Two

Ⅰ vectors

Ⅱ magnitude

Ⅲ algebra of vectors

Ⅳ unitizing vectors

Ⅴ geometry of vectors

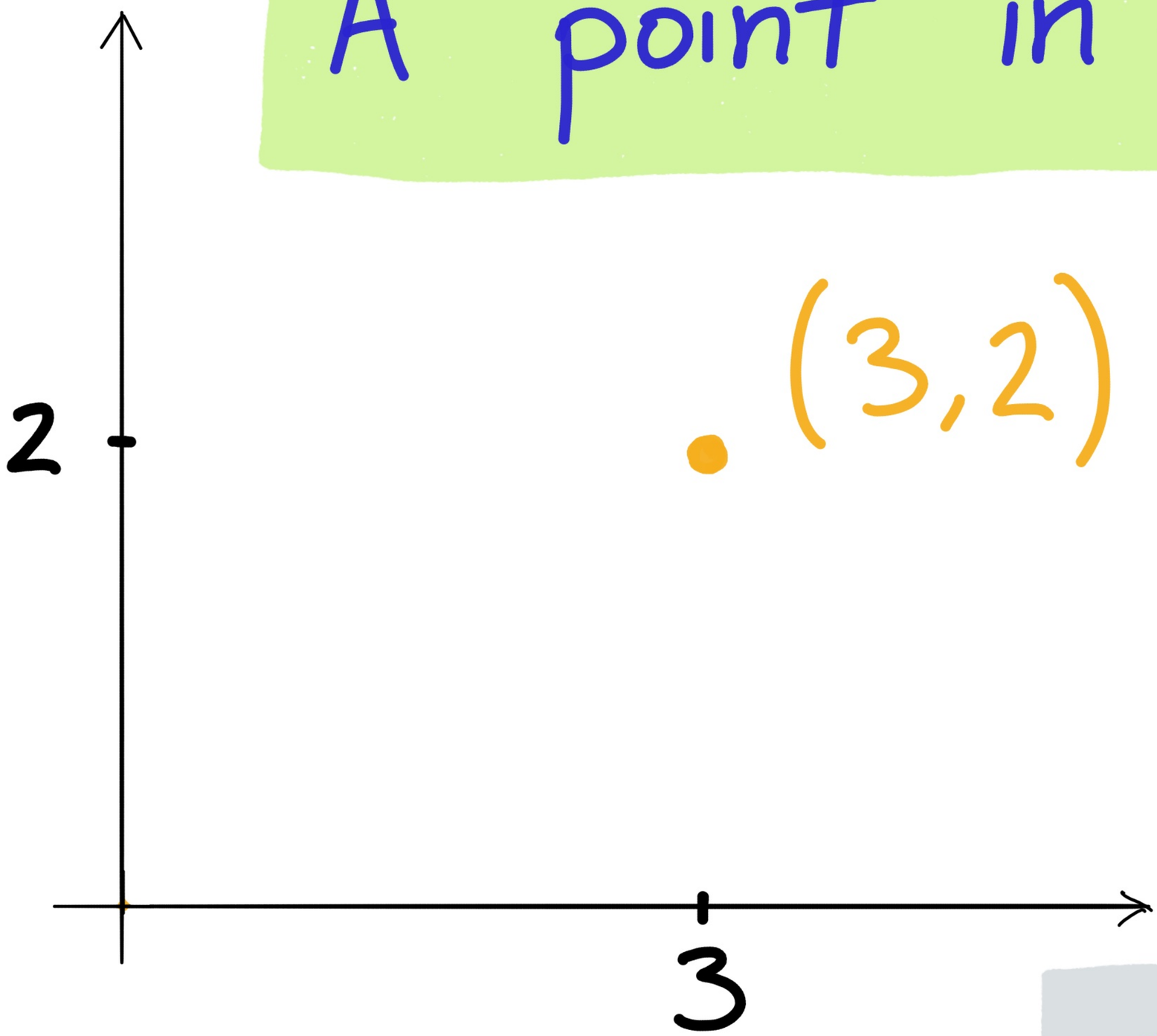
Ⓘ Vectors

$\mathbb{R}^n$

- $\mathbb{R}^1$ means $\mathbb{R}$
- $\mathbb{R}^n$ means $\mathbb{R}$, or $\mathbb{R}^2$, or $\mathbb{R}^3$.

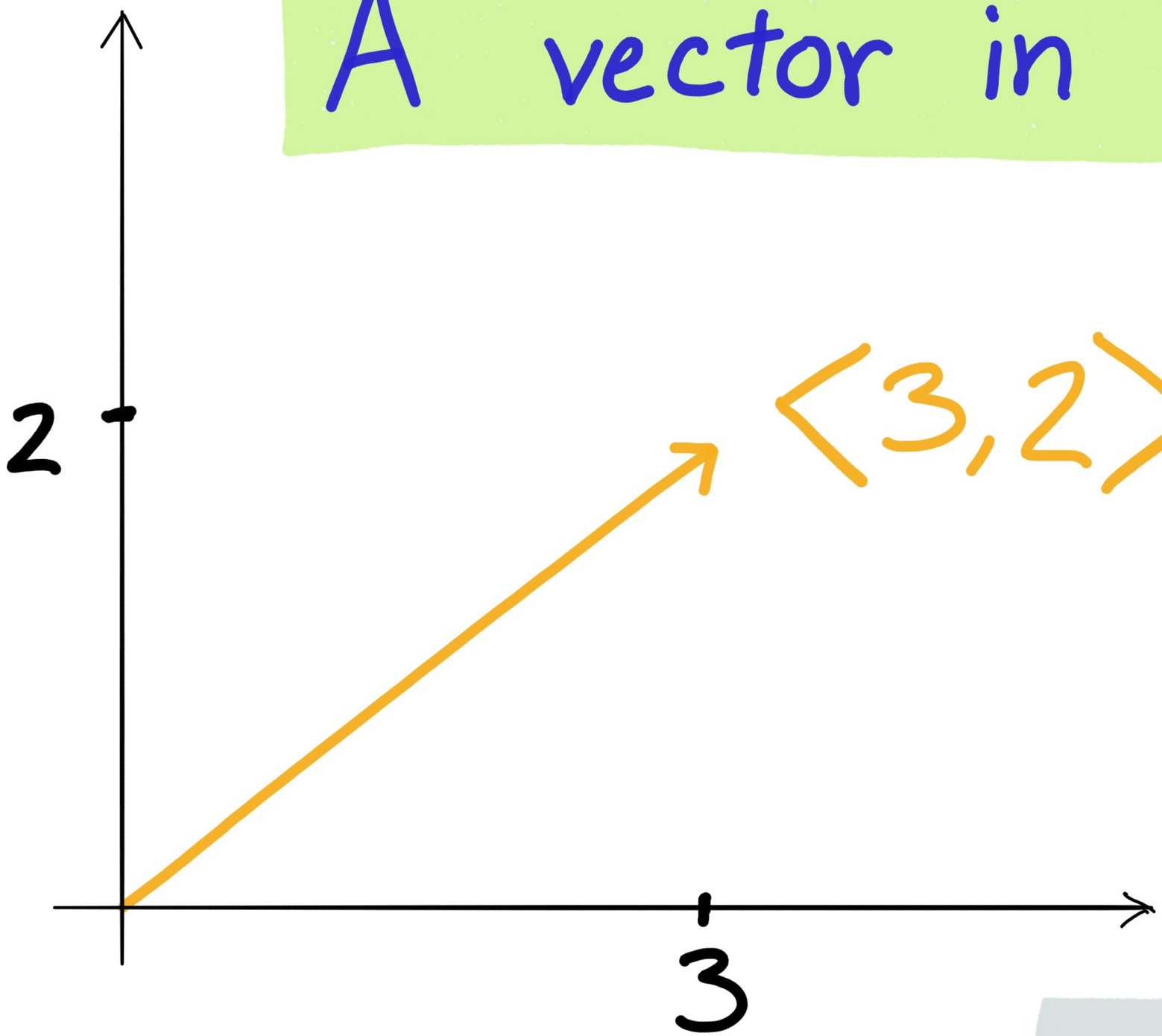
(or $\mathbb{R}^{137}$, etc.)

A point in $\mathbb{R}^2$



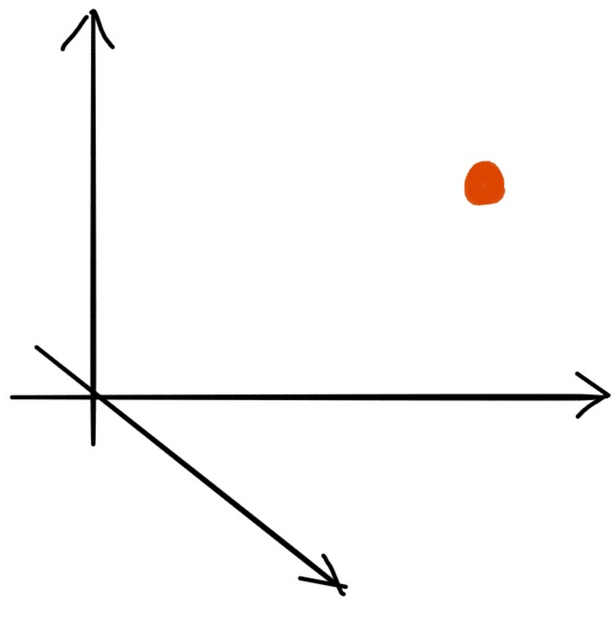
$(3,2) \in \mathbb{R}^2$

A vector in $\mathbb{R}^2$

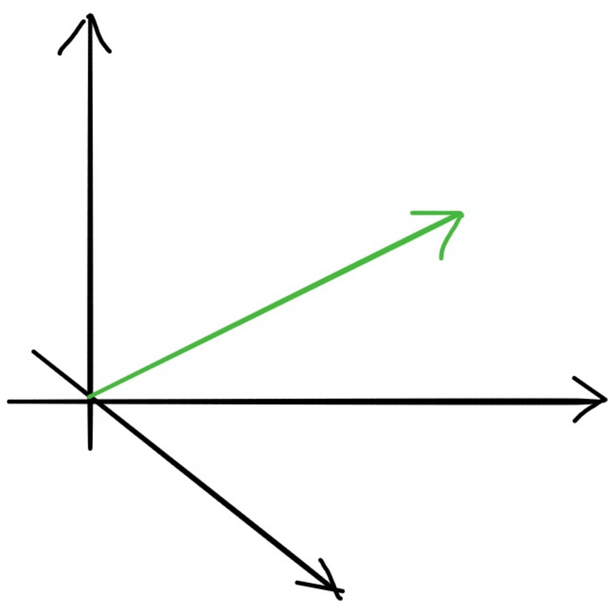


$\langle 3,2 \rangle \in \mathbb{R}^2$

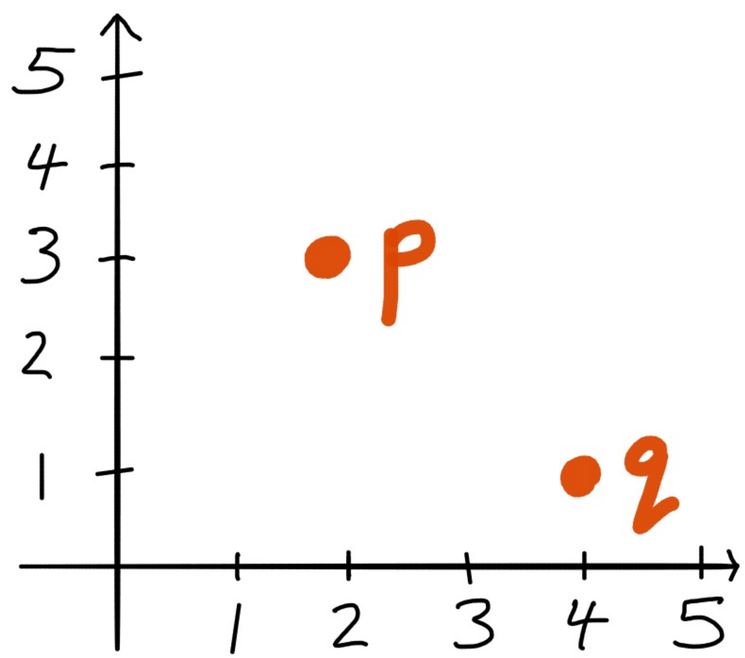
(x, y, z) is a point in $\mathbb{R}^3$.



$\langle x, y, z \rangle$ is a vector in $\mathbb{R}^3$.



p and q are common letters used to refer to points in $\mathbb{R}^n$.

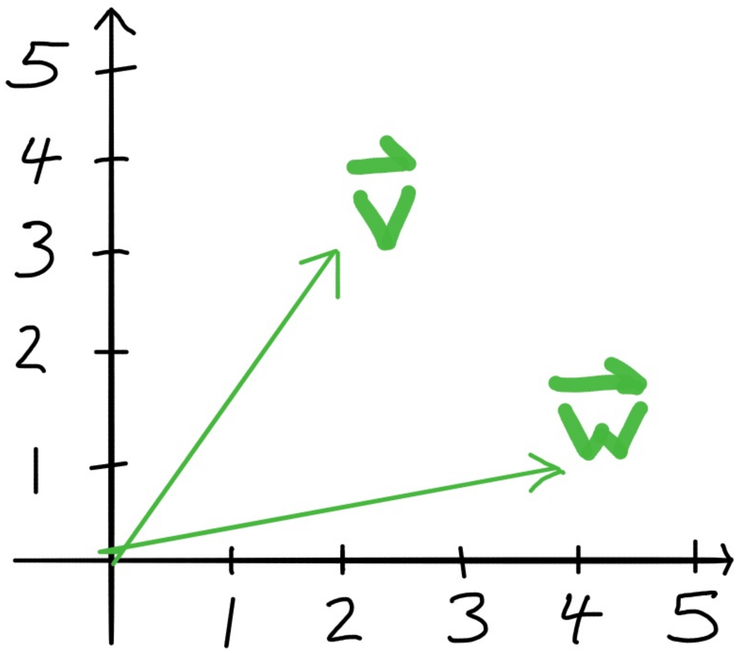


$p = (2, 3)$

$q = (4, 1)$

$p, q \in \mathbb{R}^2$

$\vec{v}$ and $\vec{w}$ are common letters used to refer to vectors in $\mathbb{R}^n$.



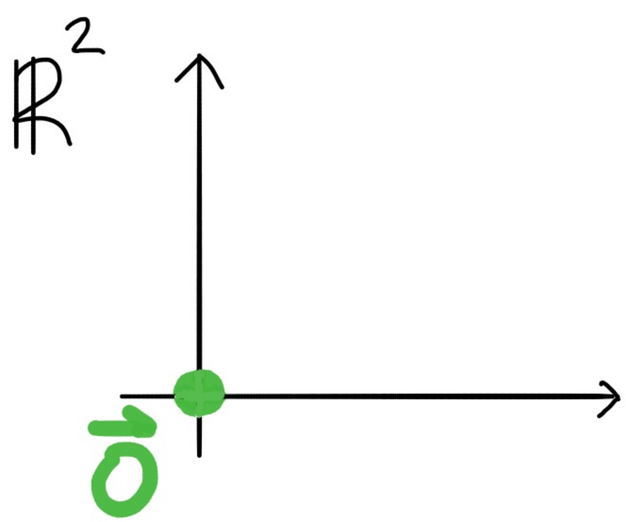
$\vec{v} = \langle 2, 3 \rangle$

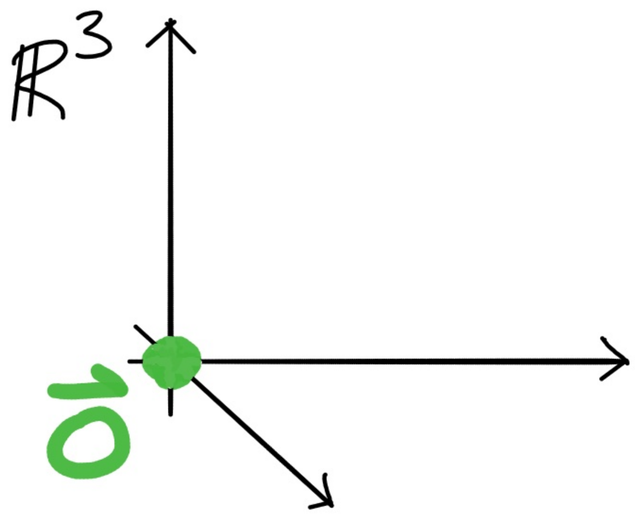
$\vec{w} = \langle 4, 1 \rangle$

$\vec{v}, \vec{w} \in \mathbb{R}^2$

The Zero vector

$\vec{0}$ is shorthand for $\langle 0,0 \rangle$ in $\mathbb{R}^2$ or $\langle 0,0,0 \rangle$ in $\mathbb{R}^3$.

