Twenty-three

Triple integrals in Cartesian coordinates

Example:

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz$$

Example:

$$\int_0^5 \int_0^4 \int_0^1 3\, dx\, dy\, dz = \int_0^5 \int_0^4 \left[3x\right]_0^1 dy\, dz$$

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$$\int_0^5\int_0^4\int_0^1 3\,dx\,dy\,dz = \int_0^5\int_0^4 \Big[3x\Big]_0^1\,dy\,dz$$

$$= \int_0^5\int_0^4 3\,dy\,dz$$

Example:

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz = \int_0^5 \int_0^4 \left[3x\right]_0^1 dy \, dz$$

$$= \int_0^5 \int_0^4 3 \, dy \, dz$$

$$= \int_0^5 \left[3y\right]_0^4 dz$$

Example:

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz = \int_0^5 \int_0^4 \left[3x\right]_0^1 dy \, dz$$

$$= \int_0^5 \int_0^4 3 \, dy \, dz$$

$$= \int_0^5 \left[3y\right]_0^4 dz$$

$$= \int_0^5 12 \, dz$$

Example:

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz = \int_0^5 \int_0^4 \left[3x\right]_0^1 dy \, dz$$

$$= \int_0^5 \int_0^4 3 \, dy \, dz$$

$$= \int_0^5 \left[3y\right]_0^4 dz$$

$$= \int_0^5 12 \, dz$$

$$= \left[12z\right]_0^5$$

Example:

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz = \int_0^5 \int_0^4 \left[3x\right]_0^1 dy \, dz$$

$$= \int_0^5 \int_0^4 3 \, dy \, dz$$

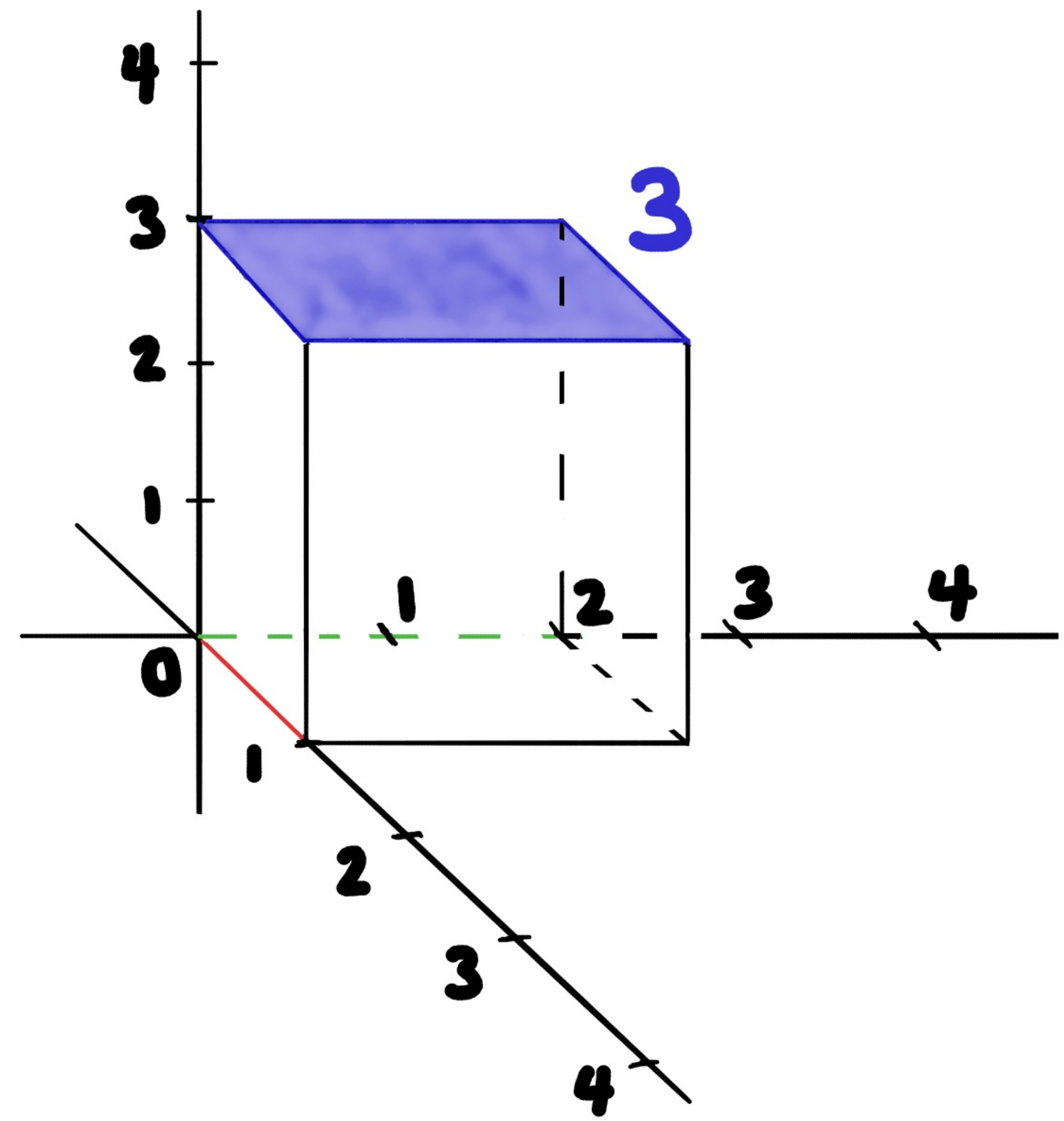
$$= \int_0^5 \left[3y\right]_0^4 dz$$

$$= \int_0^5 12 \, dz$$

$$= \left[12z\right]_0^5$$

$$= 60$$

$$\int_0^2 \int_0^1 3 \, dx \, dy$$



$$\int_0^2 \int_0^1 3 \, dx \, dy$$

height

width in y

width in x

Volume of rectangular box : $2 \cdot 1 \cdot 3 = 6$

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz$$

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz$$

width in z

width in y

width in x

height

"Volume" of 4-dimensional "rectangular box"

$$5 \cdot 4 \cdot 1 \cdot 3 = 60$$

$$\int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x,y,z)\, dx\, dy\, dz$$

$x_i, y_i, z_i \in \mathbb{R}$

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$$

width in z

width in y

width in x

varying height

$x_i, y_i, z_i \in \mathbb{R}$

Order doesn't matter

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)\,dx\,dy\,dz$$

$$=$$

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\int_{z_1}^{z_2} f(x,y,z)\,dz\,dx\,dy$$

