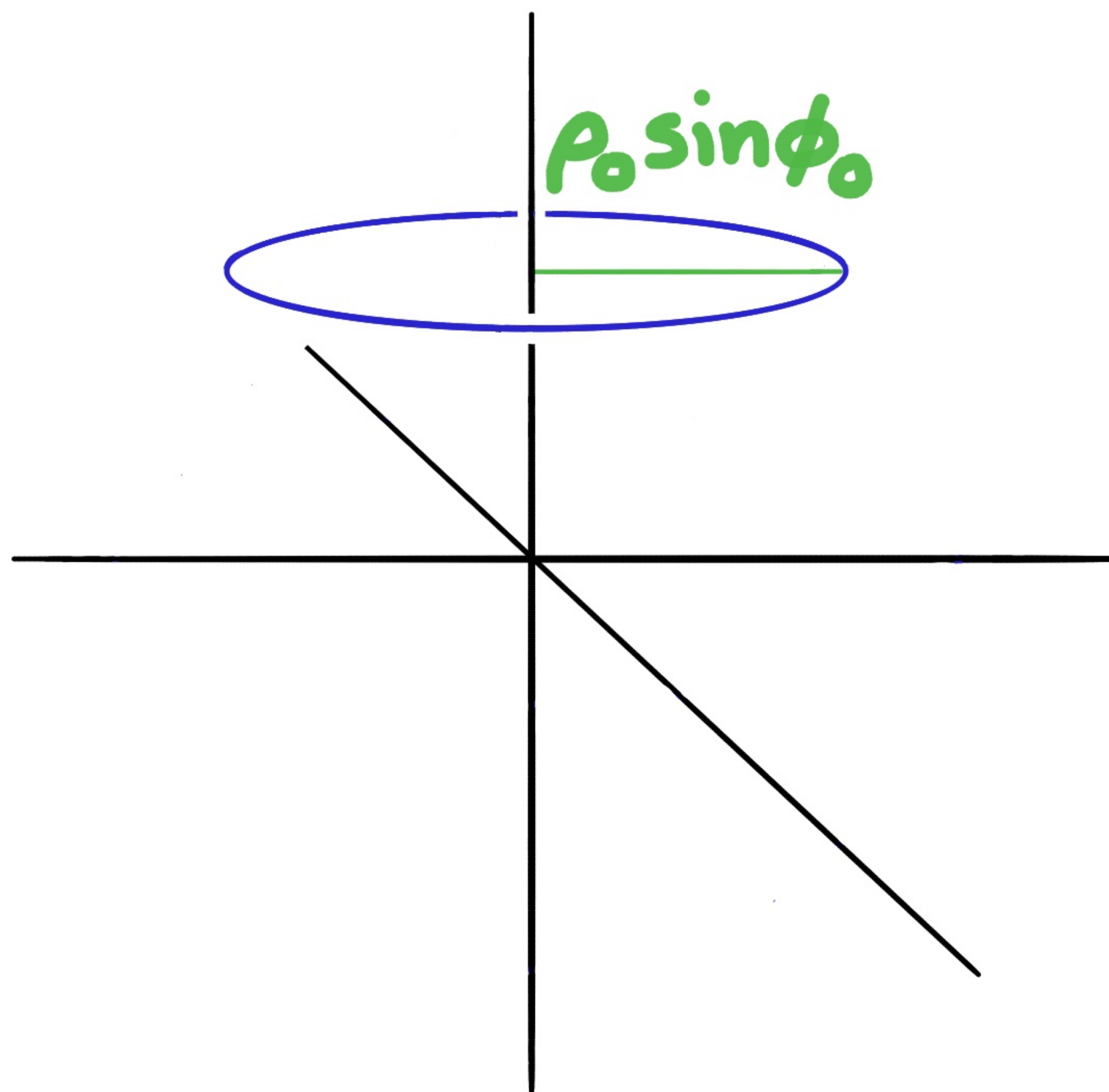
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is circle of radius $\rho_0 \sin \phi_0$



$$\rho = \rho_0$$

$$\phi = \phi_0$$

② Triple integrals

$$\iiint_R f(x,y,z) dV$$

measures a 4-dimensional volume

A diagram illustrating the components of a triple integral. The expression $\iiint_R f(x,y,z) dV$ is shown with each part highlighted in a different color: the triple integral symbol and R are in a pink oval, $f(x,y,z)$ is in a blue oval, and dV is in a green oval. A red arrow points from the text "region in $\mathbb{R}^3$ " to the pink oval. A blue arrow points from the text "height" to the blue oval. A green arrow points from the text "way to measure and multiply widths of 3 coordinates" to the green oval.

$$\iiint_R f(x,y,z) dV$$

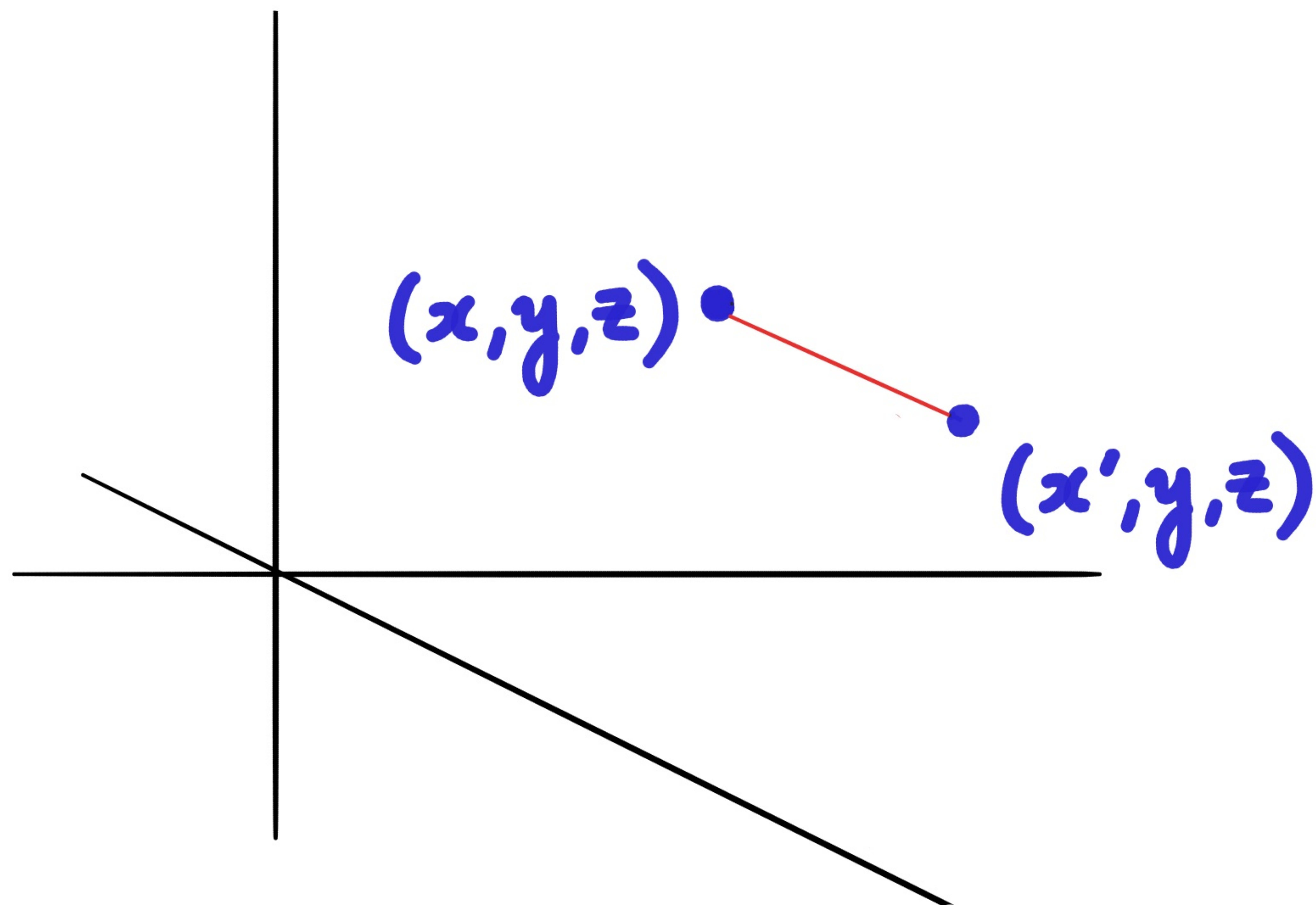
region in $\mathbb{R}^3$

height

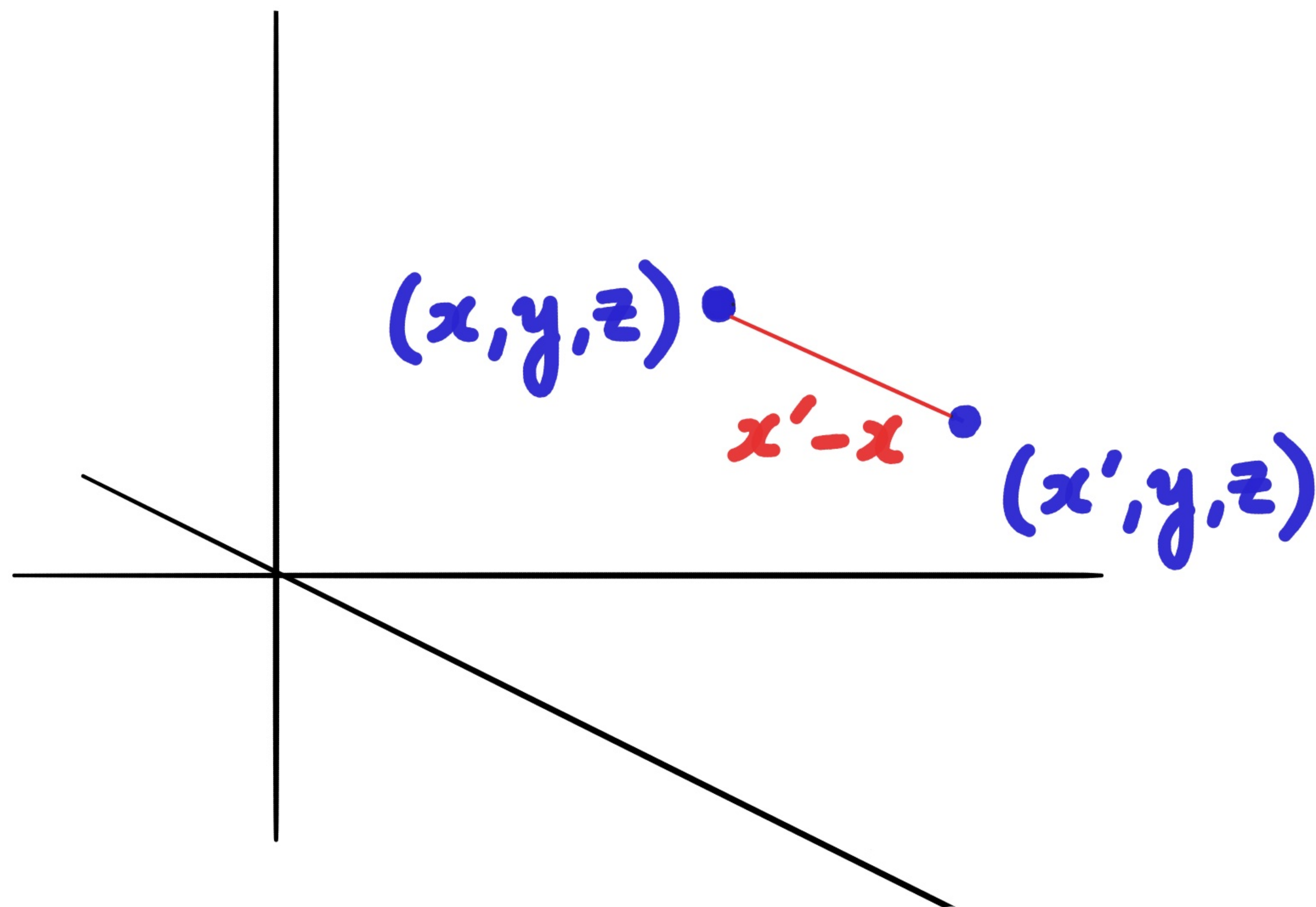
way to measure and multiply widths of 3 coordinates

