

Twenty - three

Triple integrals in Cartesian coordinates

Example:

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz$$

Example:

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz = \int_0^5 \int_0^4 [3x]_0^1 \, dy \, dz$$

Example:

$$\begin{aligned}\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz &= \int_0^5 \int_0^4 [3x]_0^1 \, dy \, dz \\ &= \int_0^5 \int_0^4 3 \, dy \, dz\end{aligned}$$

Example:

$$\begin{aligned}\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz &= \int_0^5 \int_0^4 [3x]_0^1 \, dy \, dz \\ &= \int_0^5 \int_0^4 3 \, dy \, dz \\ &= \int_0^5 [3y]_0^4 \, dz\end{aligned}$$

Example:

$$\begin{aligned}\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz &= \int_0^5 \int_0^4 [3x]_0^1 \, dy \, dz \\ &= \int_0^5 \int_0^4 3 \, dy \, dz \\ &= \int_0^5 [3y]_0^4 \, dz \\ &= \int_0^5 12 \, dz\end{aligned}$$

Example:

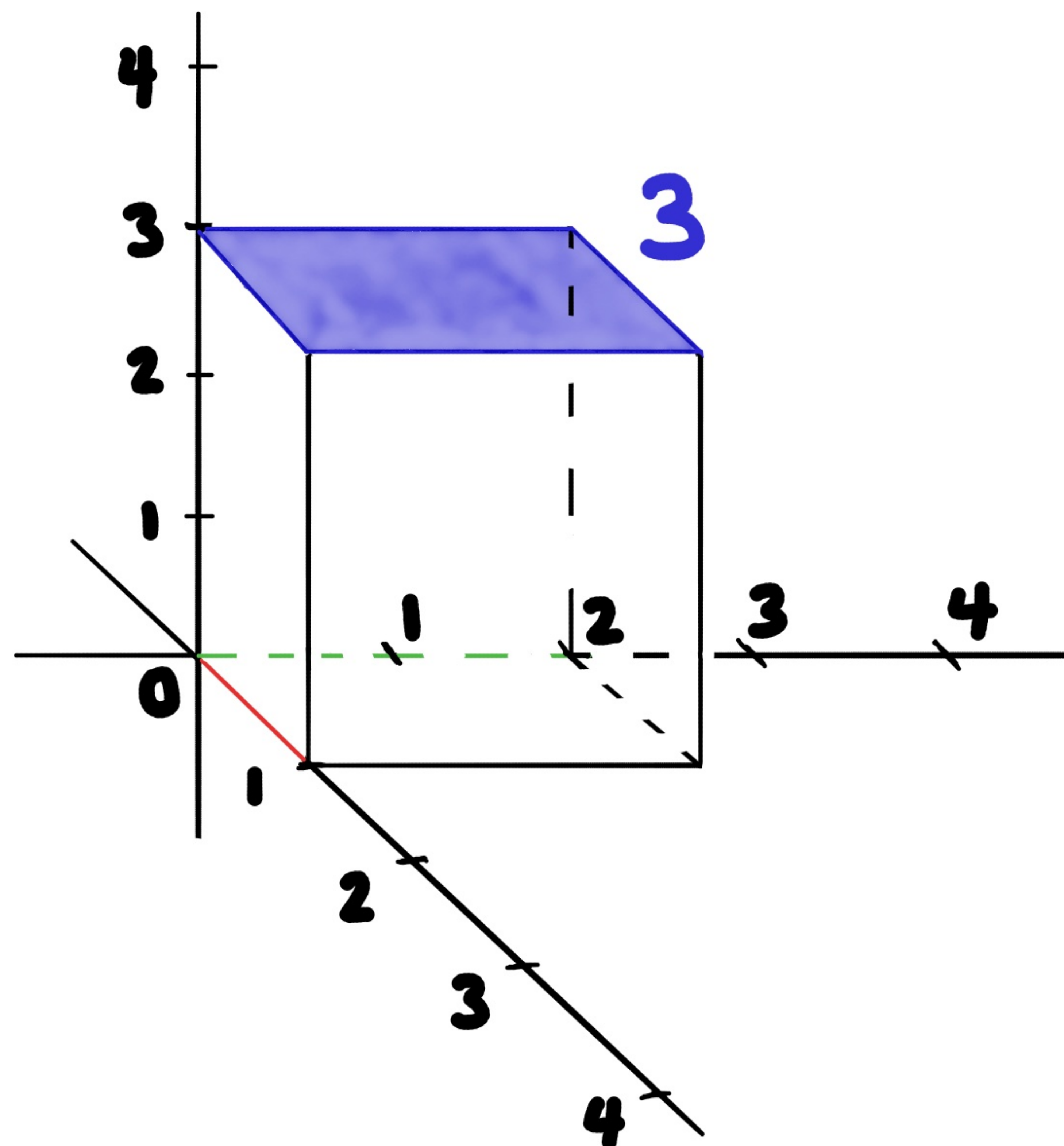
$$\begin{aligned}\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz &= \int_0^5 \int_0^4 [3x]_0' \, dy \, dz \\ &= \int_0^5 \int_0^4 3 \, dy \, dz \\ &= \int_0^5 [3y]_0^4 \, dz \\ &= \int_0^5 12 \, dz \\ &= [12z]_0^5\end{aligned}$$

Example:

$$\begin{aligned}\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz &= \int_0^5 \int_0^4 [3x]_0^1 \, dy \, dz \\&= \int_0^5 \int_0^4 3 \, dy \, dz \\&= \int_0^5 [3y]_0^4 \, dz \\&= \int_0^5 12 \, dz \\&= [12z]_0^5 \\&= 60\end{aligned}$$

$$\int_0^2 \int_0^1 3 \, dx \, dy$$

$\int_0^2 \int_0^1 3 \, dx \, dy$
 ↑ width in y
 ↑ width in x
 ↑ height



Volume of
rectangular box: $2 \cdot 1 \cdot 3 = 6$

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz$$

A diagram illustrating a 4D volume element. It consists of four vertical bars of different colors and heights, each with an integral symbol at the top and a small circle at the base. From left to right: a brown bar with the number 5, a green bar with the number 4, a red bar with the number 1, and a blue bar with the number 3. Below the brown bar is the text "width in z" with an orange arrow pointing up. Below the green bar is the text "width in y" with a green arrow pointing up. Below the red bar is the text "width in x" with a red arrow pointing up. Below the blue bar is the text "height" with a blue arrow pointing up. To the right of the bars is the expression $dx dy dz$, where dx is in red, dy is in green, and dz is in brown.

$$\int_0^5 \int_0^4 \int_0^1 3 \, dx \, dy \, dz$$

width in z width in y width in x height

"Volume" of 4-dimensional :
"rectangular box" :

$$5 \cdot 4 \cdot 1 \cdot 3 = 60$$

$$\int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y, z) \, dx \, dy \, dz$$

$$x_i, y_i, z_i \in \mathbb{R}$$

Diagram illustrating a triple integral over a volume element. The integral is written as:

$$\int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y, z) \, dx \, dy \, dz$$

The diagram shows three vertical strips representing the integration limits:

- A brown strip for z from z_1 to z_2 , labeled "width in z " (orange arrow).
- A green strip for y from y_1 to y_2 , labeled "width in y " (green arrow).
- A pink strip for x from x_1 to x_2 , labeled "width in x " (red arrow).

