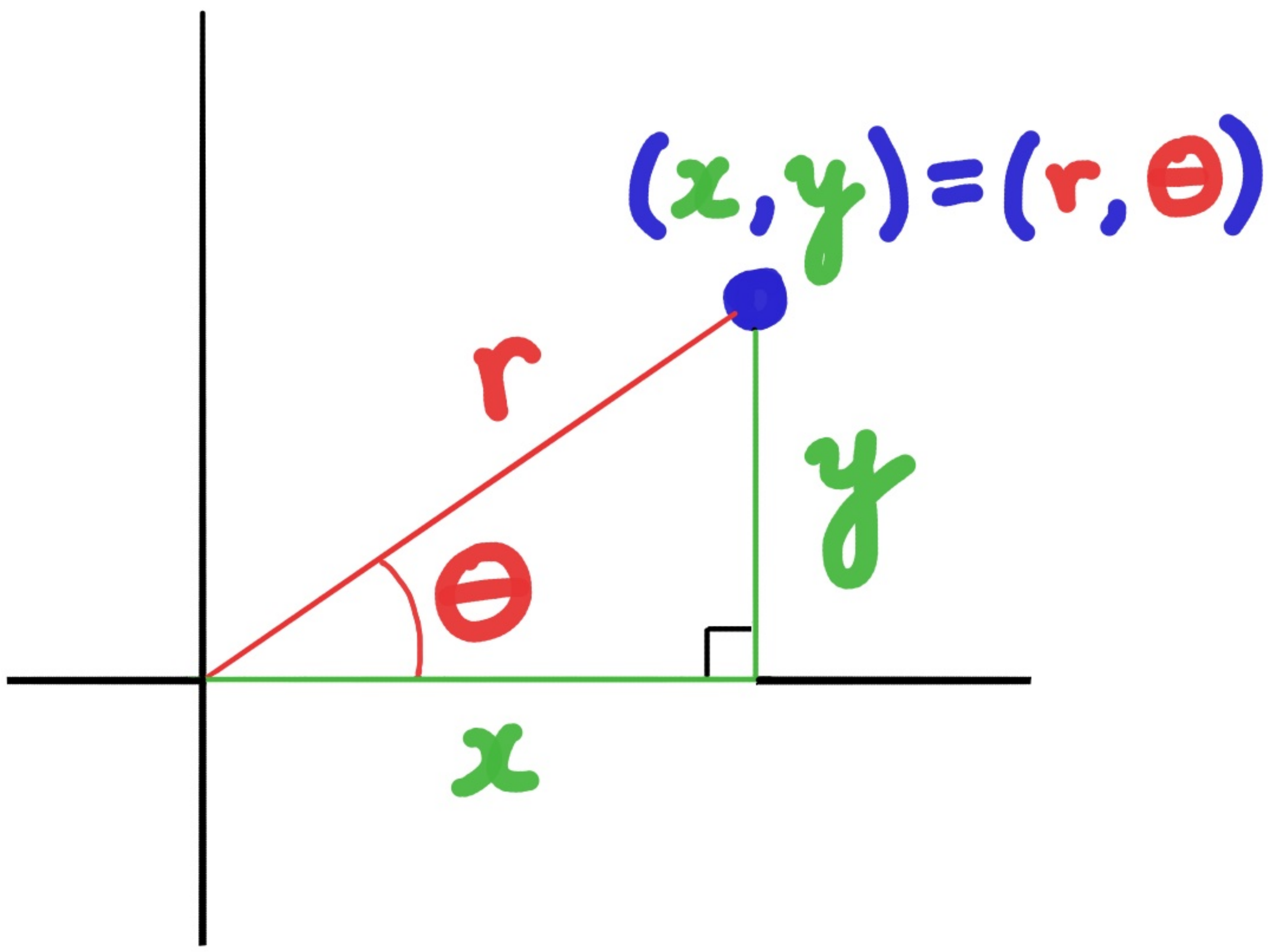
Twenty-one

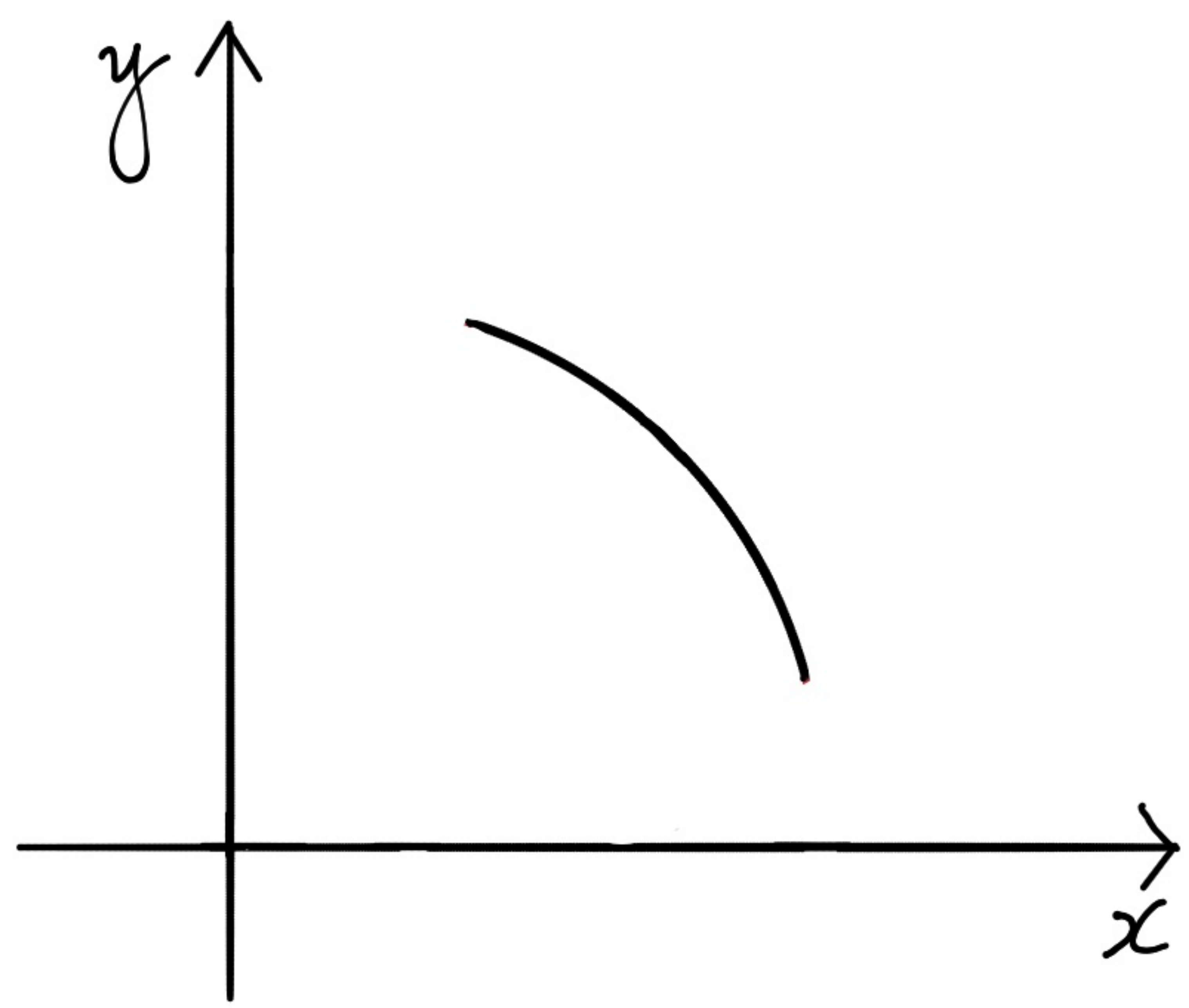
Double integrals in polar coordinates

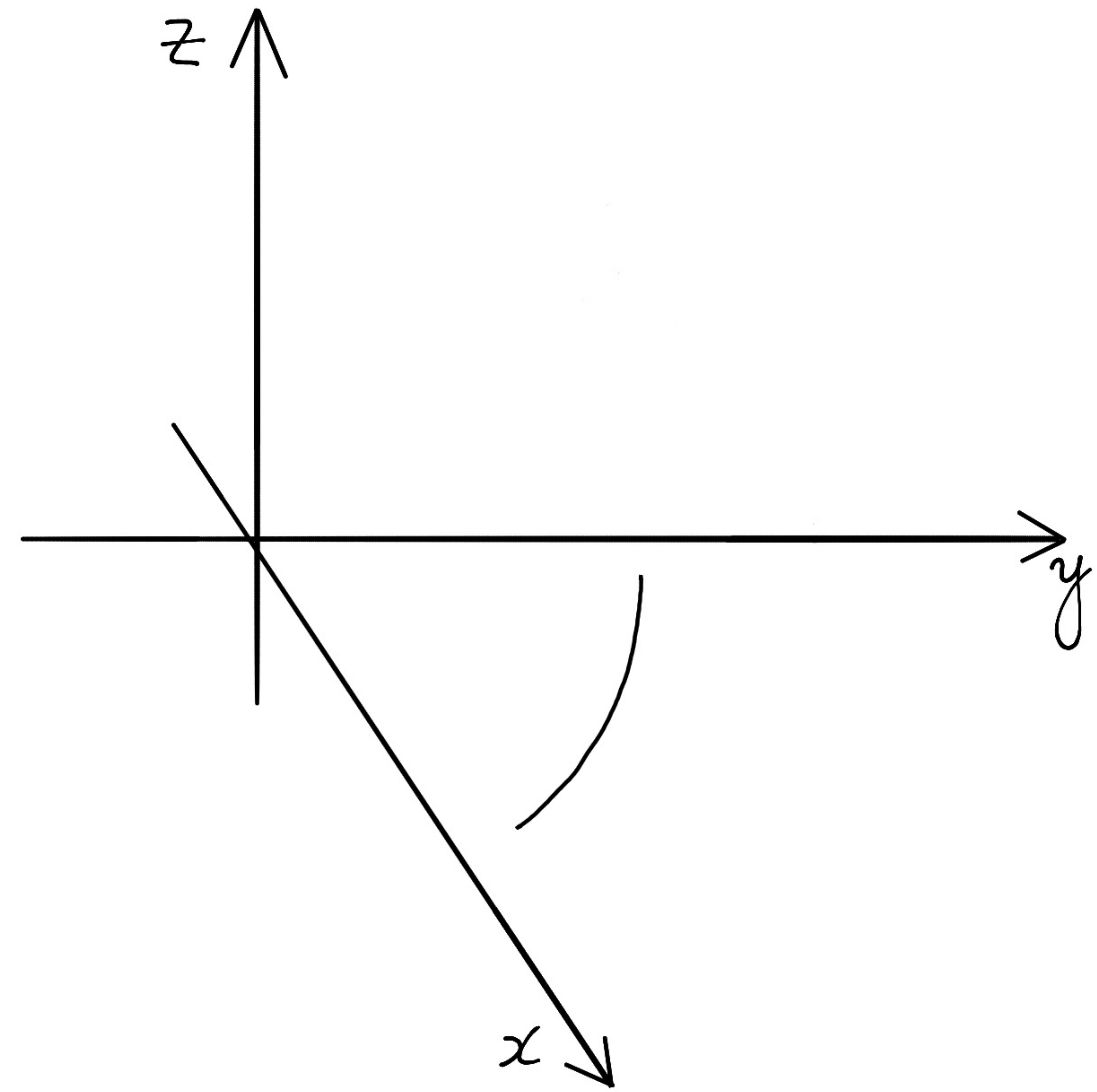
Cartesian / Polar coordinates

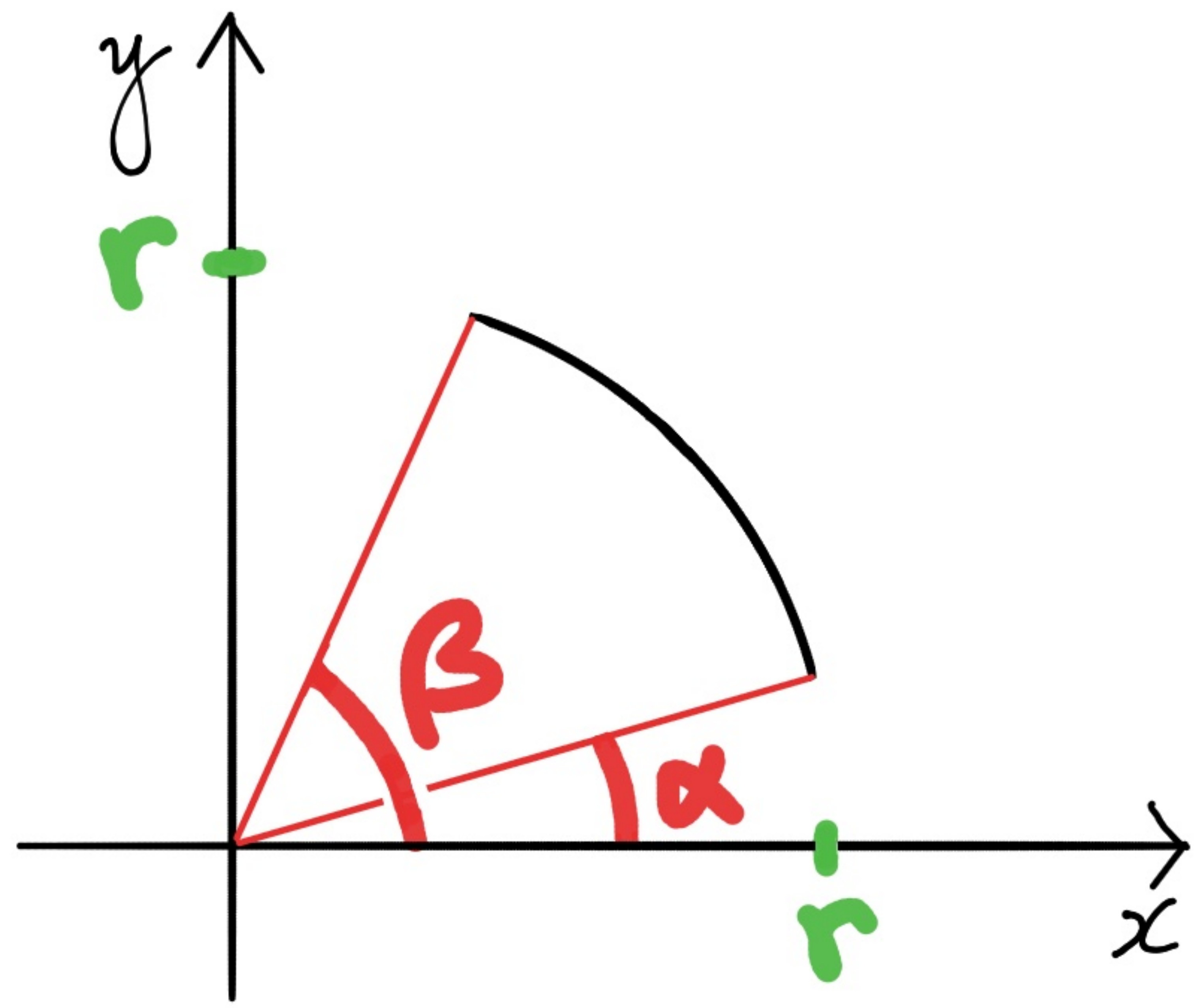


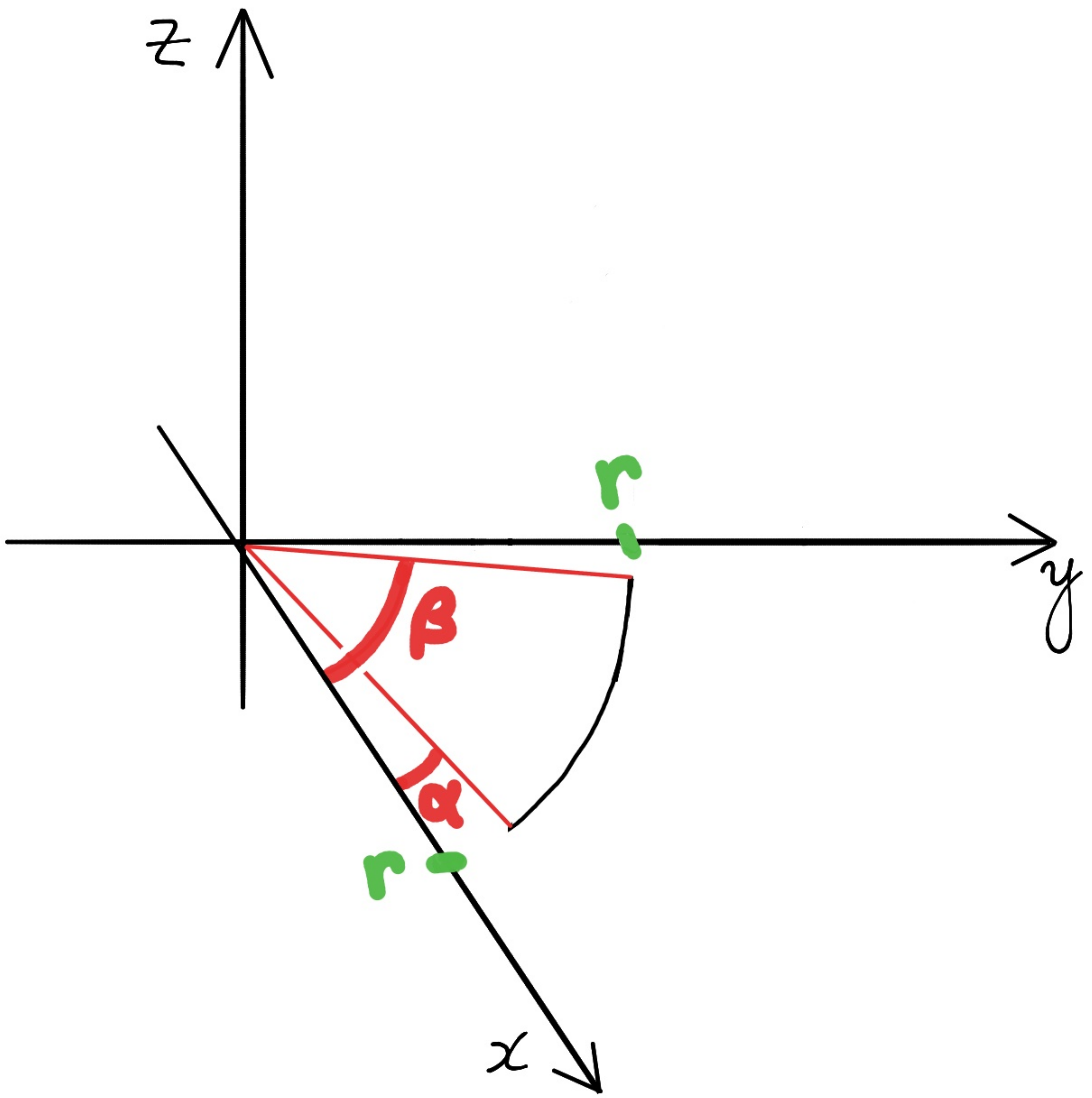
$$x = r\cos\theta$$

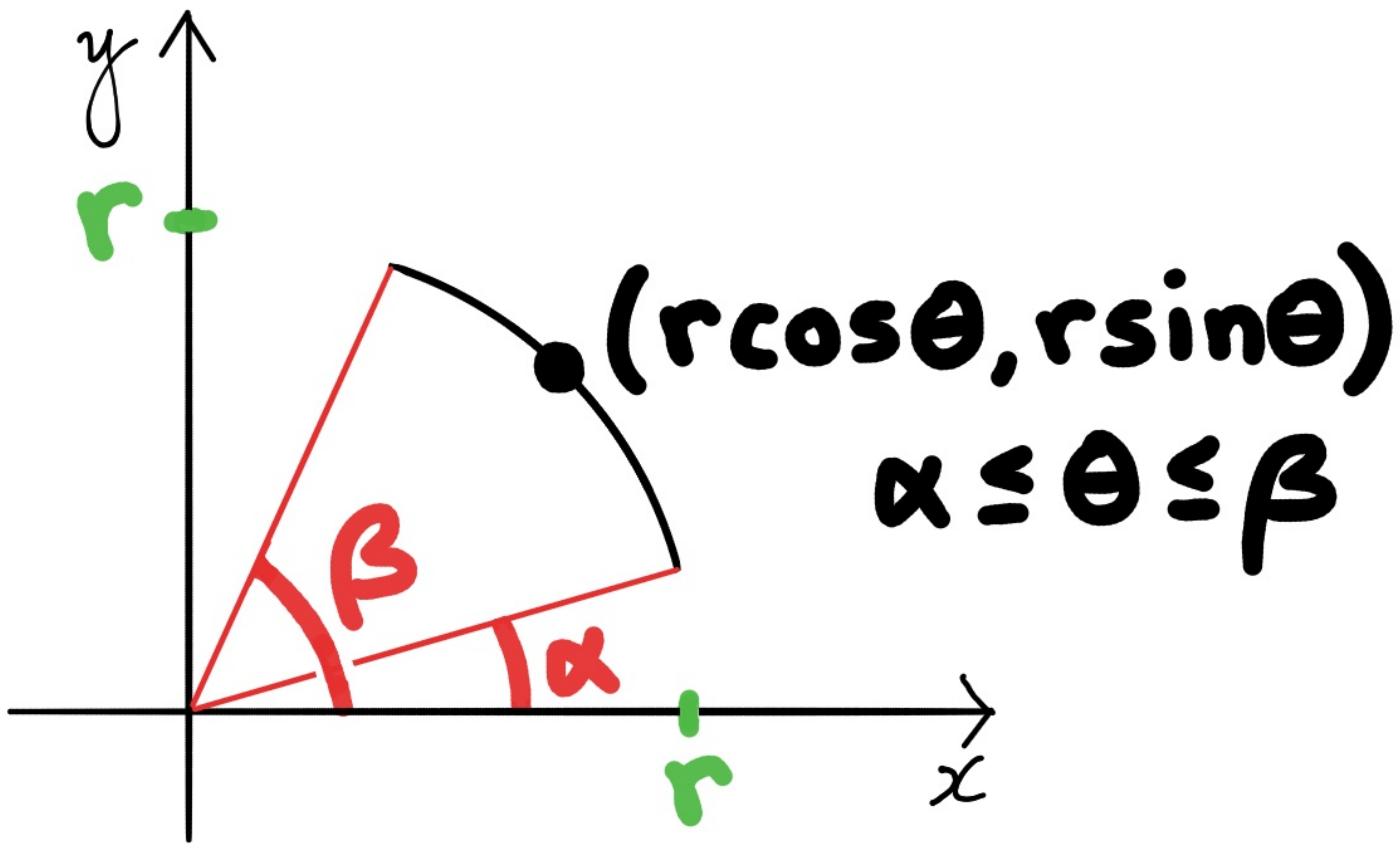
$$y = r\sin\theta$$

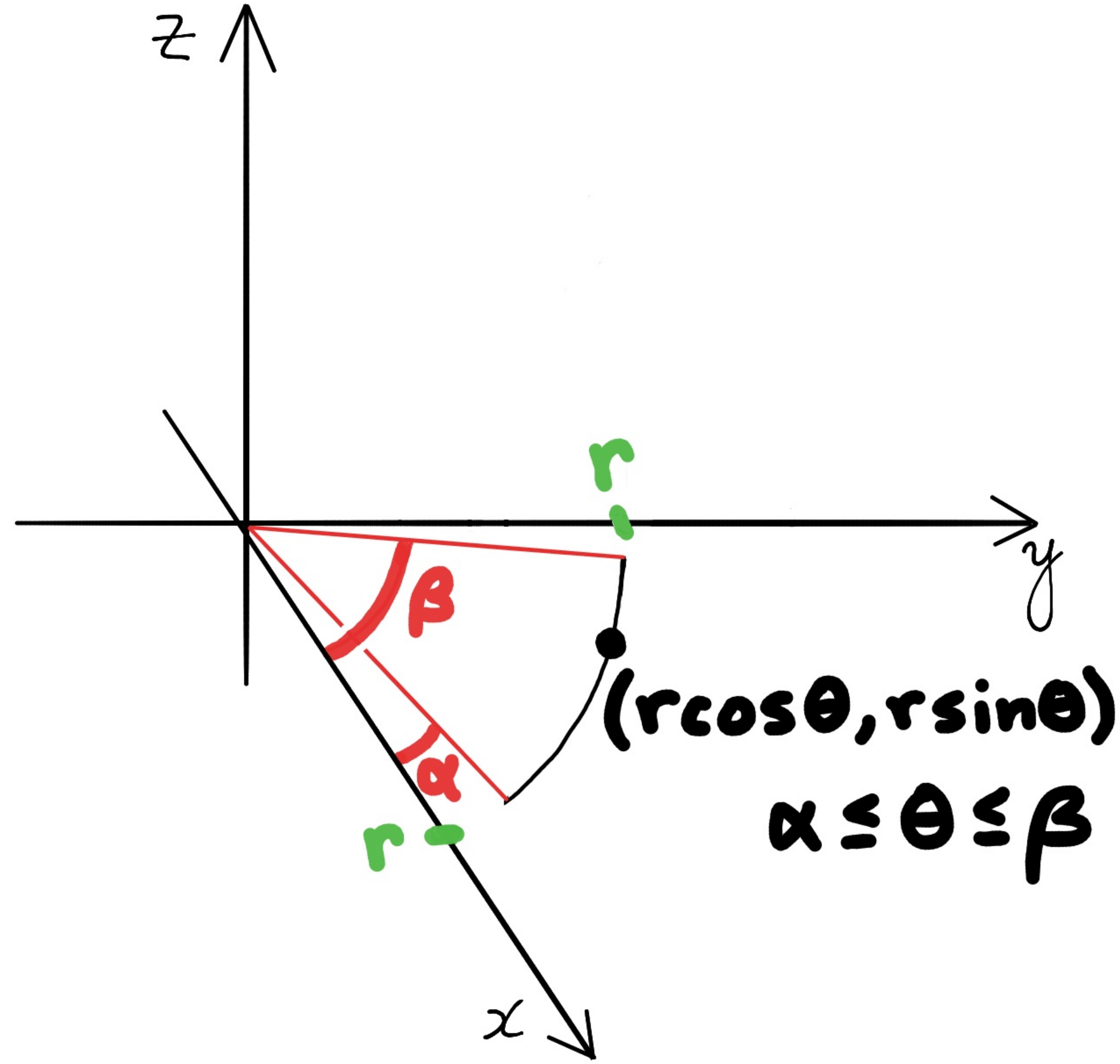

y
x

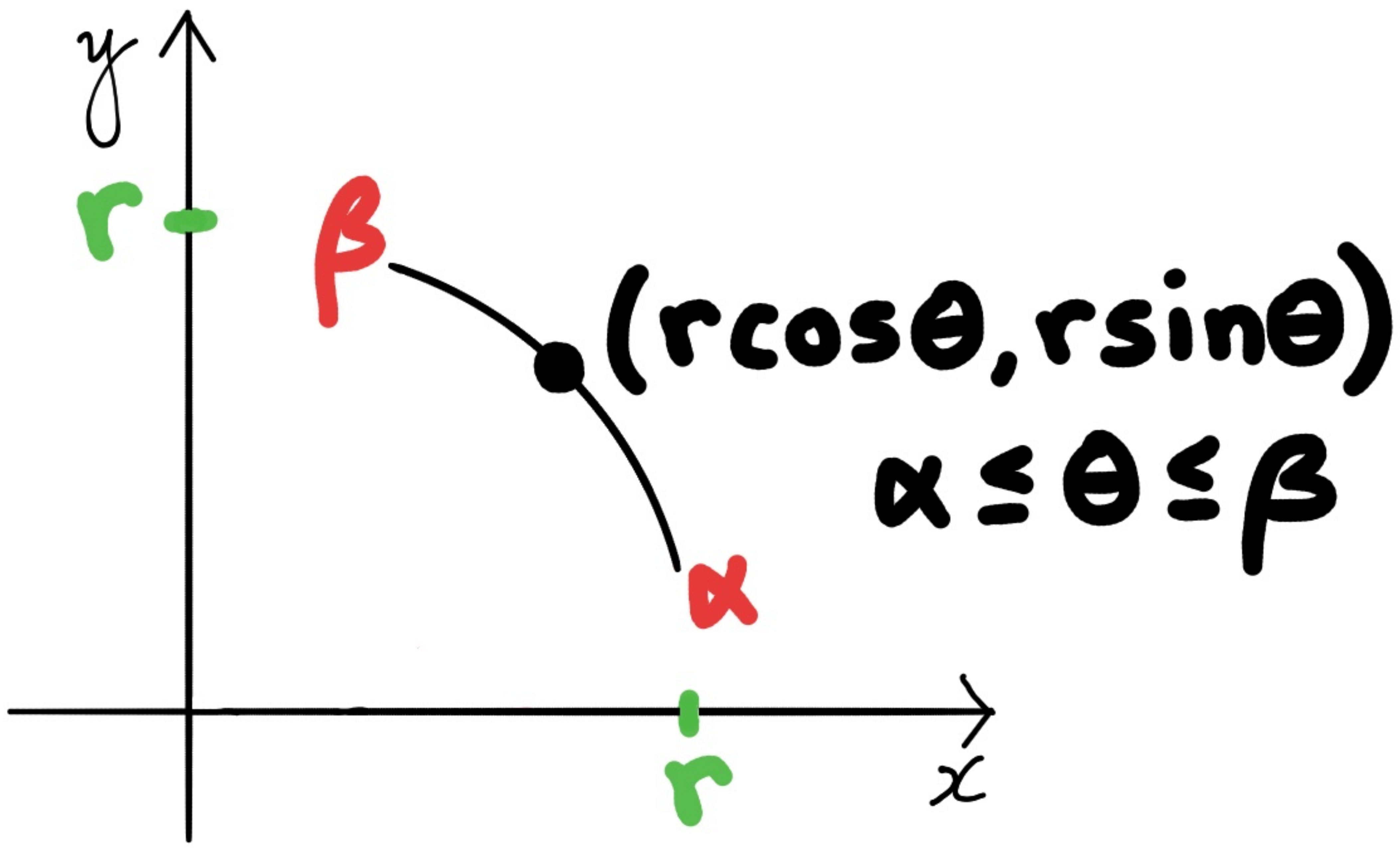

z
y
x

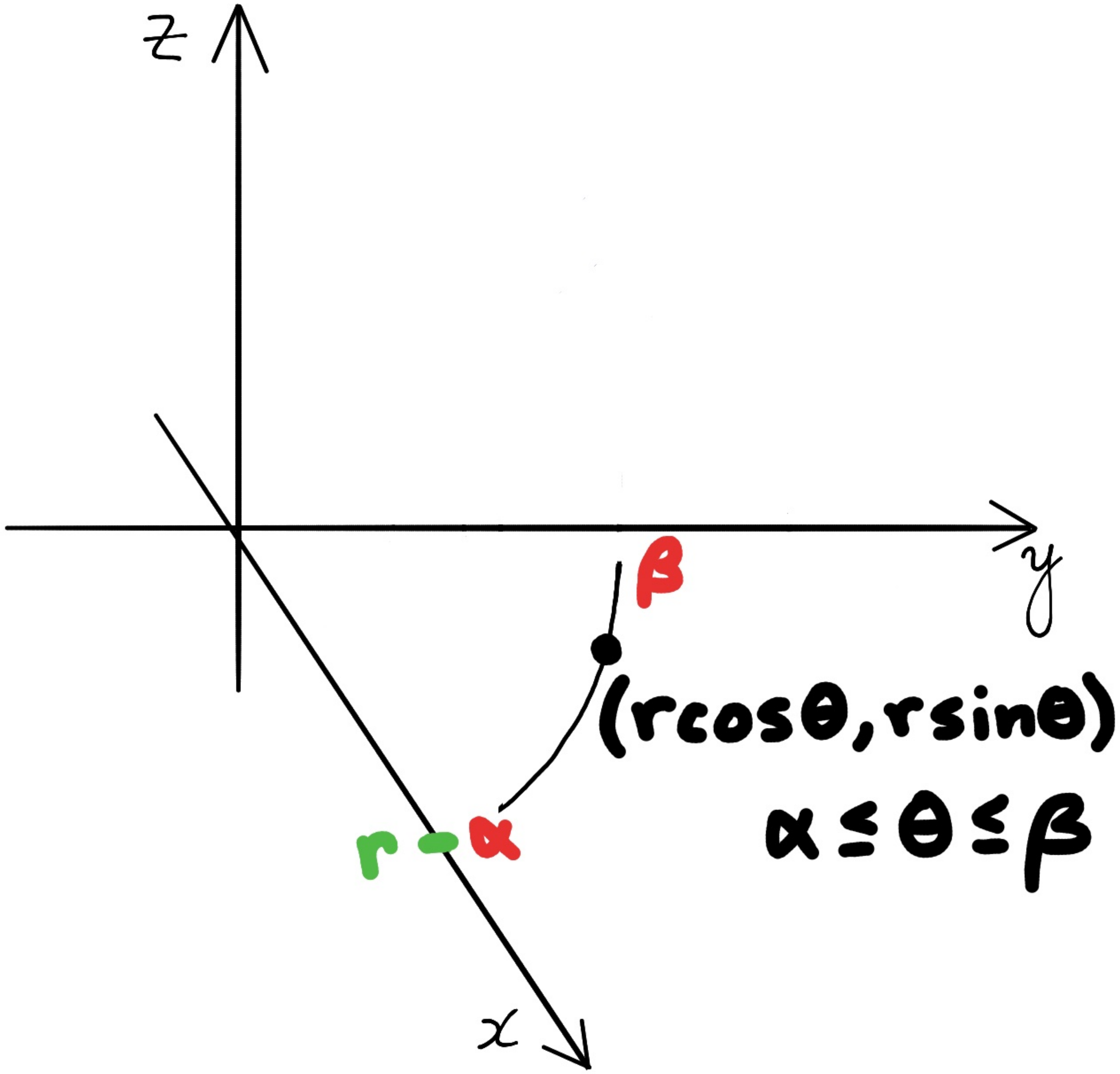