measures a 4-dimensional volume

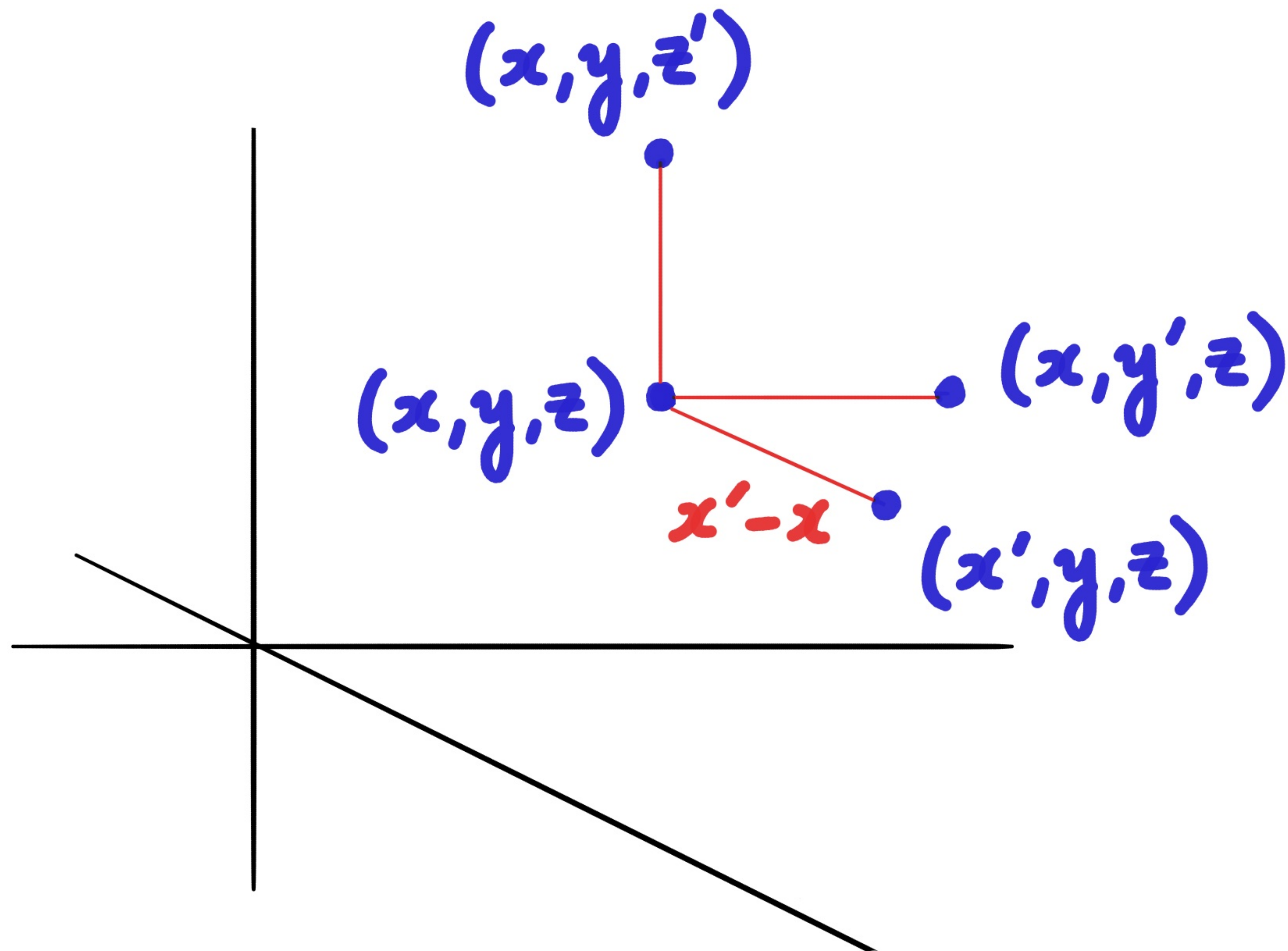
Widths and volumes in $\mathbb{R}^3$: Euclidean



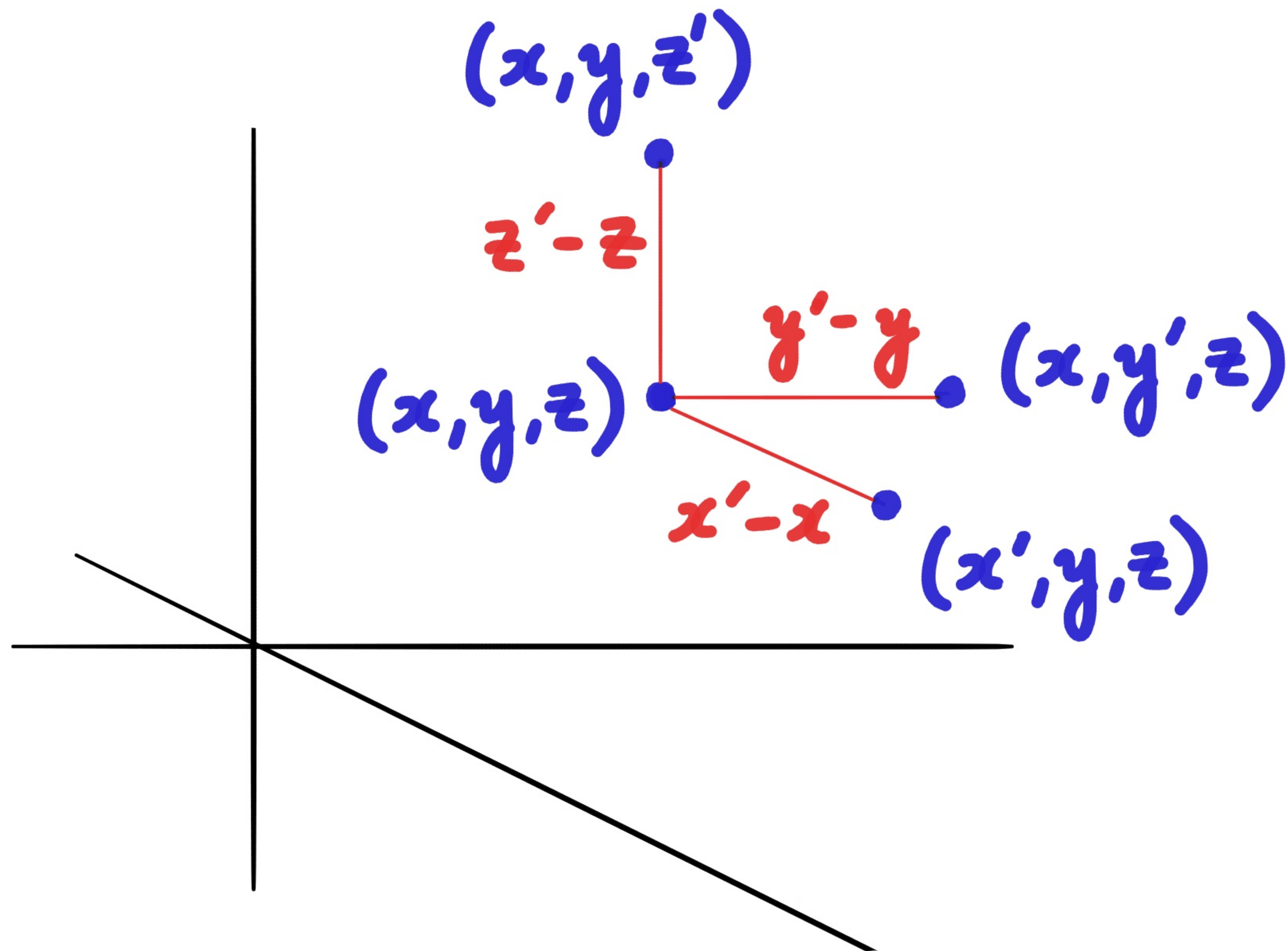
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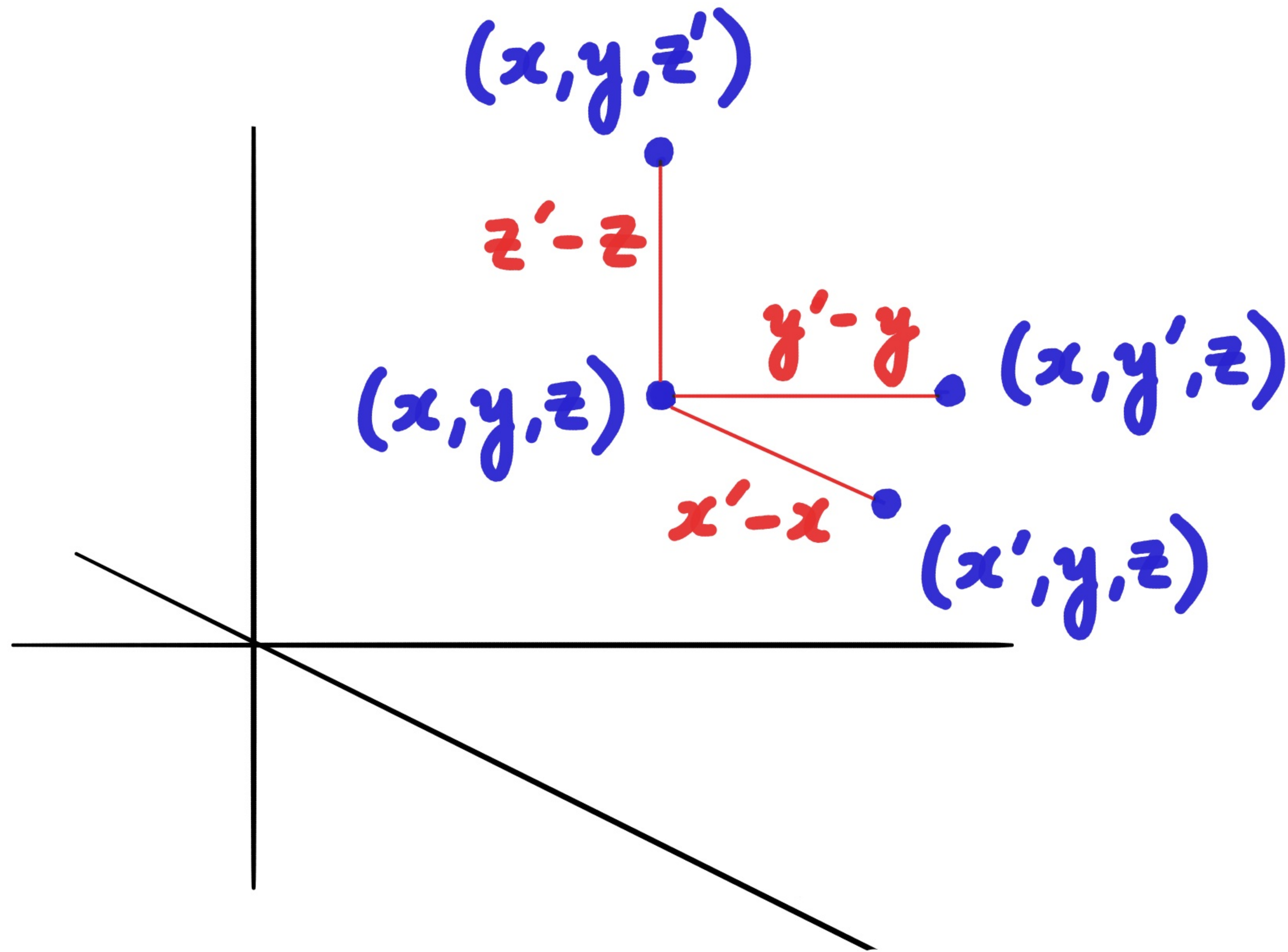
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Widths and volumes in $\mathbb{R}^3$: Euclidean