The function $f(x, y, z)$ is shown in a blue oval, with a blue arrow pointing to it labeled "varying height". The differential elements dx , dy , and dz are highlighted in pink, green, and brown respectively, matching the strips.

$$x_i, y_i, z_i \in \mathbb{R}$$

Order doesn't matter

$$\int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y, z) \, dx \, dy \, dz$$

=

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} \int_{z_1}^{z_2} f(x, y, z) \, dz \, dx \, dy$$

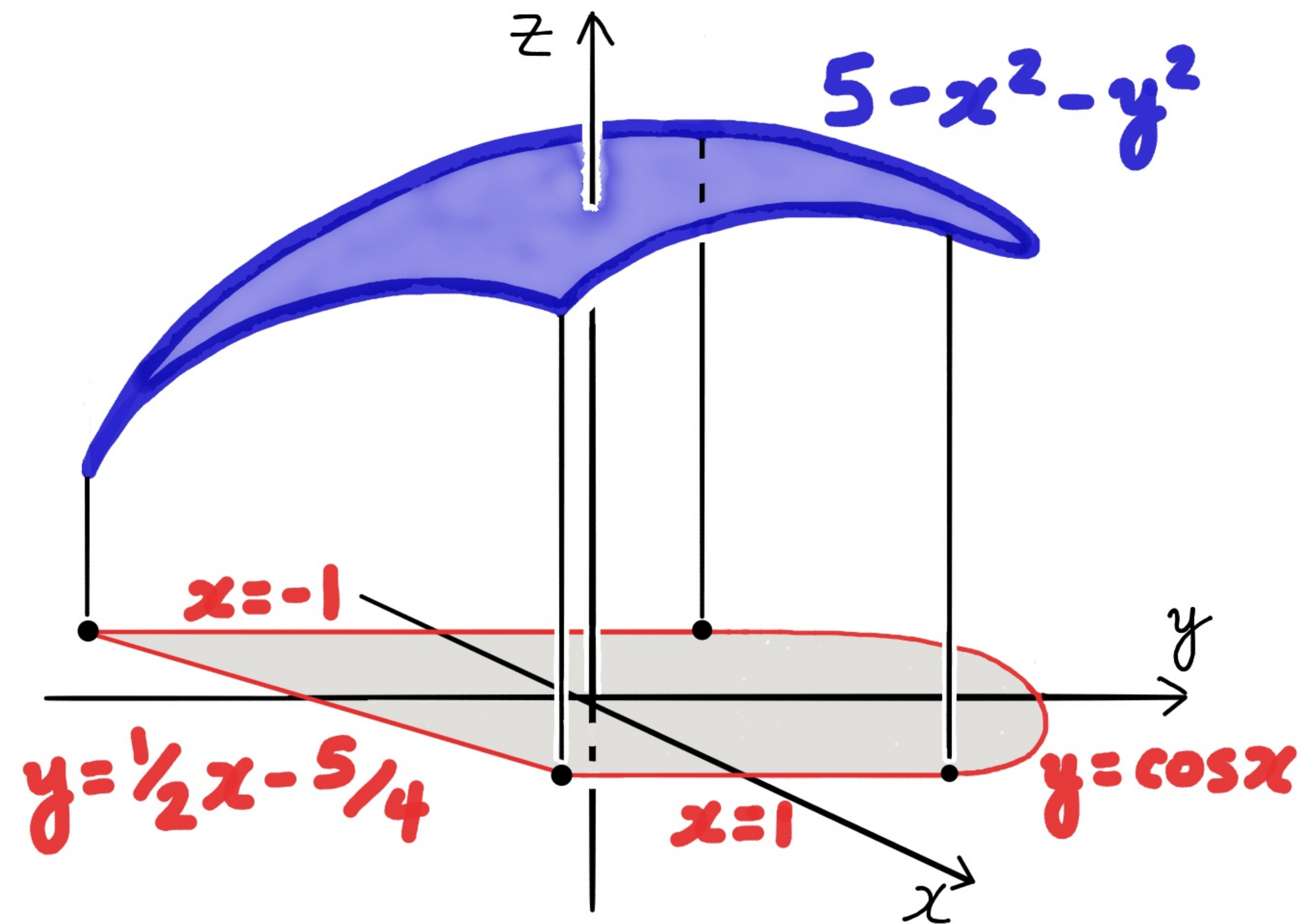
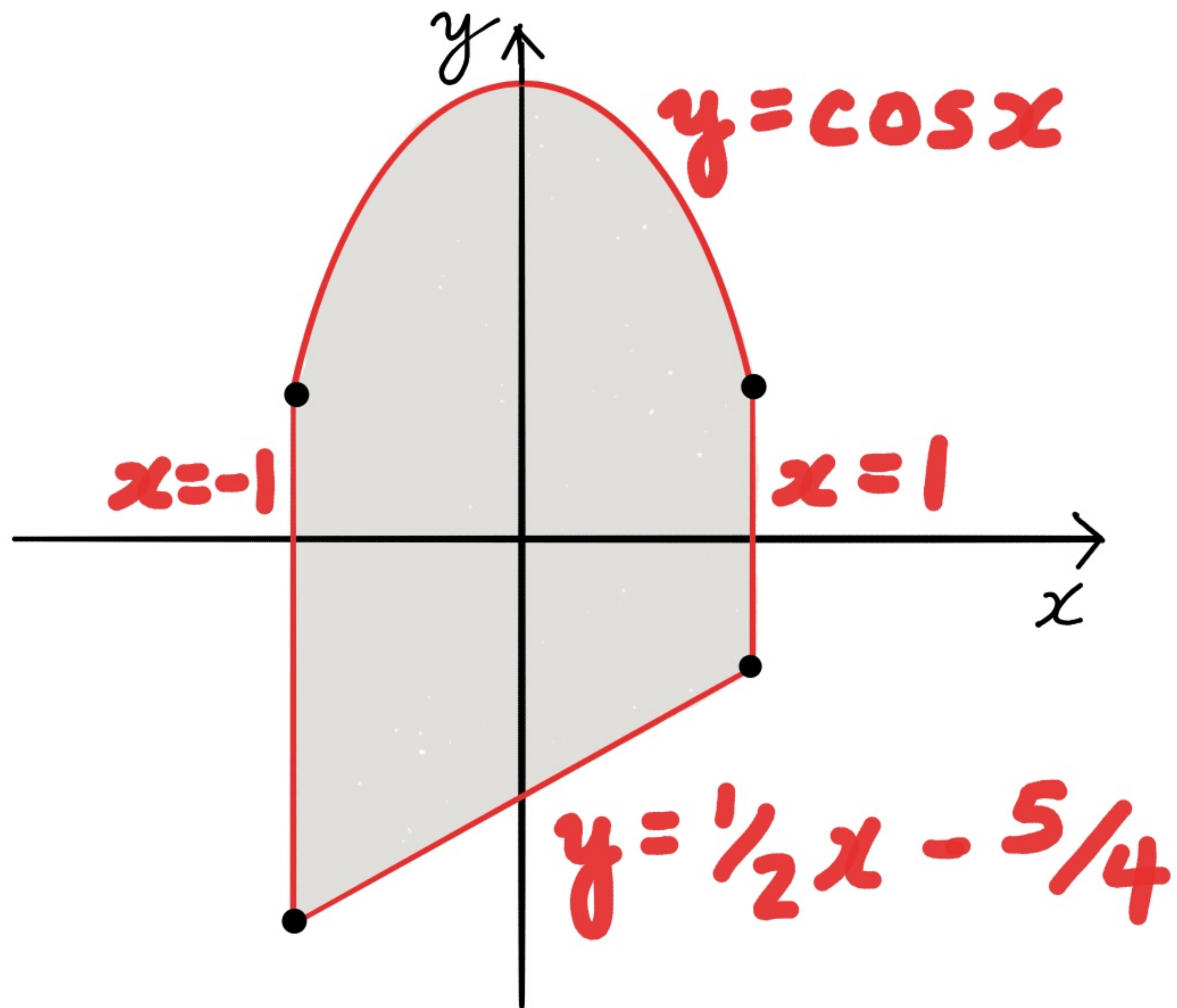
$$x_i, y_i, z_i \in \mathbb{R}$$

Example:

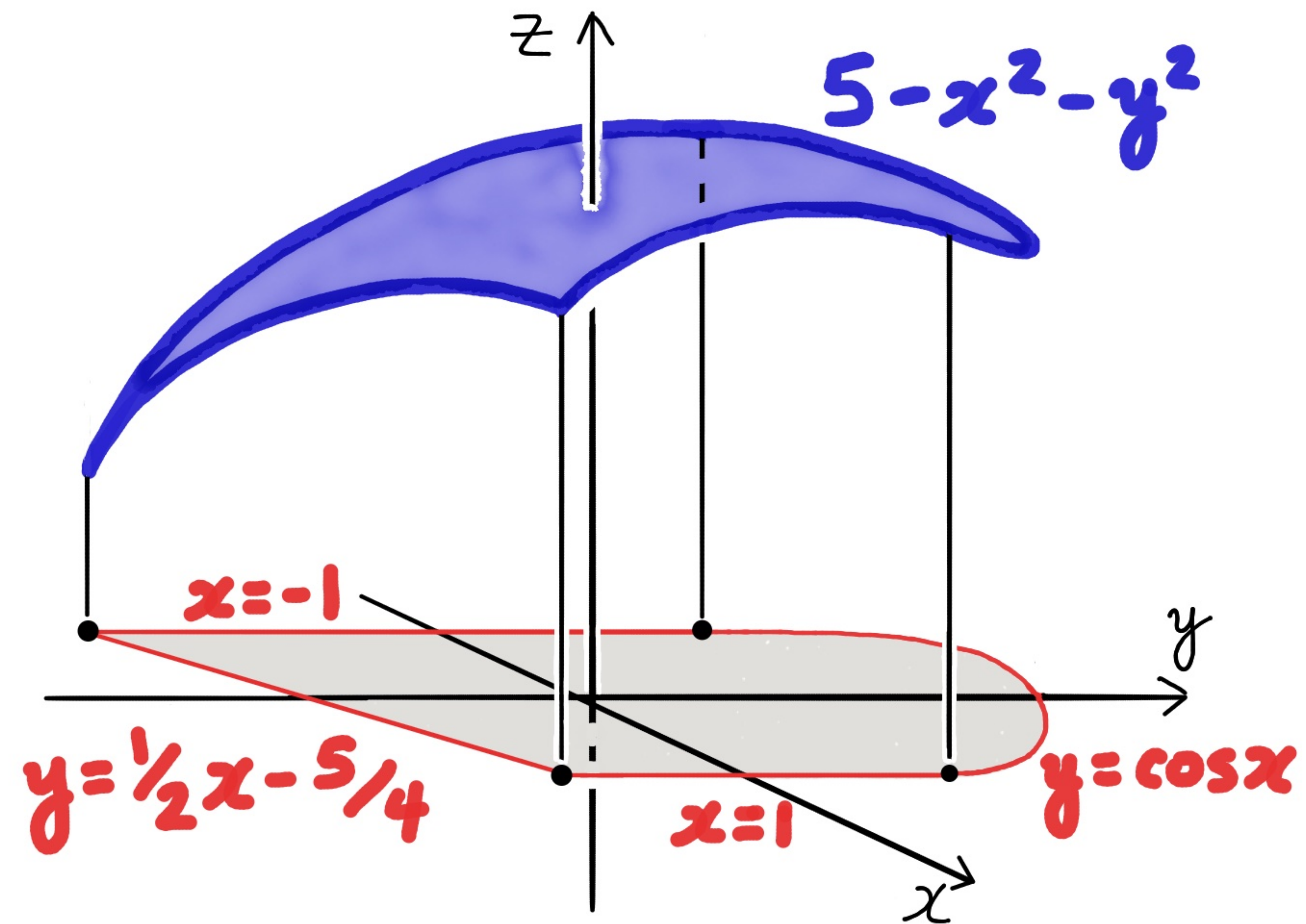
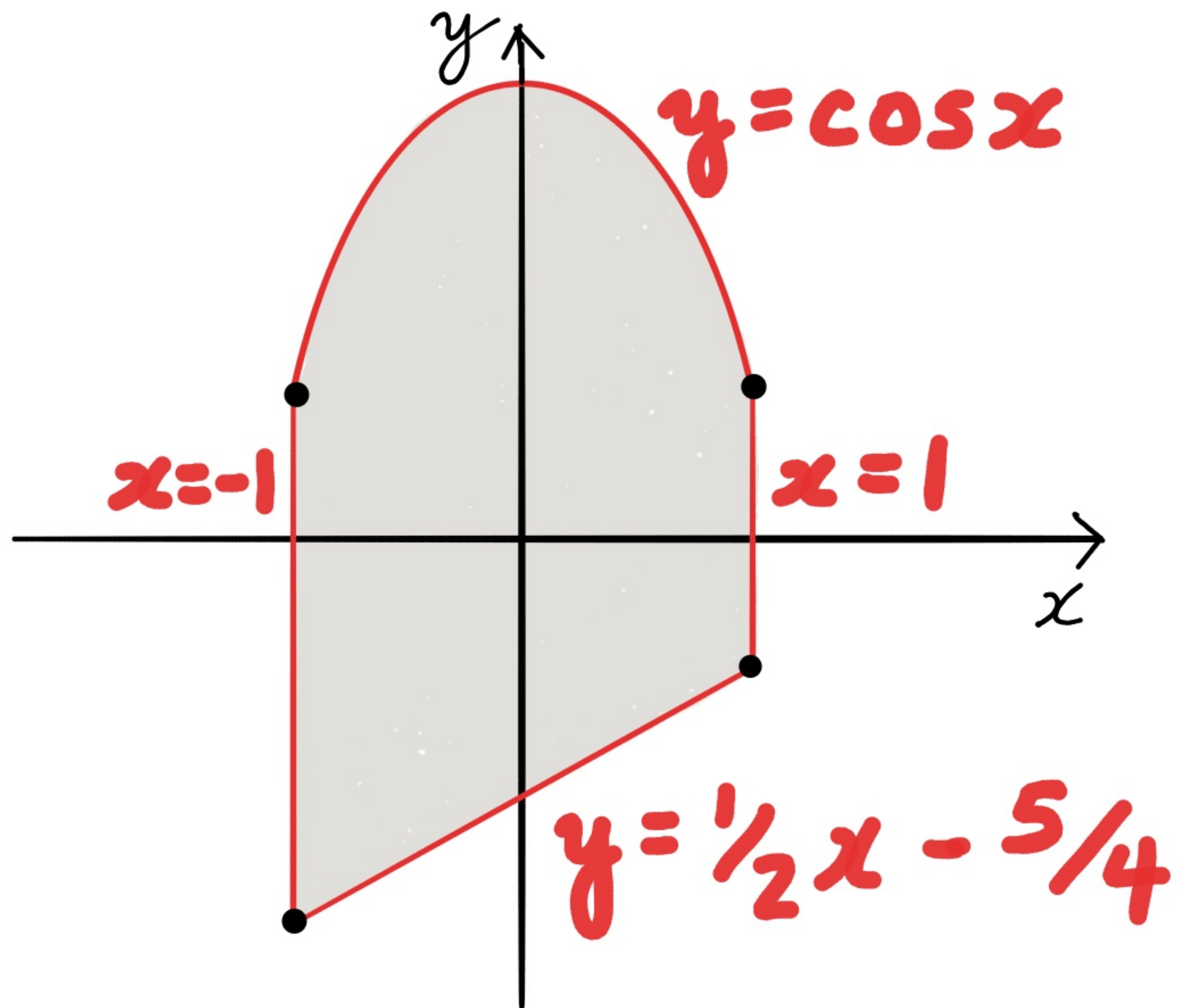
$$\int_3^{\pi} \int_4^7 \int_{-8}^6 \sin(x^2 y z + z^2) \, dx \, dy \, dz$$