y
r
β
α
r
x

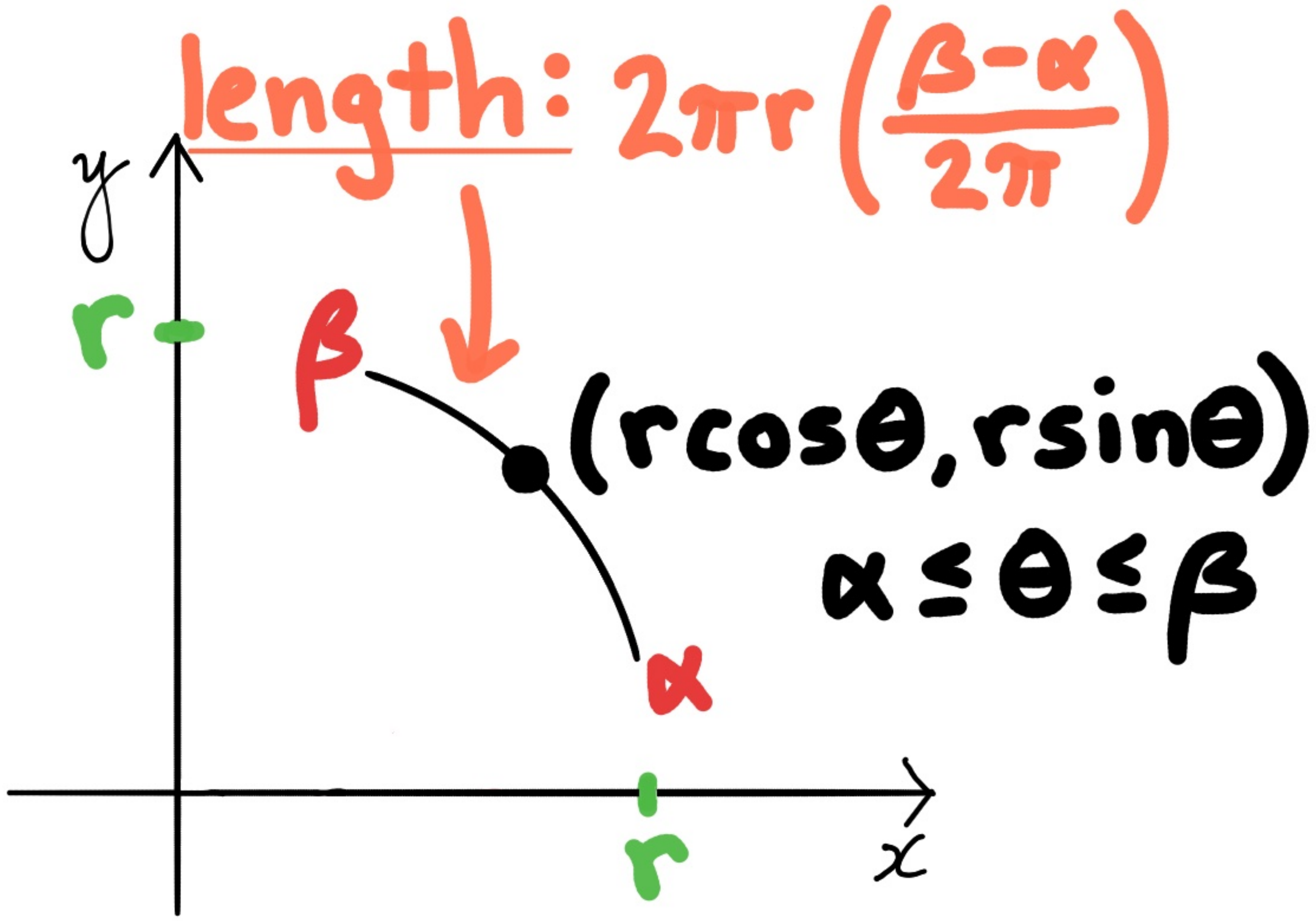

z
r
y
β
α
r
x

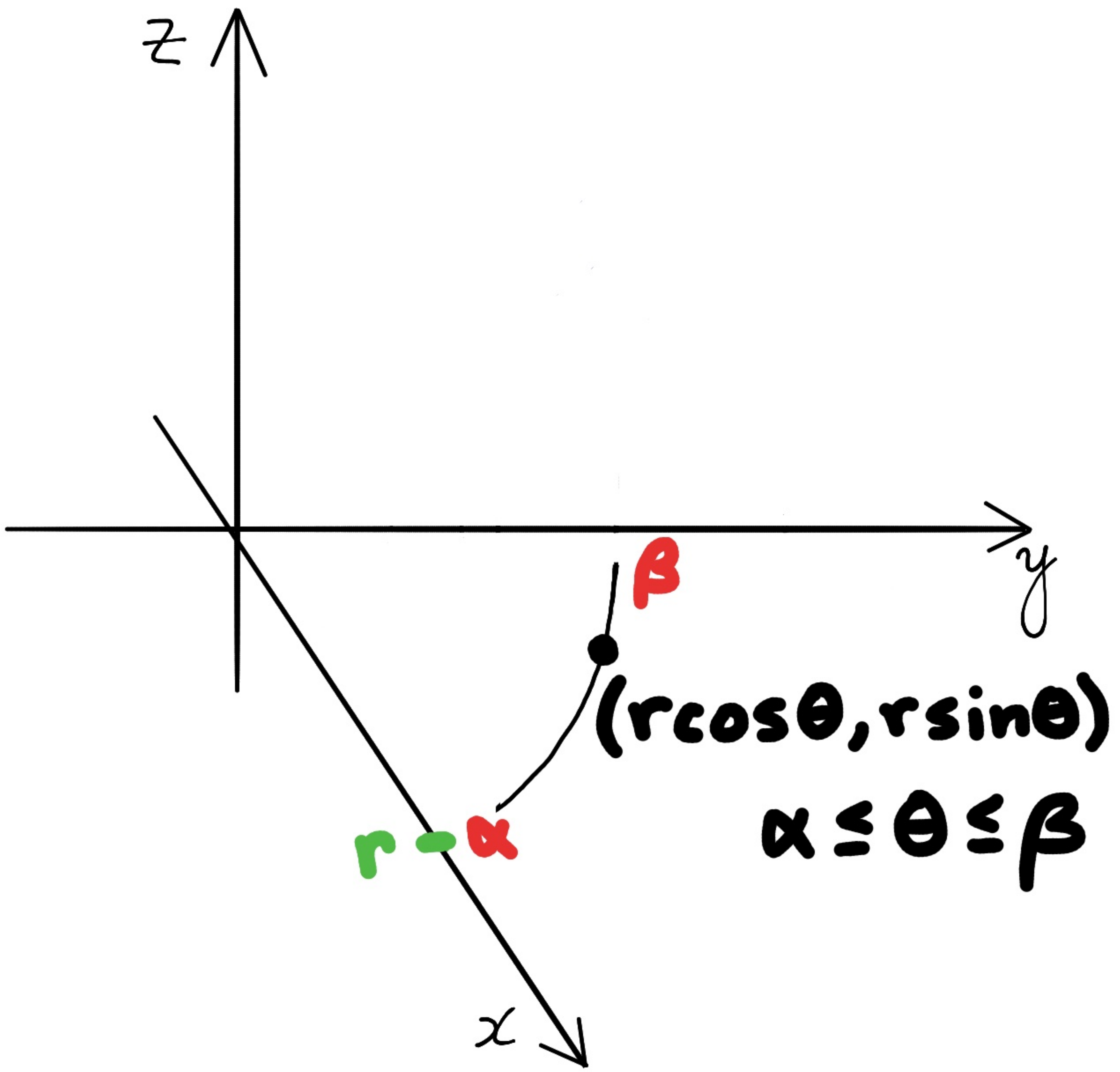

y
r
(rcosθ, rsinθ)
α ≤ θ ≤ β
β
α
r
x

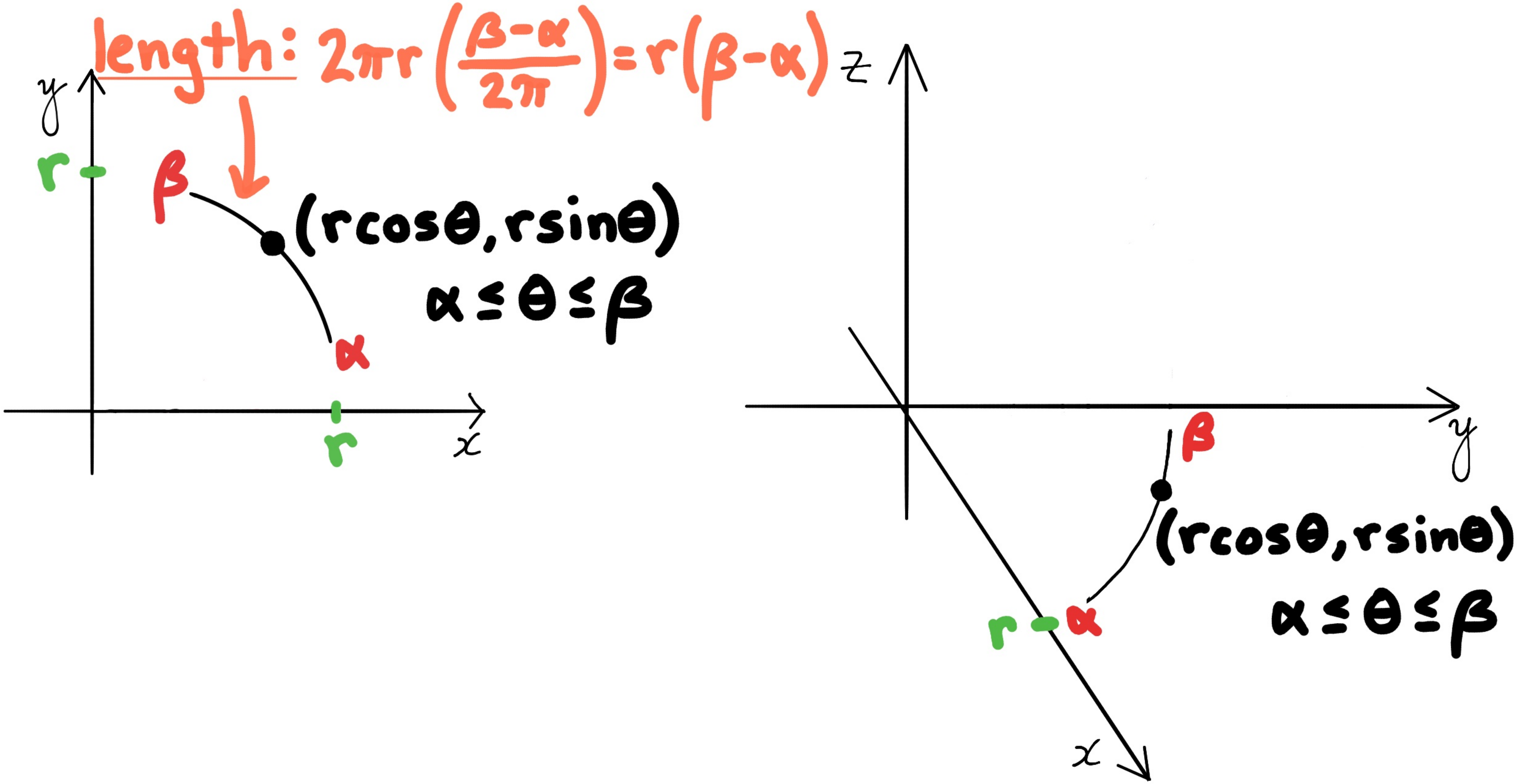

z
r
y
β
(rcosθ, rsinθ)
α
r
α ≤ θ ≤ β
x


y
r
β
(rcosθ, rsinθ)
α ≤ θ ≤ β
α
r
x

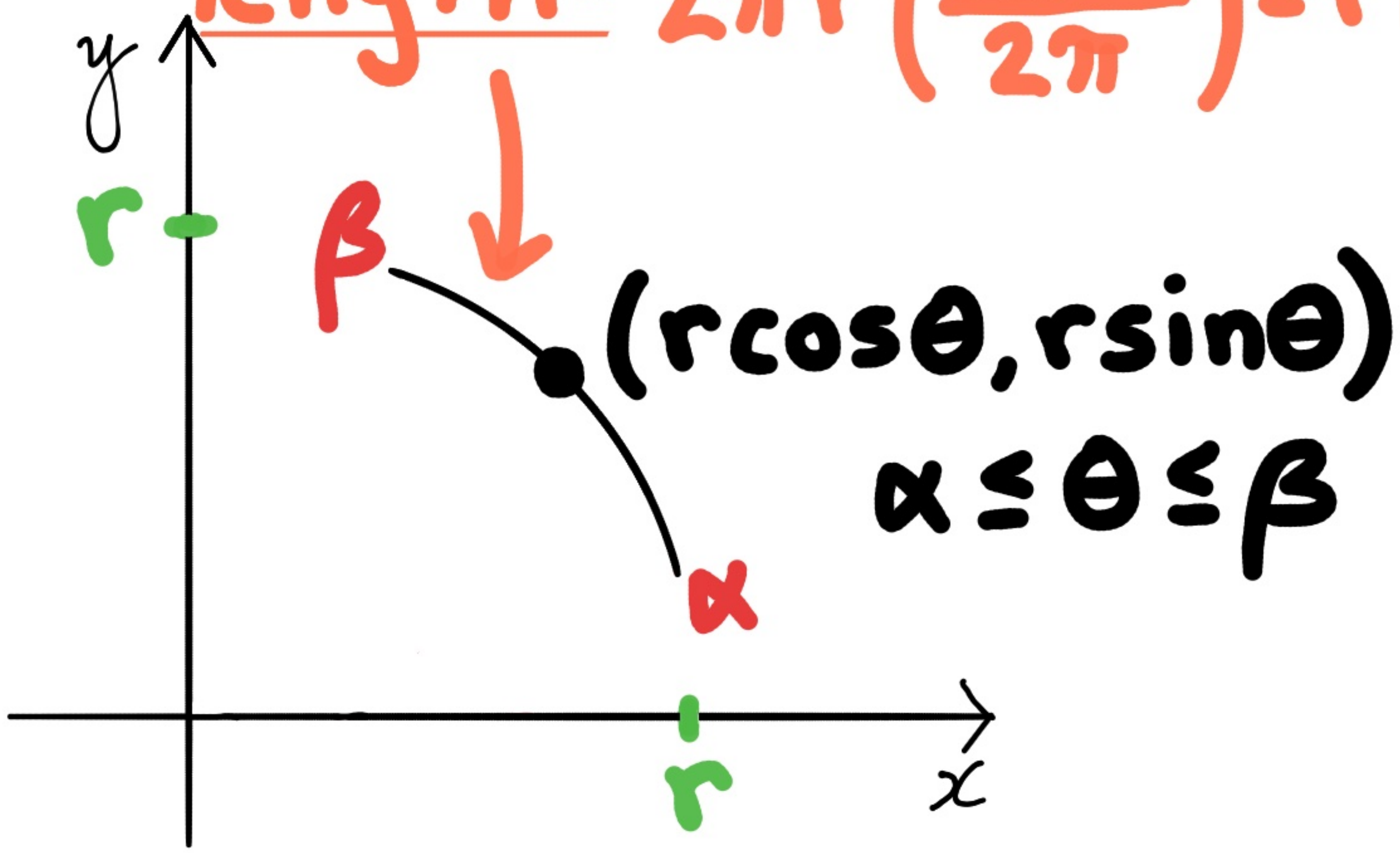

z
y
β
(rcosθ, rsinθ)
r
α
α ≤ θ ≤ β
x

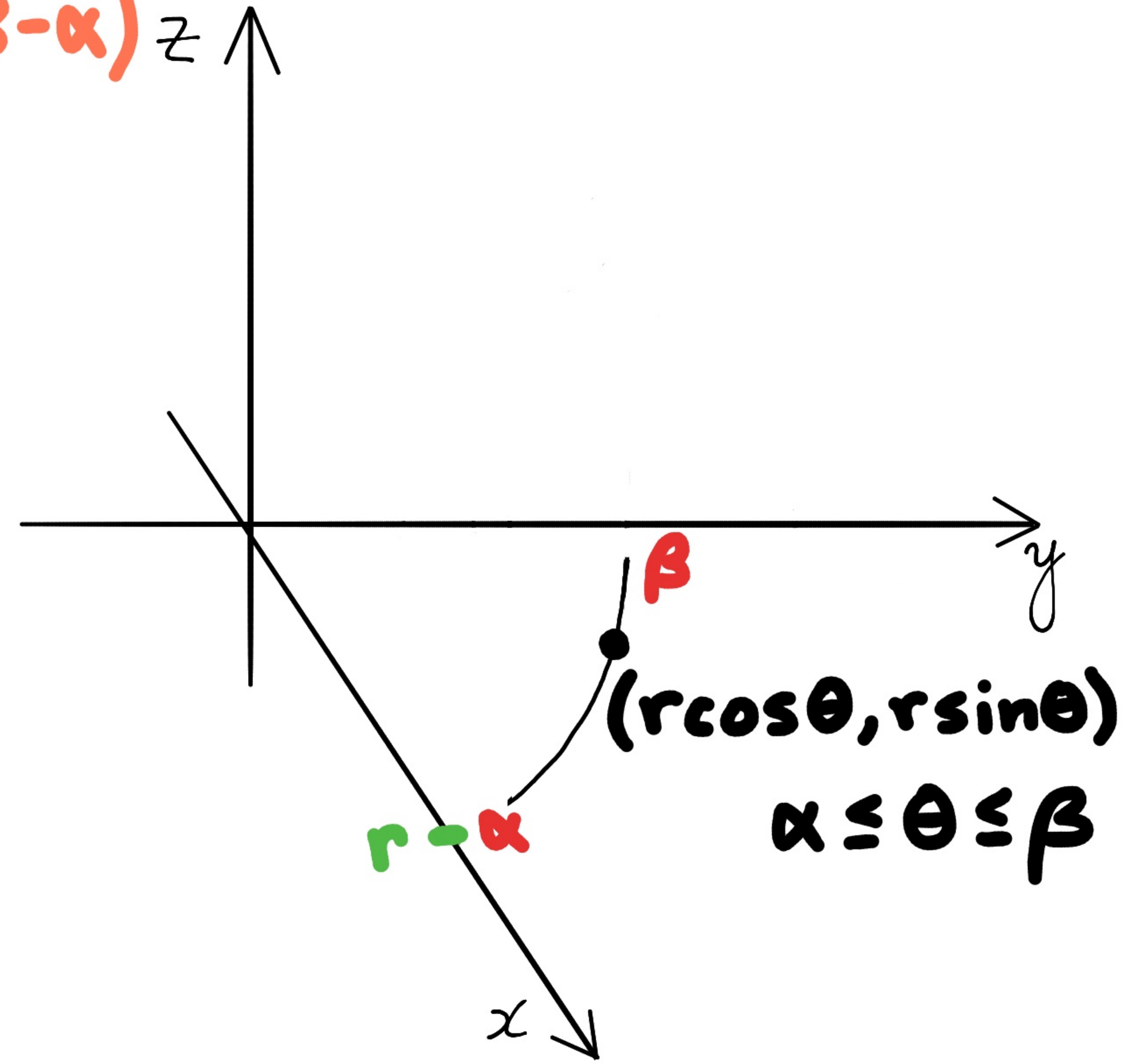

length: $2\pi r\left(\frac{\beta-\alpha}{2\pi}\right)$
y
x
r
r
β
α
(rcosθ, rsinθ)
α ≤ θ ≤ β