$$dV = dx dy dz$$

Euclidean Coordinates

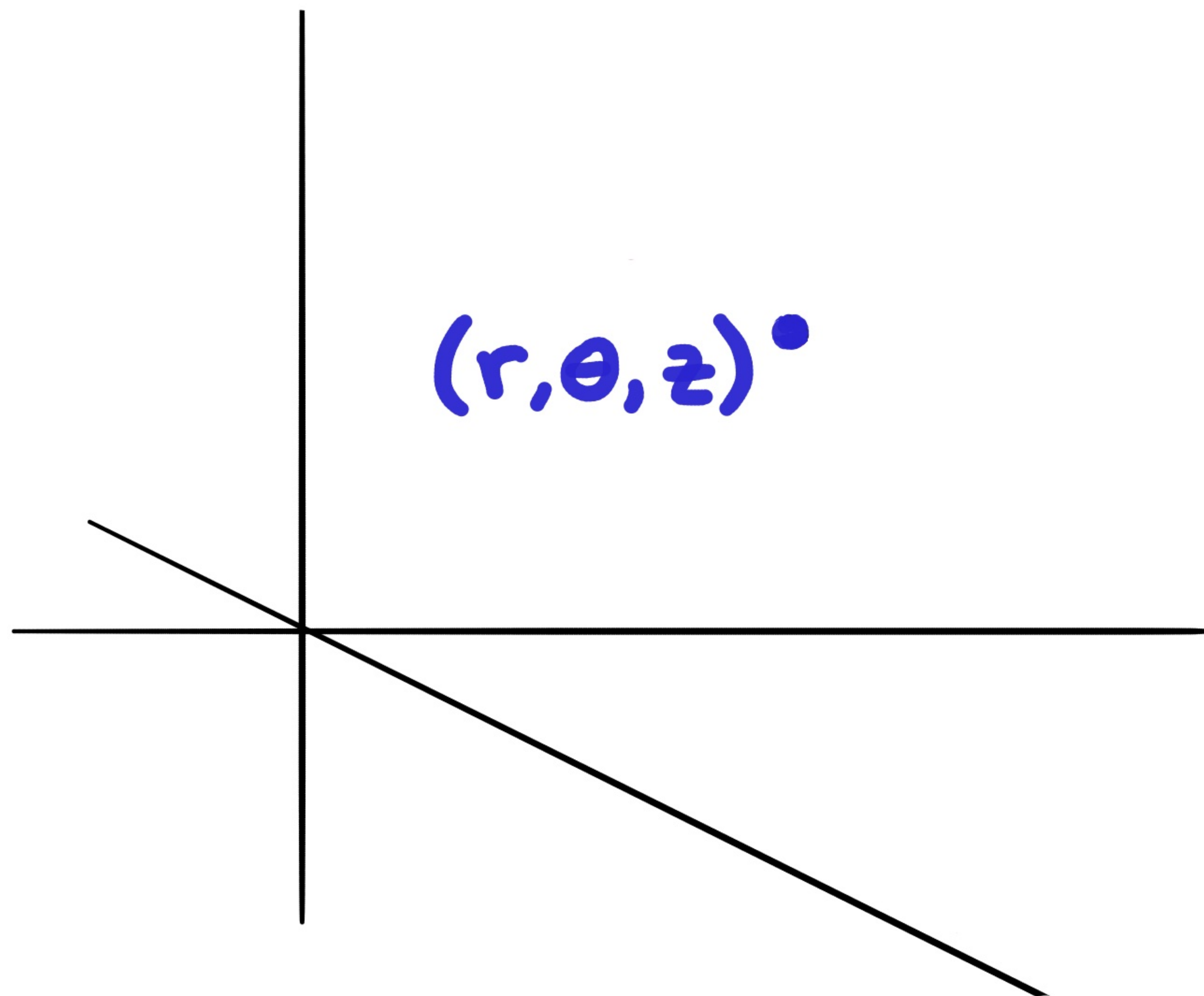
$$\iiint_R f(x,y,z) dV = \iiint_R f(x,y,z) dx dy dz$$

Euclidean Coordinates

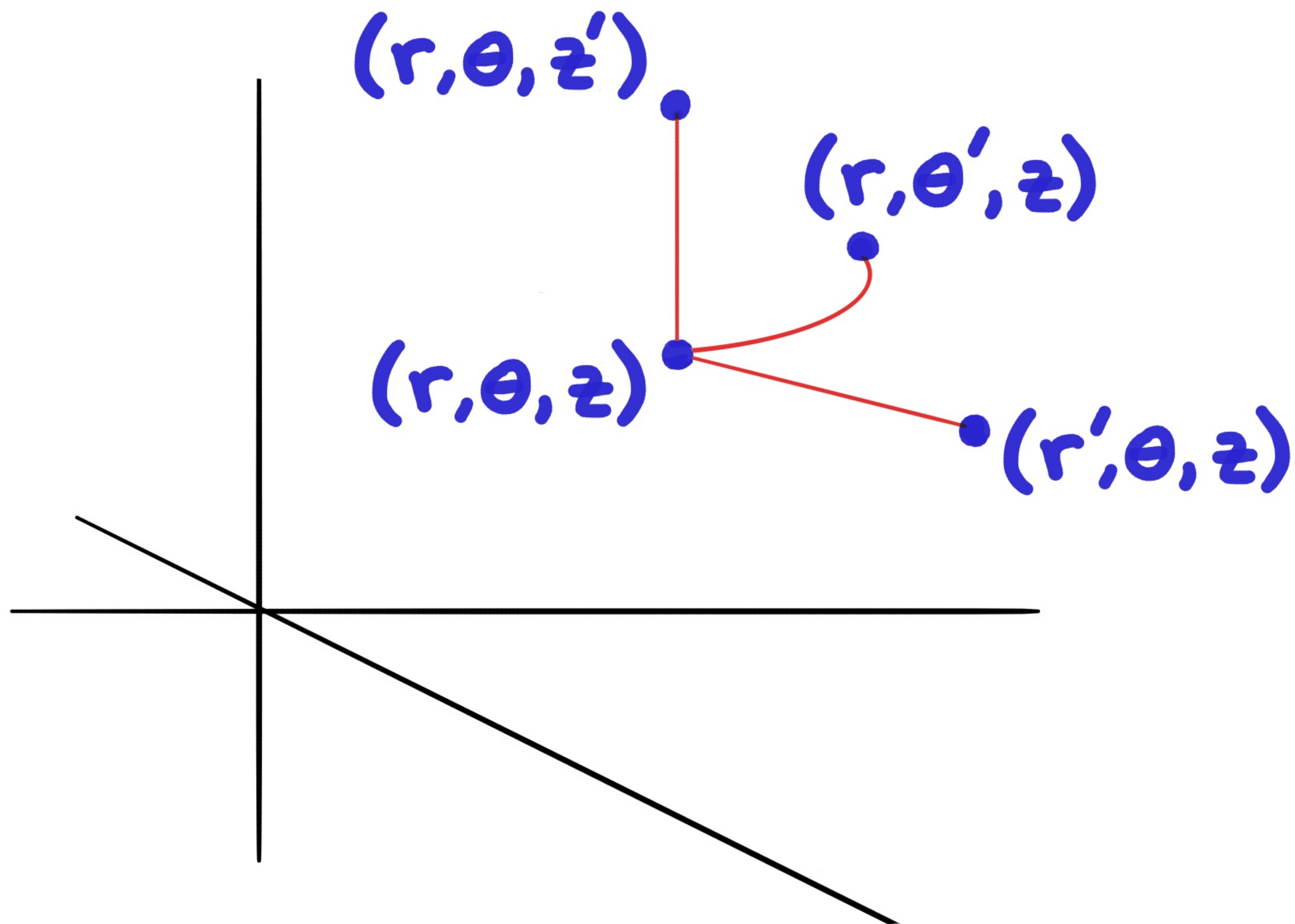
$$\iiint_R f(x,y,z) dV = \iiint_R f(x,y,z) dx dy dz$$

$$dV \mapsto dx dy dz$$

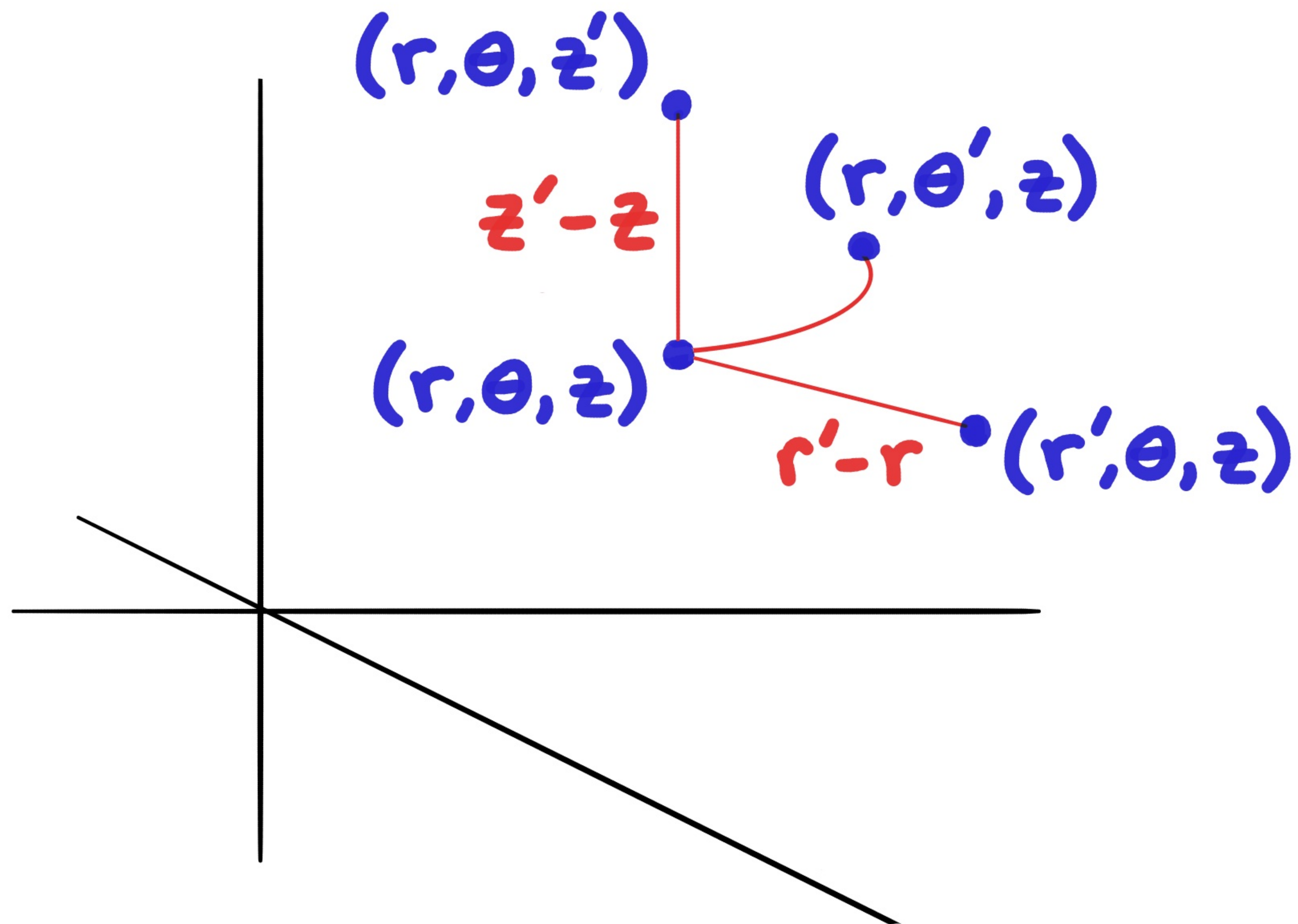
Widths and volumes in $\mathbb{R}^3$: Cylindrical