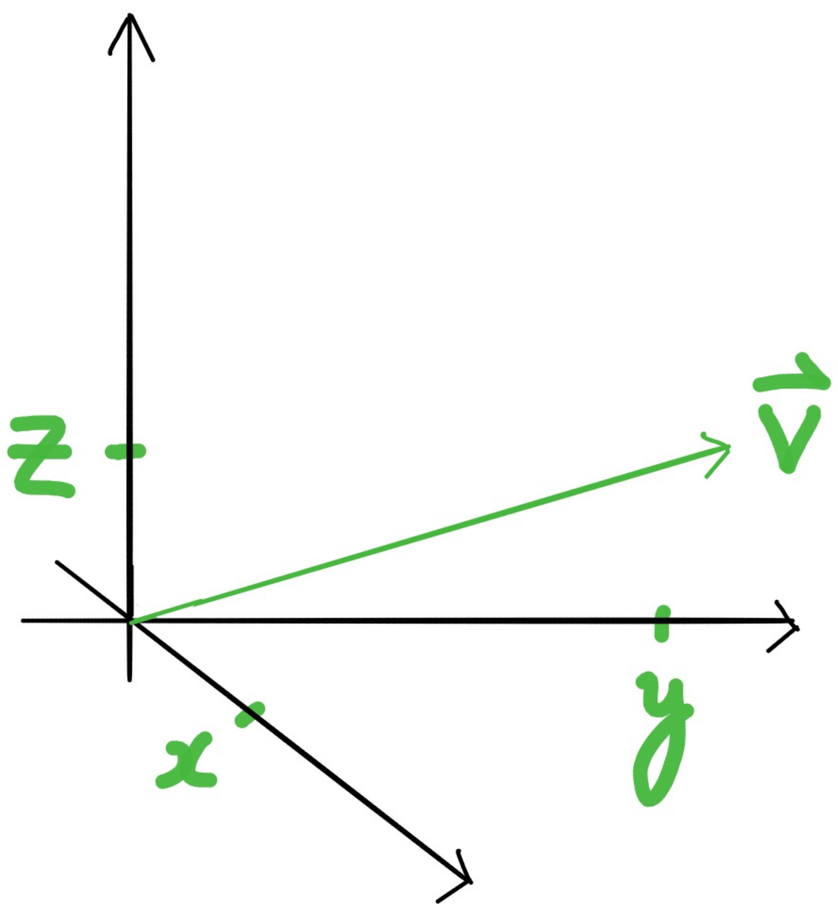




㉡ Magnitude

Magnitude (a.k.a. length or norm)

$\vec{v} = \langle x, y, z \rangle$



Magnitude (a.k.a. length or norm)

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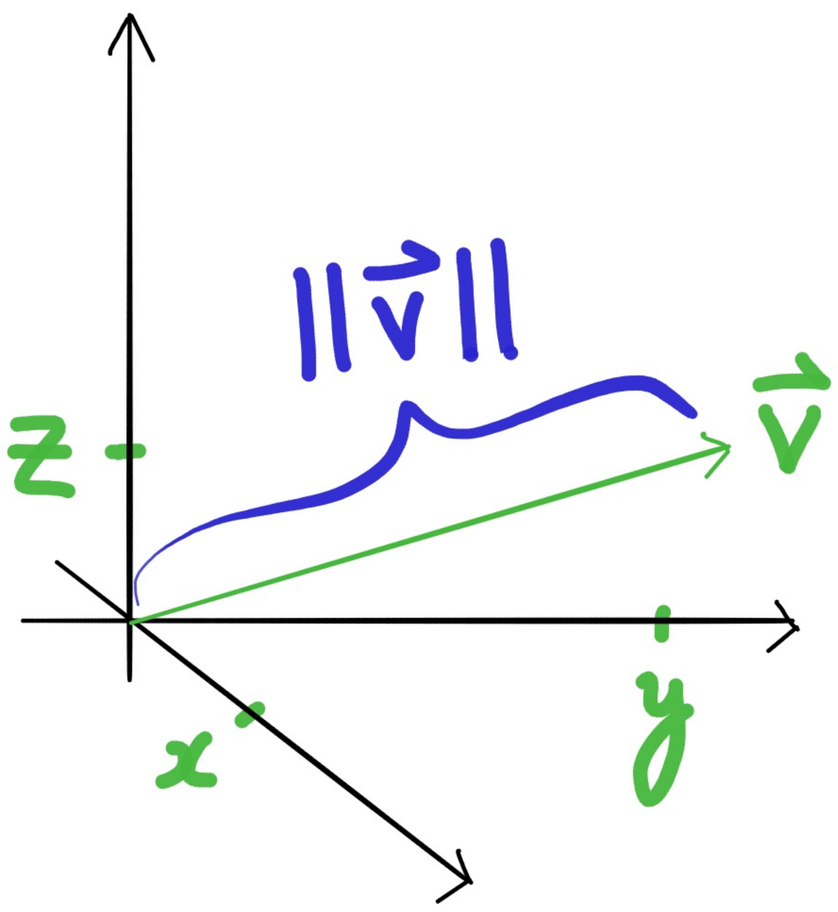
$\|\vec{v}\|$, the magnitude of $\vec{v}$, is the length of $\vec{v}$.



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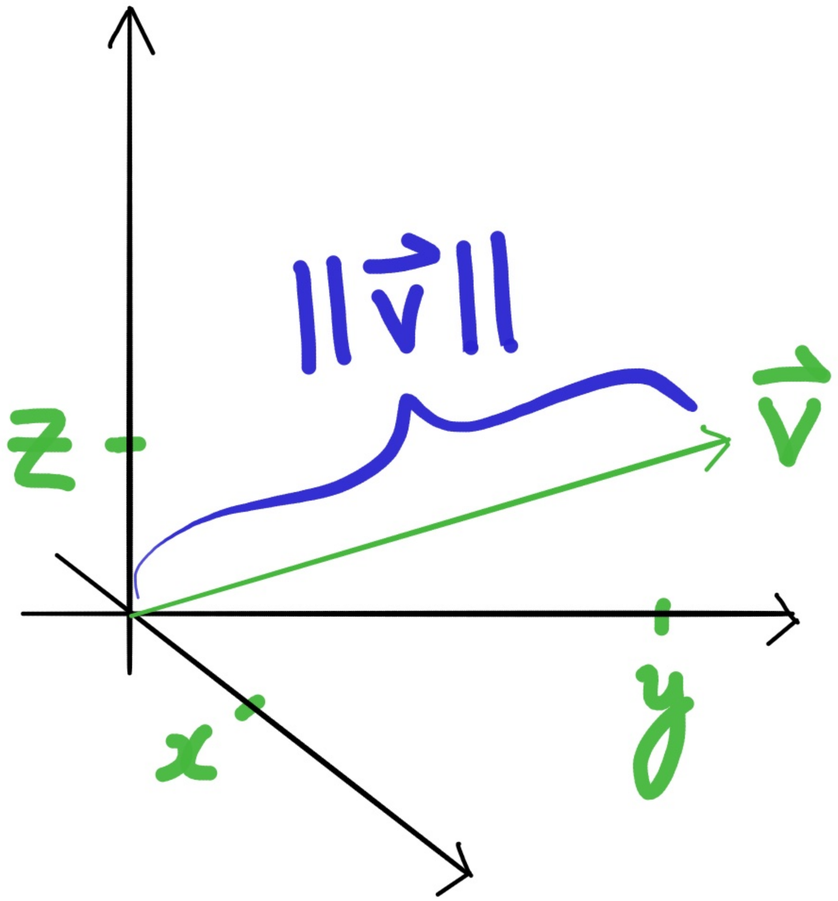

That is,

$\|\vec{v}\| = \{$ distance from (x,y,z) to $(0,0,0)\}$

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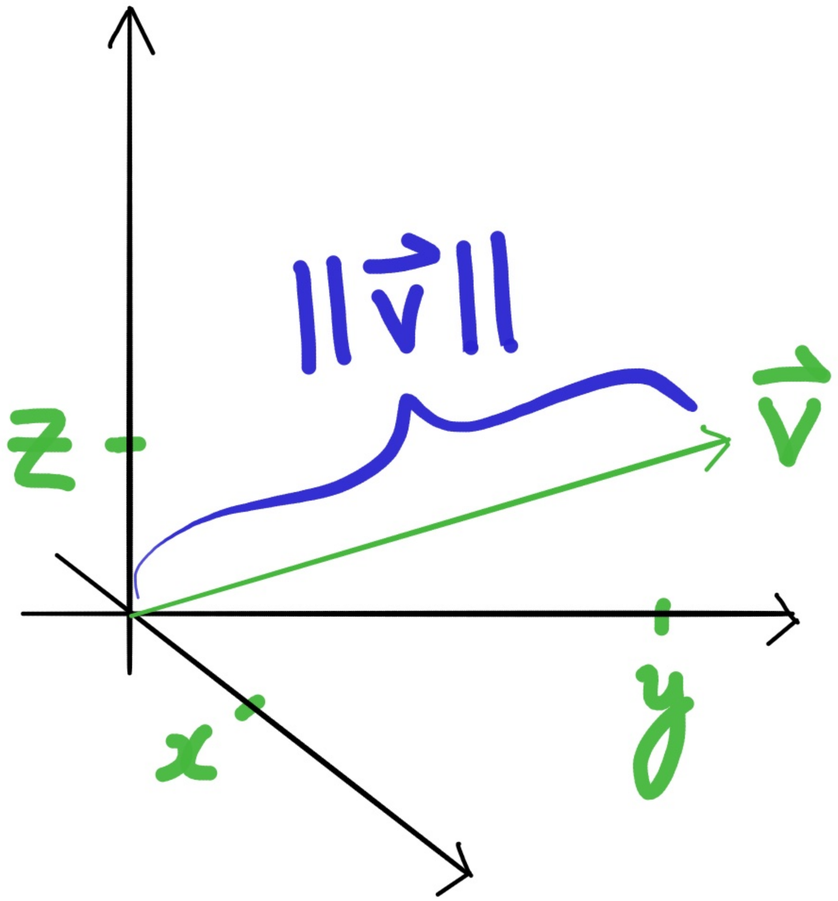

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$$= \sqrt{x^2 + y^2 + z^2}$$

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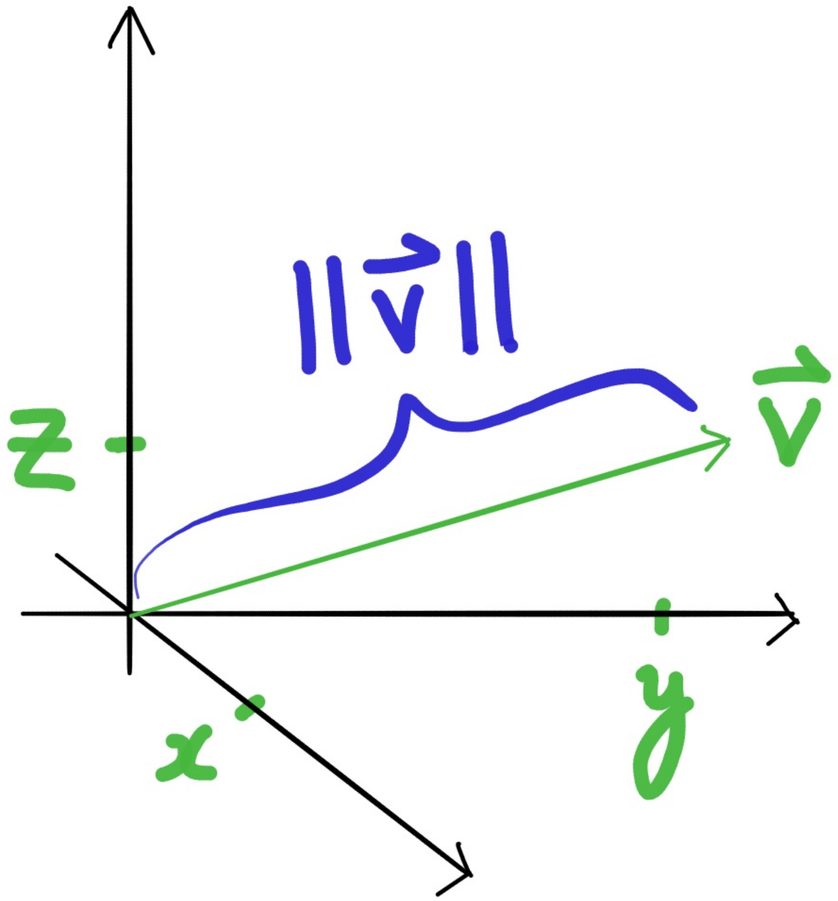

That is,

$\|\vec{v}\| = \{$ distance from (x,y,z) to $(0,0,0)\}$

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Magnitude of a vector

$$\|\langle x, y, z\rangle\| = \sqrt{x^2 + y^2 + z^2}$$

---

1) $\|\langle 1, -2, 6\rangle\|$

2) $\|\langle 2, -8\rangle\|$

Magnitude of a vector

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1) $\|\langle 1, -2, 6\rangle\| = \sqrt{1^2 + 2^2 + 6^2} = \sqrt{41}$

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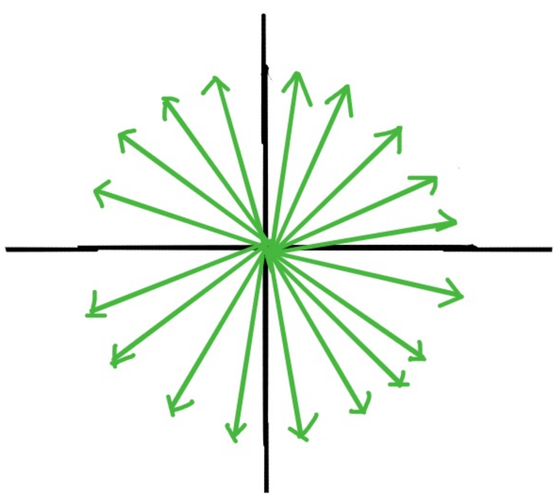
1) $\|\langle 1, -2, 6\rangle\| = \sqrt{1^2 + 2^2 + 6^2} = \sqrt{41}$

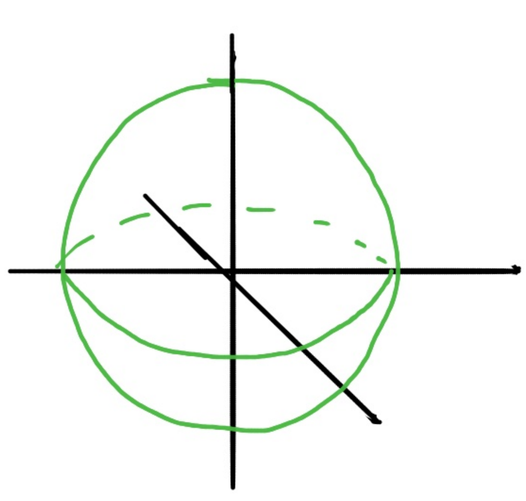
2) $\|\langle 2, -8\rangle\| = \sqrt{2^2 + 8^2} = \sqrt{68} = 2\sqrt{17}$