z
y
x
r
β
α
(rcosθ, rsinθ)
α ≤ θ ≤ β

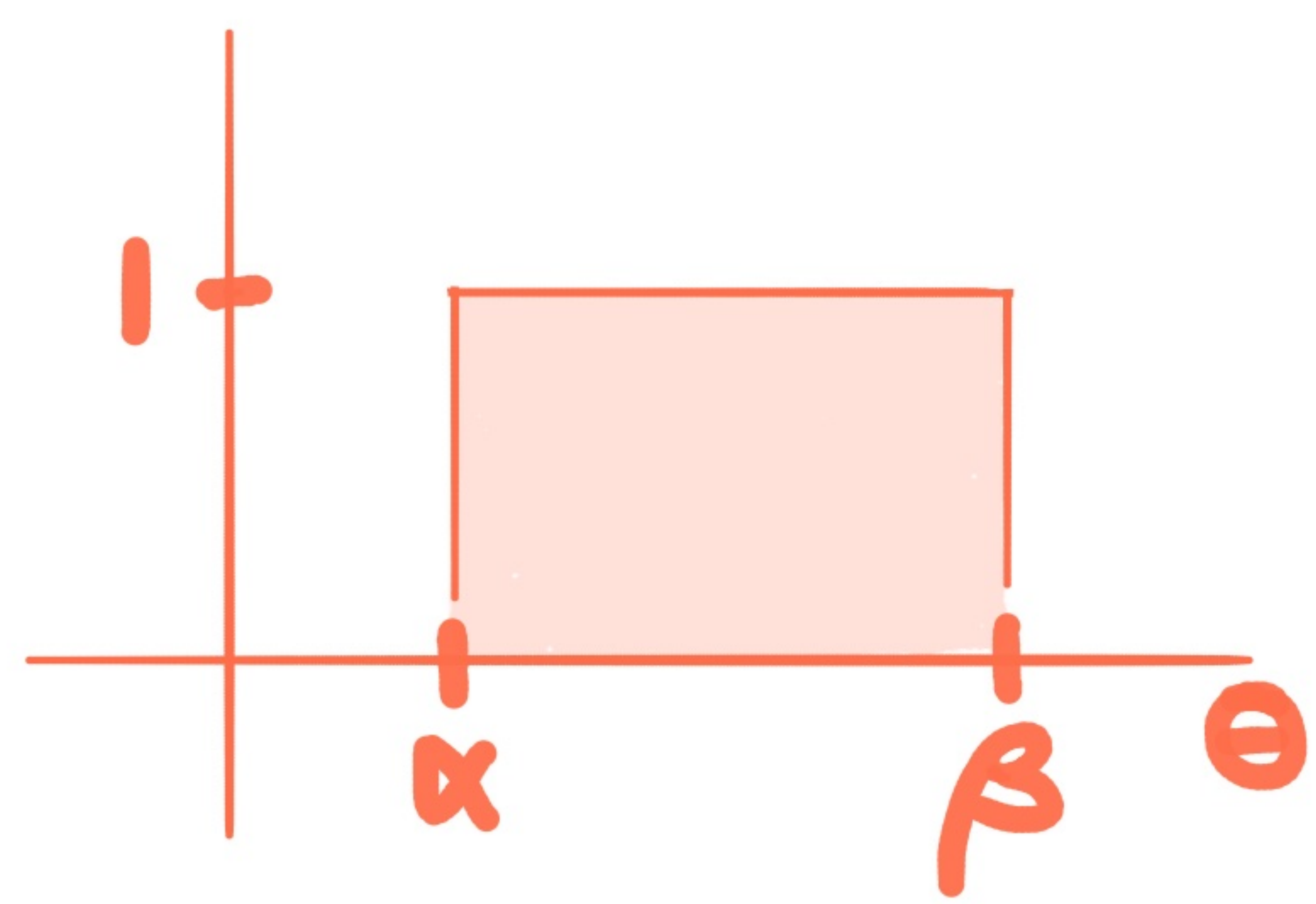

length: $2\pi r\left(\frac{\beta-\alpha}{2\pi}\right)=r(\beta-\alpha)$
y
x
r
r
β
α
$(r\cos\theta, r\sin\theta)$
$\alpha \le \theta \le \beta$
z
y
x
r
β
α
$(r\cos\theta, r\sin\theta)$
$\alpha \le \theta \le \beta$

length: $2\pi r\left(\frac{\beta-\alpha}{2\pi}\right)=r(\beta-\alpha)$

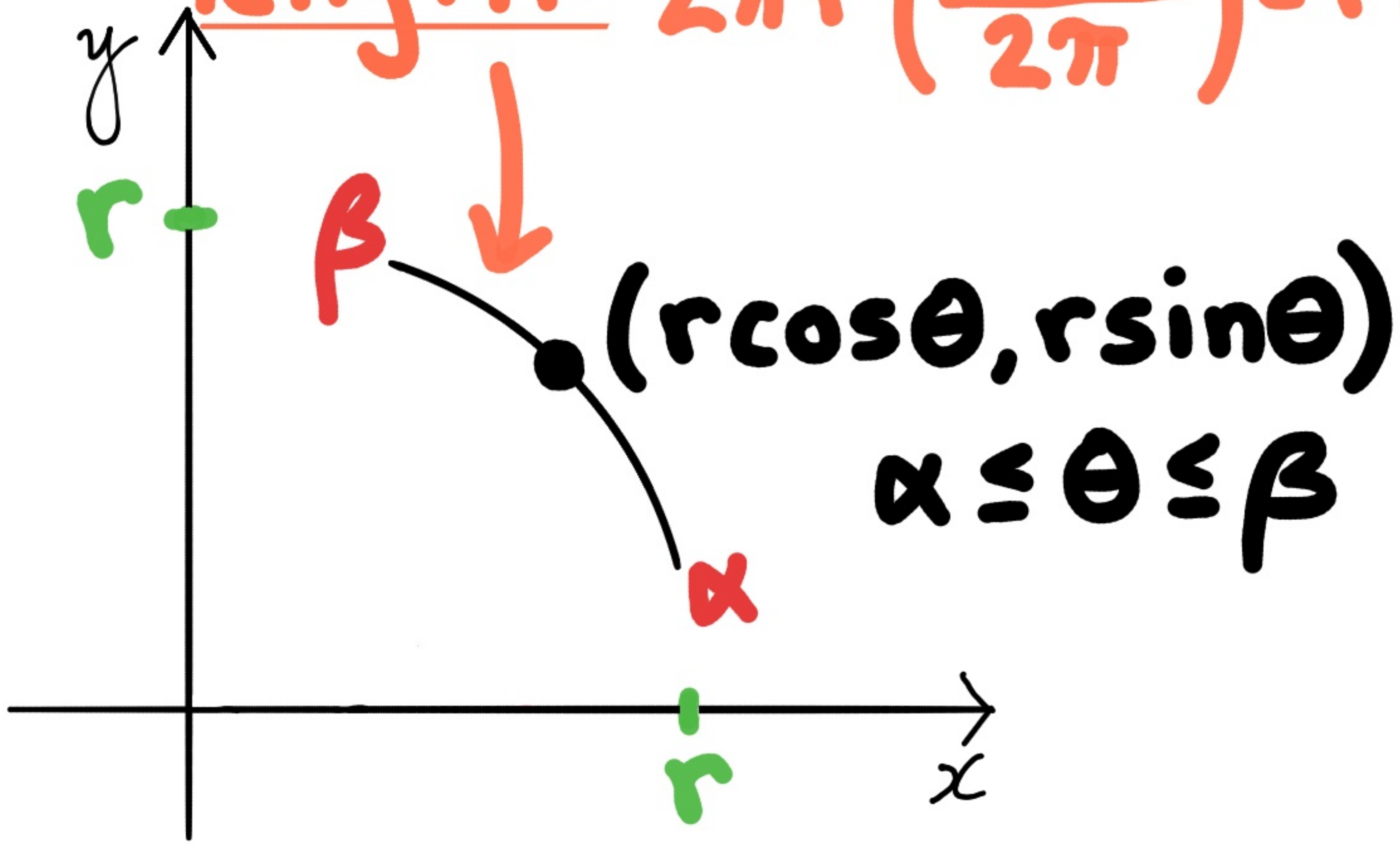


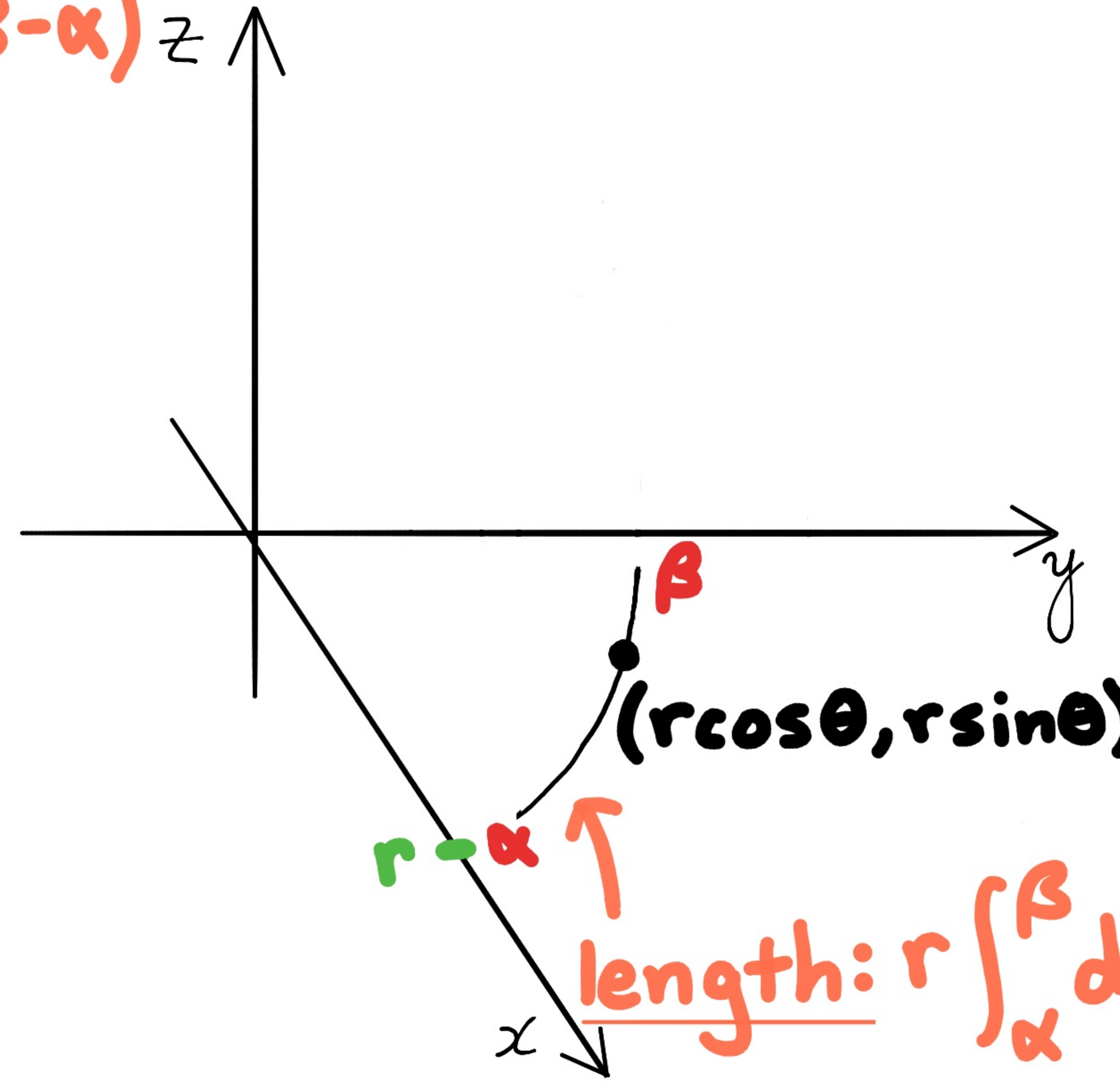


$$(\beta-\alpha)=\int_{\alpha}^{\beta} d\theta$$



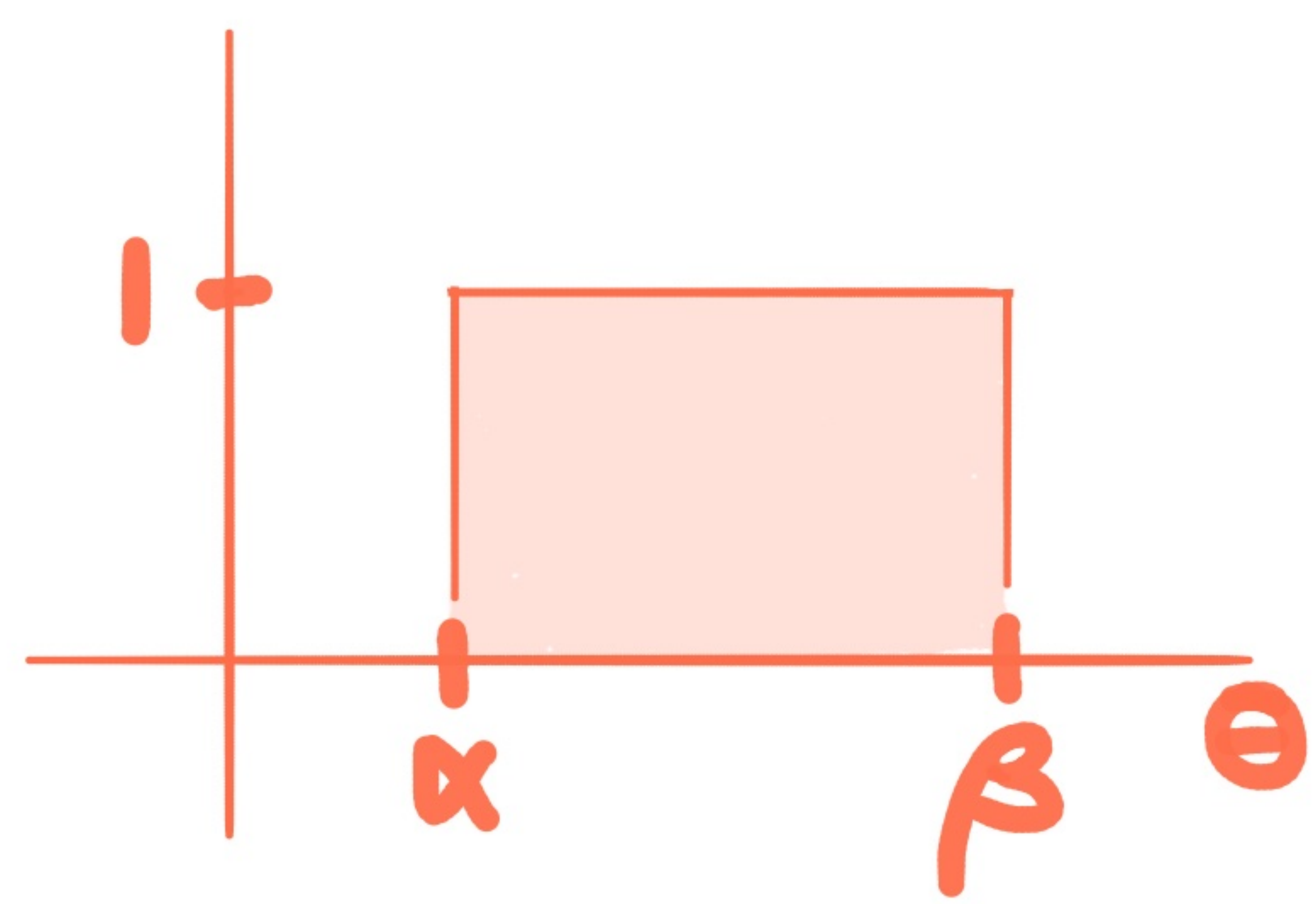
length: $2\pi r\left(\frac{\beta-\alpha}{2\pi}\right)=r(\beta-\alpha)$