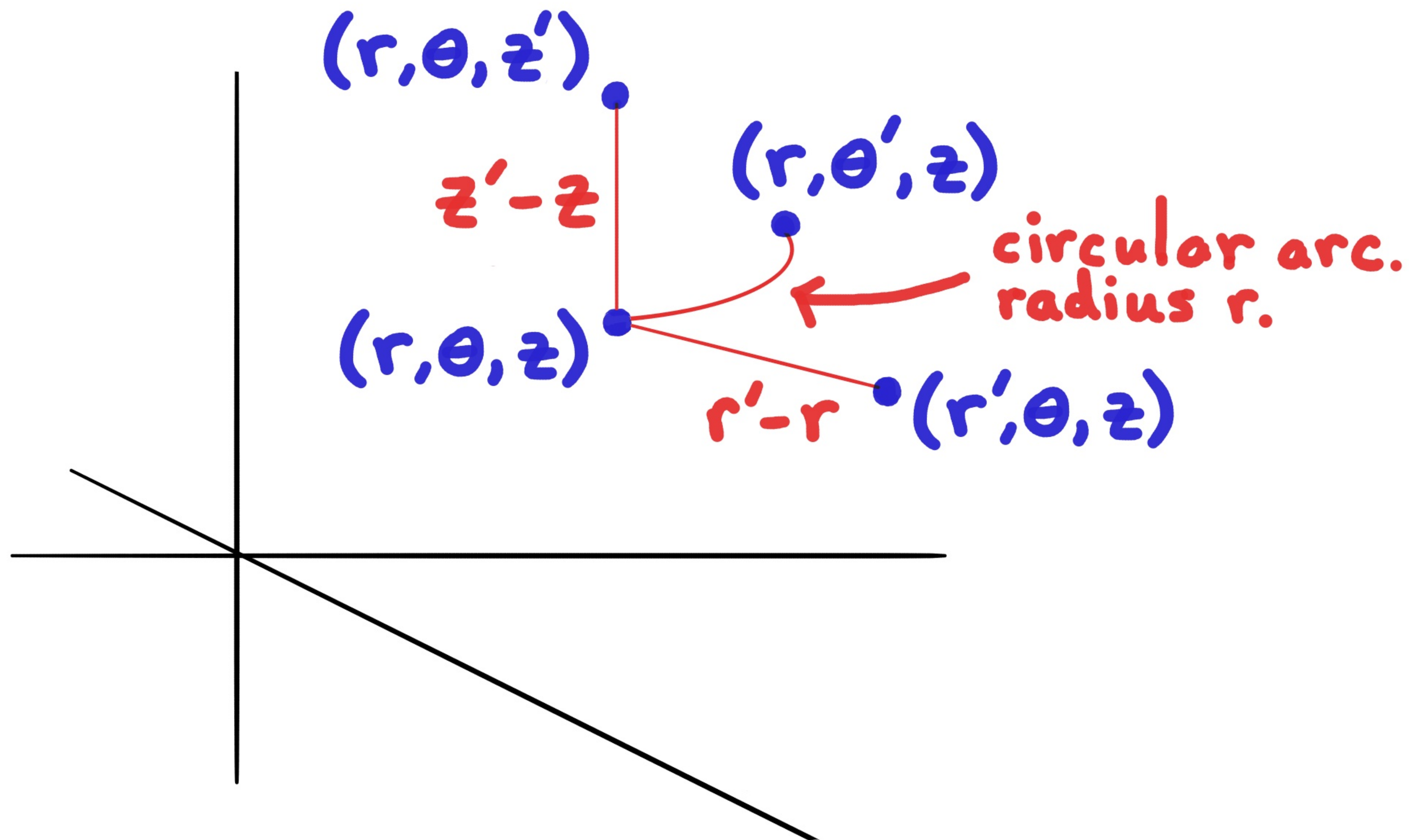
Widths and volumes in $\mathbb{R}^3$: Cylindrical



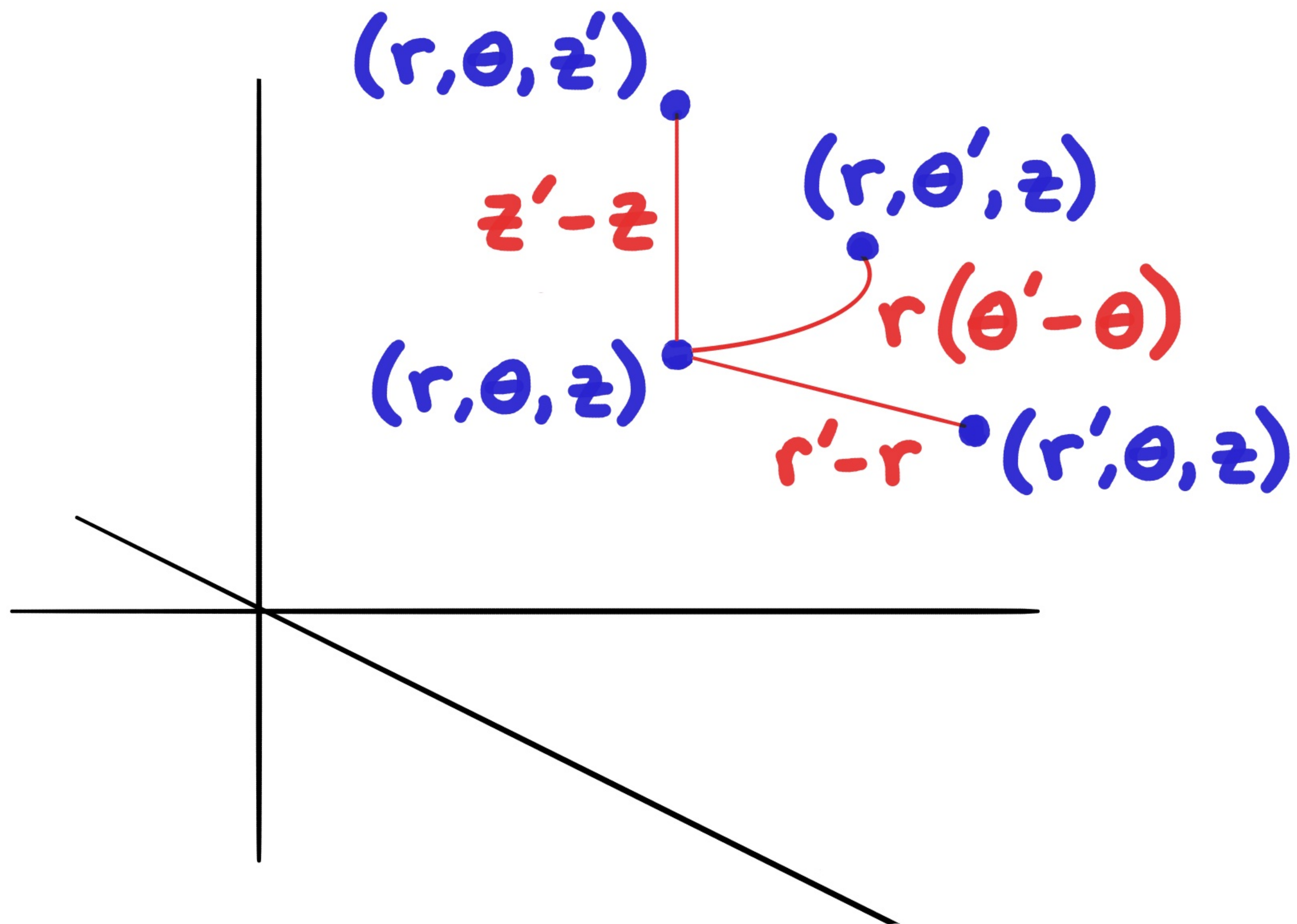
Widths and volumes in $\mathbb{R}^3$: Cylindrical



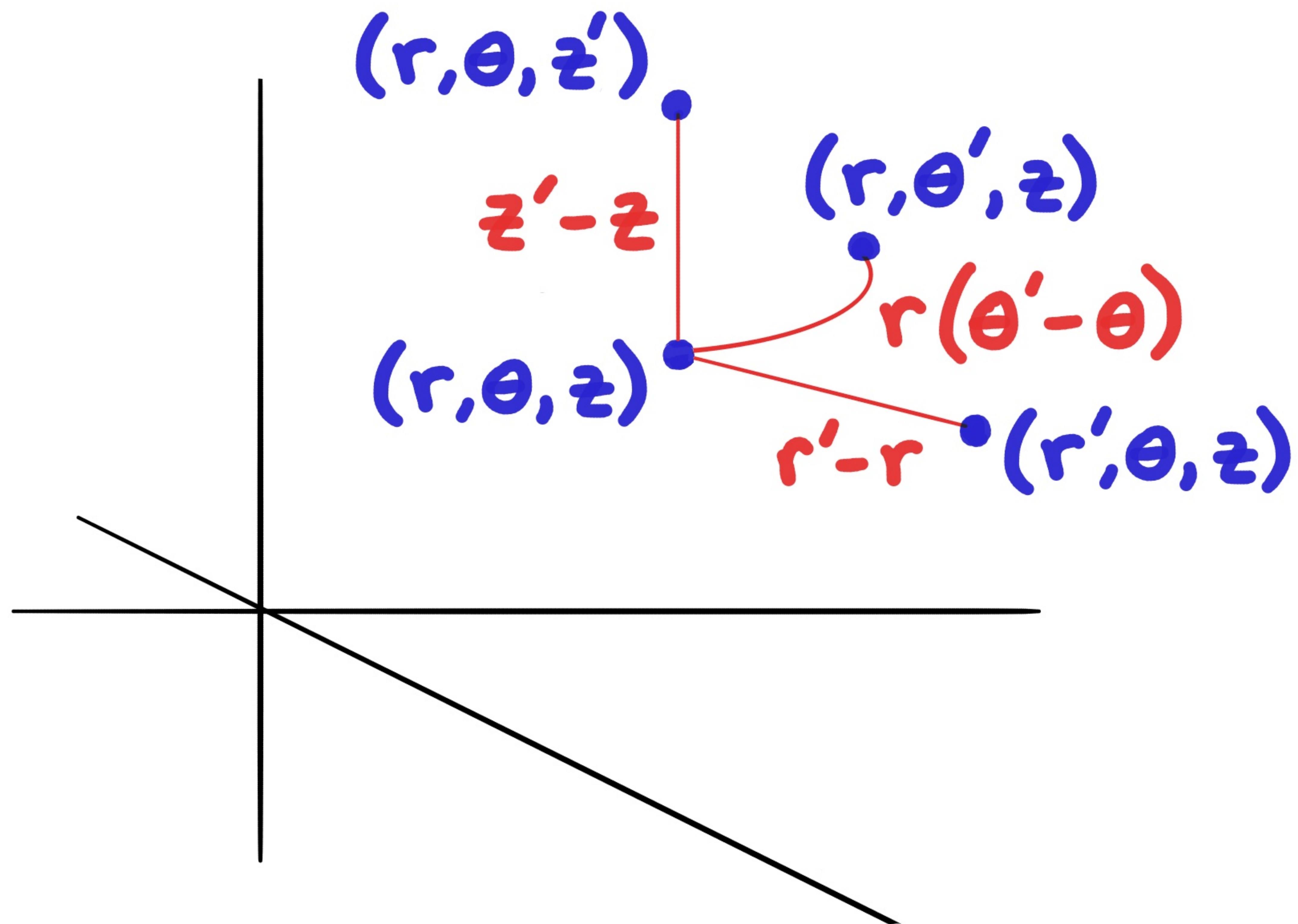
Widths and volumes in $\mathbb{R}^3$: Cylindrical



