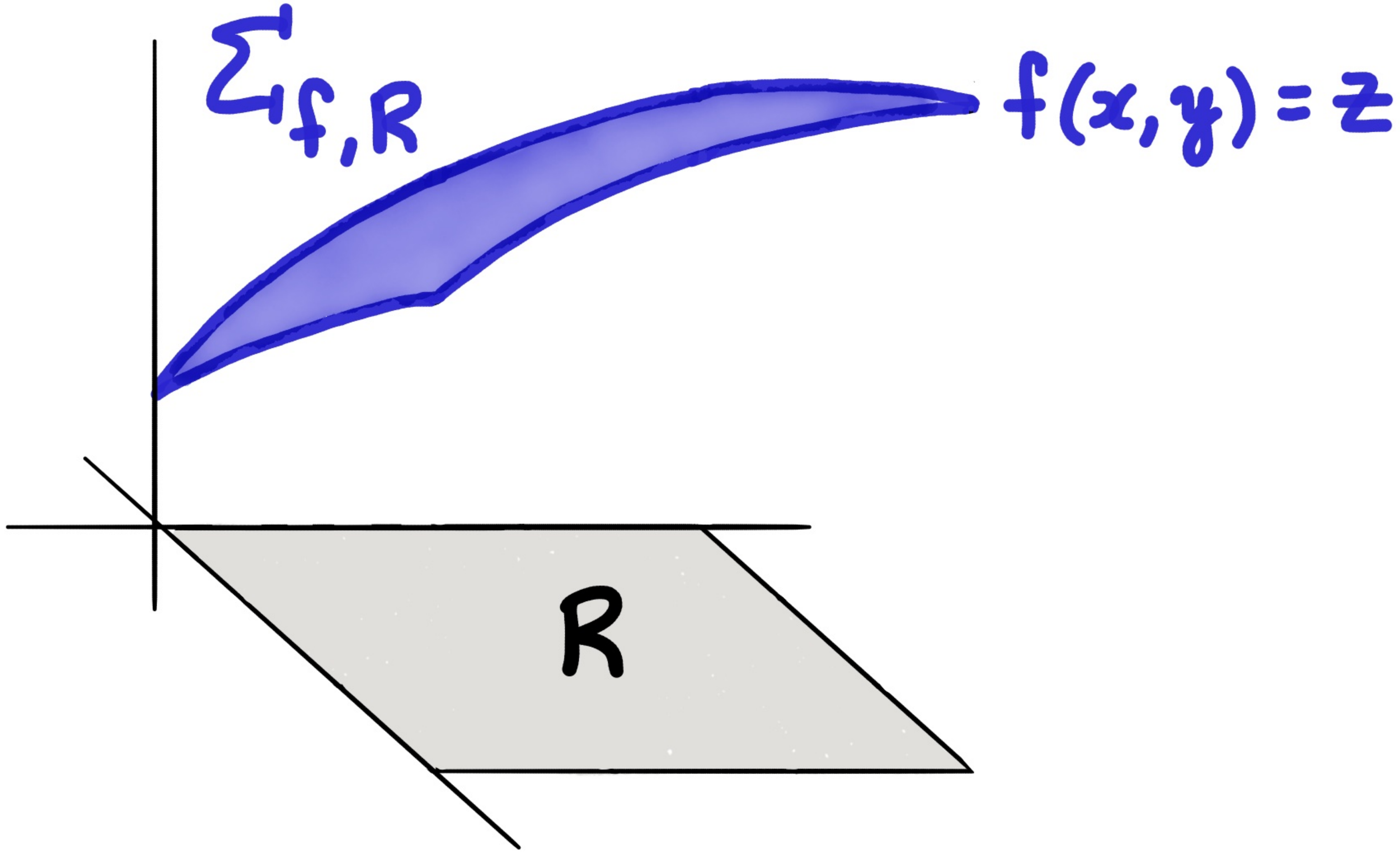
Twenty-two

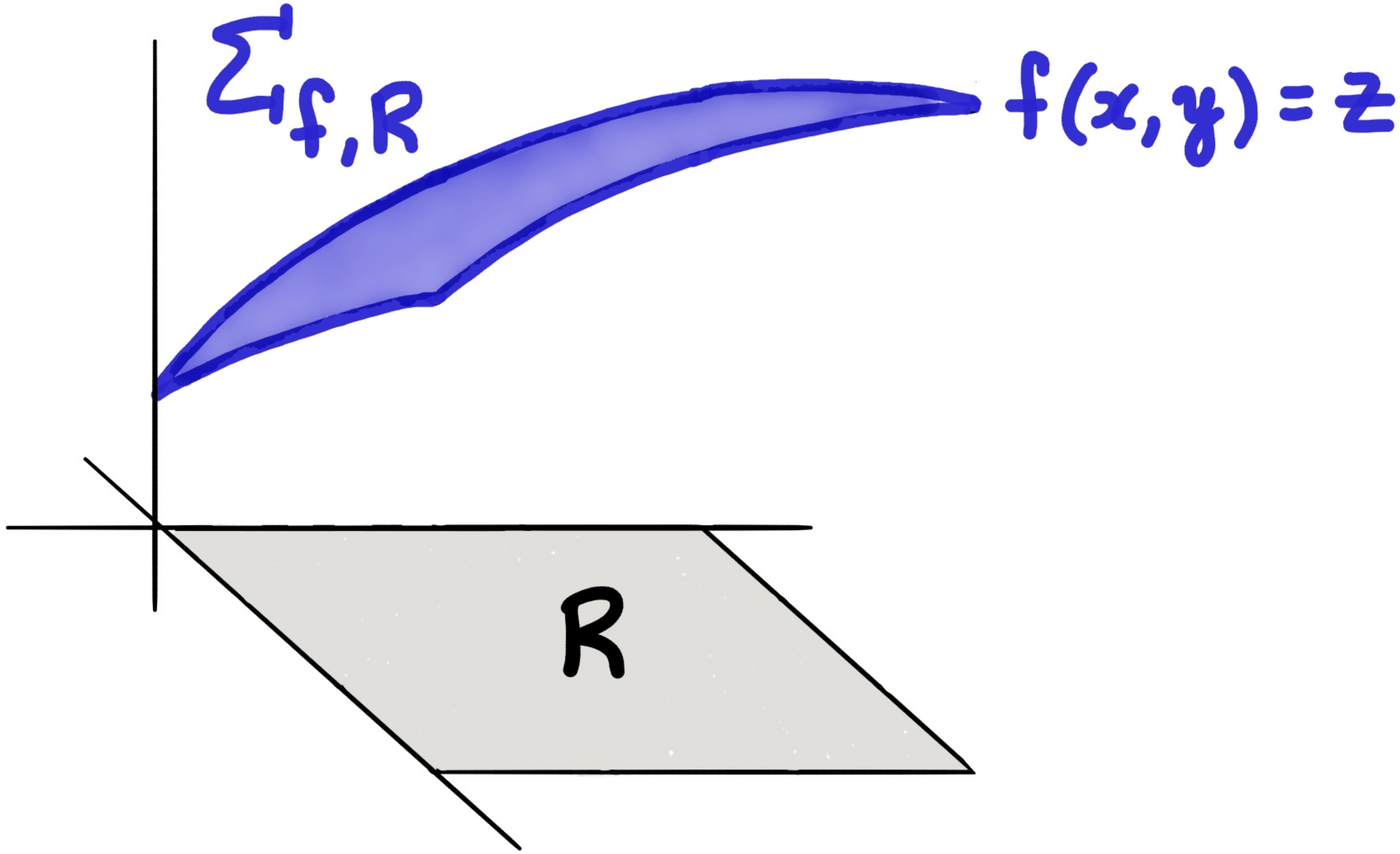
Surface Area



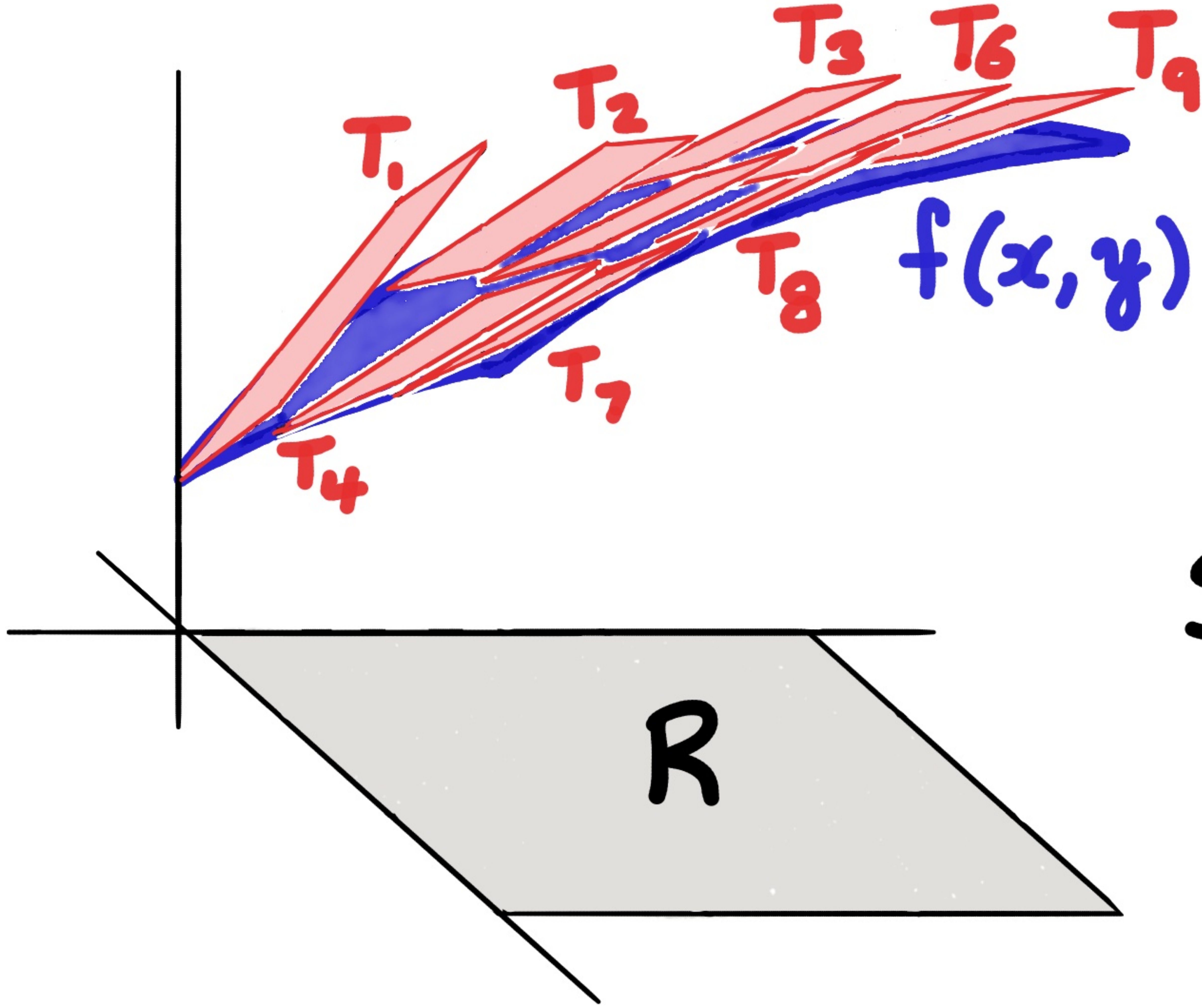
For a region R in $\mathbb{R}^2$,

and $f:\mathbb{R}^2 \to \mathbb{R}$,

let $\Sigma_{f,R}$ be the portion of the graph of $f(x,y)=z$ lying above (or below) R.