Unit vectors

$\vec{v} \in \mathbb{R}^n$ is a **unit vector** if $\|\vec{v}\| = 1$



Circle's worth of unit vectors in $\mathbb{R}^2$



Sphere's worth of unit vectors in $\mathbb{R}^3$

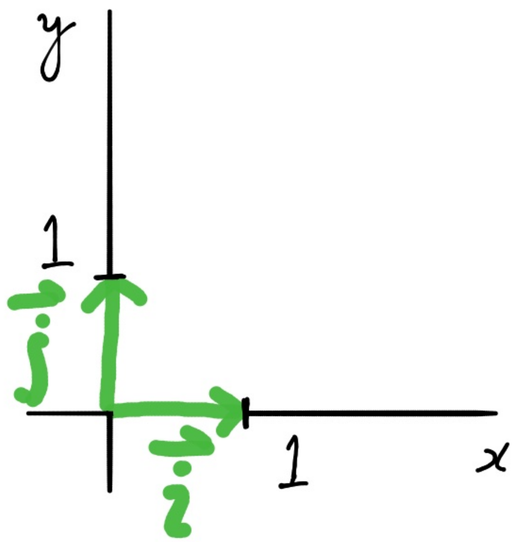
"direction away from $\vec{0}$"

Important unit vectors

$\mathbb{R}^2$

$$\vec{i} = \langle 1, 0 \rangle$$

$$\vec{j} = \langle 0, 1 \rangle$$

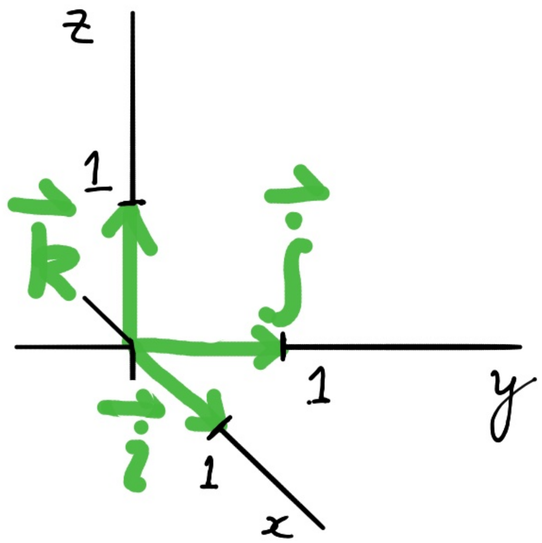


$\mathbb{R}^3$

$$\vec{i} = \langle 1, 0, 0 \rangle$$

$$\vec{j} = \langle 0, 1, 0 \rangle$$

$$\vec{k} = \langle 0, 0, 1 \rangle$$



III Algebra of vectors

Adding vectors

$$\langle x, y, z \rangle$$
$$+ \quad \langle a, b, c \rangle$$

$$\langle a+x,\ b+y,\ c+z \rangle$$

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$$\langle 1, 3, -2 \rangle + \langle 5, 10, 8 \rangle$$

Adding vectors

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$$\langle -3,4 \rangle + \langle 1,5 \rangle$$

Adding vectors

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Addition commutes!!

$$\langle -3,4 \rangle + \langle 1,5 \rangle = \langle -2,9 \rangle$$

Adding vectors

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$$= \langle 6,13,6 \rangle$$

Addition commutes!!

$$\langle -3,4 \rangle - \langle 1,5 \rangle$$

Adding vectors

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$$\langle 1,3,-2 \rangle + \langle 5,10,8 \rangle$$

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Addition commutes!!

$$\langle -3,4 \rangle - \langle 1,5 \rangle = \langle -4,-1 \rangle$$

Scaling vectors

If $a \in \mathbb{R}$,

$$a\langle x, y, z\rangle = \langle ax, ay, az\rangle$$

$$2\langle 5, -7, 3\rangle$$

$$\frac{\langle 3, 2\rangle}{2}$$

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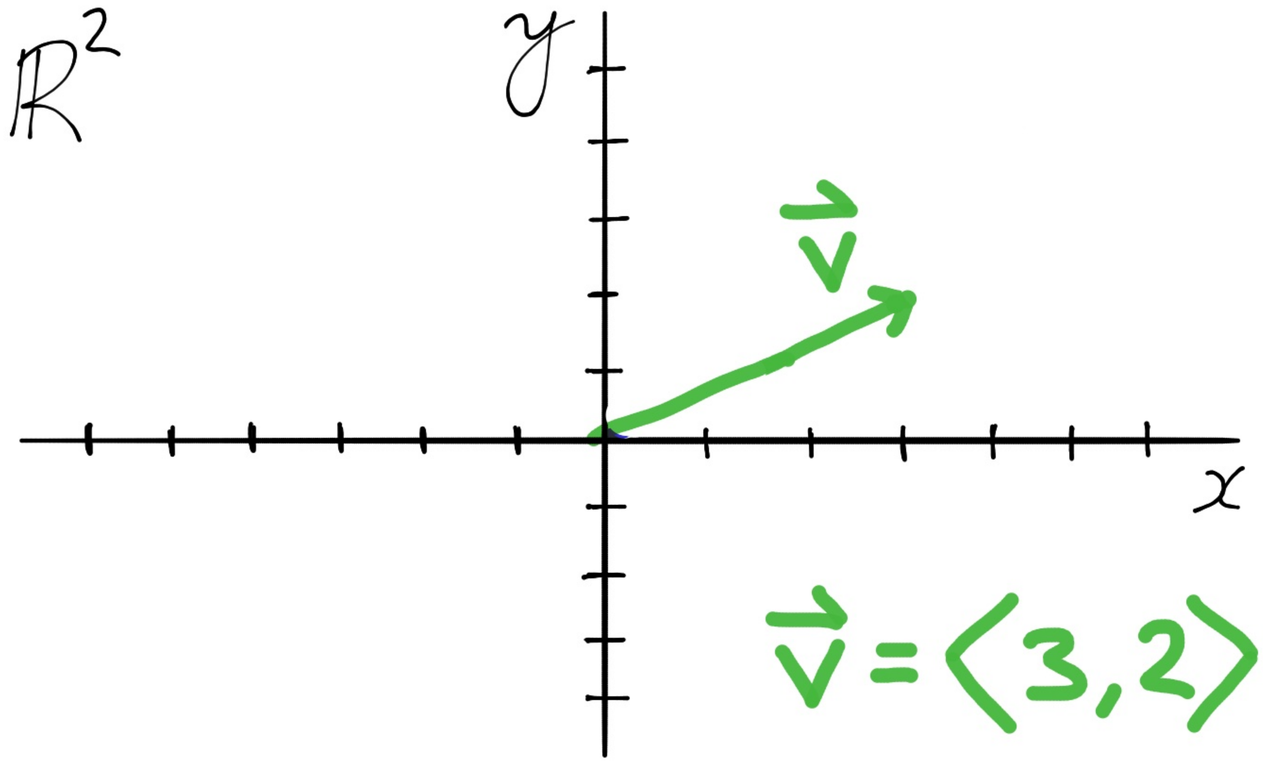

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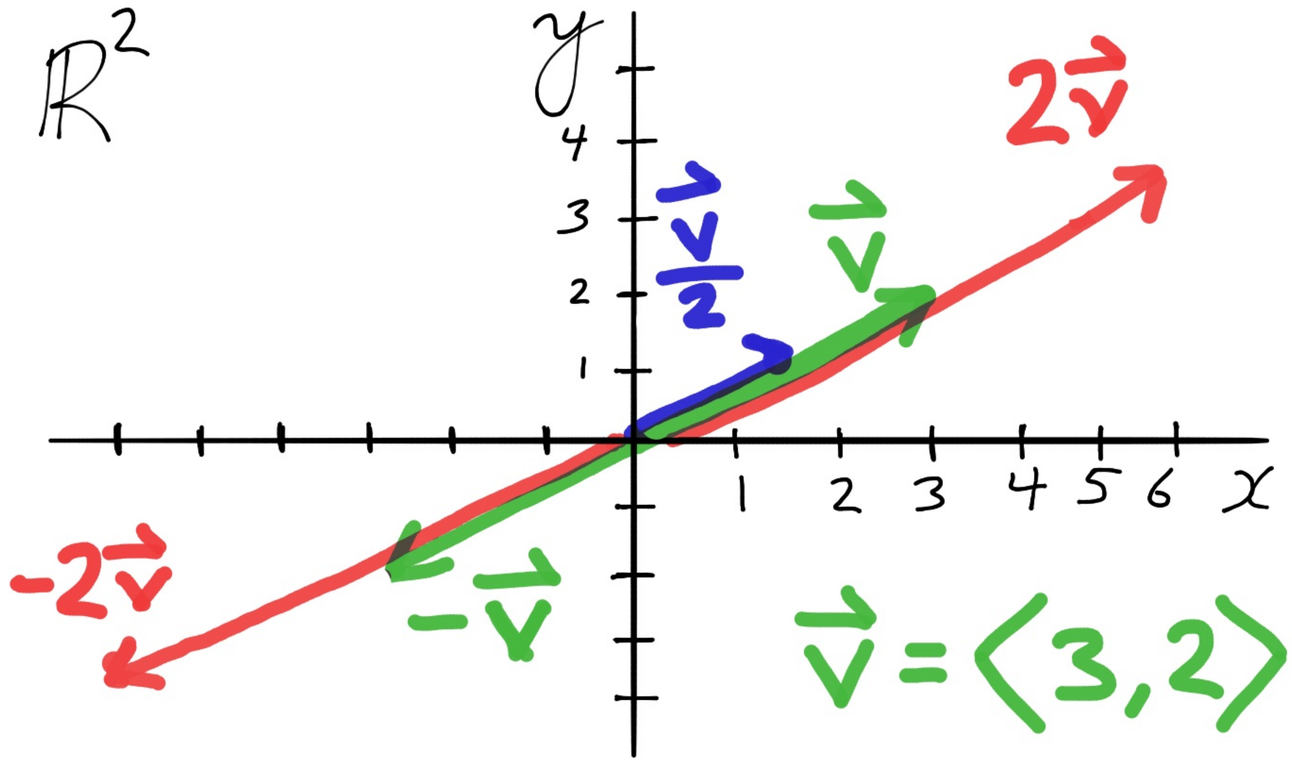
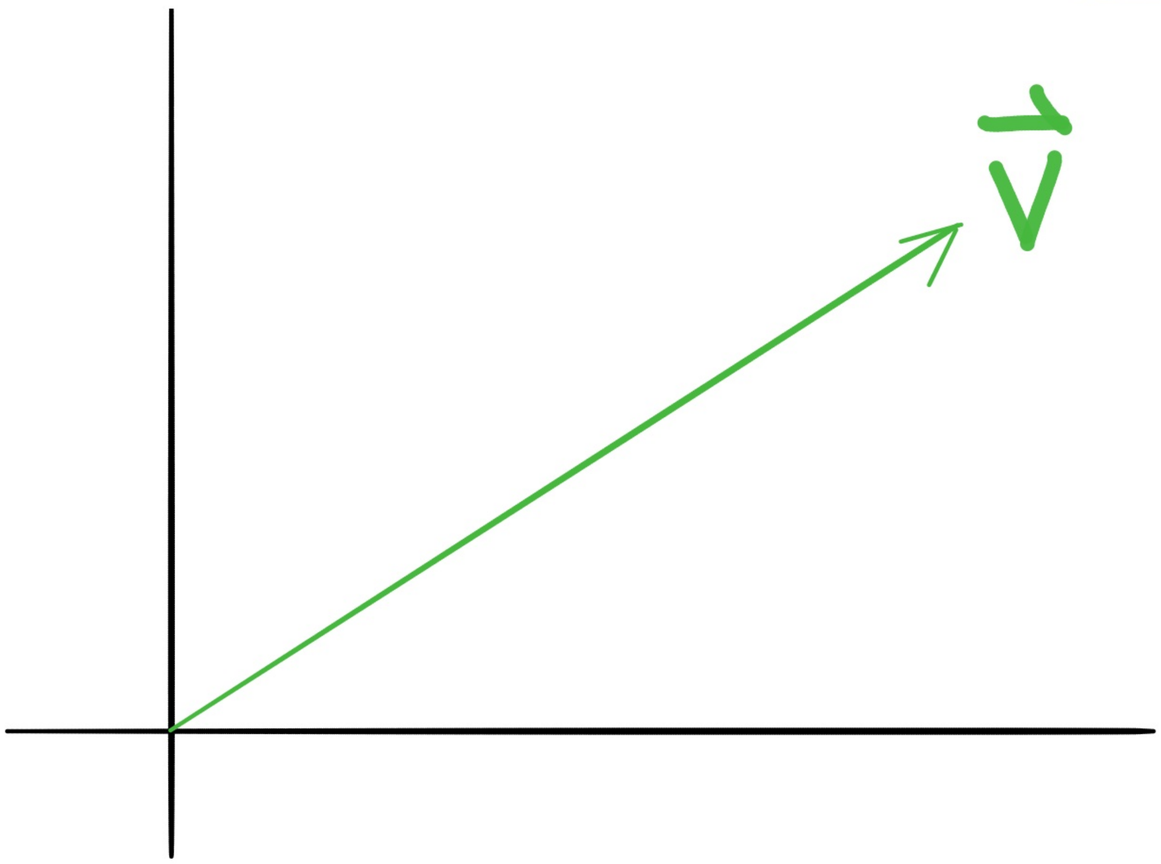


Ⅳ Unitizing vectors

Unitizing vectors



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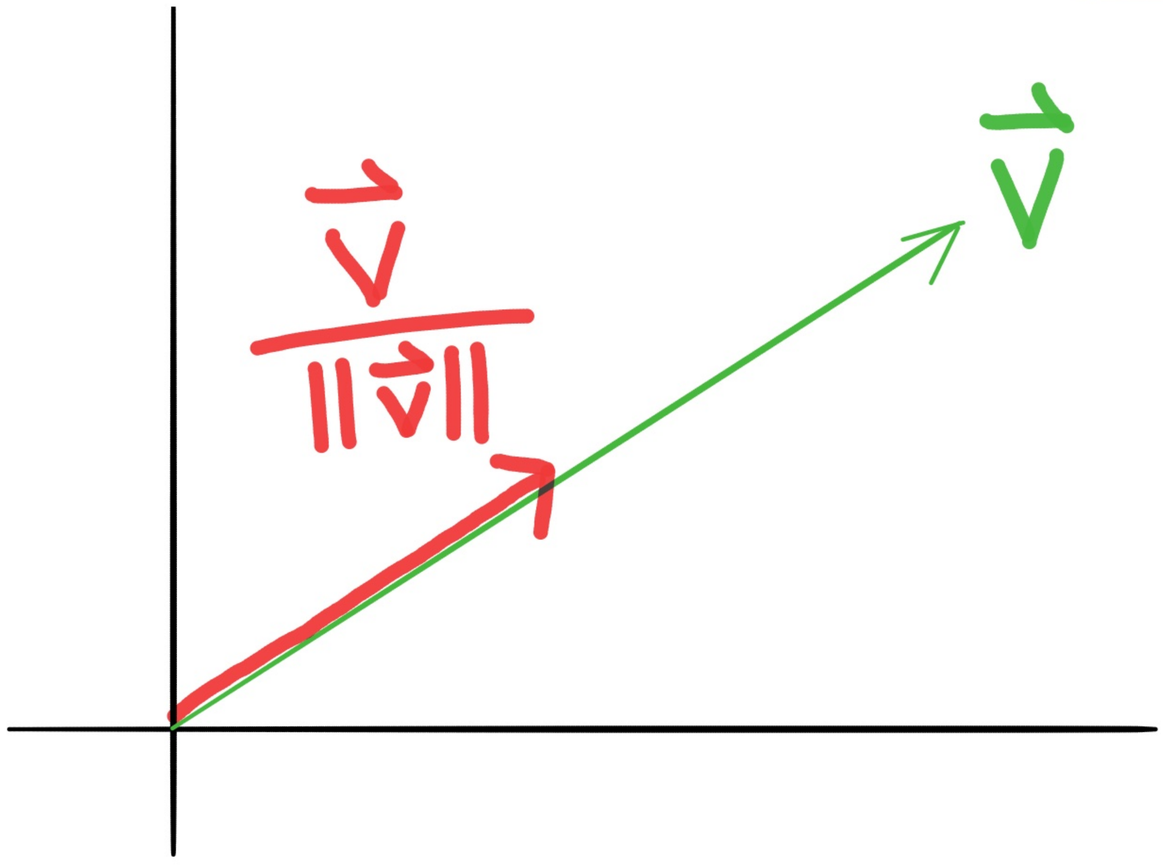