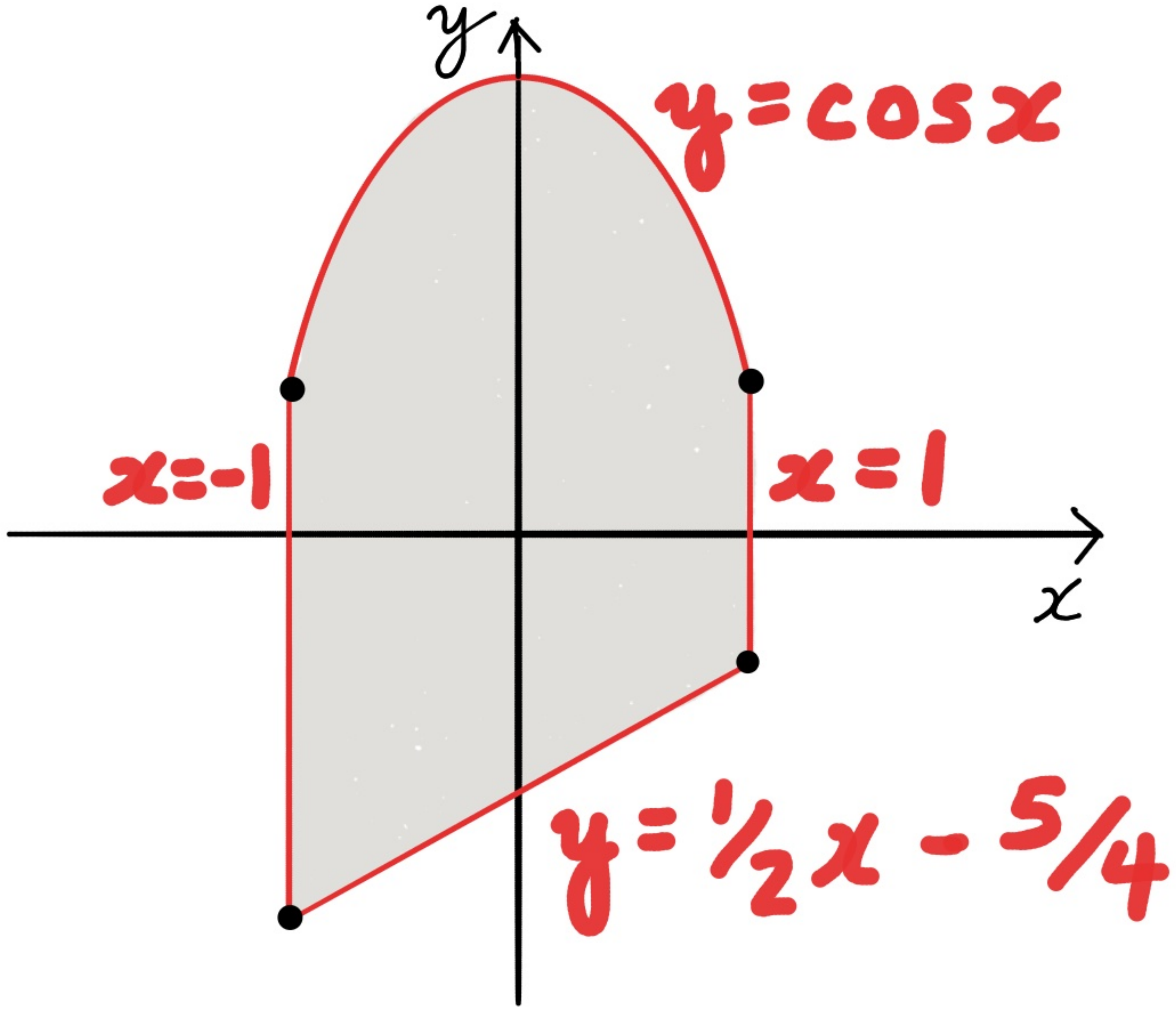
$x_i, y_i, z_i \in \mathbb{R}$

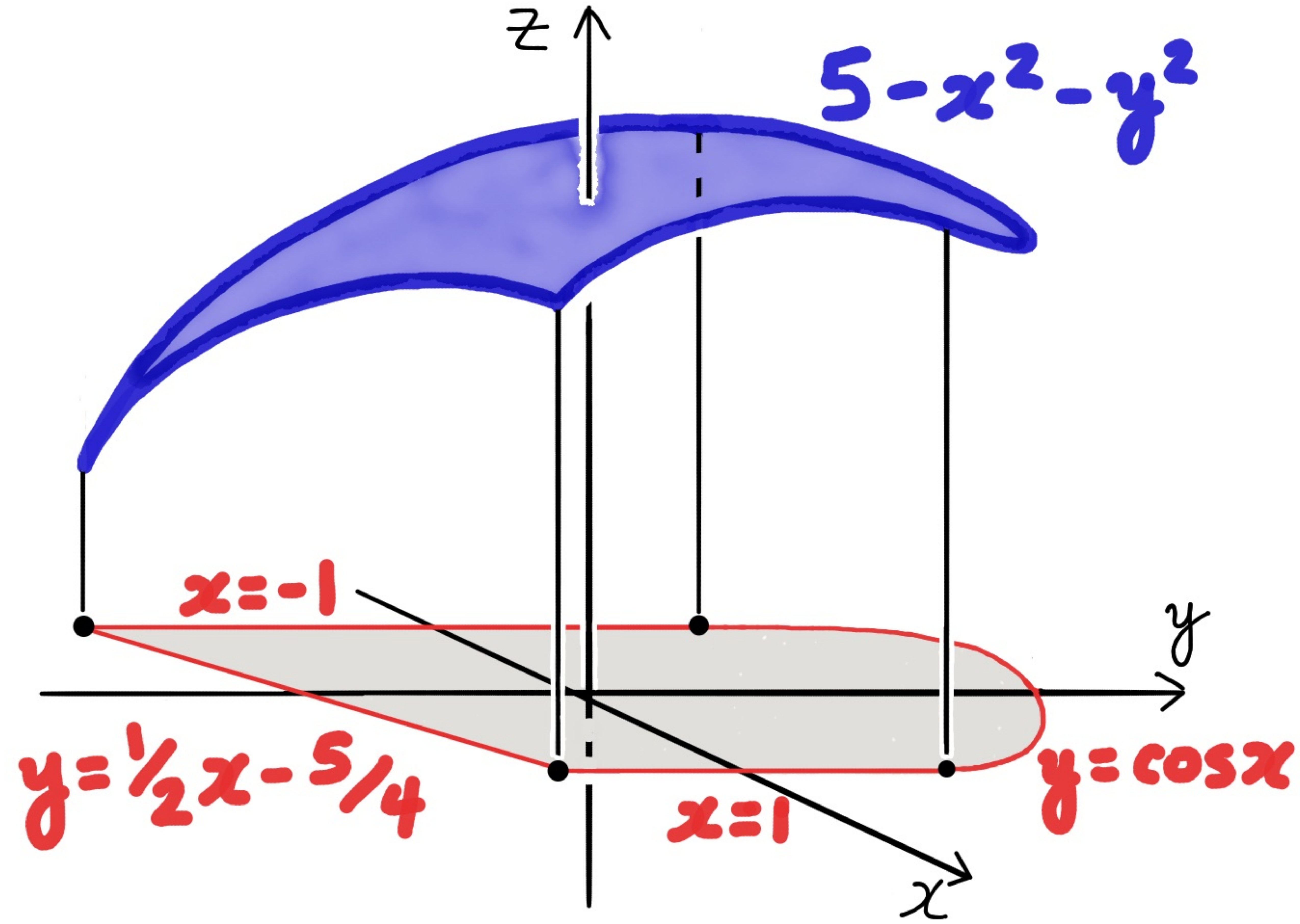
Example:

$$\int_3^{\pi}\int_4^{7}\int_{-8}^{6} \sin(x^2yz+z^2)\,dx\,dy\,dz$$

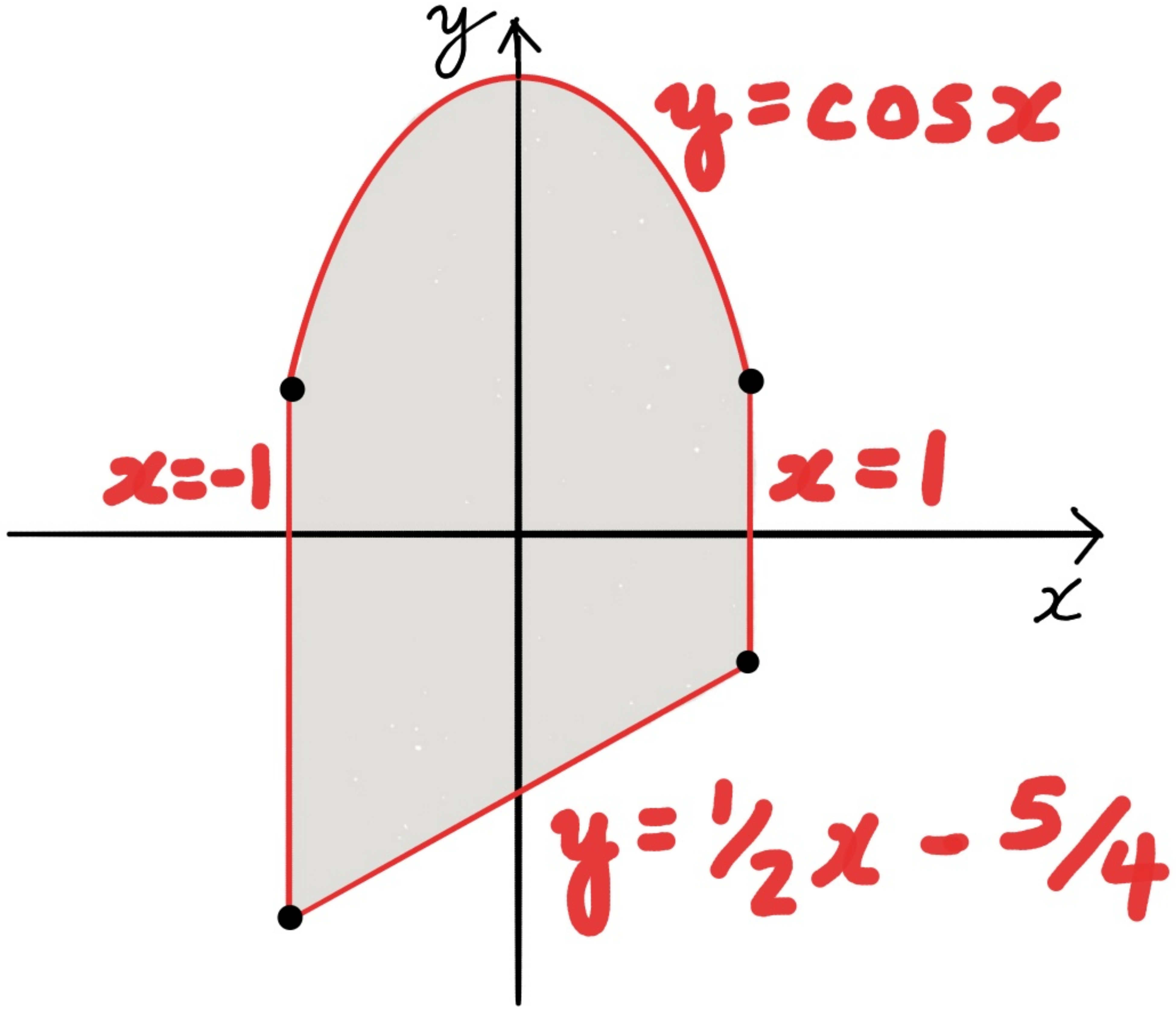
$$=$$

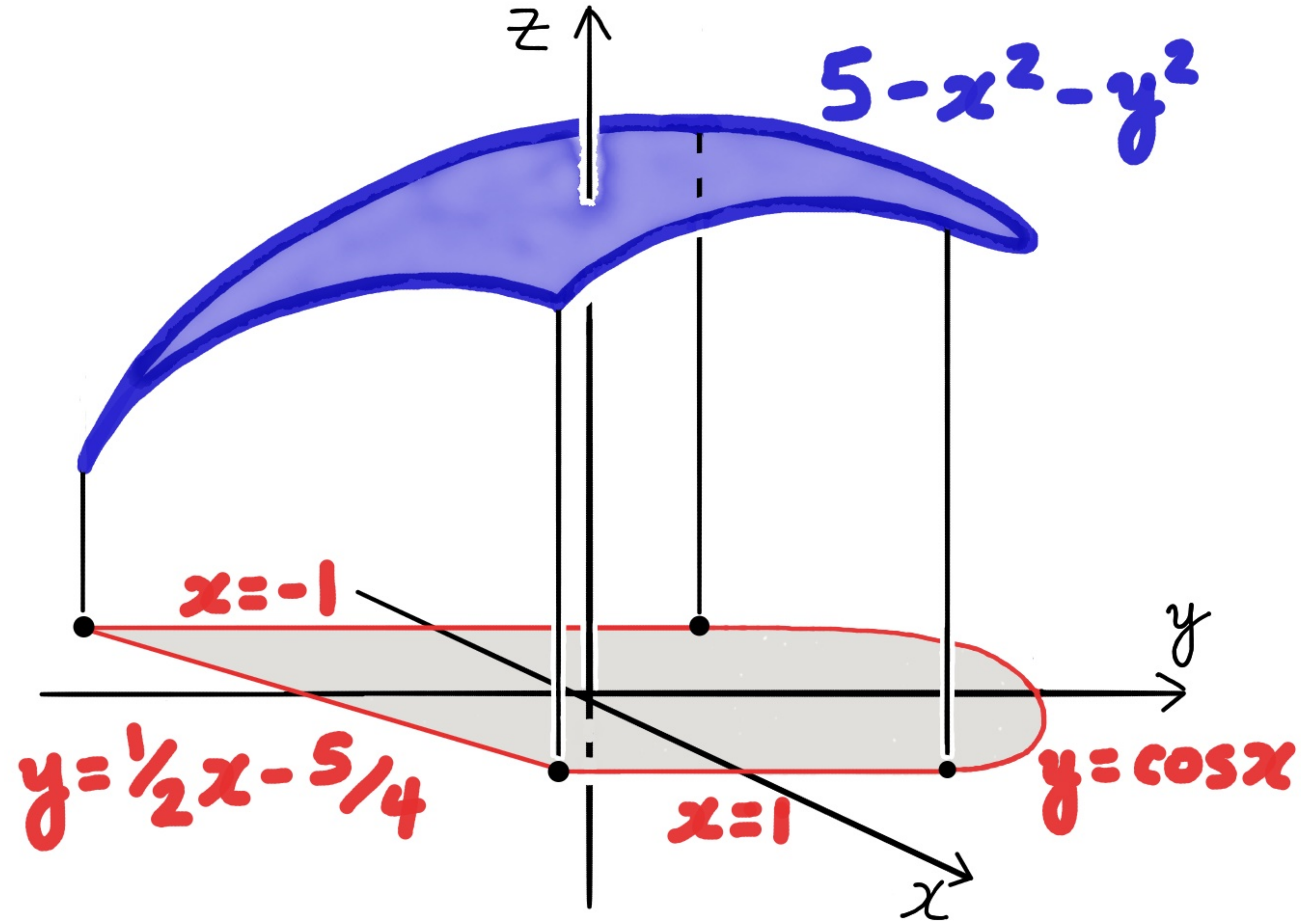
$$\int_{-8}^{6}\int_4^{7}\int_3^{\pi} \sin(x^2yz+z^2)\,dz\,dy\,dx$$





$$\int_{-1}^{1}\int_{\frac{1}{2}x-\frac{5}{4}}^{\cos x}\left(5-x^2-y^2\right)dy\,dx$$