=

$$\int_{-8}^6 \int_4^7 \int_3^{\pi} \sin(x^2 y z + z^2) \, dz \, dy \, dx$$

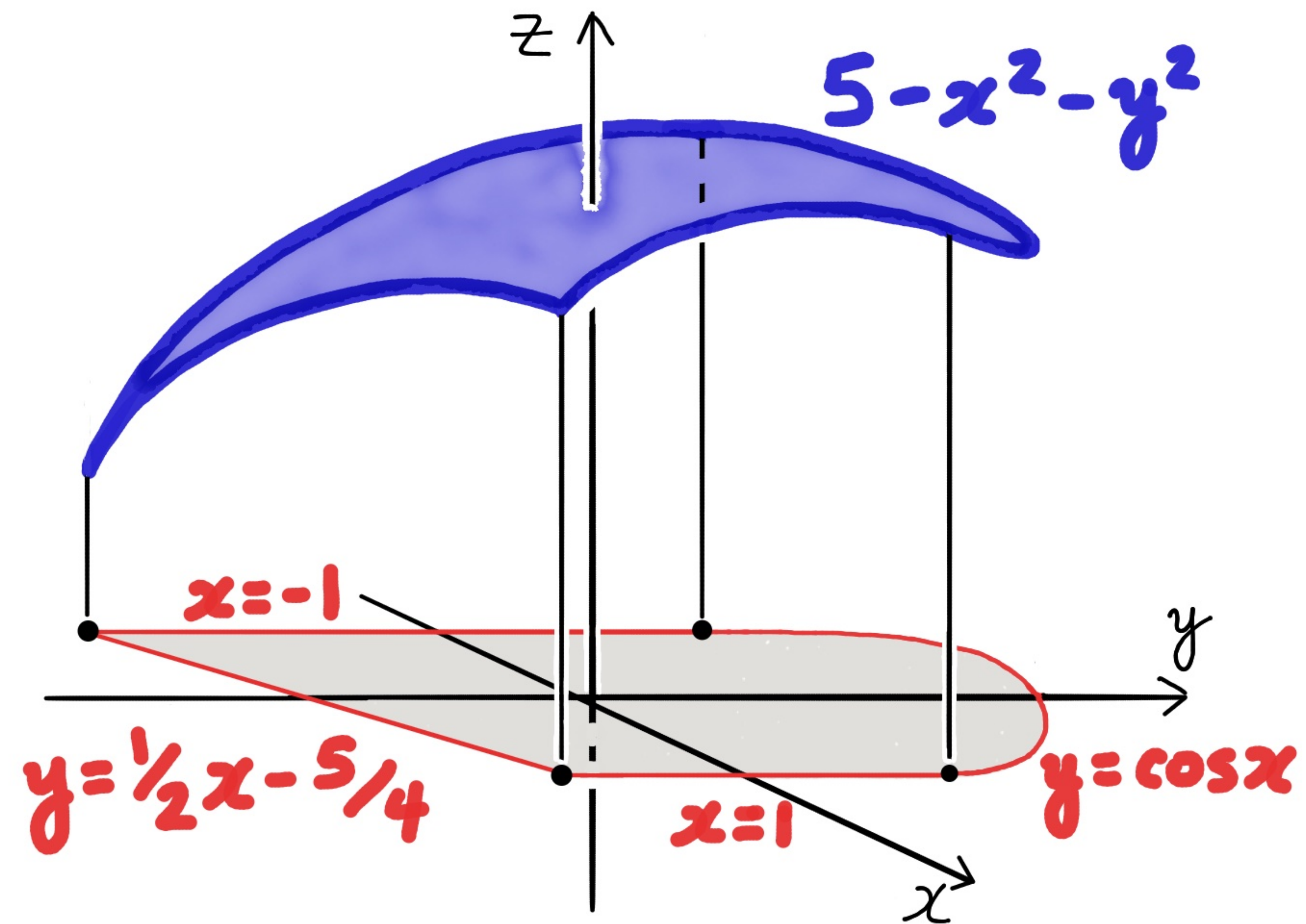
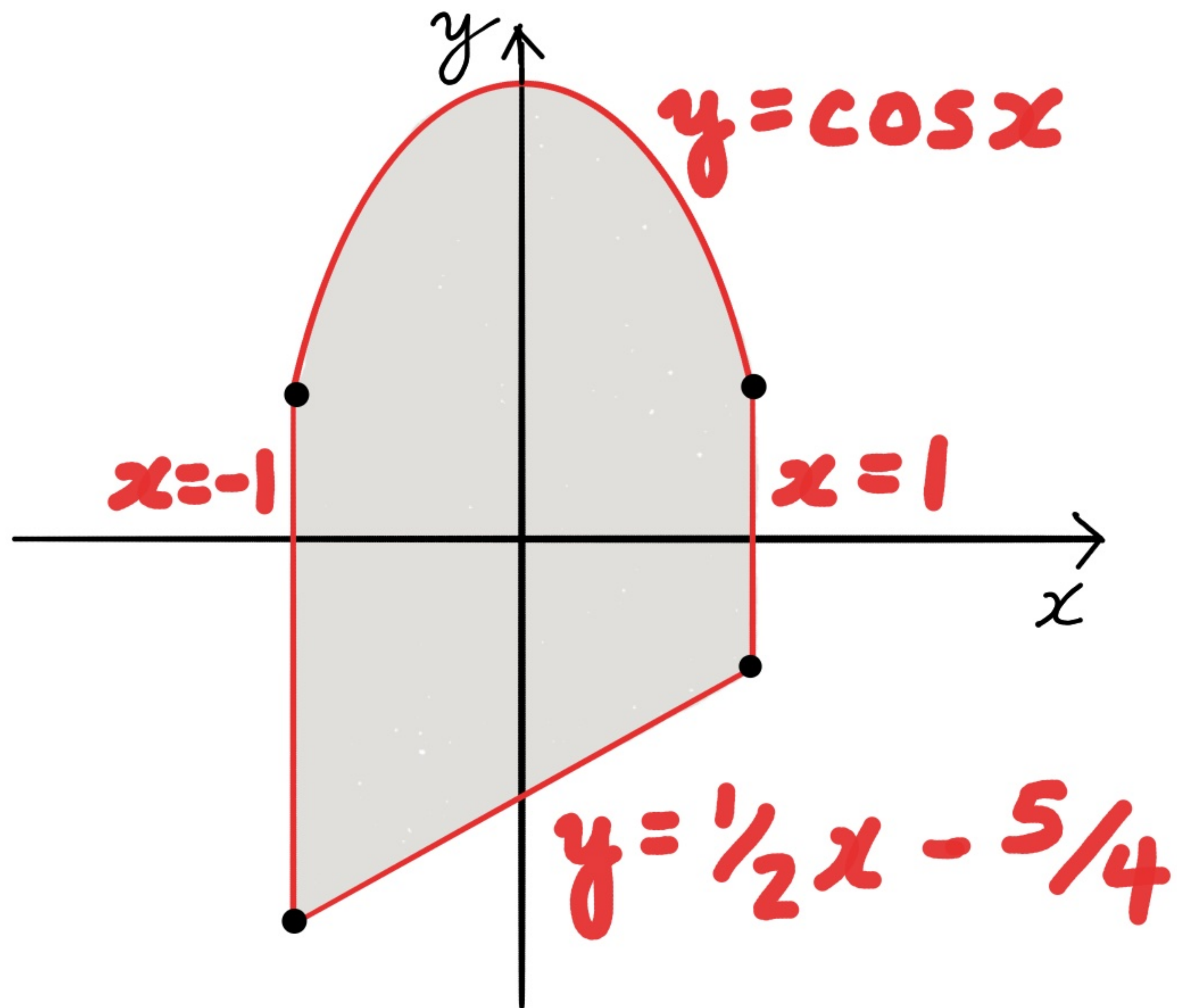


$$\int_{-1}^1 \int_{\frac{1}{2}x - \frac{5}{4}}^{\cos x} (5 - x^2 - y^2) dy dx$$



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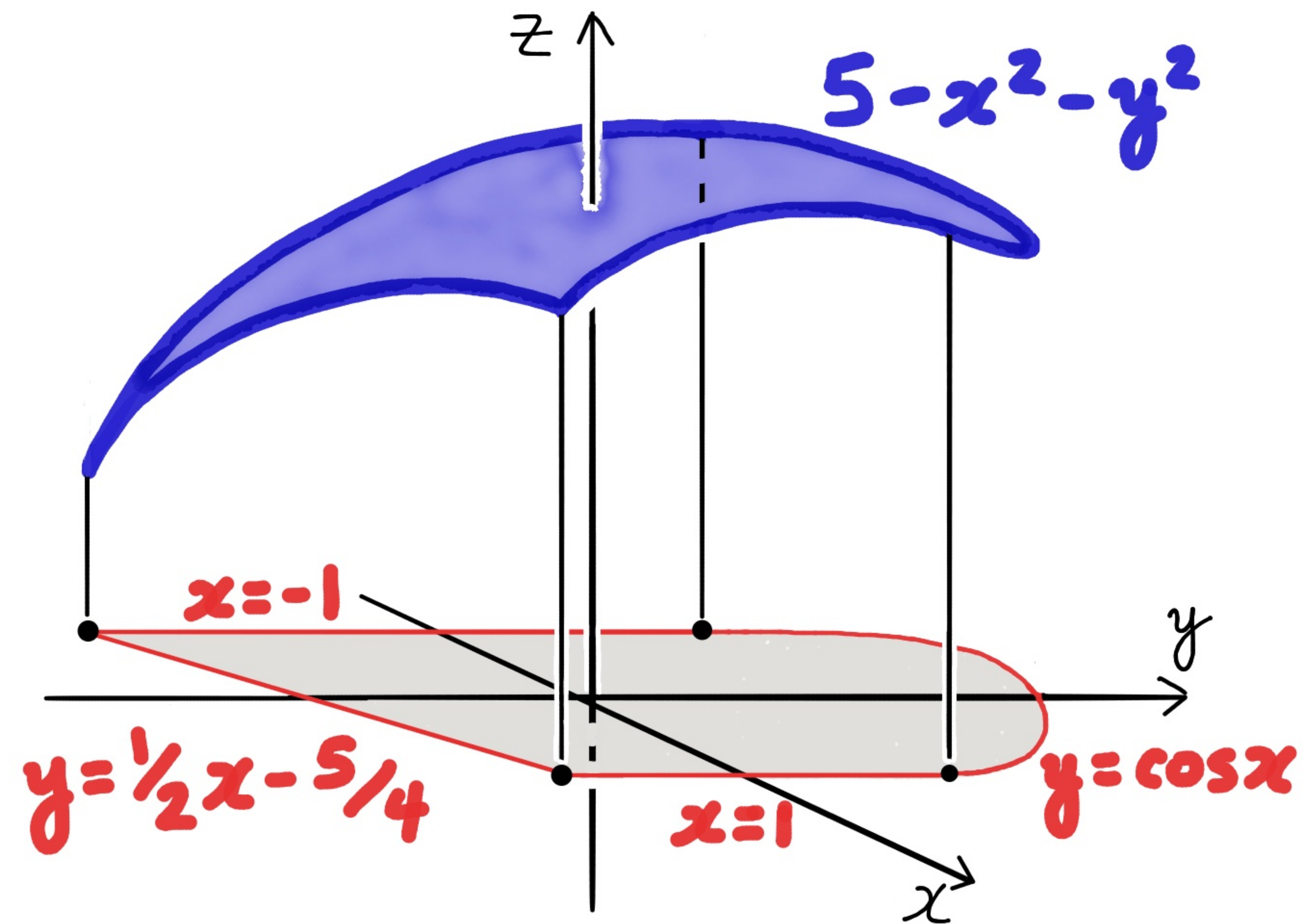
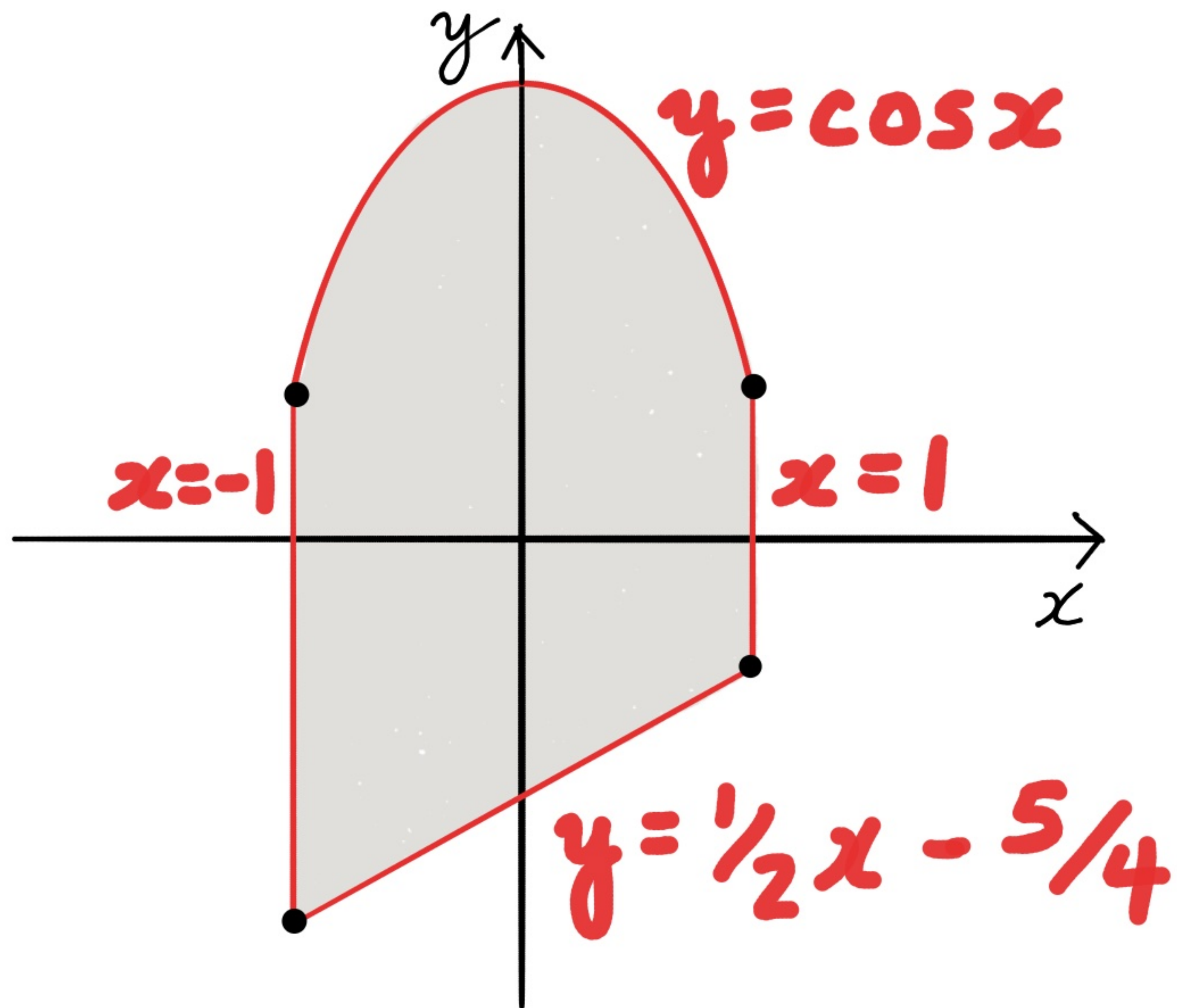
x -limits y -limits



$$\int_{-1}^1 \int_{\frac{1}{2}x - \frac{5}{4}}^{\cos x} (5 - x^2 - y^2) dy dx$$

x -limits
 constants

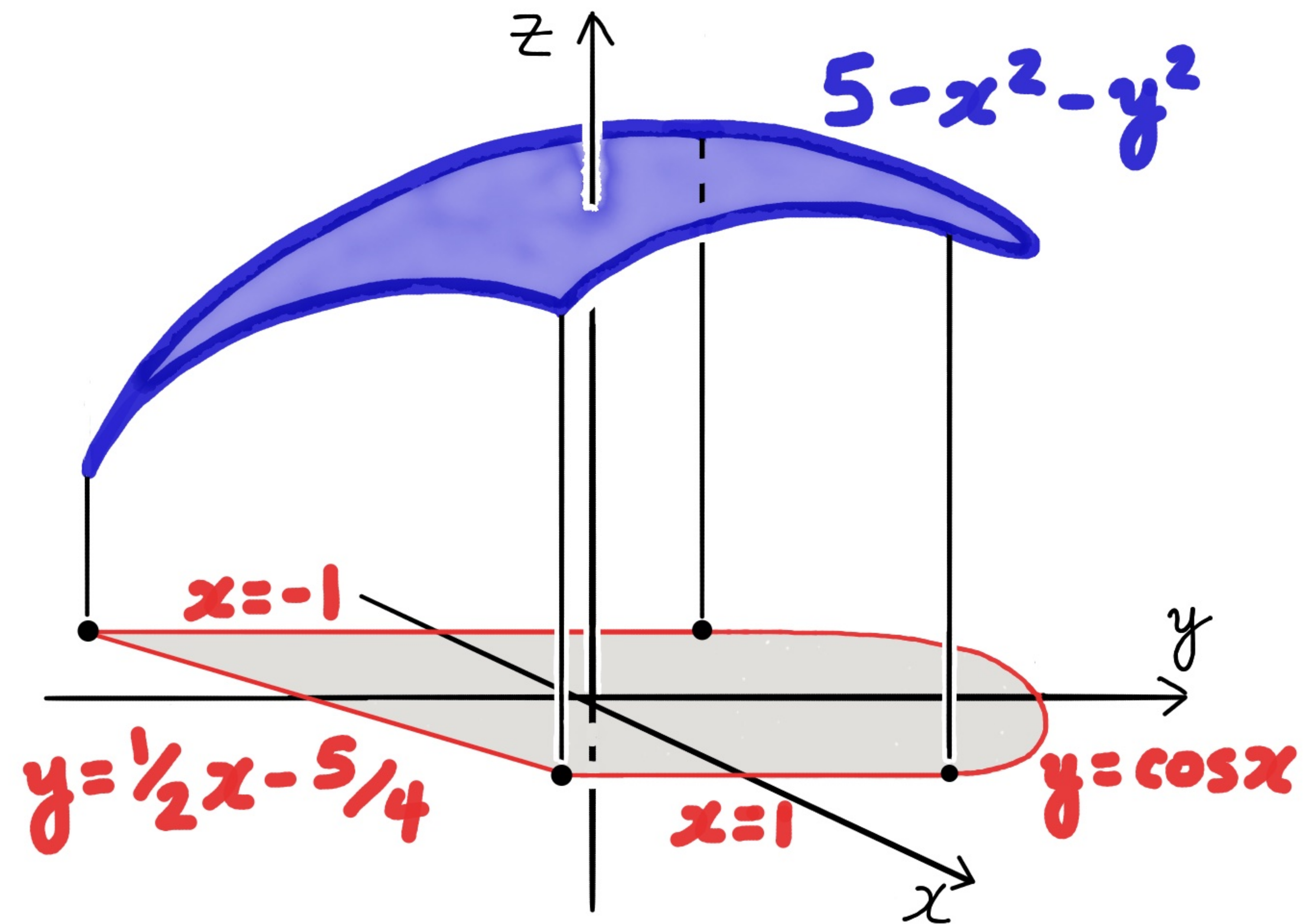
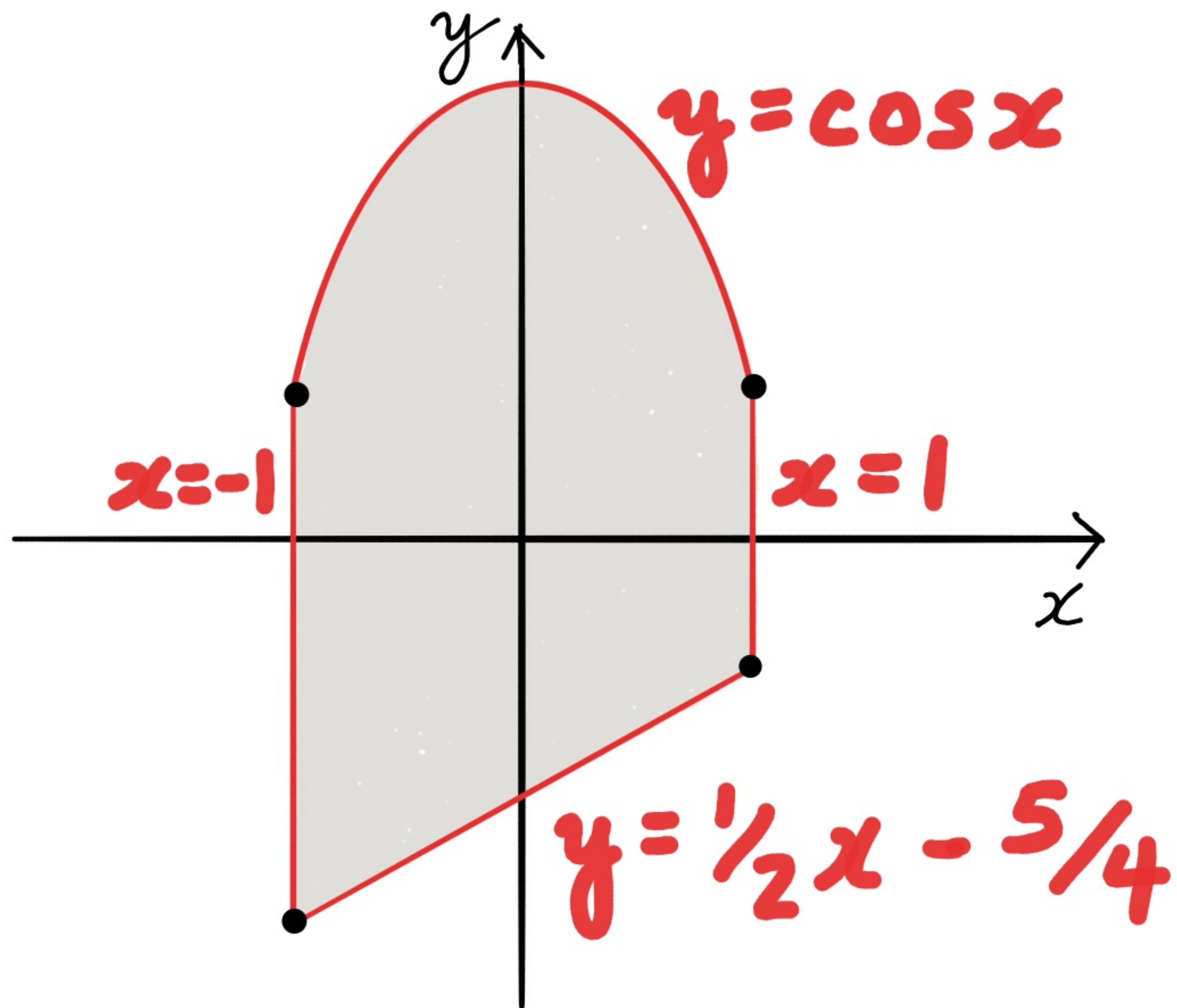
y -limits
 functions of x