Widths and volumes in $\mathbb{R}^3$: Cylindrical



Widths and volumes in $\mathbb{R}^3$: Cylindrical



$$dV = r dr d\theta dz$$

Cylindrical Coordinates

$$\iiint_R f(x, y, z) dV = \iiint_R f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

Cylindrical Coordinates

$$\iiint_R f(x, y, z) dV = \iiint_R f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

$$dV \mapsto dr d\theta dz$$

$$x \mapsto r \cos \theta$$

$$y \mapsto r \sin \theta$$

$$z \mapsto z$$

Cylindrical Coordinates

$$\iiint_R f(x, y, z) dV = \iiint_R f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

$$dV \mapsto dr d\theta dz$$

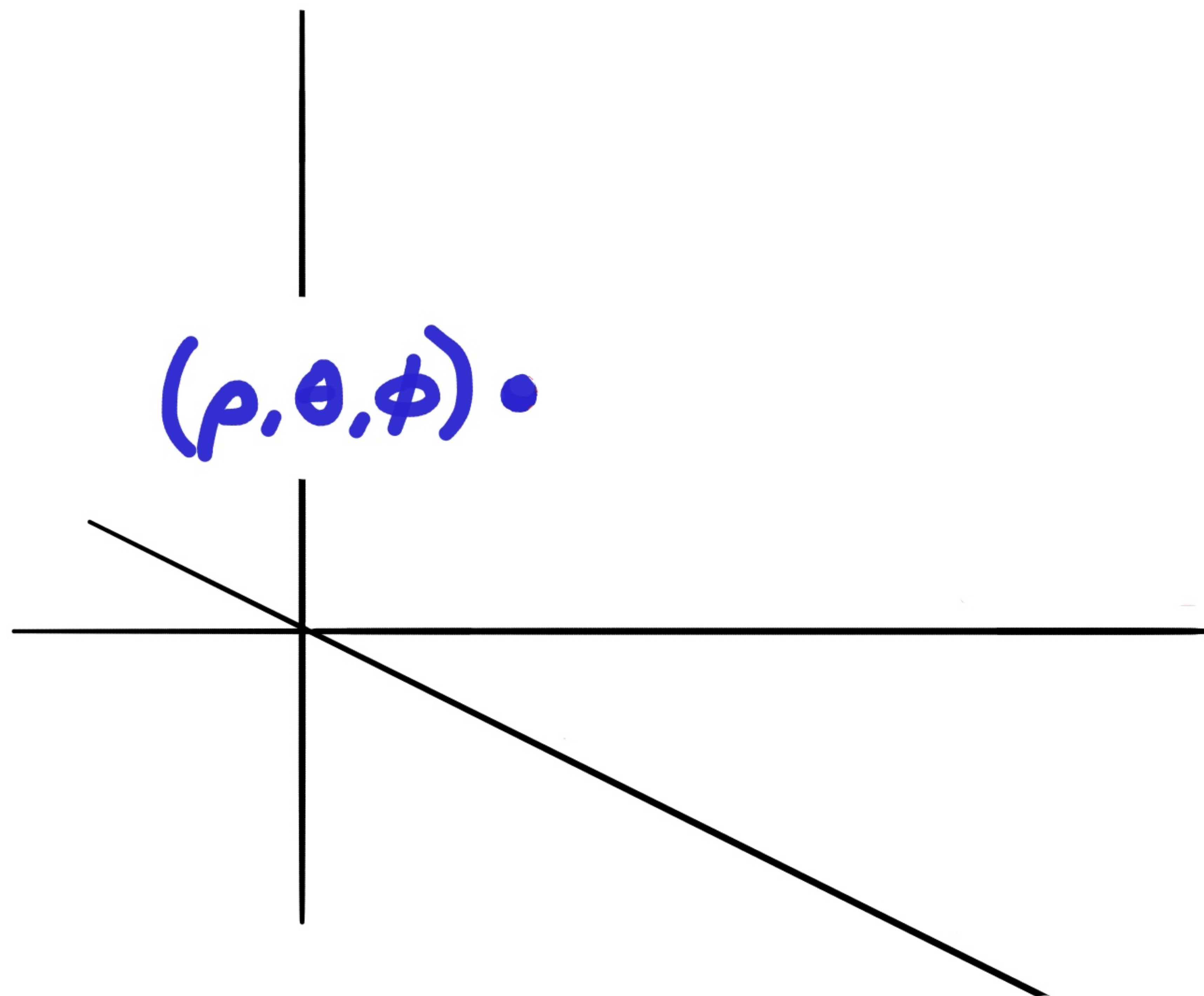
$$x \mapsto r \cos \theta$$

$$y \mapsto r \sin \theta$$

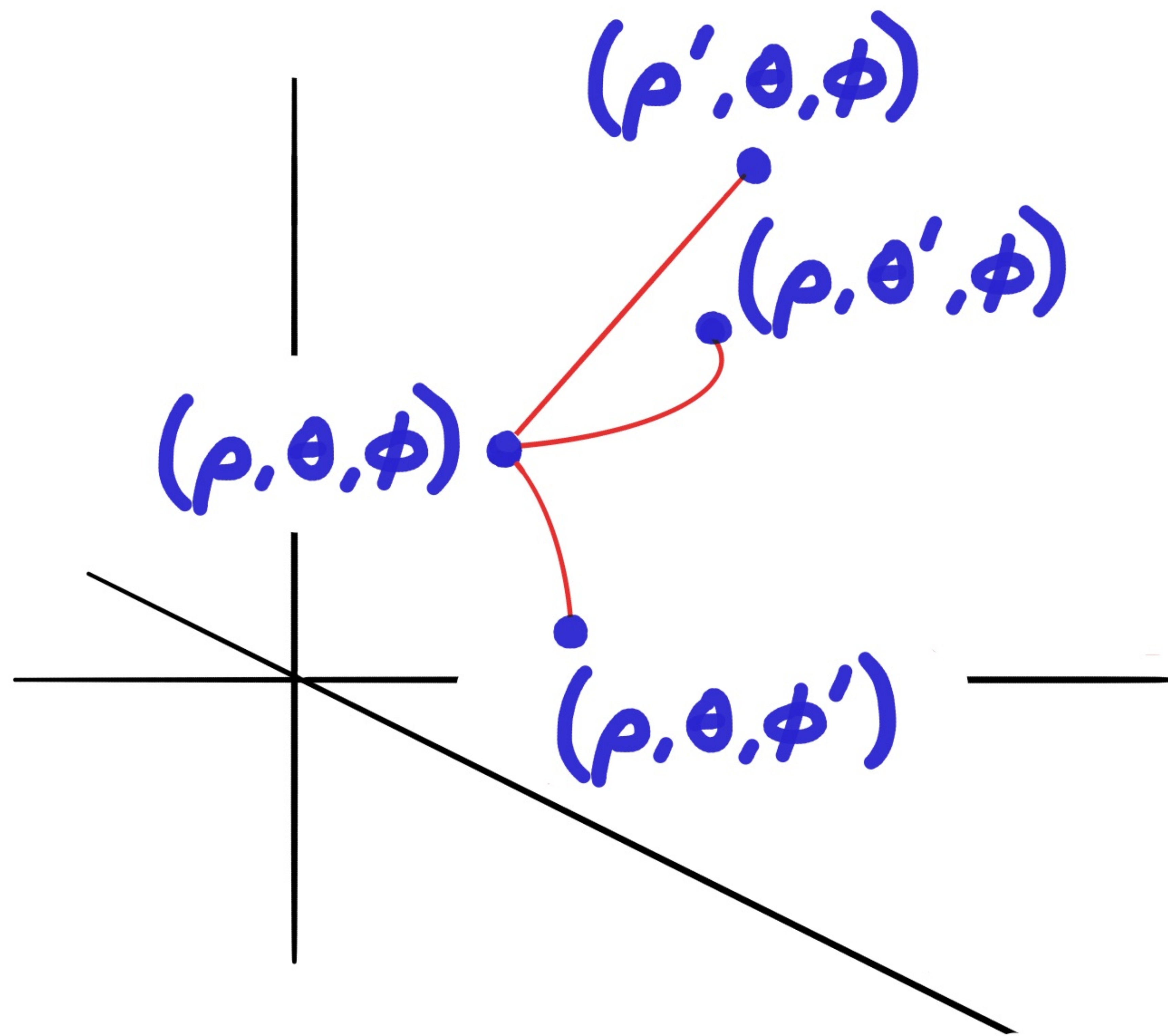
$$z \mapsto z$$

Throw in r .