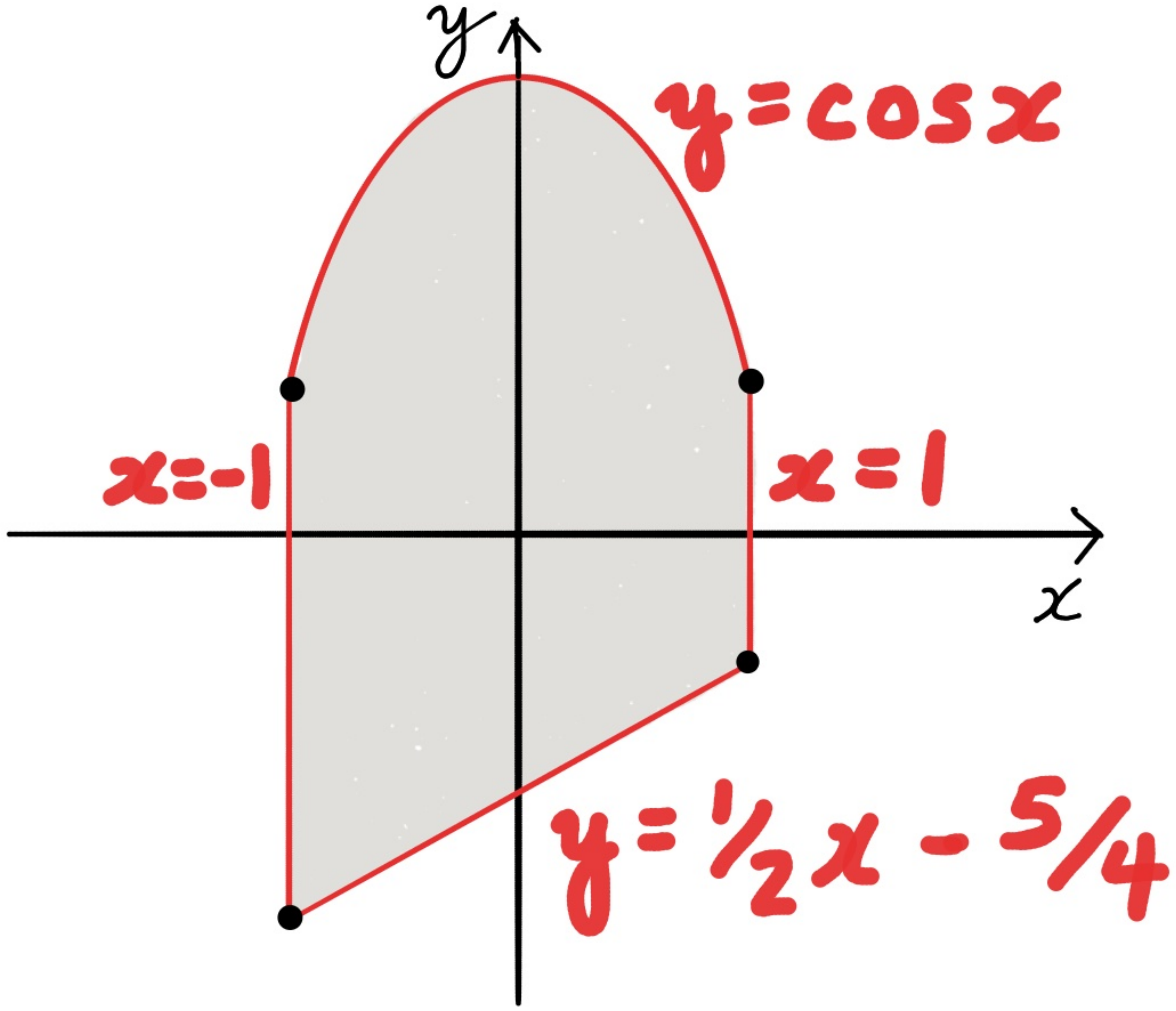


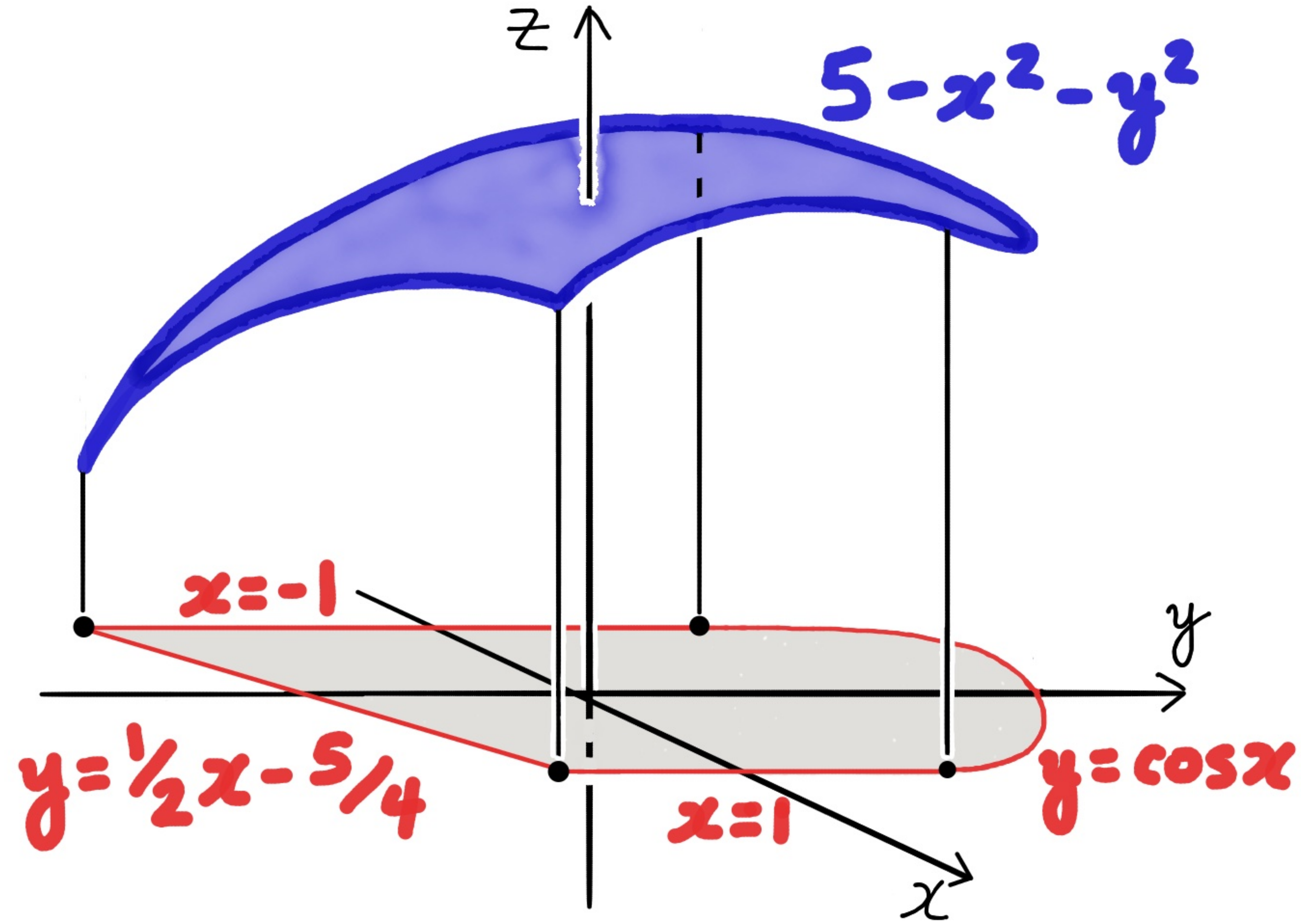


$$\int_{-1}^{1}\int_{\frac{1}{2}x-\frac{5}{4}}^{\cos x}(5-x^2-y^2)\,dy\,dx$$

x-limits

y-limits





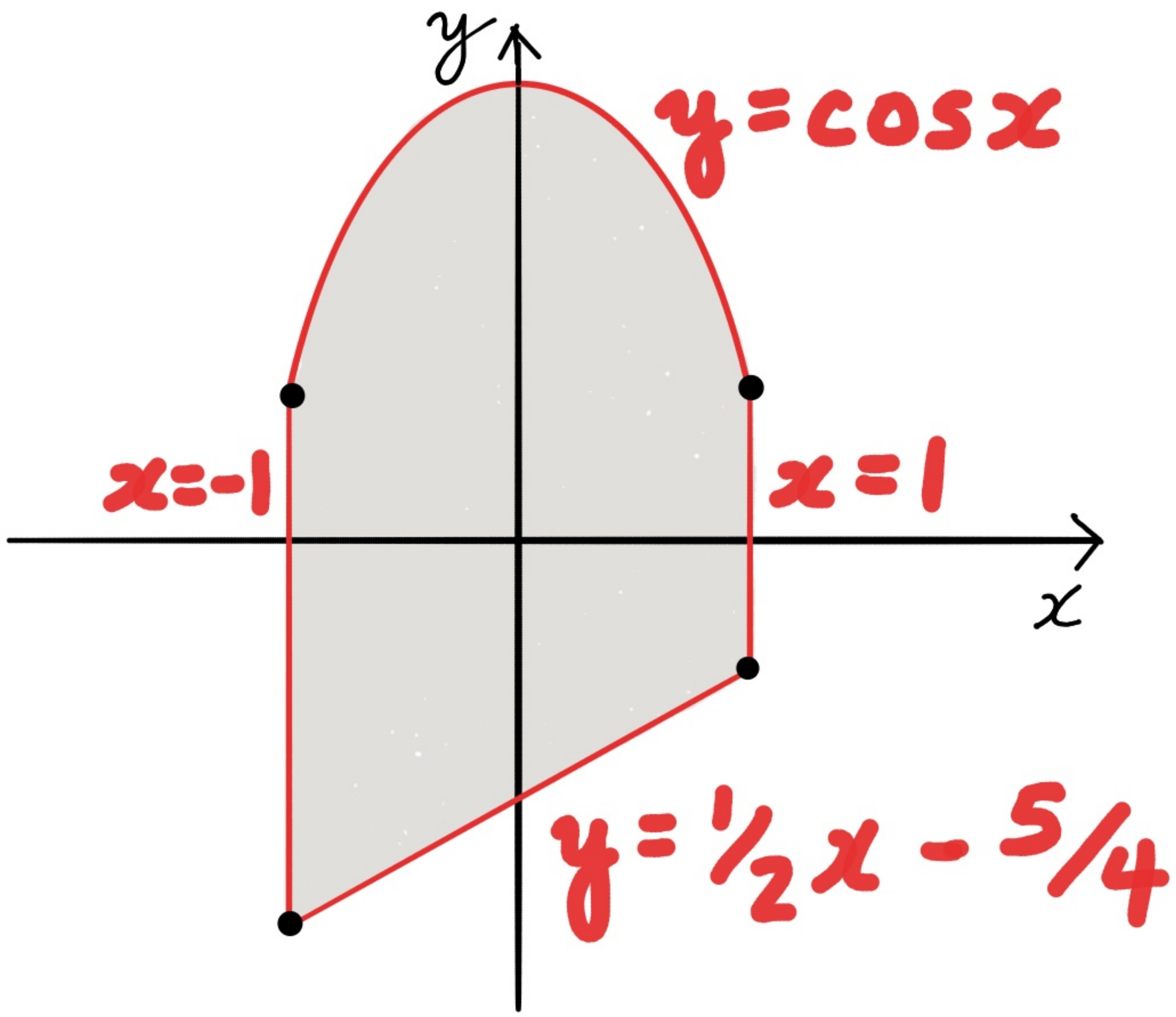
$$\int_{-1}^{1}\int_{\frac{1}{2}x-\frac{5}{4}}^{\cos x}\left(5-x^2-y^2\right)dy\,dx$$

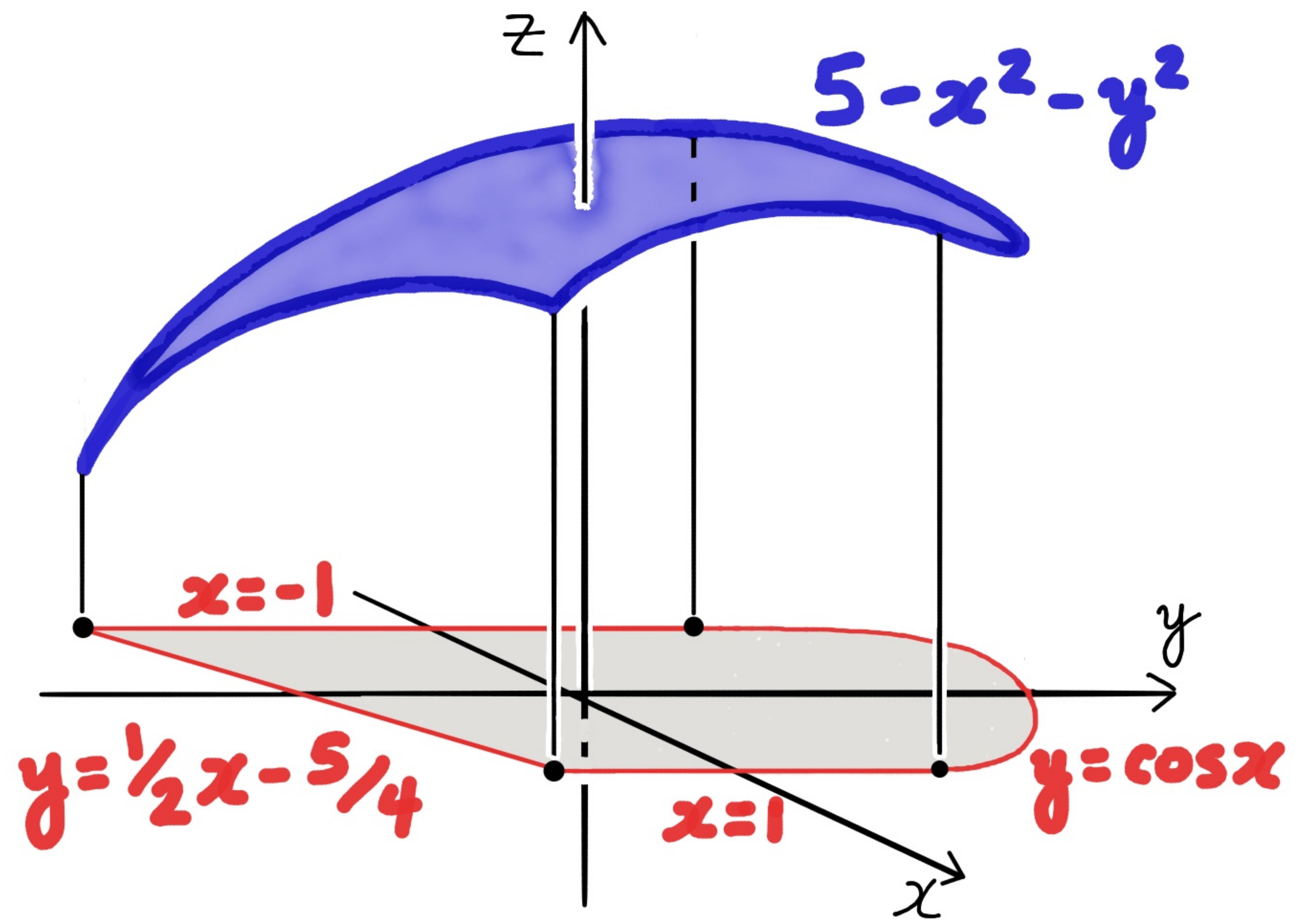
x-limits

constants

y-limits

functions of x





$$\int_{-1}^{1}\int_{\frac{1}{2}x-\frac{5}{4}}^{\cos x}\left(5-x^2-y^2\right)dy\,dx$$

x-limits

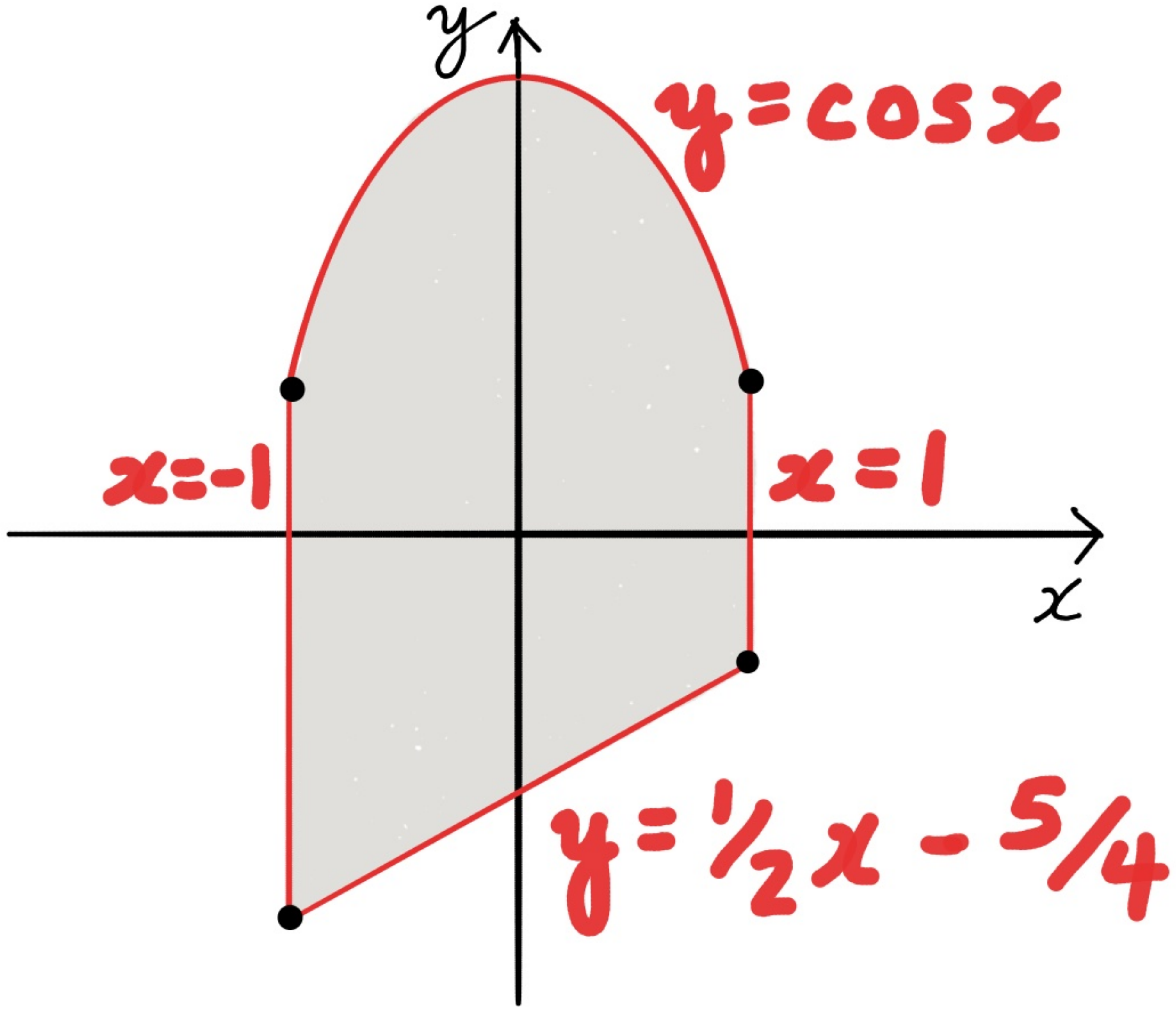
constants

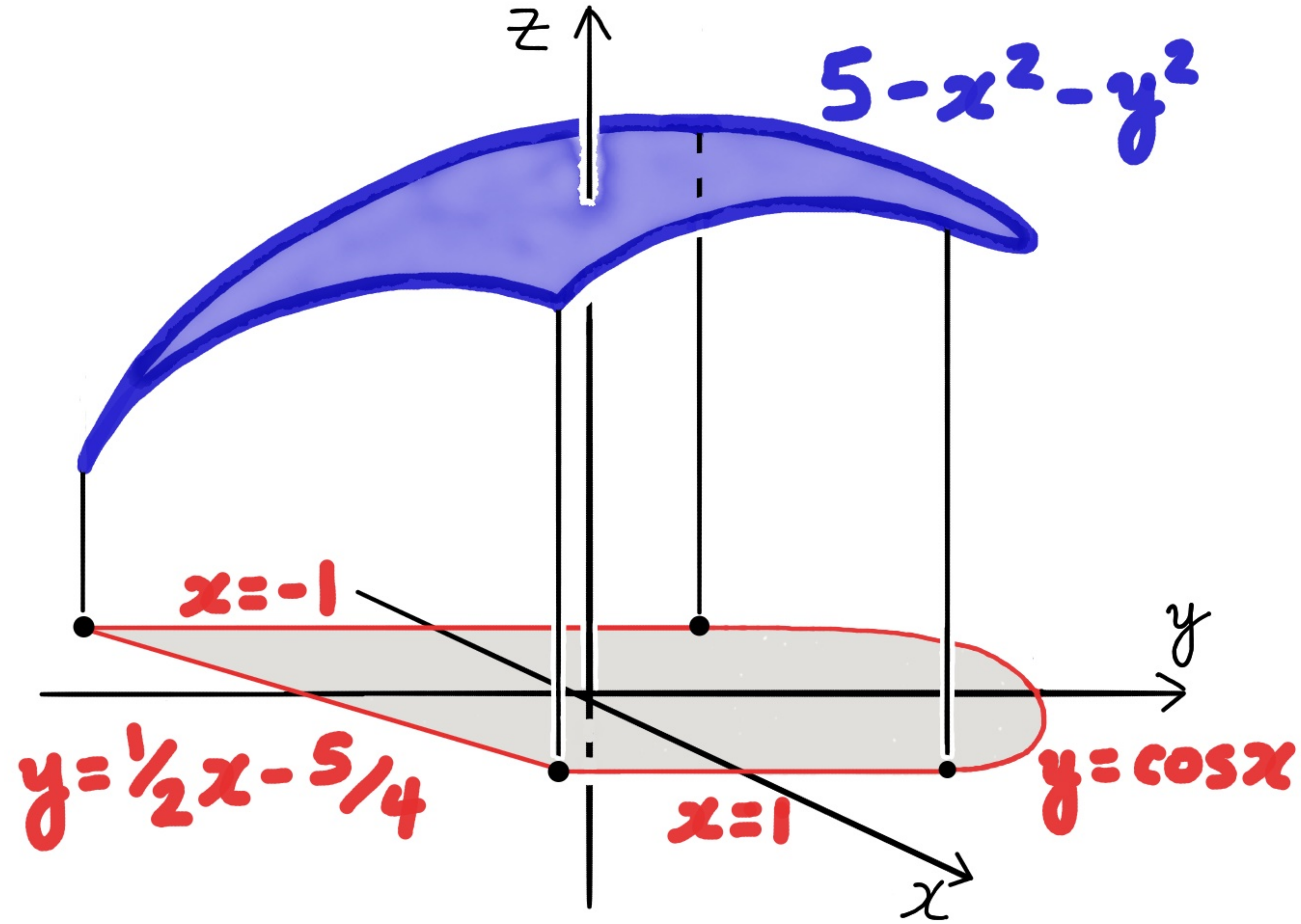
width in x

y-limits

functions of x

varying width in y





$$\int_{-1}^{1}\int_{\frac{1}{2}x-\frac{5}{4}}^{\cos x}\left(5-x^2-y^2\right)dy\,dx$$

x-limits

constants

width in x

y-limits

functions of x

varying width in y

varying height

$$\int_{2}^{7} \int_{-x}^{x^2} \int_{-\ln(x+y+1)}^{(x+y)^2} x^2 y z^3 \, dz \, dy \, dx$$

$$\int_{2}^{7} \int_{-x}^{x^2} \int_{-\ln(x+y+1)}^{(x+y)^2} x^2 y z^3 \, dz \, dy \, dx$$

varying height

x-limits

constants

width in x

y-limits

functions of x

varying width in y

z-limits

functions of x, y

varying width in z

$$\int_{2}^{7} \int_{-x}^{x^2} \int_{-\ln(x+y+1)}^{(x+y)^2} x^2 y z^3 \, dz \, dy \, dx$$

$$\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) \, dz \, dy \, dx$$

$$\int_{y_1}^{y_2} \int_{z_1(y)}^{z_2(y)} \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)\, dx\, dz\, dy$$

$$\int_{y_1}^{y_2} \int_{z_1(y)}^{z_2(y)} \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)\,dx\,dz\,dy$$