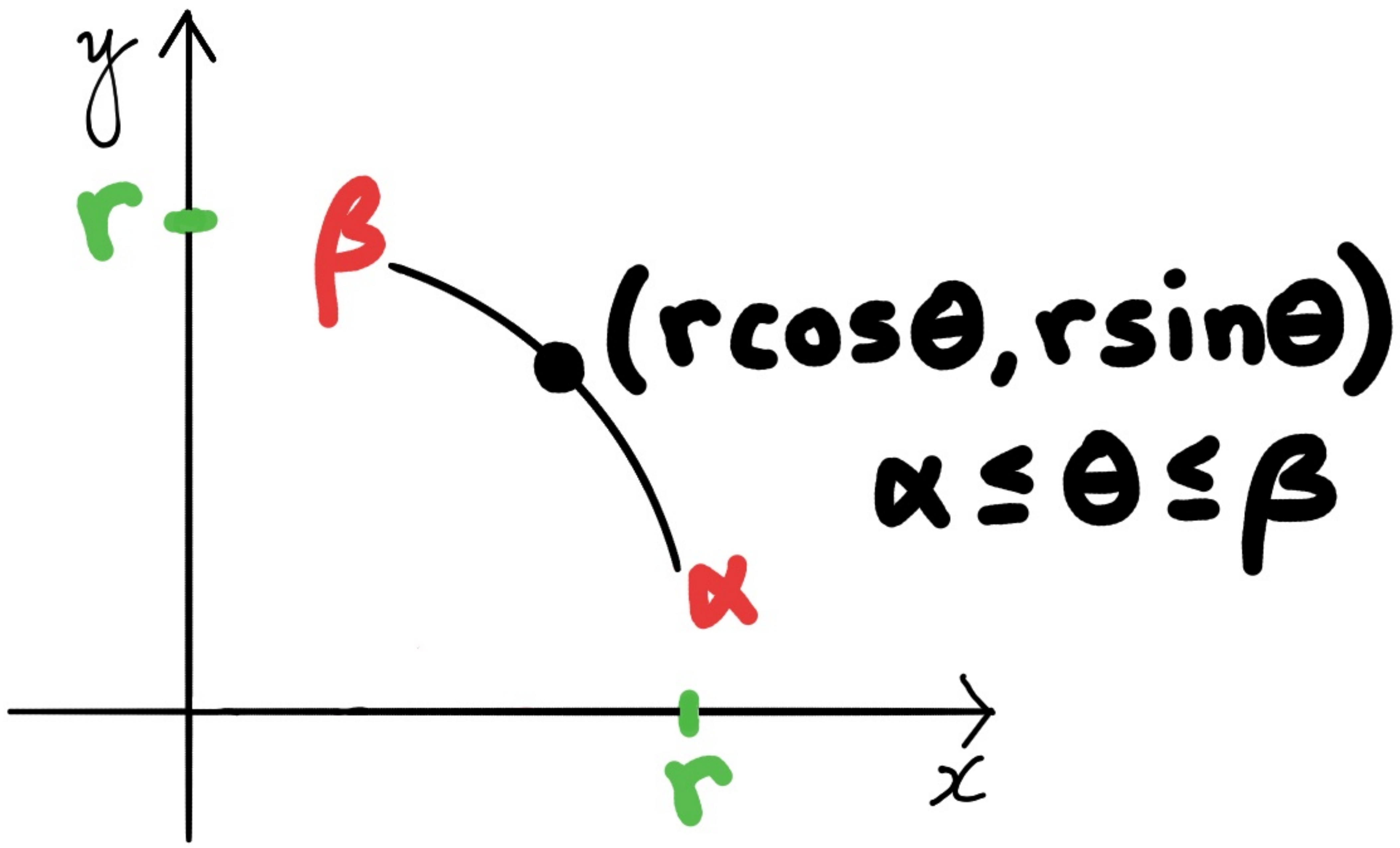


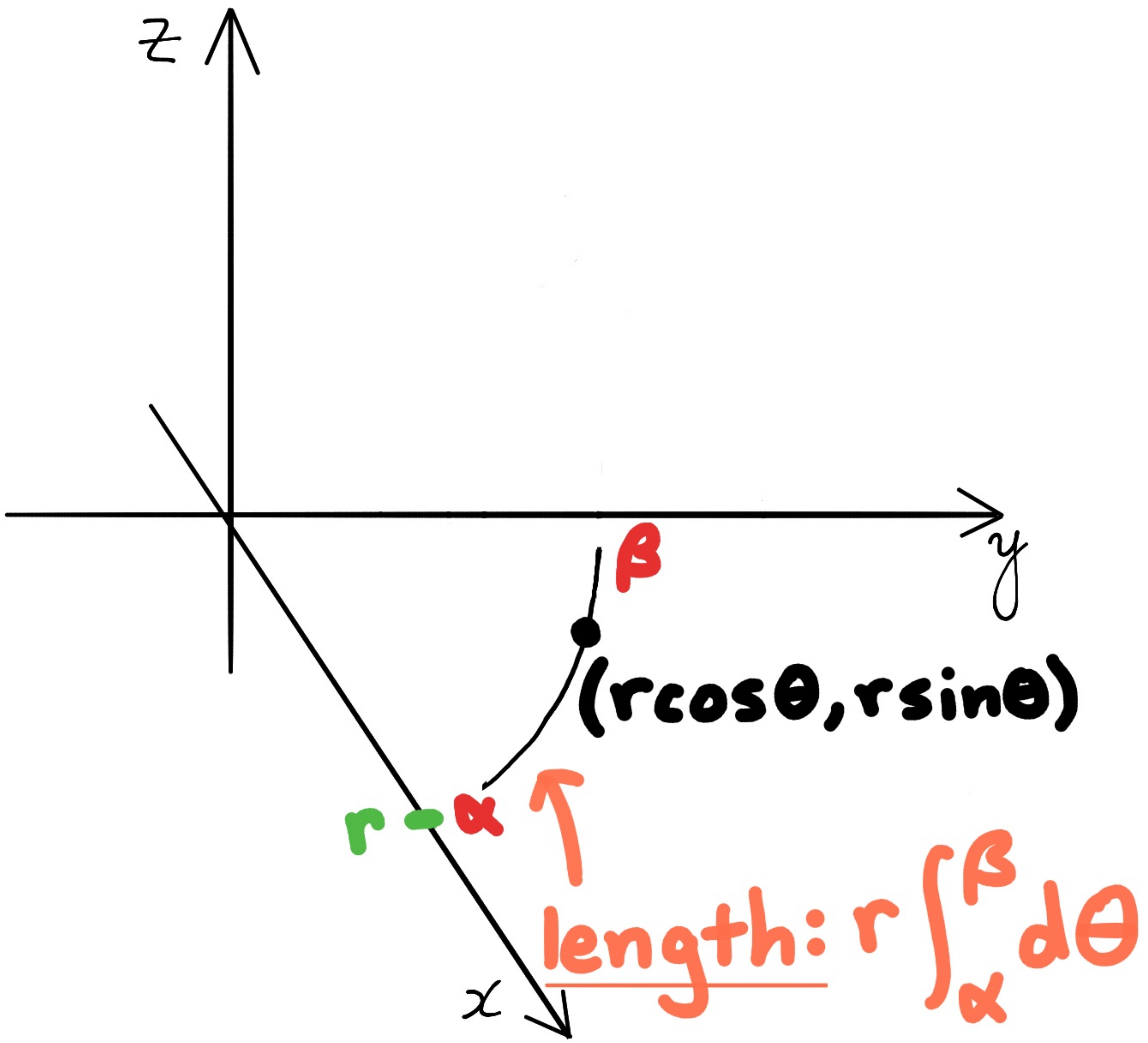


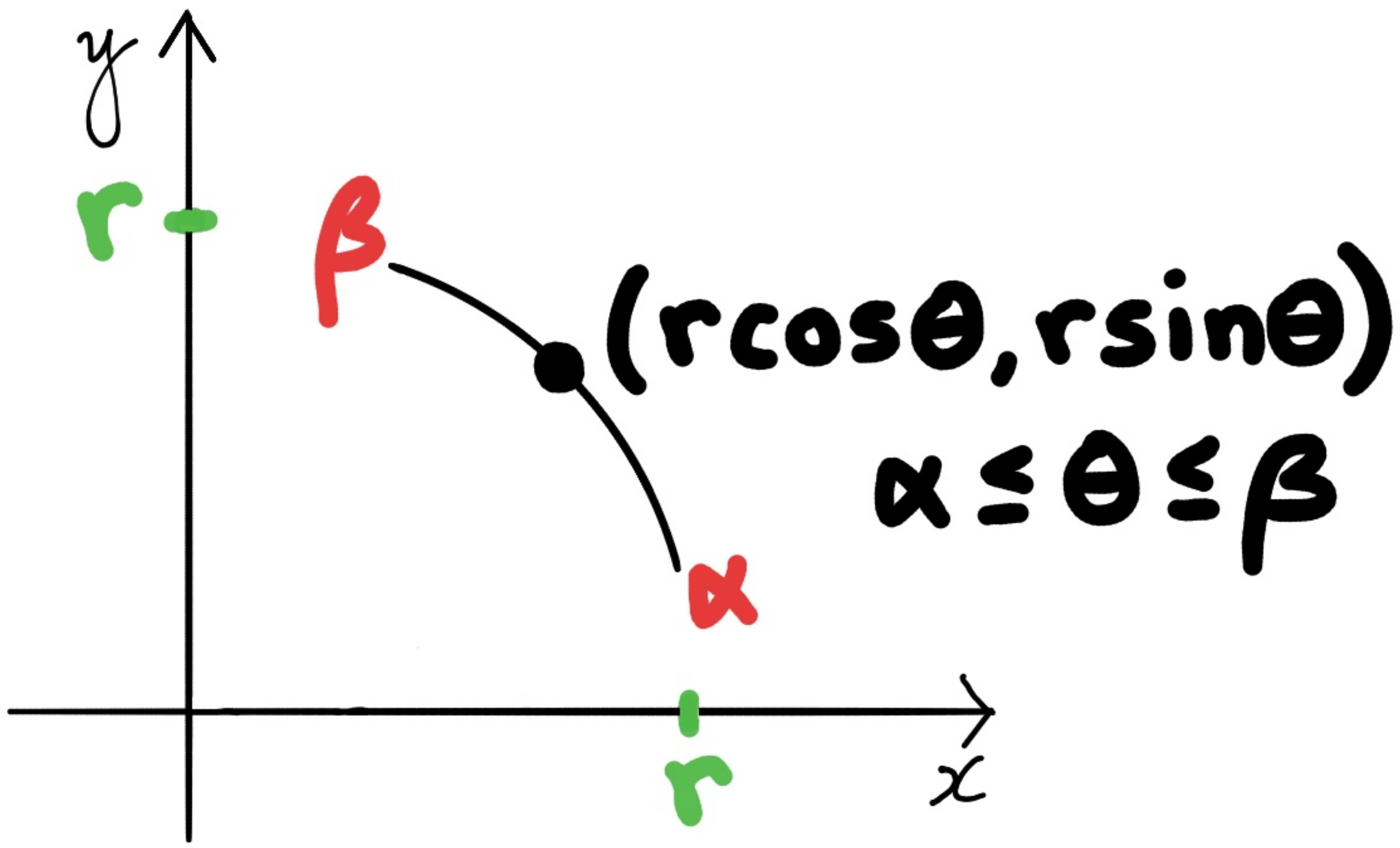
length: $r\int_{\alpha}^{\beta} d\theta$