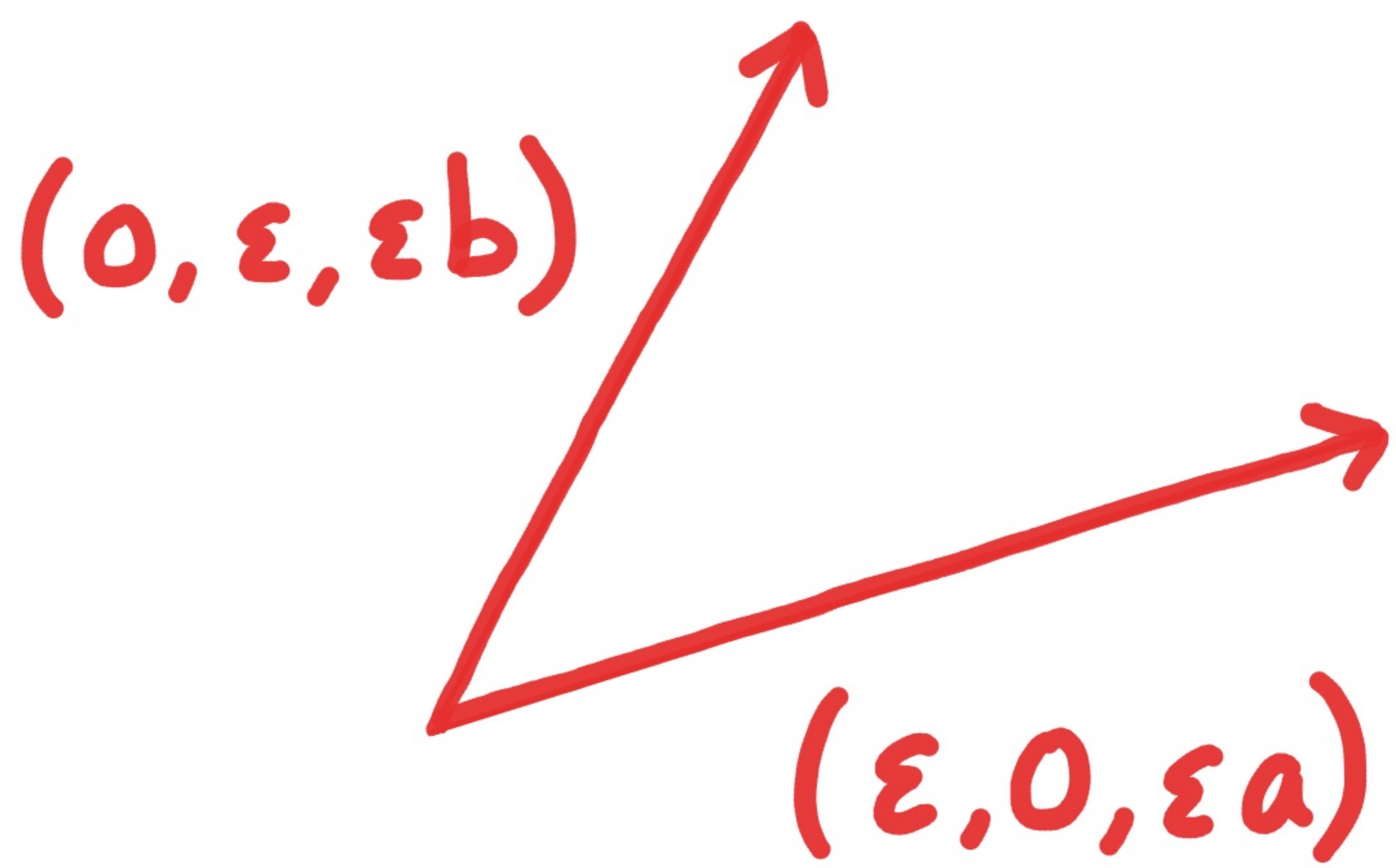
$SA(\Sigma_{f,R})$: What's the surface area of $f(x,y)=z$ above R?



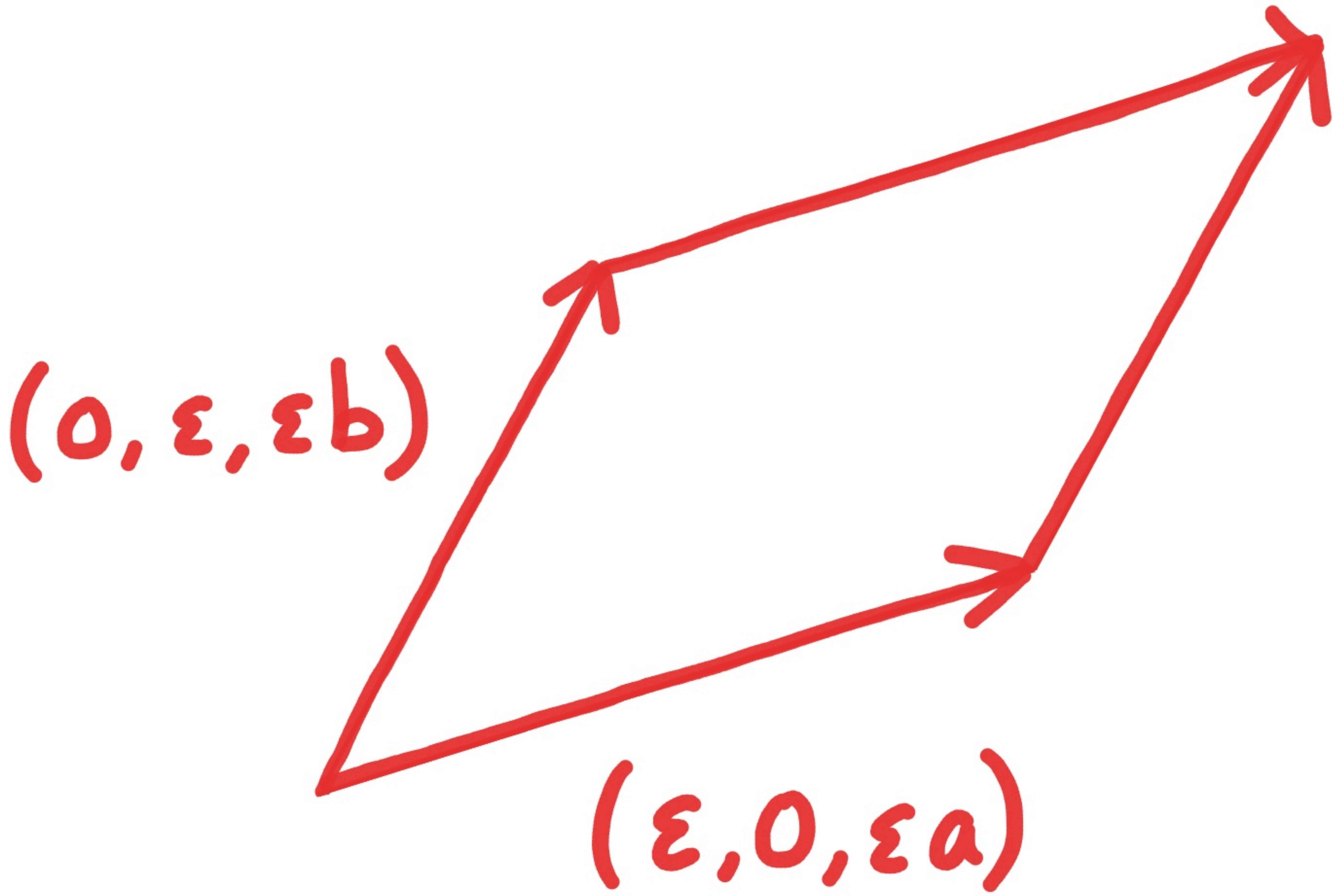
$$SA(\Sigma_{f,R}) \approx \sum_i \text{Area}(T_i)$$

$SA(\Sigma_{f,R})$: What's the surface area of $f(x,y)=z$ above R?