$$= \iiint\limits_R f(x,y,z)\,dV$$

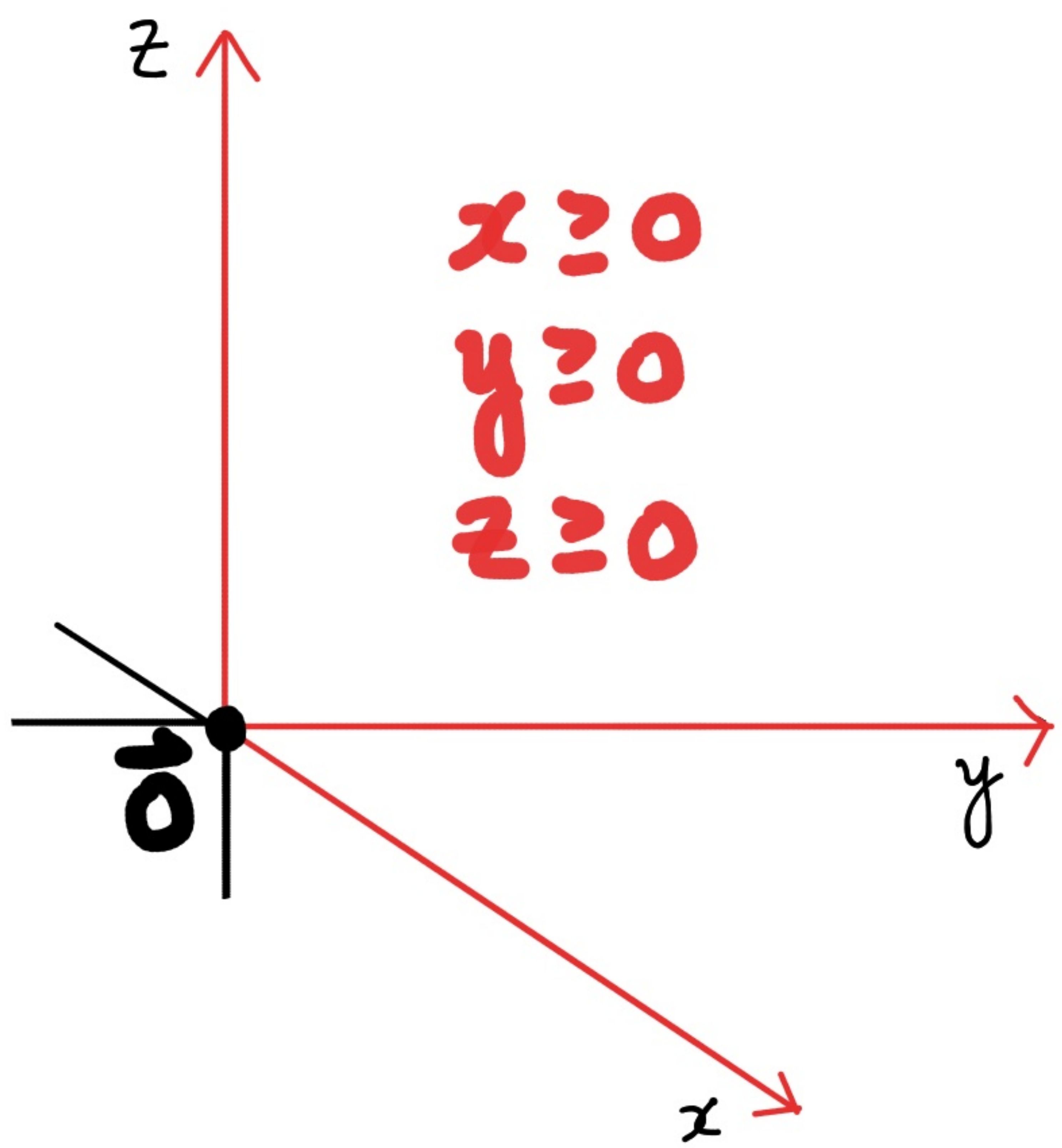
where R is region in $\mathbb{R}^3$ of points (x,y,z) with $y_1 \le y \le y_2$, $z_1(y) \le z \le z_2(y)$, and $x_1(y,z) \le x \le x_2(y,z)$.

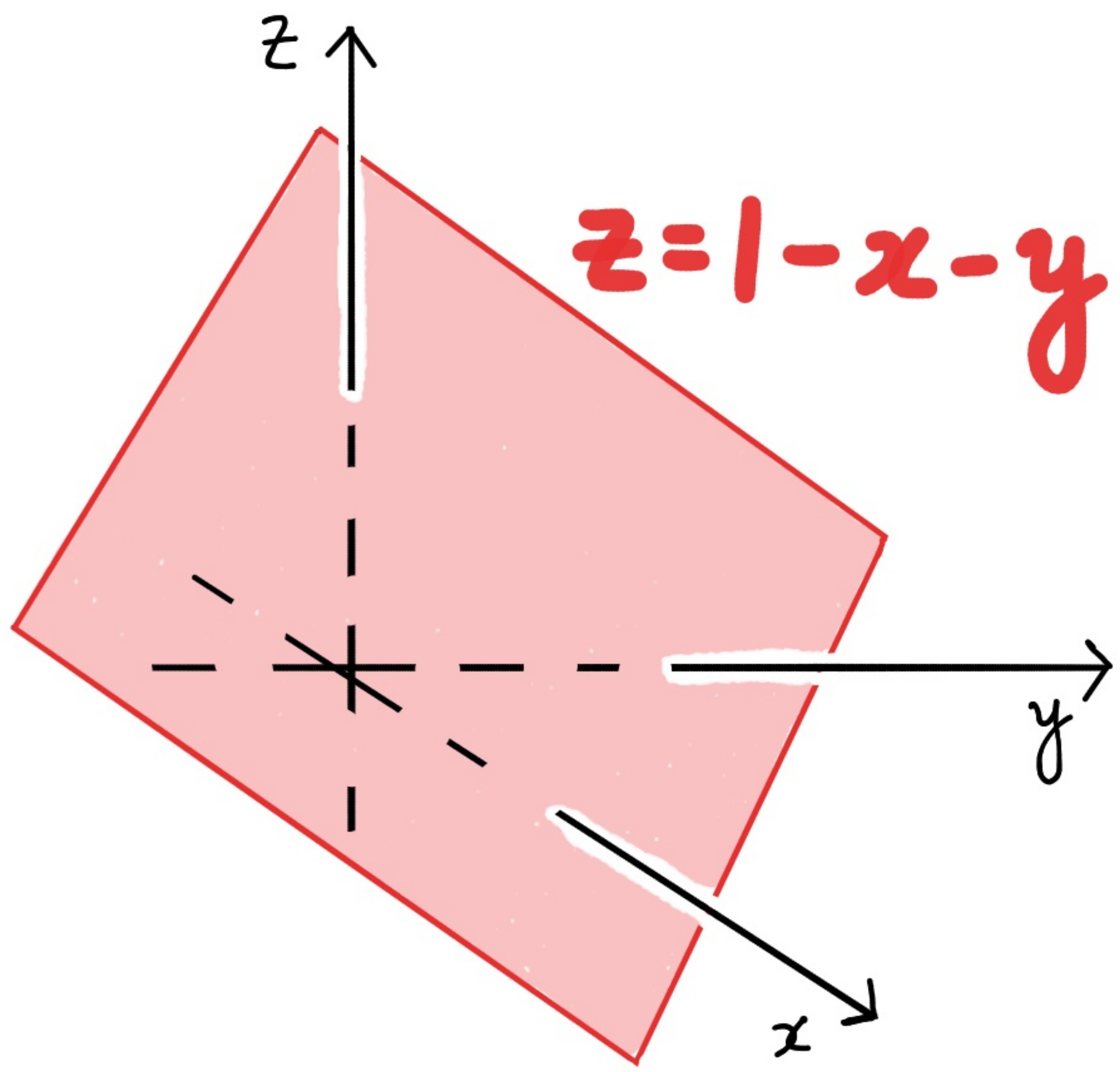
Area of R in $\mathbb{R}^2$: $\quad \displaystyle\iint_R dA$

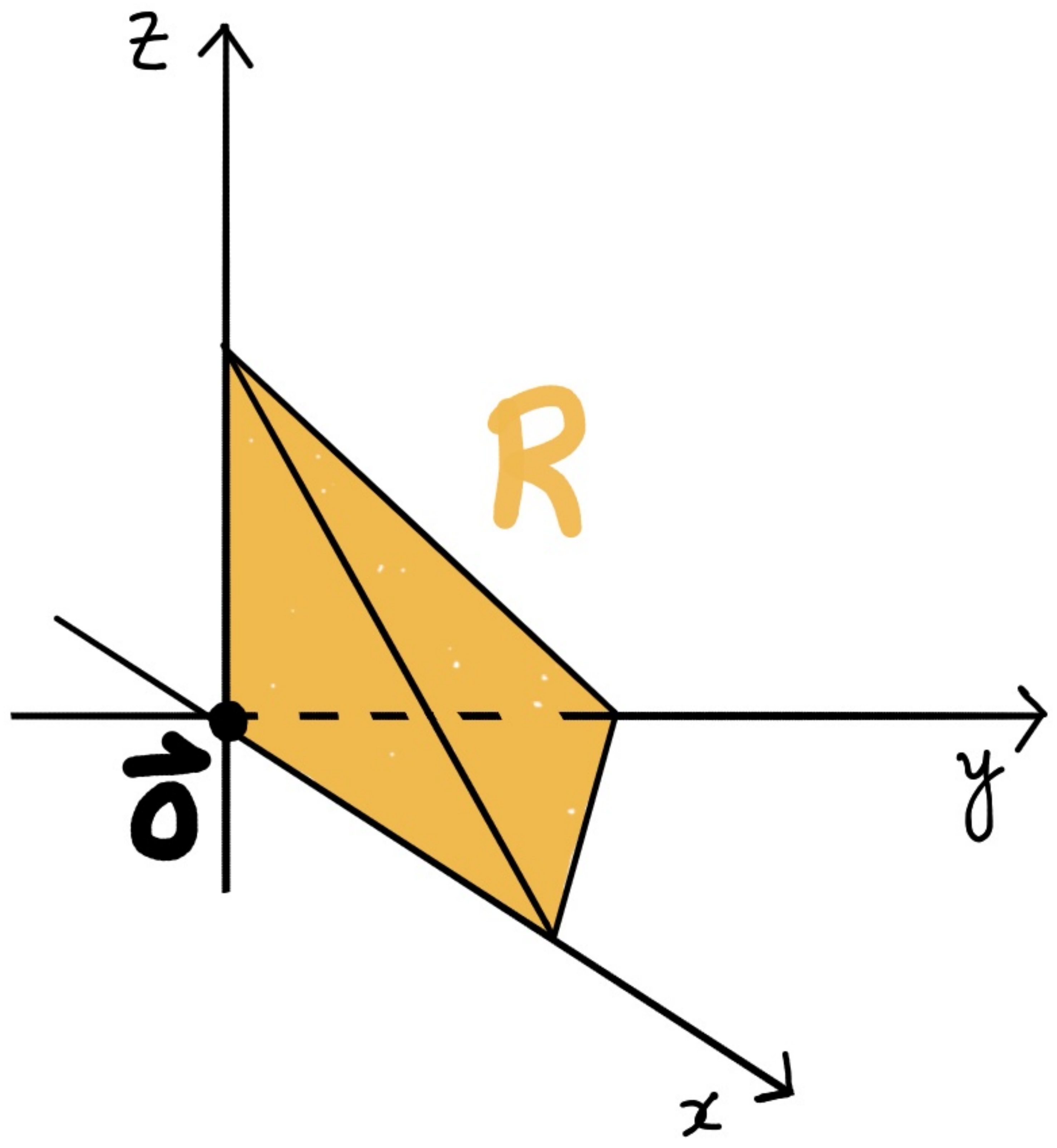
Volume of R in $\mathbb{R}^3$: $\quad \displaystyle\iiint_R dV$