$$\int_{-1}^1 \int_{\frac{1}{2}x - \frac{5}{4}}^{\cos x} (5 - x^2 - y^2) dy dx$$

x -limits
 constants
 width in x

y -limits
 functions of x
 varying width in y



$$\int_{-1}^1 \int_{\frac{1}{2}x - \frac{5}{4}}^{\cos x} (5 - x^2 - y^2) dy dx$$

x -limits
 constants
 width in x

y -limits
 functions of x
 varying width in y

varying height

$$\int_2^7 \int_{-x}^{x^2} \int_{-\ln(x+y+1)}^{(x+y)^2} x^2 y z^3 dz dy dx$$

$$\int_2^7 \int_{-x}^{x^2} \int_{-\ln(x+y+1)}^{(x+y)^2}$$

$$x^2 y z^3 dz dy dx$$

↑
varying
height

x-limits

y-limits

z-limits

constants

functions
of x

functions
of x, y

width
in x



varying
width
in y



varying
width
in z

$$\int_2^7 \int_{-x}^{x^2} \int_{-\ln(x+y+1)}^{(x+y)^2} x^2 y z^3 dz dy dx$$

$$\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) dz dy dx$$

$$\int_{y_1}^{y_2} \int_{z_1(y)}^{z_2(y)} \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z) dx dz dy$$

$$\int_{y_1}^{y_2} \int_{z_1(y)}^{z_2(y)} \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z) dx dz dy$$

$$= \iiint_R f(x,y,z) dV$$

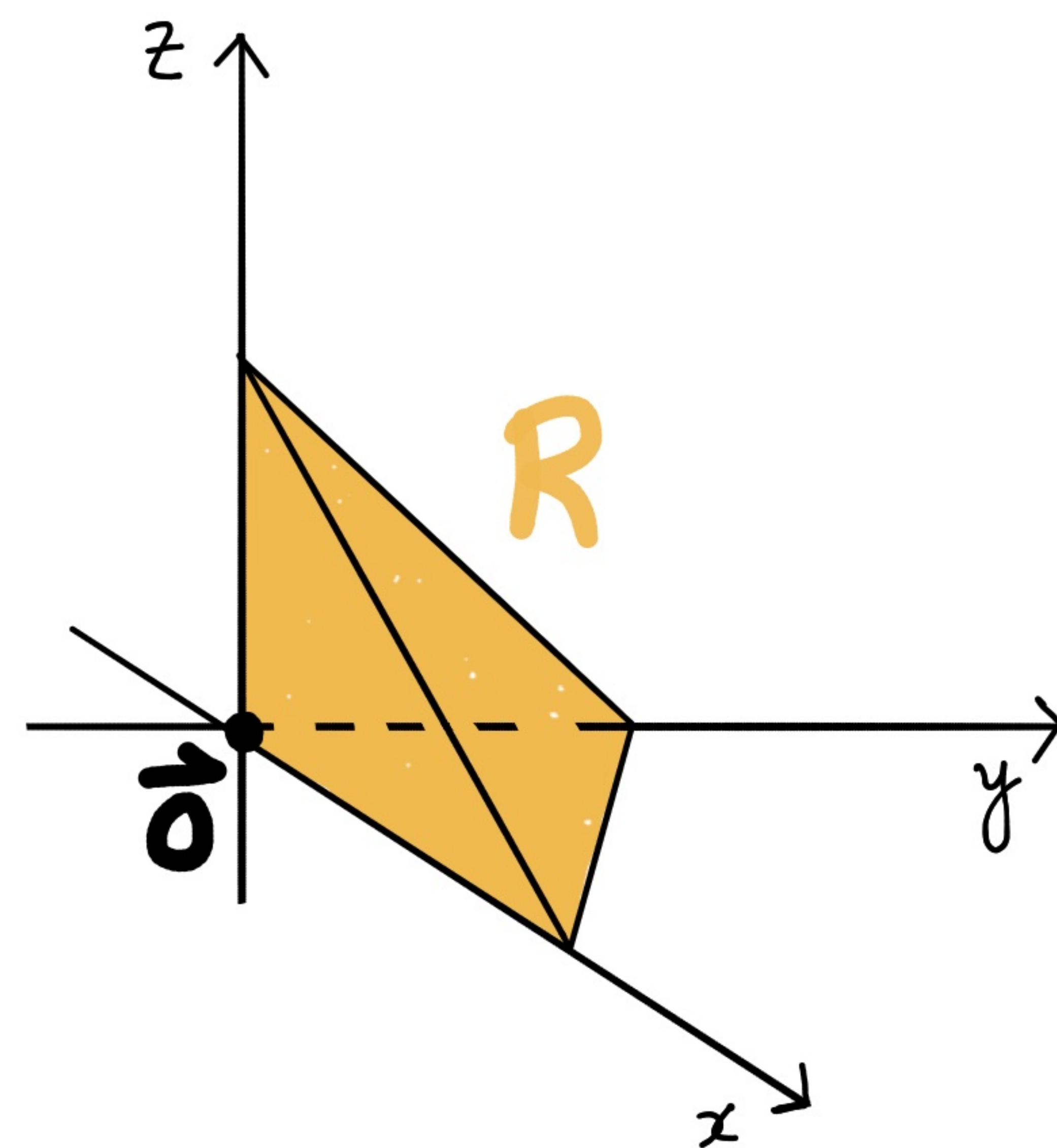
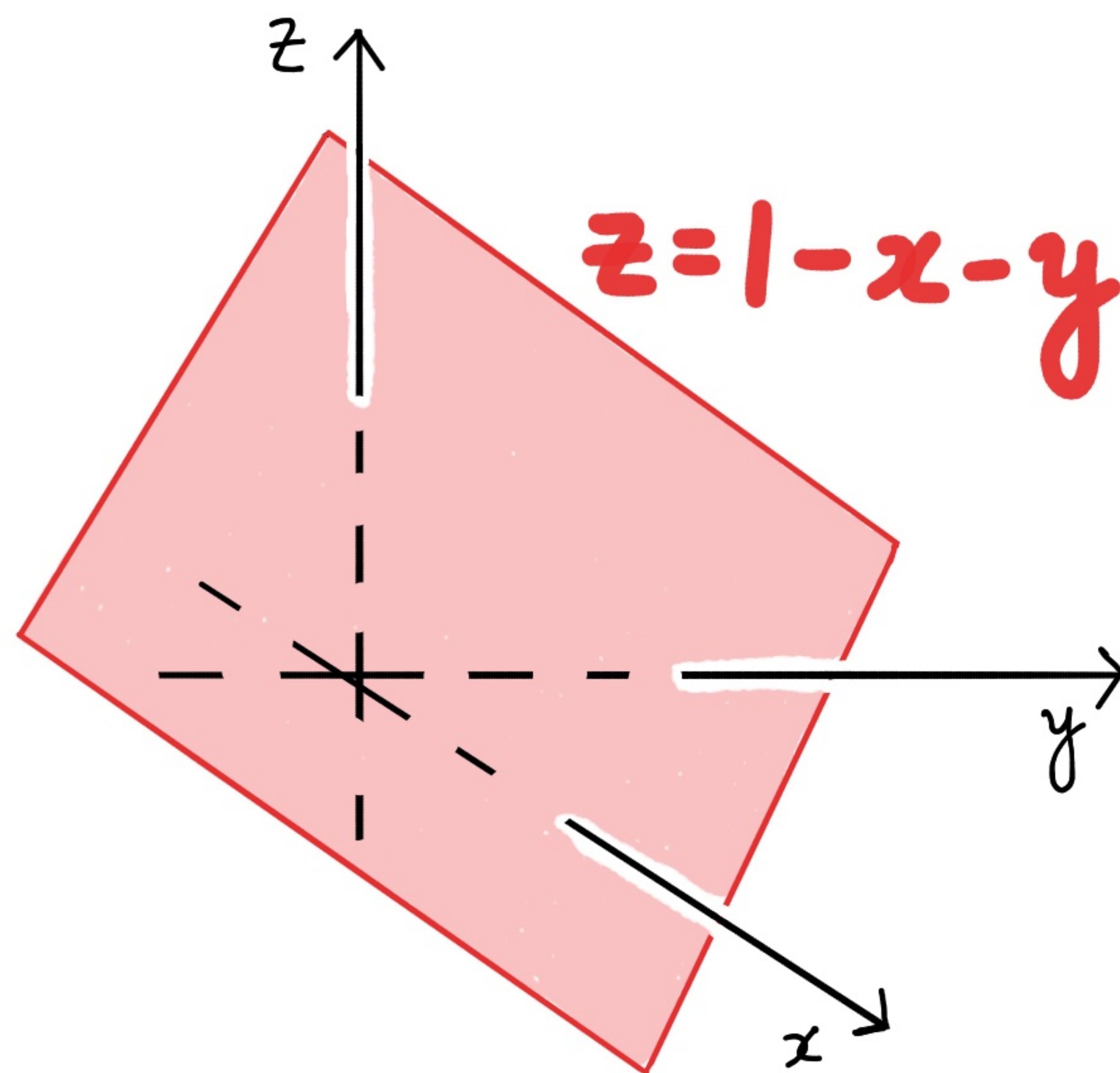
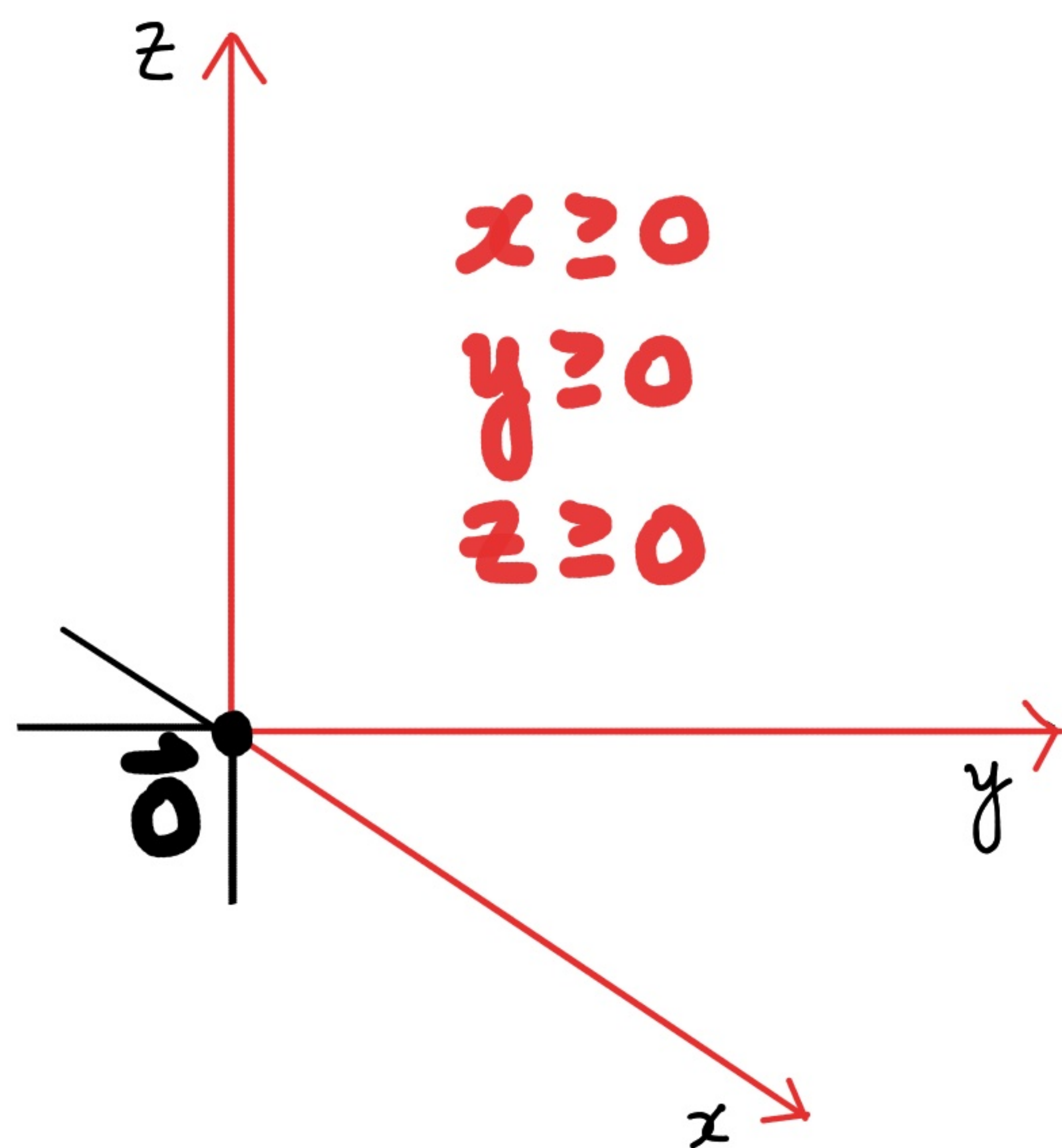
where R is region in $\mathbb{R}^3$ of points (x,y,z) with $y_1 \leq y \leq y_2$, $z_1(y) \leq z \leq z_2(y)$, and $x_1(y,z) \leq x \leq x_2(y,z)$.

Area of R in $\mathbb{R}^2$: $\iint_R dA$

Volume of R in $\mathbb{R}^3$: $\iiint_R dV$

Example:

- ① Find volume of region R in the first octant of $\mathbb{R}^3$ that contains $\vec{O}$ and is bounded by the plane $z=1-x-y$.



② Find $\iiint_R x \, dV$ where R is the region in $\mathbb{R}^3$ containing $\vec{0}$ and bounded by the following 6 surfaces:

