

Nineteen

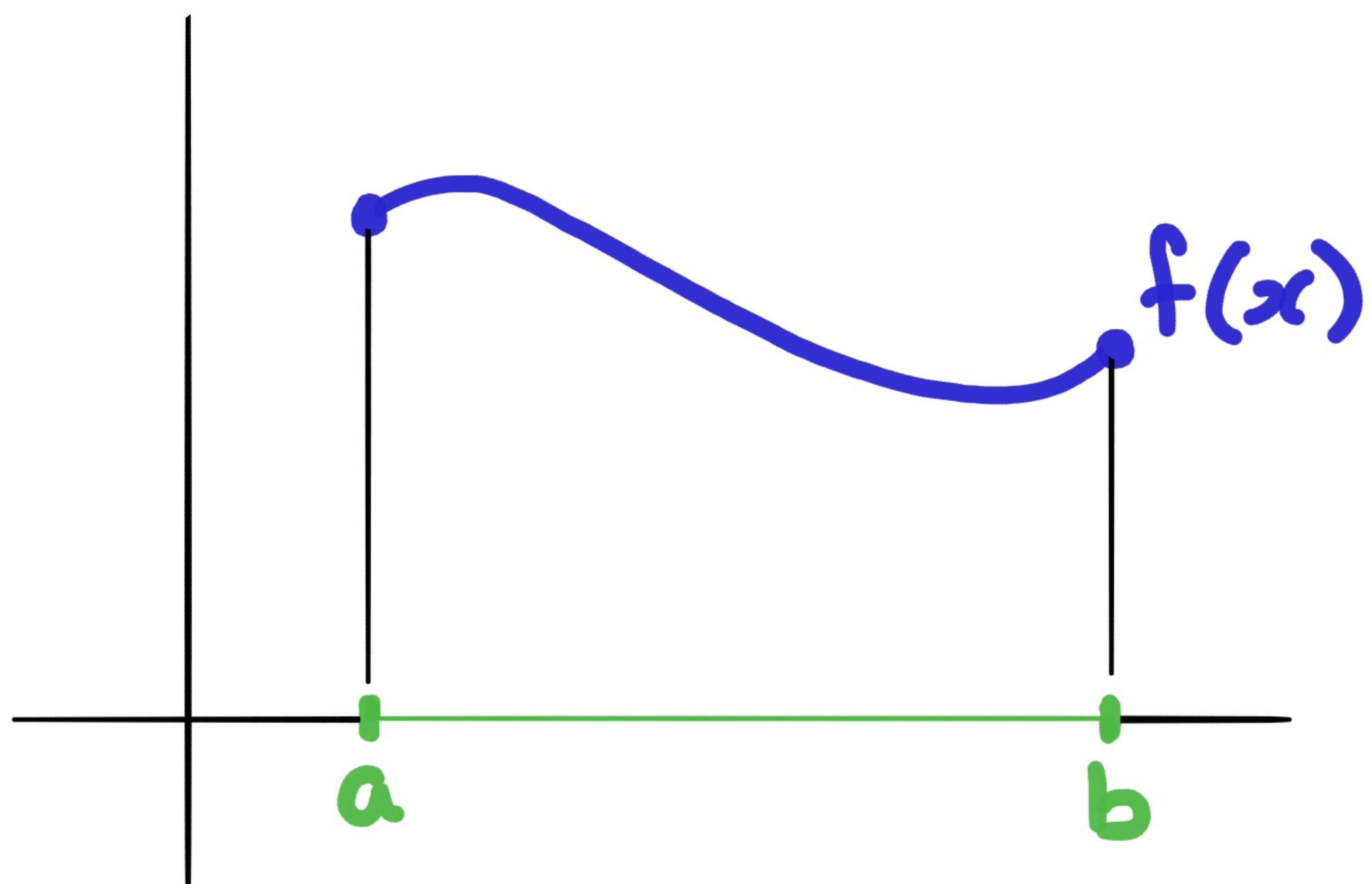
① Iterated integrals

② Simple regions in polar coordinates

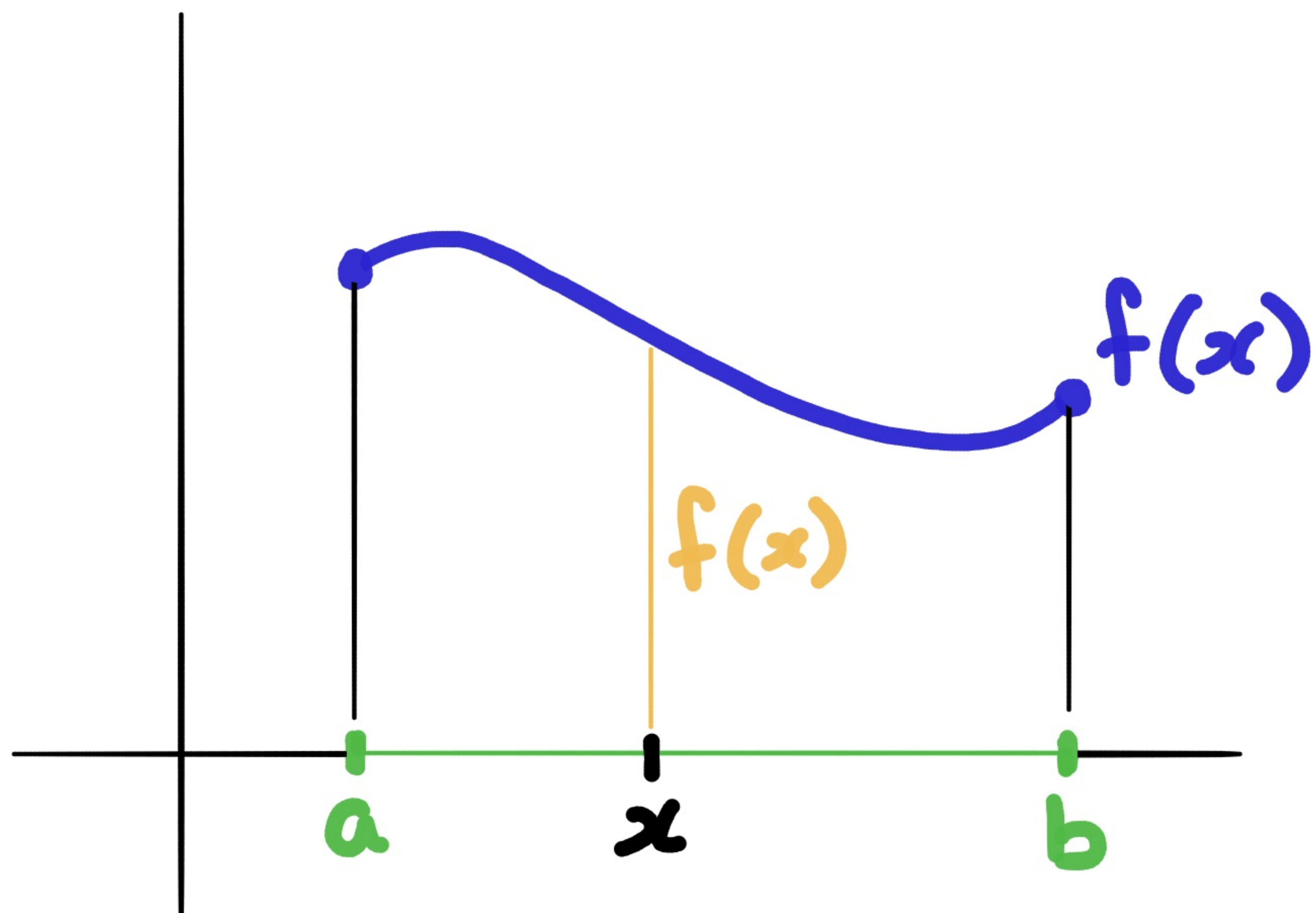
① Iterated  
integrals



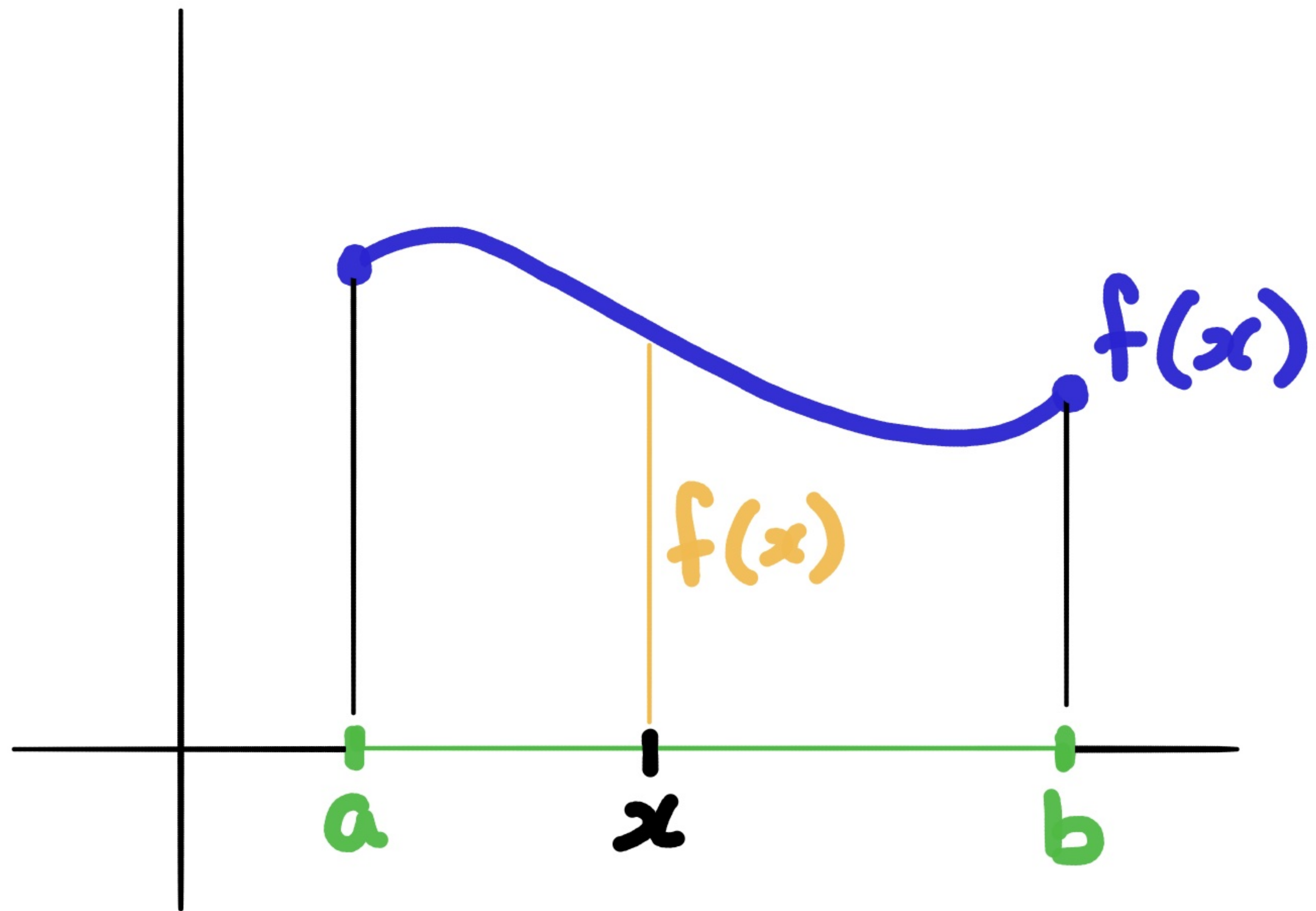
Area: width times height



$$\int_a^b f(x) dx$$



$$\int_a^b f(x) dx$$



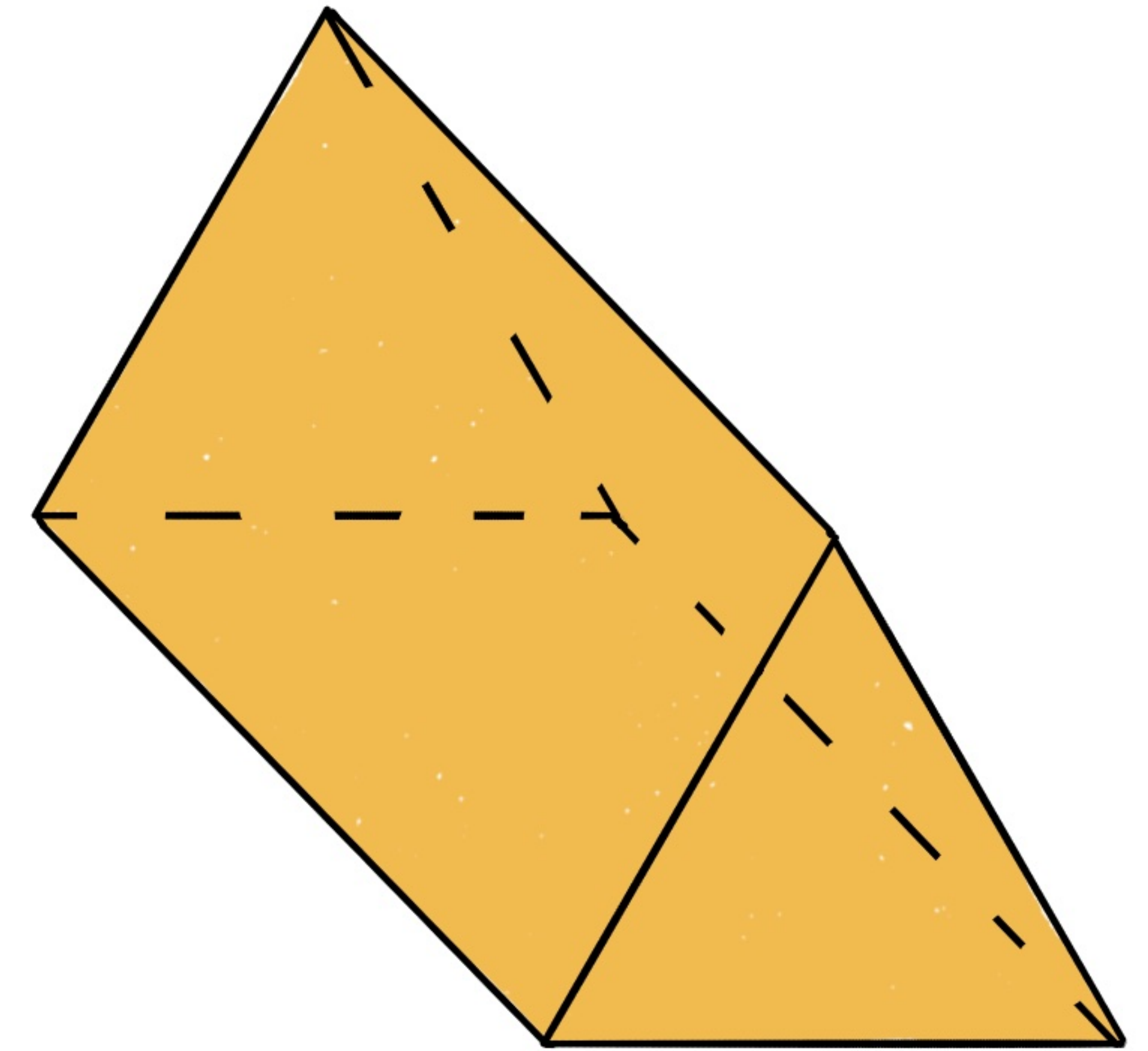
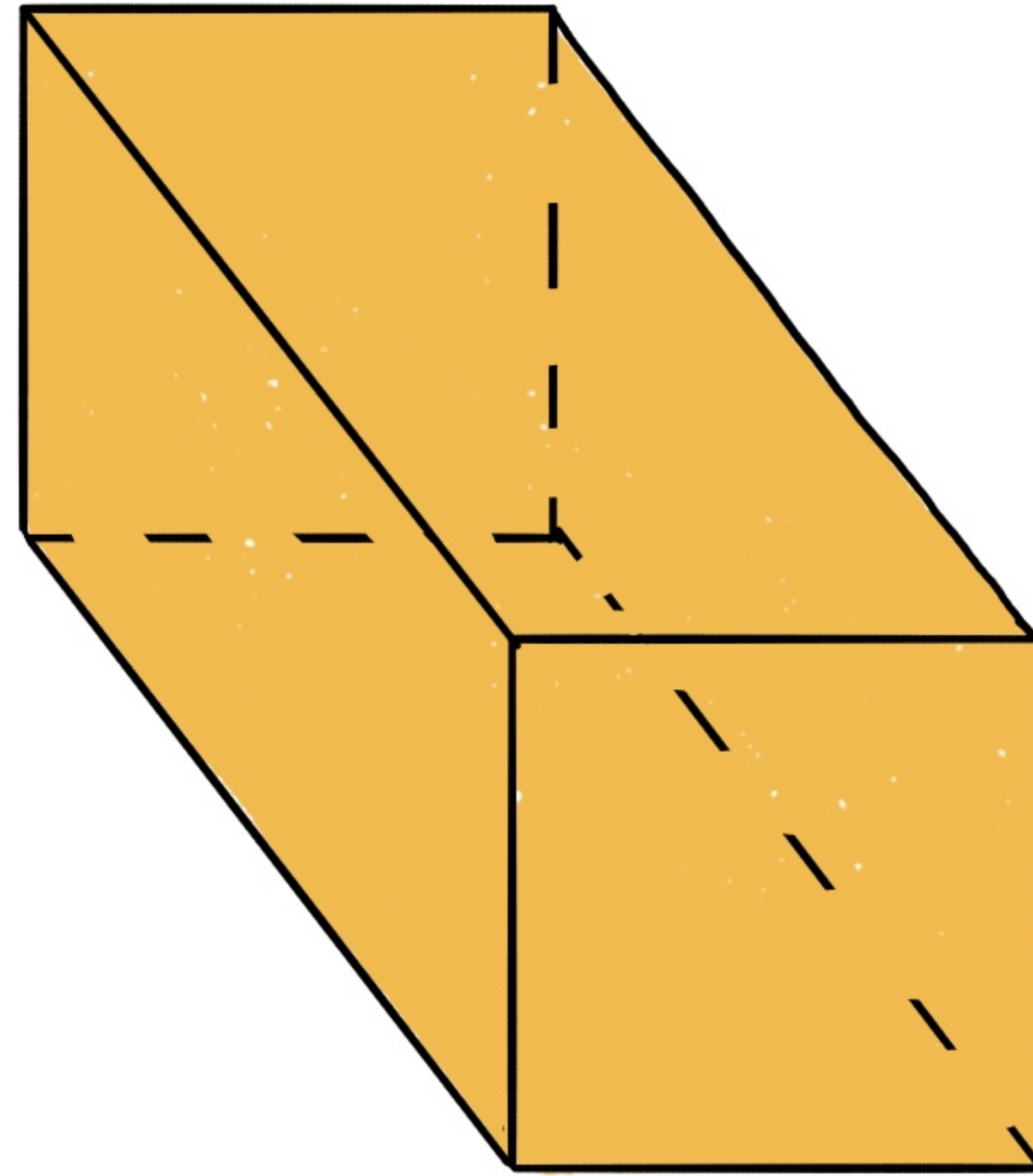
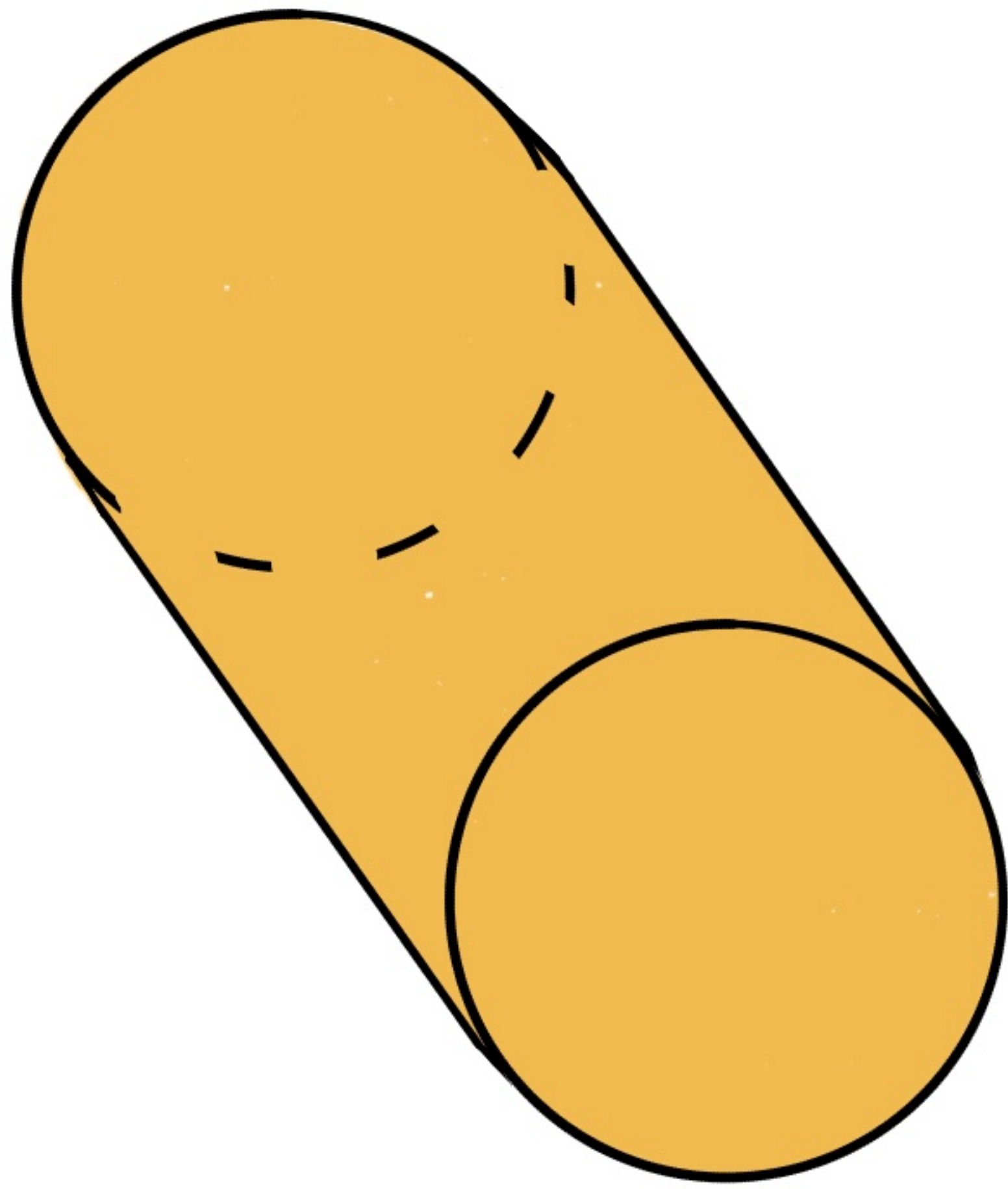
varying height