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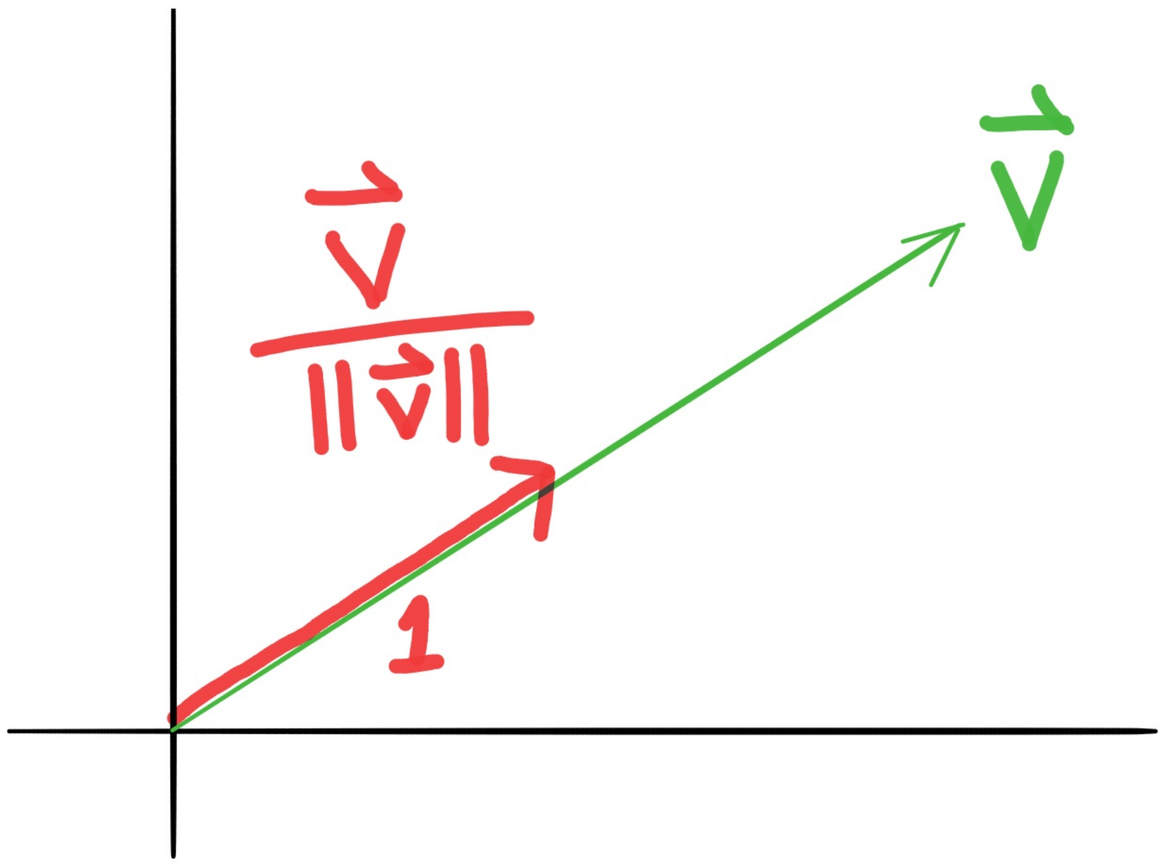

For $\vec{v} \in \mathbb{R}^n$, $(\vec{v} \neq \vec{0})$

$\frac{\vec{v}}{\|\vec{v}\|}$ is the unit vector in the direction of $\vec{v}$.

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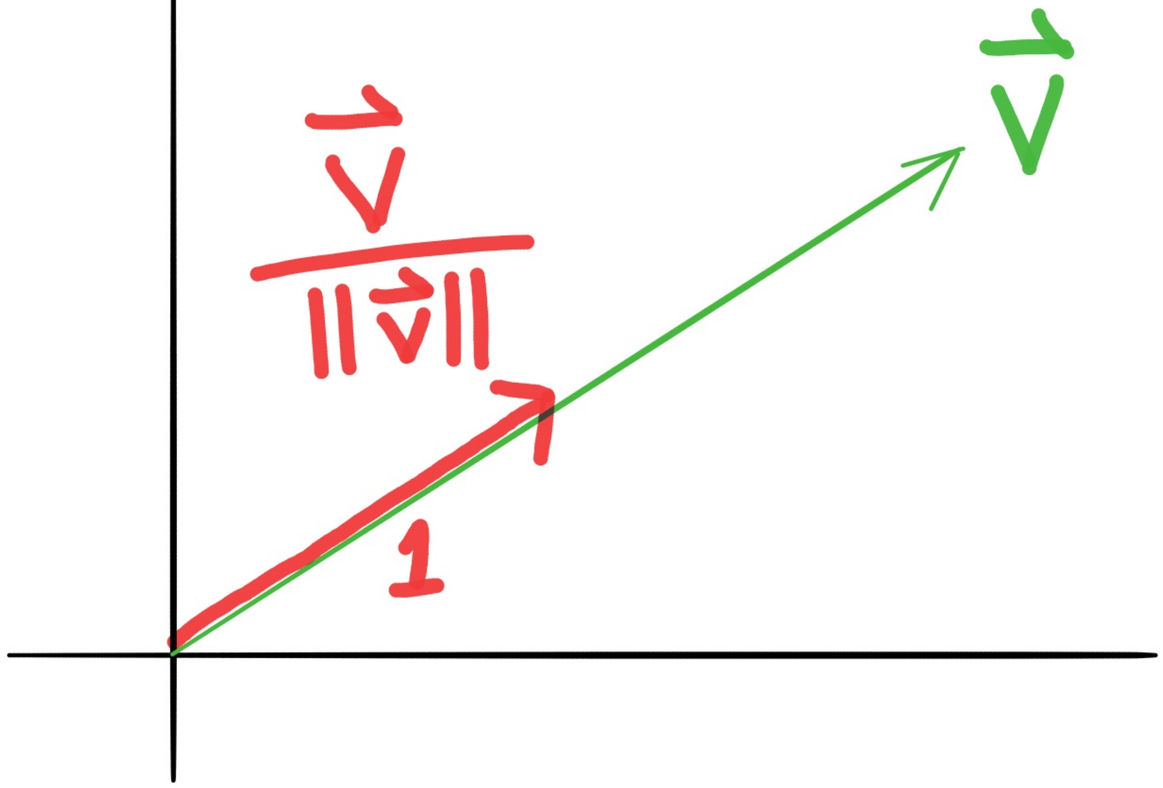

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Have to check $\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = 1$.

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$$\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \left\| \frac{\langle x, y \rangle}{\|\vec{v}\|} \right\| = \left\| \left\langle \frac{x}{\|\vec{v}\|}, \frac{y}{\|\vec{v}\|} \right\rangle \right\|$$

$$= \sqrt{\left(\frac{x}{\|\vec{v}\|}\right)^2 + \left(\frac{y}{\|\vec{v}\|}\right)^2} = \sqrt{\frac{1}{\|\vec{v}\|^2}\left(x^2 + y^2\right)}$$

$$= \frac{1}{\|\vec{v}\|}\sqrt{x^2 + y^2} = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1.$$

proof:

Have to check $\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = 1$.

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END

Example

Find the unit vector in the direction of $\langle 2,3 \rangle \in \mathbb{R}^2$.

$$\vec{v} \rightsquigarrow \frac{\vec{v}}{\|\vec{v}\|}$$

Example

Find the unit vector in the direction of $\langle 2,3 \rangle \in \mathbb{R}^2$.

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1.) Find $\|\vec{v}\|$.

2.) Find $\frac{\vec{v}}{\|\vec{v}\|}$.

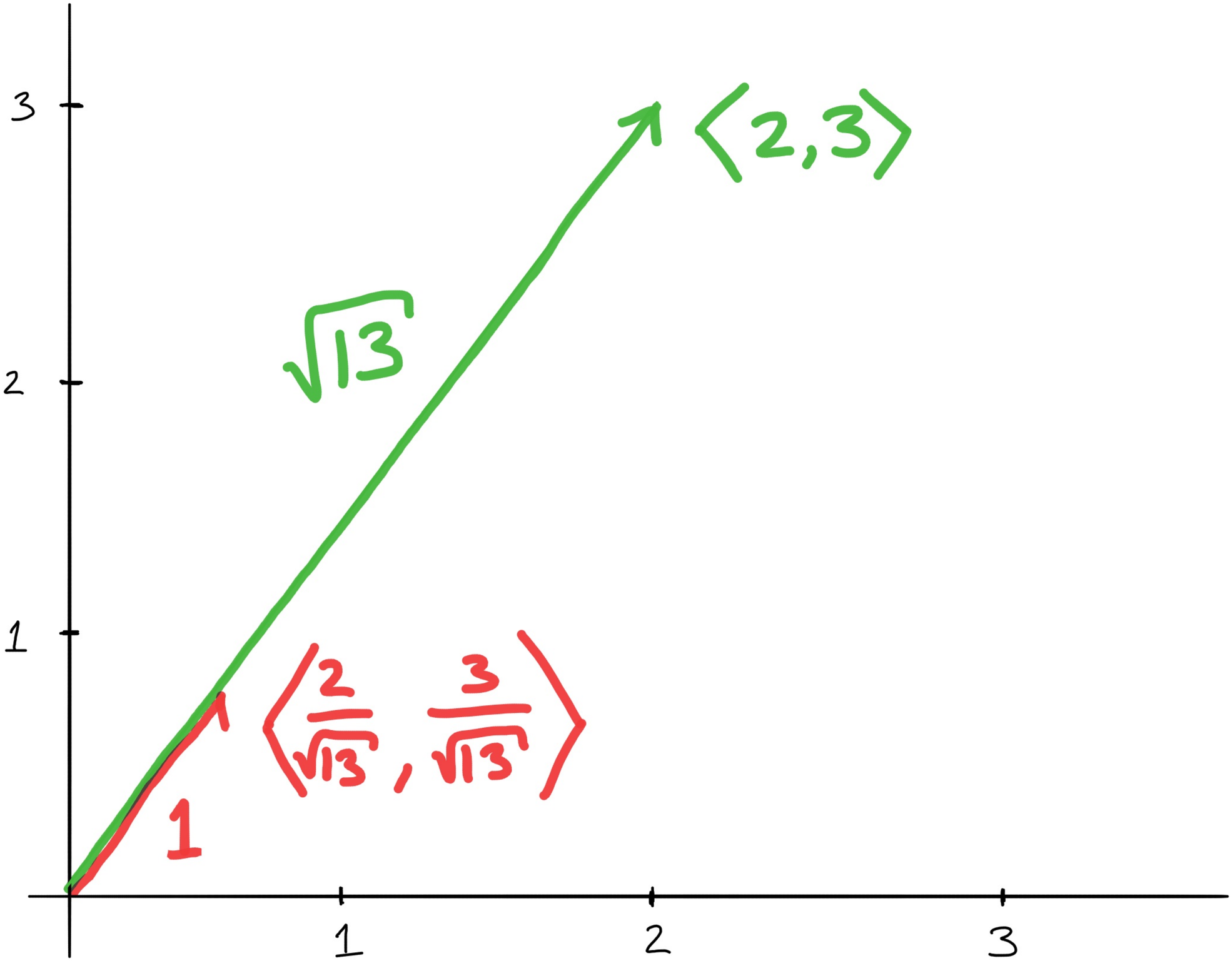
Example

Find the unit vector in the direction of $\langle 2,3 \rangle \in \mathbb{R}^2$.

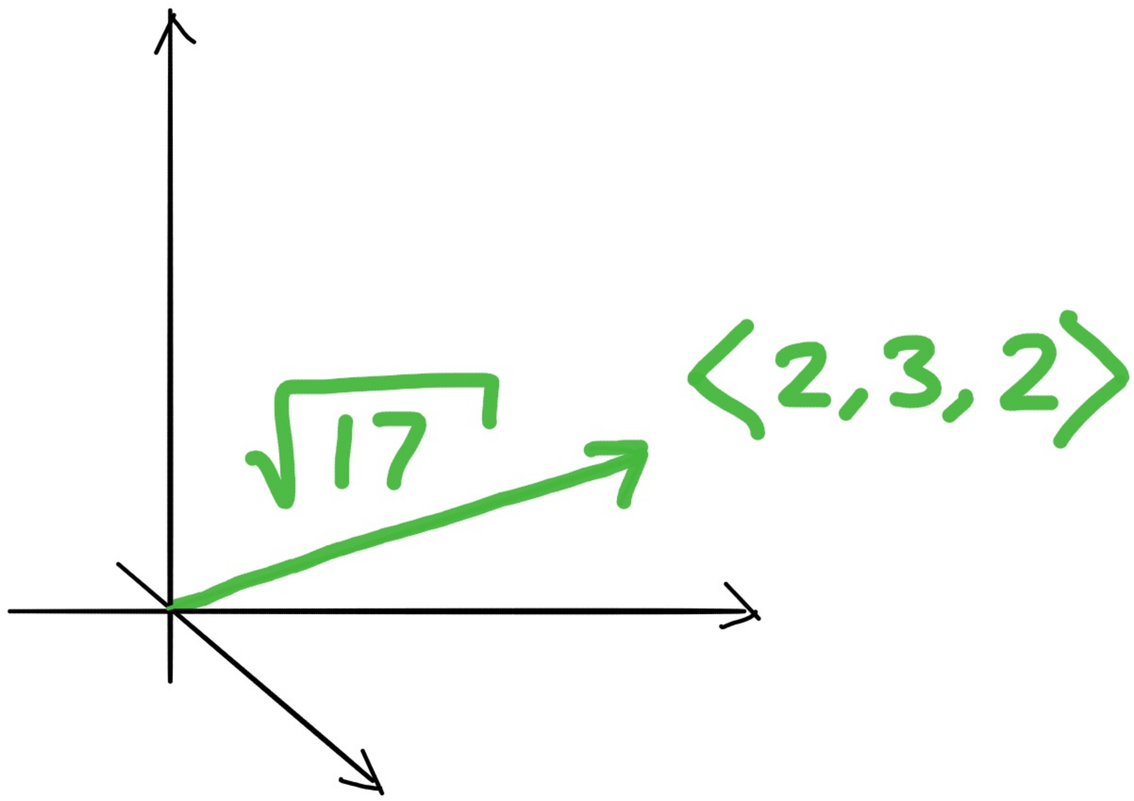
$$\vec{v} \rightsquigarrow \frac{\vec{v}}{\|\vec{v}\|}$$

$$\|\langle 2,3 \rangle\| = \sqrt{2^2+3^2} = \sqrt{13}, \text{ so}$$

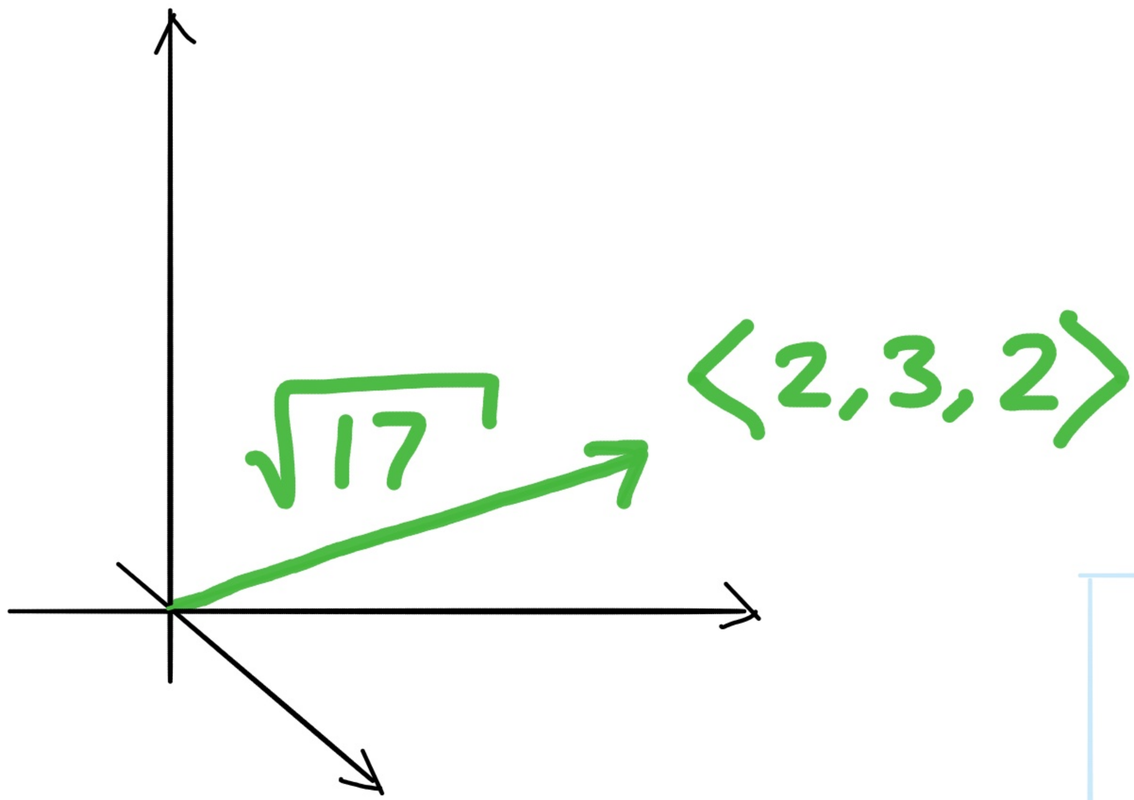
$$\frac{\langle 2,3 \rangle}{\|\langle 2,3 \rangle\|} = \frac{\langle 2,3 \rangle}{\sqrt{13}} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$


3
2
1
1
2
3
$\sqrt{13}$
$\langle 2,3 \rangle$
$\left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$
1

Example: Find unit vector in direction of vector below.



Example: Find unit vector in direction of vector below.

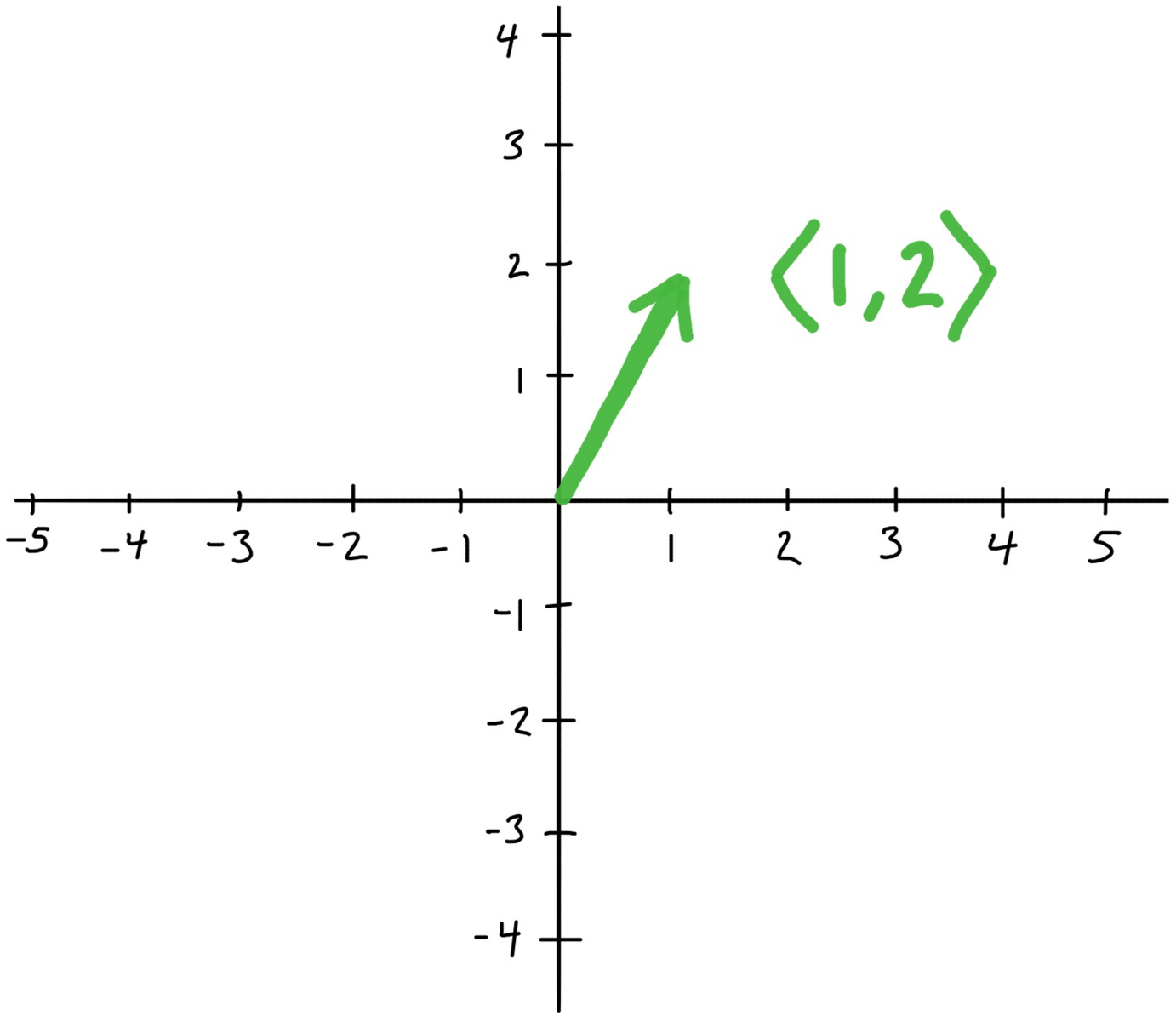


Solution:

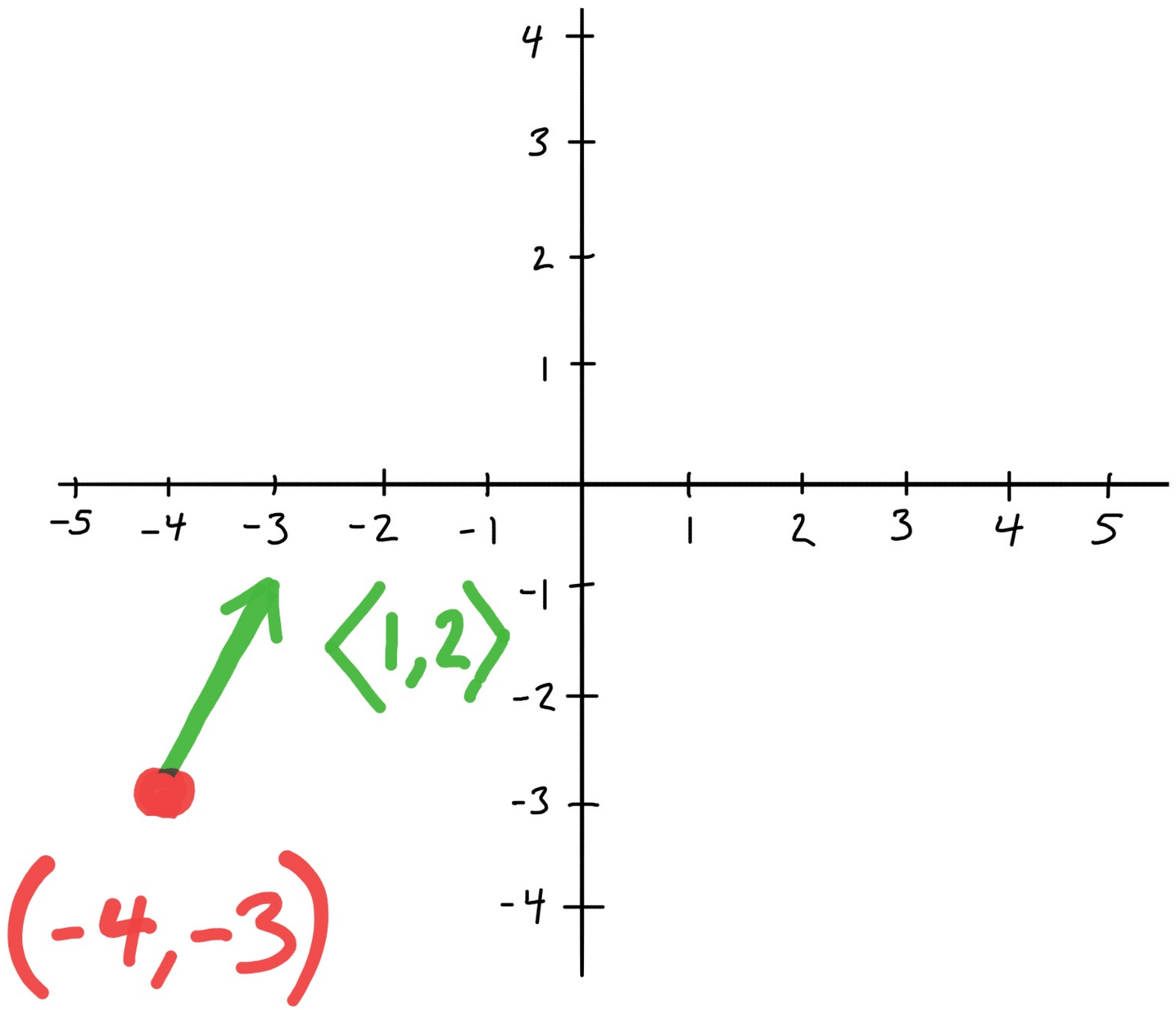
$$\left\langle \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}} \right\rangle$$

Ⅴ Geometry of vectors

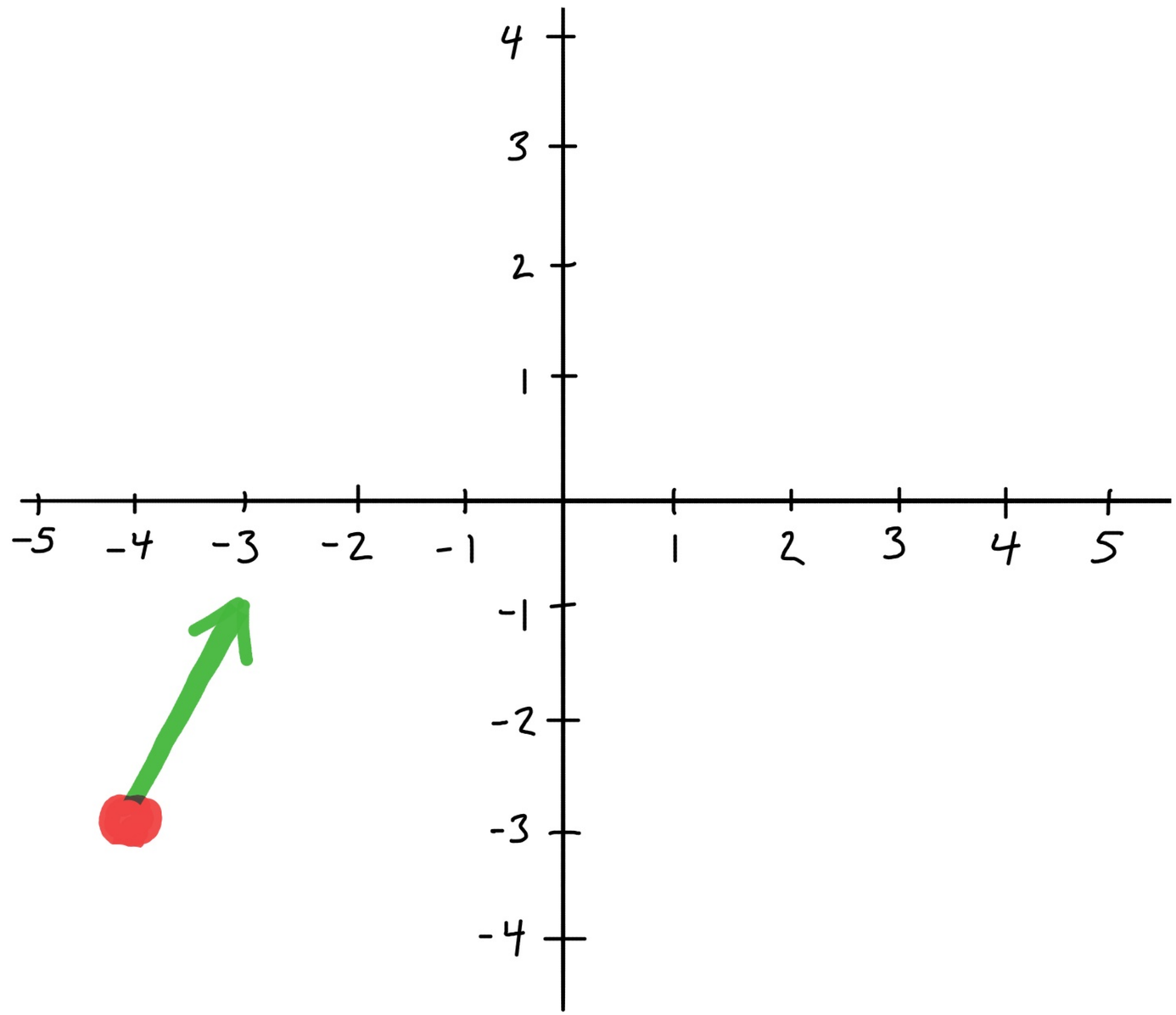
Equivalent vectors



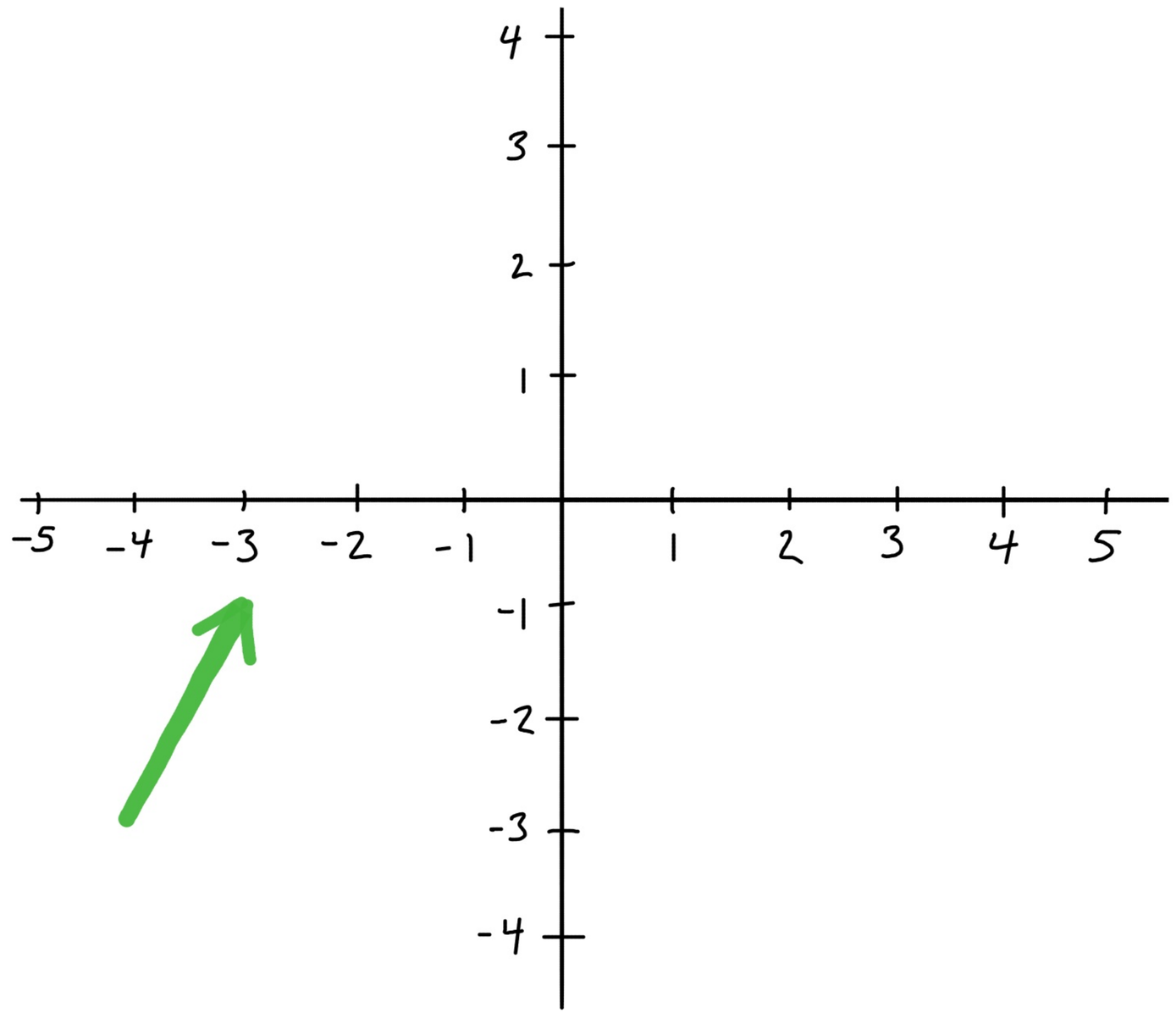
Equivalent vectors



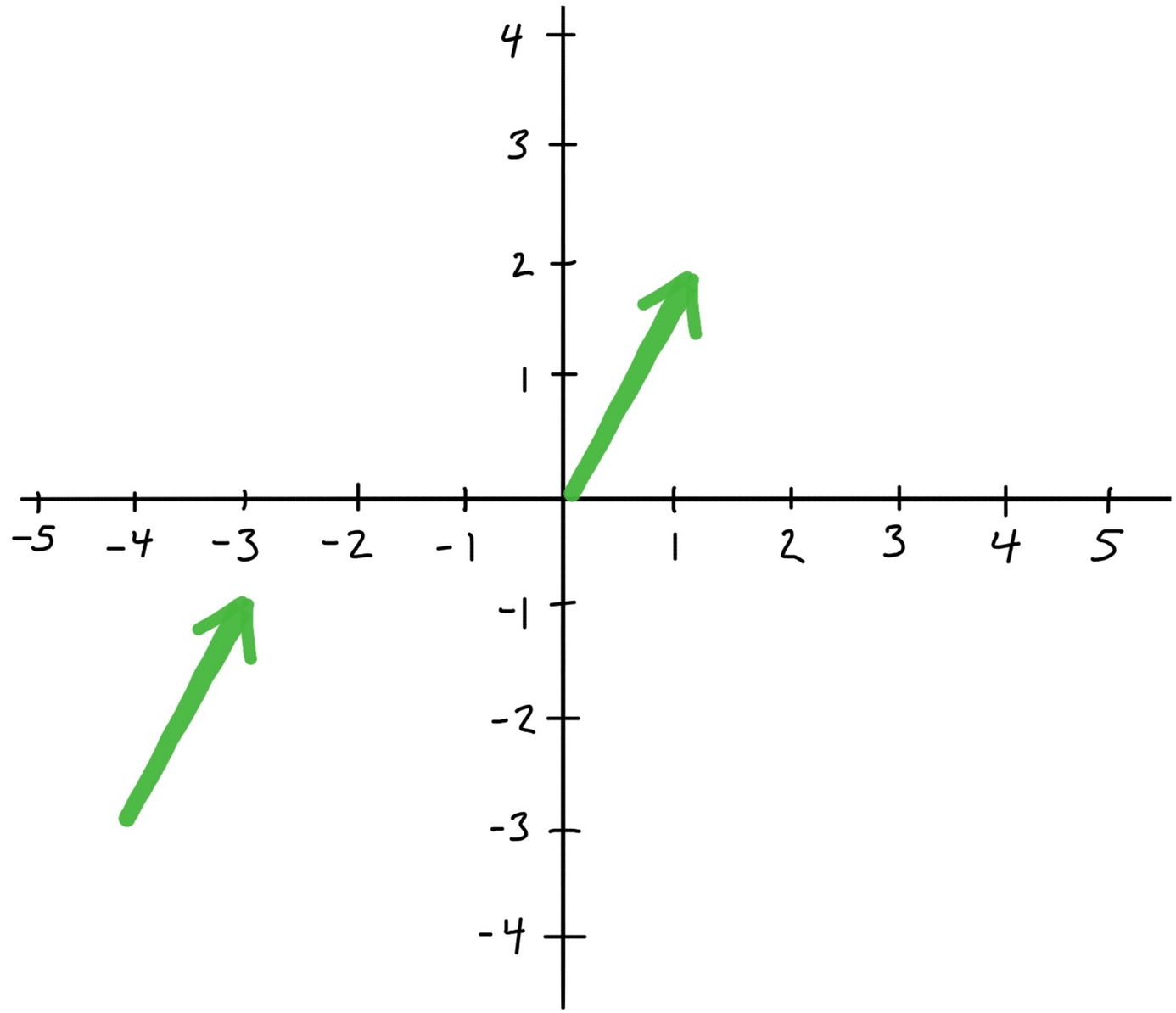
Equivalent vectors



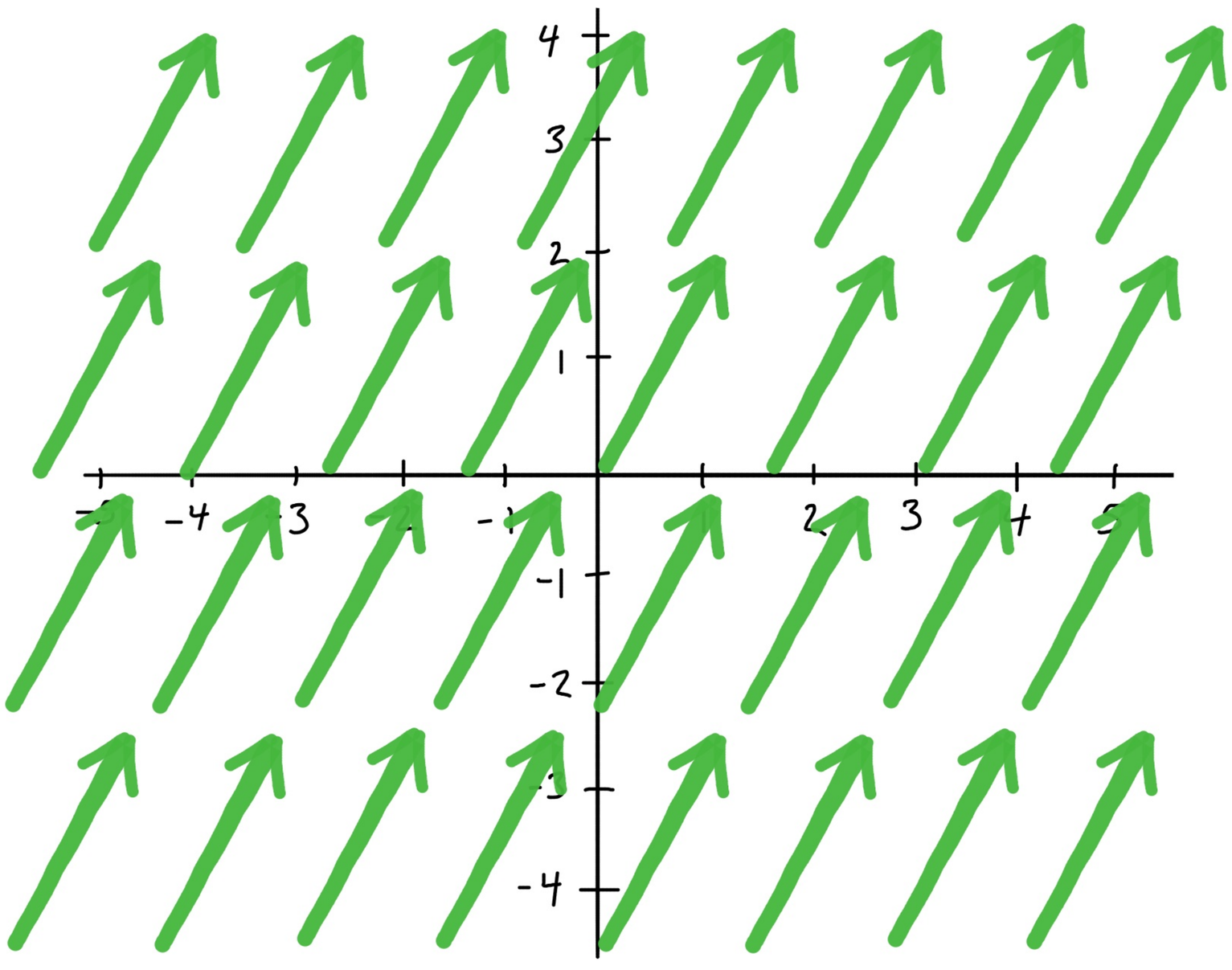
Equivalent vectors



Equivalent vectors

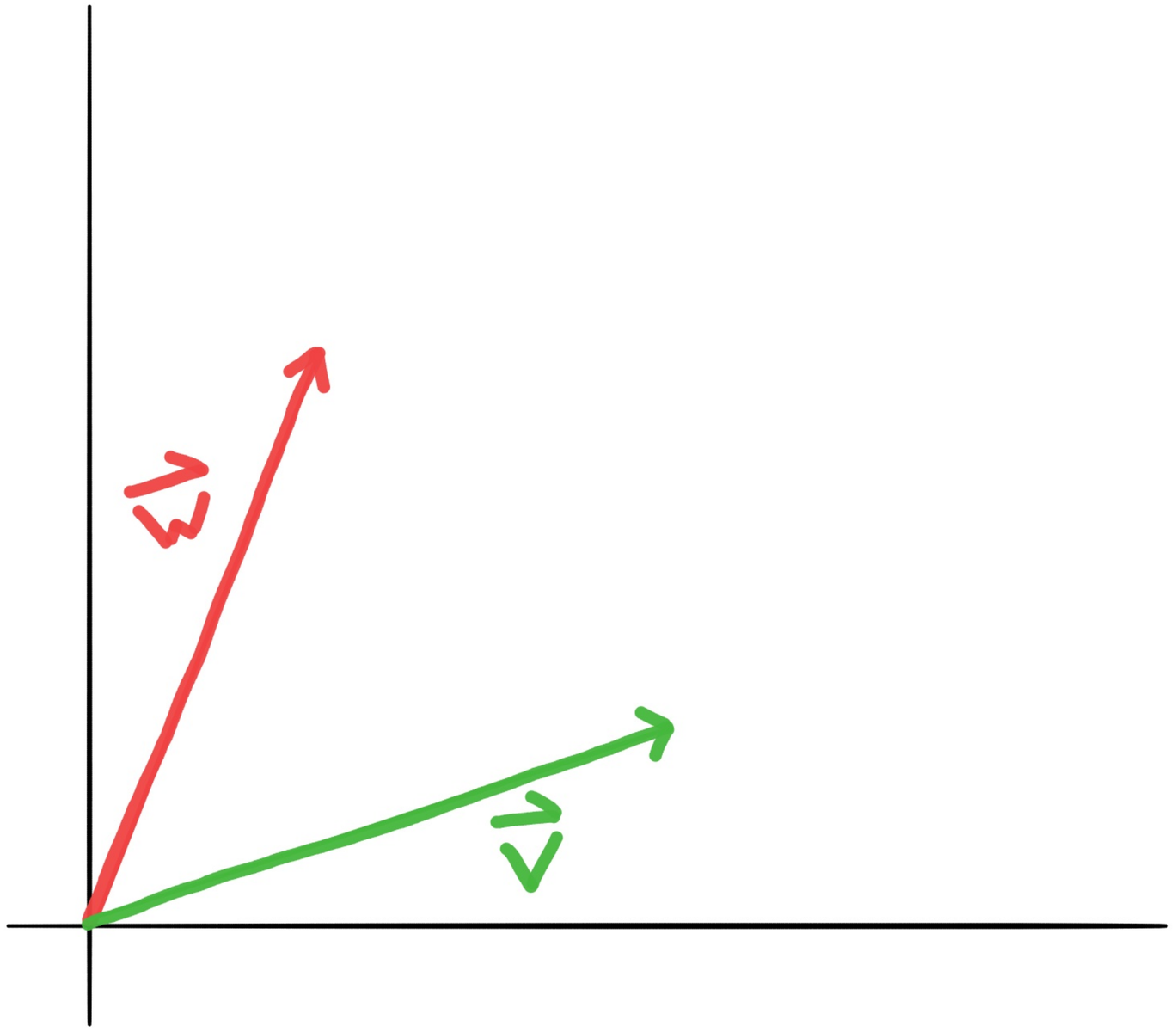


Equivalent vectors

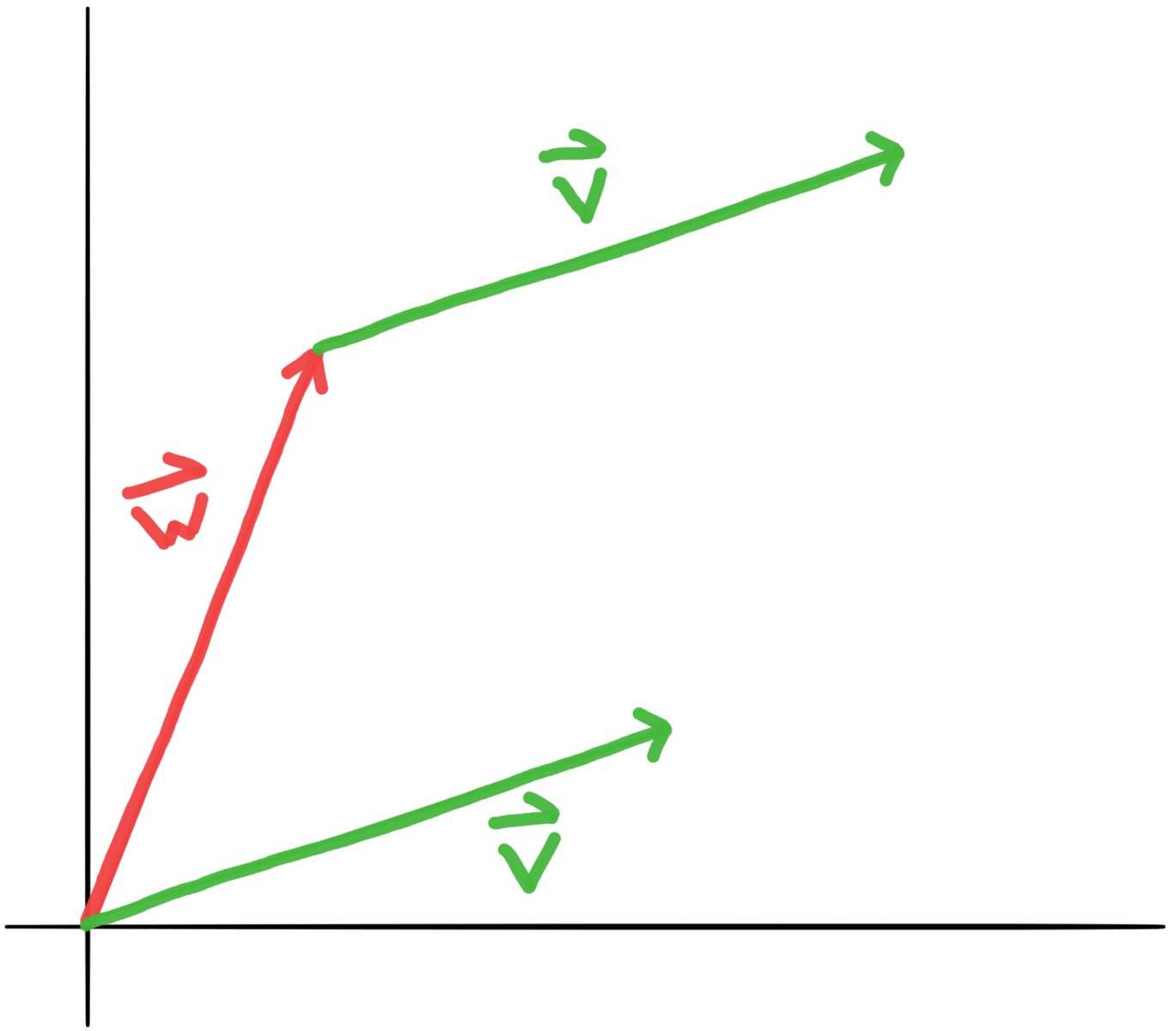


$\langle 1, 2 \rangle$

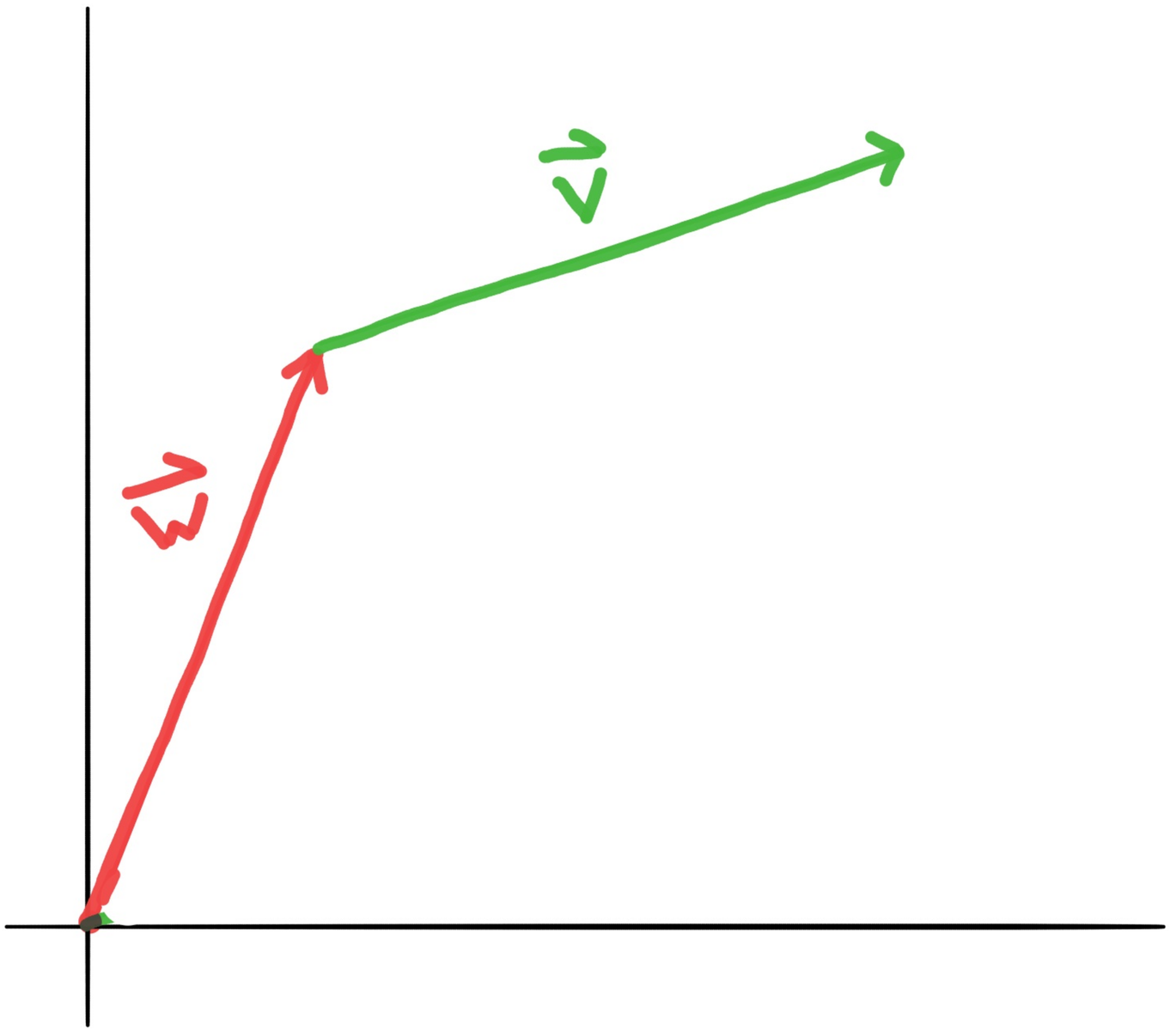
Vector addition



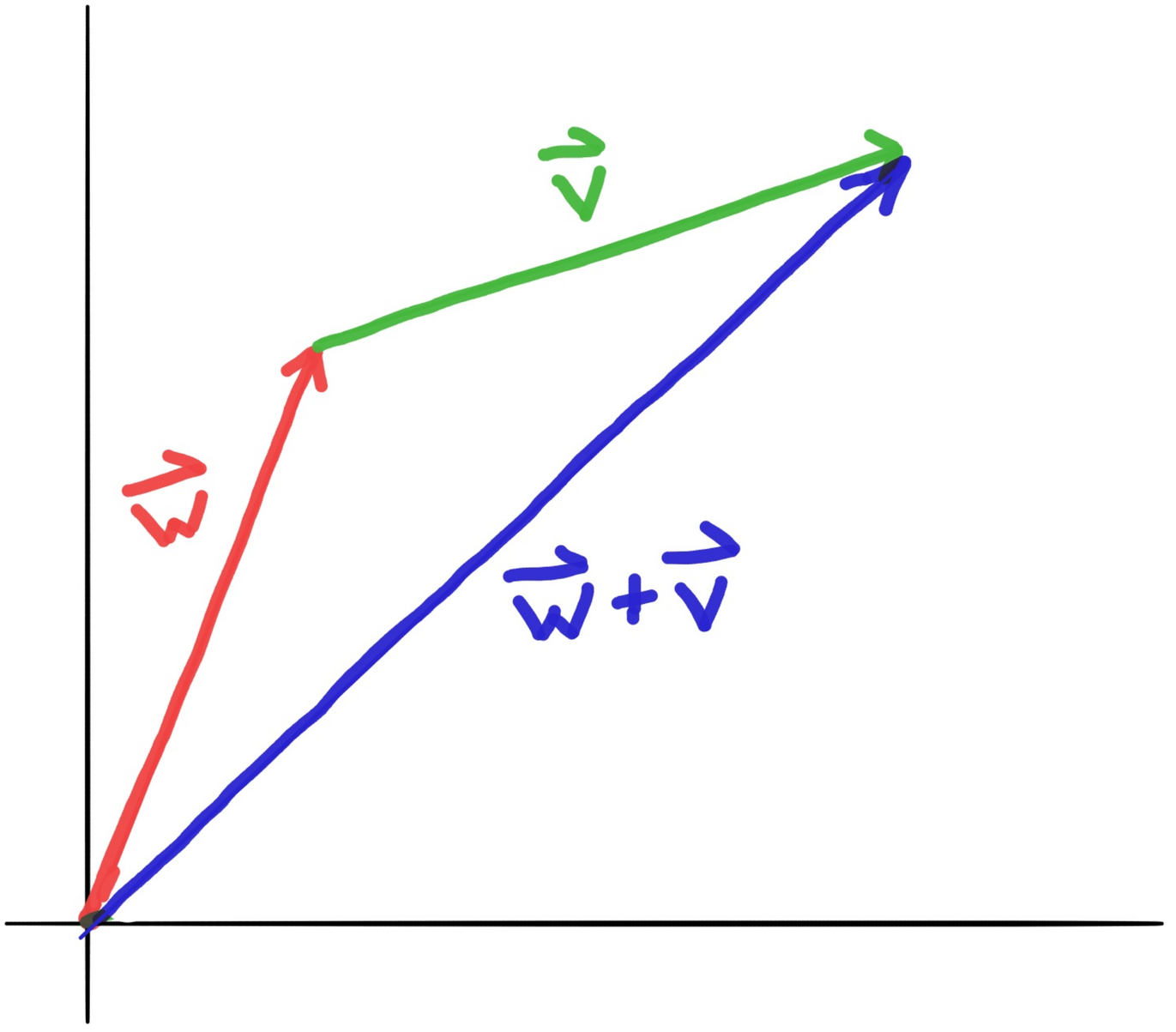
Vector addition



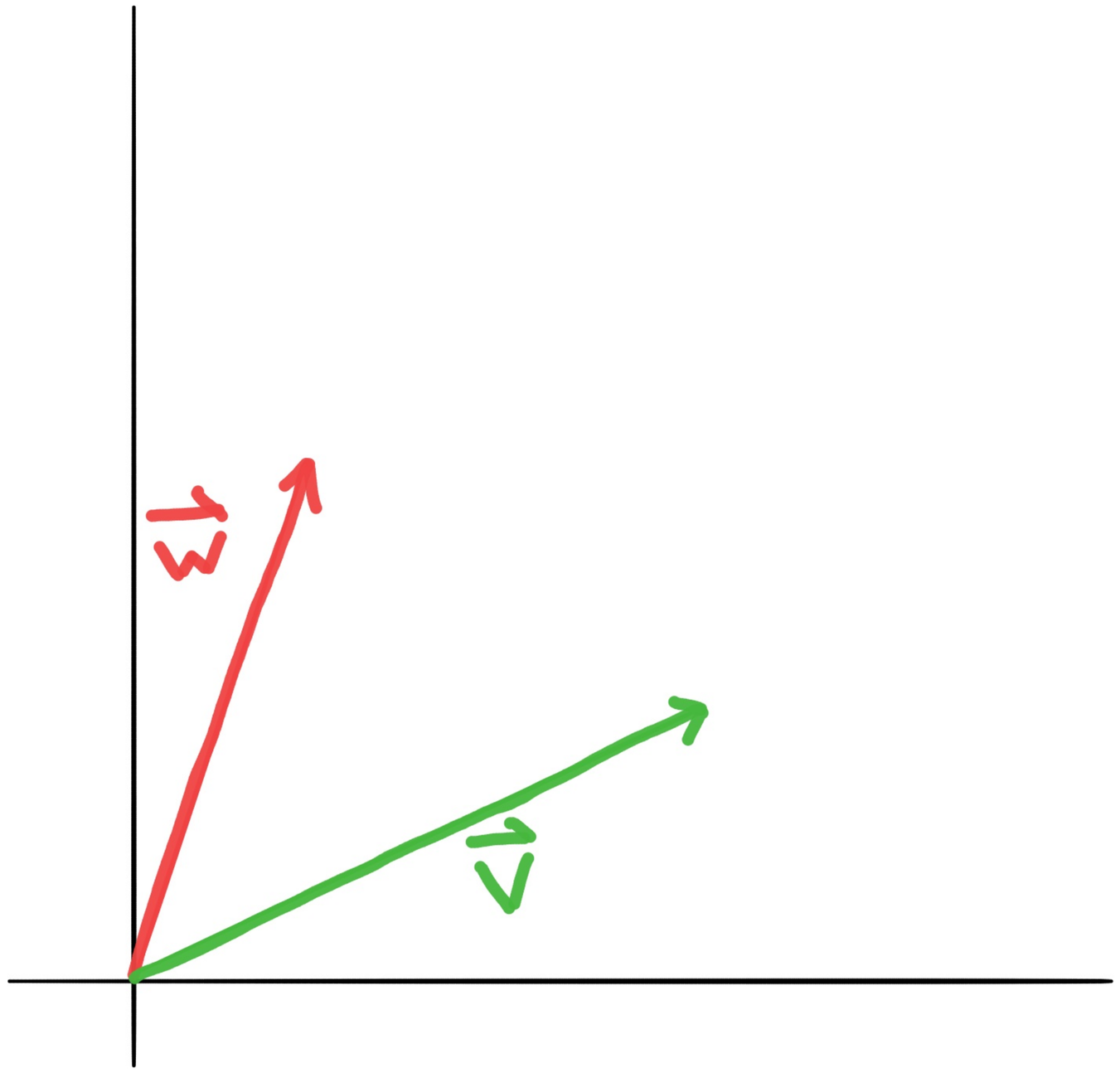
Vector addition



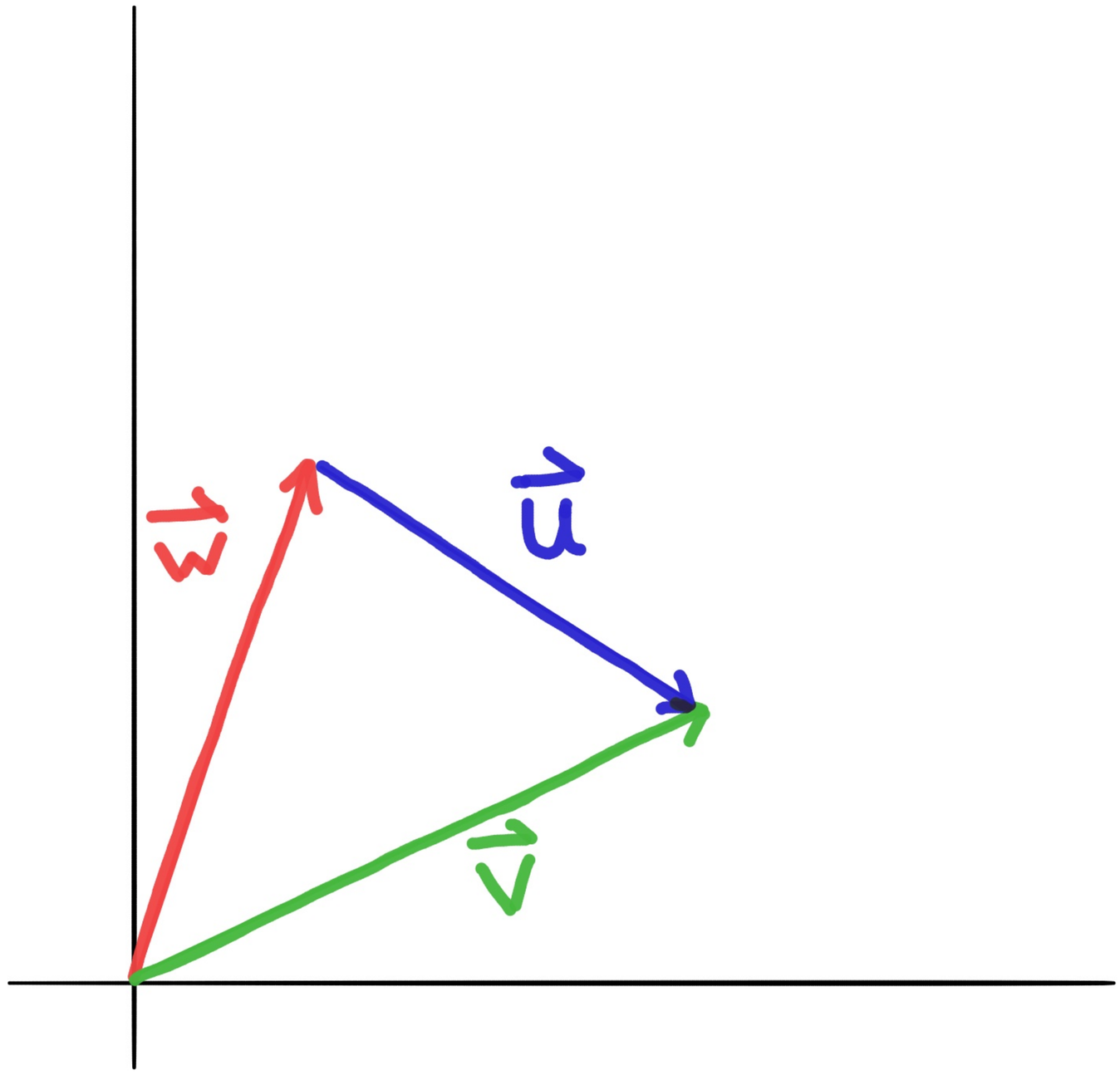
Vector addition



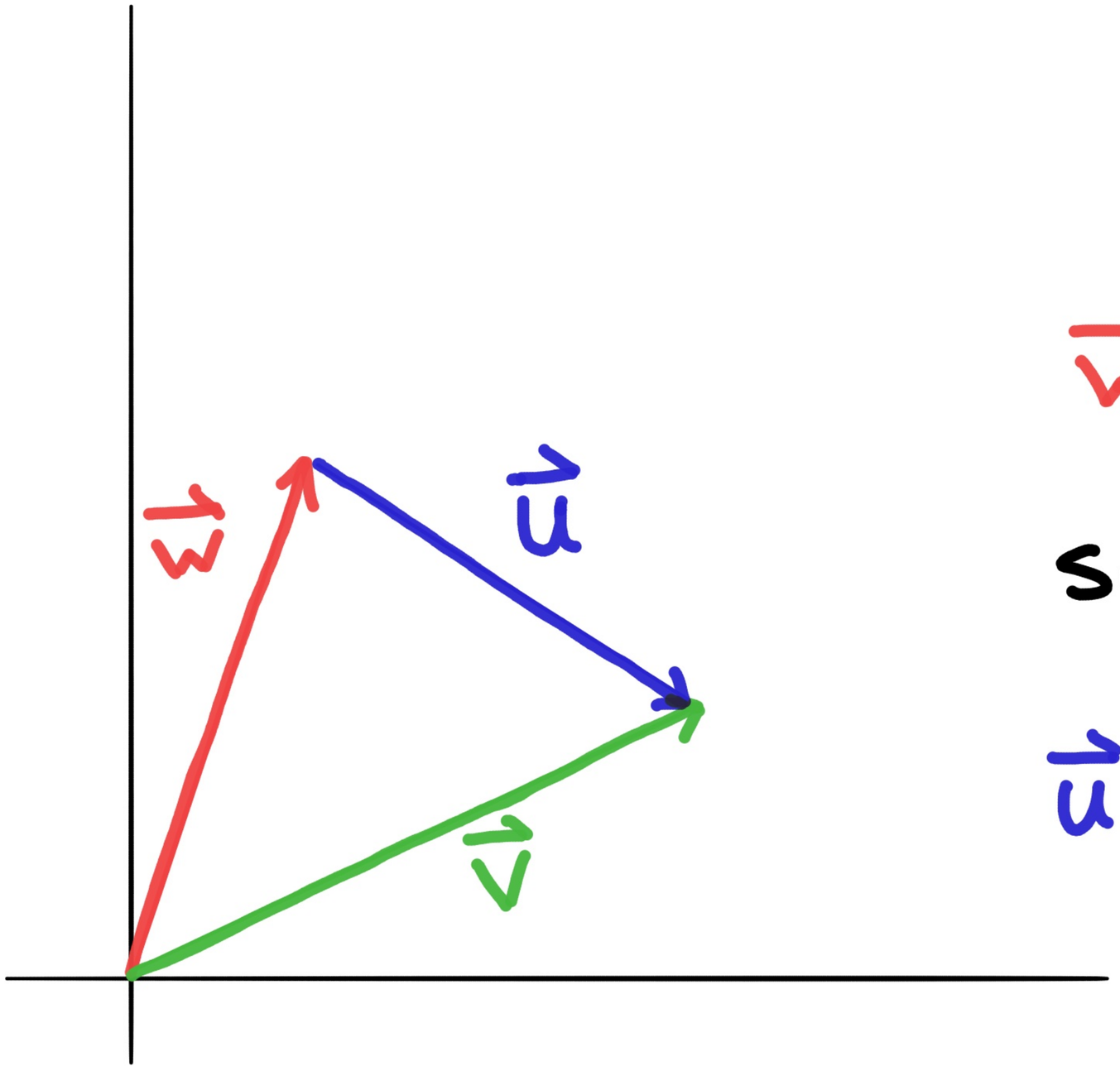
Vector subtraction