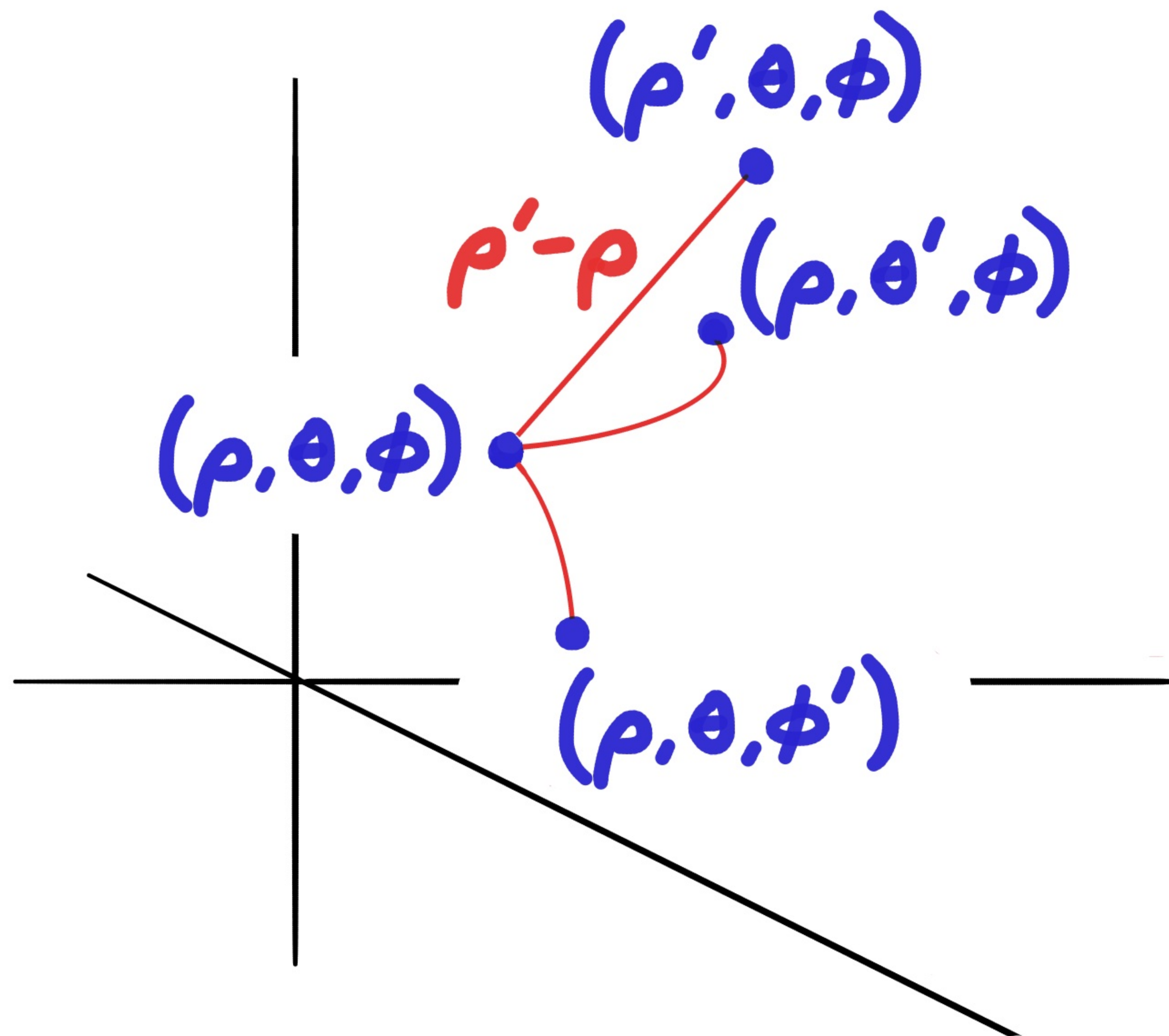
Widths and volumes in $\mathbb{R}^3$: Spherical



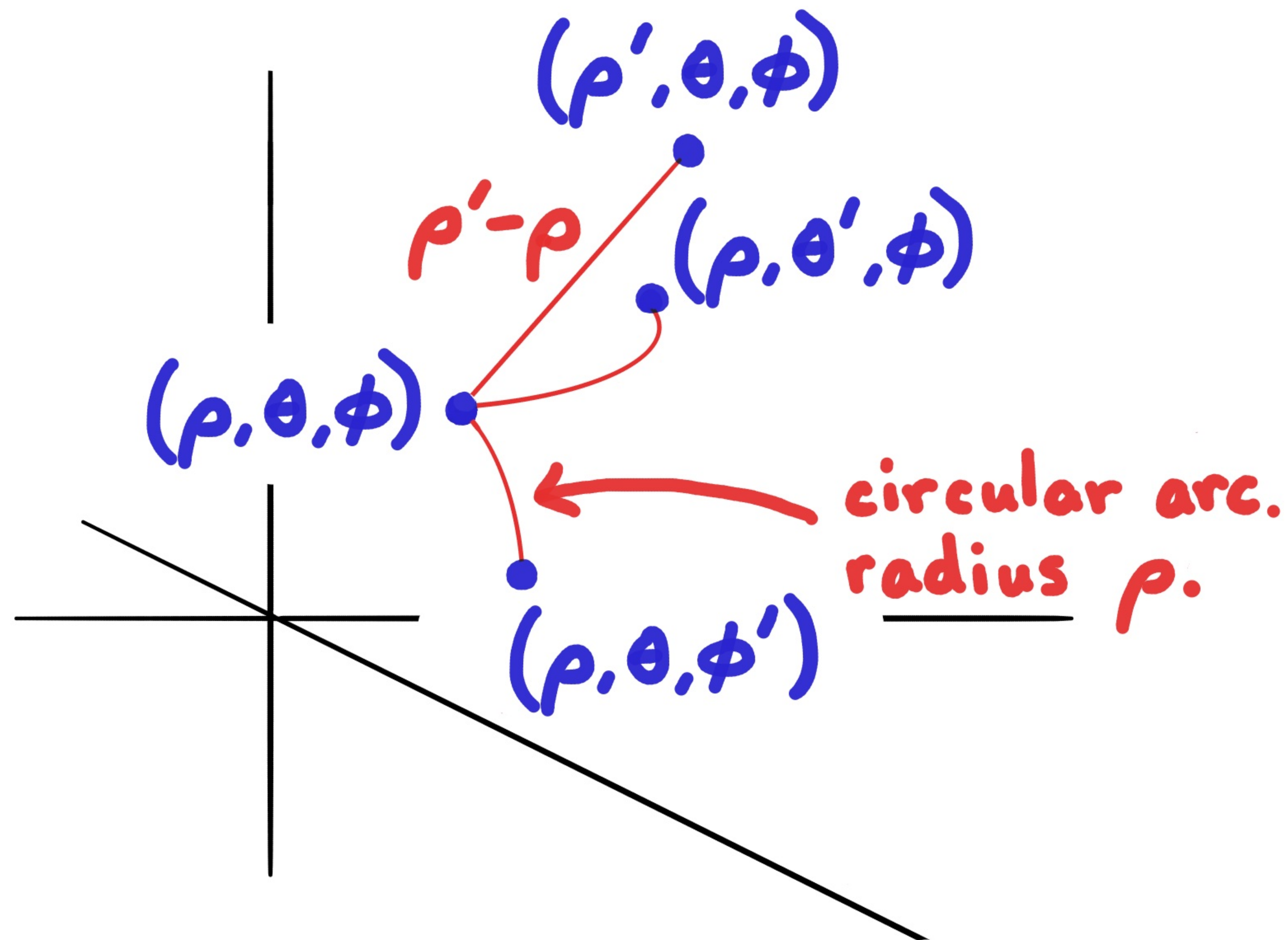
Widths and volumes in $\mathbb{R}^3$: Spherical



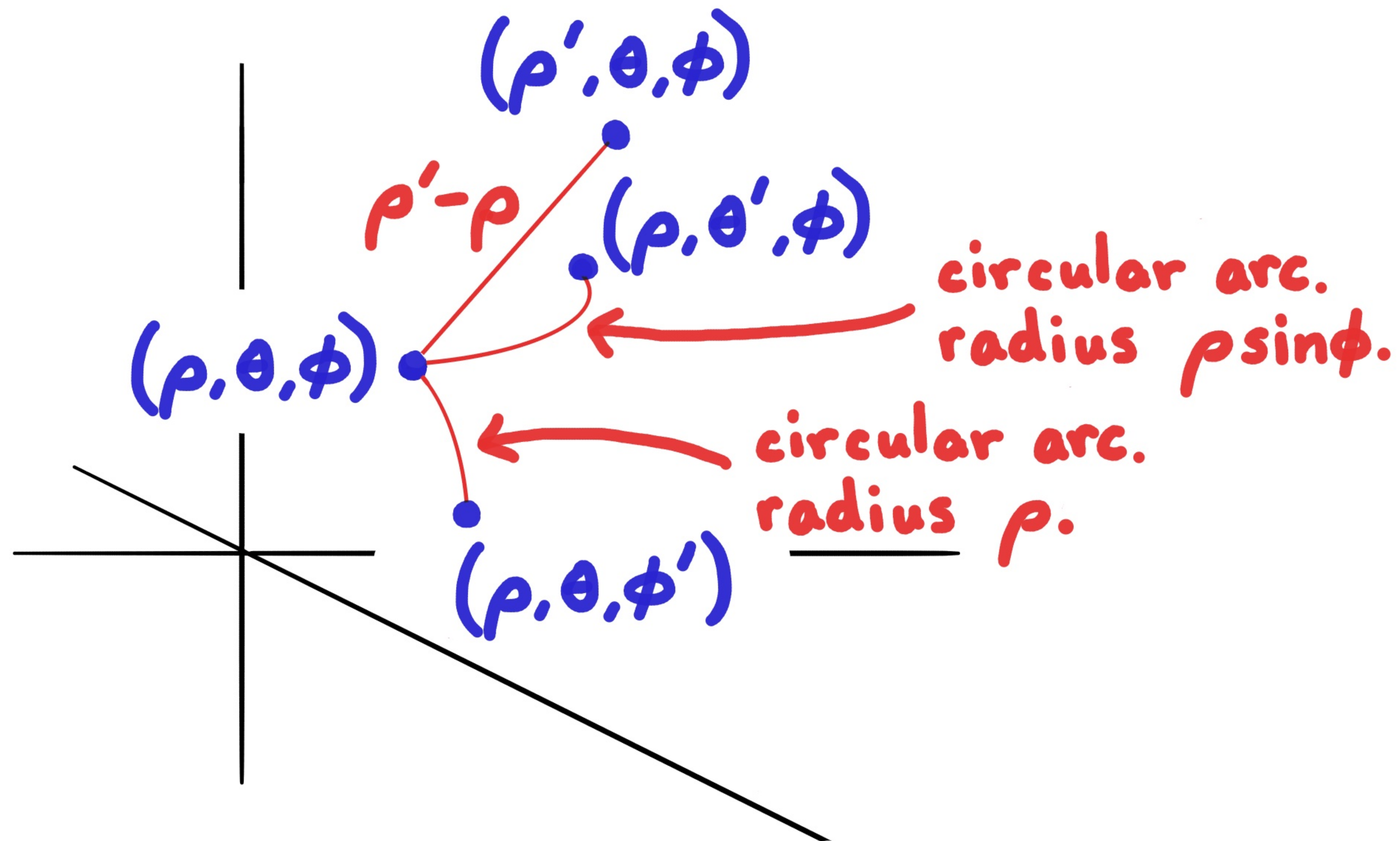
Widths and volumes in $\mathbb{R}^3$: Spherical