↓

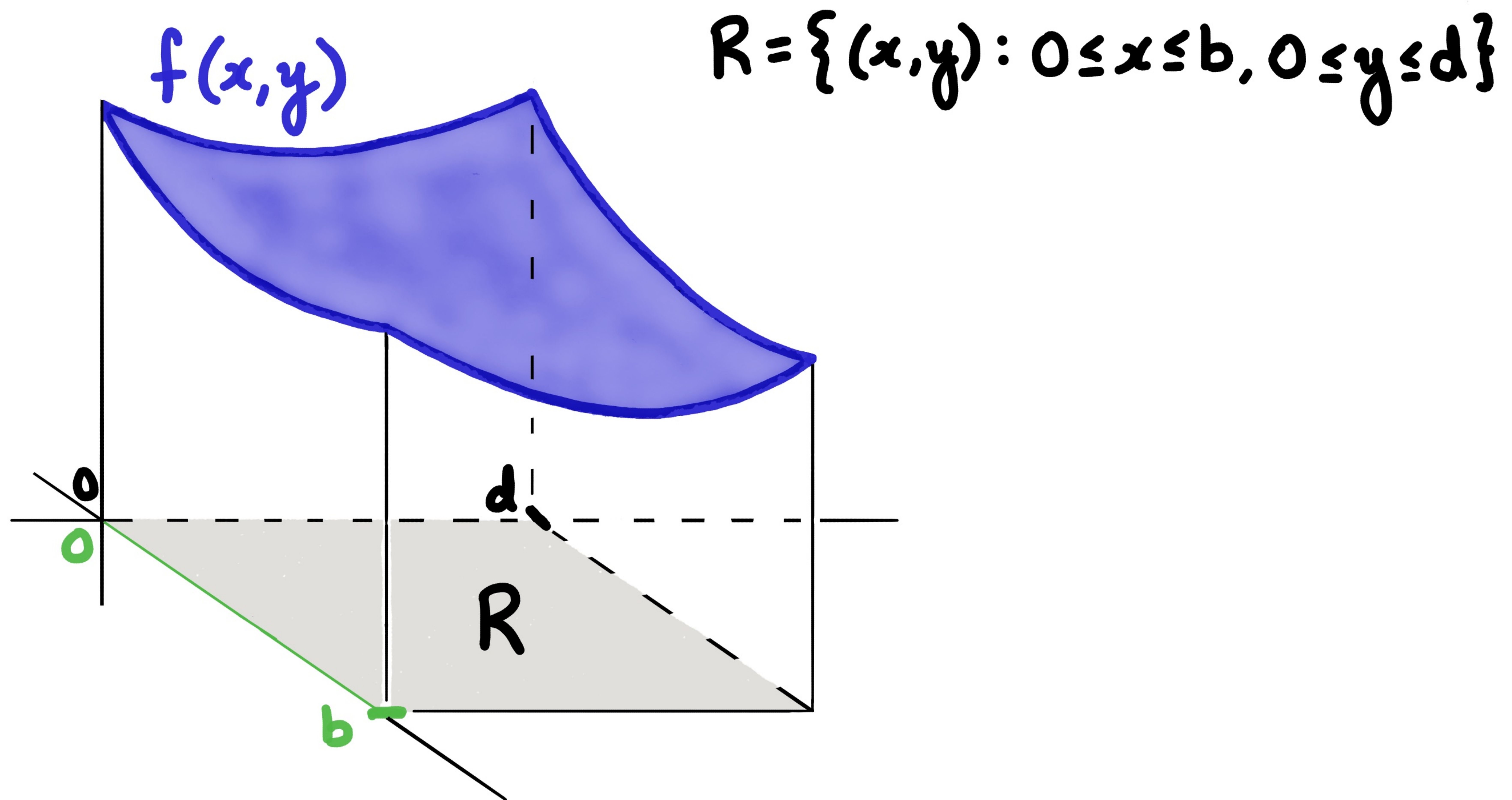
$$\int_a^b f(x) dx$$

width

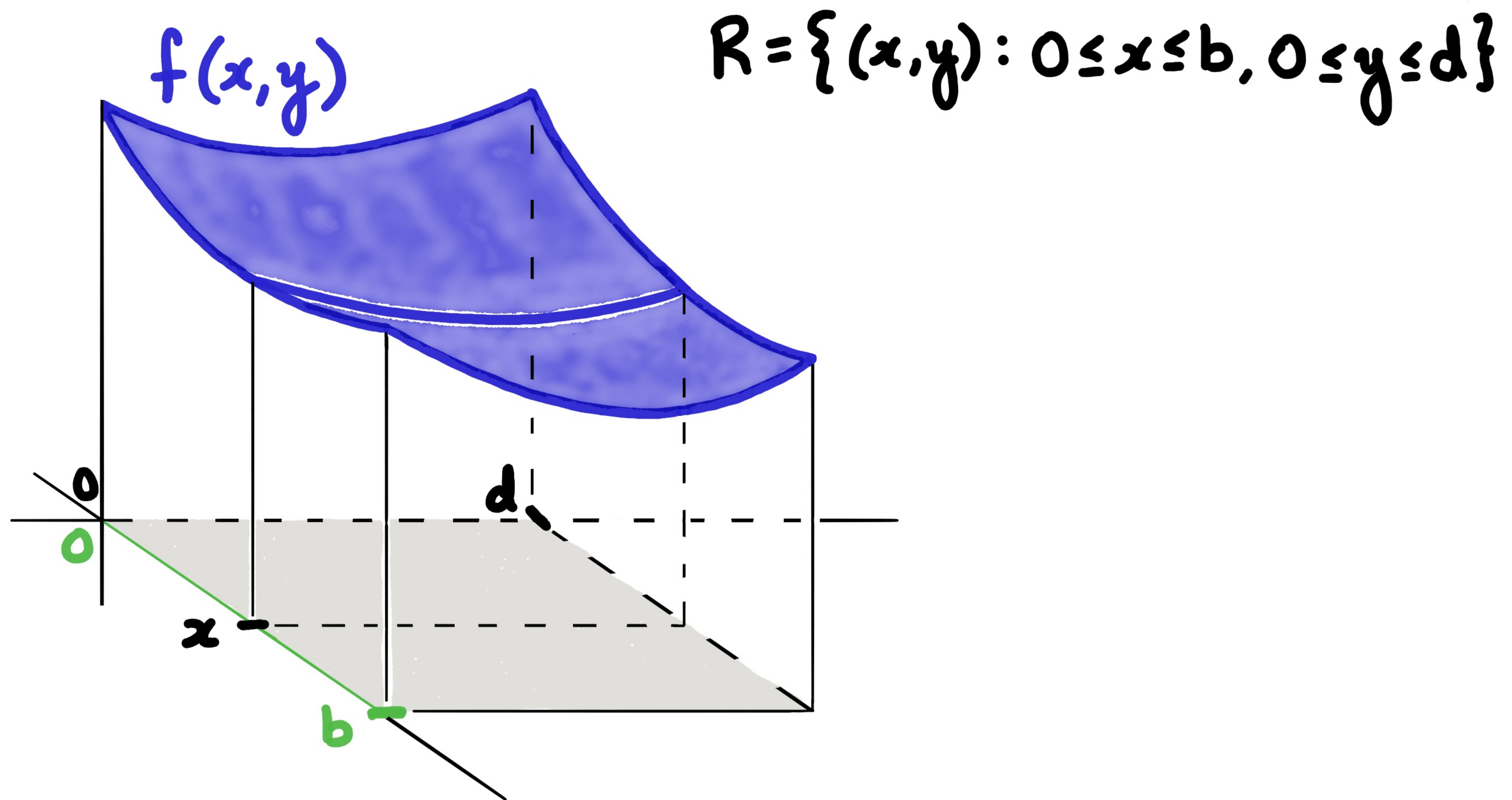
The diagram illustrates the components of the definite integral  $\int_a^b f(x) dx$ . The expression is written with the limits  $a$  and  $b$  enclosed in a light green oval, the function  $f(x)$  enclosed in a light orange oval, and the differential  $dx$  enclosed in a light green oval. An orange arrow points from the text "varying height" to the function  $f(x)$ . Two green arrows point from the text "width" to the limits  $a$  and  $b$ .



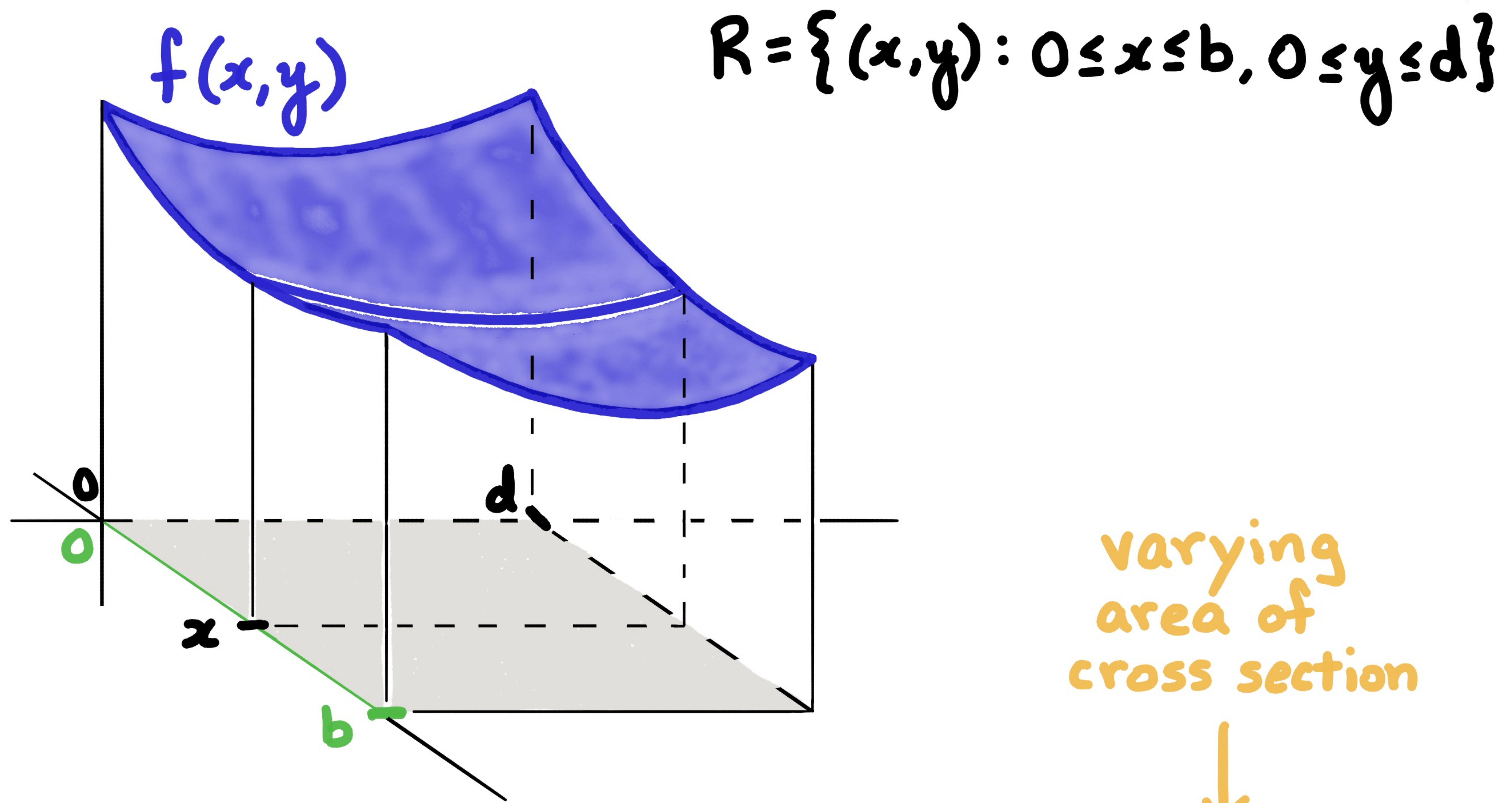
Volume: width times area of cross section