$$(\beta-\alpha)=\int_{\alpha}^{\beta} d\theta$$

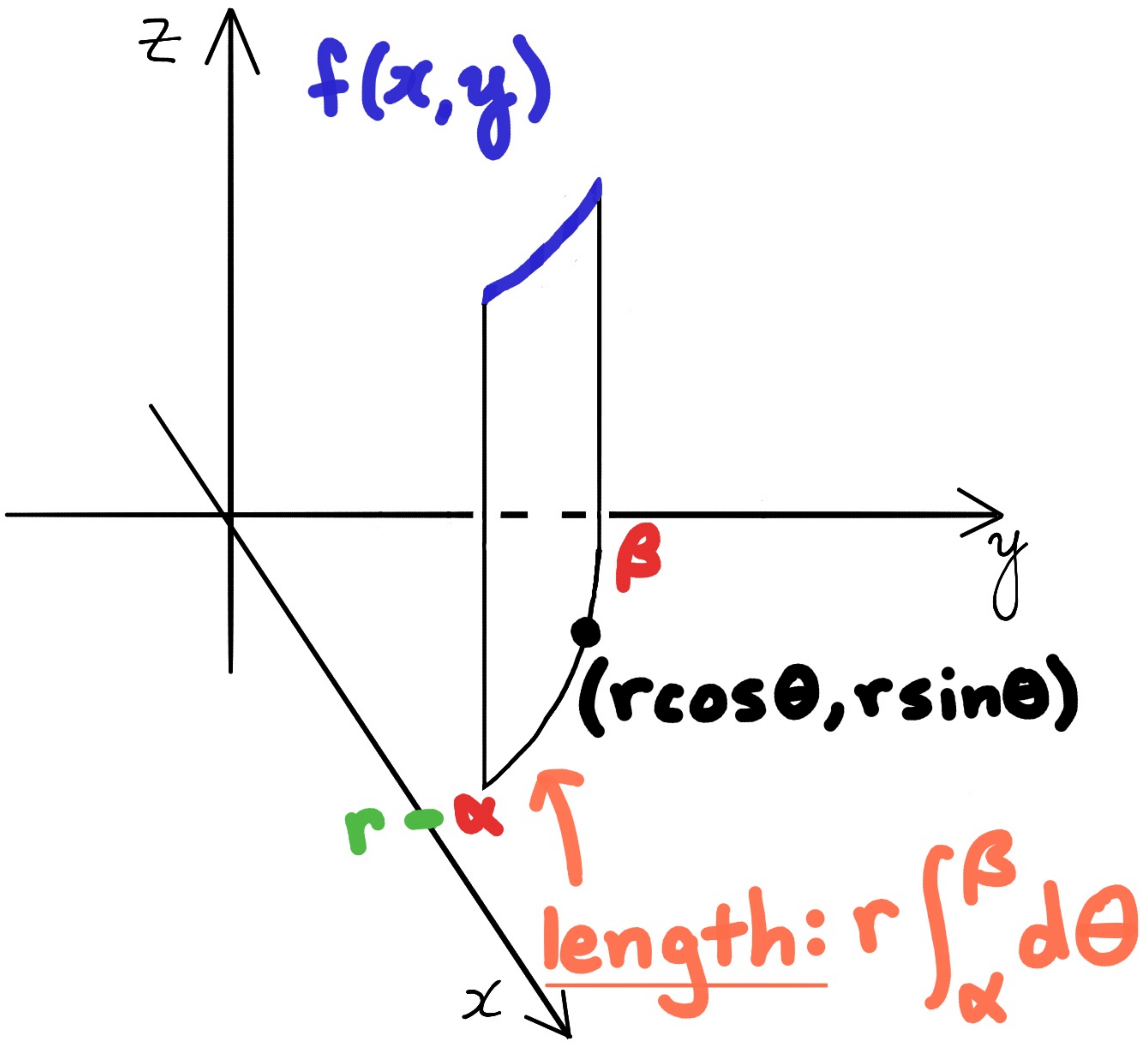



y
r
β
(rcosθ, rsinθ)
α ≤ θ ≤ β
α
r
x

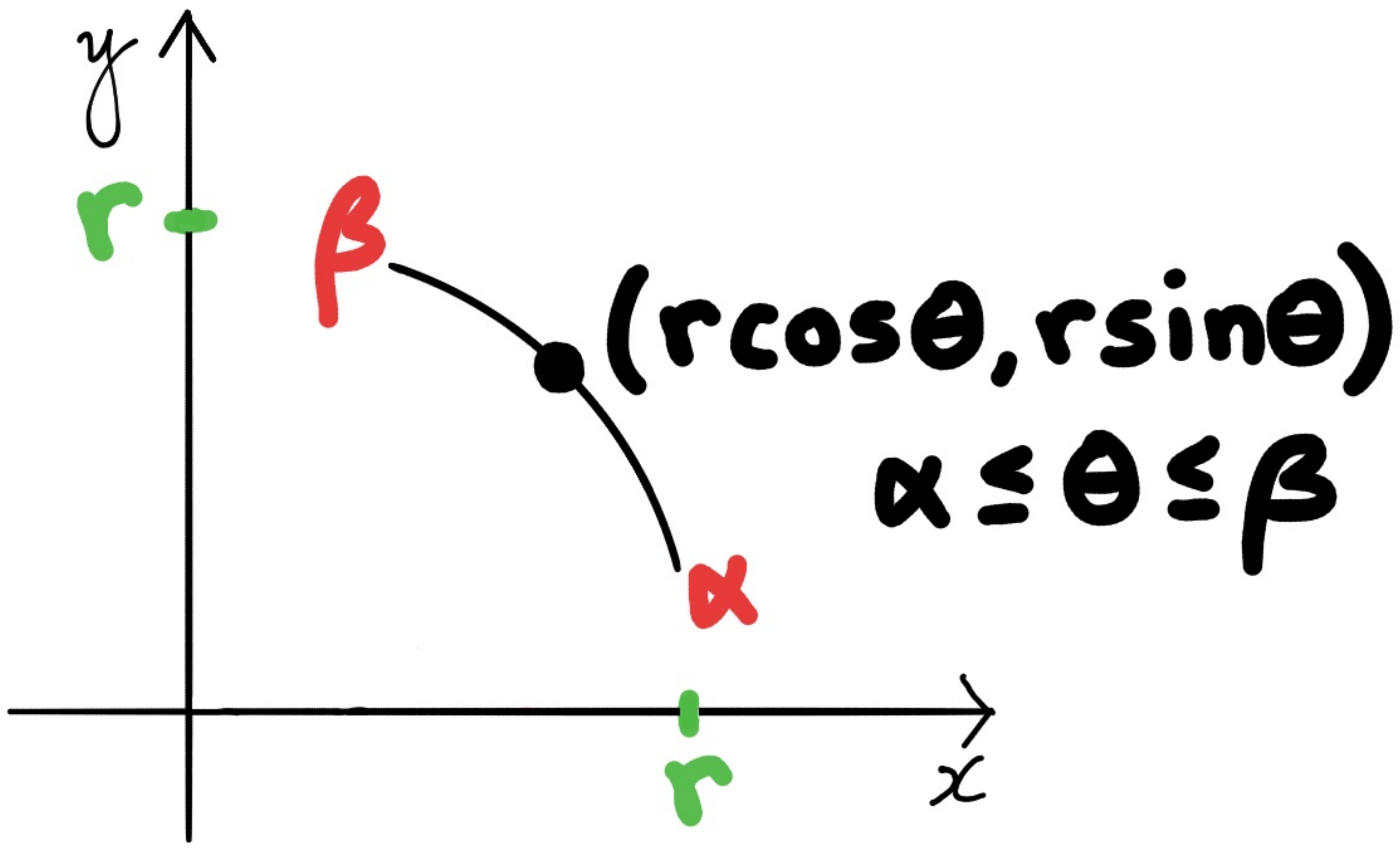

z
y
β
(rcosθ, rsinθ)
r
α
length: $r\int_\alpha^\beta d\theta$
x

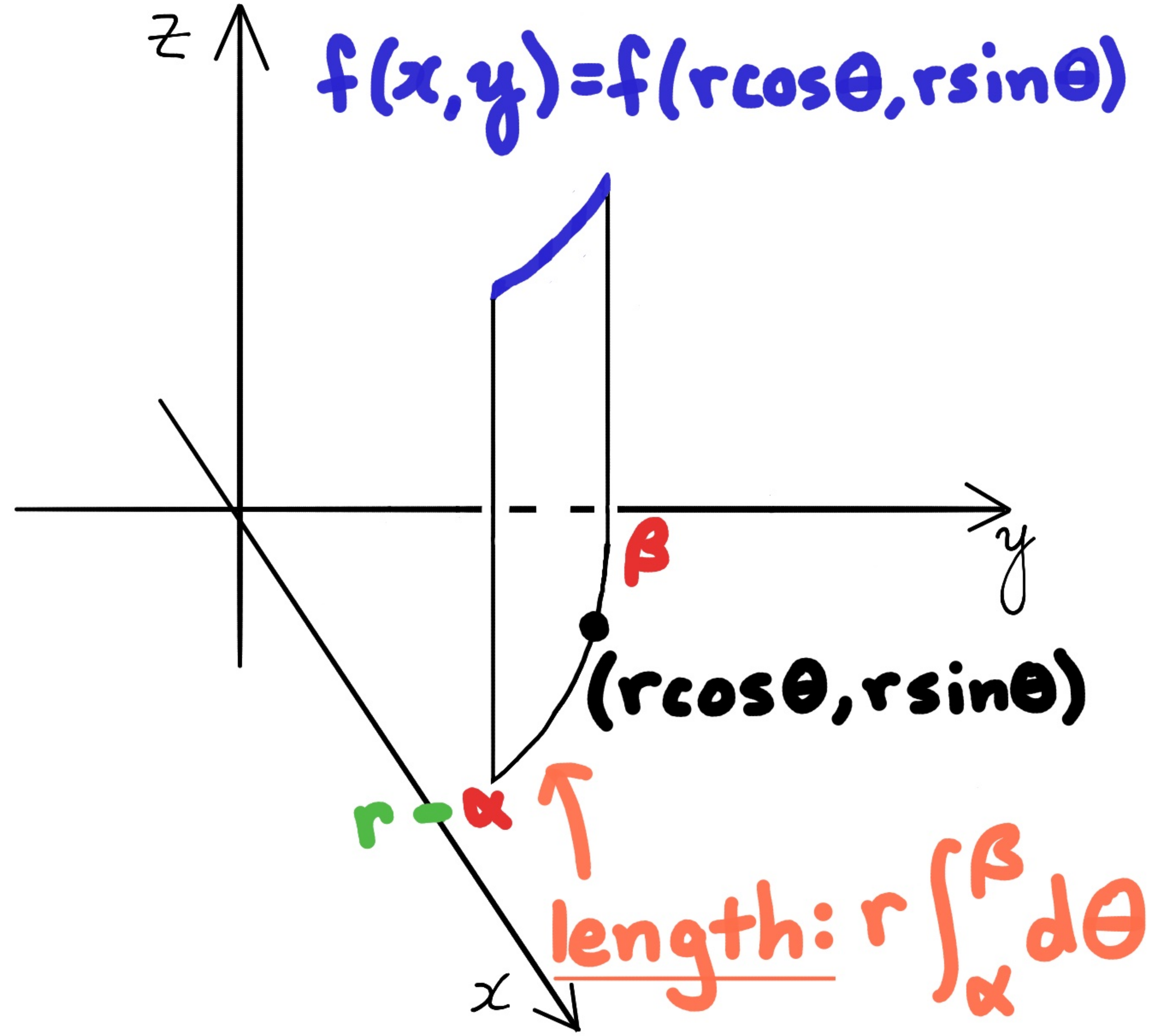


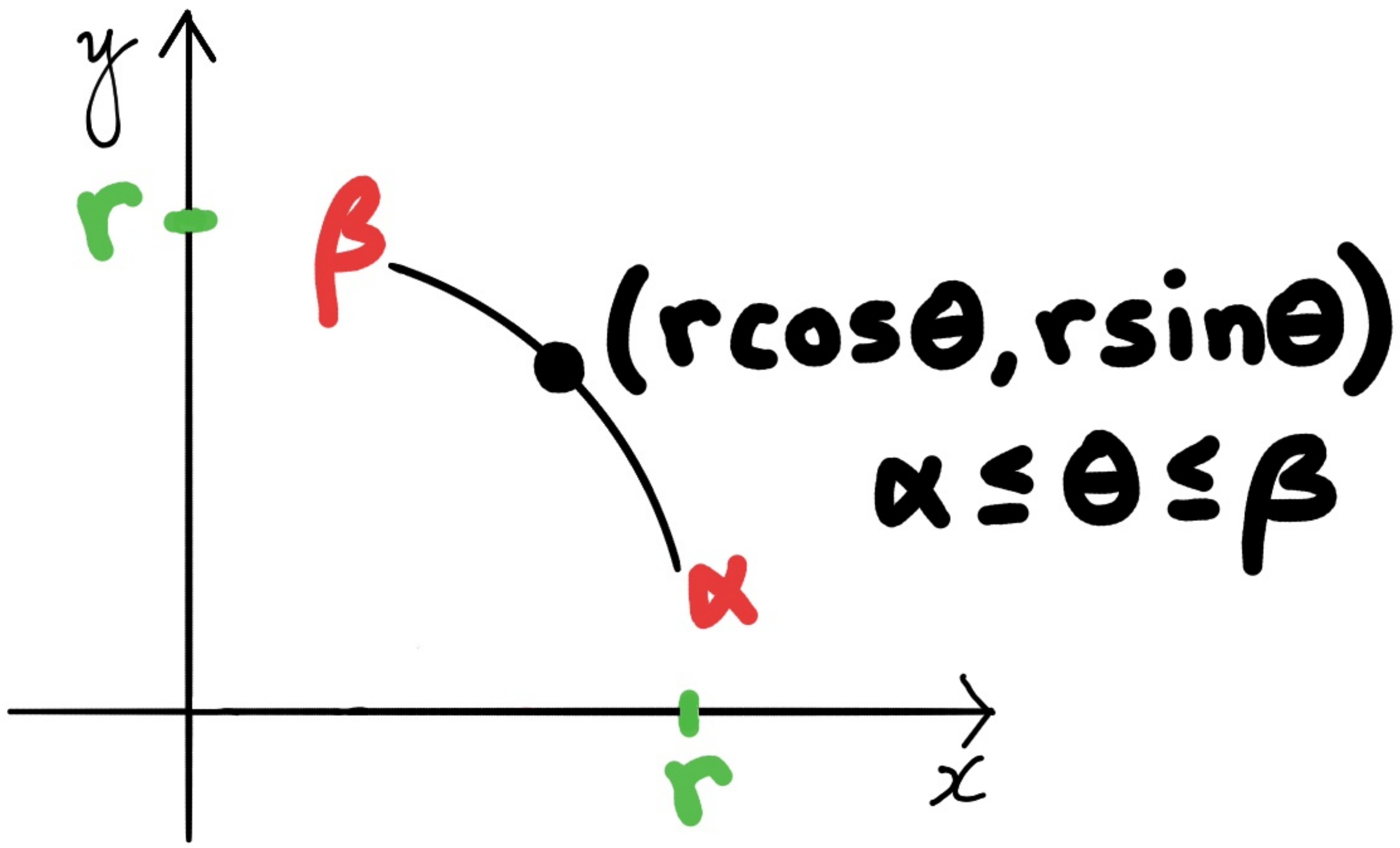
y
r
β
(rcosθ, rsinθ)
α ≤ θ ≤ β
α
r
x

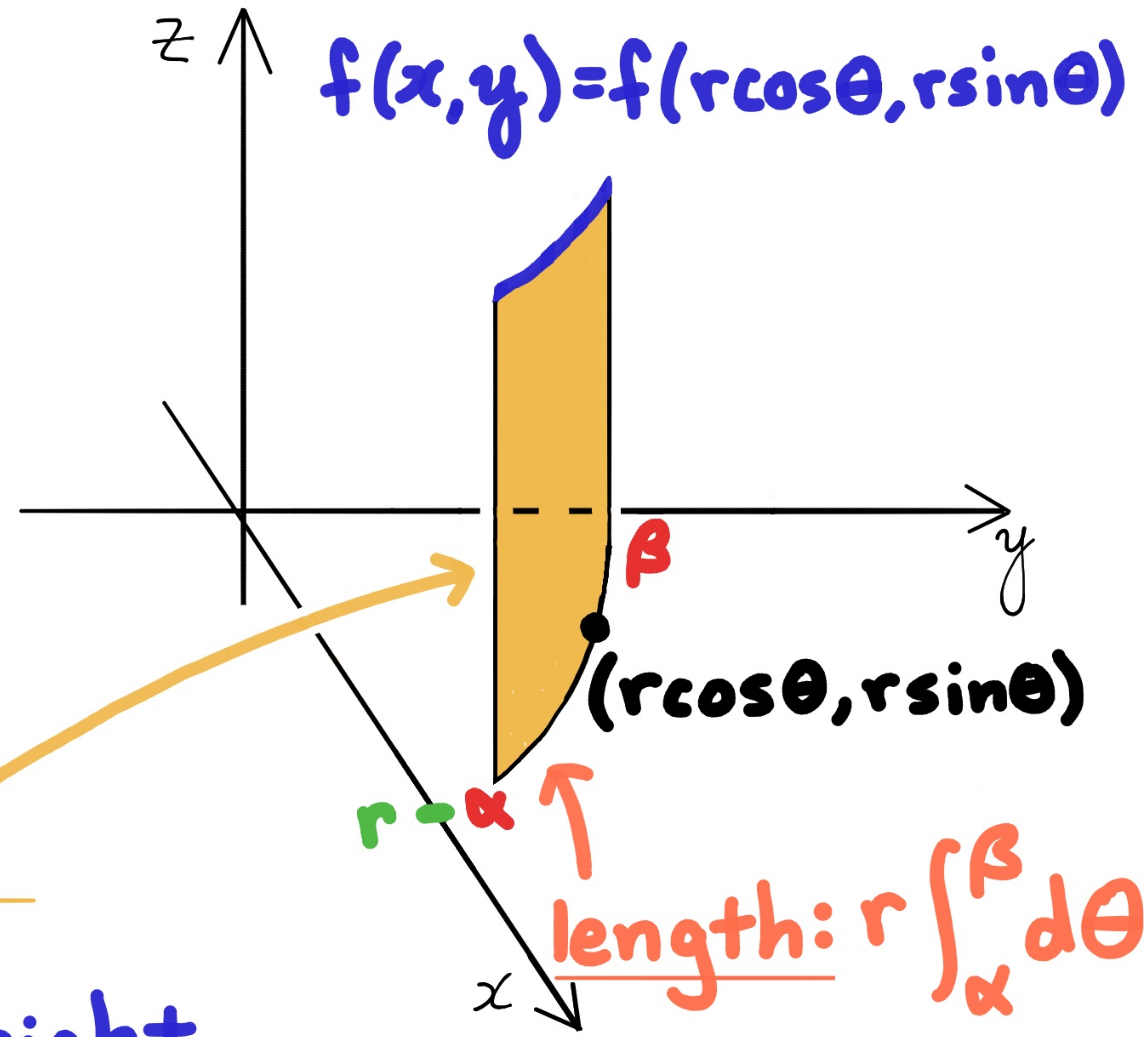


z
f(x,y)
β
y
(rcosθ, rsinθ)
r
α
length: $r\int_\alpha^\beta d\theta$
x


y
r
β
(rcosθ,rsinθ)
α≤θ≤β
α
r
x

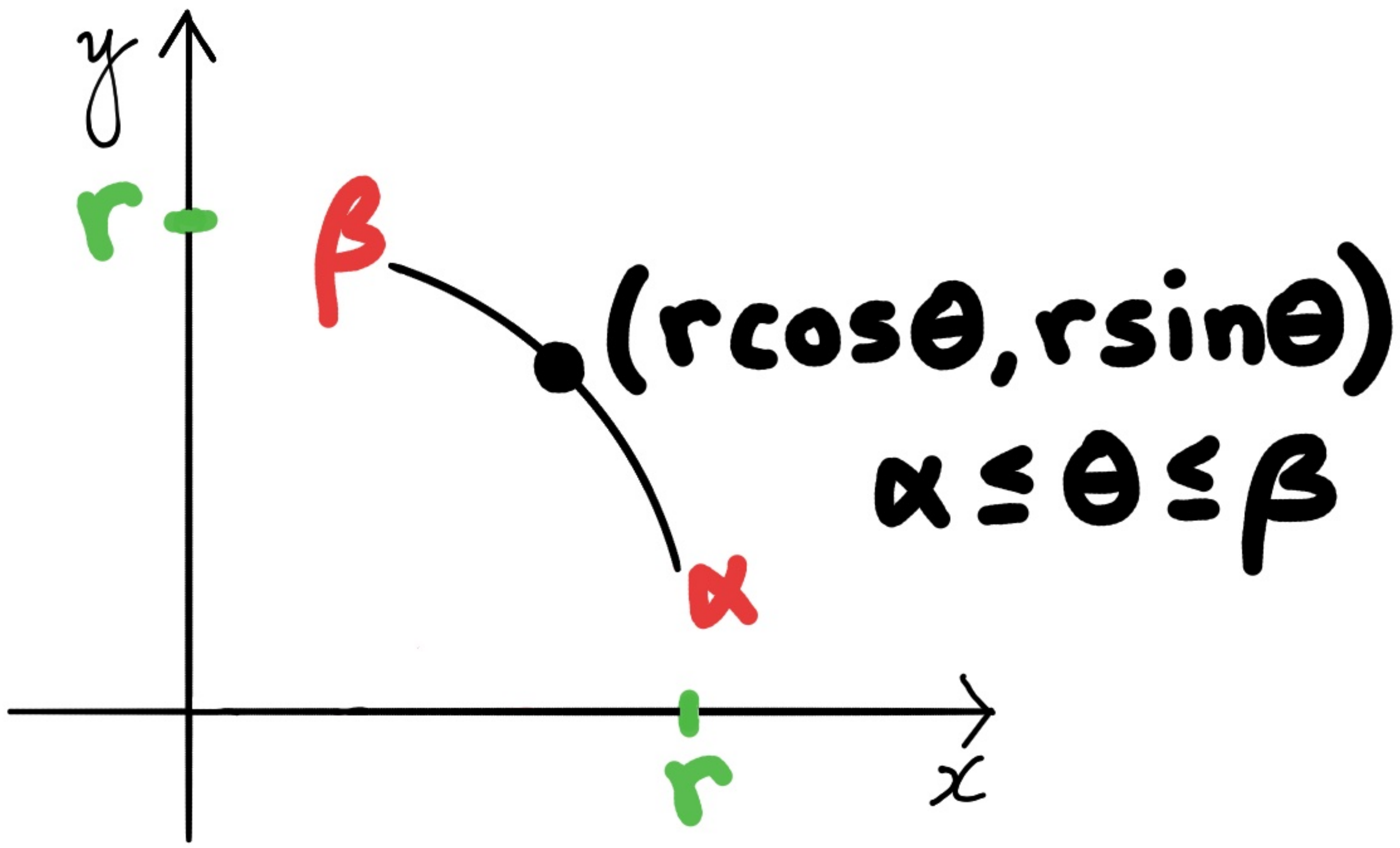

z
f(x,y)=f(rcosθ,rsinθ)
y
β
(rcosθ,rsinθ)
r
α
length: $r\int_{\alpha}^{\beta} d\theta$
x

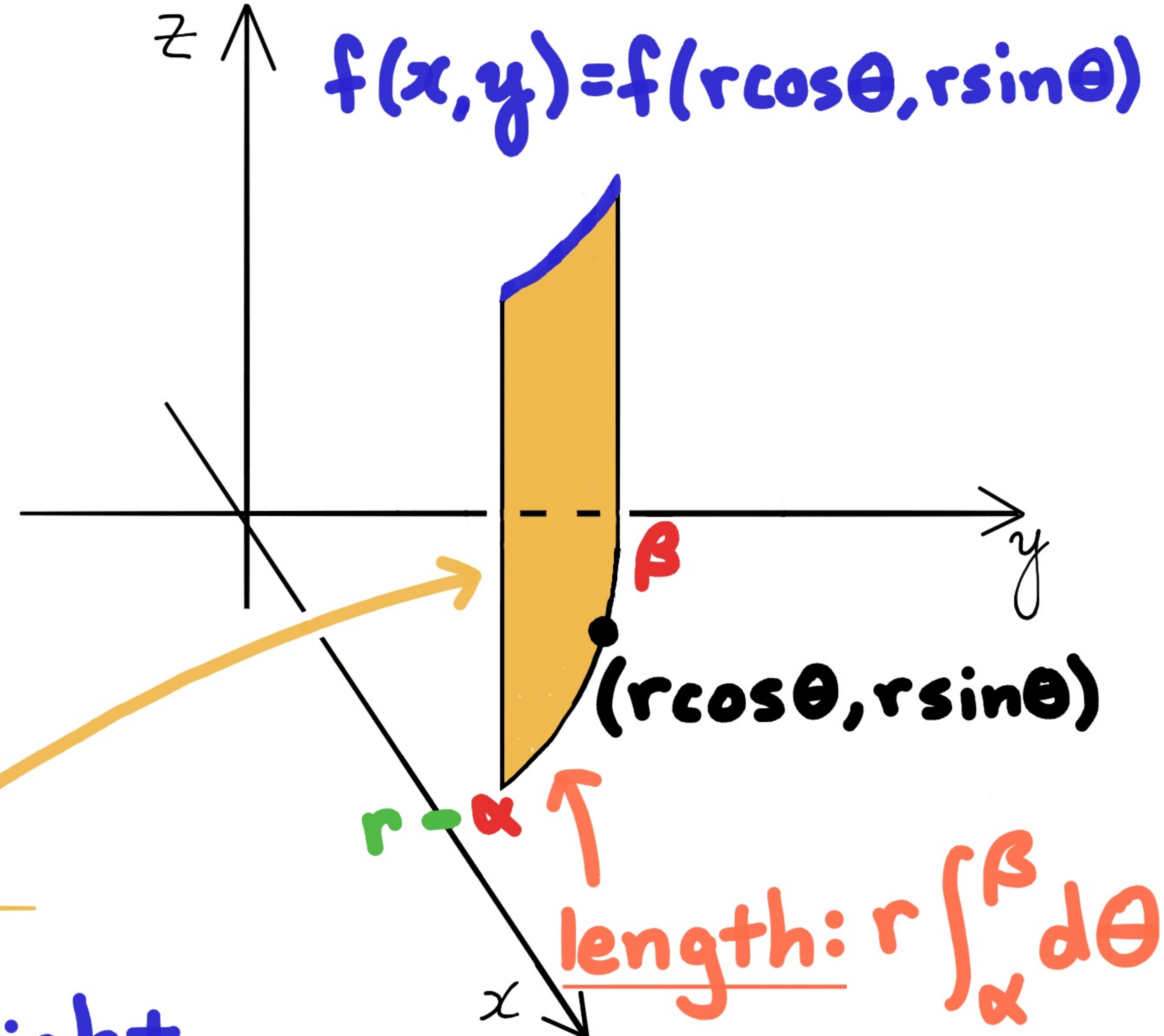




Area:

length × height



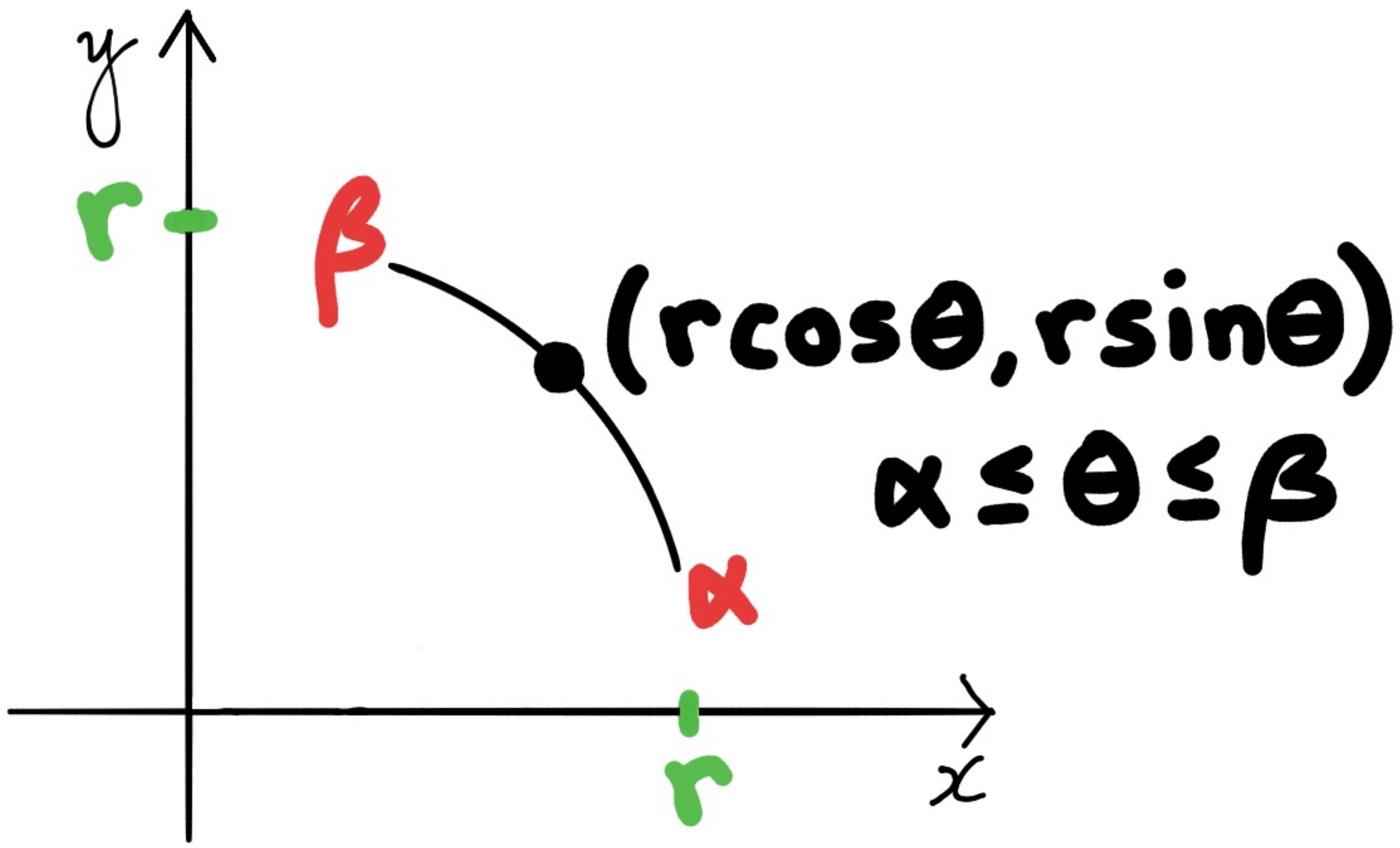


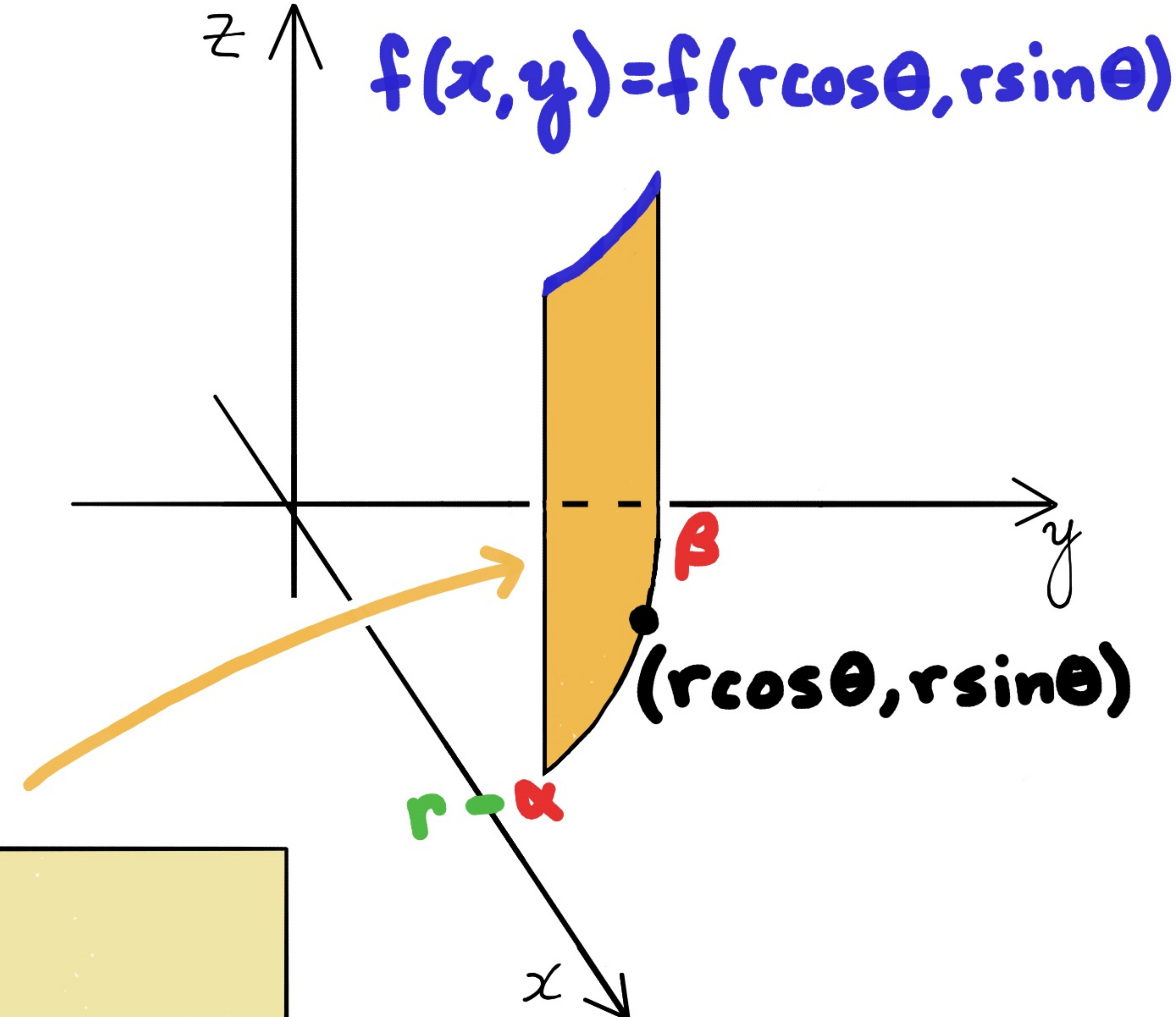
Area:

varying height

$$r\int_{\alpha}^{\beta} f(r\cos\theta, r\sin\theta)\, d\theta$$

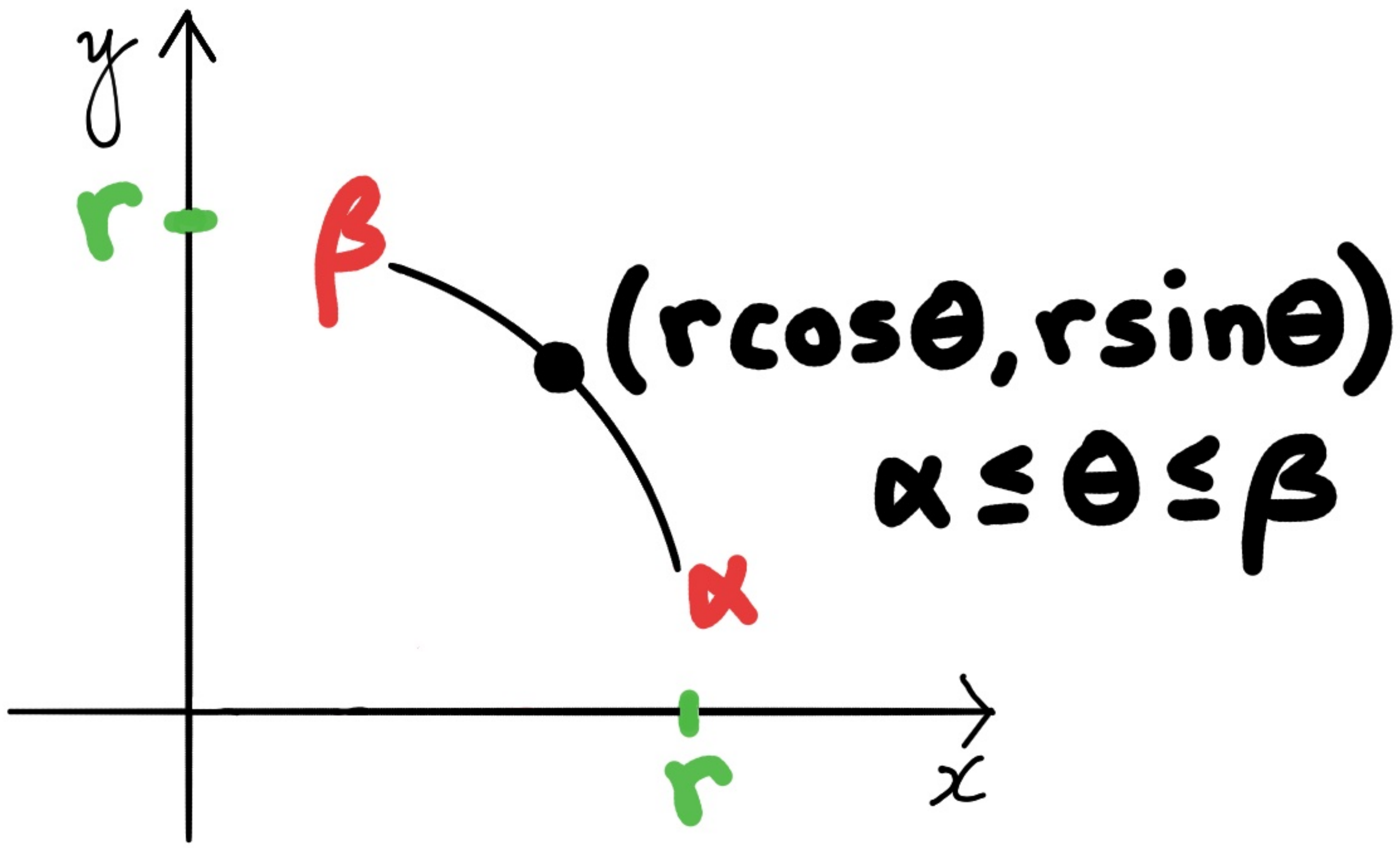
length

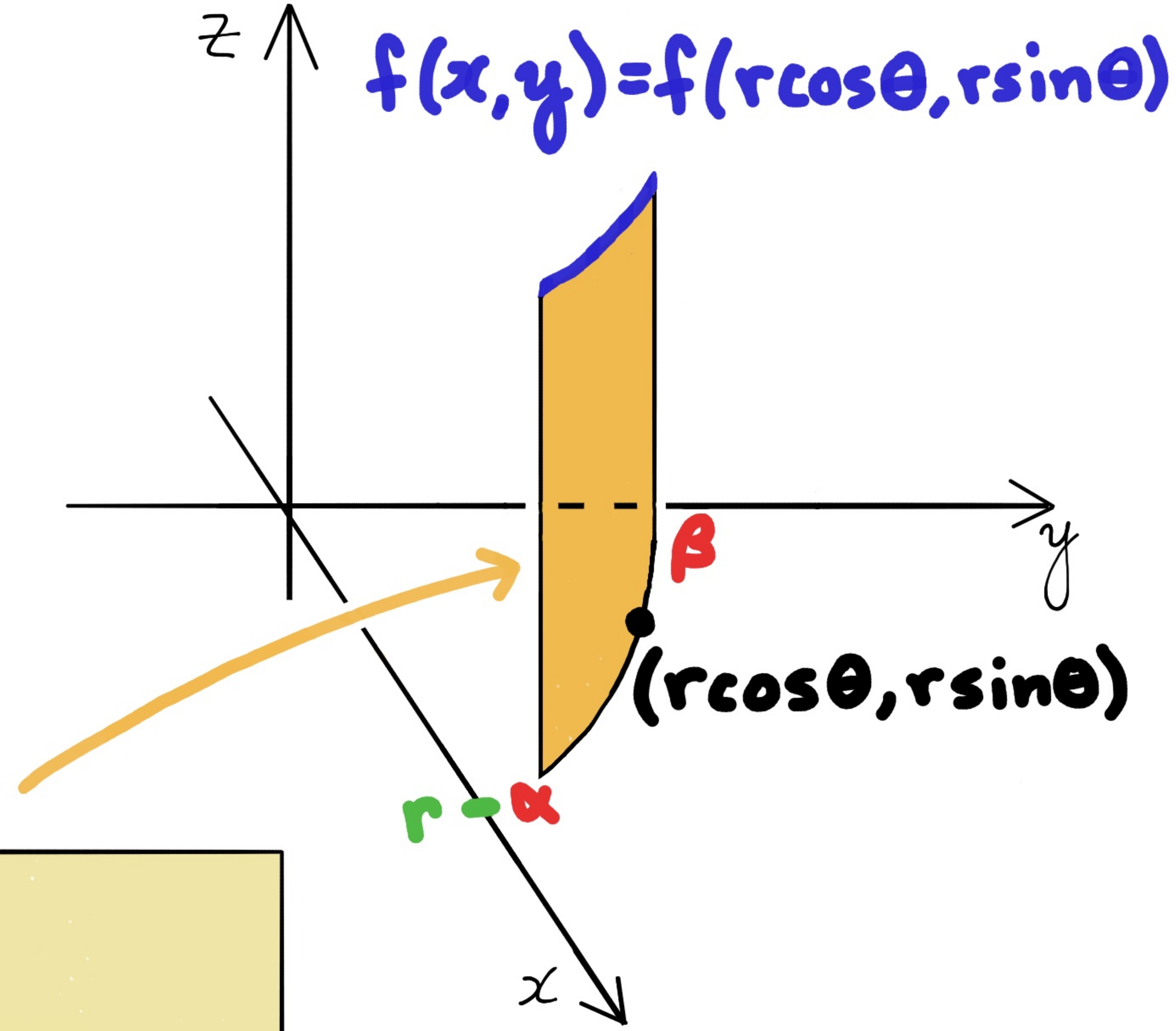




Area:

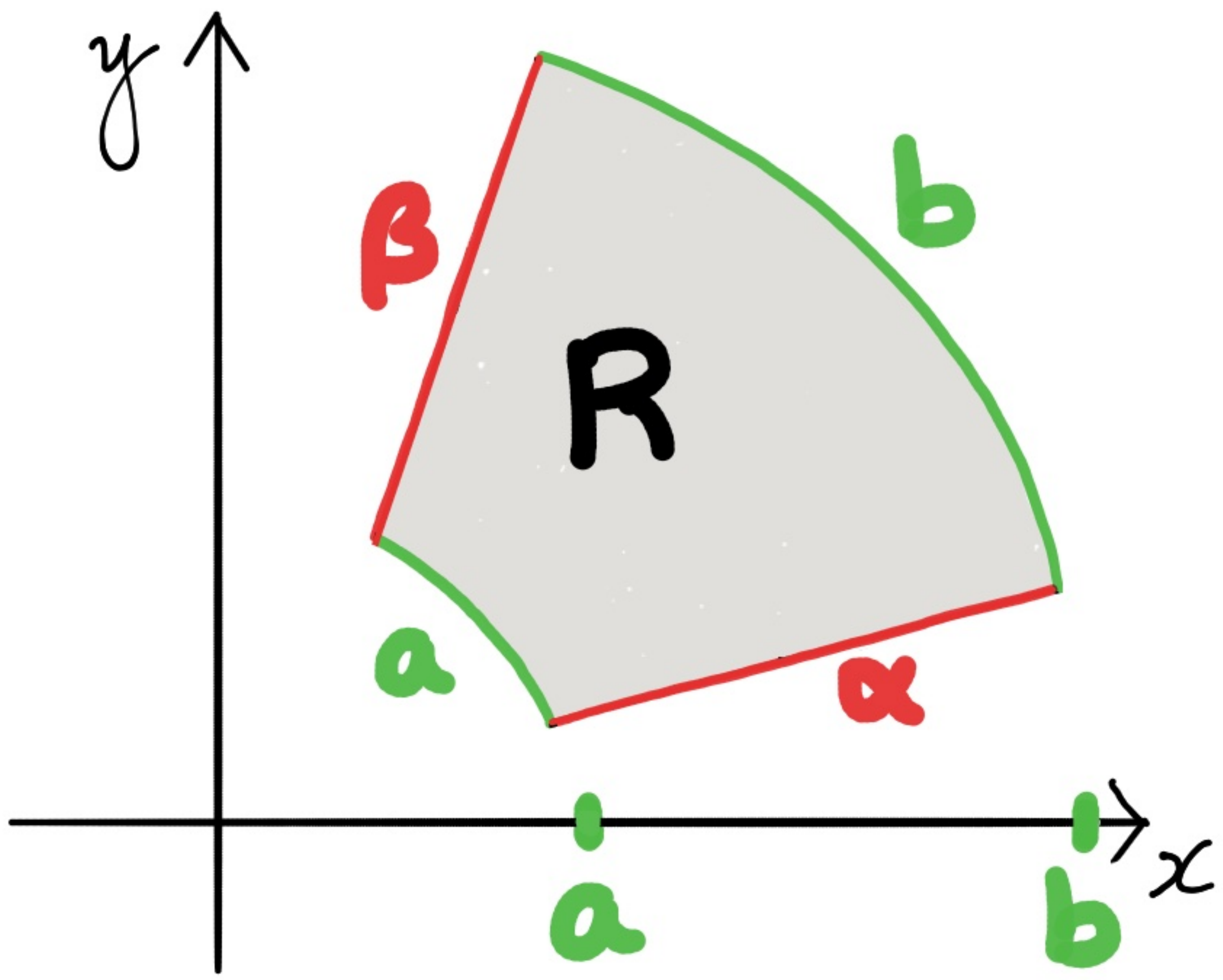
$$r\int_{\alpha}^{\beta} f(r\cos\theta, r\sin\theta)\, d\theta$$