Vector subtraction



Vector subtraction

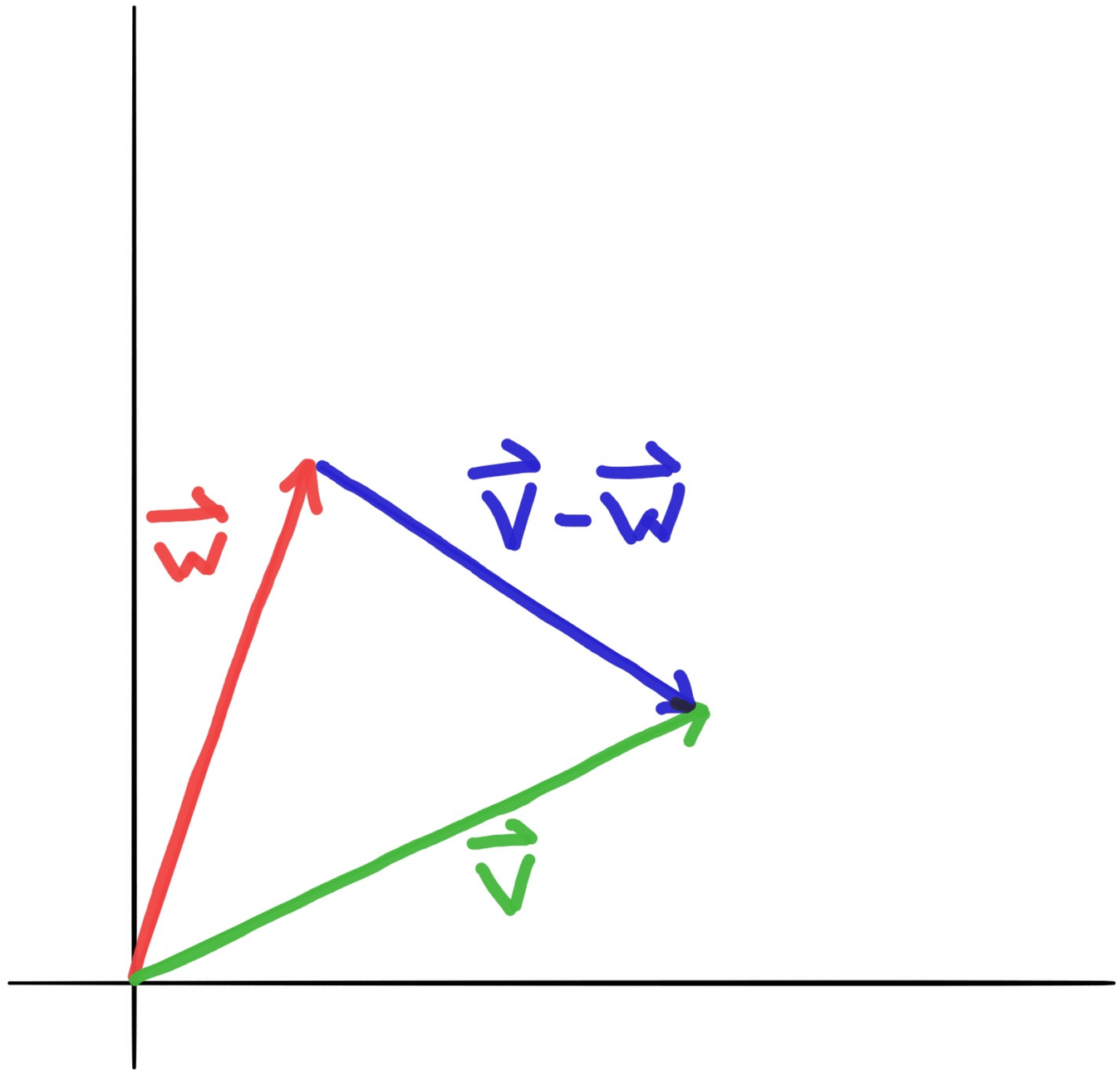


$$\vec{w} + \vec{u} = \vec{v}$$

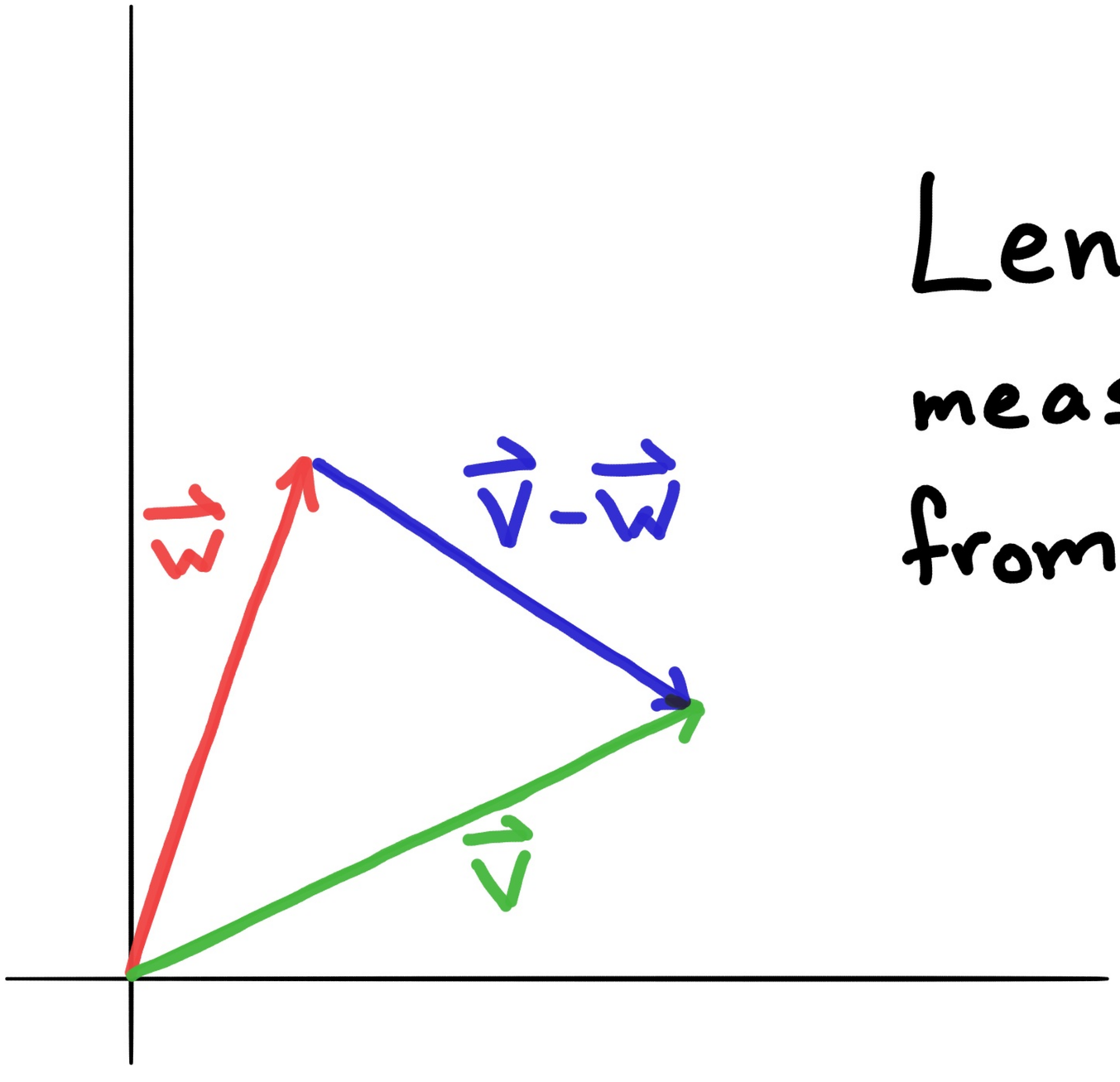
so

$$\vec{u} = \vec{v} - \vec{w}$$

Vector subtraction

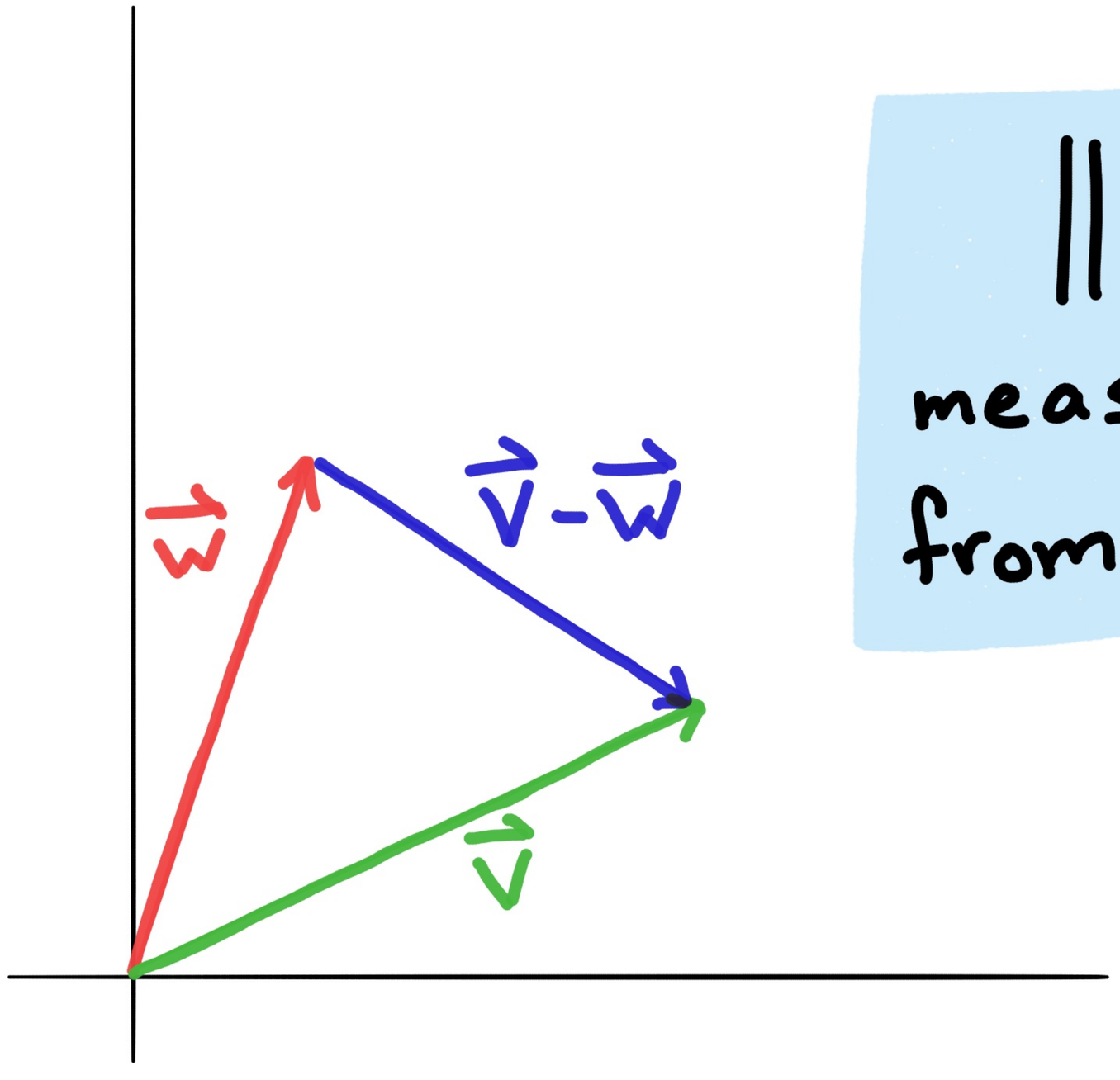


Vector subtraction



Length of $\vec{v}-\vec{w}$
measures distance
from $\vec{w}$ to $\vec{v}$

Vector subtraction



$\|\vec{v}-\vec{w}\|$

measures distance from $\vec{w}$ to $\vec{v}$

$\|\vec{v}-\vec{w}\|$ is distance between $\vec{v}$ and $\vec{w}$.

Example: Find distance between $\langle 2,7\rangle$ and $\langle 1,3\rangle$.

$\|\vec{v}-\vec{w}\|$ is distance between $\vec{v}$ and $\vec{w}$.

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1.) $\langle 2,7\rangle - \langle 1,3\rangle = \textcircled{?}$

2.) $\|\textcircled{?}\| = ??$

$\|\vec{v}-\vec{w}\|$ is distance between $\vec{v}$ and $\vec{w}$.

Example: Find distance between $\langle 2,7\rangle$ and $\langle 1,3\rangle$.

1.) $\langle 2,7\rangle - \langle 1,3\rangle = \langle 1,4\rangle$

2.) $\|\langle 1,4\rangle\| = \sqrt{1^2+4^2} = \sqrt{17}$

$\sqrt{17}$ is distance.

Note:

$$a\vec{i} + b\vec{j} + c\vec{k} = \langle a, b, c \rangle$$

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$$= b\vec{j} + c\vec{k} + a\vec{i}$$

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$$a\vec{i} + b\vec{j} + c\vec{k} = \langle a, b, c \rangle$$

$$= b\vec{j} + c\vec{k} + a\vec{i}$$

$$\langle 2, 0, -1 \rangle = 2\vec{i} - \vec{k}$$