$$\iint_R f(x,y) dA$$

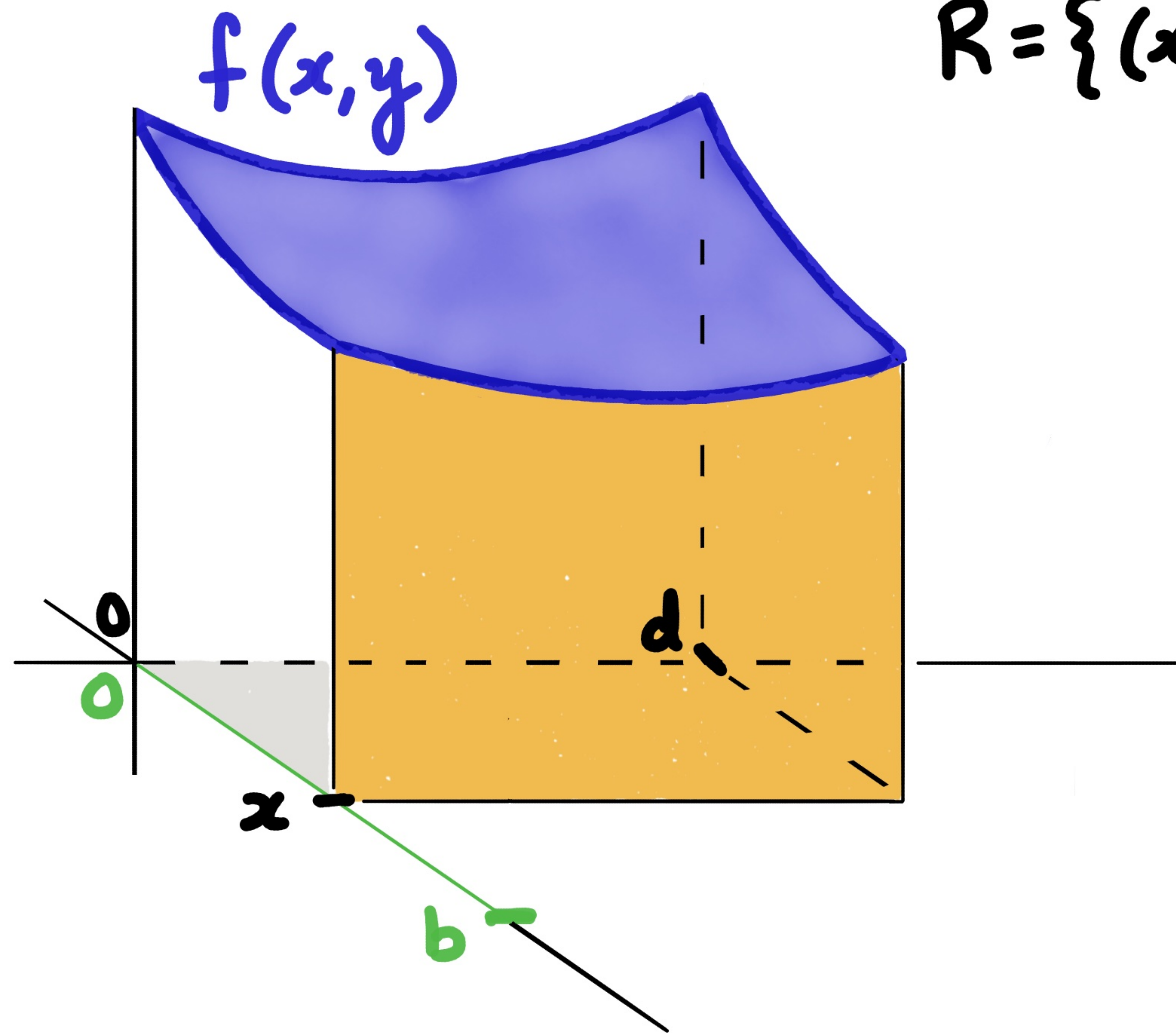


$$\iint_R f(x,y) dA$$



varying  
area of  
cross section

$$\iint_R f(x,y) dA = \int_0^b \text{width} dx$$



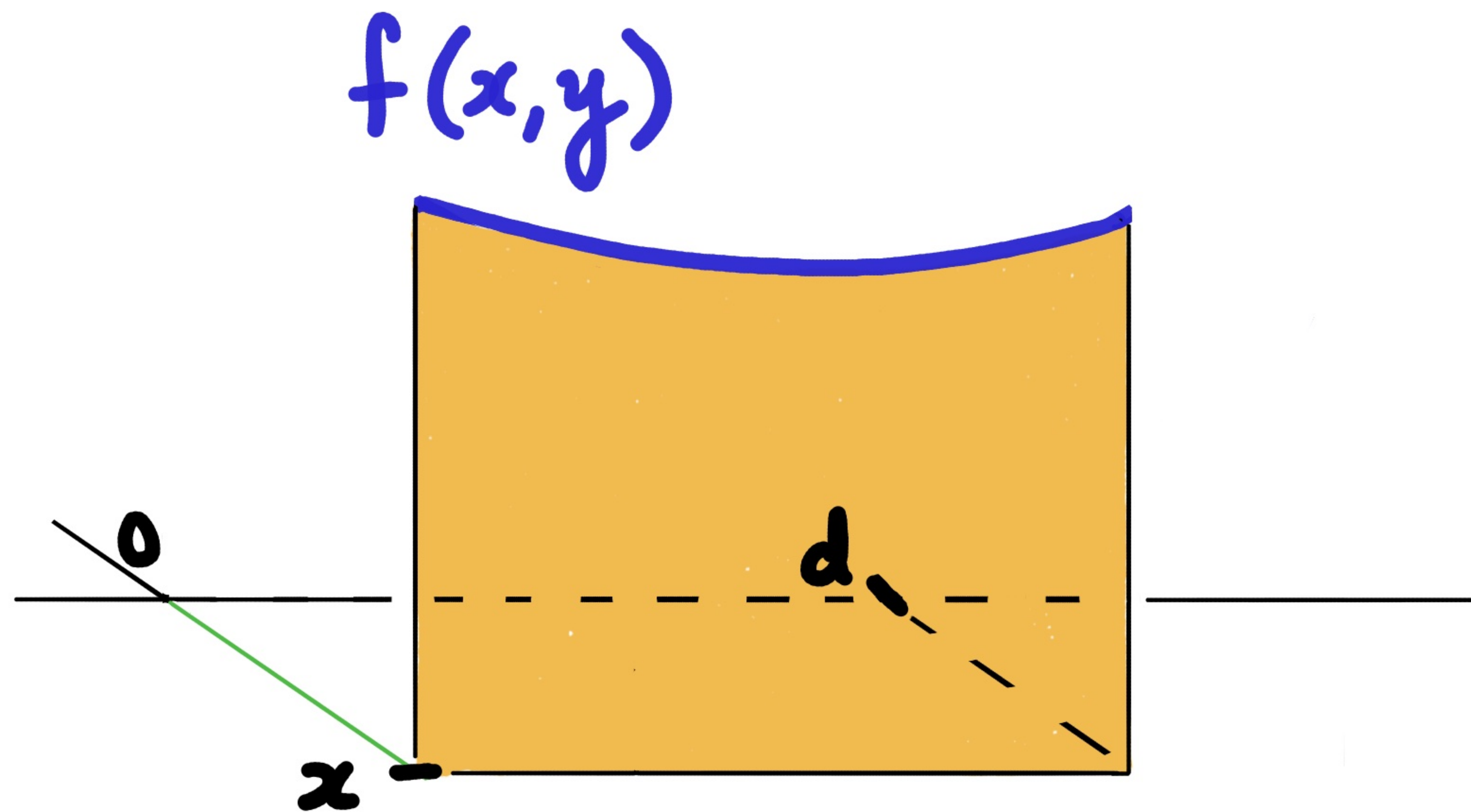
$$R = \{(x, y) : 0 \leq x \leq b, 0 \leq y \leq d\}$$

varying  
area of  
cross section



$$\iint_R f(x, y) dA = \int_0^b \text{width} dx$$

$$R = \{(x, y) : 0 \leq x \leq b, 0 \leq y \leq d\}$$

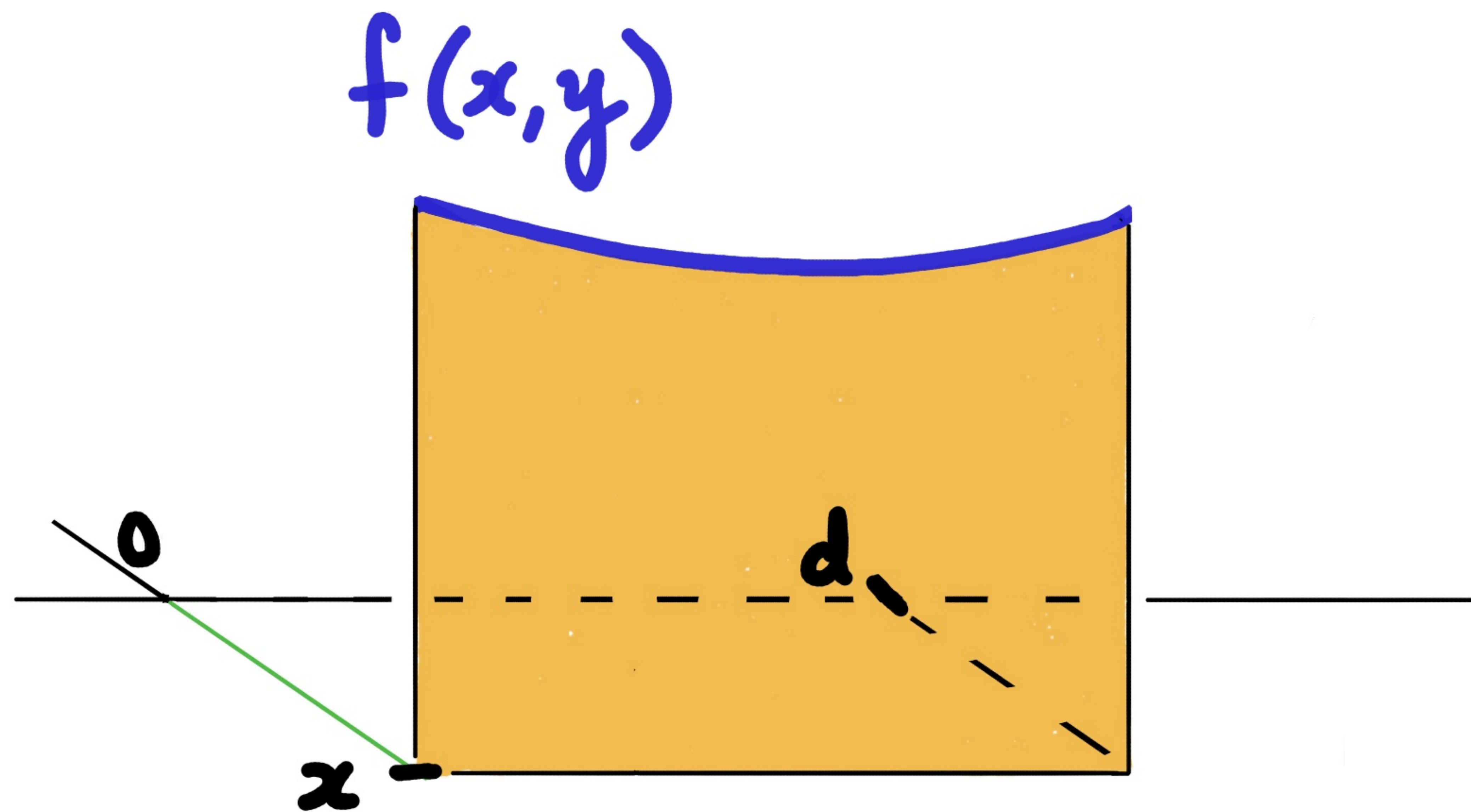


varying  
area of  
cross section

$$\iint_R f(x, y) dA = \int_0^b \text{width} \, dx$$

A 2D diagram illustrating the region  $R$  in the  $xy$ -plane. The region is colored orange. The  $x$ -axis is labeled  $x$ , and the  $y$ -axis is labeled  $y$ . A dashed line indicates the boundary  $y = d$ . The width of the region is labeled width, and the differential element  $dx$  is shown.

$$R = \{(x, y) : 0 \leq x \leq b, 0 \leq y \leq d\}$$



$x = \text{const.}$

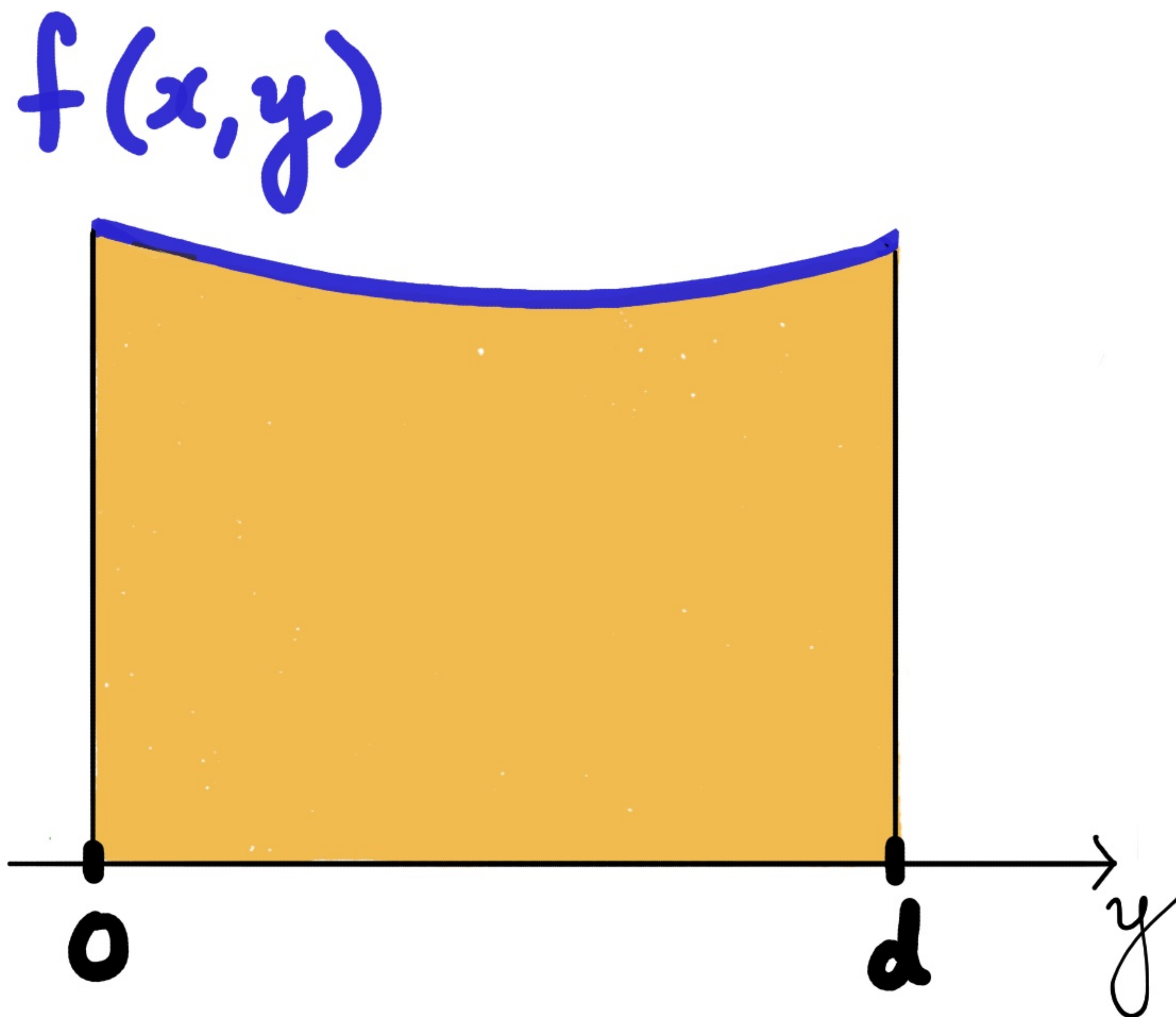
varying  
area of  
cross section

$$\iint_R f(x, y) dA = \int_0^b \boxed{dx}$$

width

A 2D diagram showing a rectangular region  $R$  in the  $xy$ -plane. The region is colored orange. The  $x$ -axis is labeled with  $0$  and  $b$ , and the  $y$ -axis is labeled with  $d$ . A green line connects the origin to the point  $(x, 0)$  on the  $x$ -axis. The region is labeled with "width" and  $dx$ .

$$R = \{(x, y) : 0 \leq x \leq b, 0 \leq y \leq d\}$$



$x = \text{const.}$

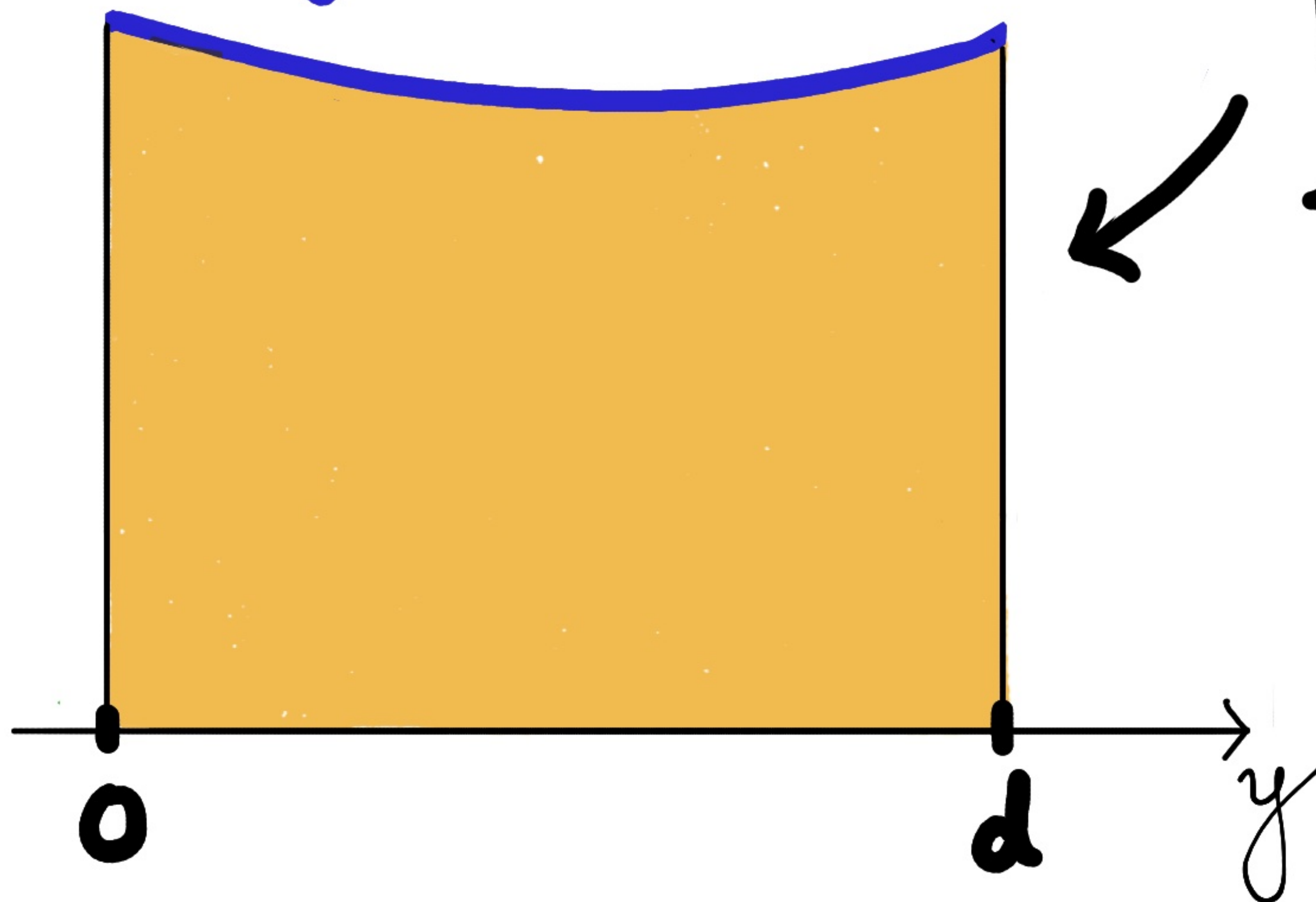
varying  
area of  
cross section

$$\iint_R f(x, y) dA = \int_0^b \text{width} dx$$

A diagram illustrating the integration process. A yellow rectangle represents the cross-section at a fixed  $x$ . The width of the rectangle is labeled "width" and the height is labeled  $dx$ . The x-axis is labeled with 0 and  $b$ . An arrow points from the text "varying area of cross section" to the rectangle.

$$R = \{(x, y) : 0 \leq x \leq b, 0 \leq y \leq d\}$$

$f(x, y)$



$$\int_0^d f(x, y) dy$$

varying  
area of  
cross section

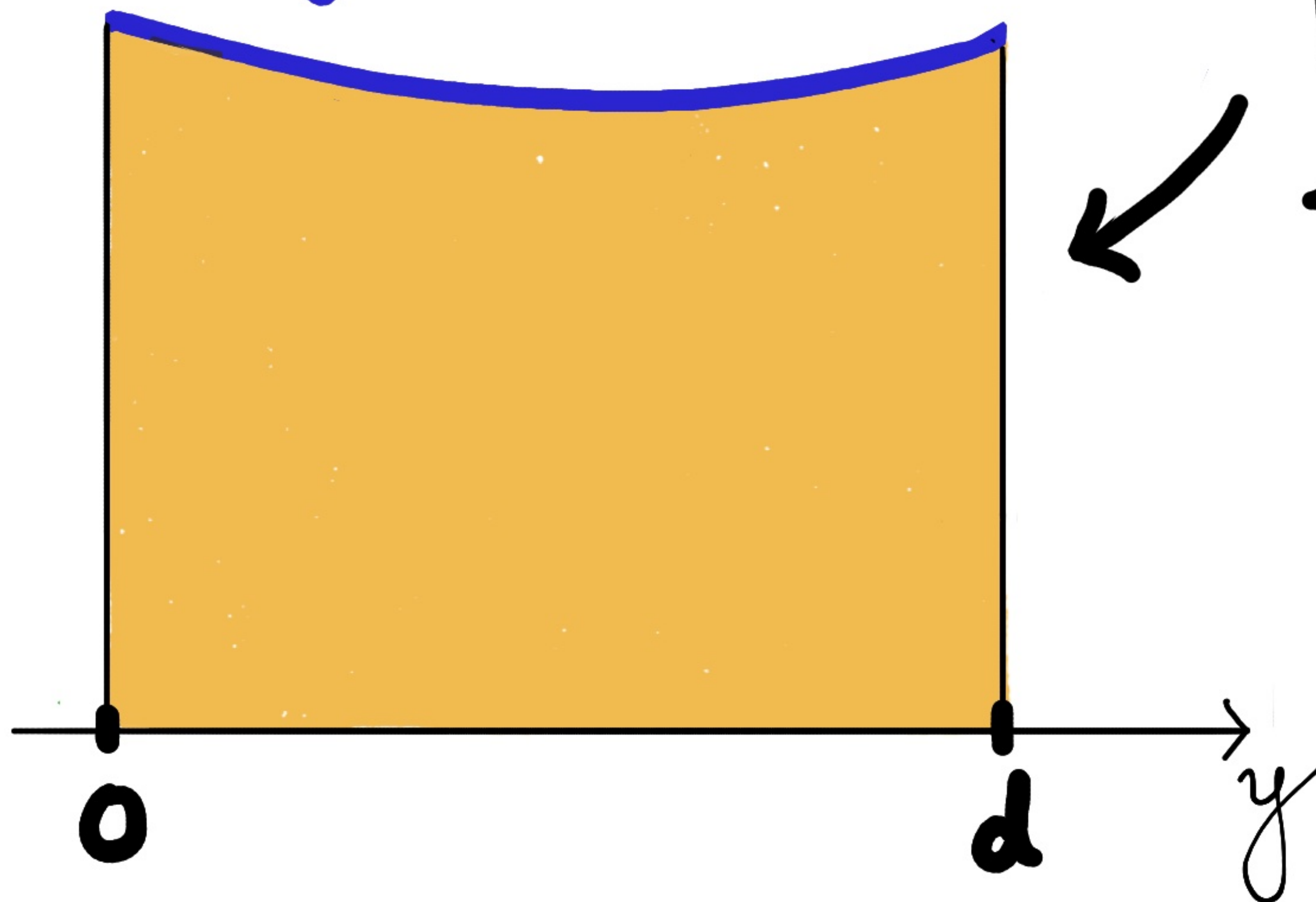
$x = \text{const.}$

$$\iint_R f(x, y) dA = \int_0^b dx$$

width

$$R = \{(x, y) : 0 \leq x \leq b, 0 \leq y \leq d\}$$

$f(x, y)$



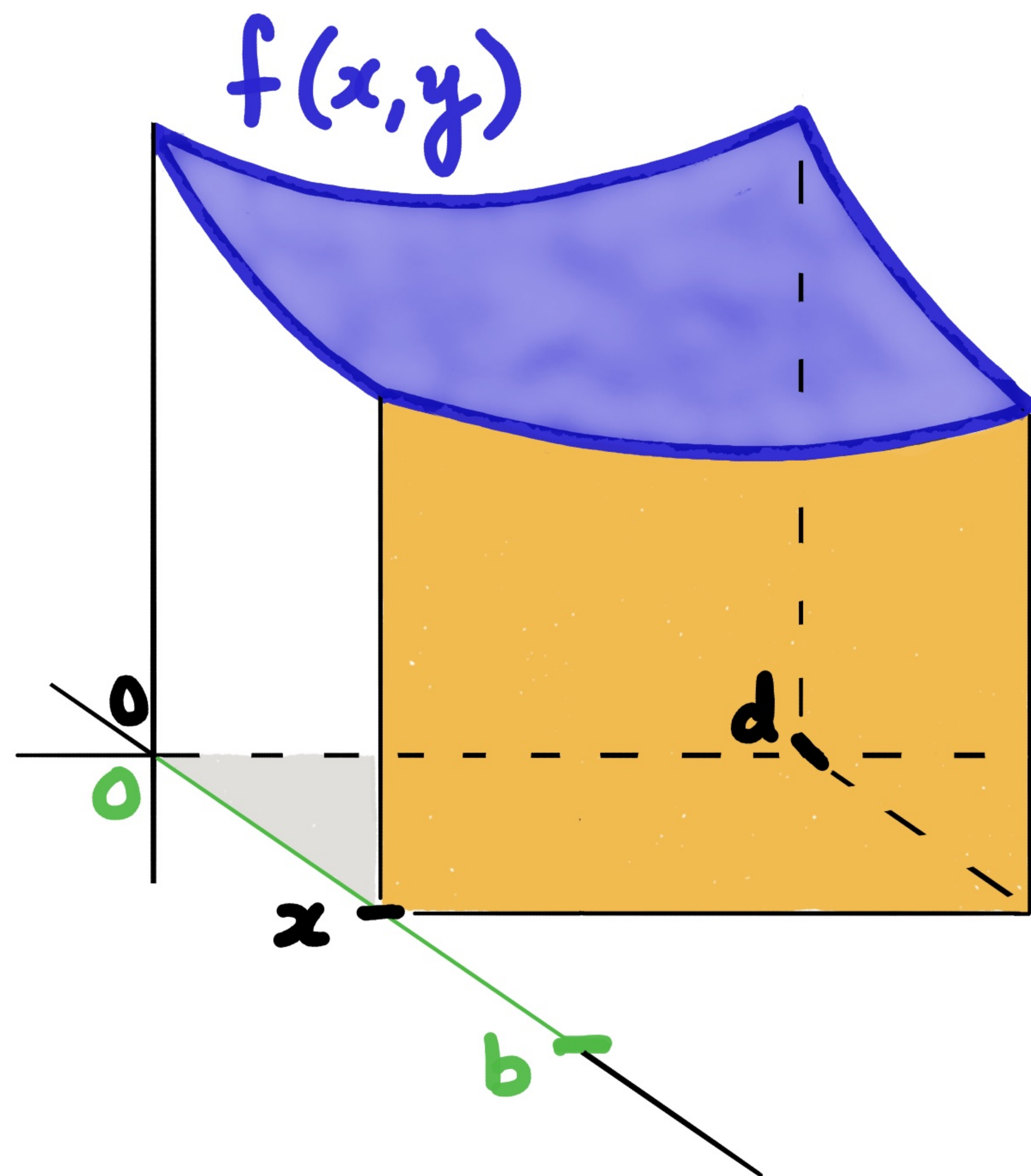
$$\int_0^d f(x, y) dy$$

varying  
area of  
cross section

$x = \text{const.}$

$$\iint_R f(x, y) dA = \int_0^b \left( \int_0^d f(x, y) dy \right) dx$$

width



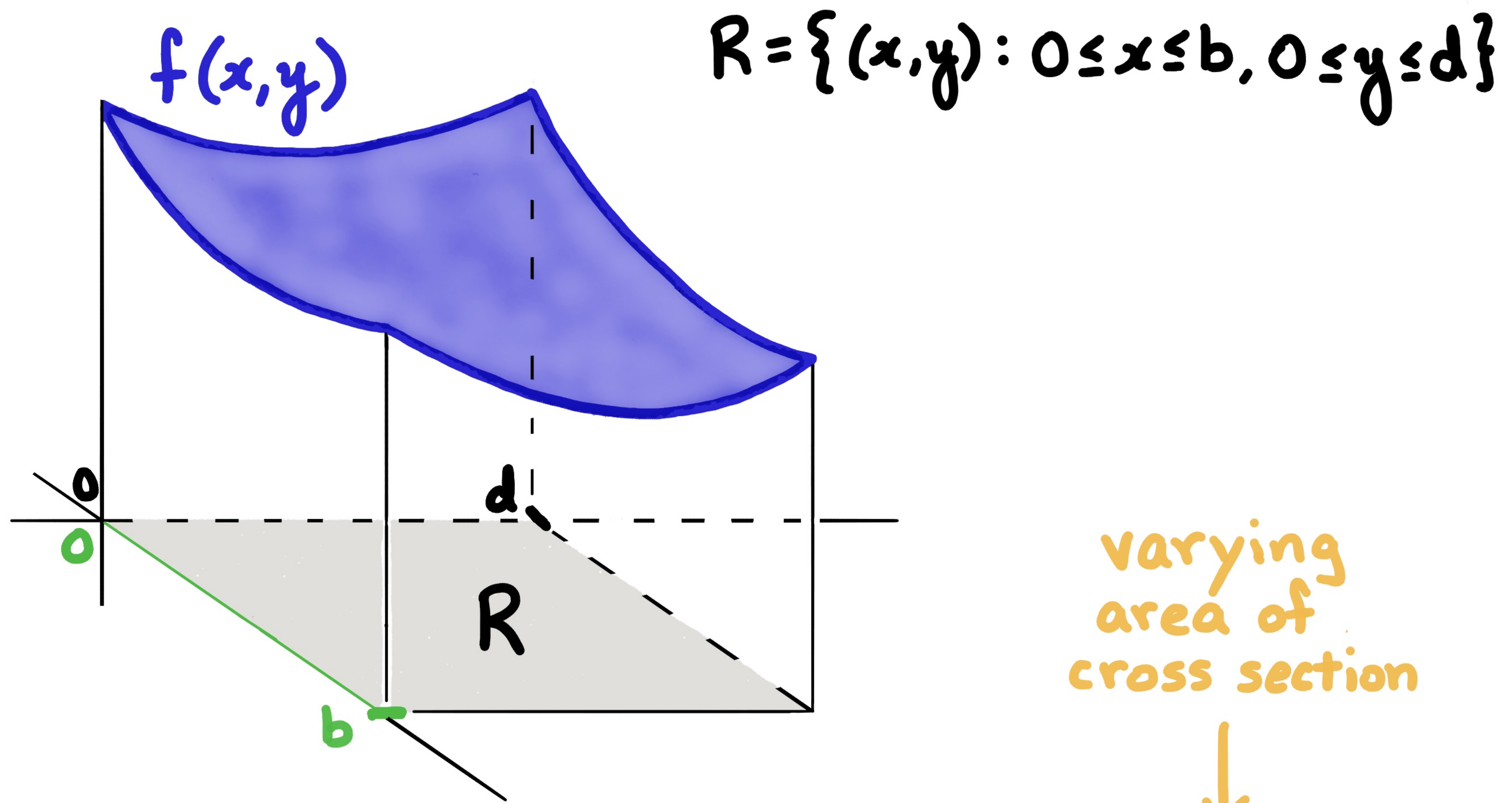
$$R = \{(x, y) : 0 \leq x \leq b, 0 \leq y \leq d\}$$

$$\int_0^d f(x, y) dy$$

varying  
area of  
cross section

$$\iint_R f(x, y) dA = \int_0^b \int_0^d f(x, y) dy dx$$

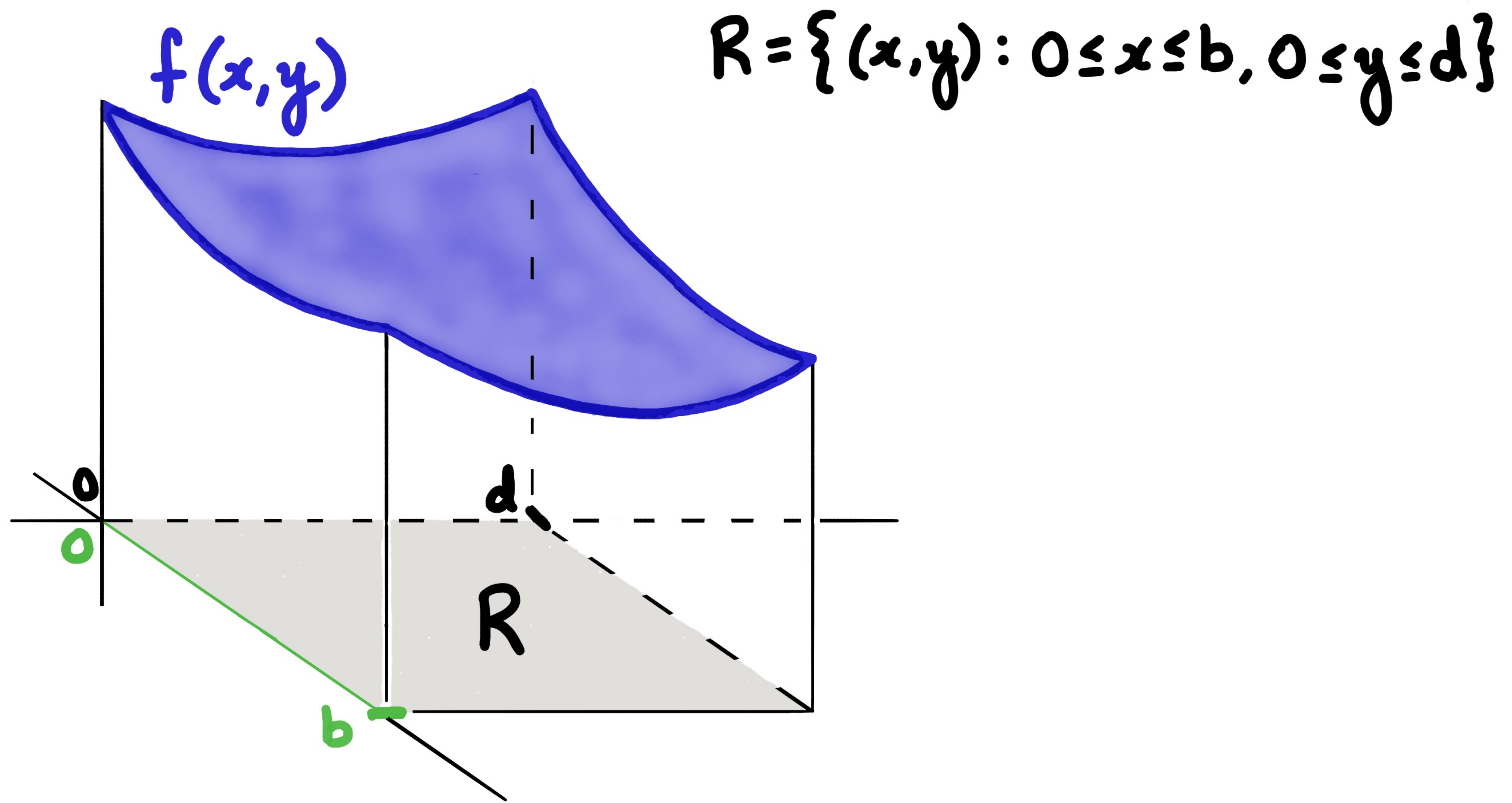
width



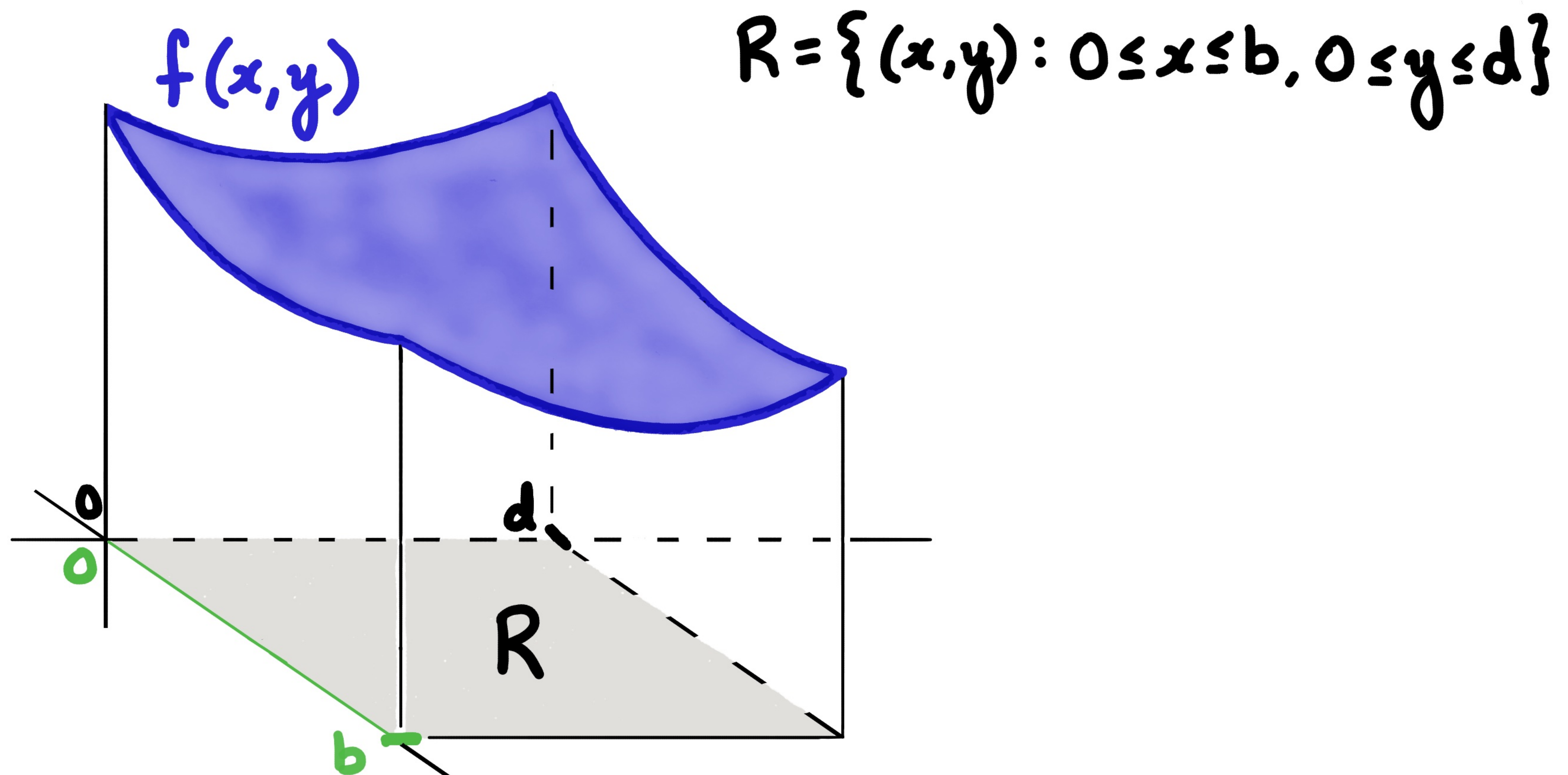
varying  
area of  
cross section

$$\iint_R f(x,y) dA = \int_0^b \int_0^d f(x,y) dy dx$$

width



$$\iint_R f(x,y) dA = \int_0^b \int_0^d f(x,y) dy dx$$

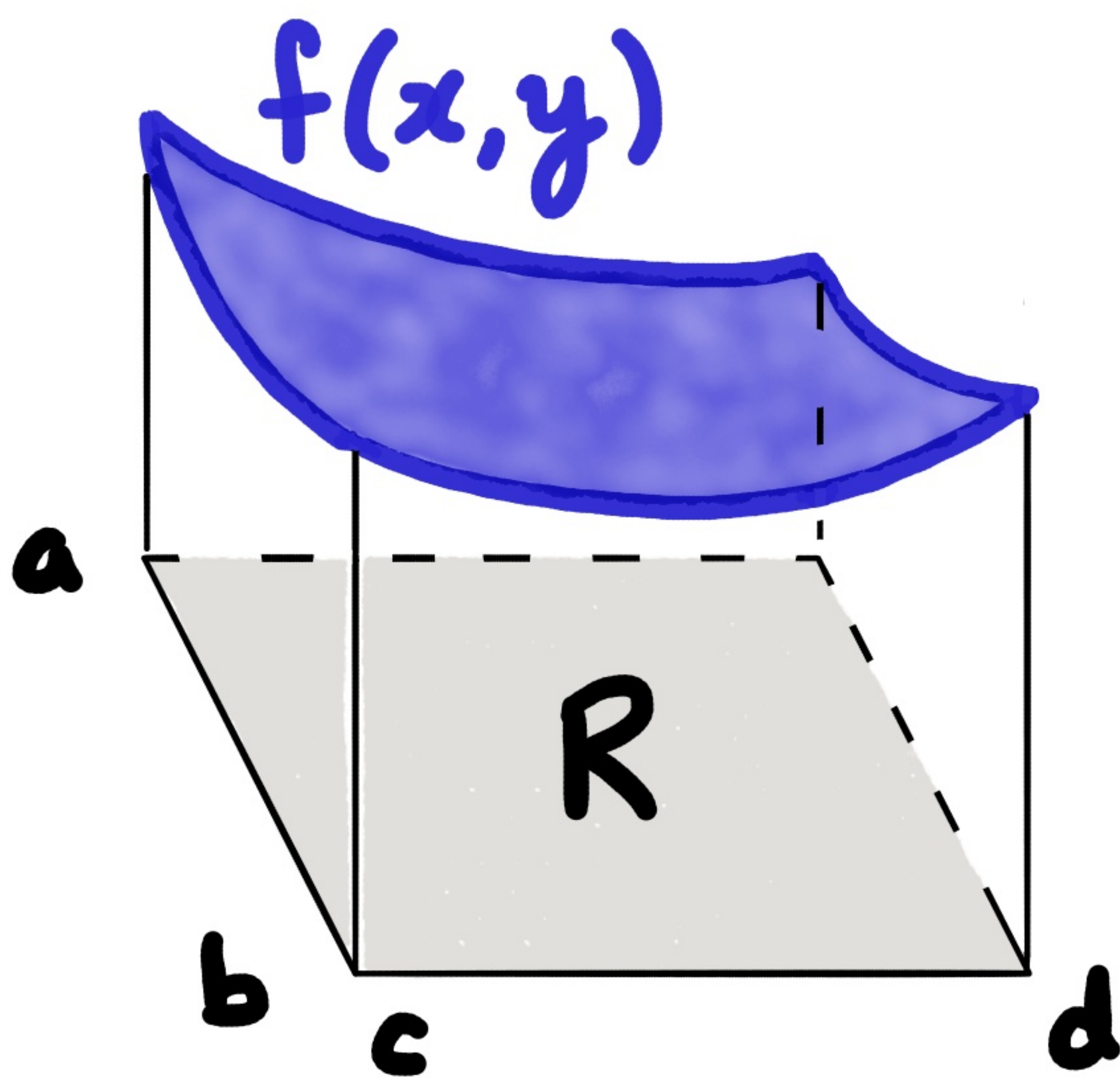


$$\iint_R f(x,y) dA = \int_0^b \int_0^d f(x,y) dy dx$$

$$dA \mapsto dy dx$$

# Iterated integrals

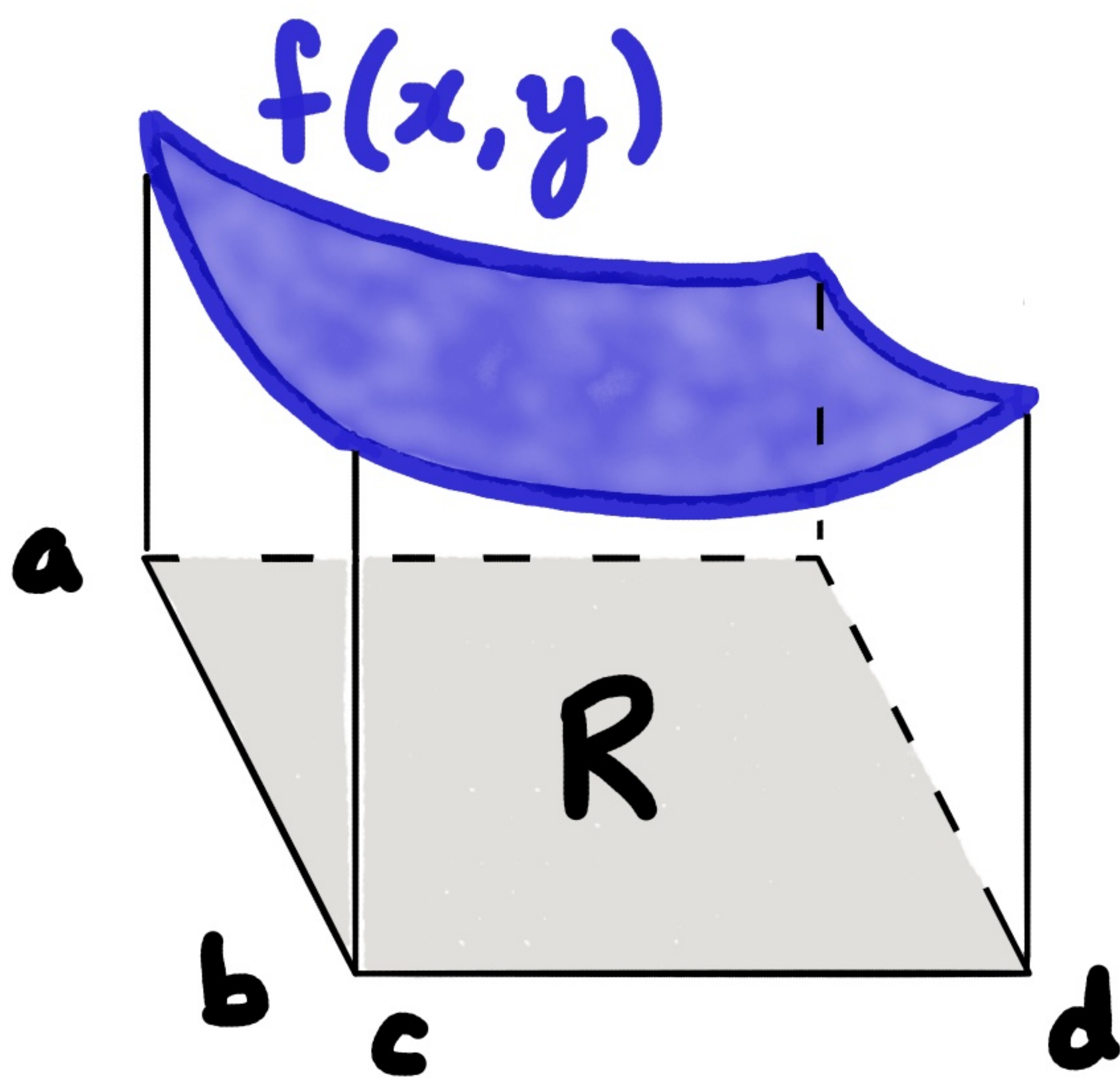
$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx$$



$$R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\} \quad f \text{ is continuous}$$

# Iterated integrals

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx$$



$$= \int_c^d \int_a^b f(x,y) dx dy$$

$$R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$$

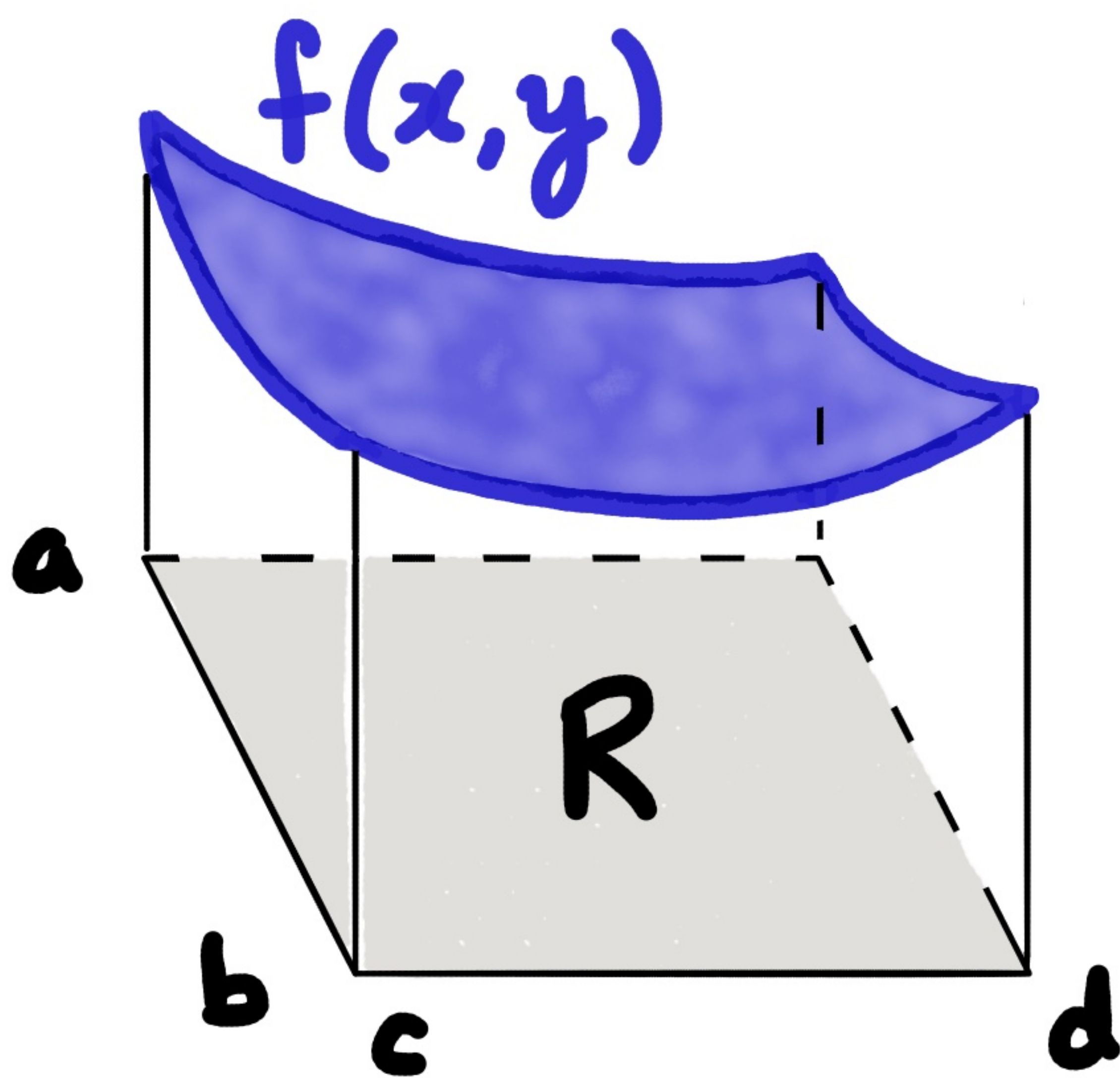
$f$  is continuous

# Iterated integrals

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx$$

inner integral

$$= \int_c^d \int_a^b f(x,y) dx dy$$



$$R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$$

$f$  is continuous

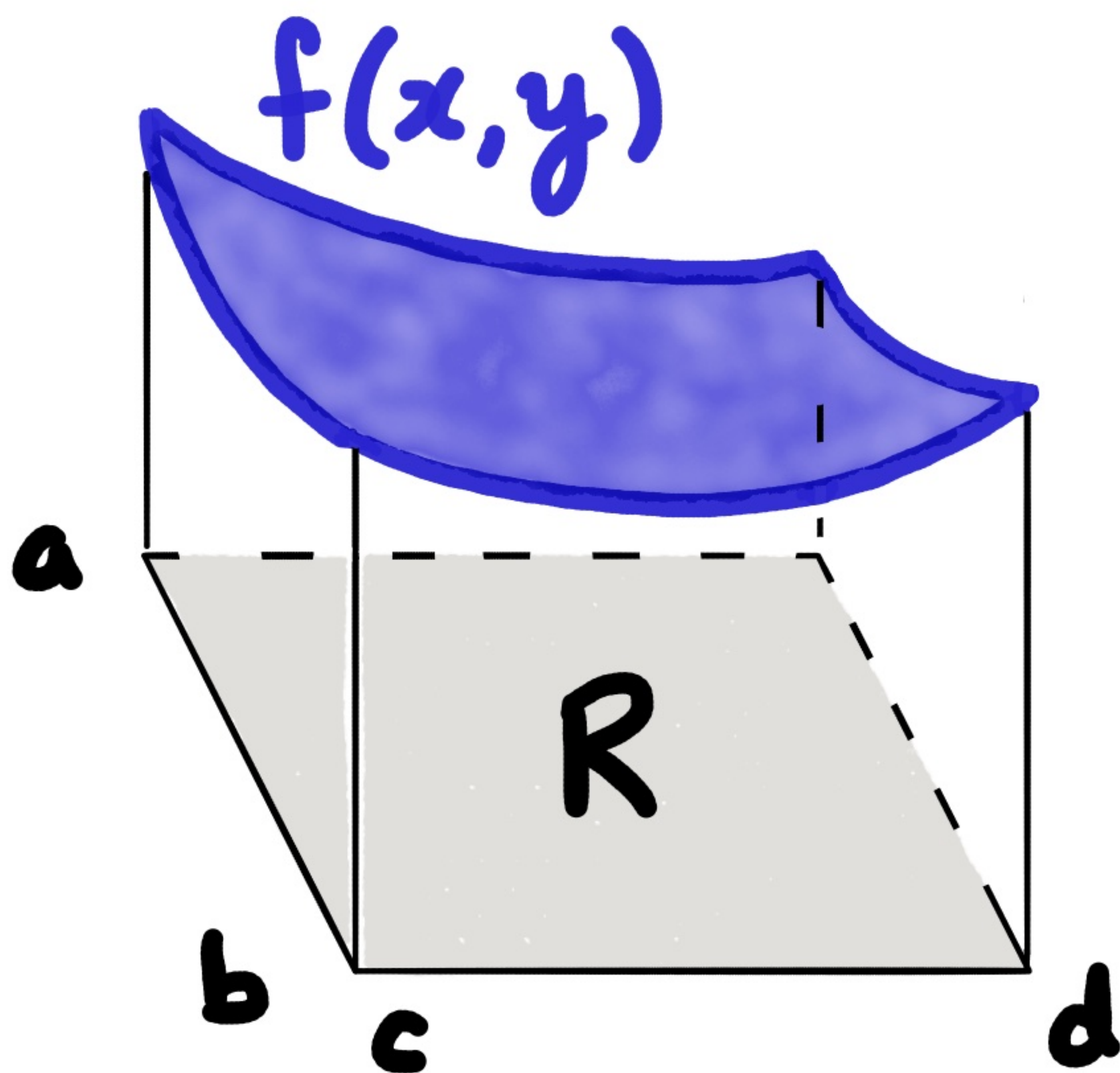
# Iterated integrals

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx$$

inner integral

$$= \int_c^d \int_a^b f(x,y) dx dy$$

outer integral



$$R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$$

$f$  is continuous

## Examples:

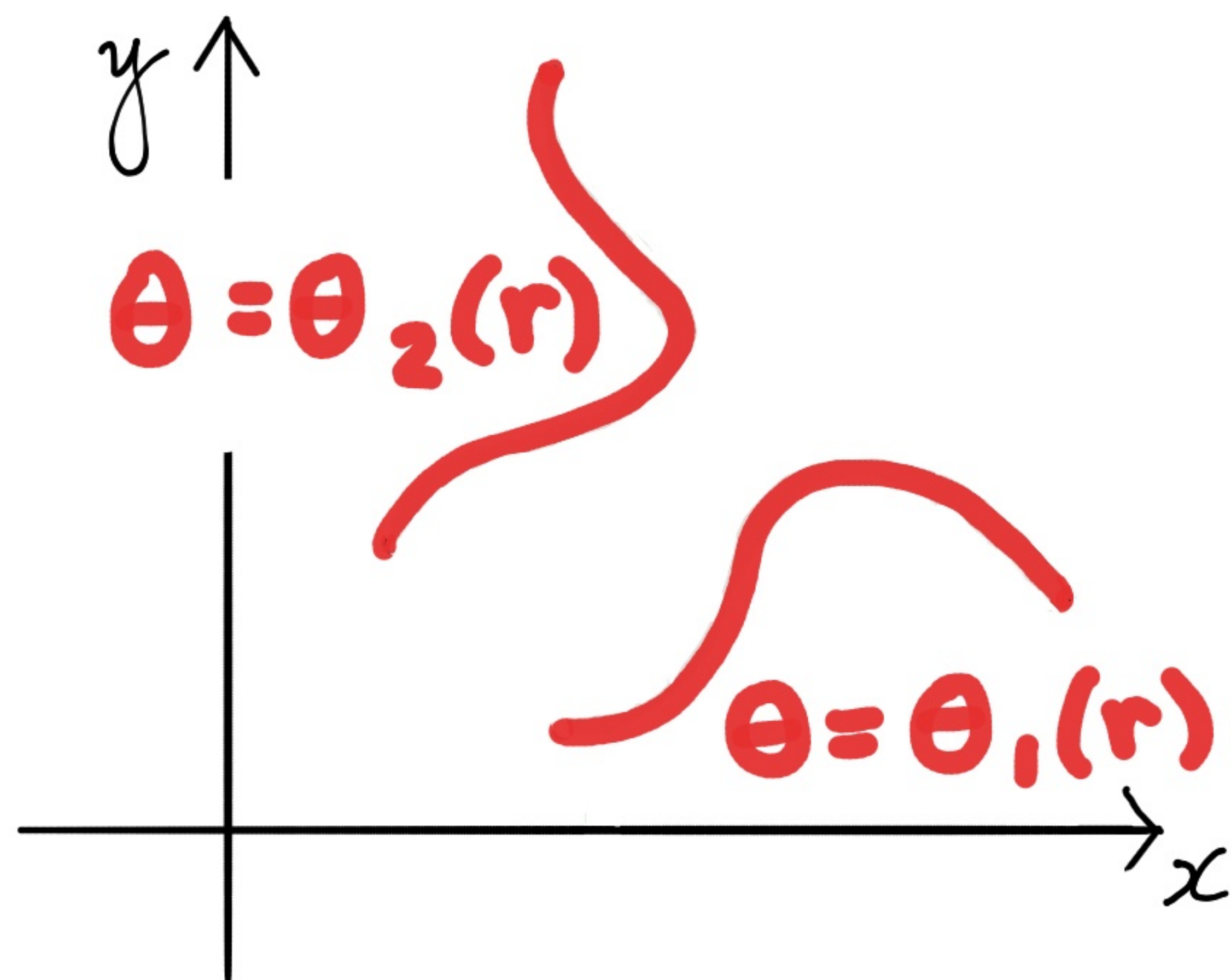
$$\textcircled{1} \int_0^2 \int_1^3 (x-y) dx dy$$

$$\textcircled{2} \int_1^3 \int_0^2 (x-y) dy dx$$

$$\textcircled{3} \iint_R x e^{xy} dA \quad \text{where } R = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

② Simple regions  
in polar coordinates

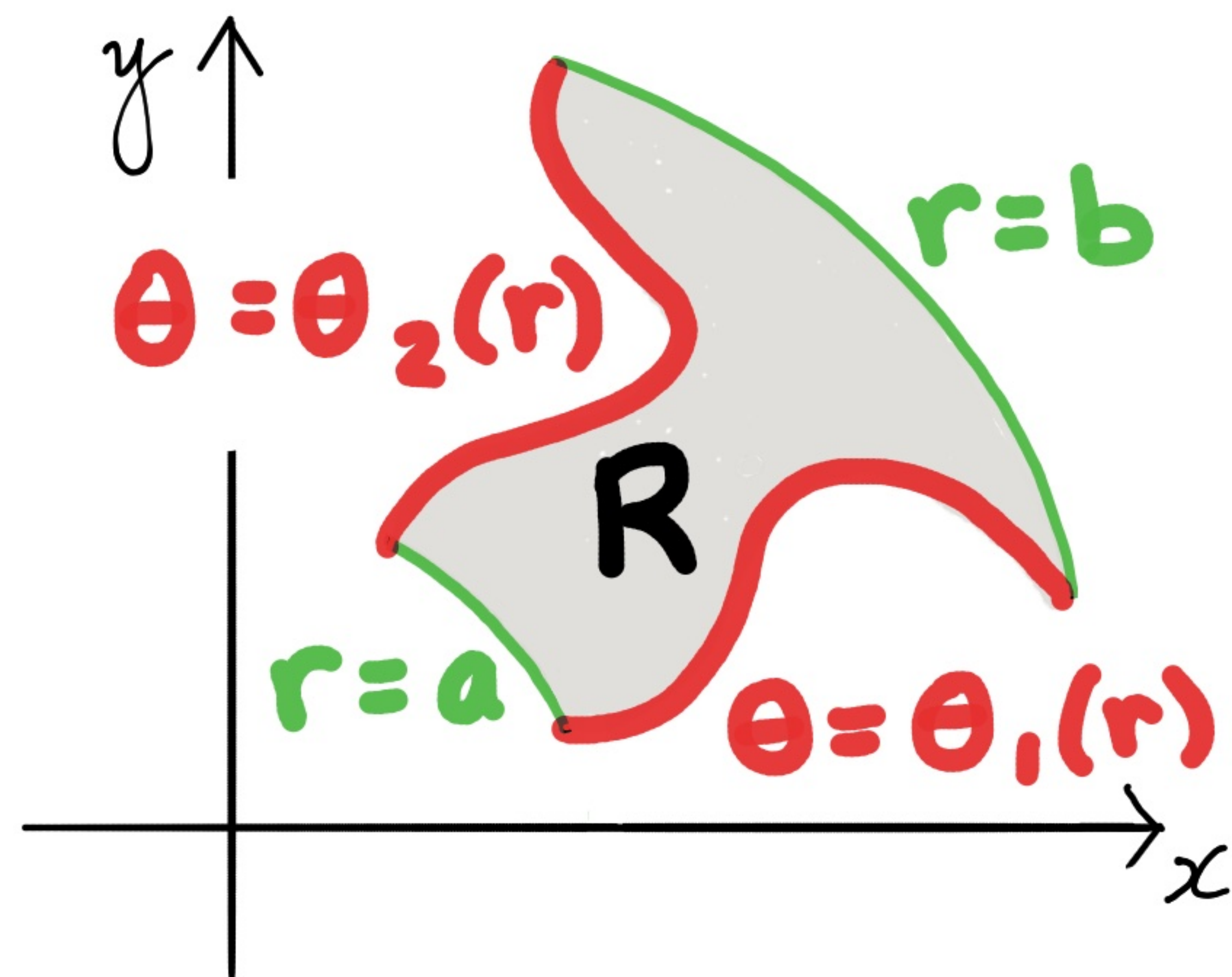
# $\theta$ -simple regions



$$\theta_1: \mathbb{R} \rightarrow \mathbb{R}$$

$$\theta_2: \mathbb{R} \rightarrow \mathbb{R}$$

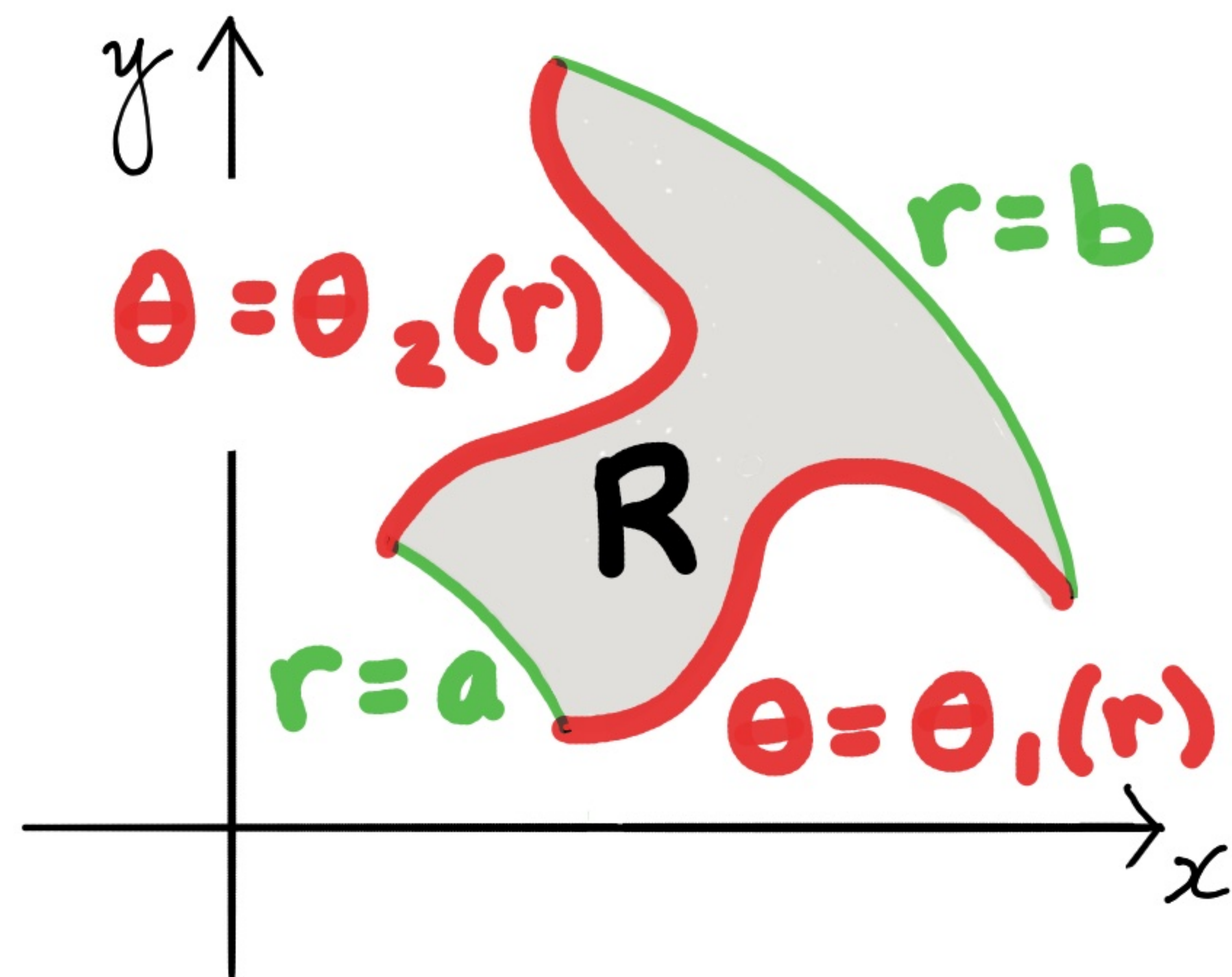
# $\theta$ -simple regions



$$\theta_1: \mathbb{R} \rightarrow \mathbb{R}$$

$$\theta_2: \mathbb{R} \rightarrow \mathbb{R}$$

# $\theta$ -simple regions



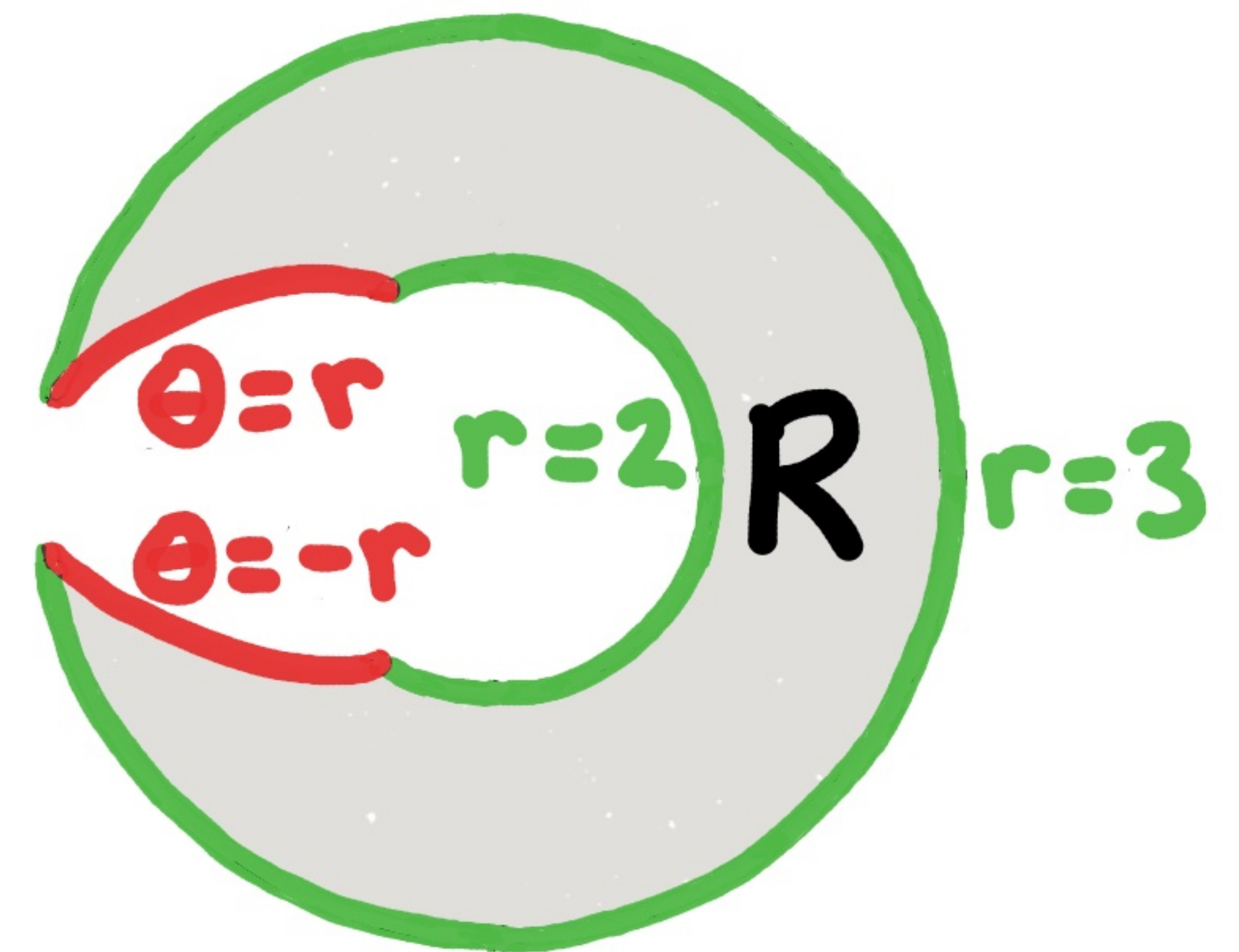
$$\theta_1: \mathbb{R} \rightarrow \mathbb{R}$$

$$\theta_2: \mathbb{R} \rightarrow \mathbb{R}$$

$$R = \{(r, \theta) : a \leq r \leq b, \theta_1(r) \leq \theta \leq \theta_2(r)\}$$

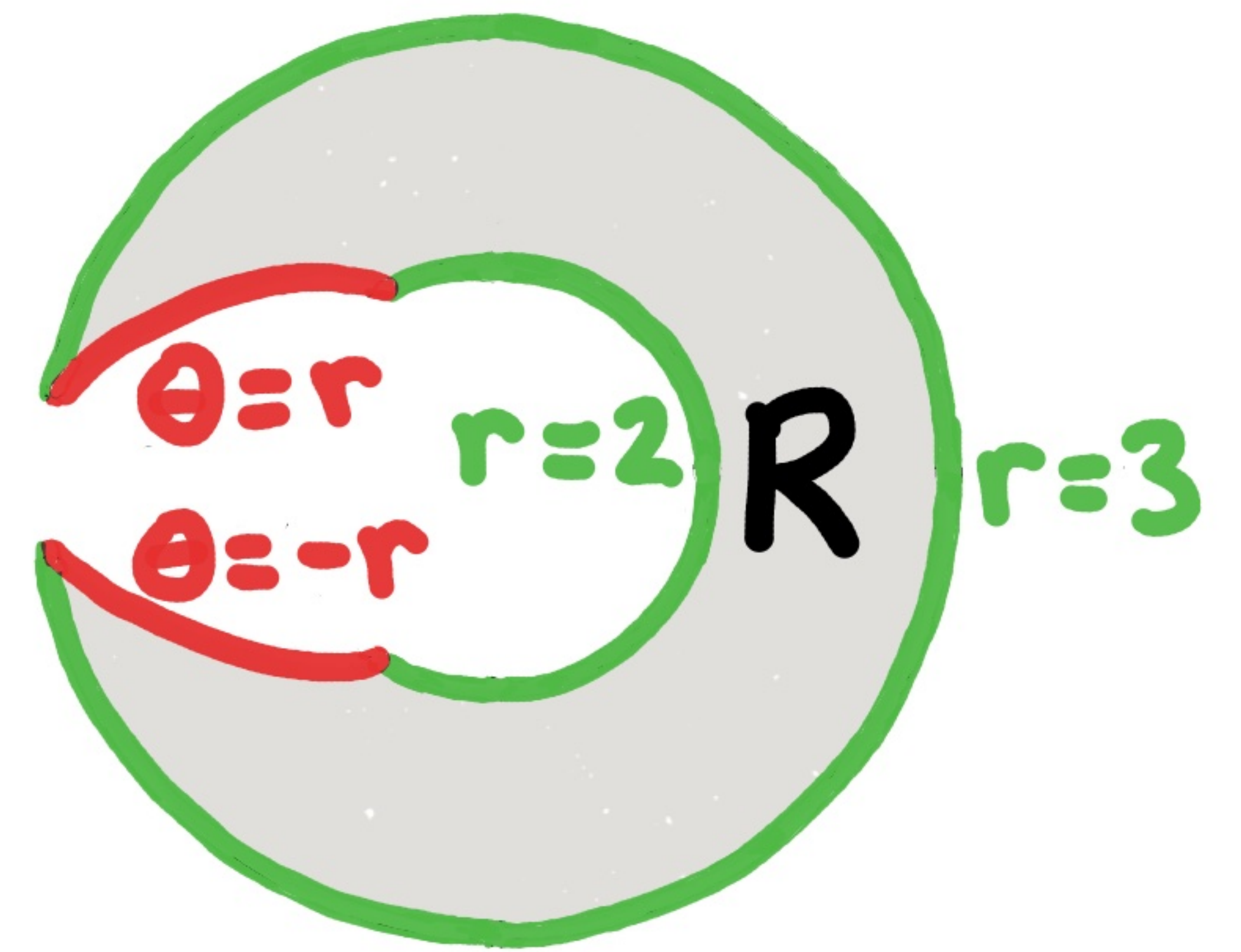
## Problem:

Write the region  $R$   
as a  $\theta$ -simple set.



## Problem:

Write the region  $R$   
as a  $\theta$ -simple set.

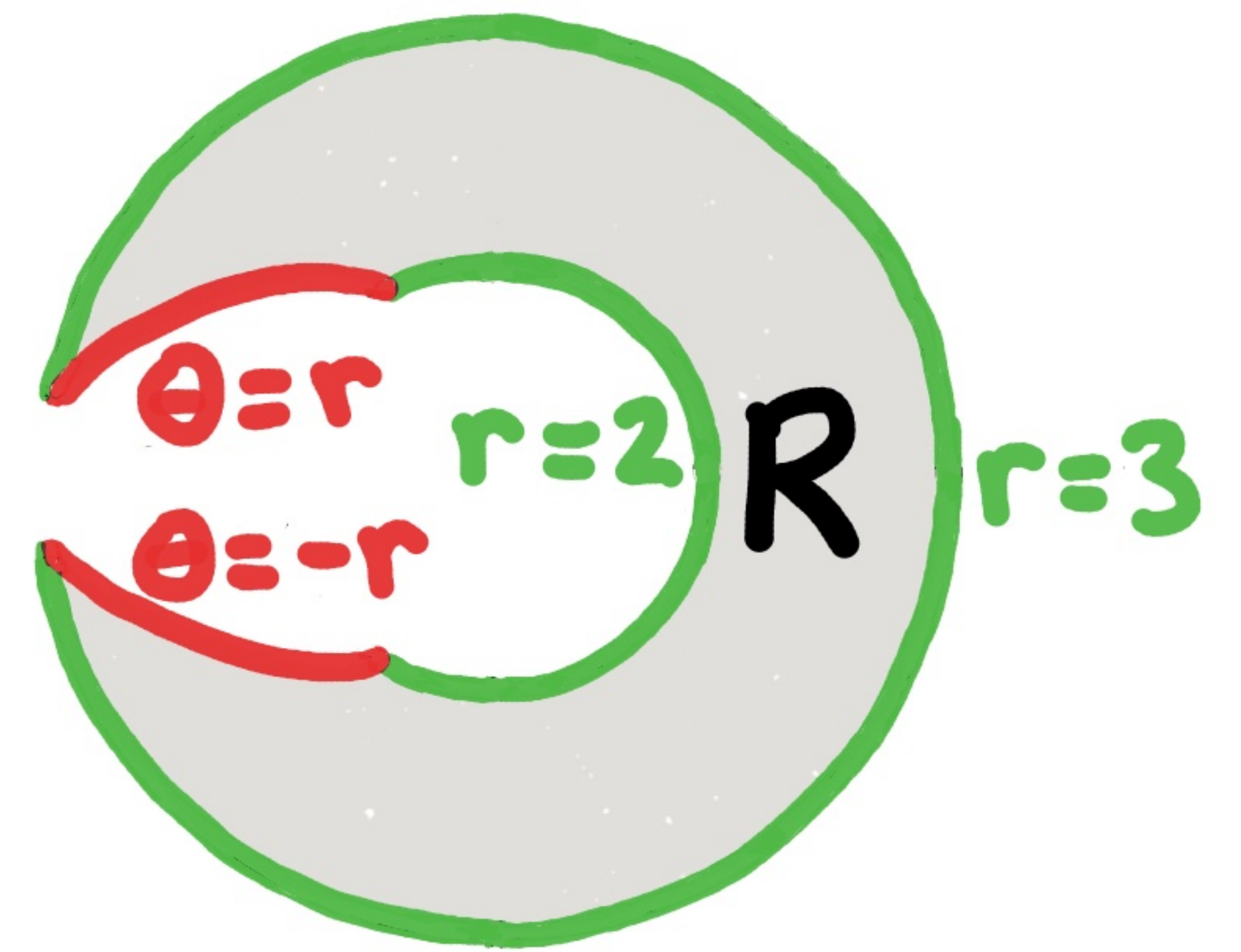


## Answer:

$$R = \{ (r, \theta) : 2 \leq r \leq 3, \theta = r, \theta = -r \}$$

## Problem:

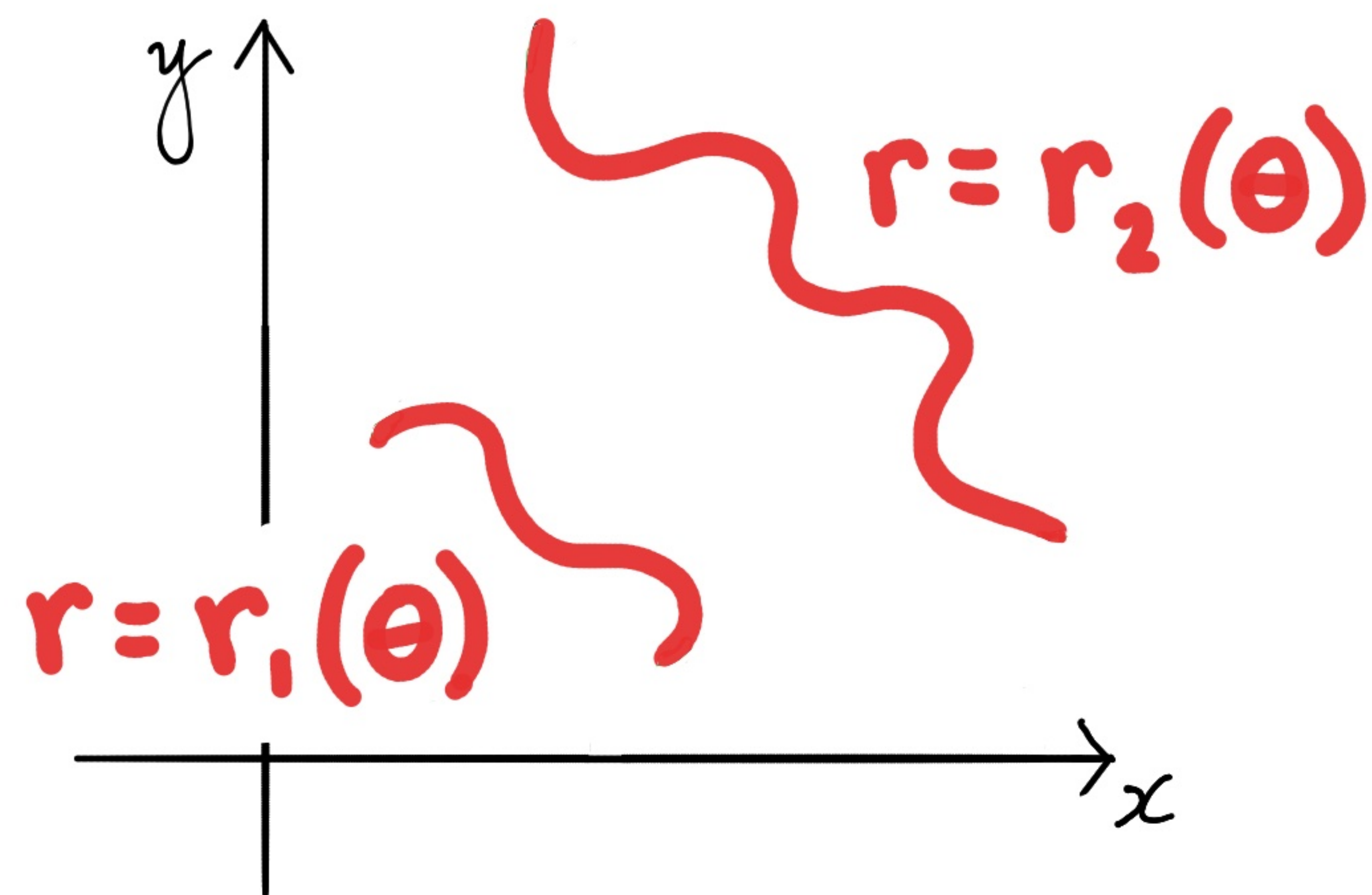
Write the region  $R$   
as a  $\theta$ -simple set.