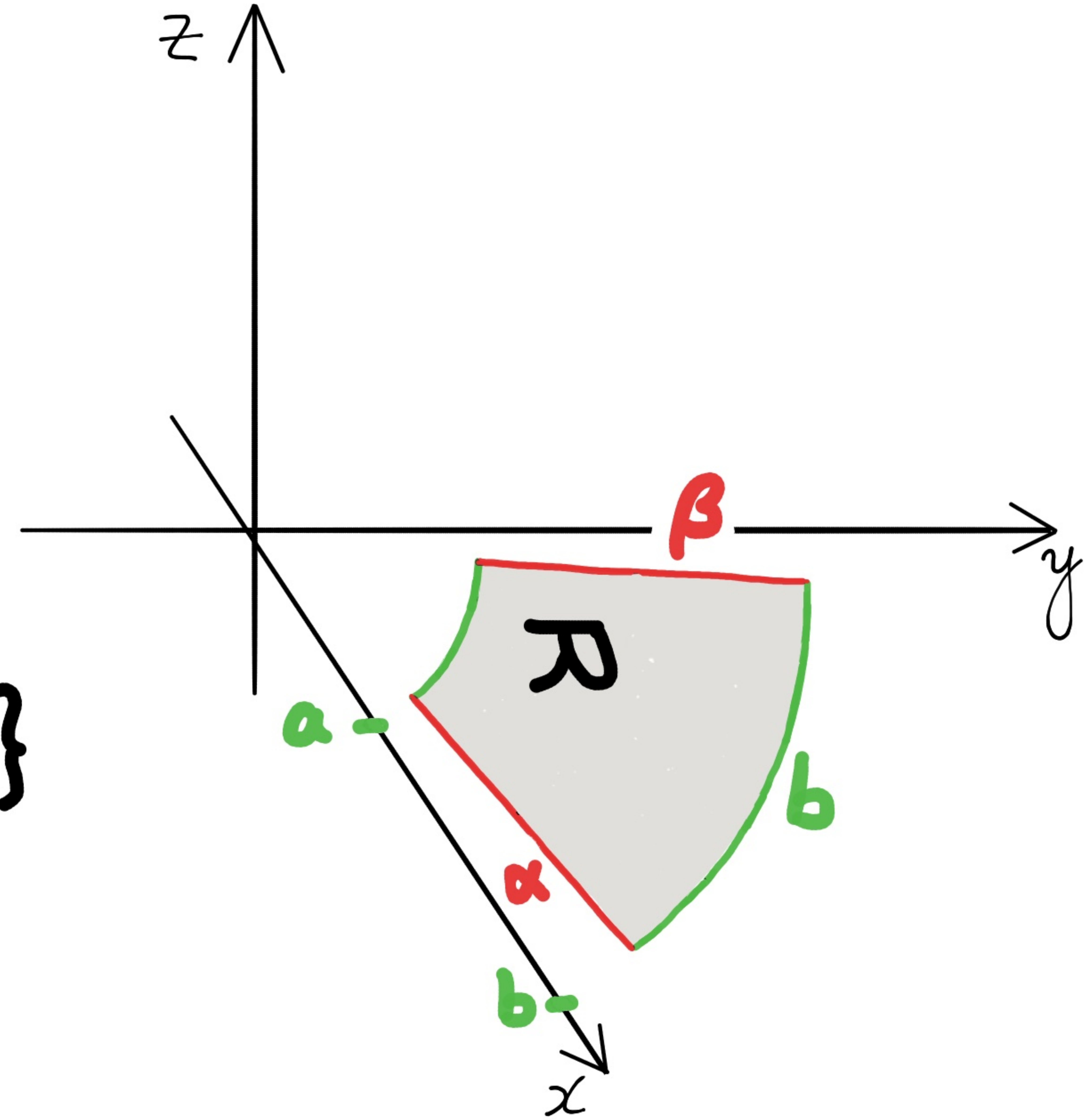


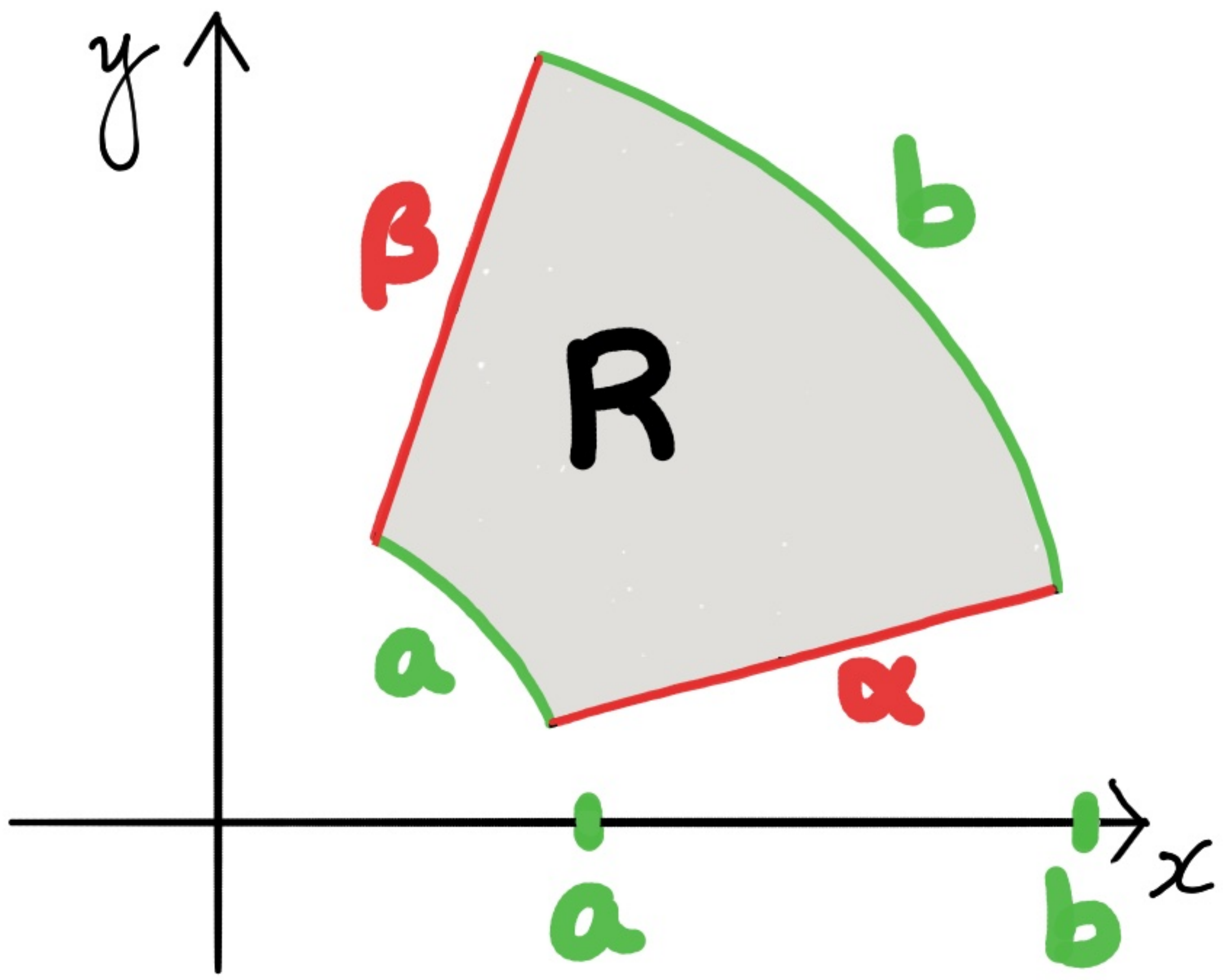
Area:

$$\int_{\alpha}^{\beta} f(r\cos\theta, r\sin\theta)\, r\, d\theta$$

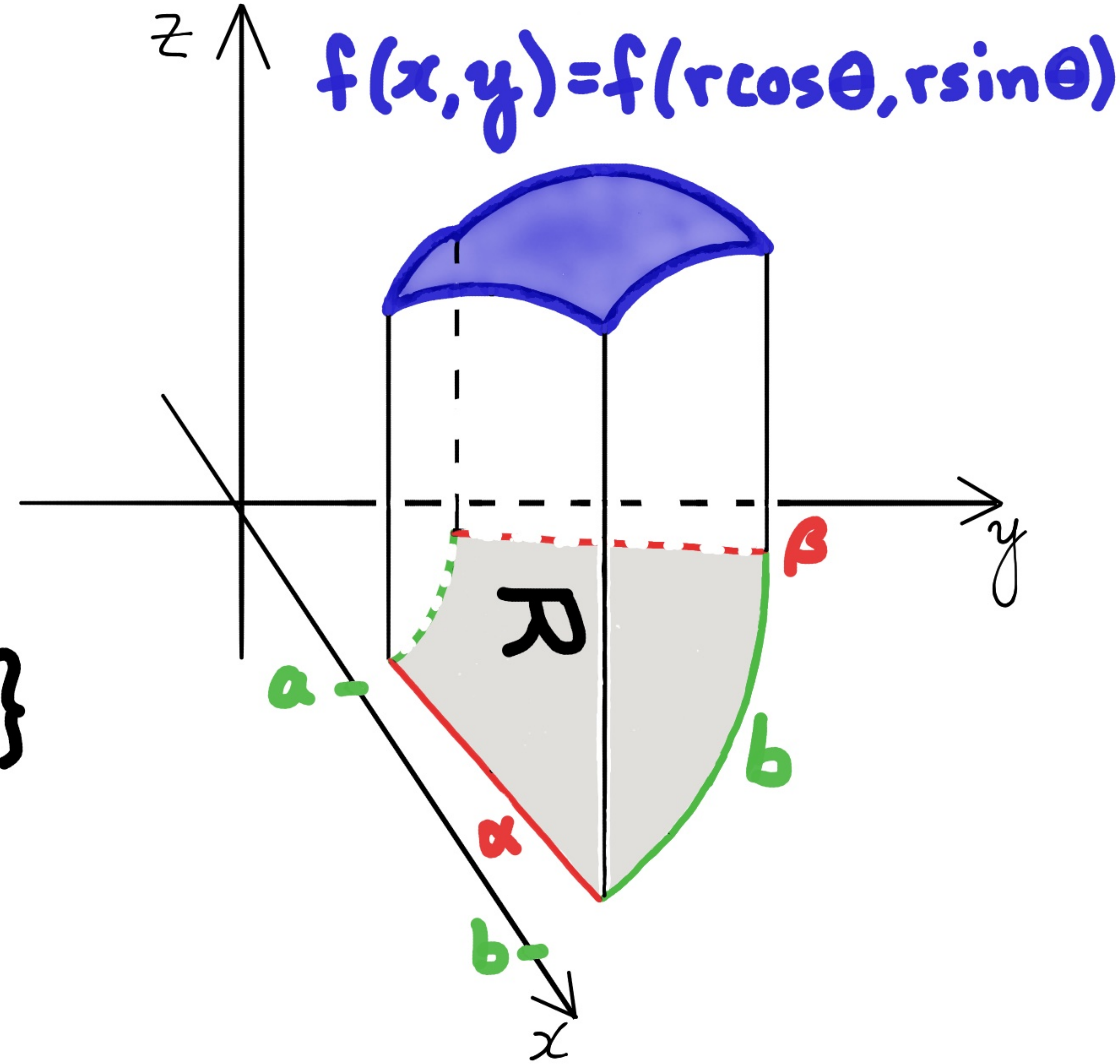


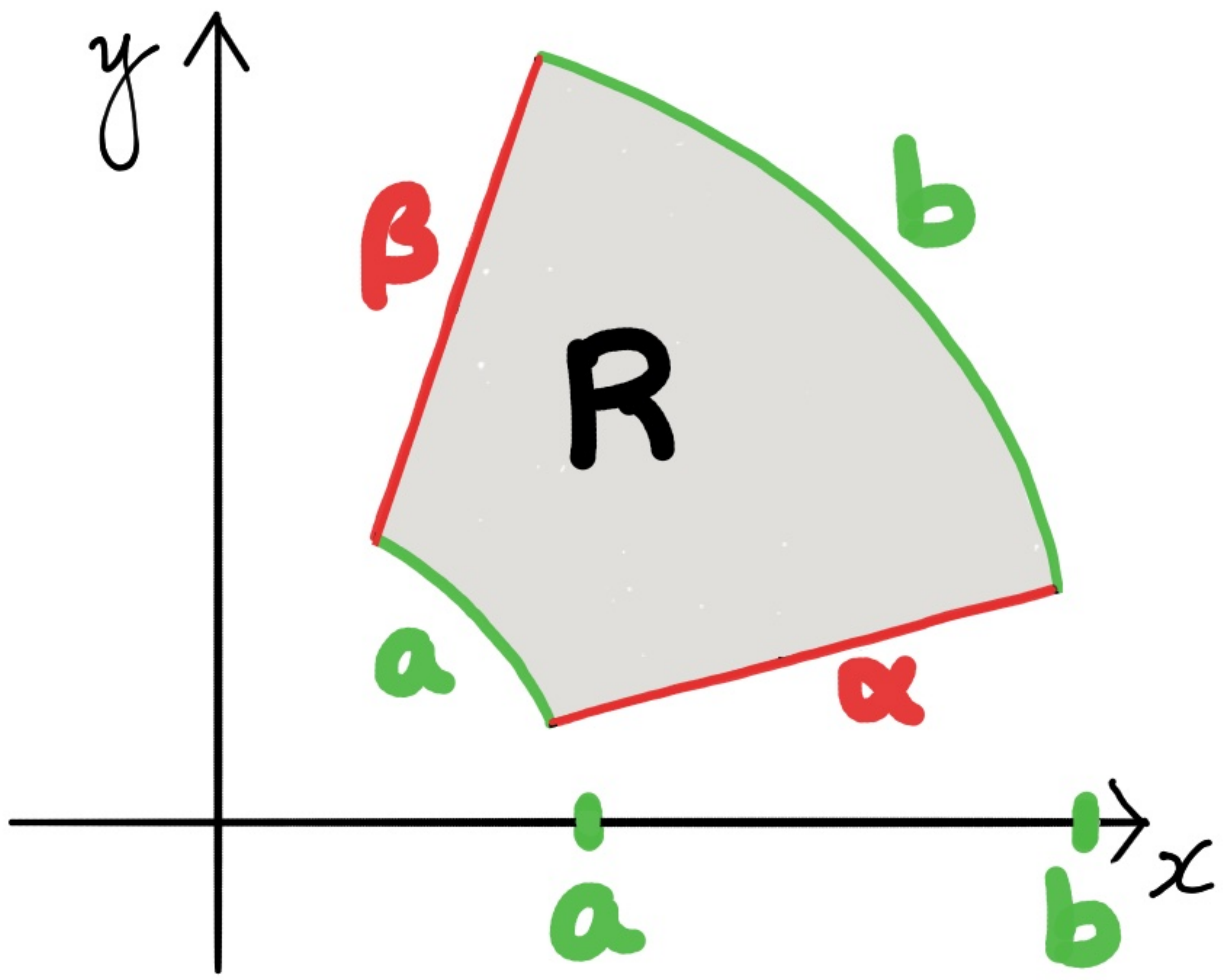
$$R = \{(r, \theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

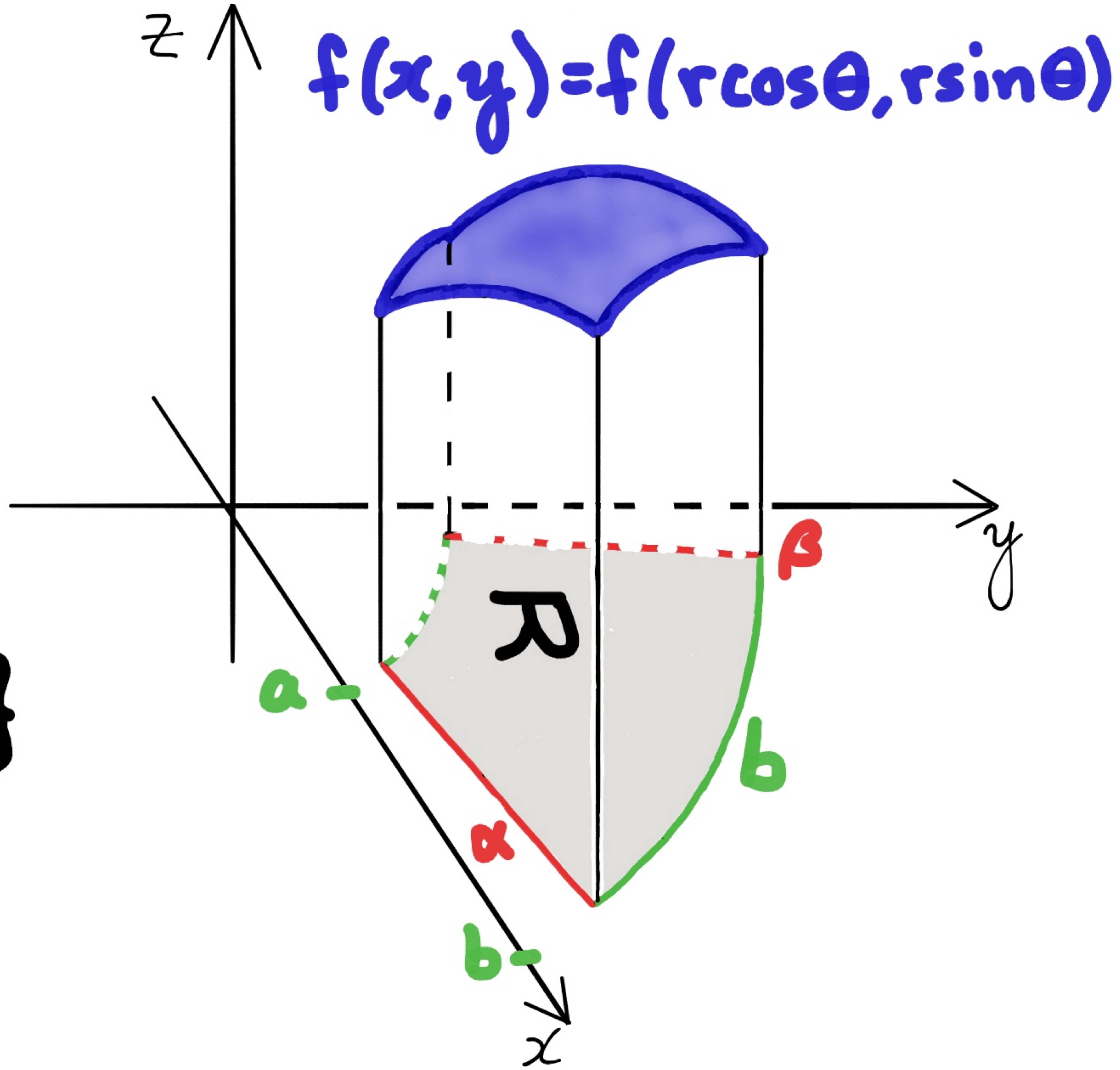




$$R=\{(r,\theta): a \le r \le b,\ \alpha \le \theta \le \beta\}$$

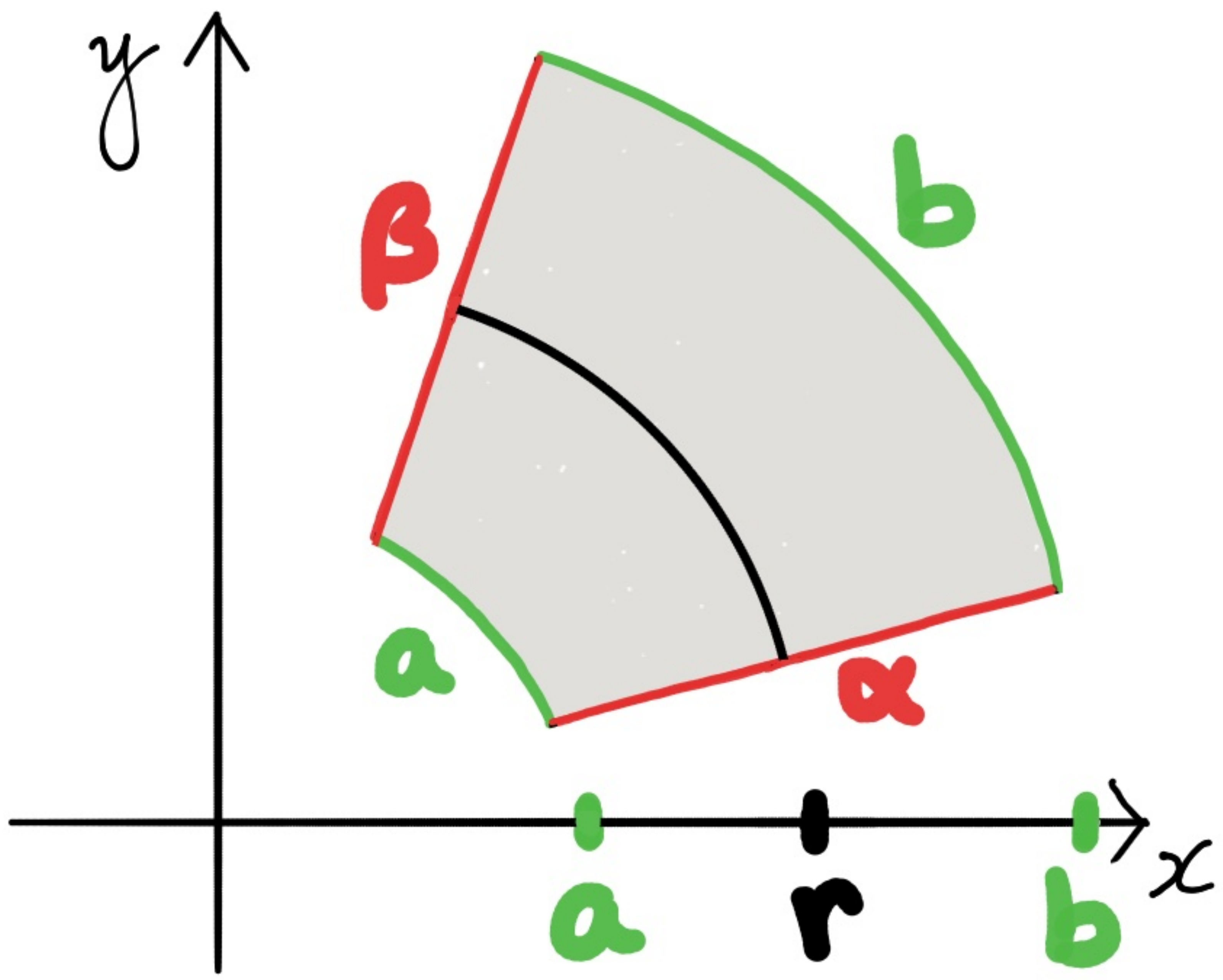


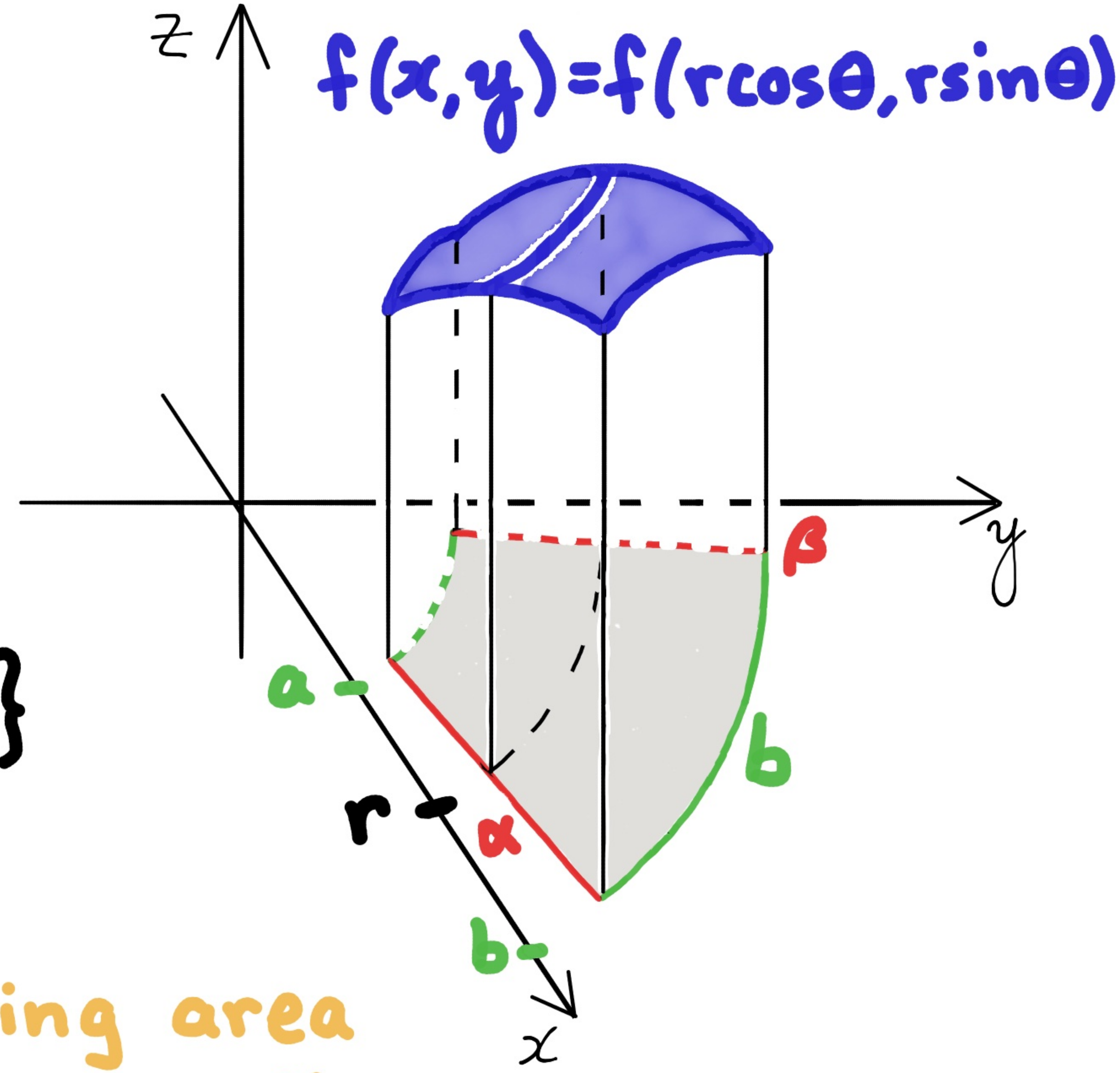




$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

$$\iint_R f(x,y)\, dA$$



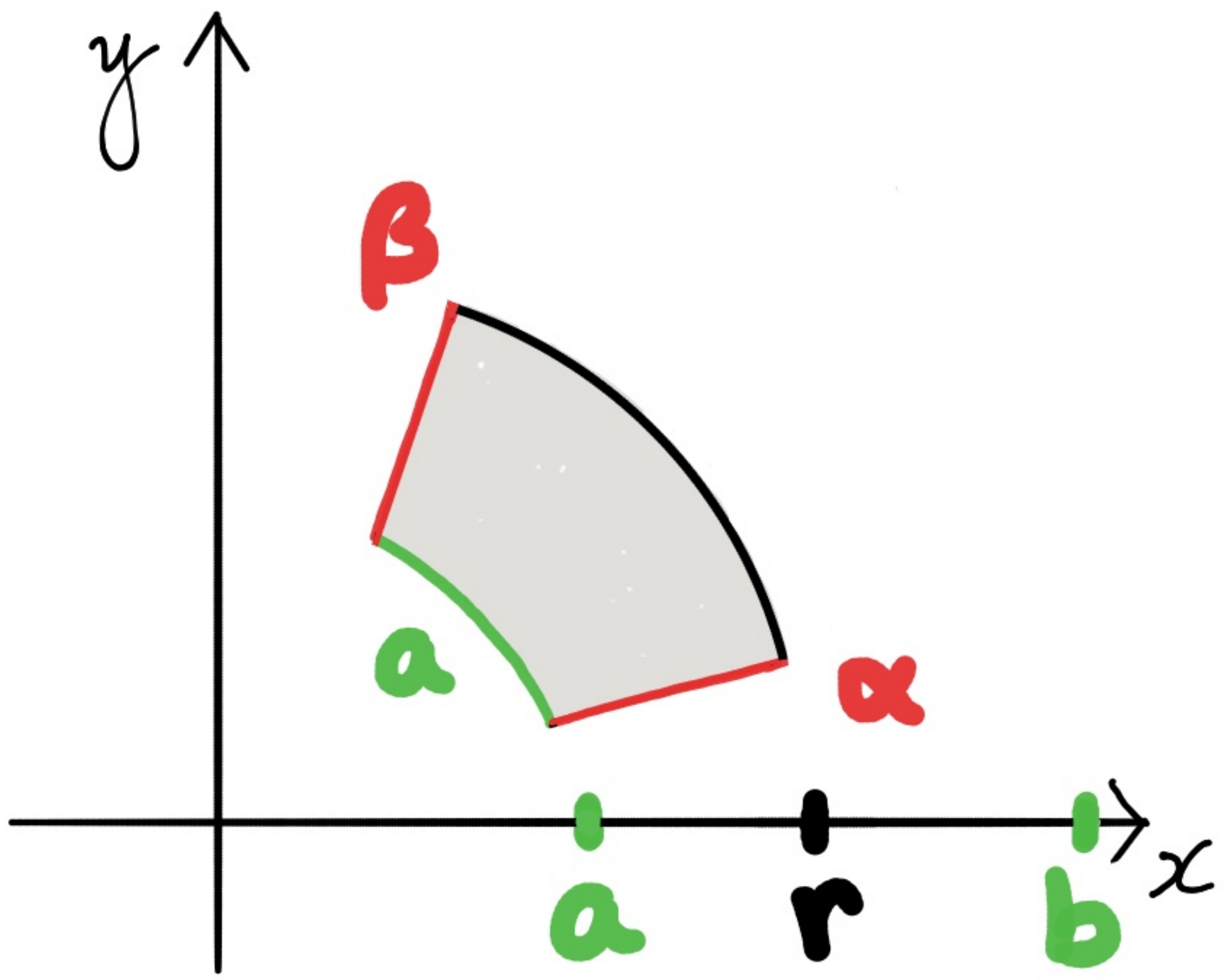


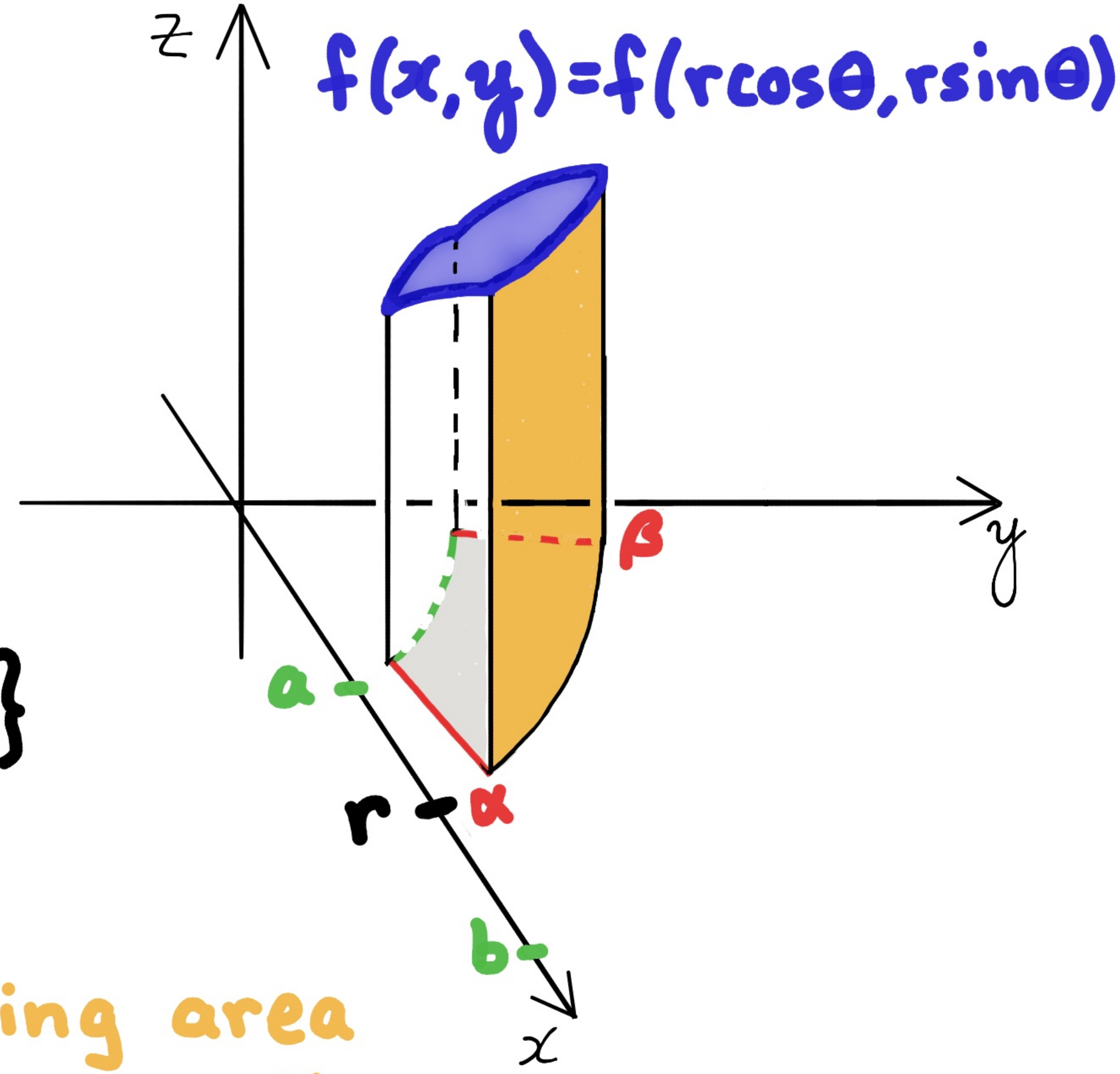
$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

varying area of cross section

$$\iint_R f(x,y)\,dA = \int_a^b \qquad dr$$

width



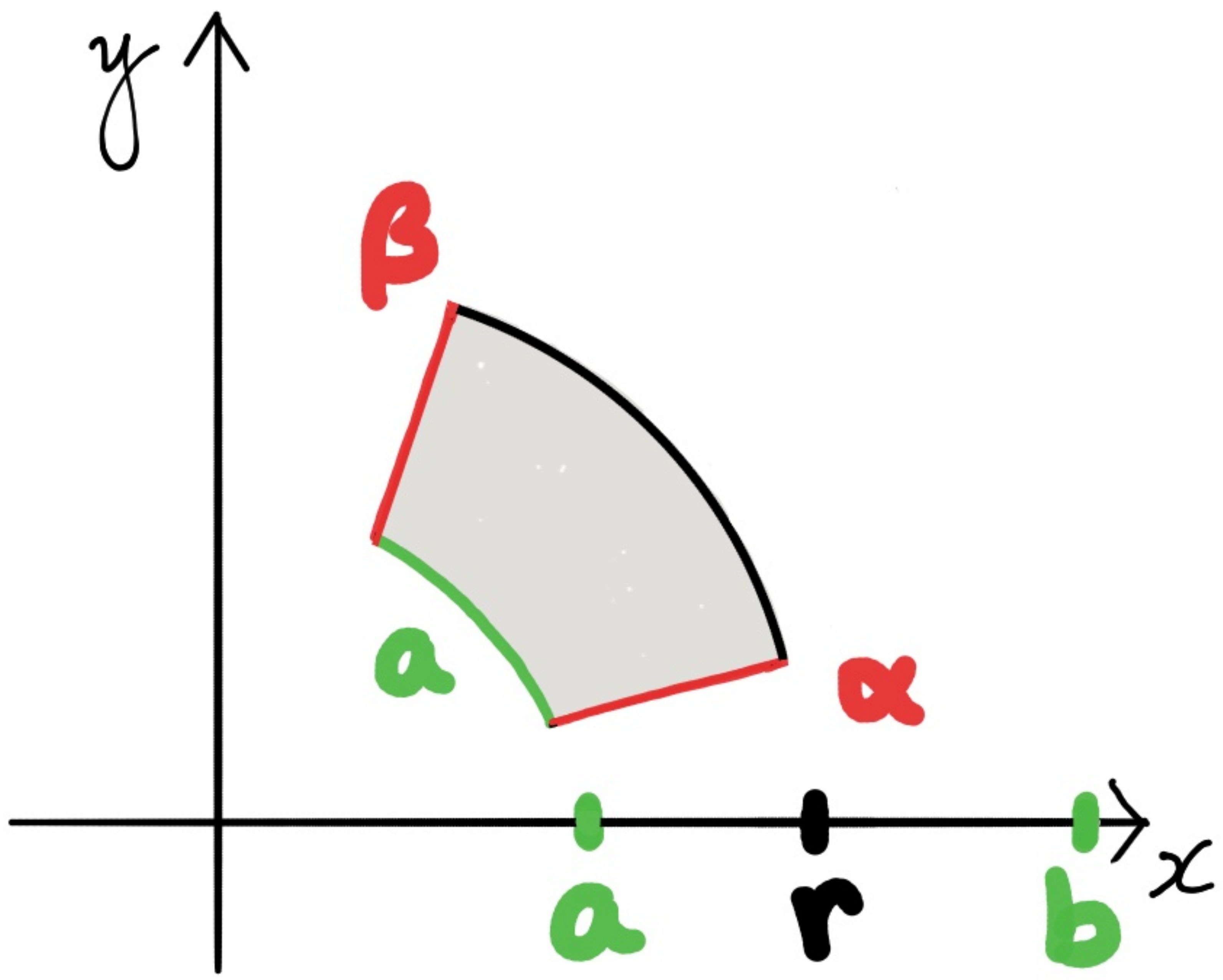


$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

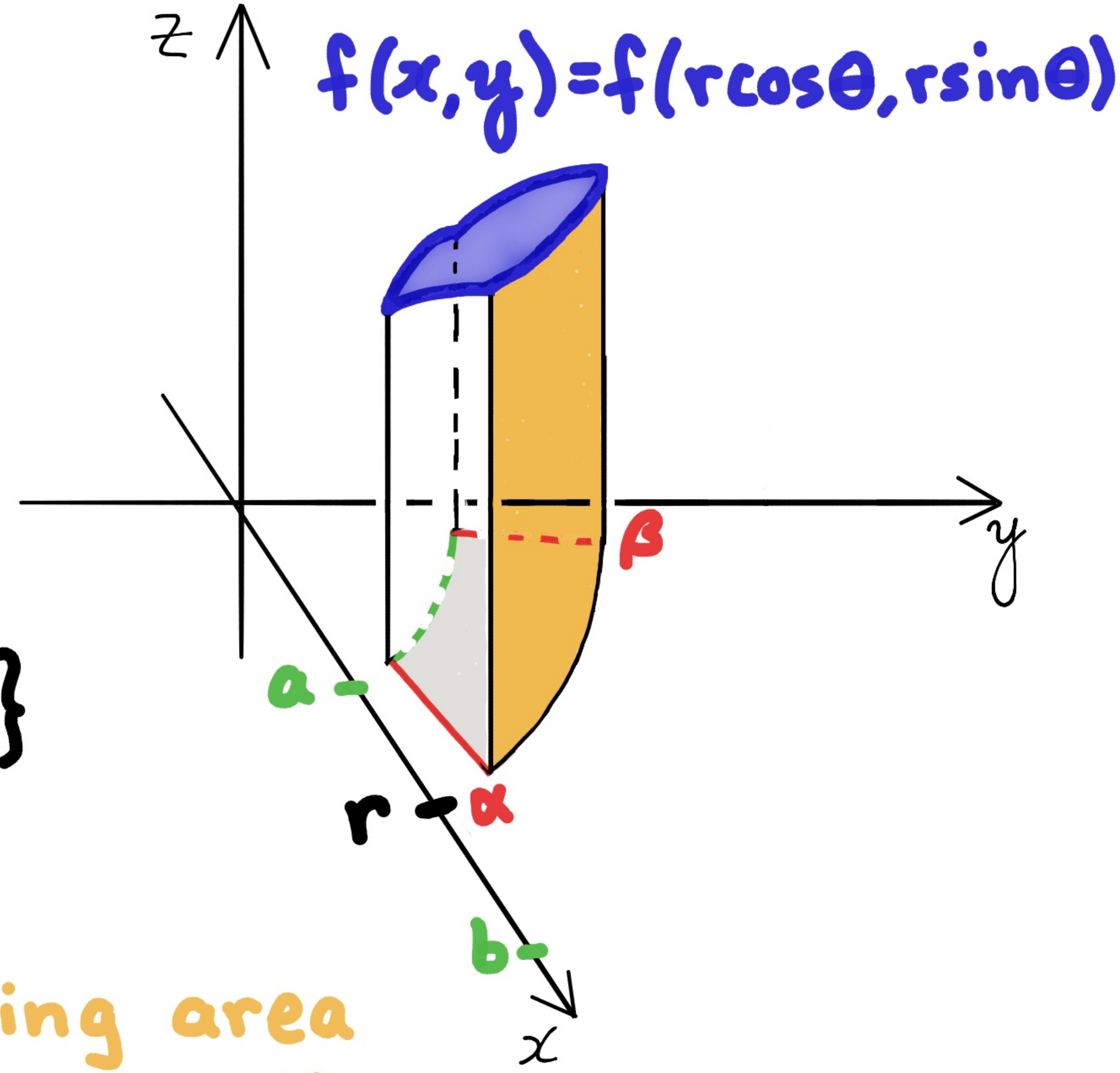
varying area of cross section

$$\iint_R f(x,y)\,dA = \int_a^b \qquad\qquad dr$$

width



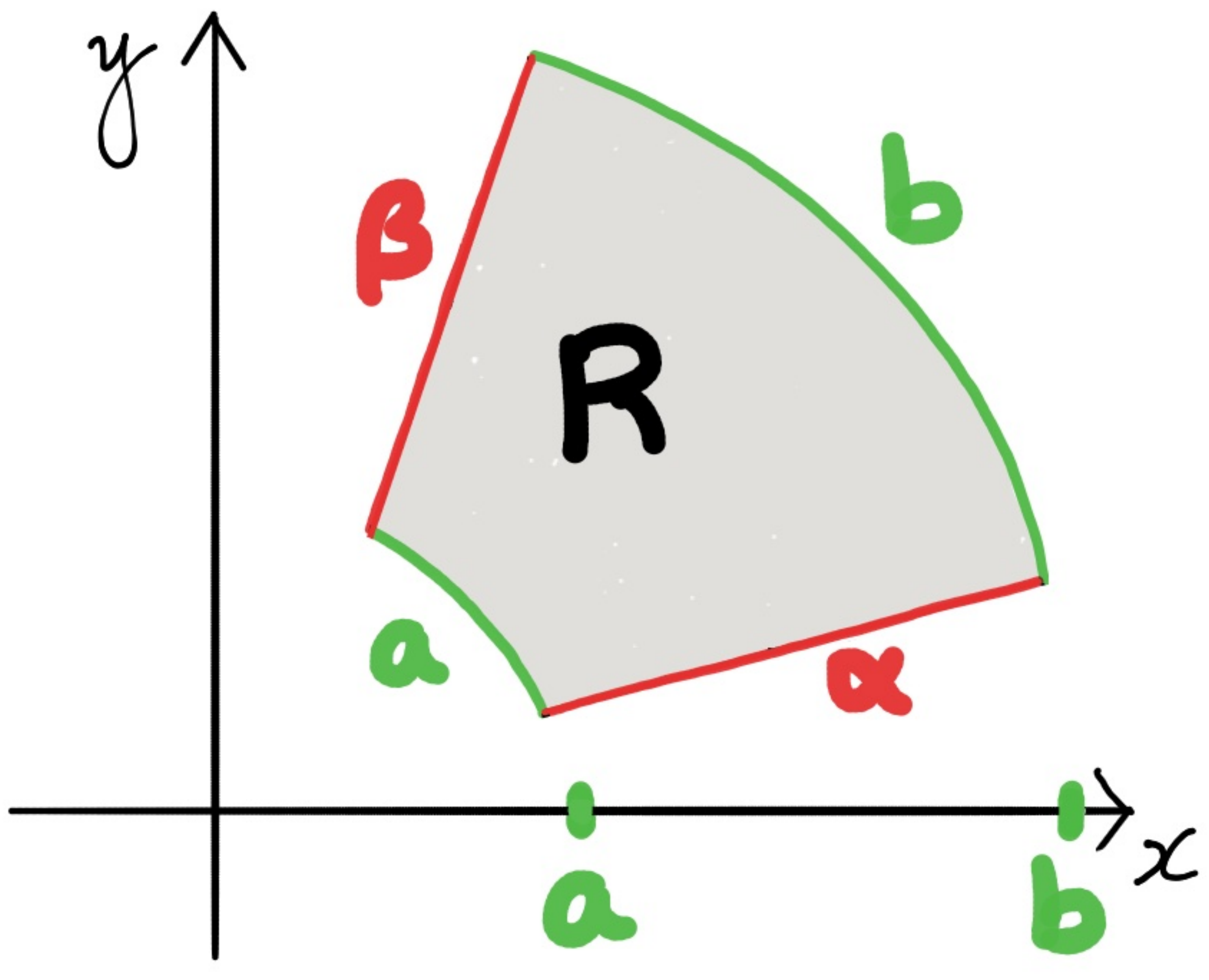
$$R = \{(r, \theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