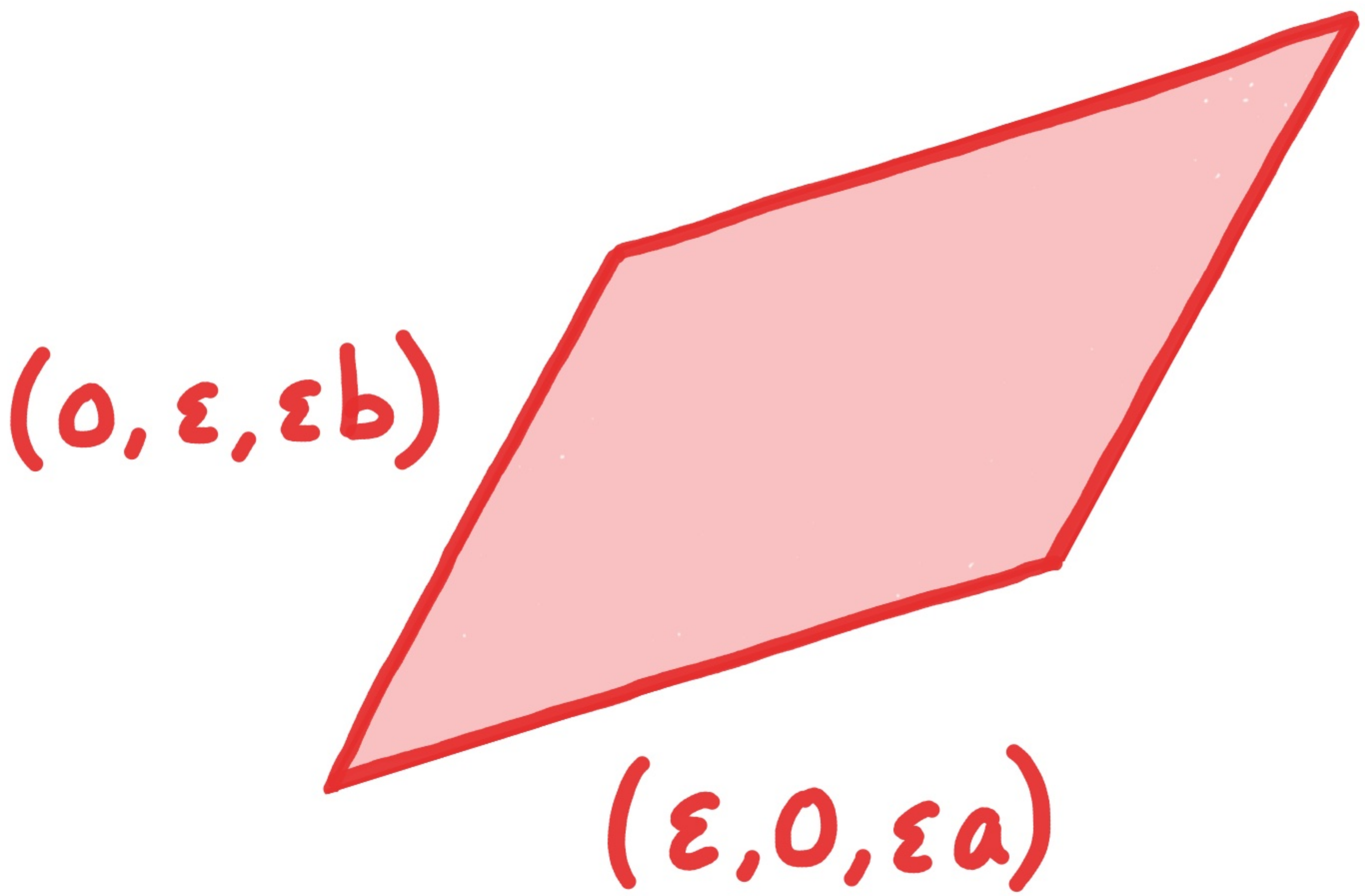
$\mathbb{R}^3$



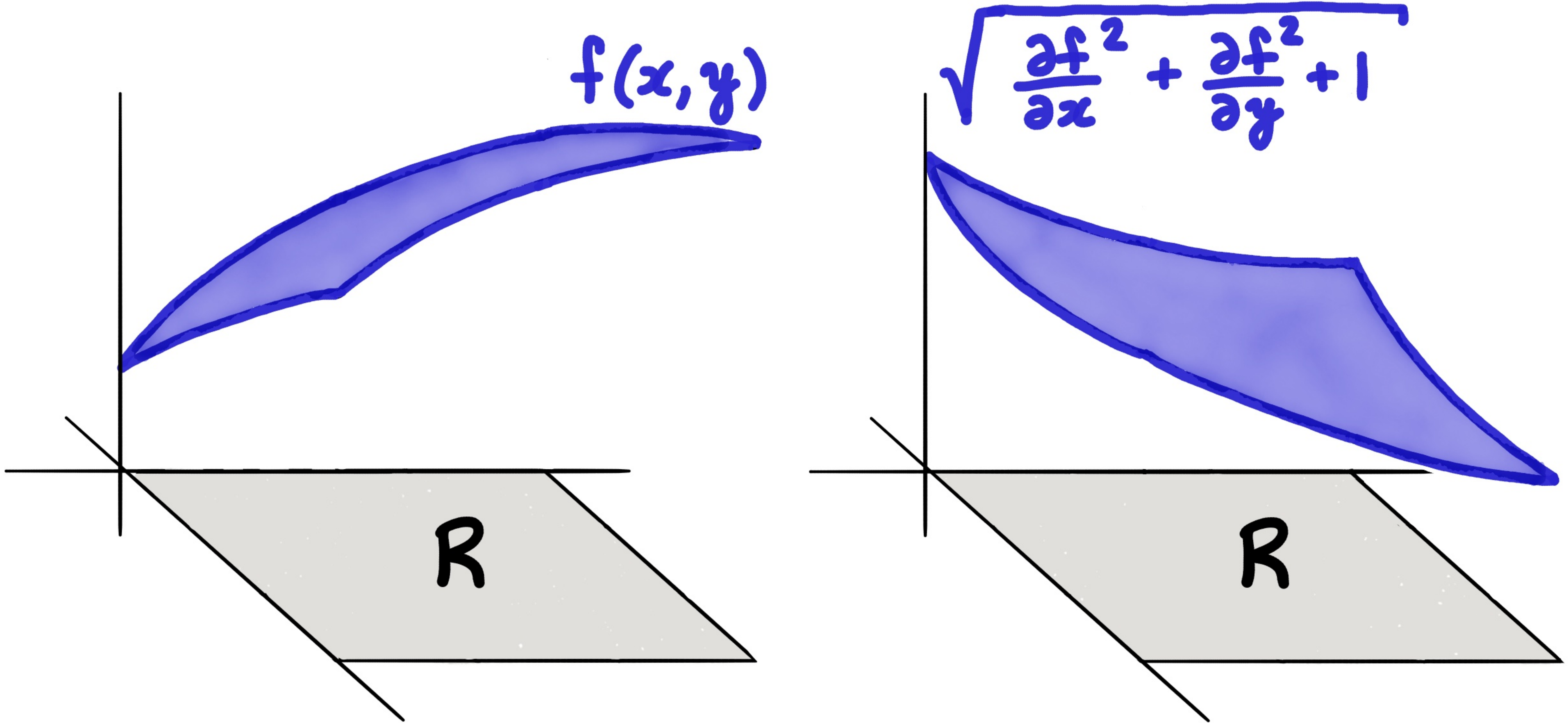
$\mathbb{R}^3$



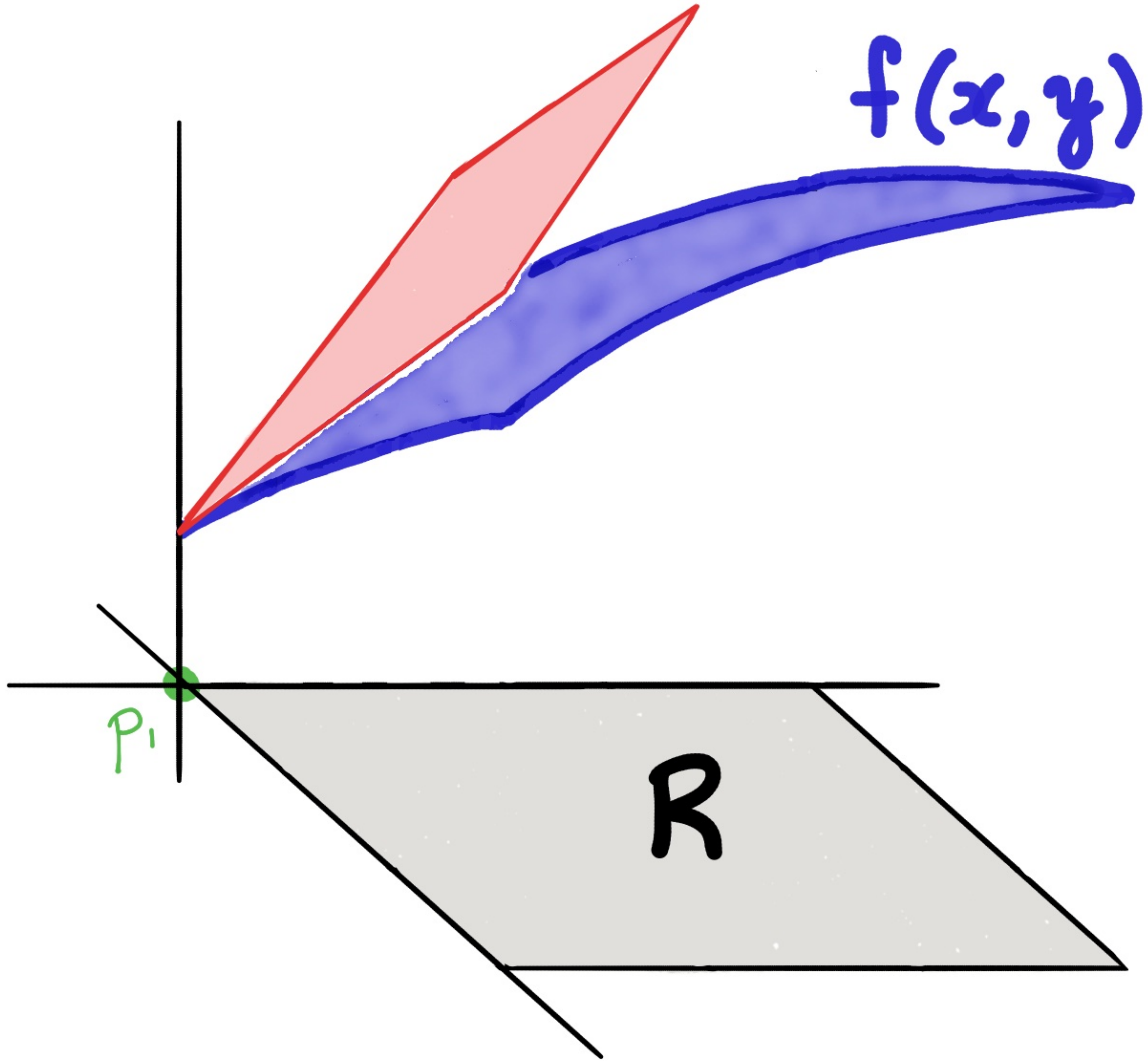
$\mathbb{R}^3$



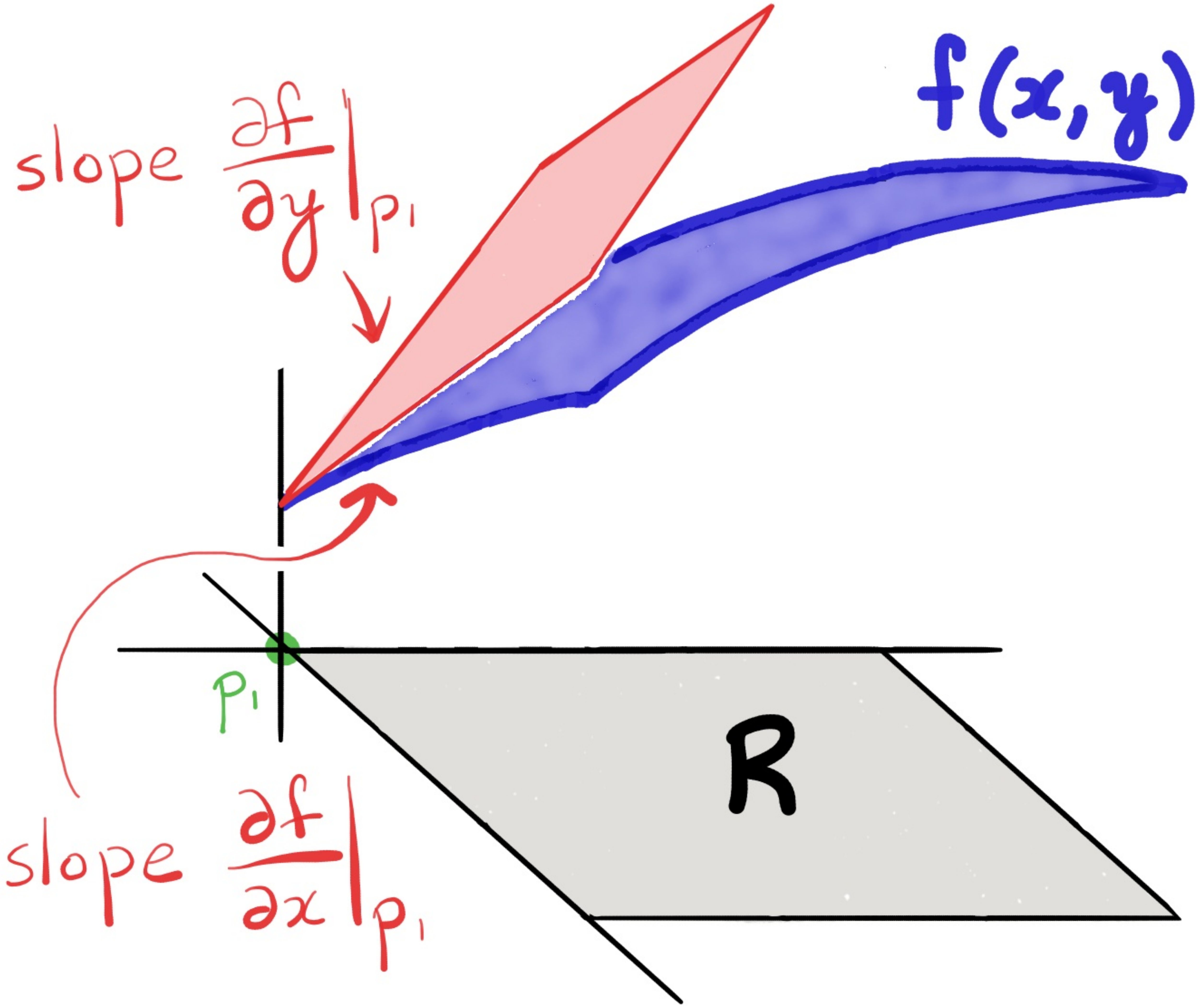
<u>Area</u>: $\varepsilon^2 \sqrt{a^2 + b^2 + 1}$



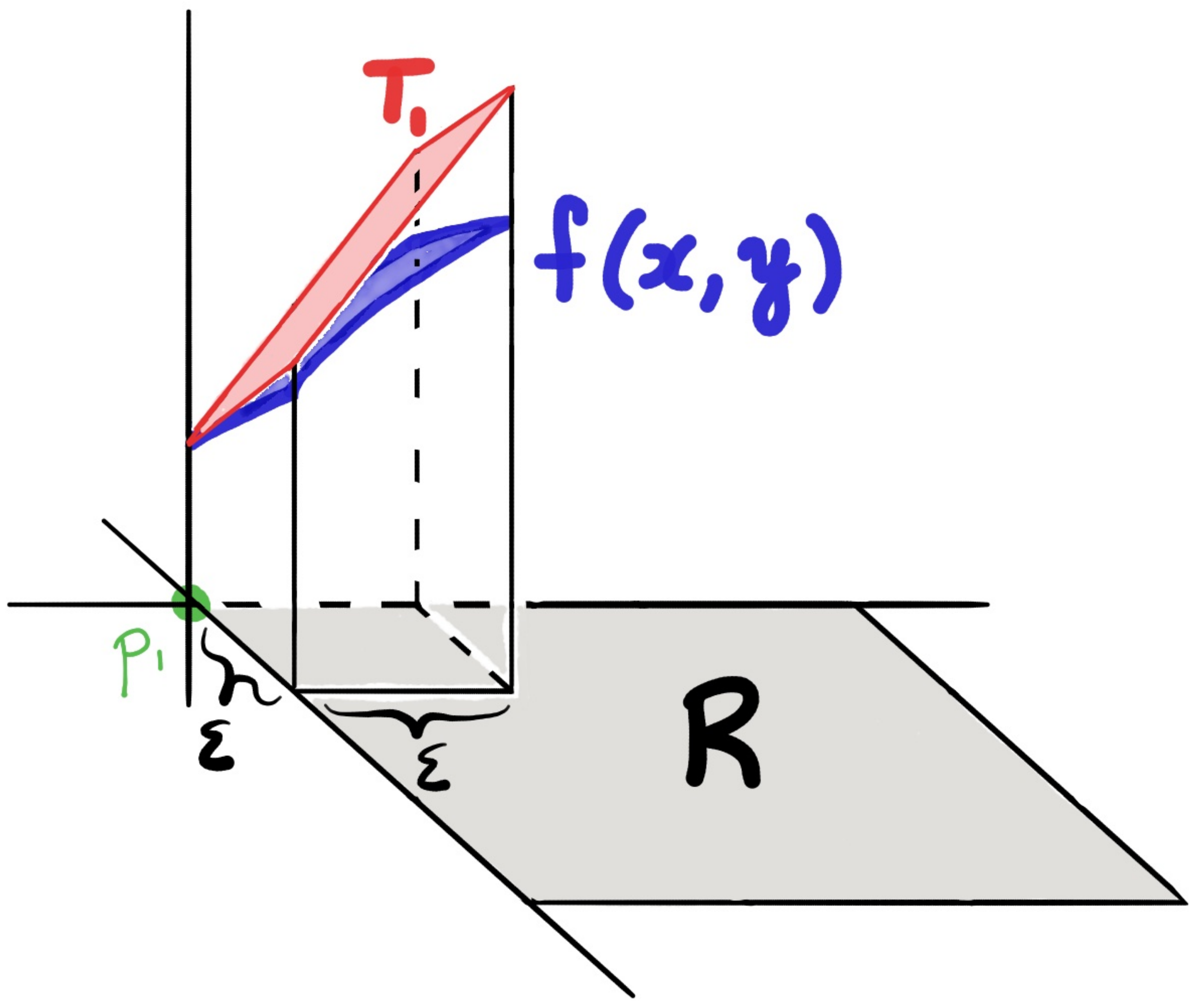
$SA(\Sigma_{f,R})$: What's the surface area of $f(x,y)=z$ above R?