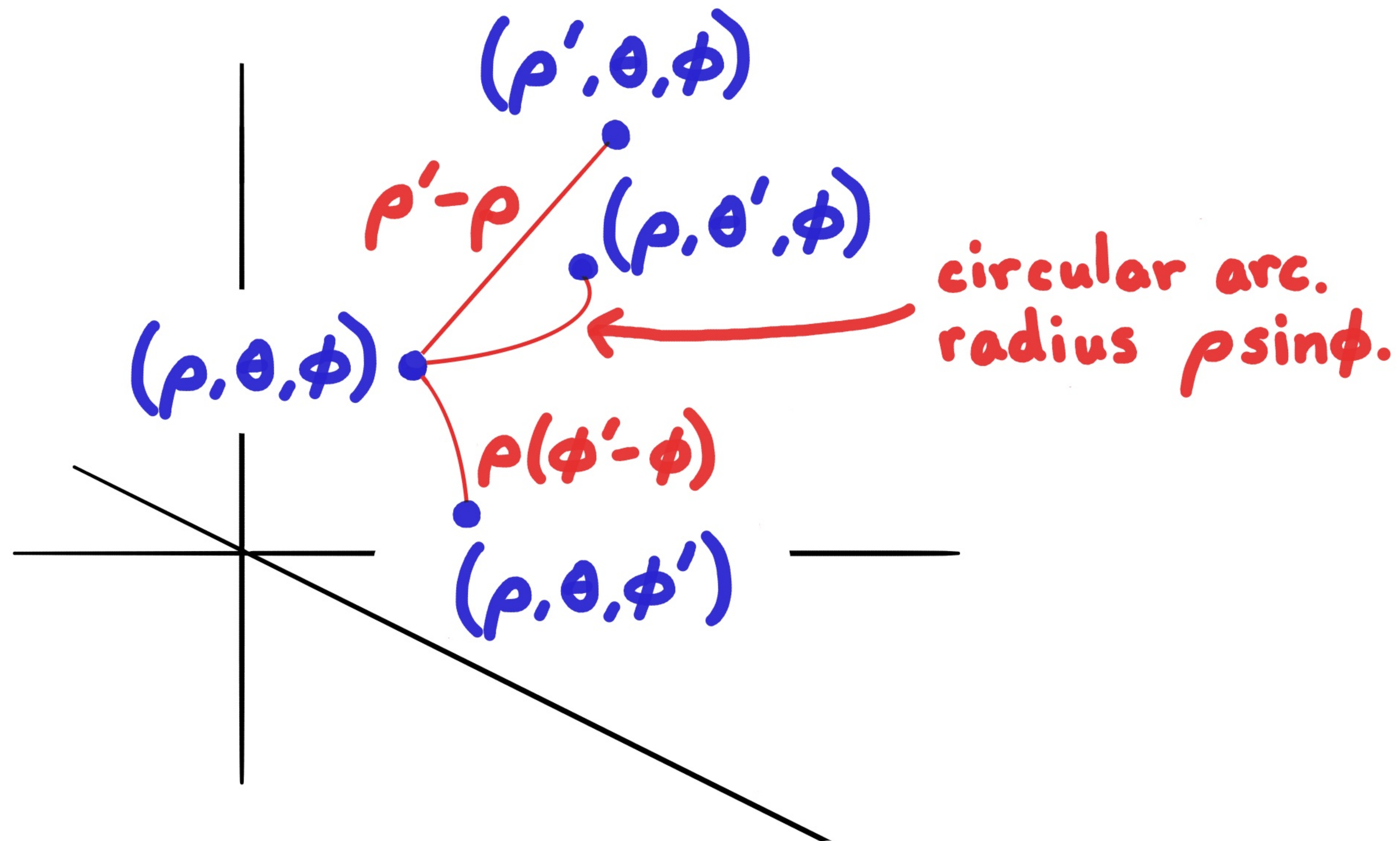
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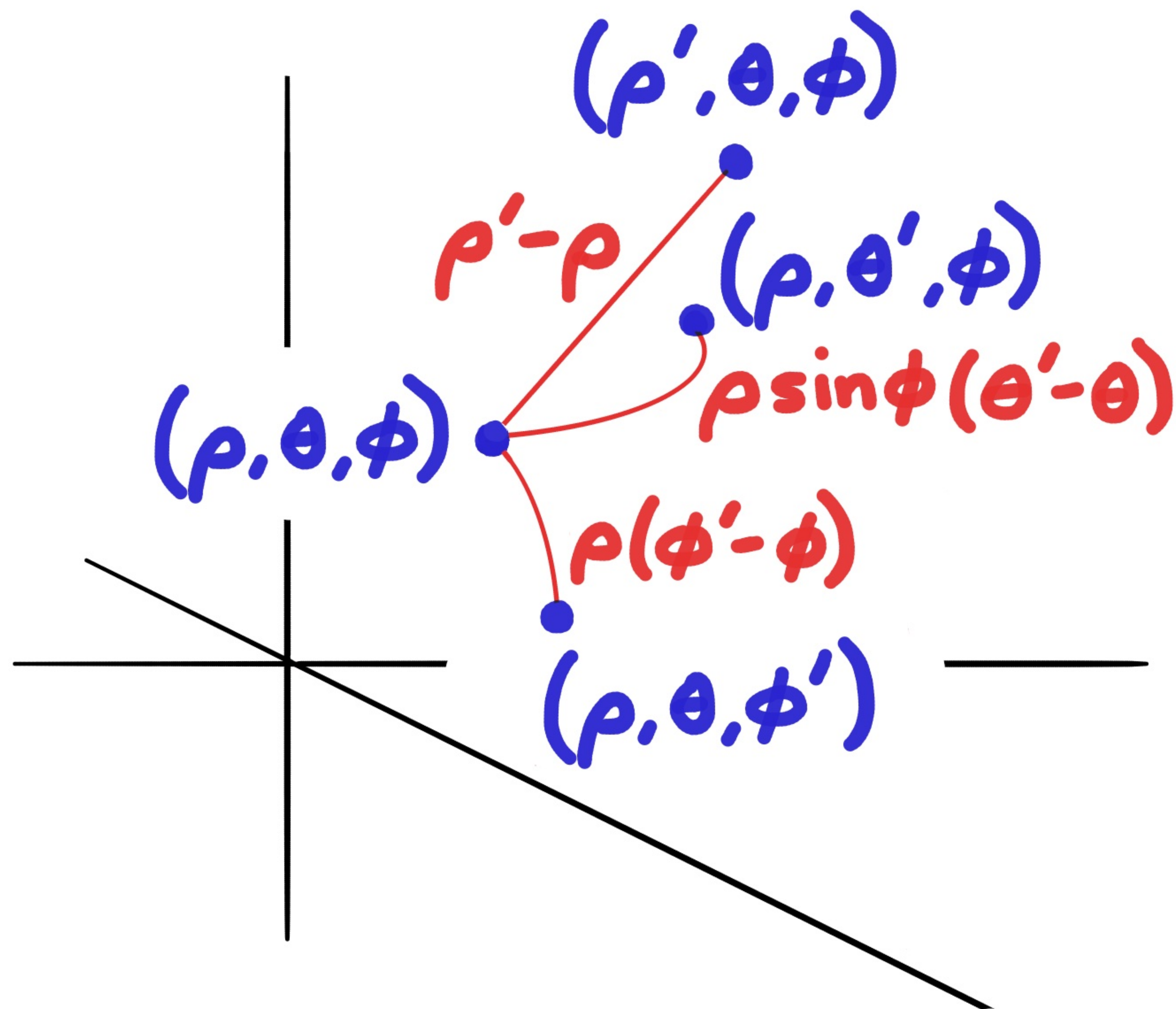
Widths and volumes in $\mathbb{R}^3$: Spherical



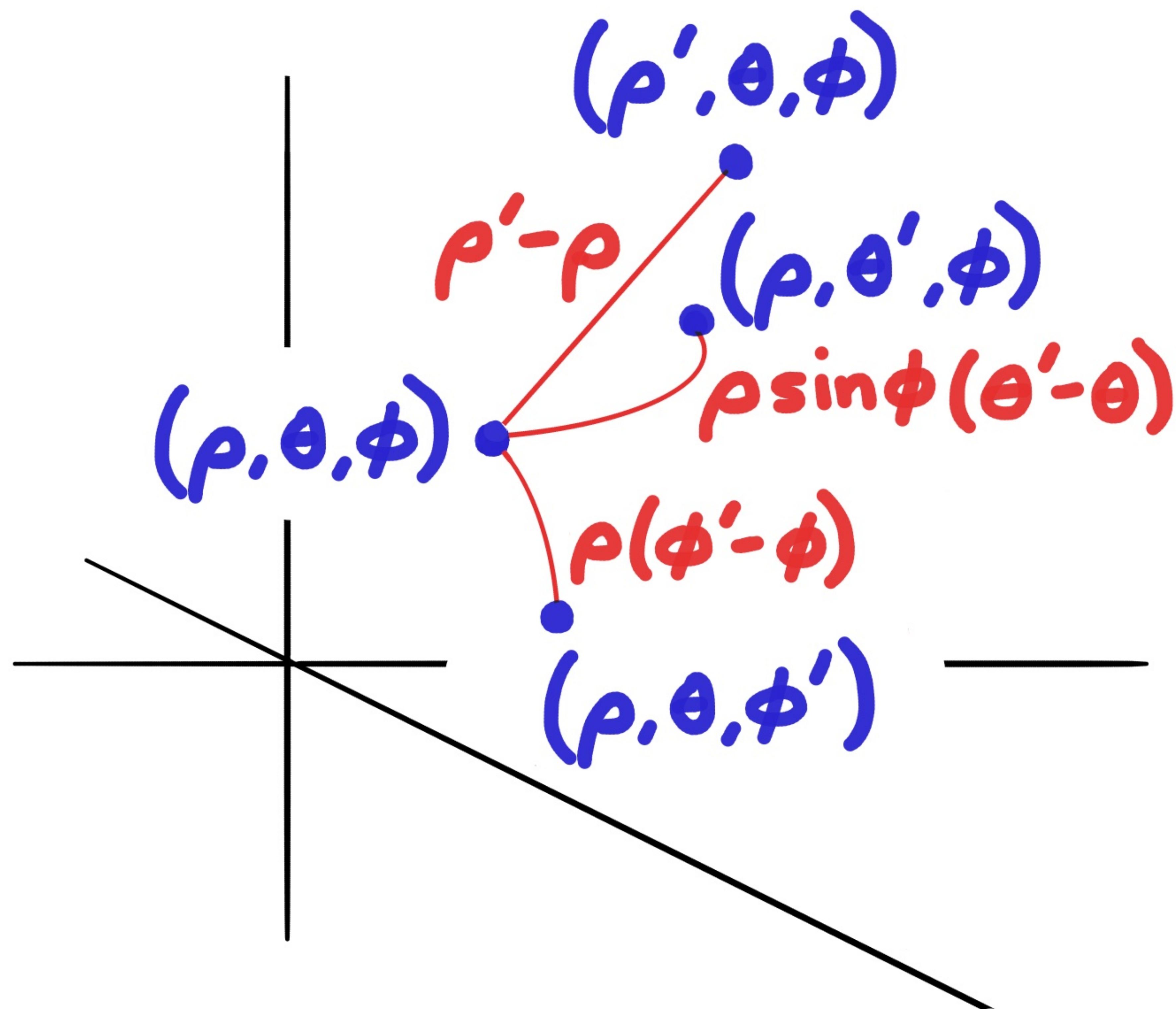
Widths and volumes in $\mathbb{R}^3$: Spherical



Widths and volumes in $\mathbb{R}^3$: Spherical



Widths and volumes in $\mathbb{R}^3$: Spherical



$$dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

Spherical Coordinates

$$\iiint_R f(x, y, z) dV$$

$$= \iiint_R f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho d\theta d\phi$$

Spherical Coordinates

$$\iiint_R f(x, y, z) dV$$

$$= \iiint_R f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho d\theta d\phi$$

$$dV \mapsto d\rho d\theta d\phi$$

$$x \mapsto \rho \sin \phi \cos \theta$$

$$y \mapsto \rho \sin \phi \sin \theta$$

$$z \mapsto \rho \cos \phi$$

Spherical Coordinates

$$\iiint_R f(x, y, z) dV$$

$$= \iiint_R f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho d\theta d\phi$$

$$dV \mapsto d\rho d\theta d\phi$$

$$x \mapsto \rho \sin \phi \cos \theta$$

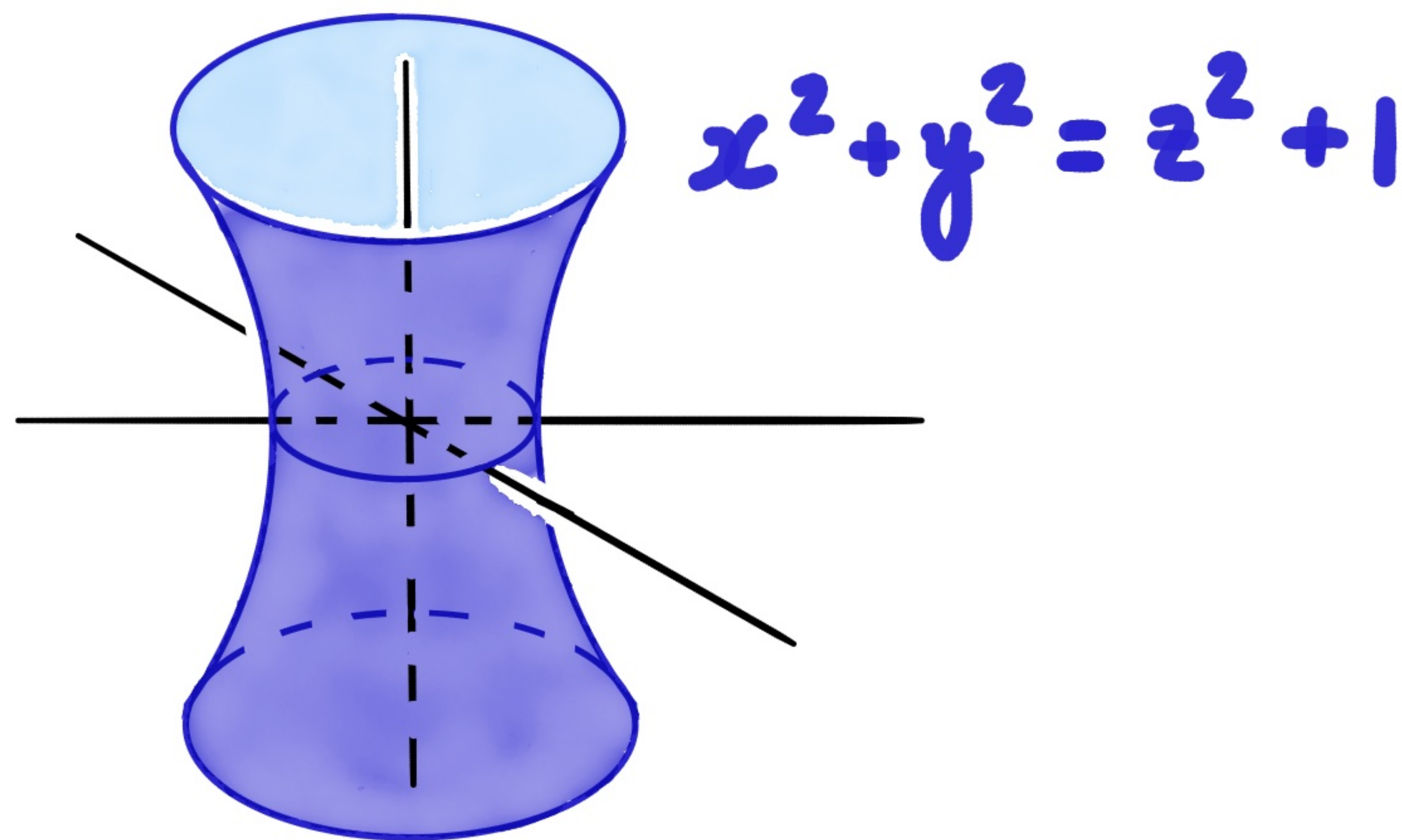
$$y \mapsto \rho \sin \phi \sin \theta$$

$$z \mapsto \rho \cos \phi$$

Throw in $\rho^2 \sin \phi$.

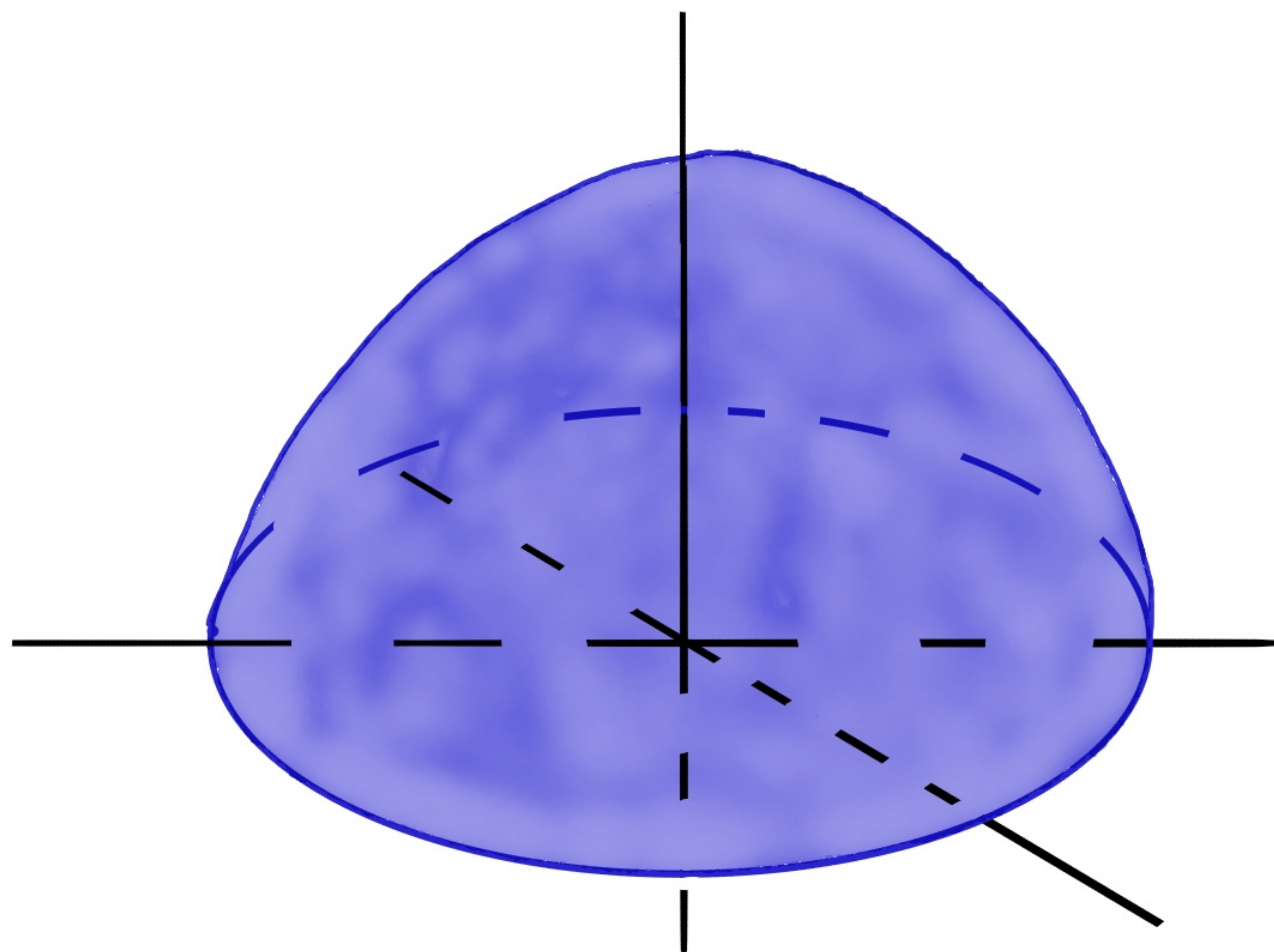
Examples:

- ① Find $\iiint_R y \, dV$ where R is the region above the xy -plane, with $z \leq 4$, and bounded by the surface $x^2 + y^2 = z^2 + 1$.

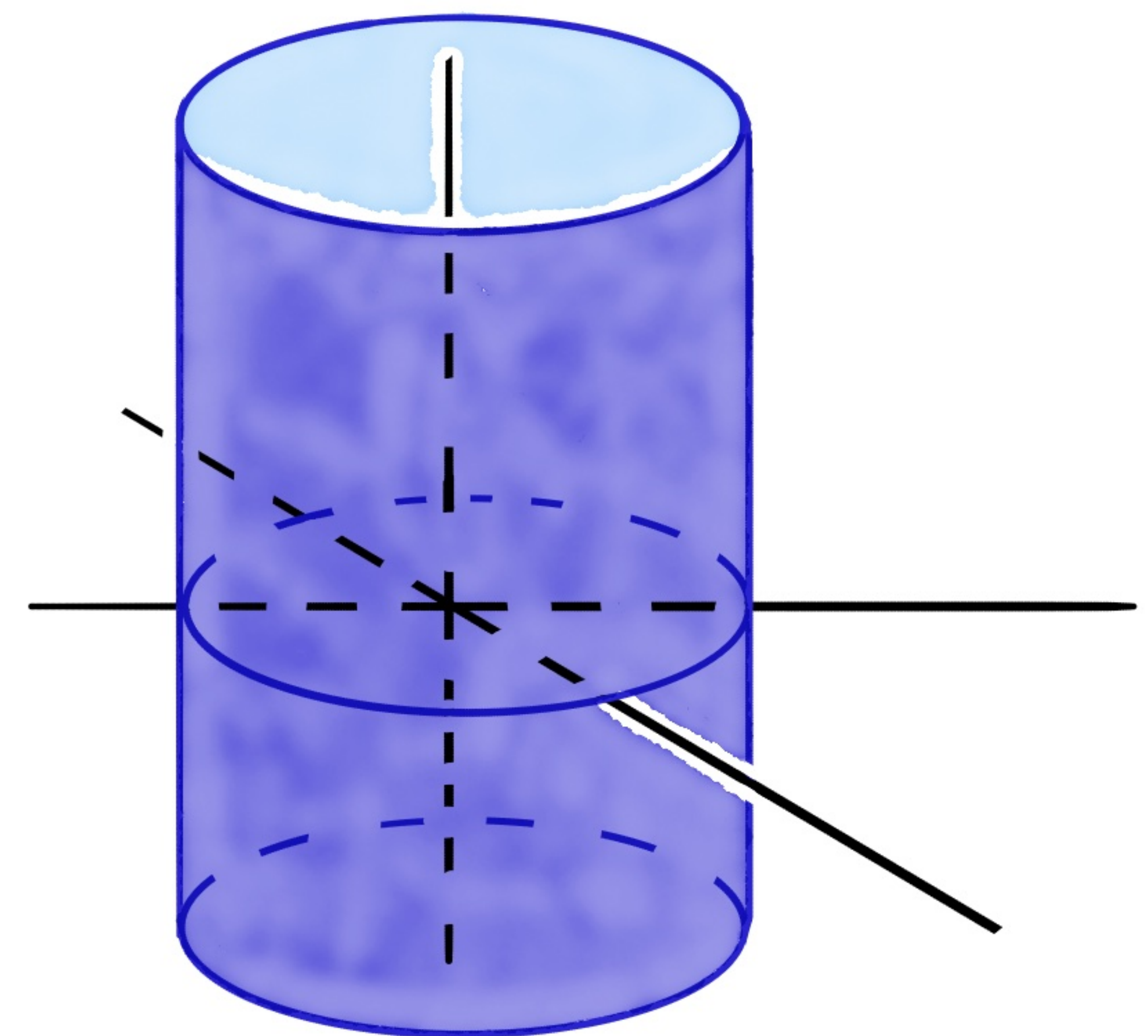


② Find the volume of the region in $\mathbb{R}^3$ enclosed by a sphere of radius a .

③ Find the volume of the region in $\mathbb{R}^3$ above the xy -plane, below the surface $z = 9 - x^2 - y^2$, and contained in the cylinder $x^2 + y^2 = 4$.



$$z = 9 - x^2 - y^2$$



$$x^2 + y^2 = 4$$