Example:

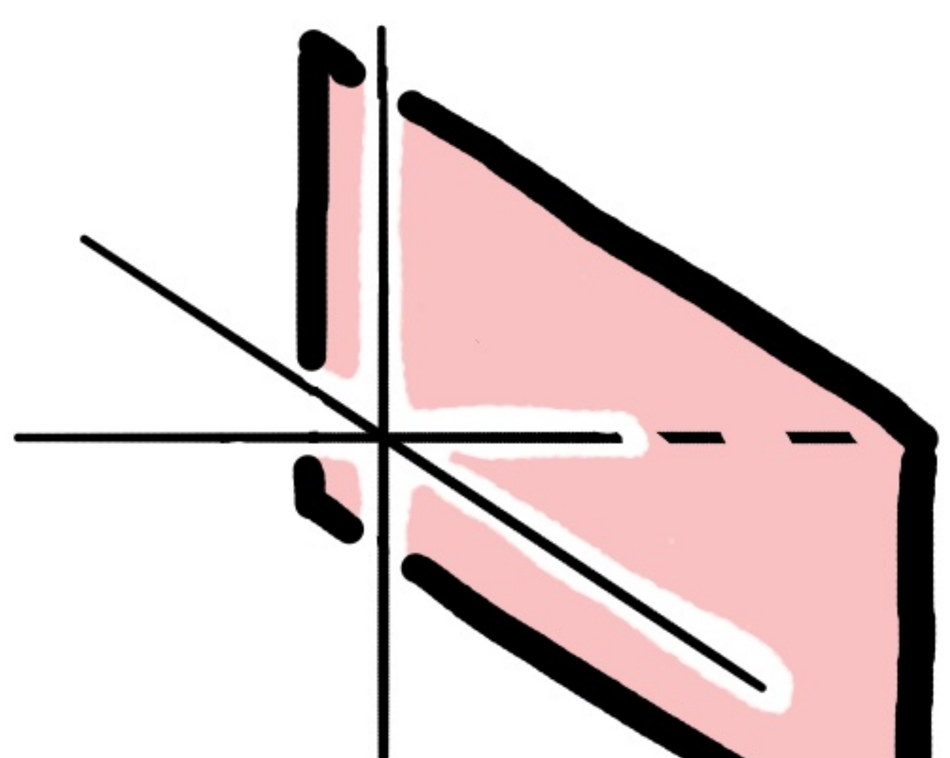
① Find volume of region R in the first octant of $\mathbb{R}^3$ that contains $\vec{0}$ and is bounded by the plane $z=1-x-y$.



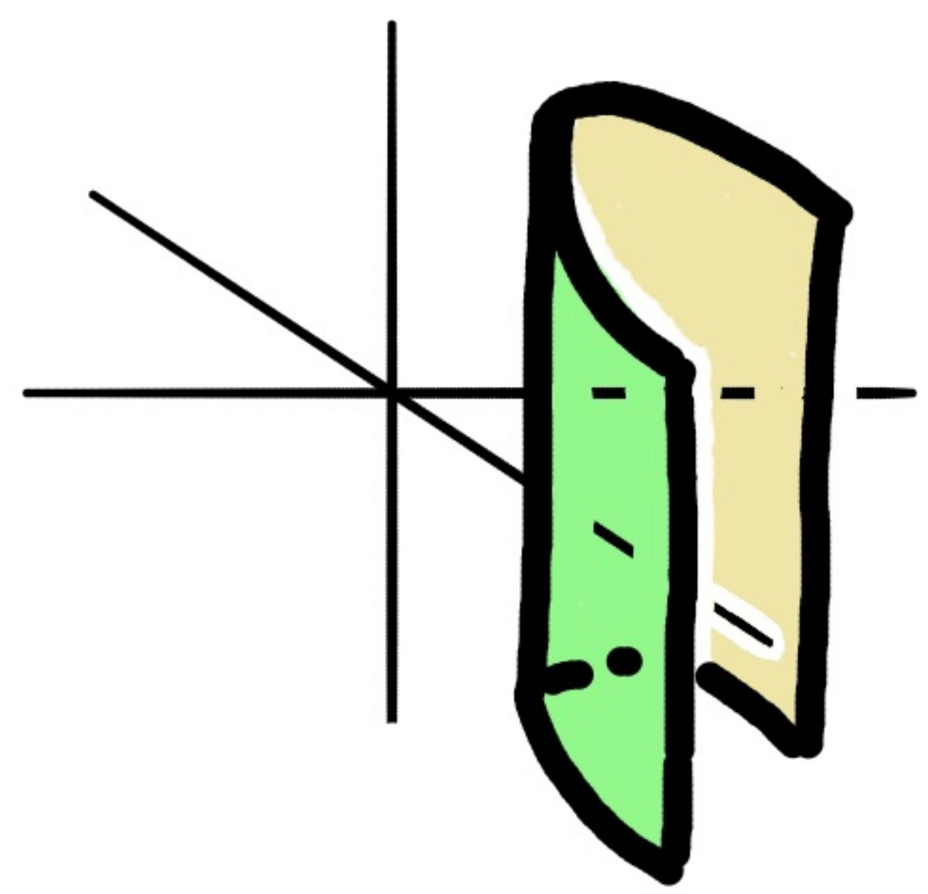




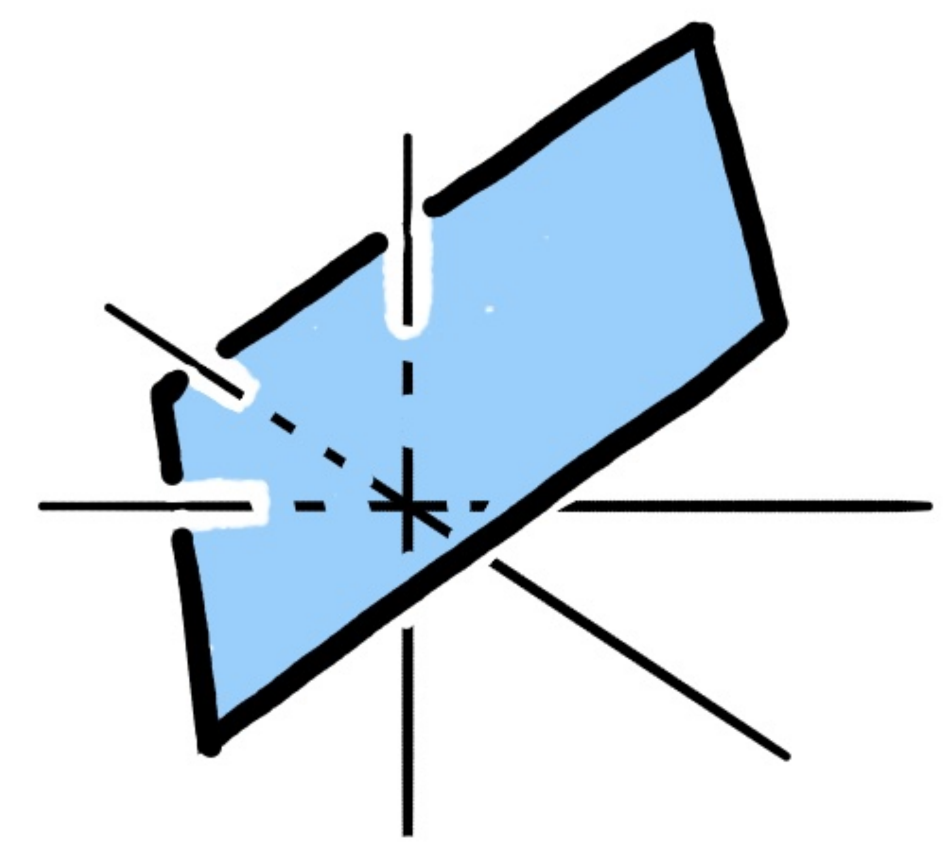
② Find $\iiint_R x\,dV$ where R is the region in $\mathbb{R}^3$ containing $\vec{0}$ and bounded by the following 6 surfaces:



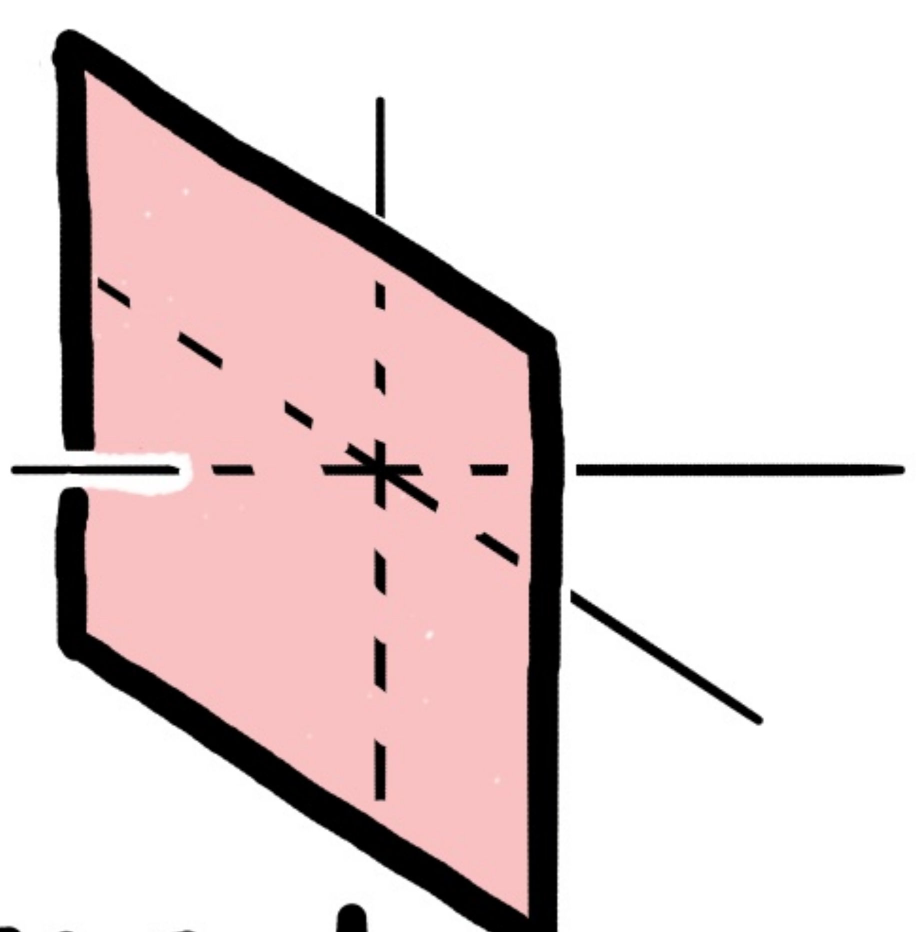
$y=1$



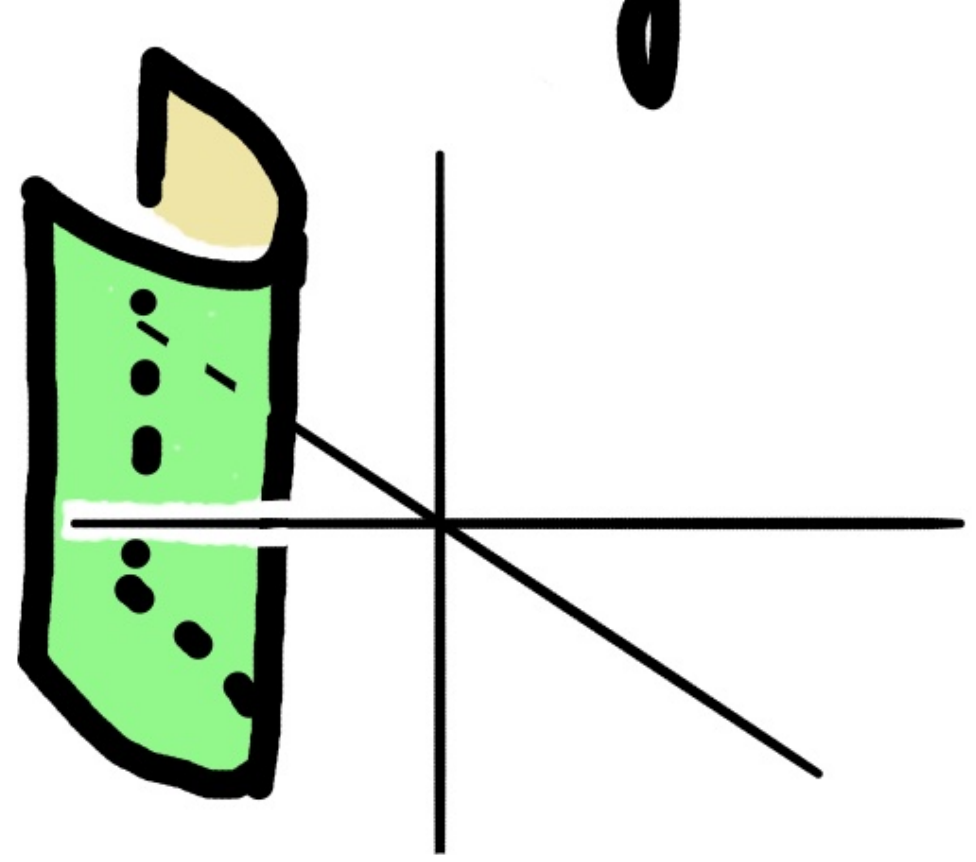
$x=y^2+1$



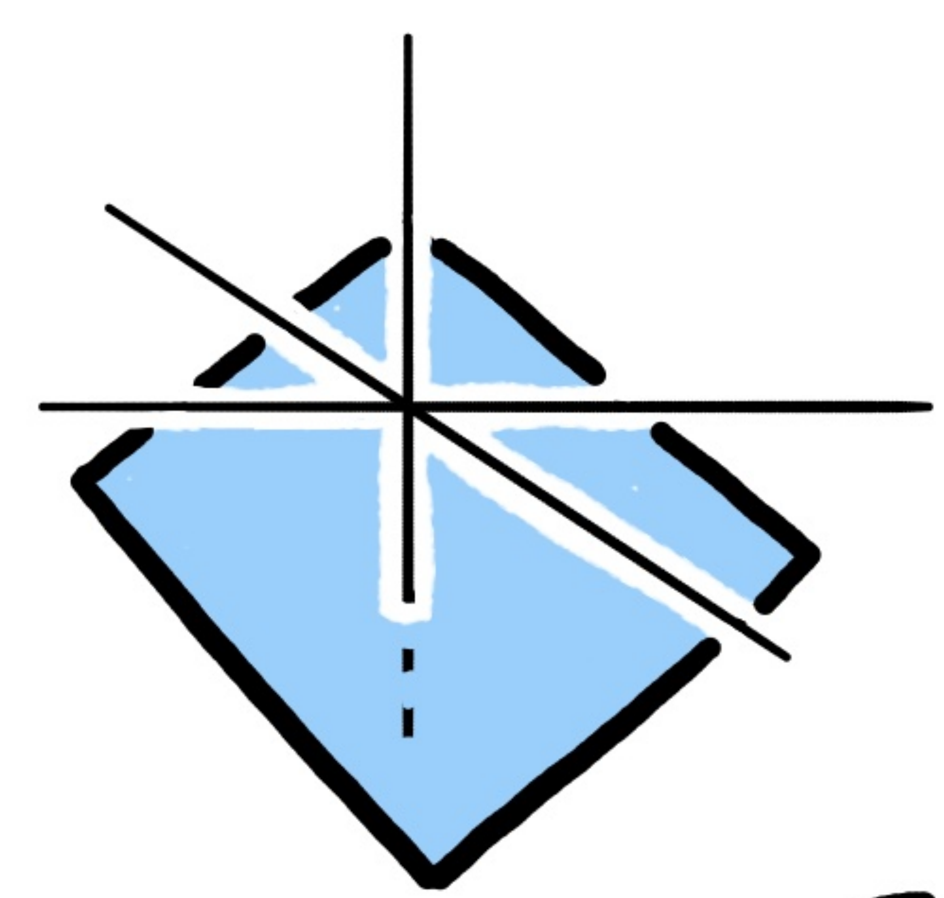
$x+y-z=-5$



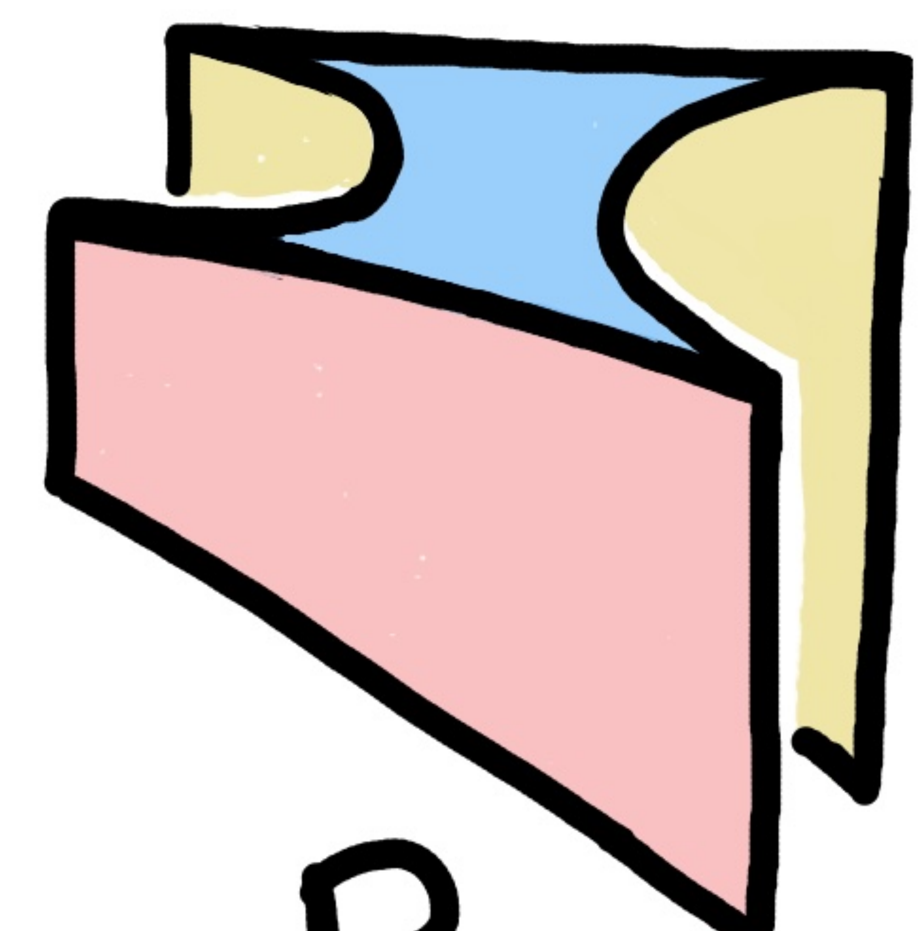
$y=-1$



$x=-y^2-1$



$z+x+y=-5$



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