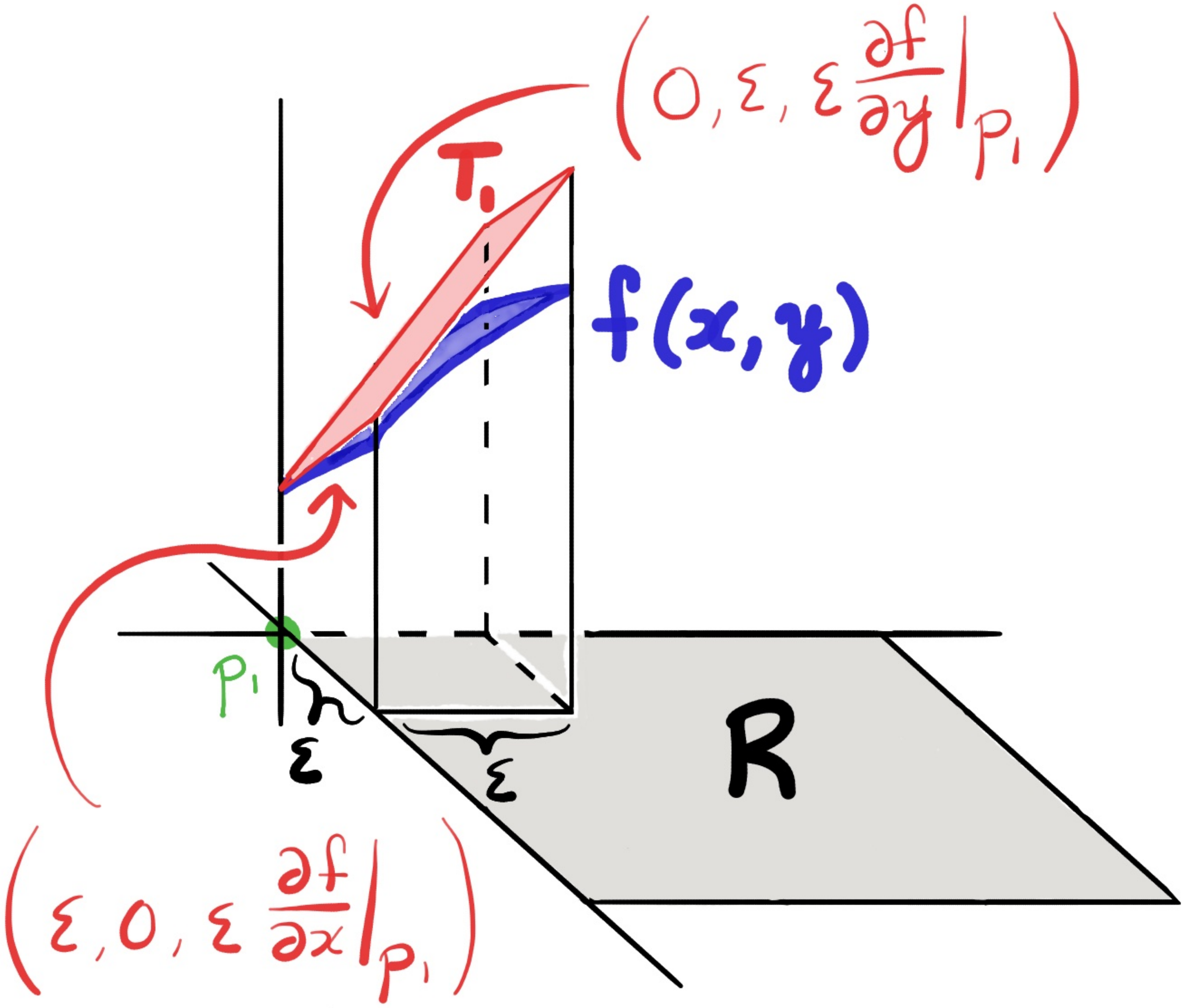
f(x,y)
R
P1



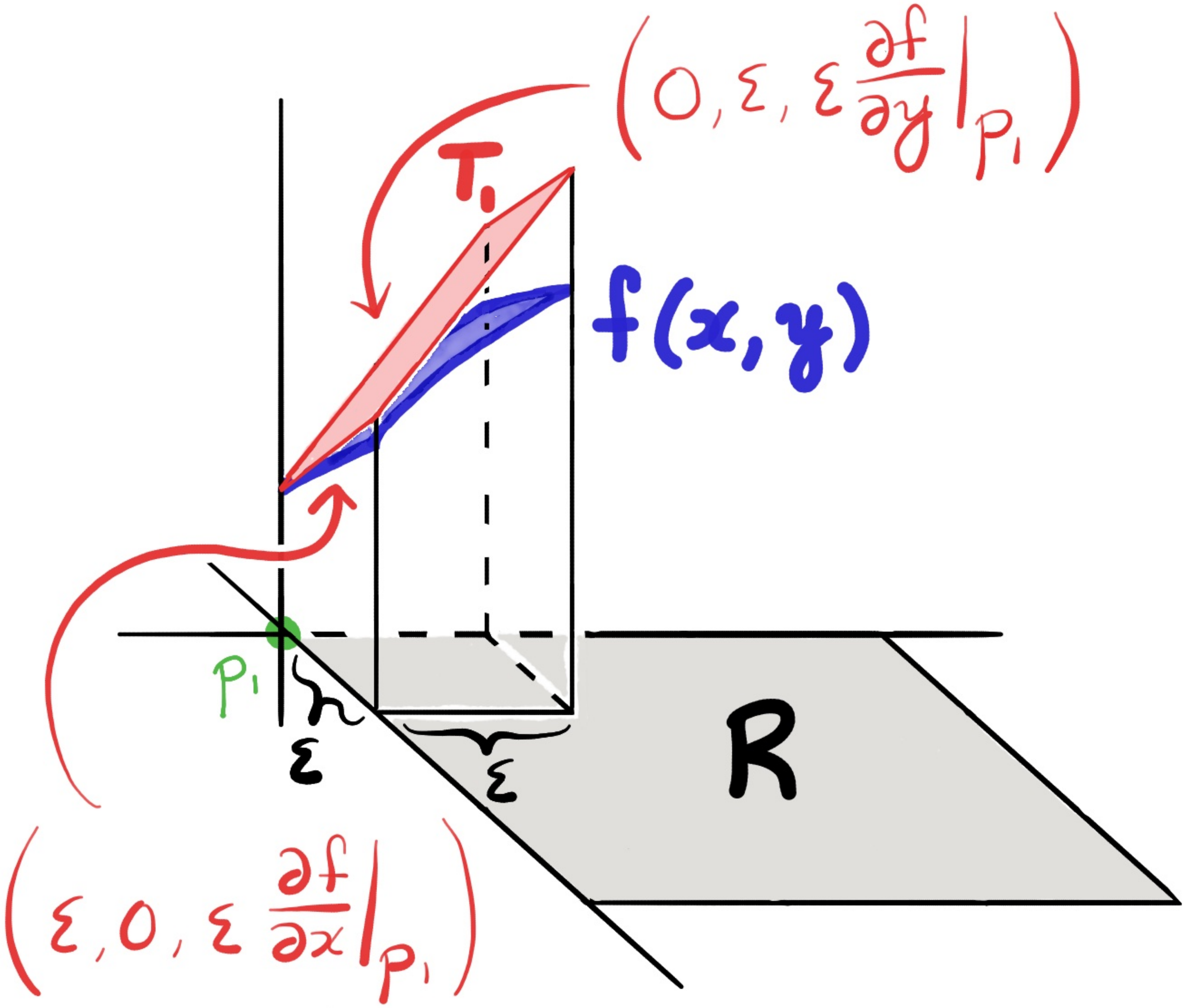
f(x, y)
slope $\frac{\partial f}{\partial y}\Big|_{p_1}$
p_1
R
slope $\frac{\partial f}{\partial x}\Big|_{p_1}$



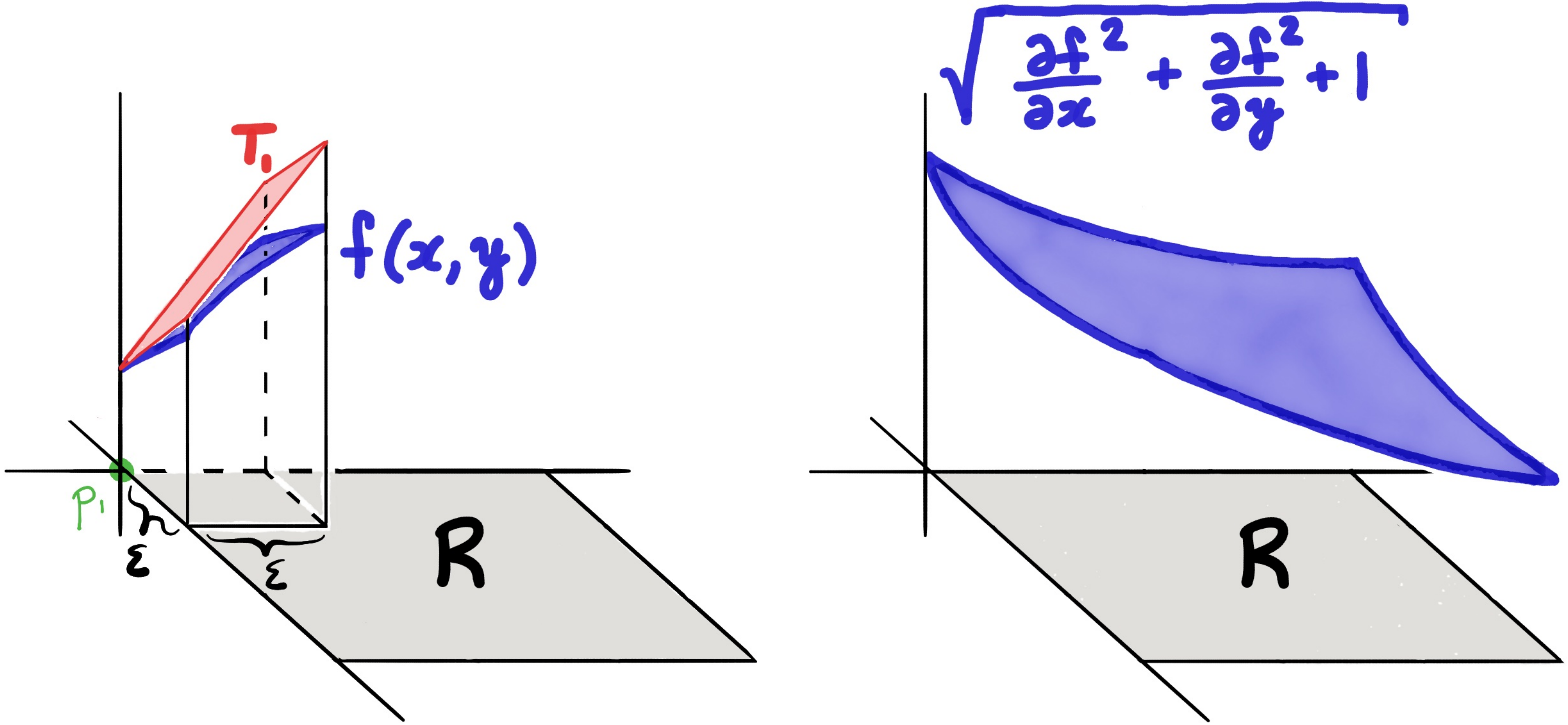
T_1
$f(x,y)$
P_1
ε
ε
R



$\left(0, \varepsilon, \varepsilon \frac{\partial f}{\partial y}\Big|_{p_1}\right)$
T_1
$f(x,y)$
p_1
ε
ε
R
$\left(\varepsilon, 0, \varepsilon \frac{\partial f}{\partial x}\Big|_{p_1}\right)$



$$\text{Area}(T_1) = \varepsilon^2 \sqrt{\left.\frac{\partial f}{\partial x}\right|_{p_1}^2 + \left.\frac{\partial f}{\partial y}\right|_{p_1}^2 + 1}$$



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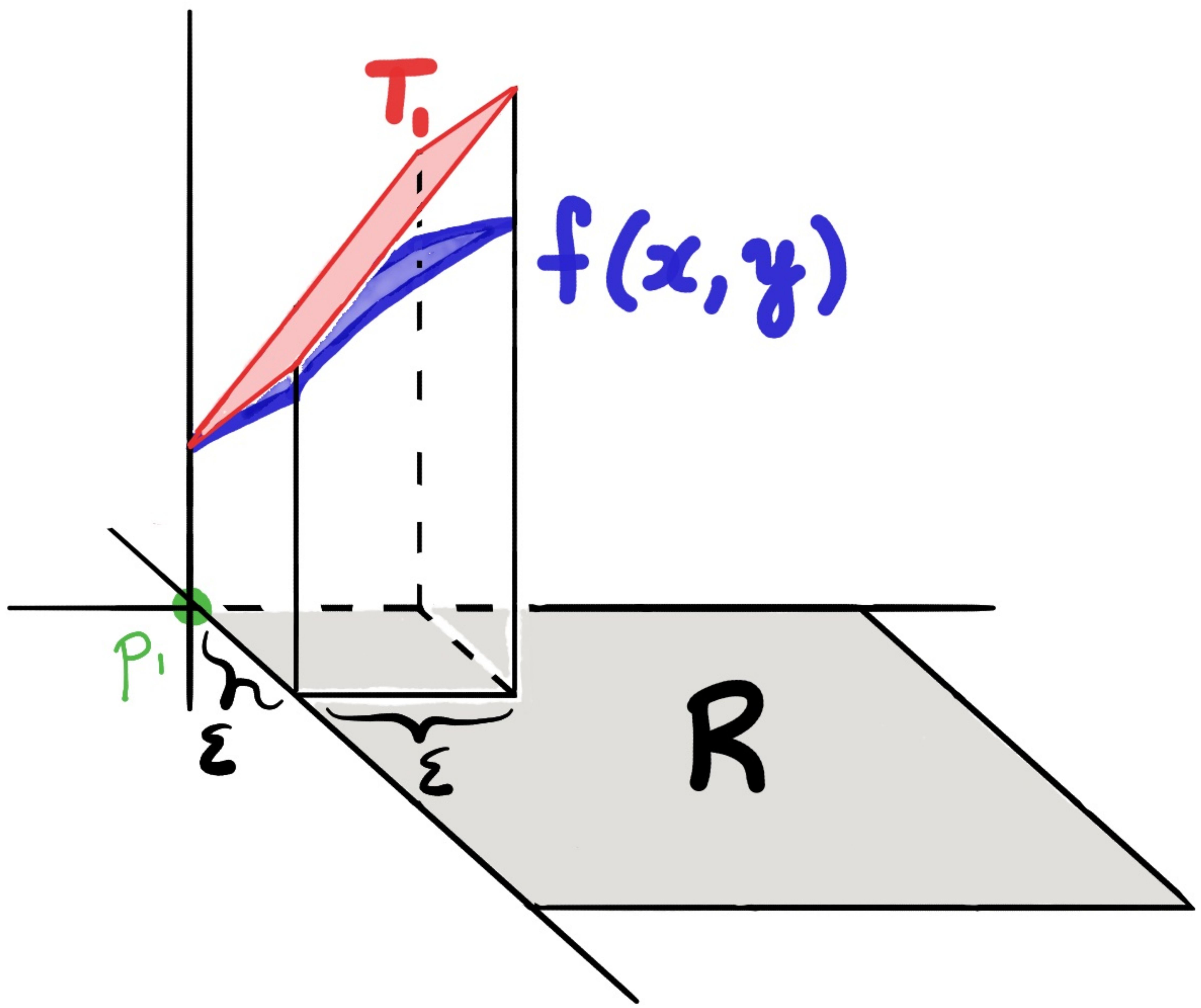

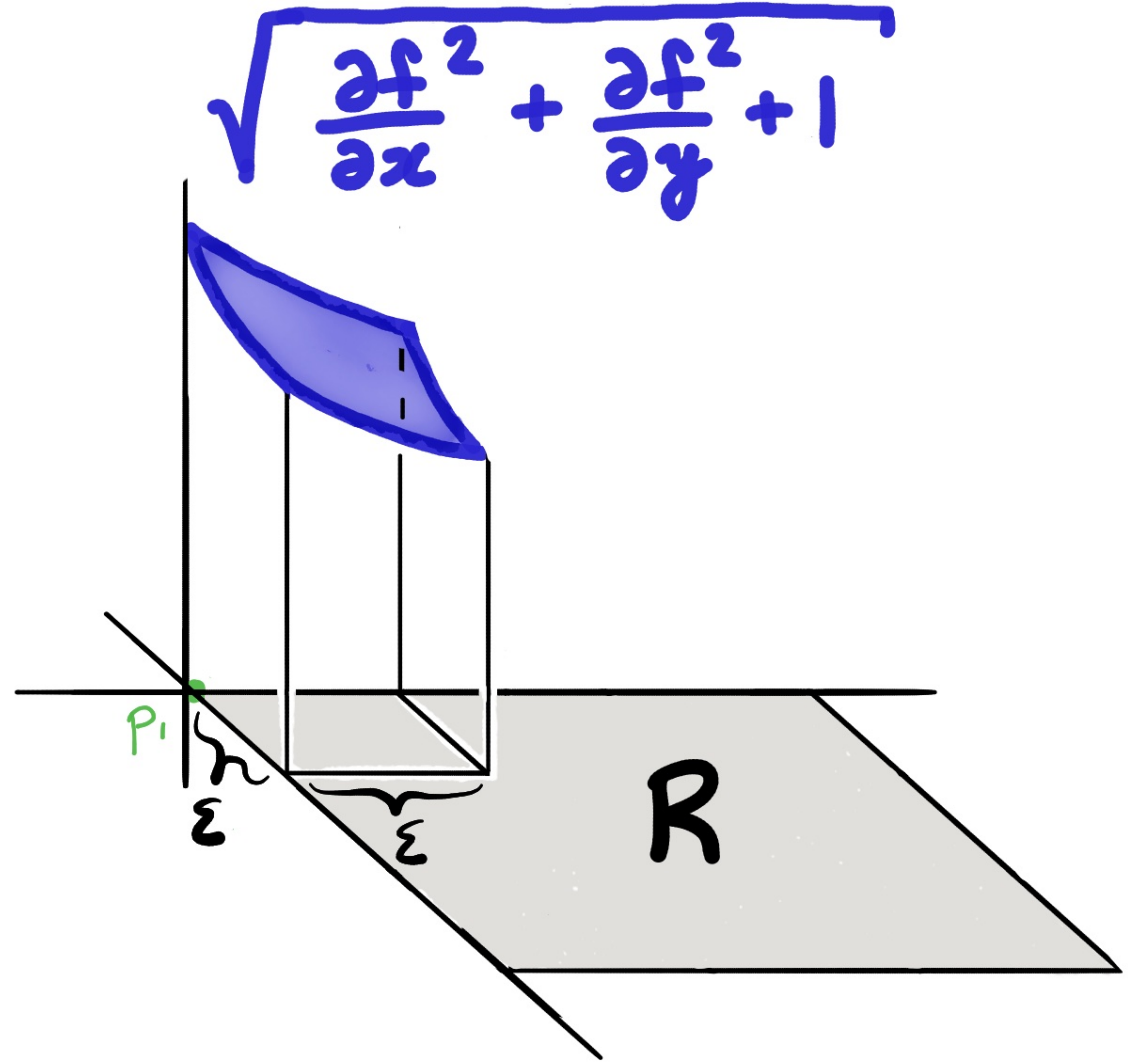

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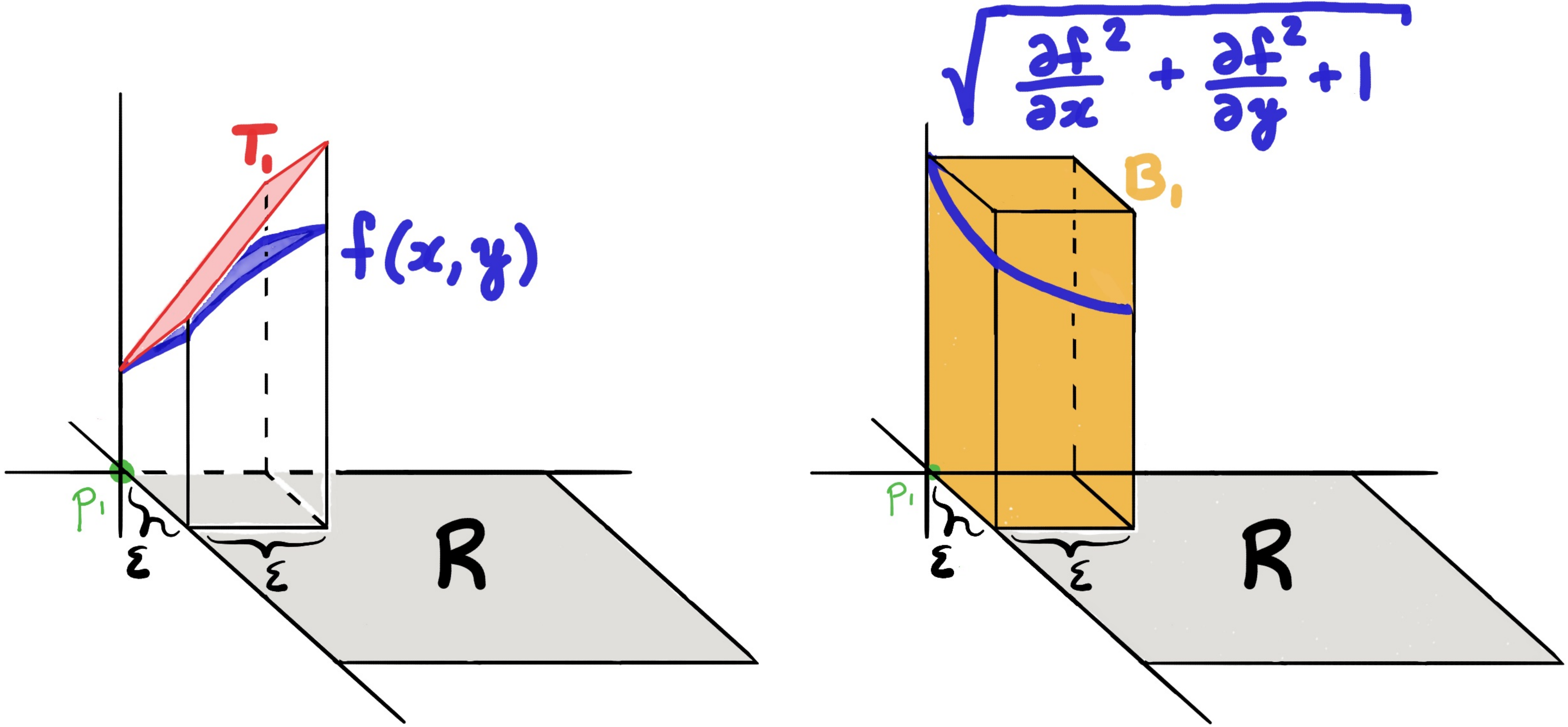

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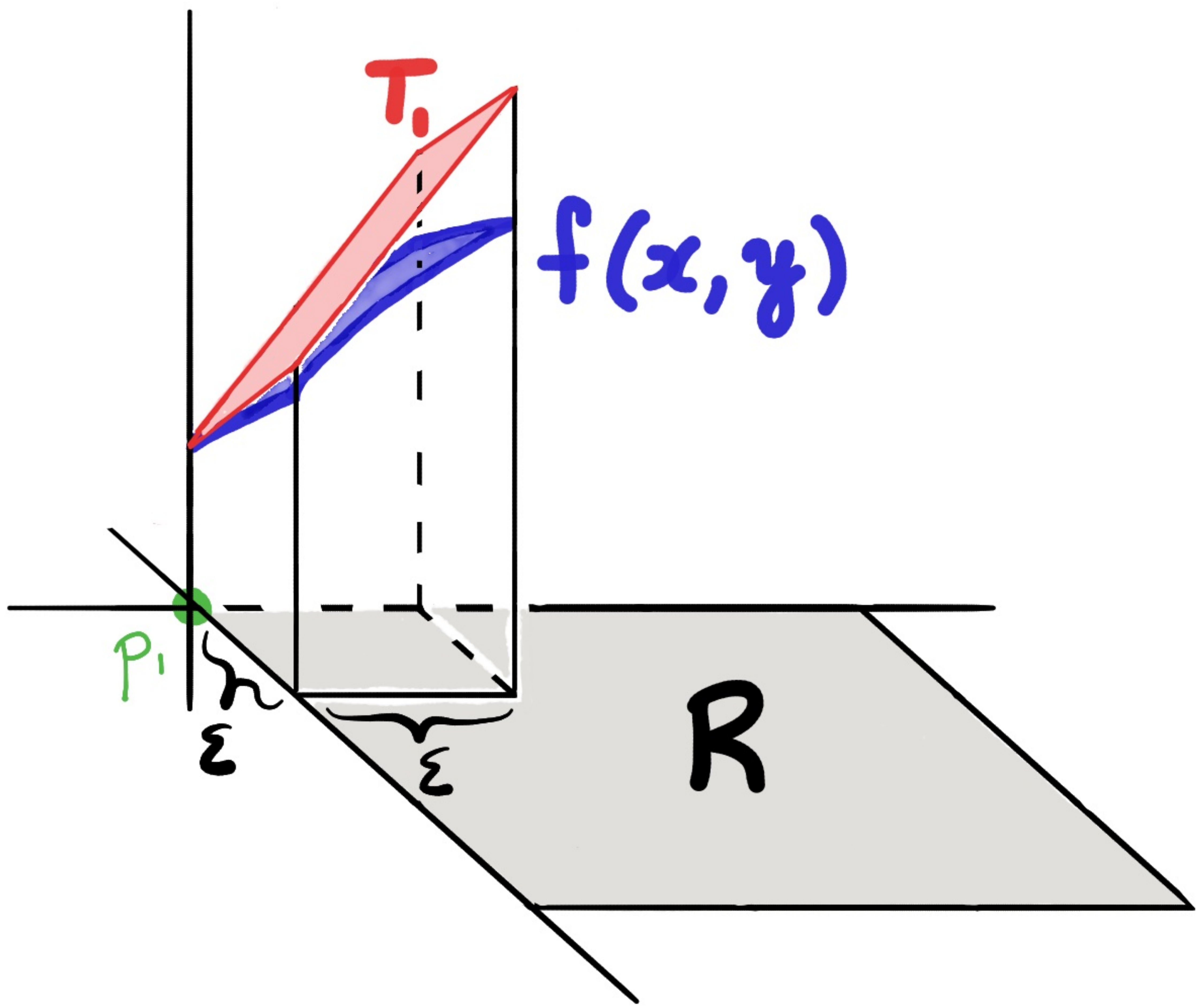
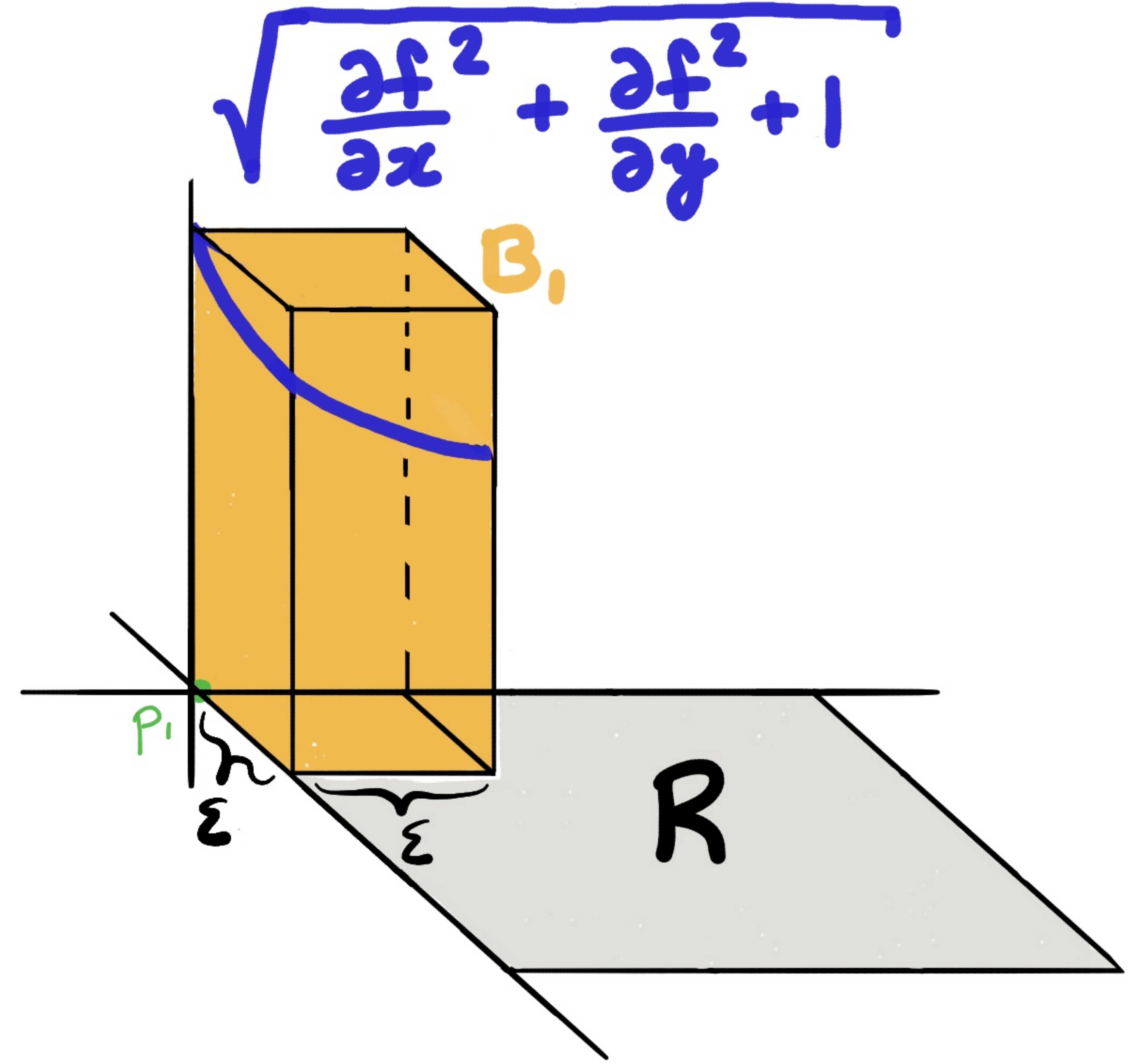
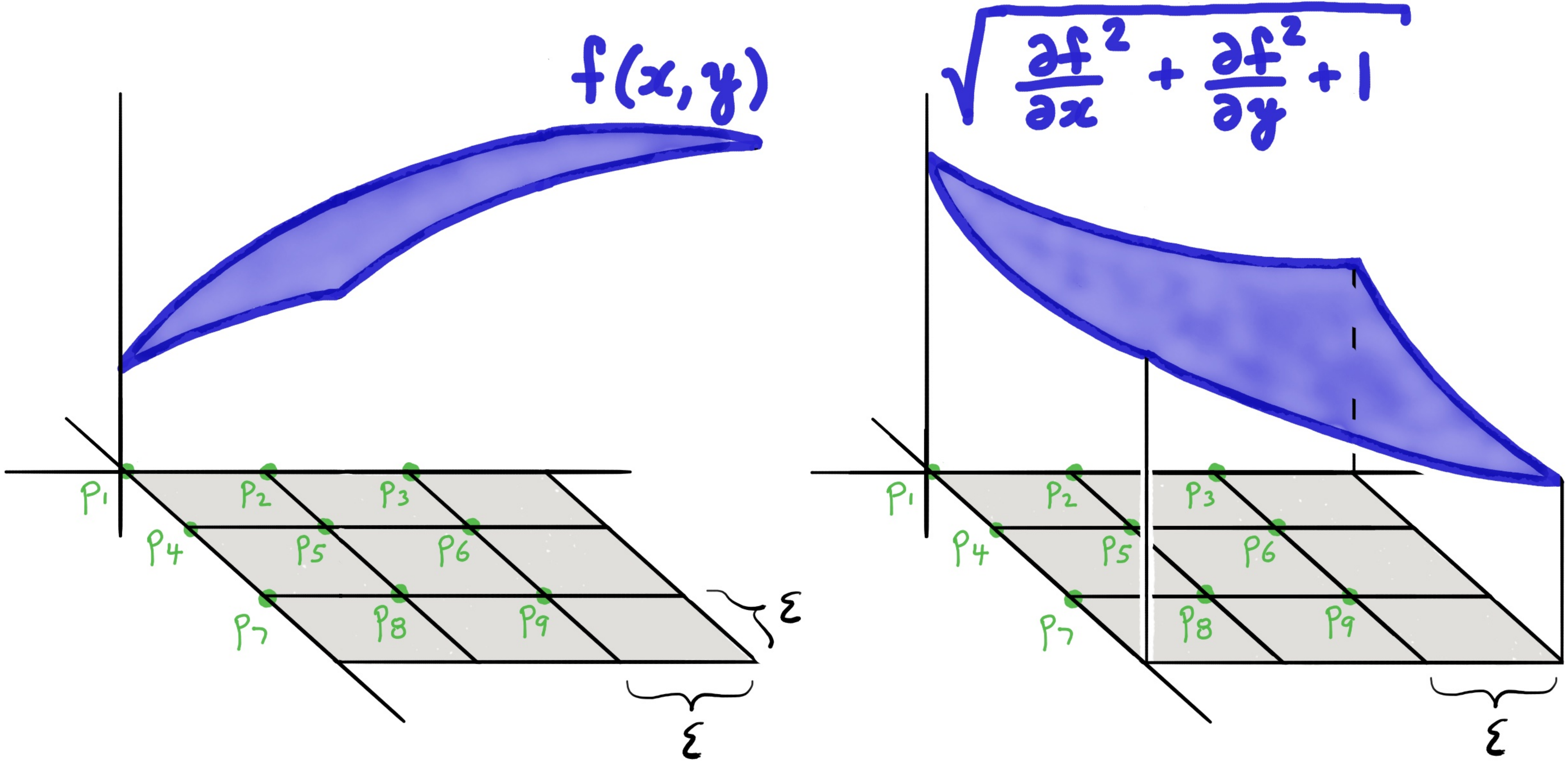




$$\text{Area}(T_1) = \varepsilon^2 \sqrt{\left.\frac{\partial f}{\partial x}\right|_{p_1}^2 + \left.\frac{\partial f}{\partial y}\right|_{p_1}^2 + 1} = \text{Volume}(B_1)$$



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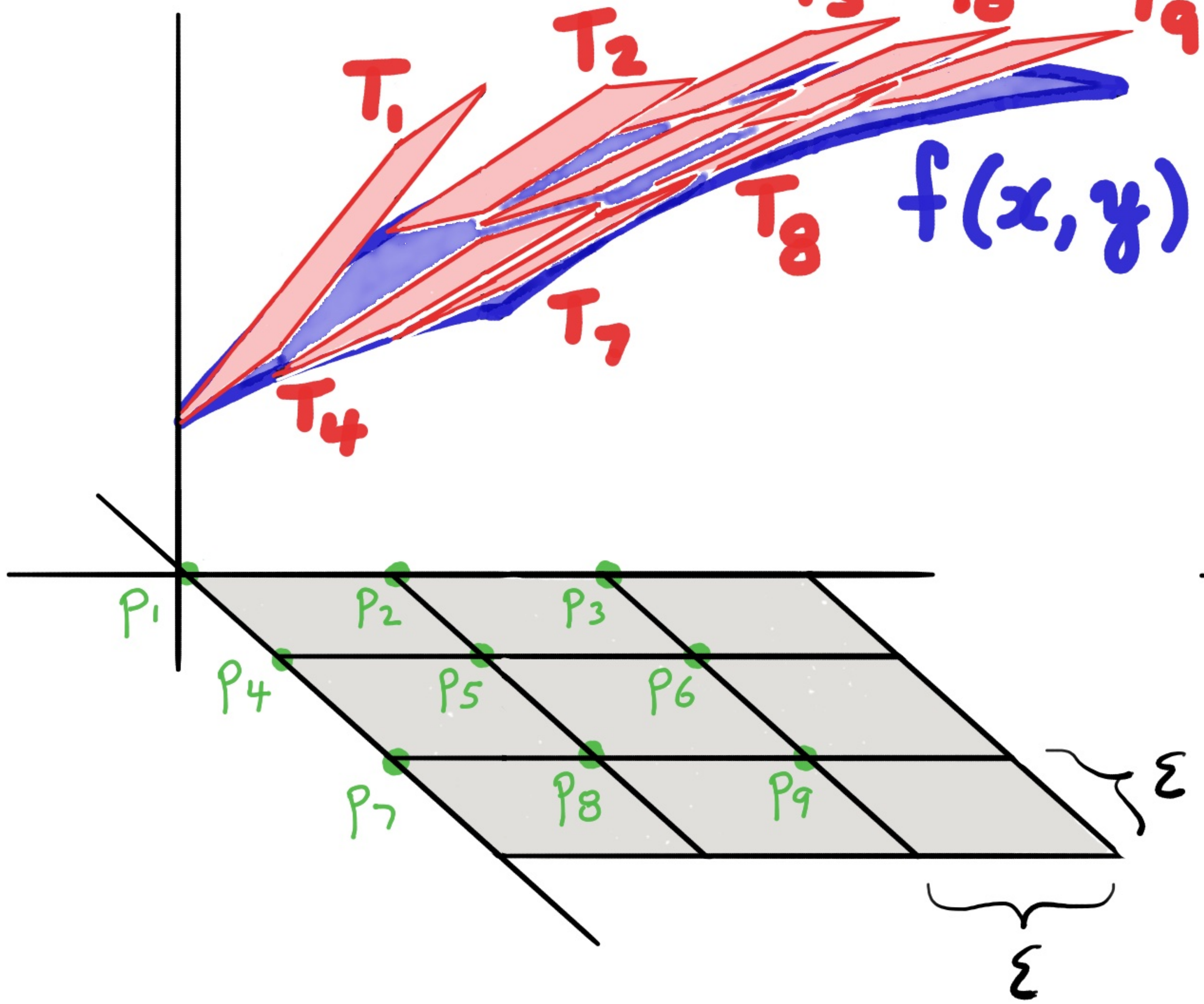

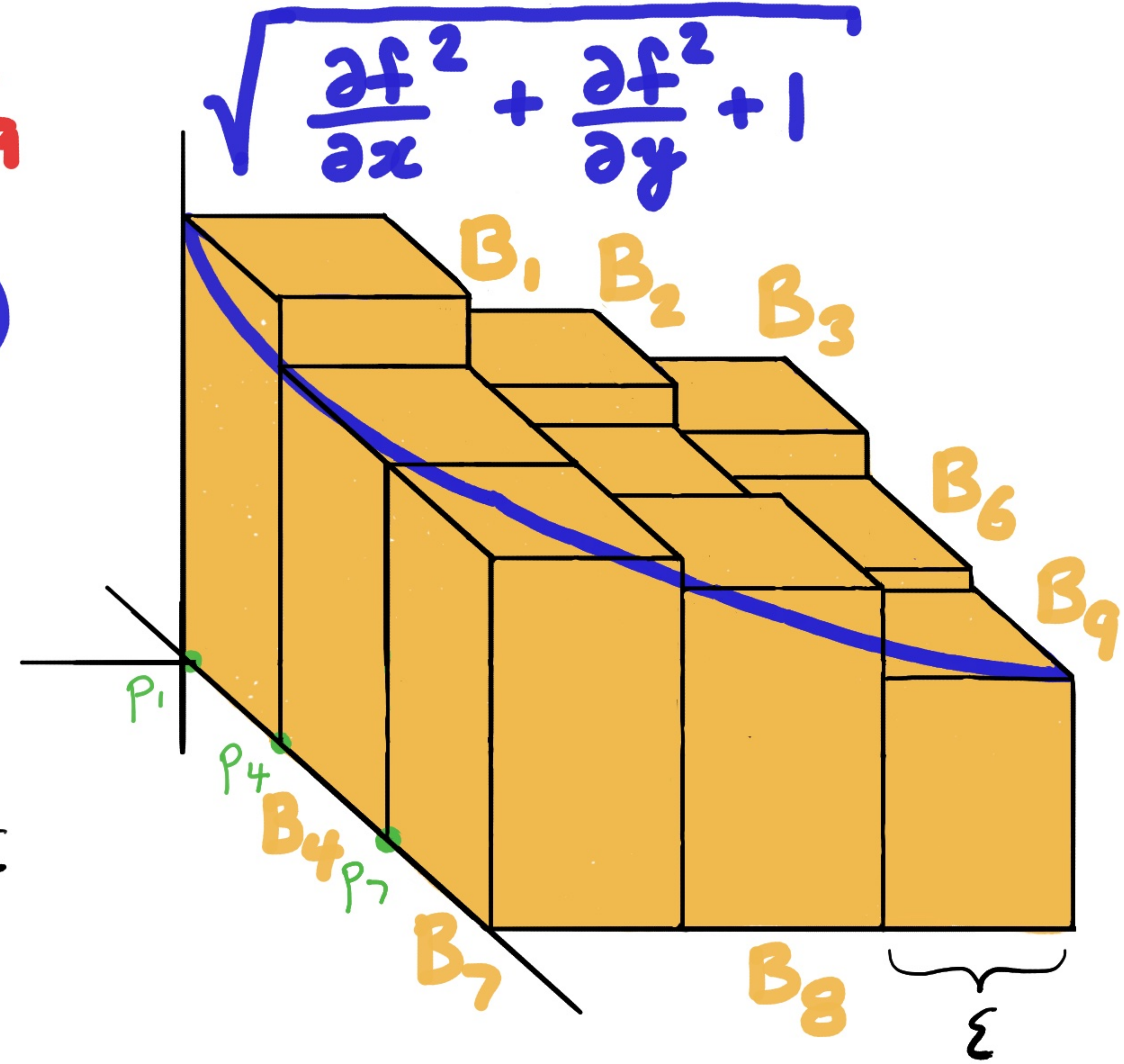

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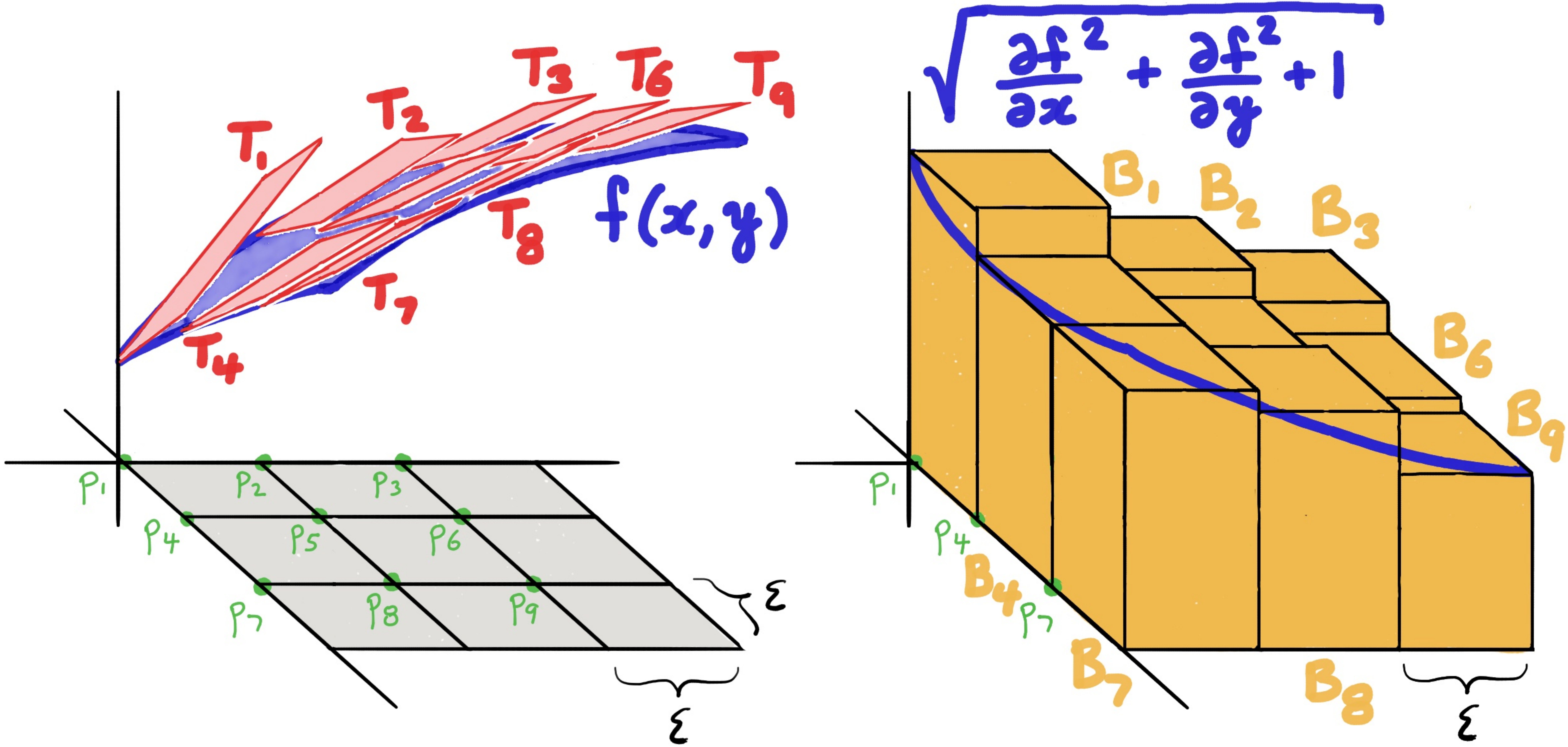
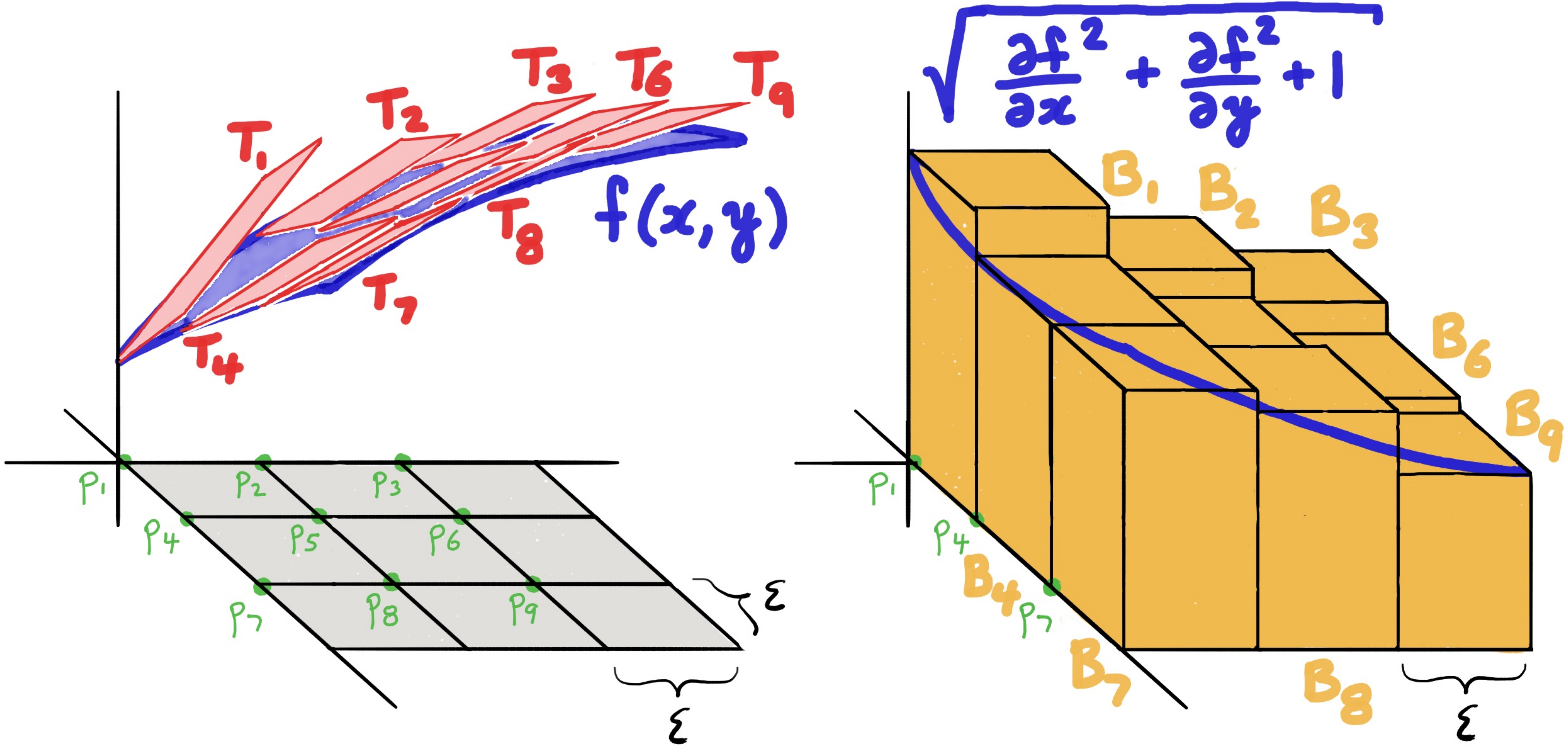
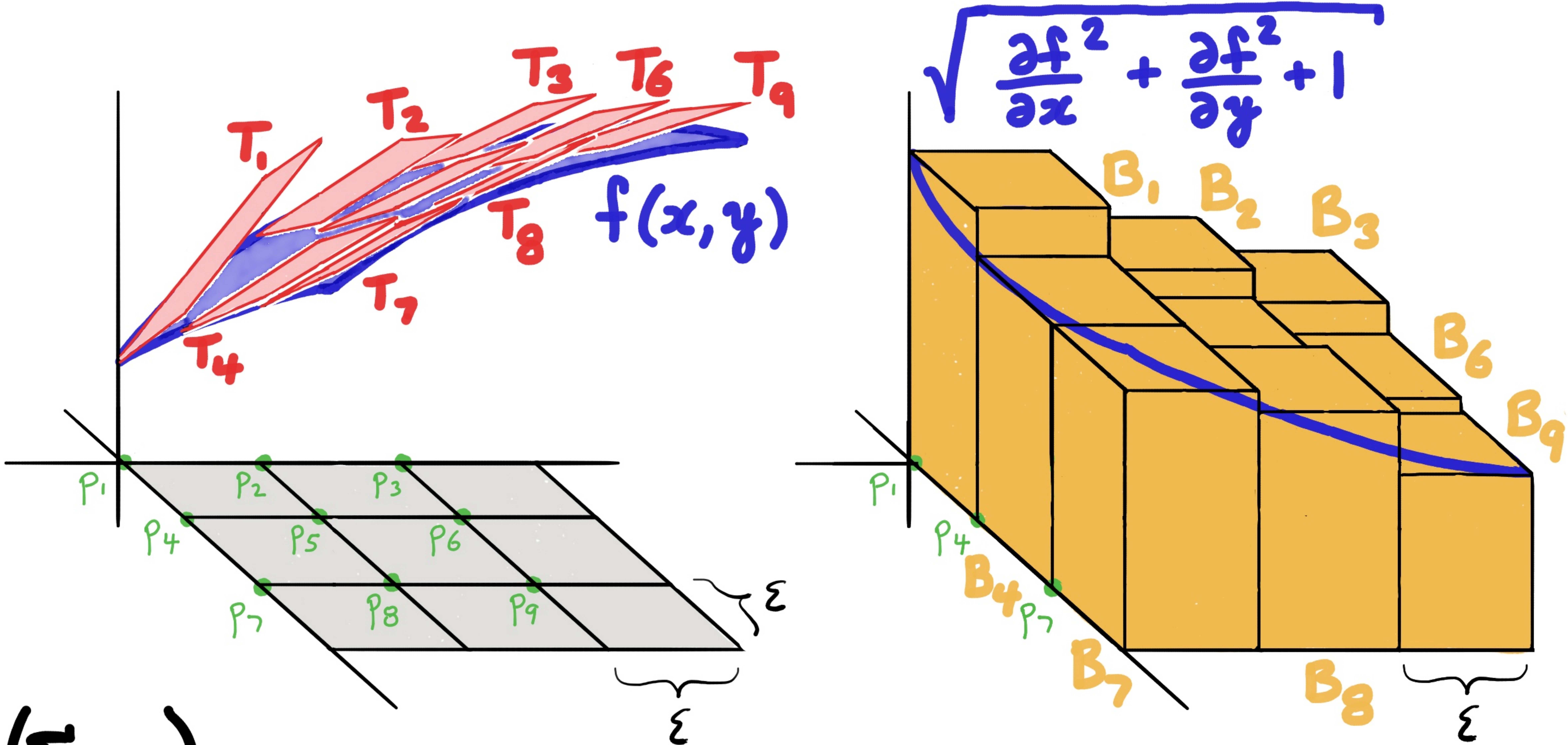


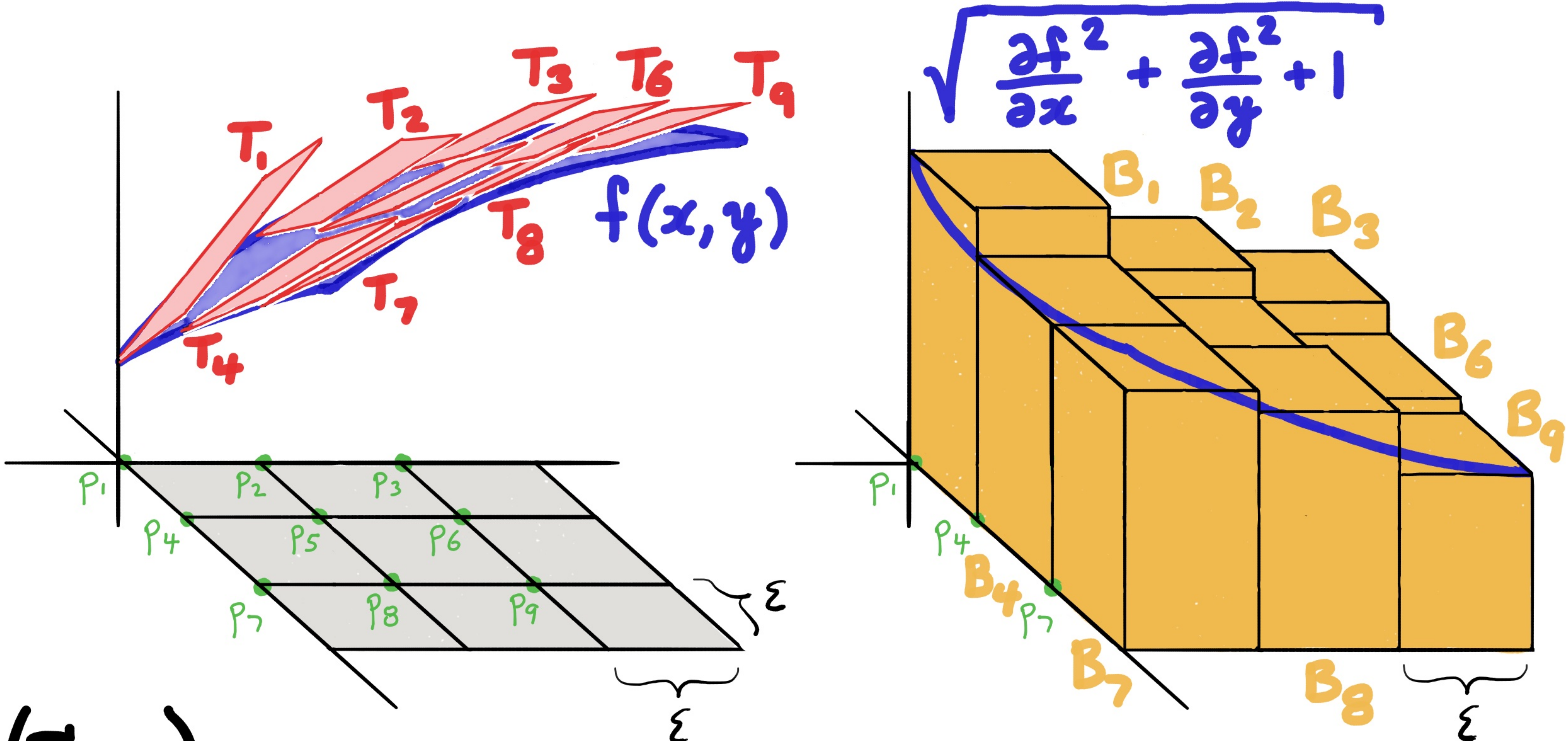
$$\text{Area}(T_i) = \varepsilon^2 \sqrt{\left.\frac{\partial f}{\partial x}\right|_{p_i}^2 + \left.\frac{\partial f}{\partial y}\right|_{p_i}^2 + 1} = \text{Volume}(B_i)$$



$$\sum_i \text{Area}(T_i) = \sum_i \text{Volume}(B_i)$$



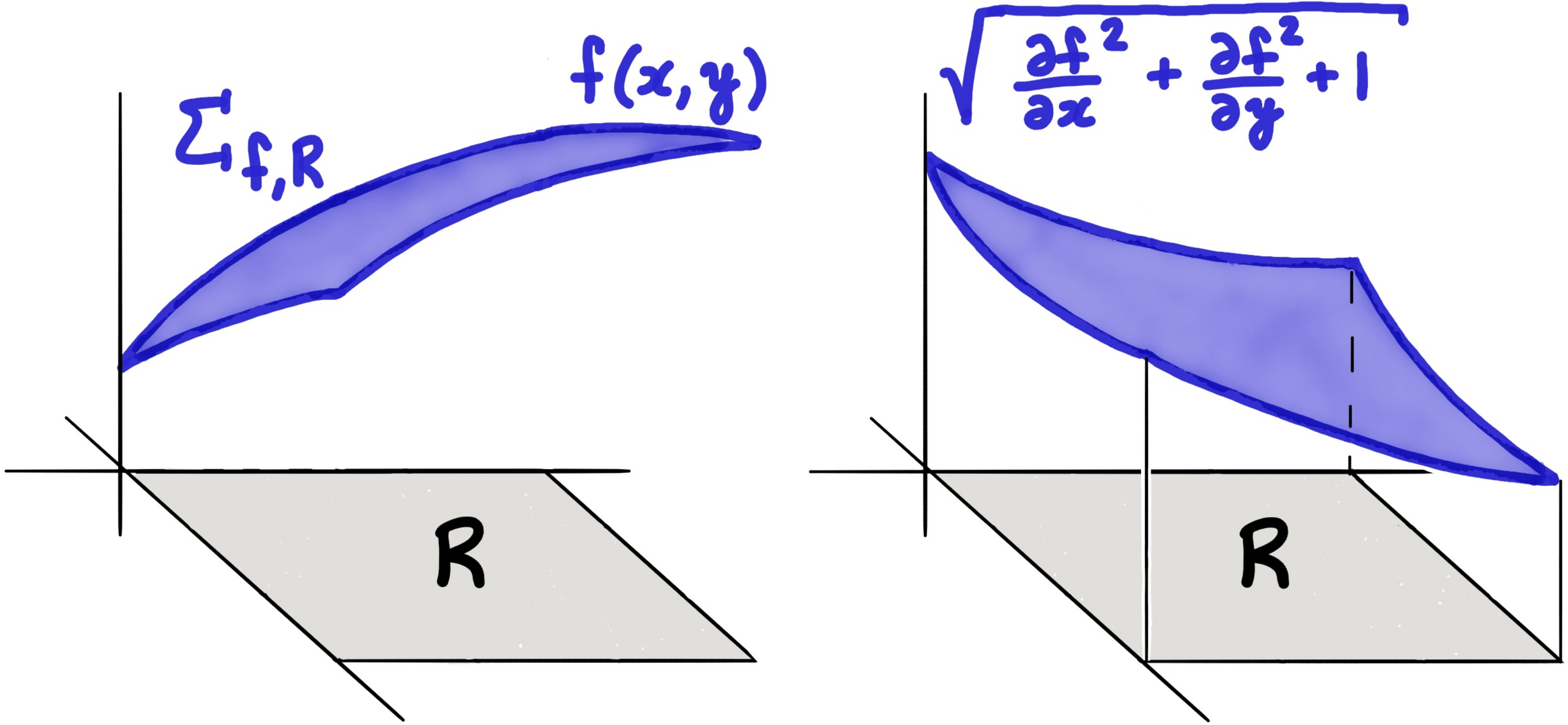
$$SA(\Sigma_{f,R})$$
$$\approx \sum_i \mathrm{Area}(T_i) = \sum_i \mathrm{Volume}(B_i) \approx \iint_R \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 + 1}\, dA$$



$$SA(\Sigma_{f,R})$$

$$\approx \sum_i \text{Area}(T_i) = \sum_i \text{Volume}(B_i) \approx \iint_R \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 + 1}\, dA$$

In the limit, $$SA(\Sigma_{f,R}) = \iint_R \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 + 1}\, dA$$



$$\text{Surface Area}(\Sigma_{f,R}) = \iint_R \sqrt{\frac{\partial f}{\partial x}^2 + \frac{\partial f}{\partial y}^2 + 1}\, dA$$

Examples:

① Find surface area of portion of plane $2x-4y+2z=4$ that lies above the triangle in the xy-plane with vertices $(0,1)$, $(1,0)$, and $(0,0)$.

② Find surface area of $z=x^2+y^2$, $0 \le z \le 4$.

③ Find surface area of a sphere of radius a by finding surface area of its upper hemisphere.