## Answer:

$$R = \{(r, \theta) : 2 \leq r \leq 3, -r \leq \theta \leq r\}$$

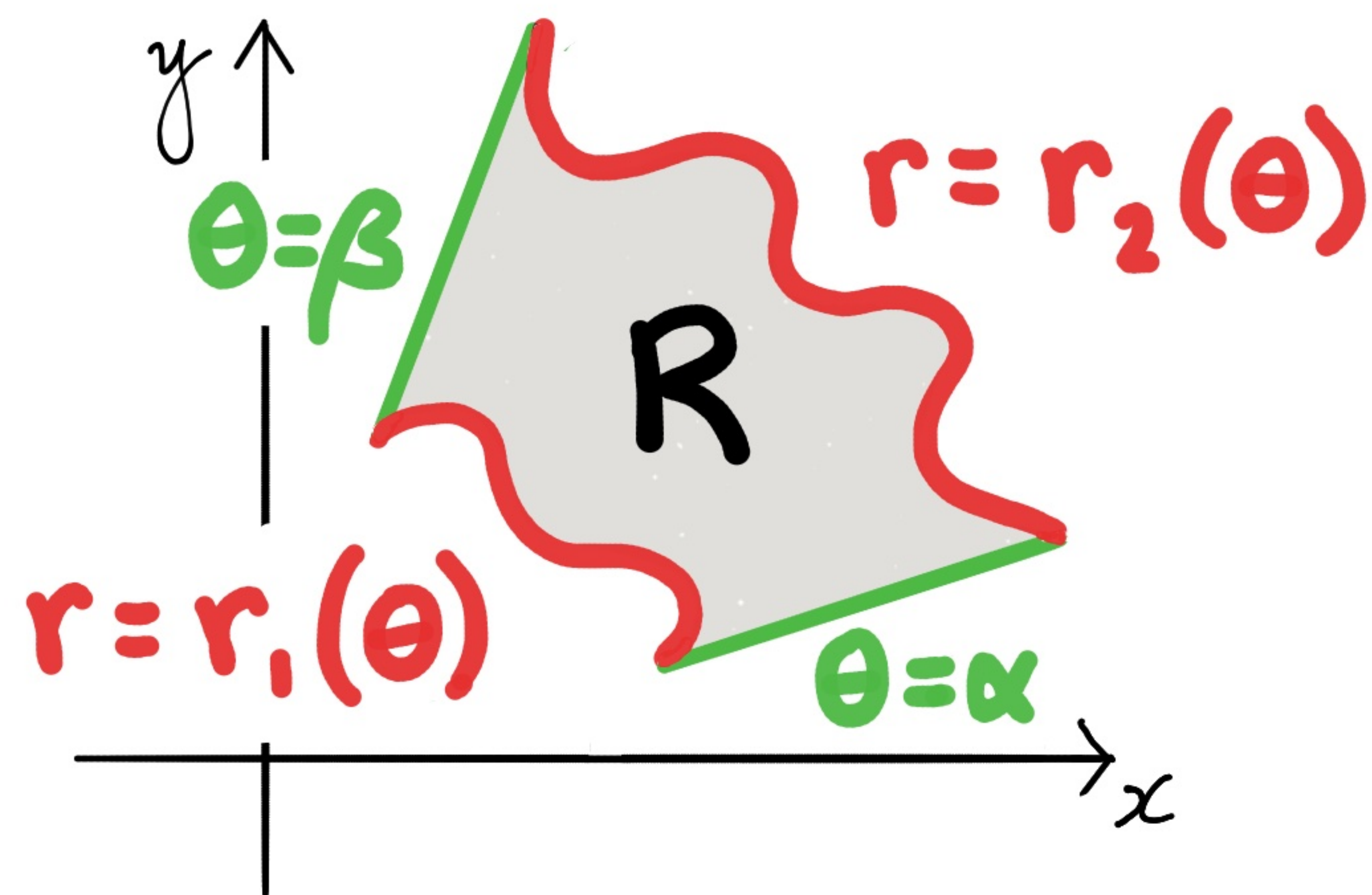
# $r$ -simple regions



$$r_1: \mathbb{R} \rightarrow \mathbb{R}$$

$$r_2: \mathbb{R} \rightarrow \mathbb{R}$$

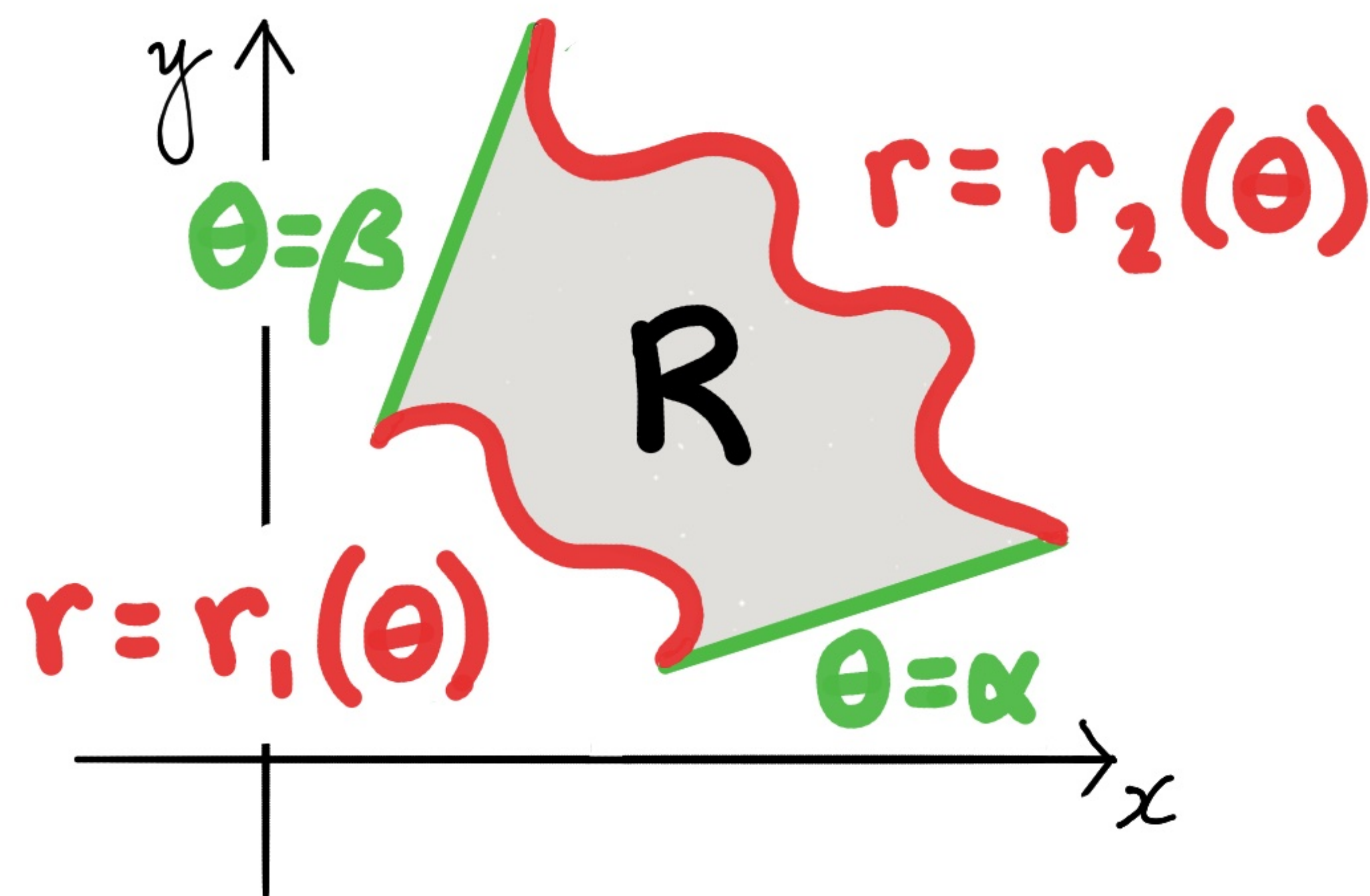
# $r$ -simple regions



$$r_1: \mathbb{R} \rightarrow \mathbb{R}$$

$$r_2: \mathbb{R} \rightarrow \mathbb{R}$$

# $r$ -simple regions



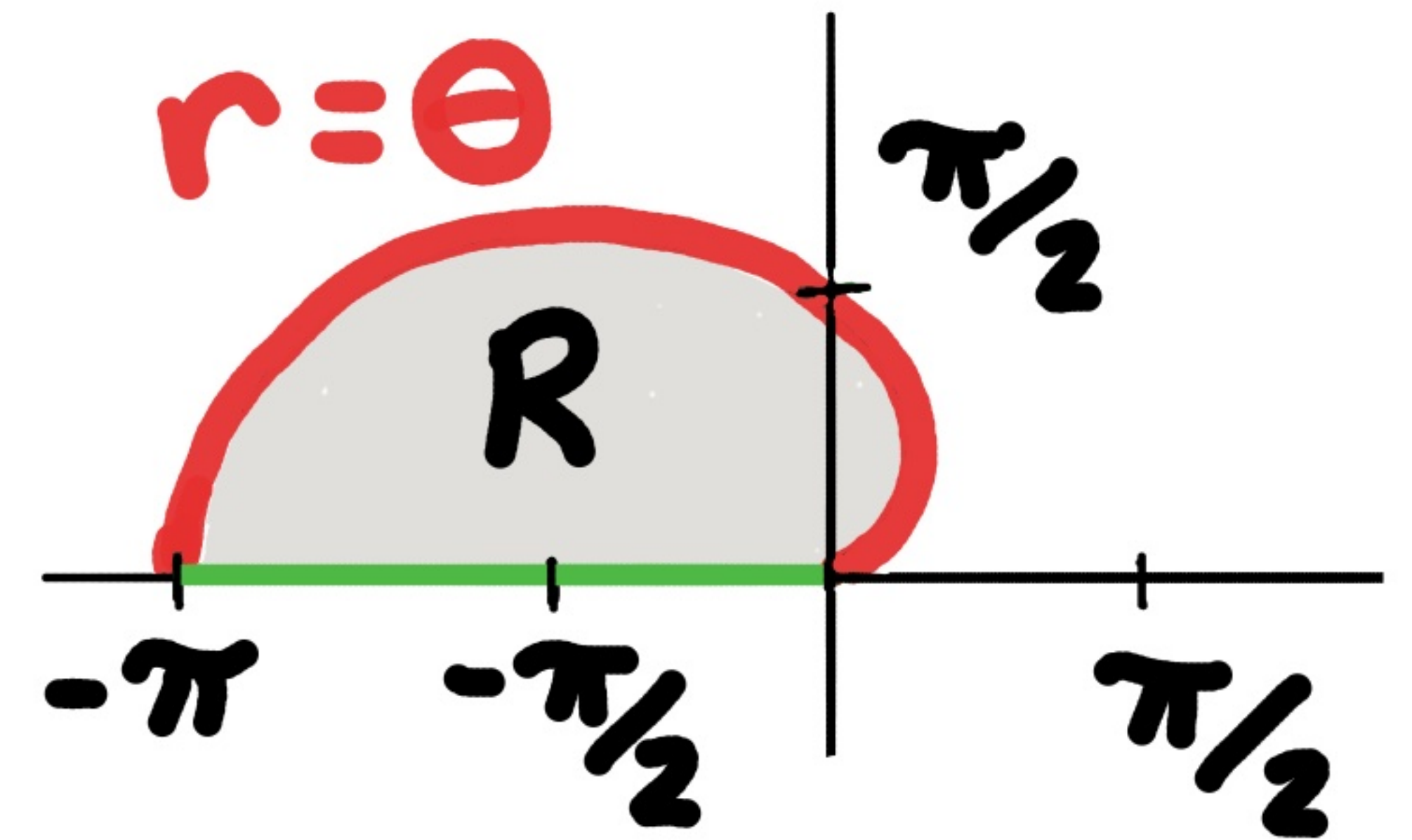
$$r_1: \mathbb{R} \rightarrow \mathbb{R}$$

$$r_2: \mathbb{R} \rightarrow \mathbb{R}$$

$$R = \{(r, \theta) : \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)\}$$

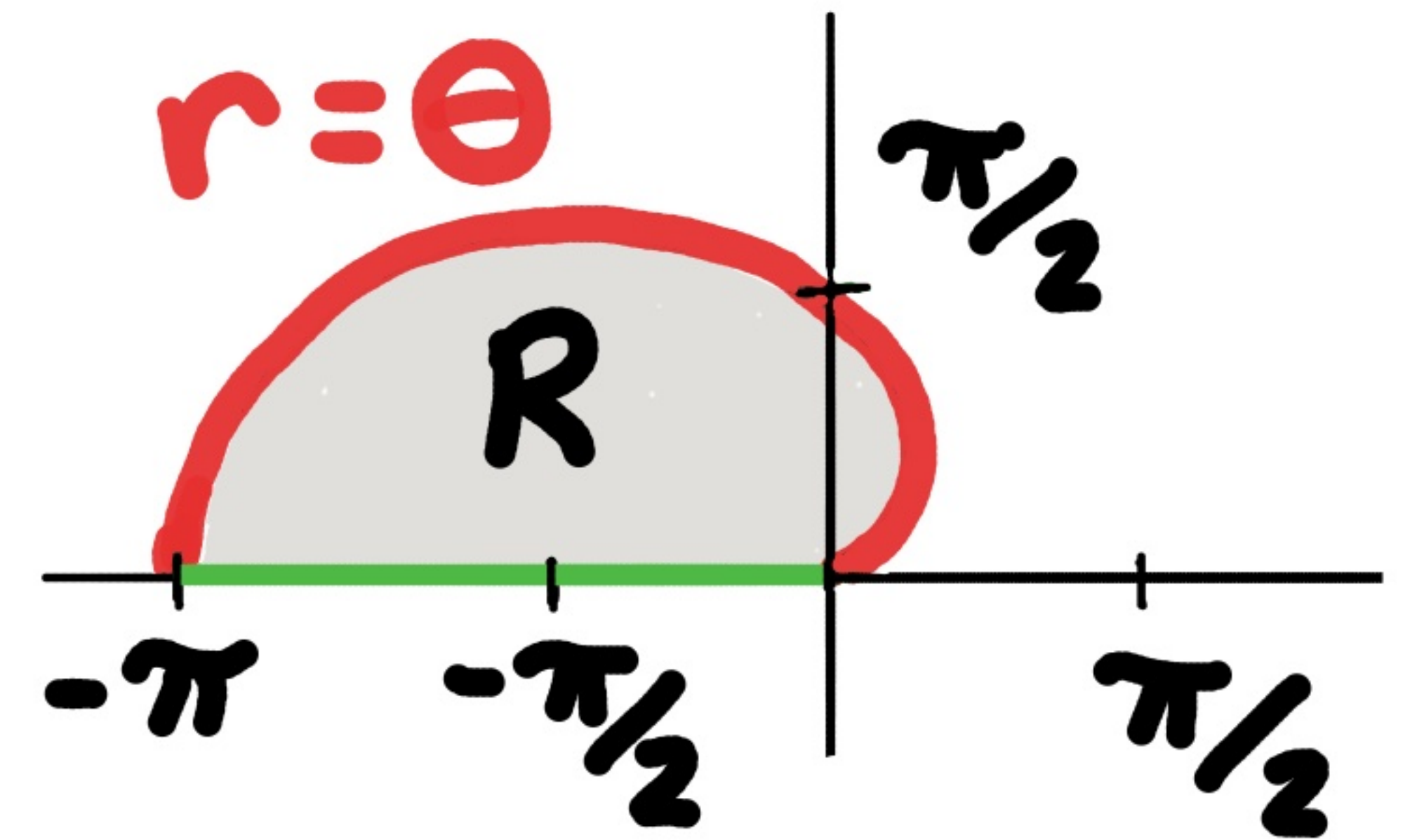
## Problem:

Write the region  $R$   
as an  $r$ -simple set.



## Problem:

Write the region  $R$   
as an  $r$ -simple set.

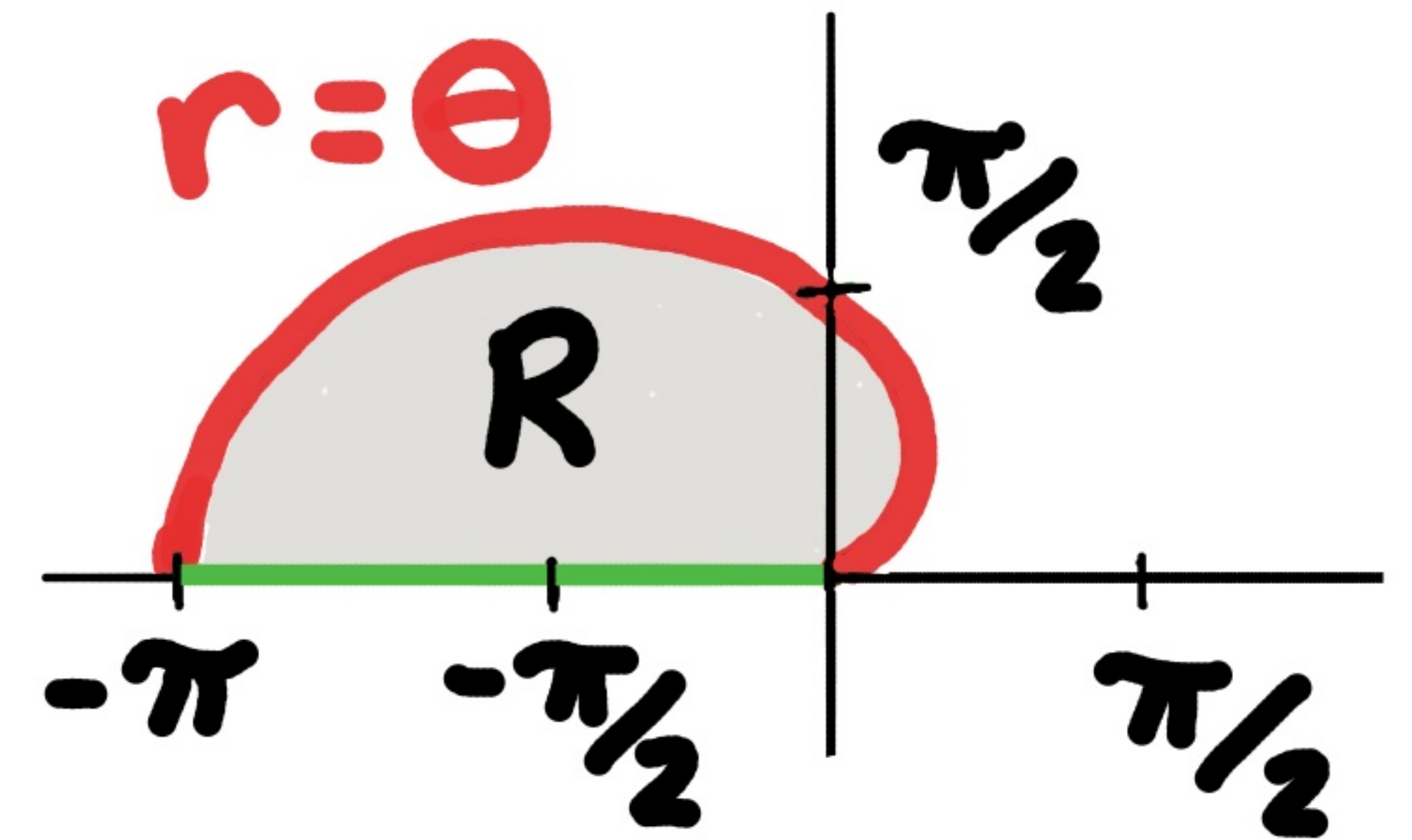


## Answer:

$$R = \{(r, \theta) : 0 \leq \theta \leq \pi,$$

Problem:

Write the region  $R$   
as an  $r$ -simple set.



Answer:

$$R = \{(r, \theta) : 0 \leq \theta \leq \pi, 0 \leq r \leq \theta\}$$