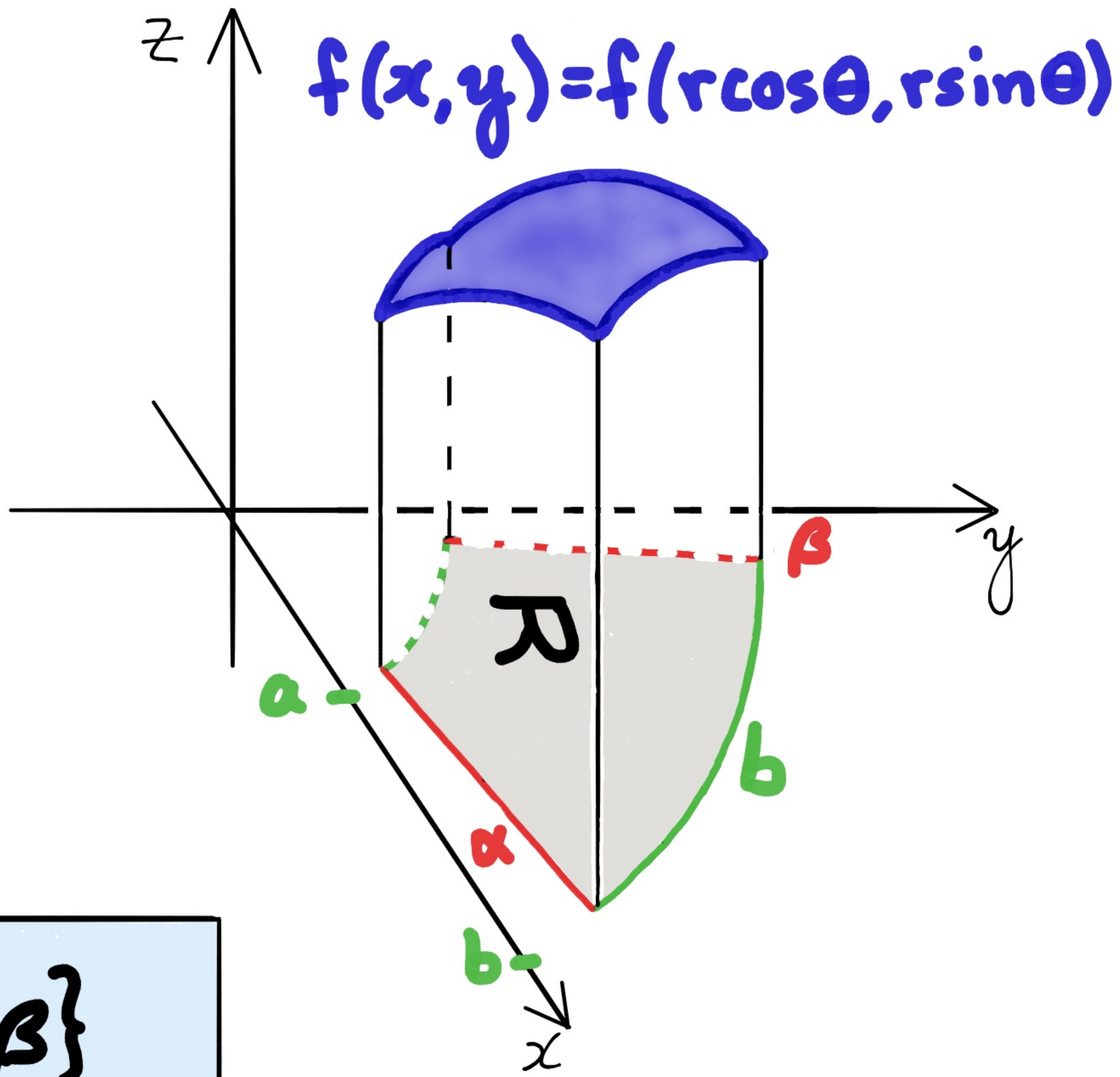


varying area of cross section

$$\iint_R f(x,y)\,dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

width





$$R = \{(r,\theta): a \le r \le b,\ \alpha \le \theta \le \beta\}$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$$dA \mapsto d\theta\, dr$$

$$x \mapsto r\cos\theta$$

$$y \mapsto r\sin\theta$$

$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

$$\iint_R f(x,y)\,dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$$dA \mapsto d\theta\, dr$$

$$x \mapsto r\cos\theta$$

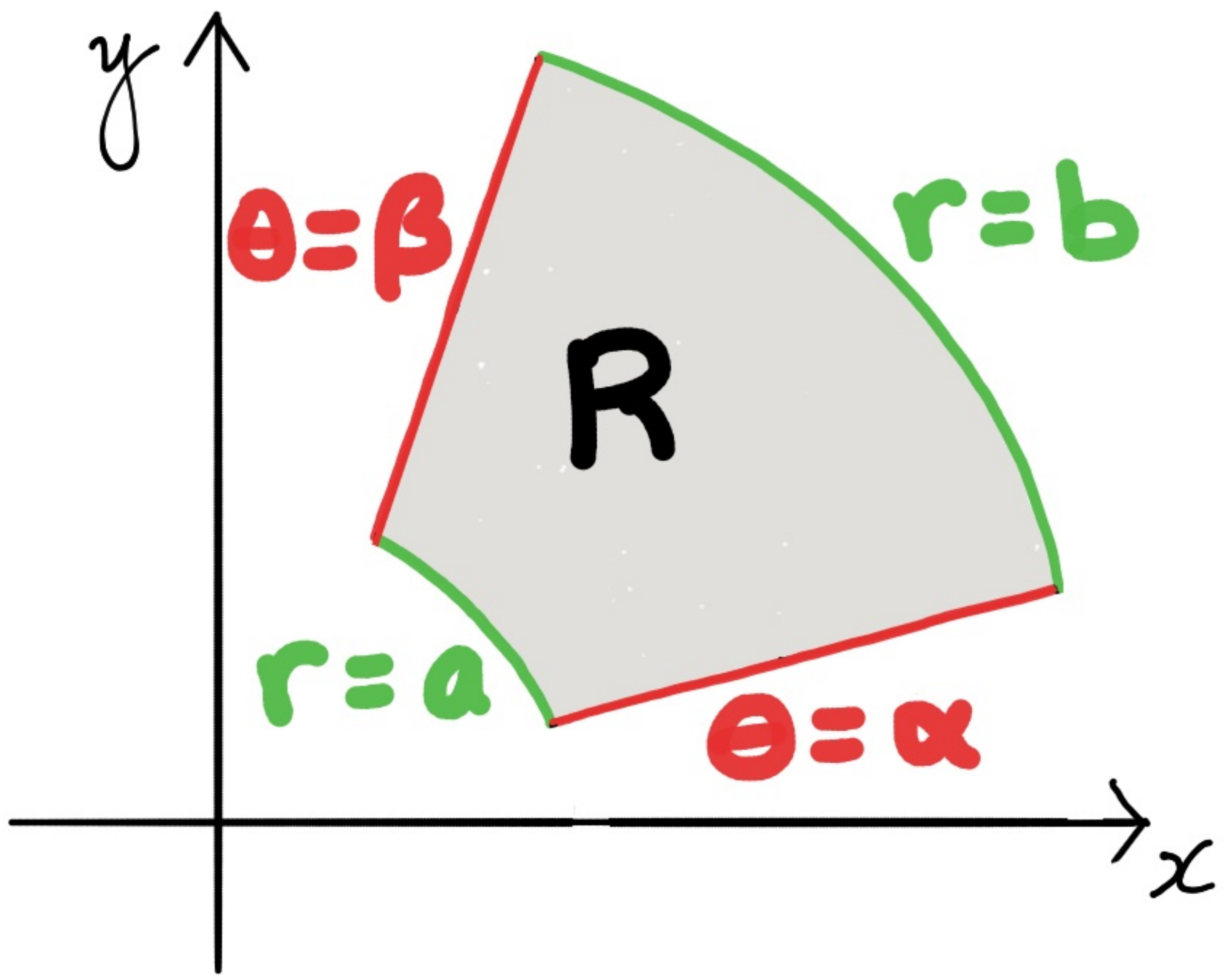
$$y \mapsto r\sin\theta$$

Throw in r.

Examples:

① $\iint_R e^{x^2+y^2}\,dA$ where R is the region bounded by the circle of radius 5, centered at $\vec{0} \in \mathbb{R}^2$.

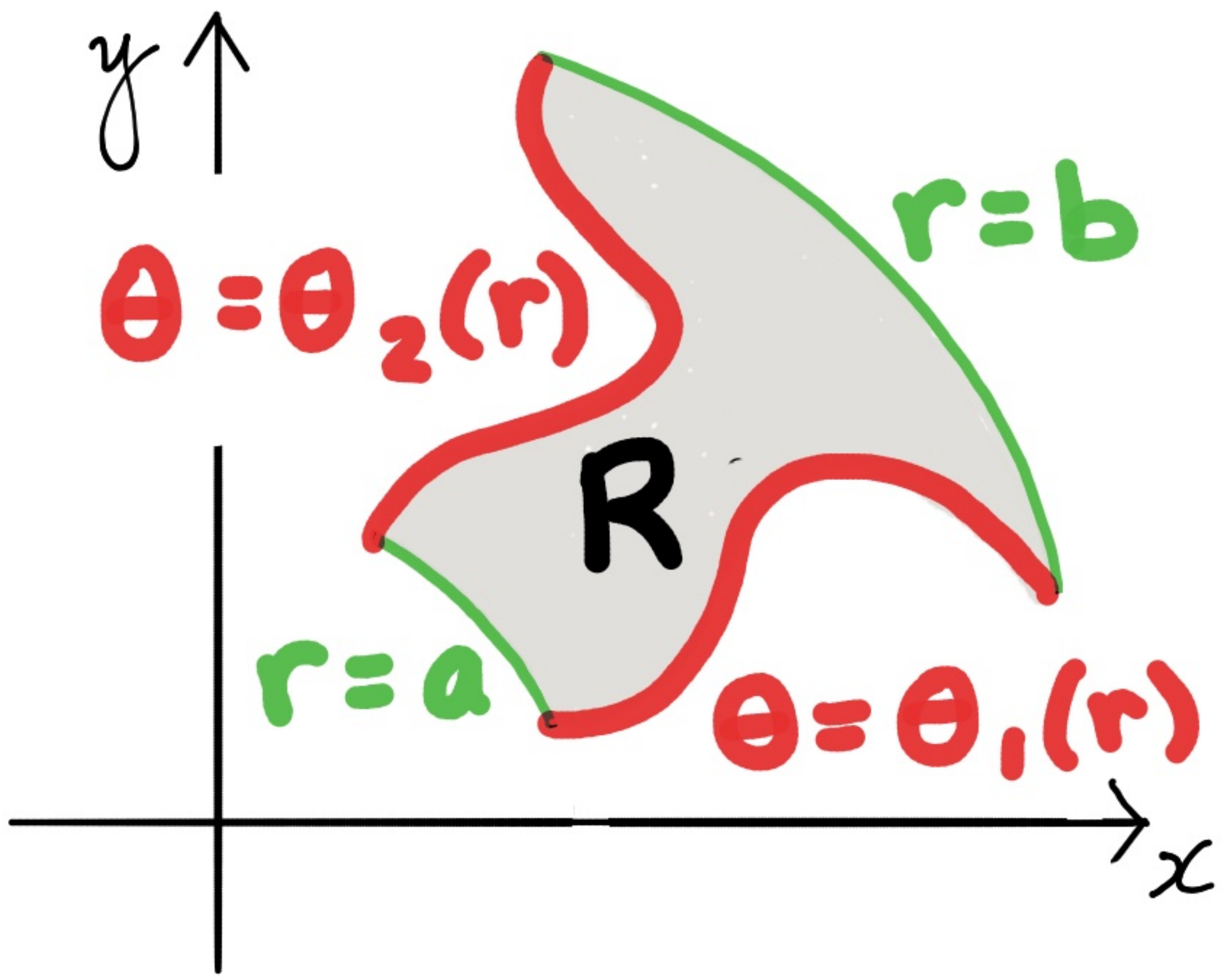
② $\iint_R x\,dA$ where $R = \{(r,\theta) : 3 \le r \le 6,\ 0 \le \theta \le \pi/2\}$.



$$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$$

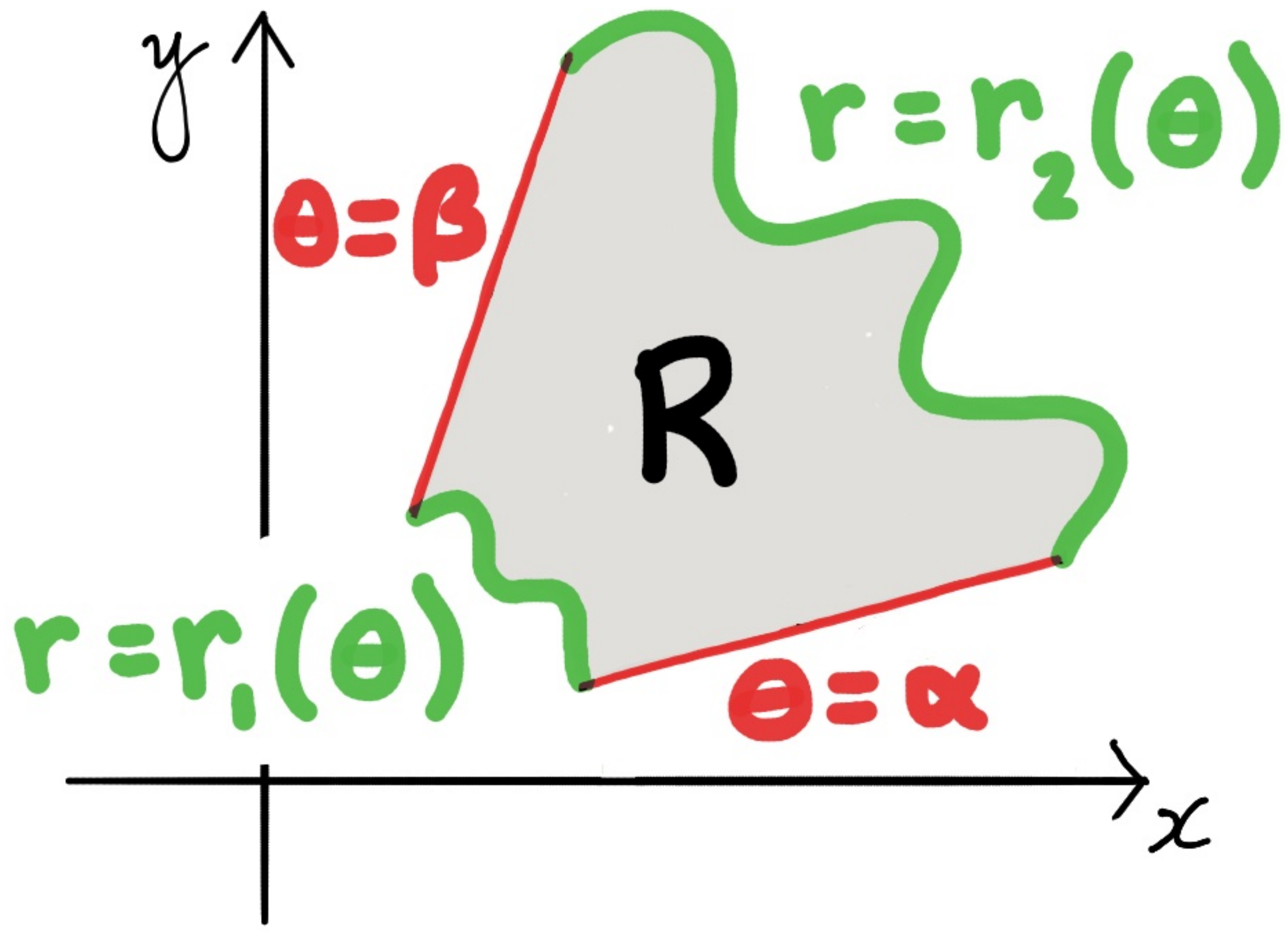
$$\iint_R f(x,y)\,dA = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



$$R=\{(r,\theta): a\le r\le b,\ \theta_1(r)\le\theta\le\theta_2(r)\}$$

$$\iint_R f(x,y)\,dA=\int_a^b\int_{\theta_1(r)}^{\theta_2(r)} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$



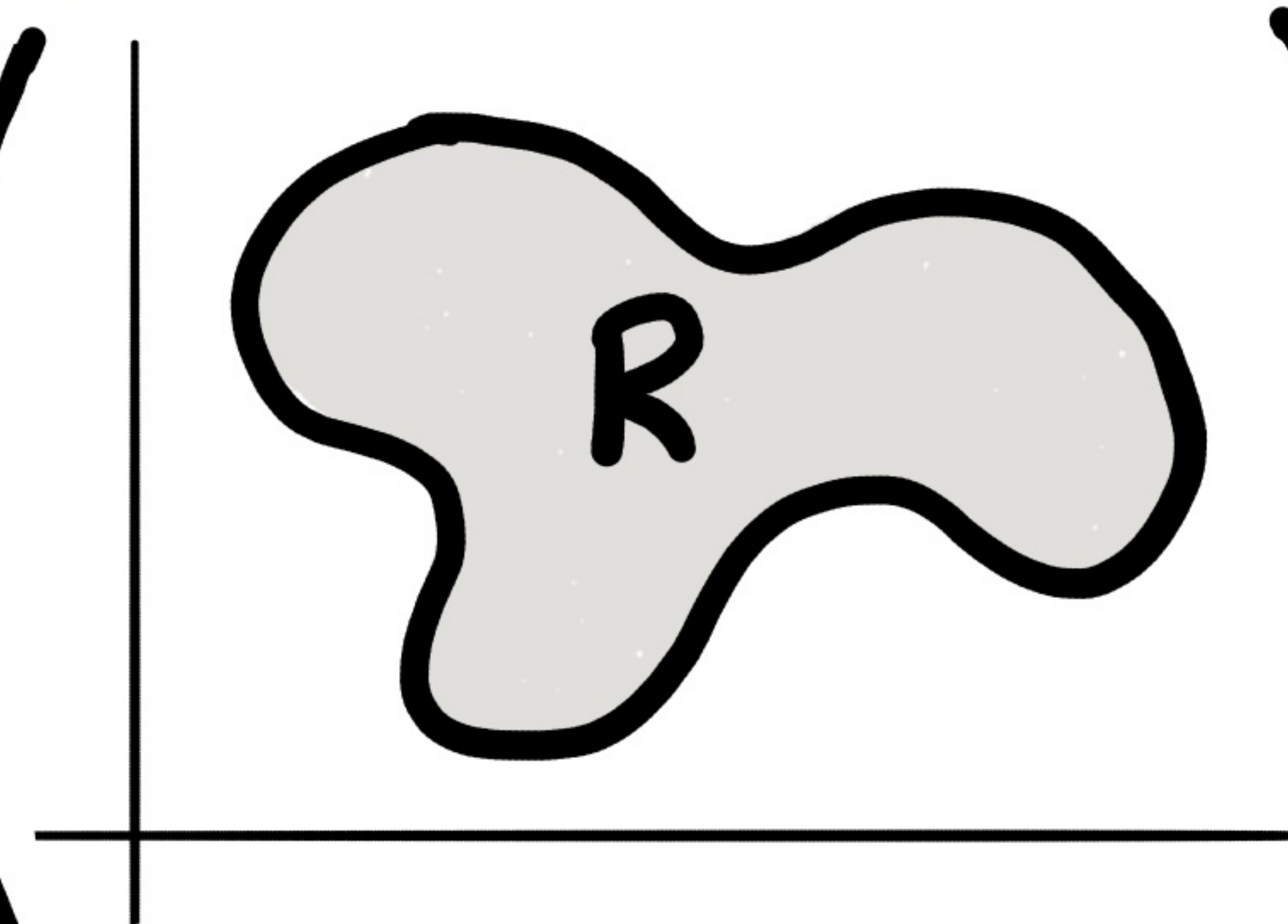
$$R=\{(r,\theta):\alpha\le\theta\le\beta,\ r_1(\theta)\le r\le r_2(\theta)\}$$

$$\iint_R f(x,y)\,dA=\int_\alpha^\beta\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Area formula

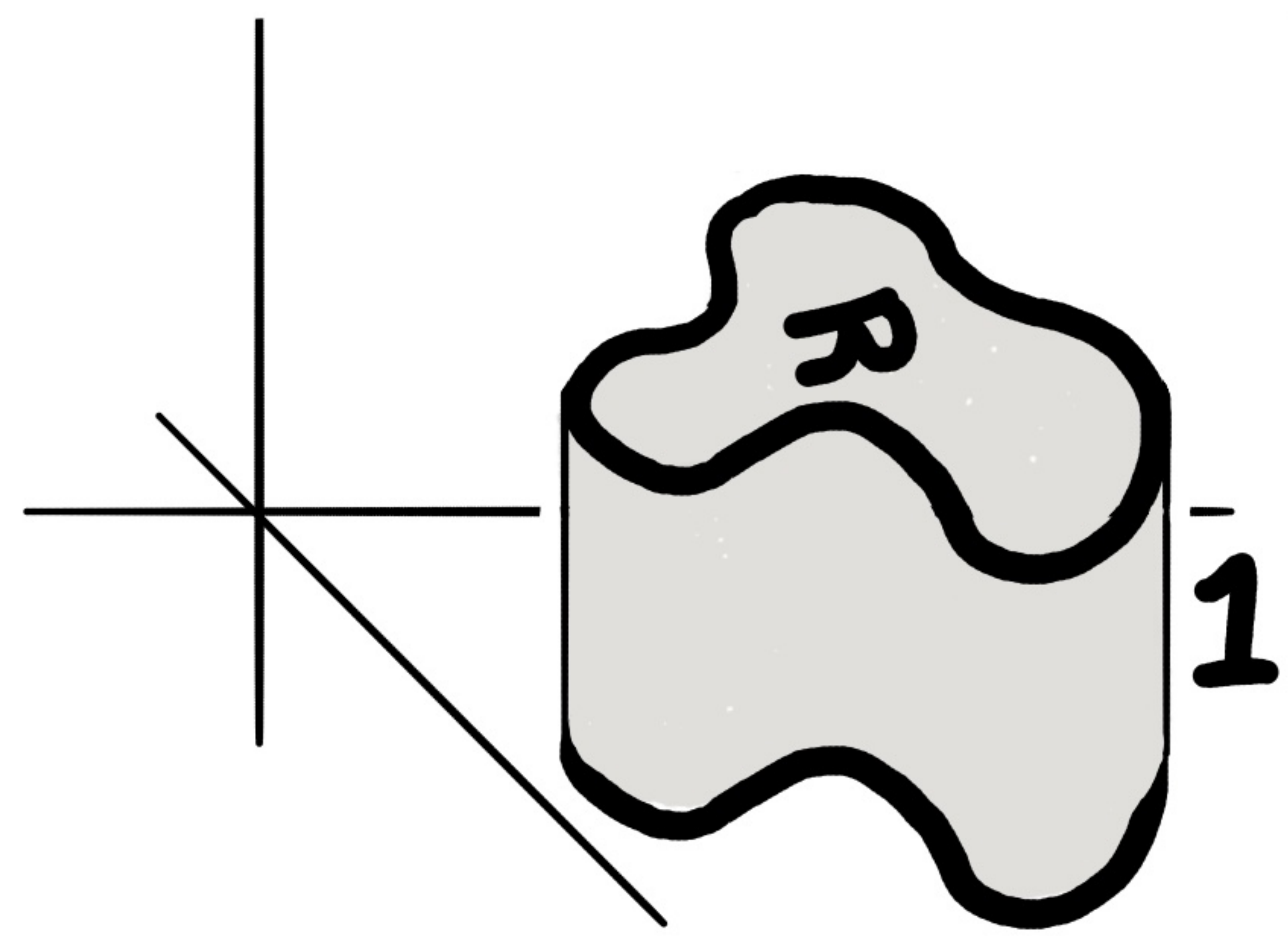
R a region in $\mathbb{R}^2$.

Area(R) = Area(



)

= Volume(



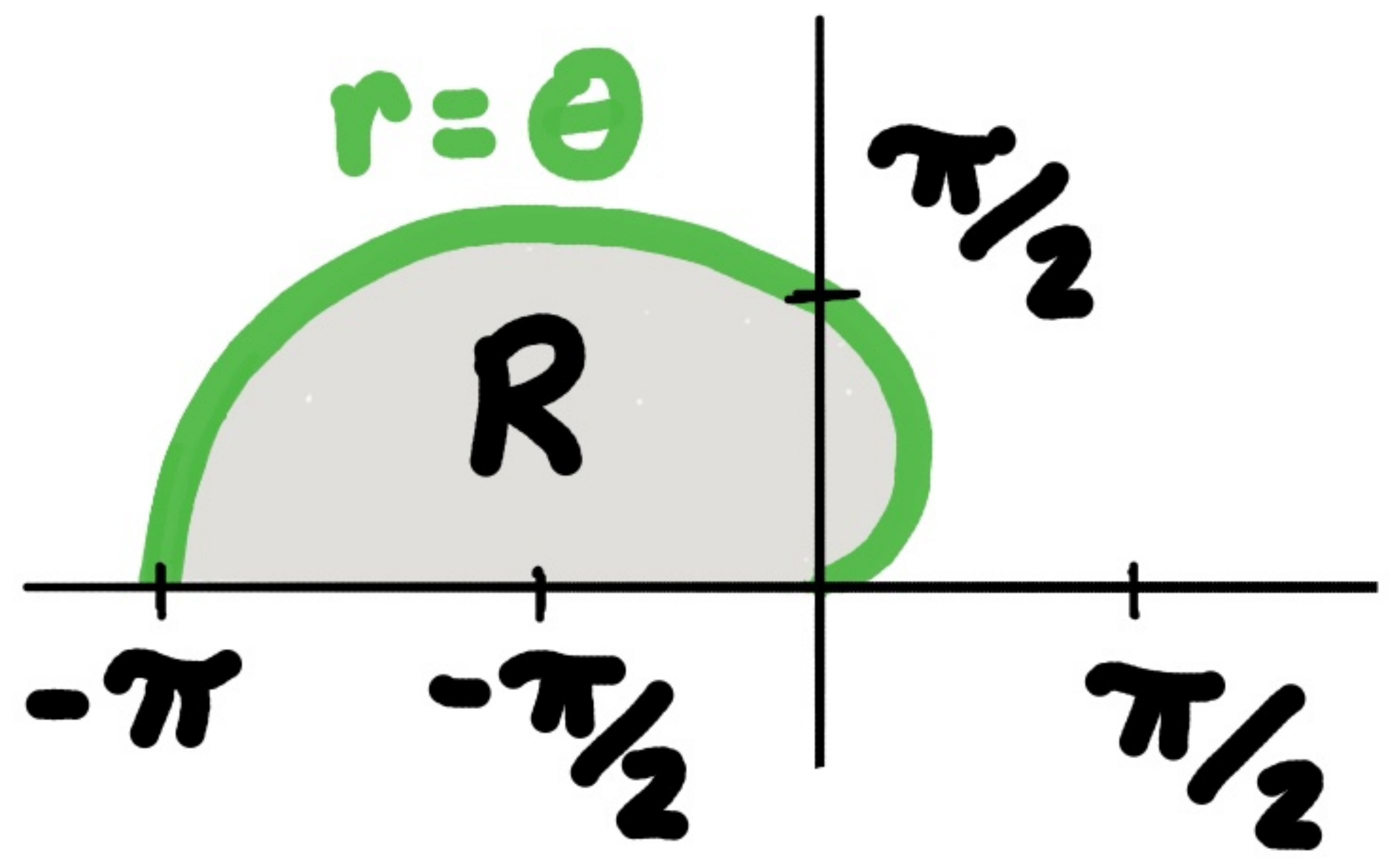
)

$$= \iint_R 1\,dA$$

$$= \iint_R dA$$

Examples:

③ Find the area of the region in the plane above the x-axis and below the curve $r=\theta$ for $0 \le \theta \le \pi$.



④ Find the area of region of the plane drawn below: