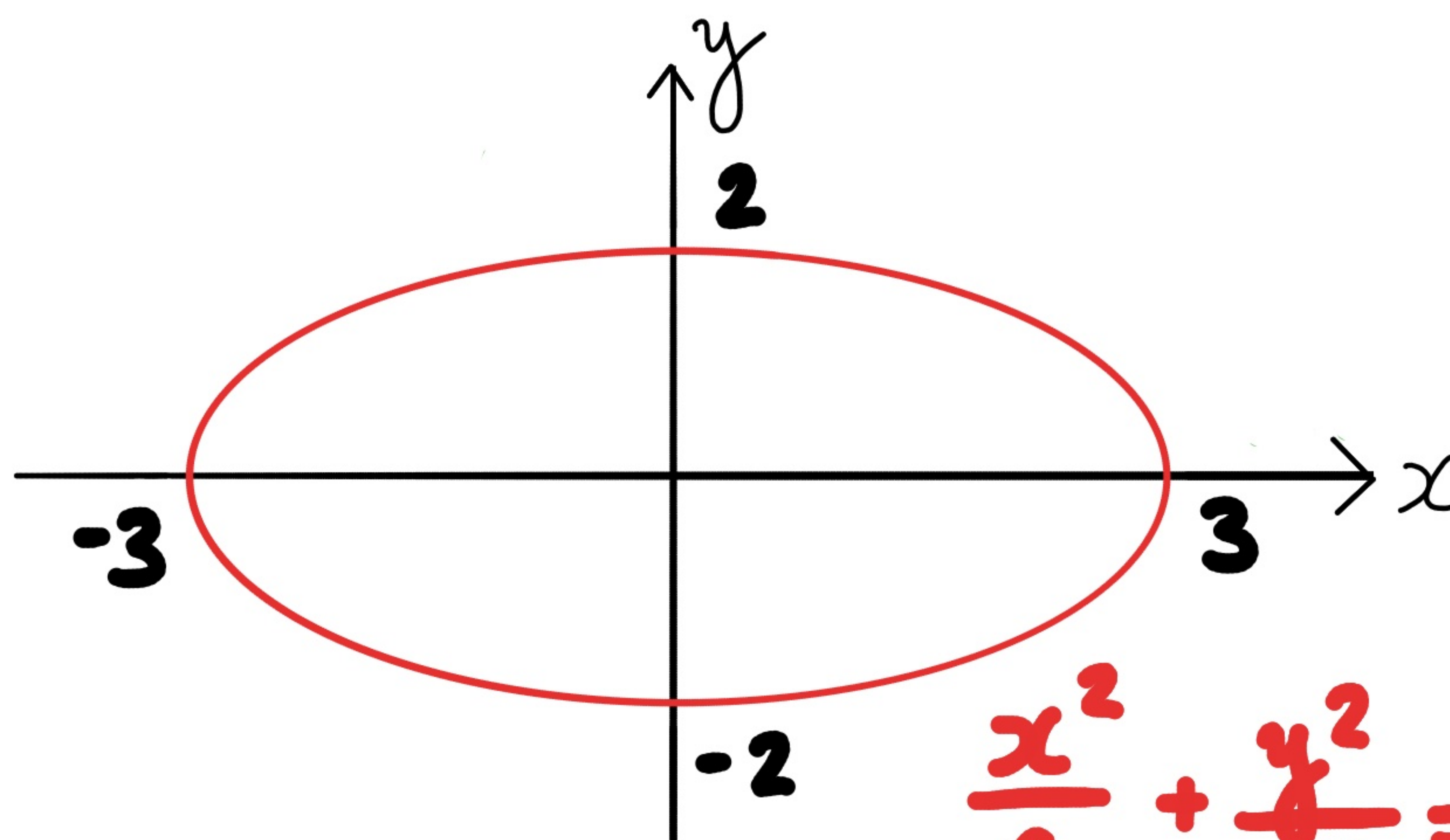
Seventeen

Method of Langrange multipliers

Example: Find max/min for $\sqrt{x^2+y^2}$ given that $\frac{x^2}{9}+\frac{y^2}{4}=1$.



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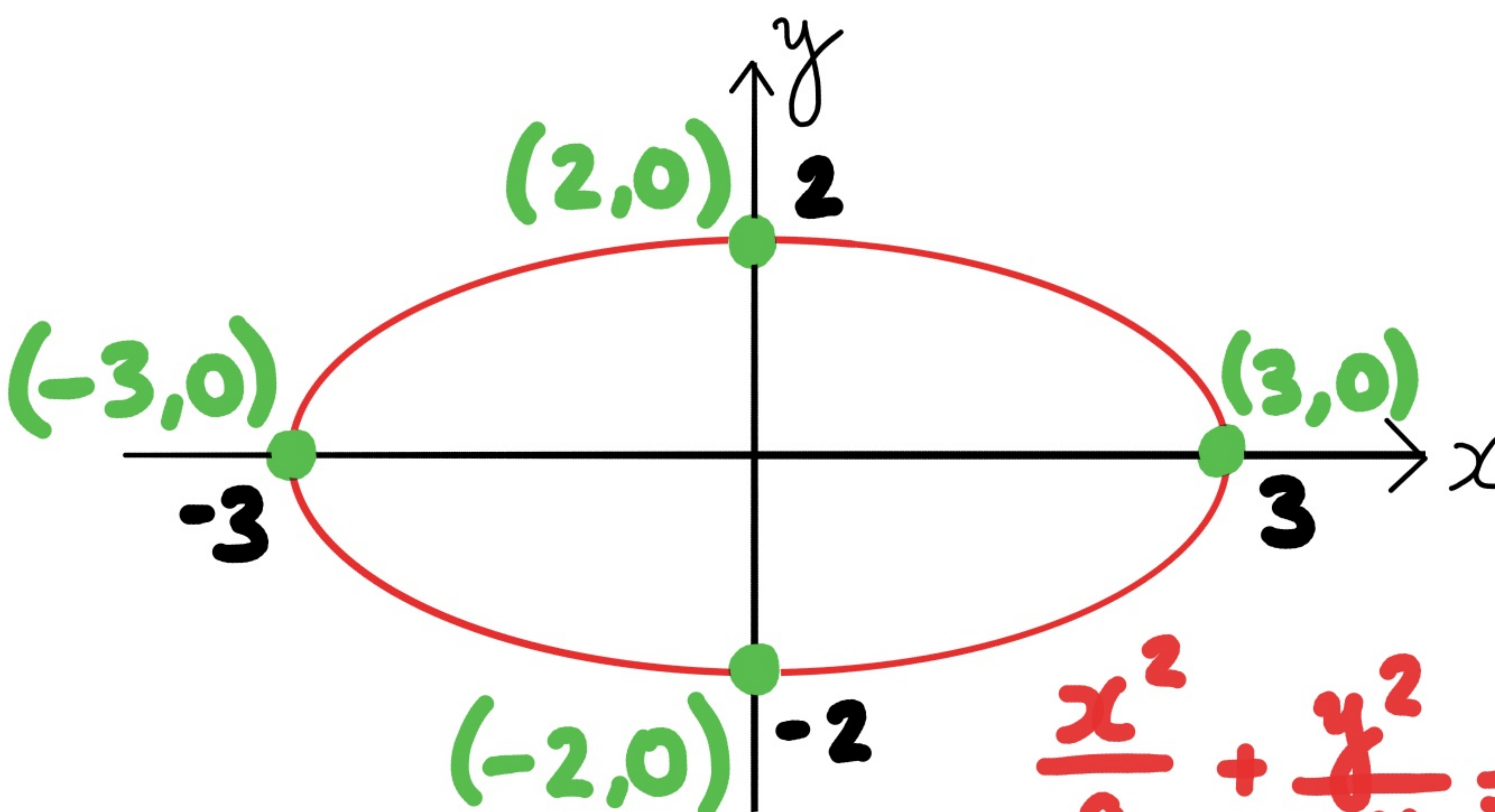

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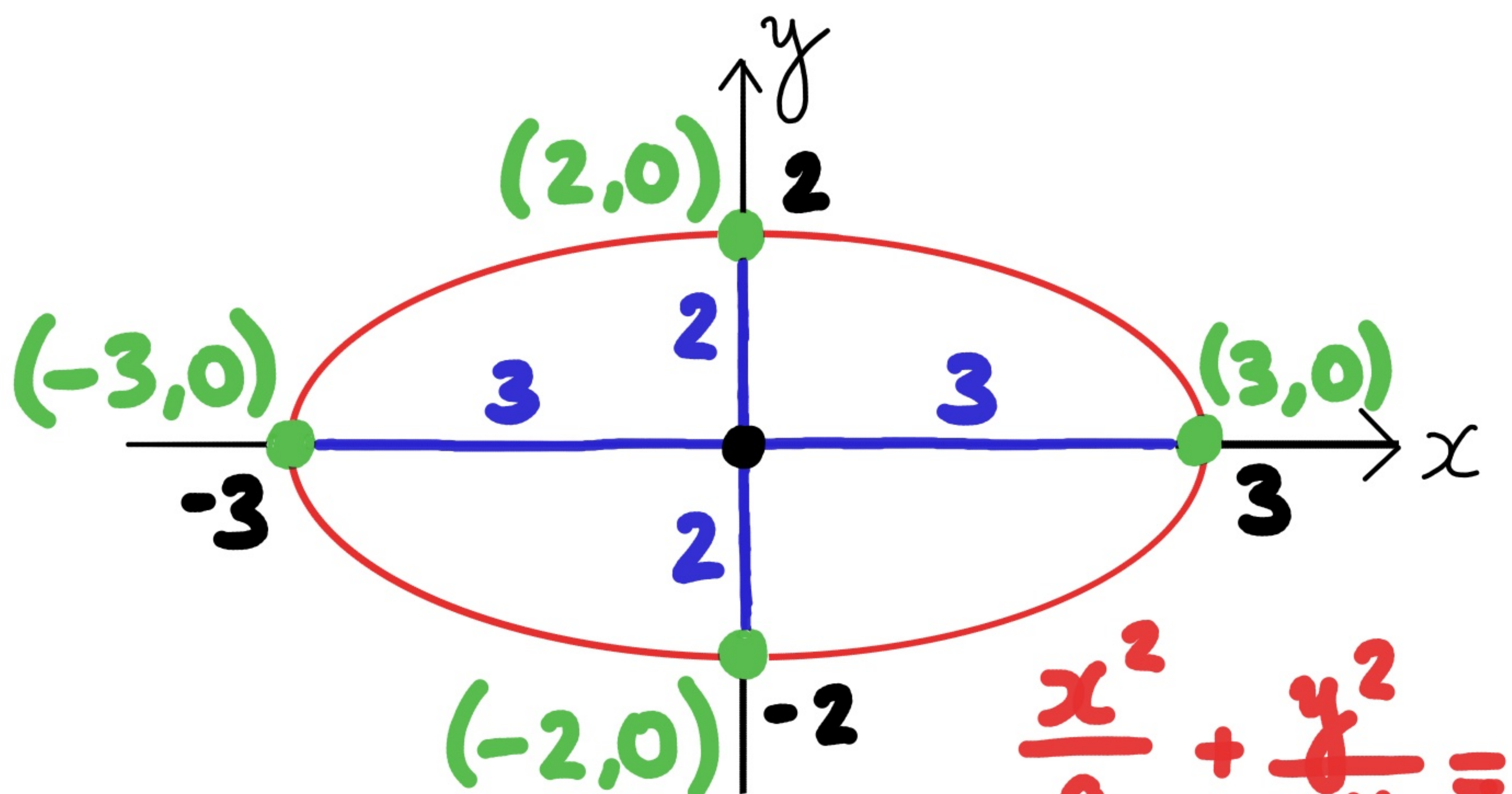

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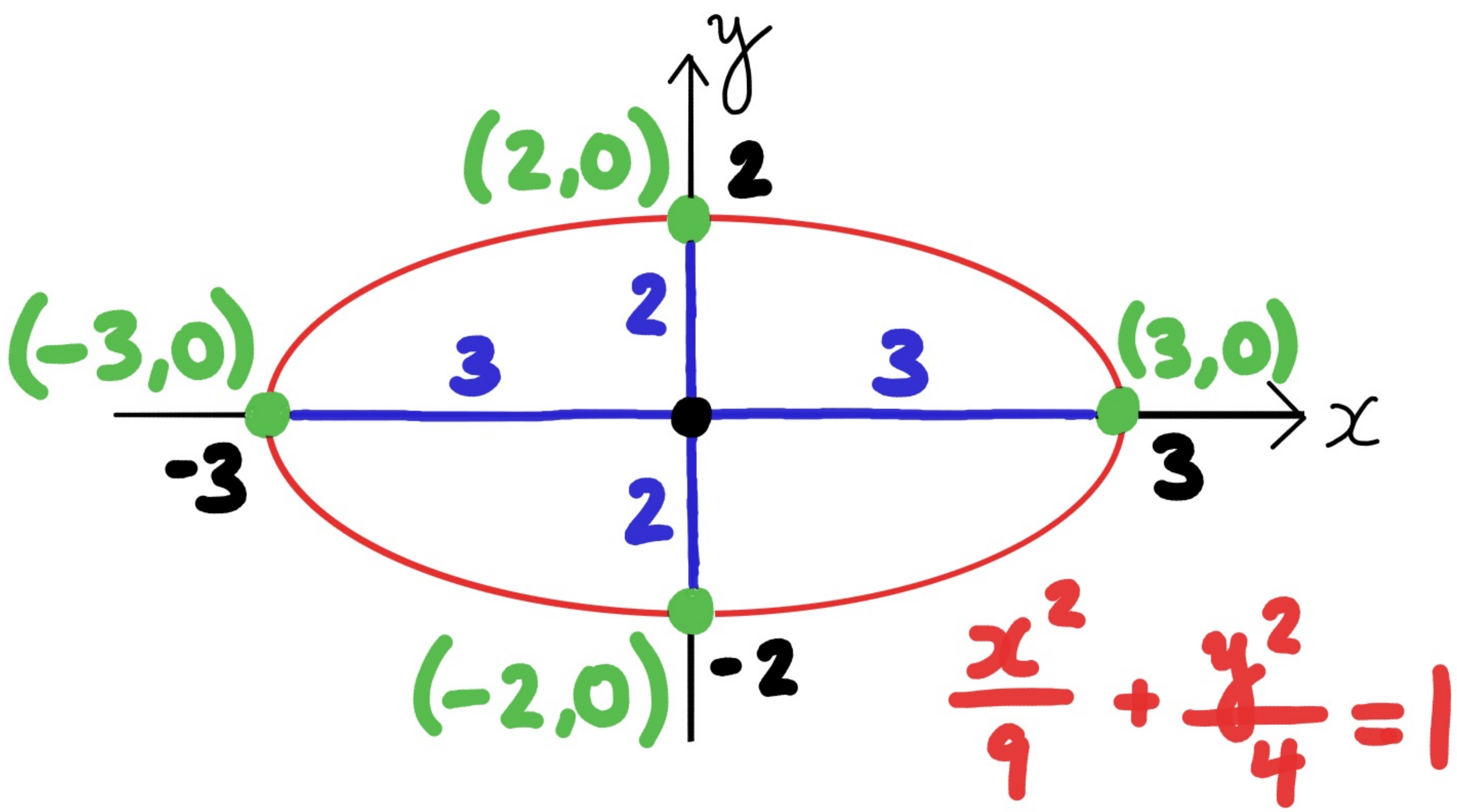

max: 3

min: 2

Example: Find max/min for $\sqrt{x^2+y^2}$ given that $\frac{x^2}{9}+\frac{y^2}{4}=1$.

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Meta-example: Find max/min for $f(x,y)$ given that $g(x,y)=c$.

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Meta-example: Find max/min for $f(x,y)$ given that $g(x,y)=c$.

Meta-example: Find max/min for $f(p)$ subject to the constraint $g(p)=c$.

$f : \mathbb{R}^2 \to \mathbb{R}$

$g : \mathbb{R}^2 \to \mathbb{R}$

$p_0 \in \mathbb{R}^2$

If $\nabla f(p_0) \neq \lambda \nabla g(p_0)$ for any $\lambda \in \mathbb{R}$, then:

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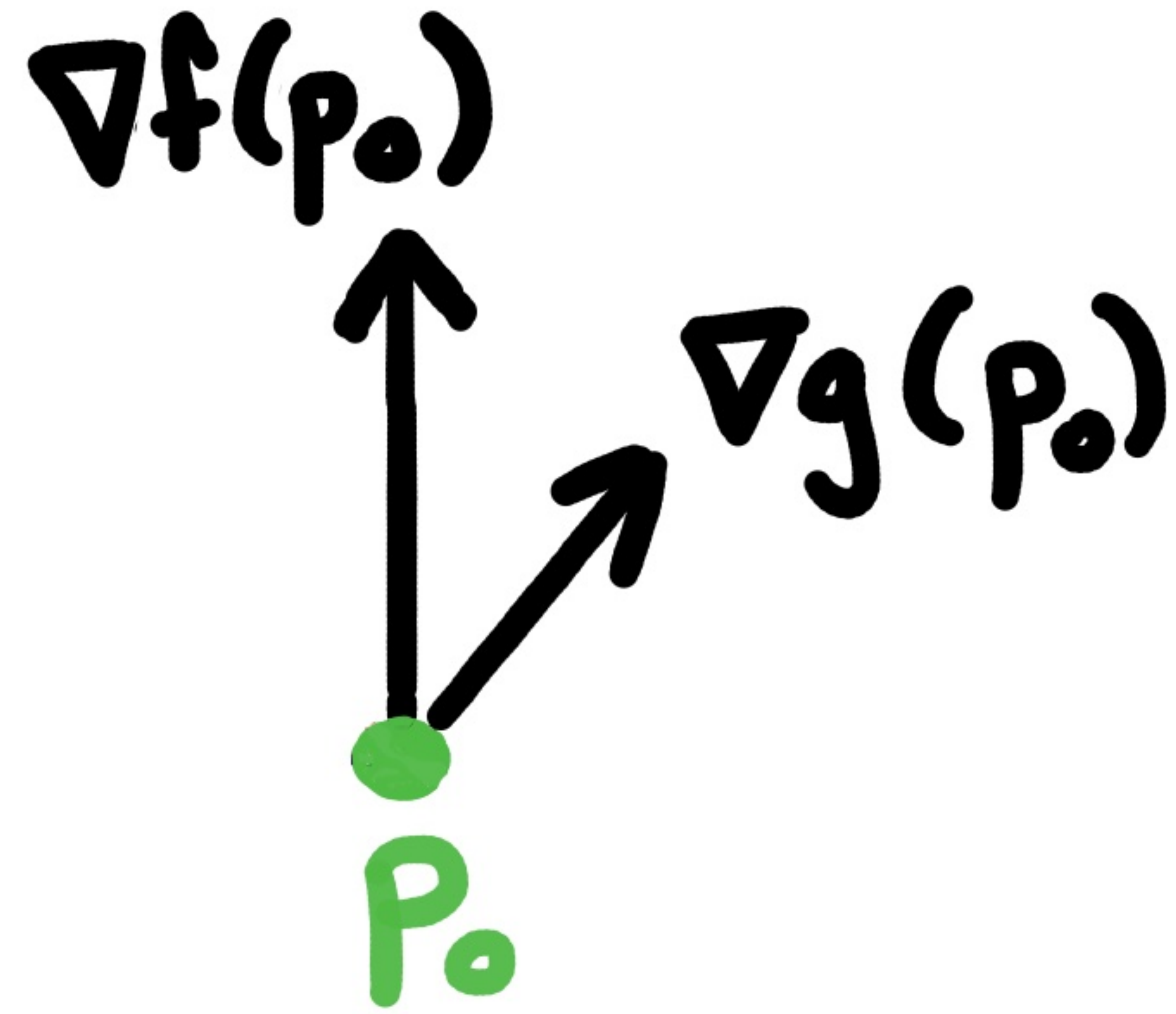

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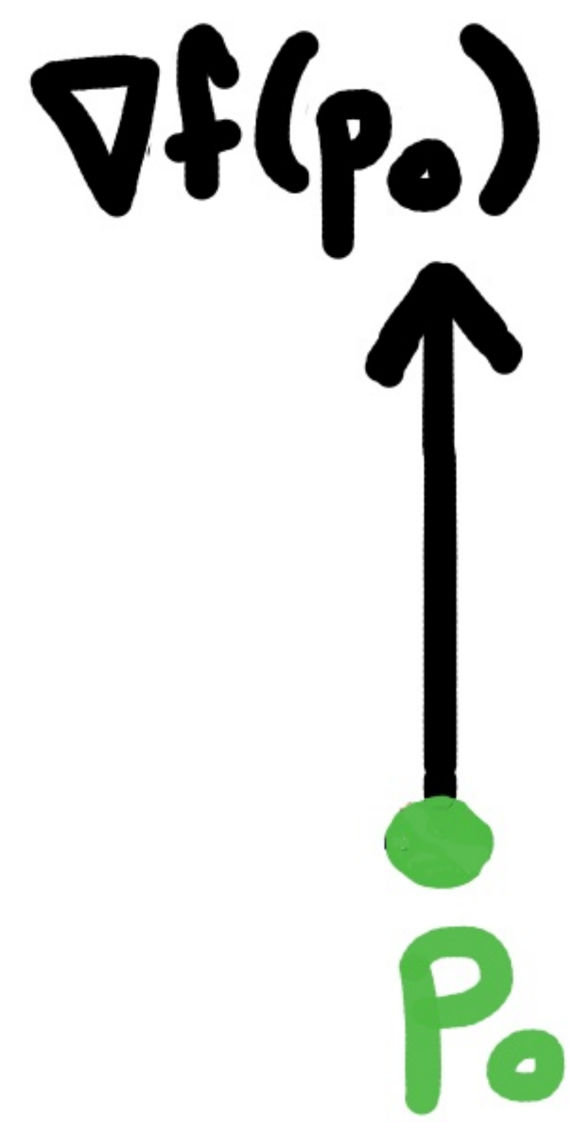

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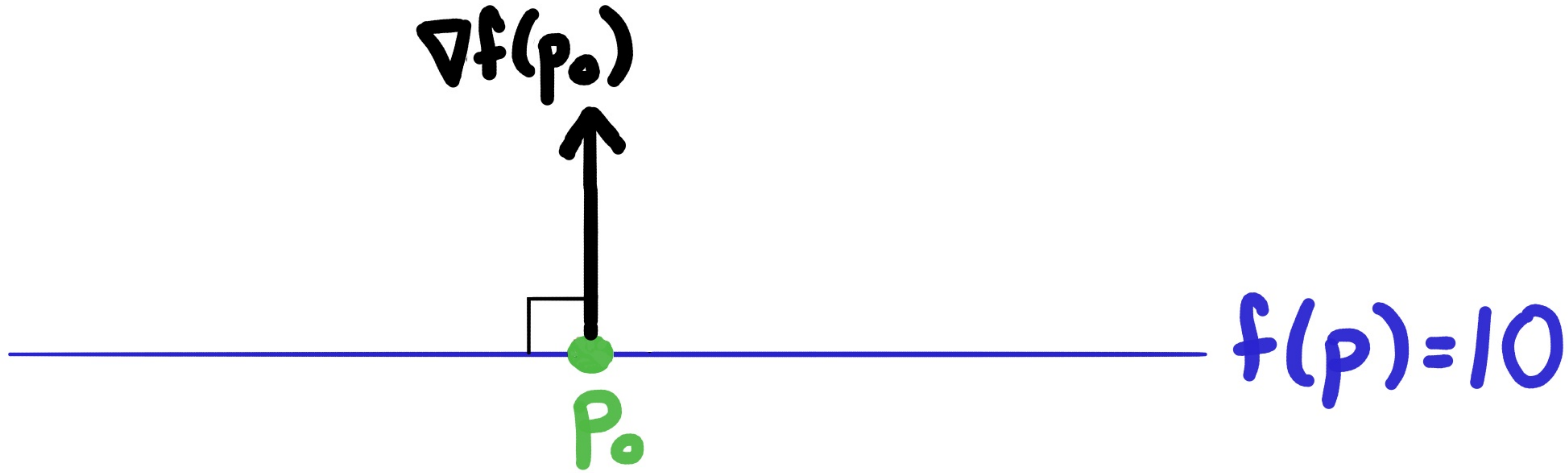

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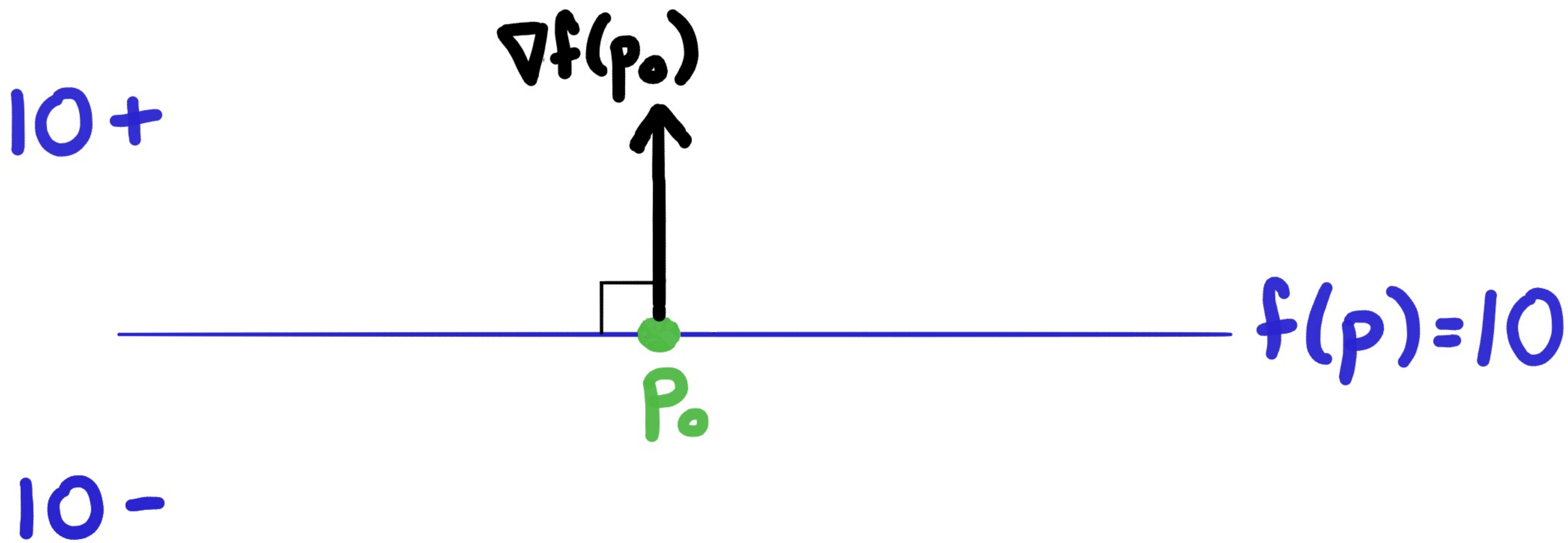
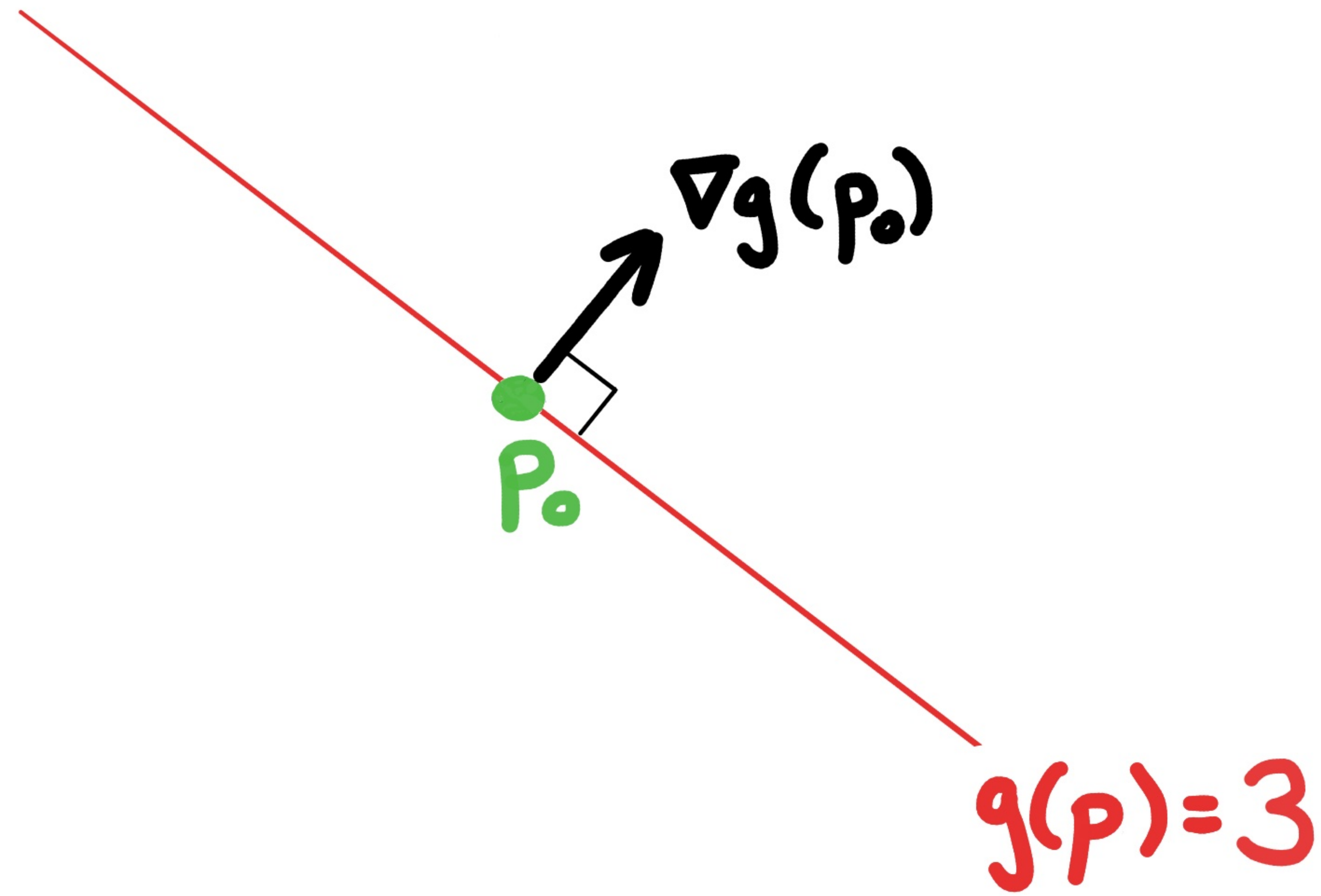


$\color{blue}{f}: \mathbb{R}^2 \to \mathbb{R}$

$\color{red}{g}: \mathbb{R}^2 \to \mathbb{R}$

$\color{green}{p_0} \in \mathbb{R}^2$

If $\nabla f(p_0) \neq \lambda \nabla g(p_0)$ for any $\lambda \in \mathbb{R}$, then:



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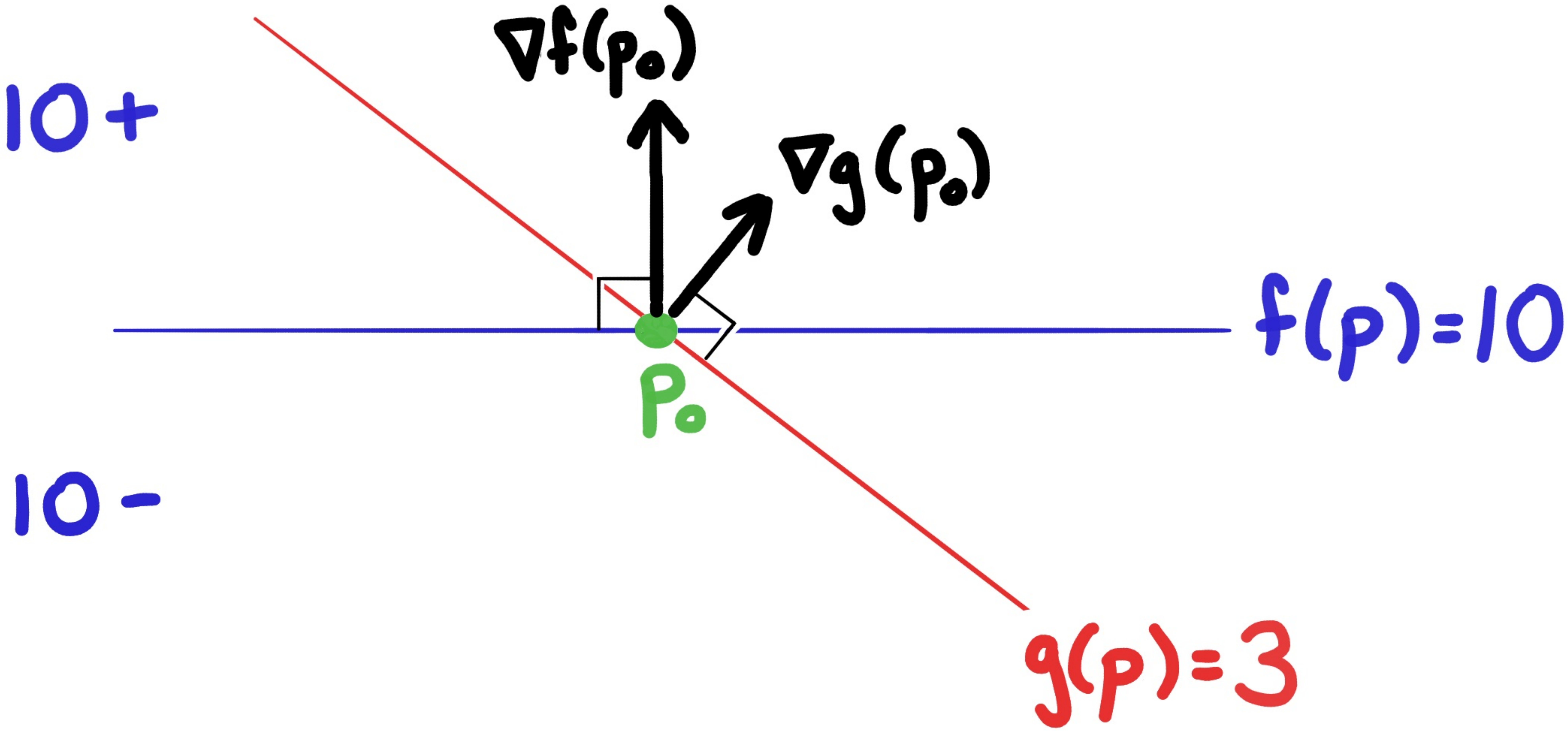

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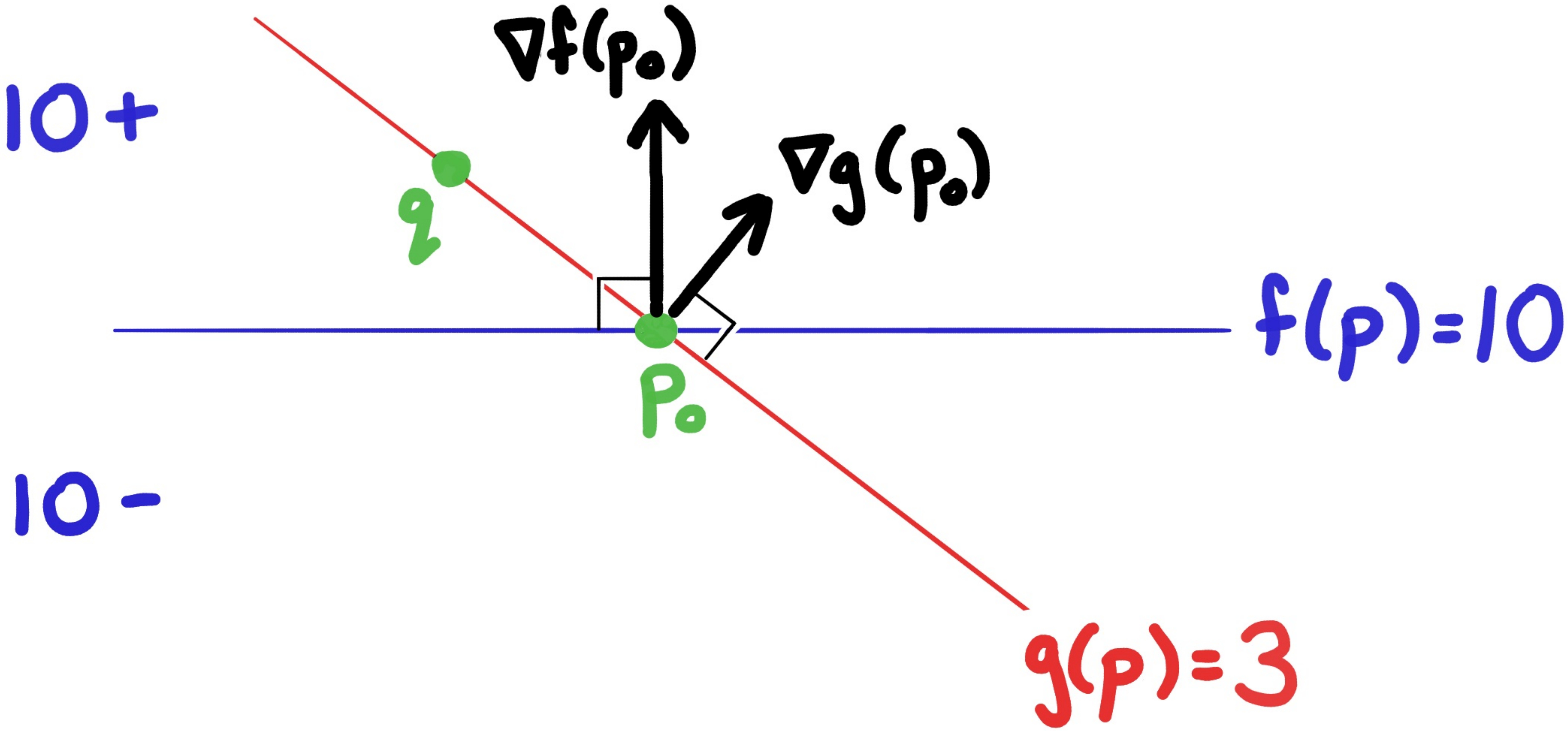

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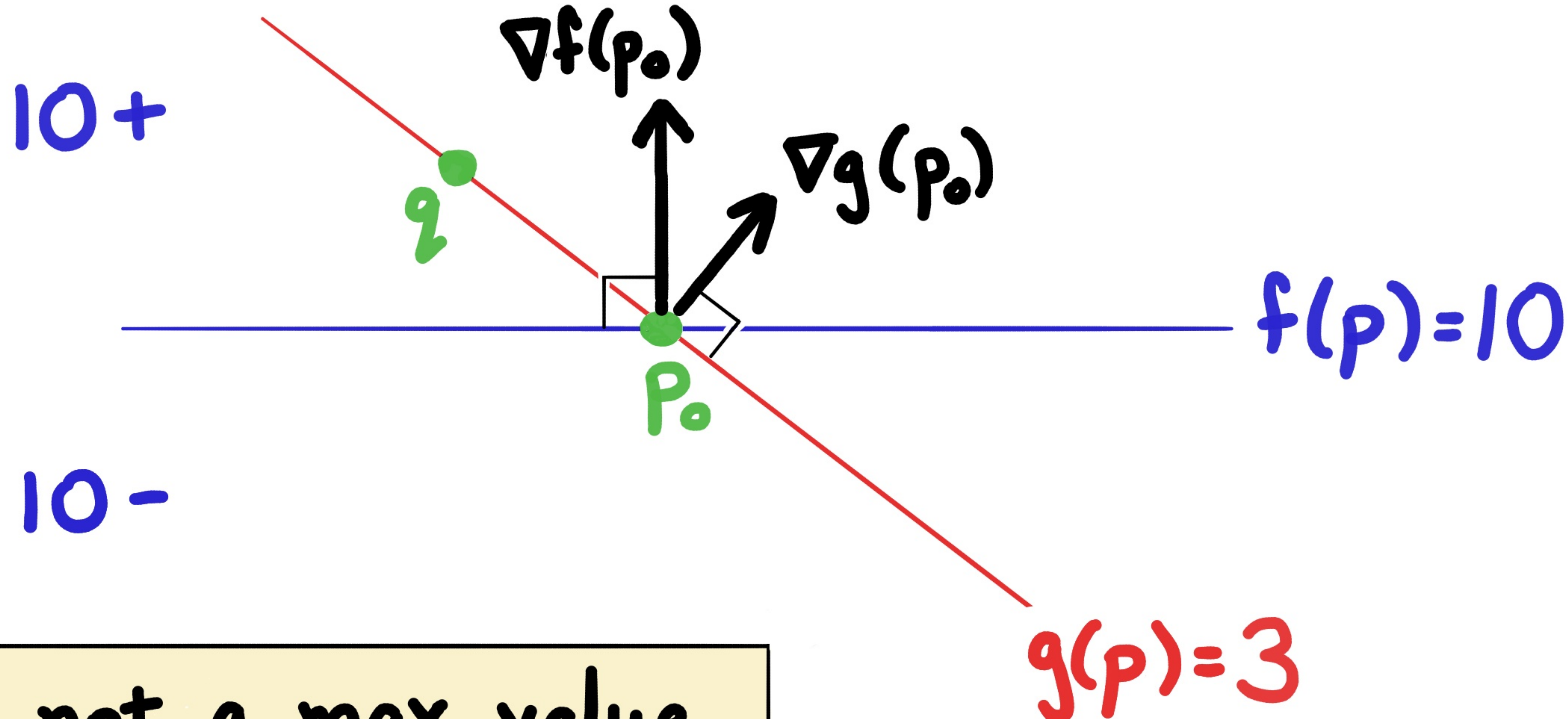

$f(p_0)$ is not a max value of $f(p)$ subject to $g(p)=3$. Indeed, $f(q) > f(p_0)$ and $g(q)=3$.

Summary:

If $\nabla f(p_0) \neq \lambda \nabla g(p_0)$ for some $\lambda \in \mathbb{R}$, then $f(p_0)$ is neither max nor min for $f(p)$ subject to $g(p) = c$.

Summary:

If $\nabla f(p_0) \neq \lambda \nabla g(p_0)$ for some $\lambda \in \mathbb{R}$, then $f(p_0)$ is neither max nor min for $f(p)$ subject to $g(p)=c$.

Conclusion:

To find max/min for $f(p)$ subject to $g(p)=c$, find p_0 with $\nabla f(p_0) = \lambda \nabla g(p_0)$ for some $\lambda \in \mathbb{R}$.

Method of Lagrange multipliers

To find max/min for $f(p)$ subject to the constraint $g(p)=c$, find p_0 such that

(i) $\nabla f(p_0) = \lambda \nabla g(p_0)$ for some $\lambda \in \mathbb{R}$, and

(ii) $g(p_0) = c$.

Then check $f(p_0)$ for each such p_0.

(The points p_0 are called <u>critical points</u>.)

Examples:

1. Let $f(x,y) = -4x - 2y$ and $g(x,y) = x^2 + y^2$. Find min and max values of $f(x,y)$ subject to the constraint $g(x,y) = 1$.

2. Let $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y,z) = -x - y + z$. Find min value of $f(x,y,z)$ subject to the constraint $g(x,y,z) = 6$.

3. Find min and max values for $f(x,y) = 3x^2 + y^2$ on the domain $S = \{(x,y) : x^2 + y^2 \leq 25\}$.

1. Let $f(x,y) = -4x - 2y$ and $g(x,y) = x^2 + y^2$.
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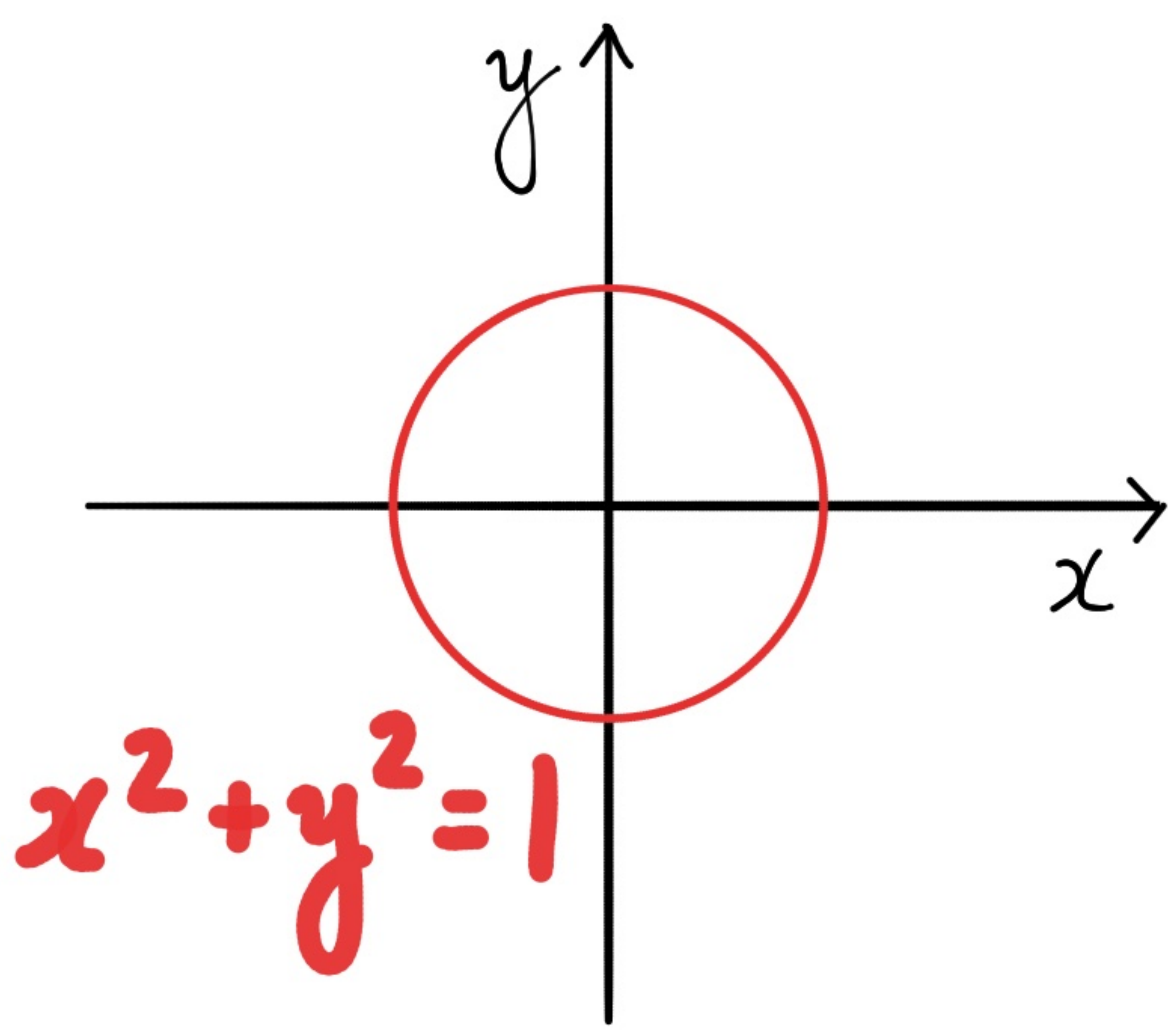
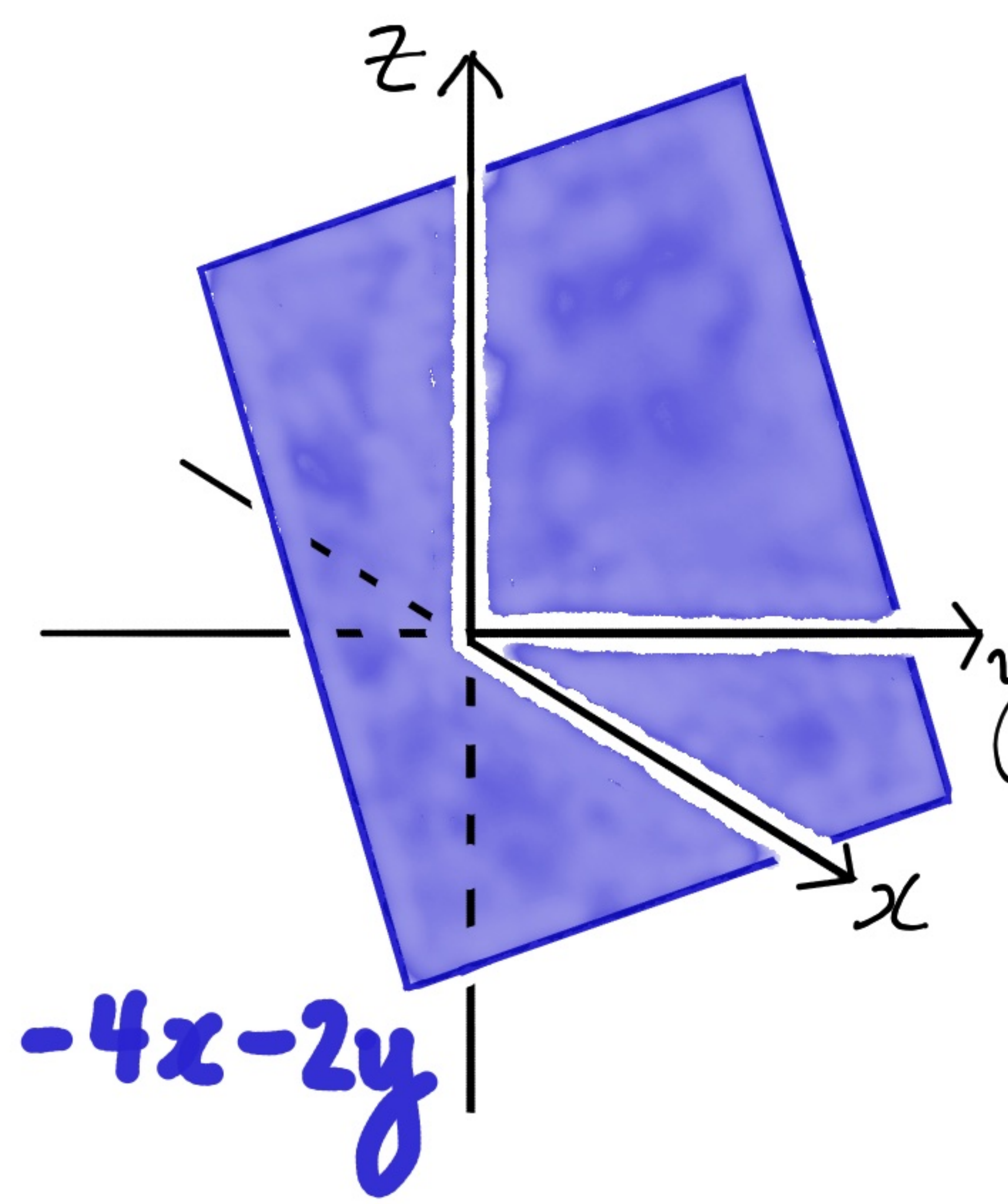
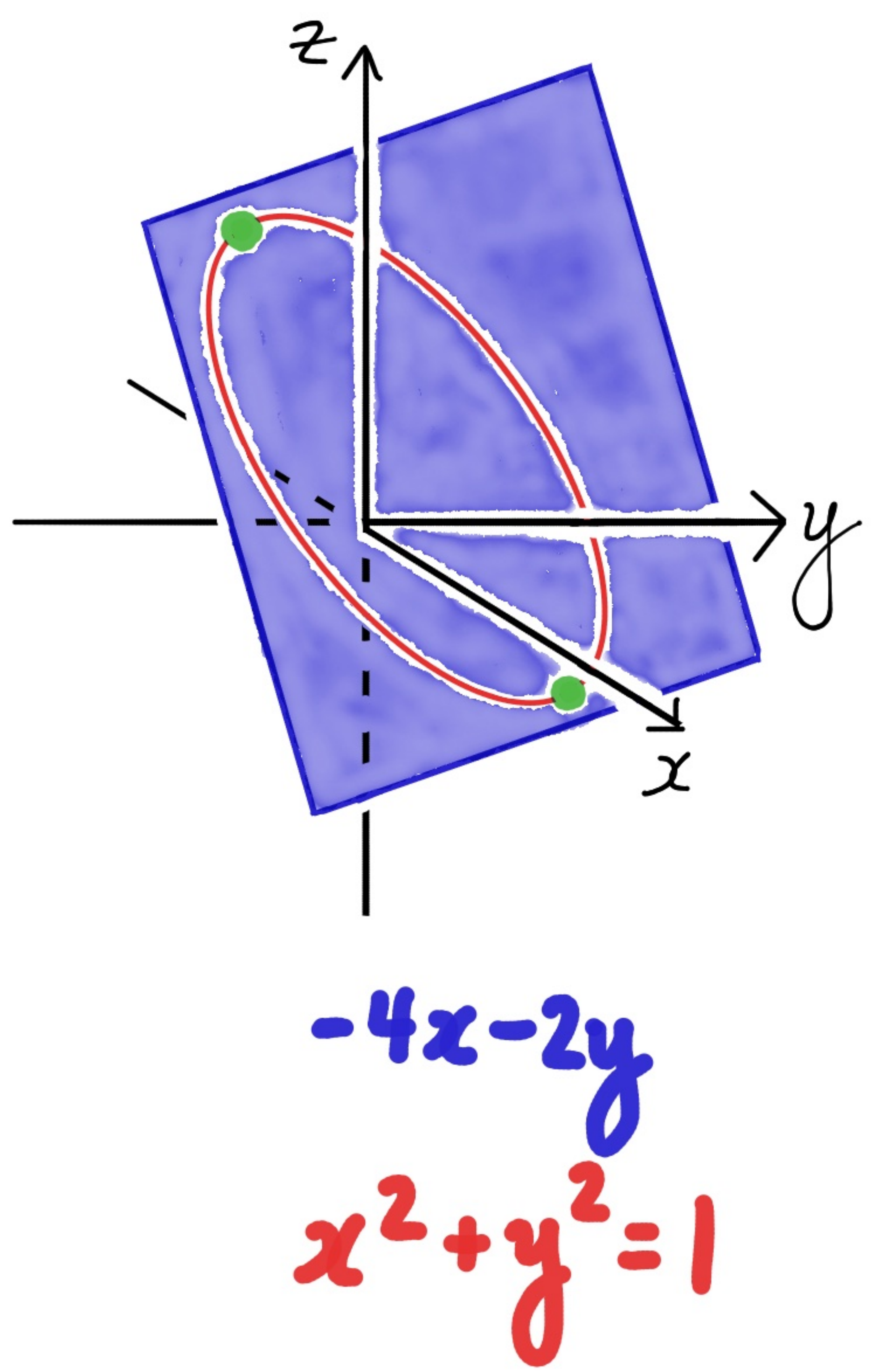
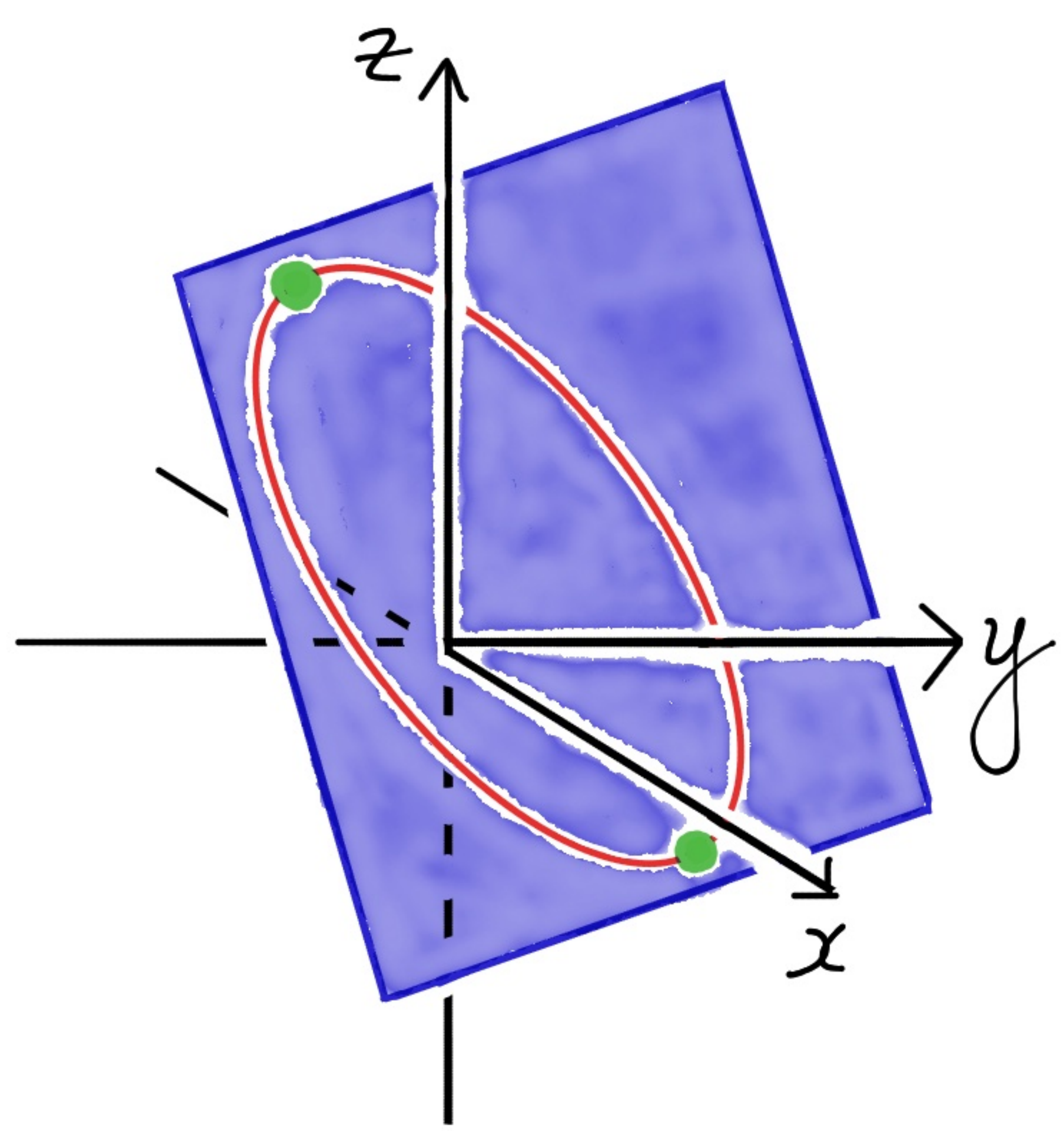


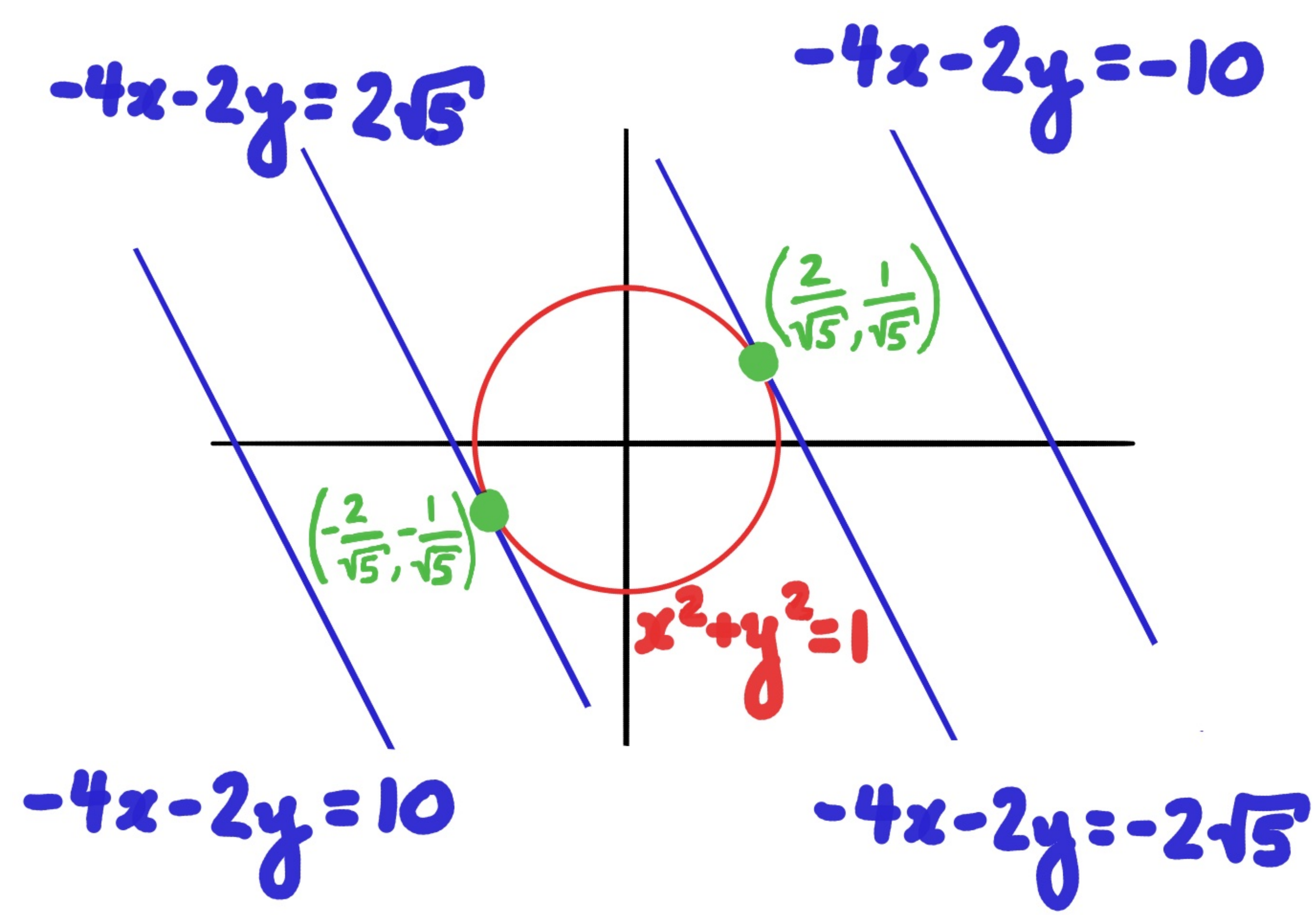




(closed and bounded)

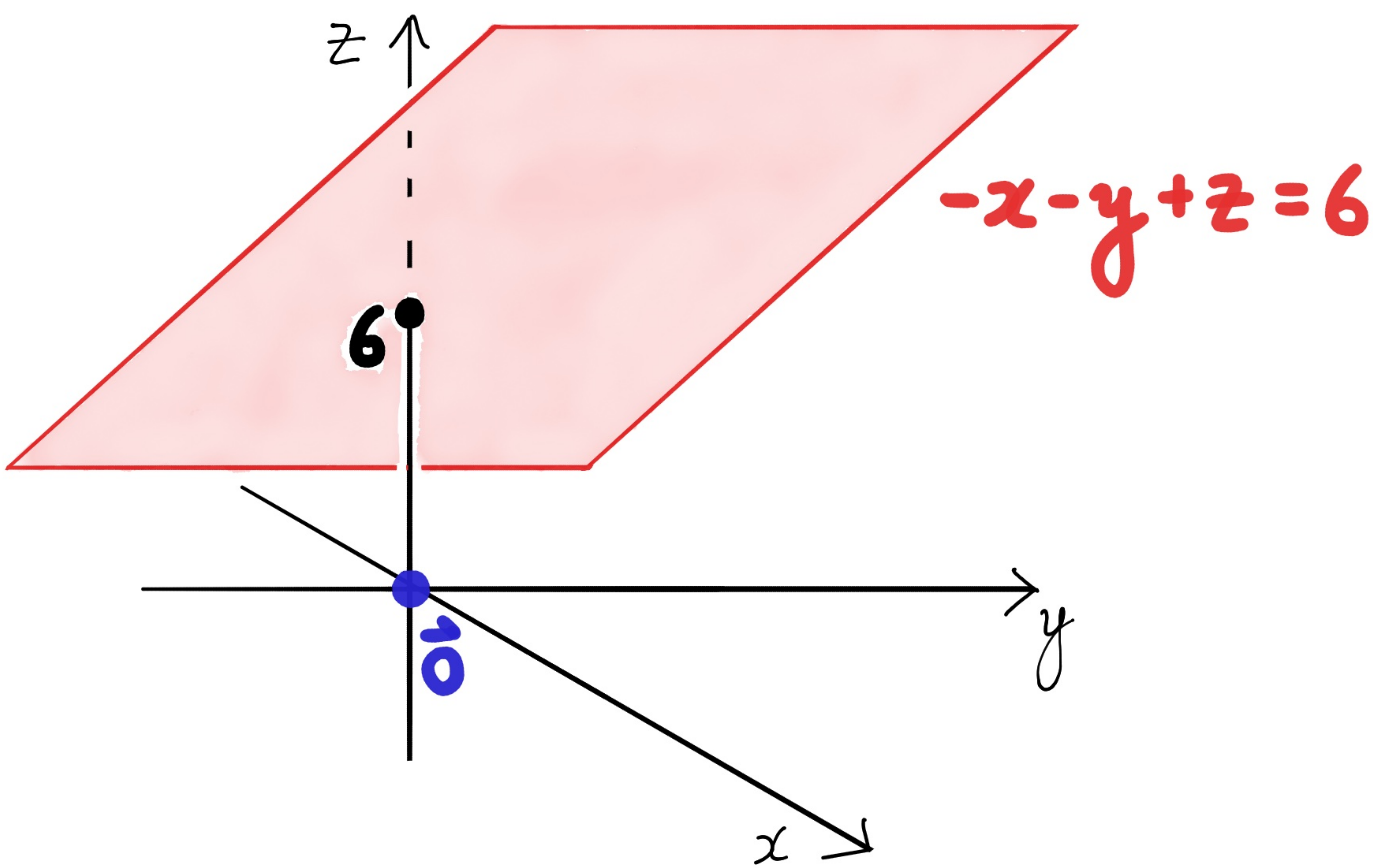
$$\left.\begin{array}{r l}\text{min/max} & -4x-2y \\ \text{w.r.t.} & x^2+y^2=1\end{array}\right\} \Rightarrow \begin{cases} -4\frac{2}{\sqrt{5}}-2\frac{1}{\sqrt{5}} = -2\sqrt{5} \\ -4\frac{-2}{\sqrt{5}}-2\frac{-1}{\sqrt{5}} = 2\sqrt{5}\end{cases}$$

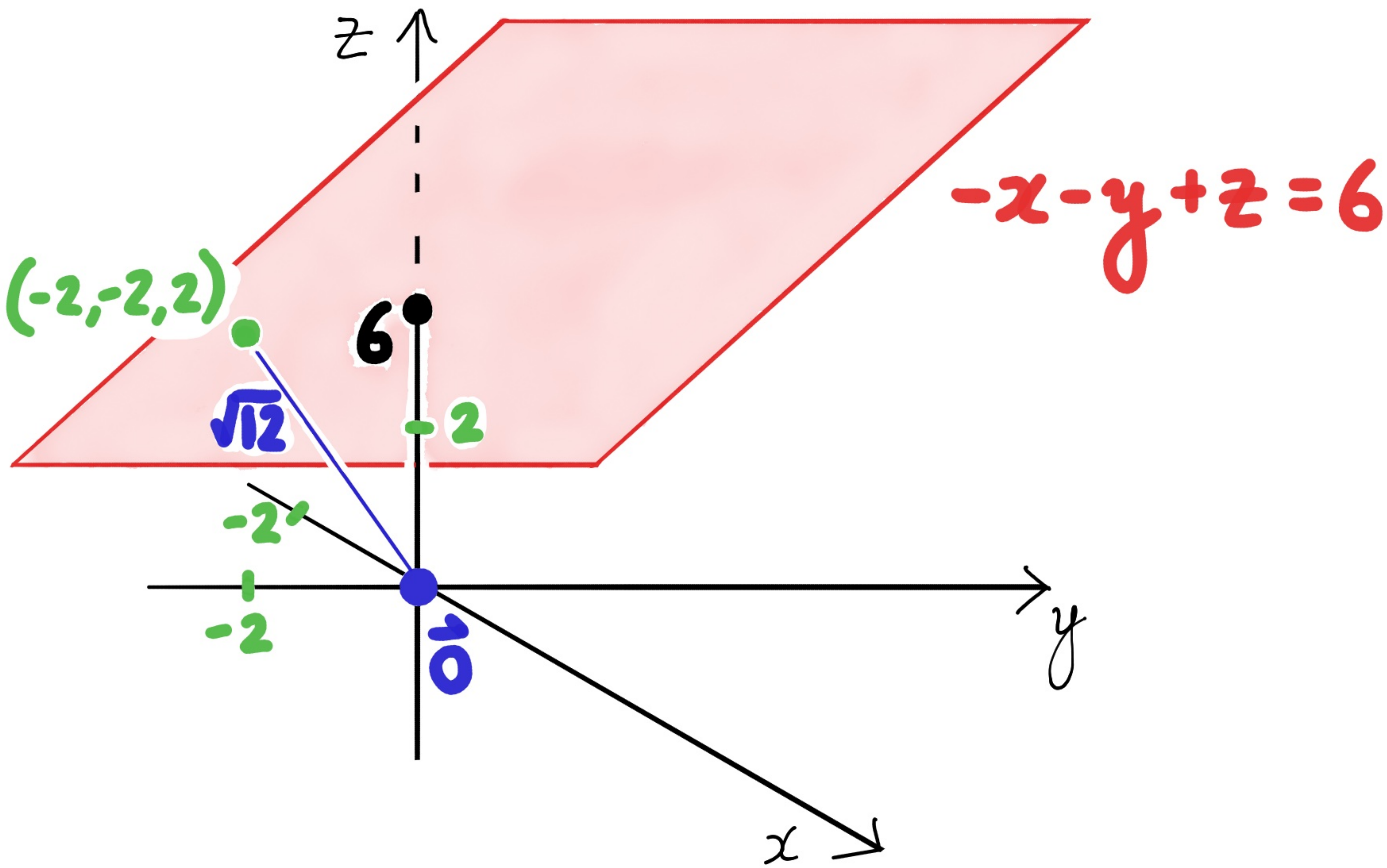




② Let $f(x,y,z)=x^2+y^2+z^2$ and $g(x,y,z)=-x-y+z$. Find min value of $f(x,y,z)$ subject to the constraint $g(x,y,z)=6$.

② Let $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = -x-y+z$.
Find min value of $f(x,y,z)$ subject to the constraint $g(x,y,z)=6$.





$$\min \quad x^2+y^2+z^2 \quad \text{w.r.t.} \quad -x-y+z=6$$

$$\text{is} \quad (-2)^2+(-2)^2+(2)^2=12.$$

③ Find min and max values for $f(x,y) = 3x^2 + y^2$ on the domain $S = \{(x,y) : x^2 + y^2 \le 25\}$.

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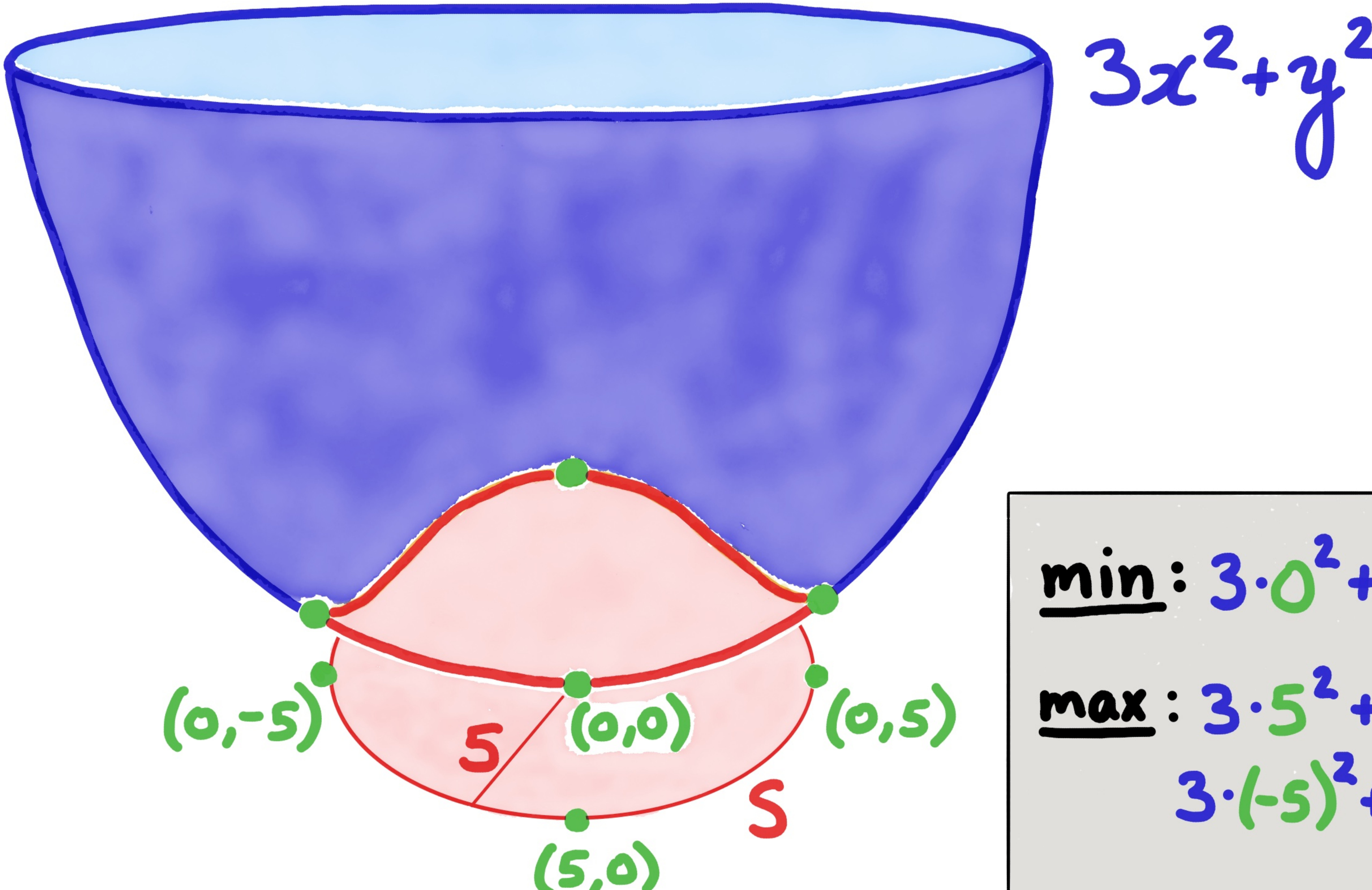

min: $3 \cdot 0^2 + 0^2 = 0$

max: $3 \cdot 5^2 + 0^2 = 75$

$3 \cdot (-5)^2 + 0^2 = 75$

3 Find min and max values for $f(x,y)=3x^2+y^2$ on the domain $S=\{(x,y): x^2+y^2\le 25\}$.

③ Find min and max values for $f(x,y) = 3x^2 + y^2$ on the domain $S = \{(x,y) : x^2 + y^2 \leq 25\}$.

Solution:

I: Interior of S: single critical point at $(0,0)$: $\nabla f(0,0) = \vec{0}$.

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For $g(x,y) = x^2 + y^2$,

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For $g(x,y) = x^2 + y^2$, solve

- $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 25$

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For $g(x,y)=x^2+y^2$, solve

- $\nabla f(x,y)=\lambda\nabla g(x,y)$ and $g(x,y)=25$
- $f_x=\lambda g_x$, $f_y=\lambda g_y$, and $g(x,y)=25$

③ Find min and max values for $f(x,y) = 3x^2 + y^2$ on the domain $S = \{(x,y) : x^2 + y^2 \le 25\}$.

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- $f_x = \lambda g_x$, $f_y = \lambda g_y$, and $g(x,y) = 25$
- $6x = \lambda 2x$, $2y = \lambda 2y$, and $x^2 + y^2 = 25$

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If $x \neq 0$,

- $3x = \lambda x$, $y = \lambda y$, and $x^2 + y^2 = 25$

Since $x^2 + y^2 = 25$, either $x \neq 0$ or $y \neq 0$.

If $x \neq 0$, $3 = \lambda$

- $3x=\lambda x$, $y=\lambda y$, and $x^2+y^2=25$

Since $x^2+y^2=25$, either $x\neq 0$ or $y\neq 0$.

If $x\neq 0$, $3=\lambda$, $y=0$

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Since $x^2 + y^2 = 25$, either $x \neq 0$ or $y \neq 0$.

If $x \neq 0$, $3 = \lambda$, $y = 0$, and $x = \pm 5$.

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Since $x^2 + y^2 = 25$, either $x \neq 0$ or $y \neq 0$.

If $x \neq 0$, $3 = \lambda$, $y = 0$, and $x = \pm 5$.

<u>critical points:</u> $(5,0)$ and $(-5,0)$

- $3x = \lambda x$, $y = \lambda y$, and $x^2 + y^2 = 25$

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If $x \neq 0$, $3 = \lambda$, $y = 0$, and $x = \pm 5$.

critical points: $(5,0)$ and $(-5,0)$

If $y \neq 0$,

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Since $x^2+y^2=25$, either $x \neq 0$ or $y \neq 0$.

If $x \neq 0$, $3 = \lambda$, $y = 0$, and $x = \pm 5$.

<u>critical points:</u> $(5,0)$ and $(-5,0)$

If $y \neq 0$, $1 = \lambda$

- $3x = \lambda x$, $y = \lambda y$, and $x^2 + y^2 = 25$

Since $x^2 + y^2 = 25$, either $x \neq 0$ or $y \neq 0$.

If $x \neq 0$, $3 = \lambda$, $y = 0$, and $x = \pm 5$.

critical points: $(5,0)$ and $(-5,0)$

If $y \neq 0$, $1 = \lambda$, $x = 0$

- $3x = \lambda x$, $y = \lambda y$, and $x^2 + y^2 = 25$

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If $y \neq 0$, $1 = \lambda$, $x = 0$, and $y = \pm 5$.

<u>critical points:</u> $(0,5)$ and $(0,-5)$

Critical points from interior and boundary of S

$(0,0)$

$(5,0)$

$(-5,0)$

$(0,5)$

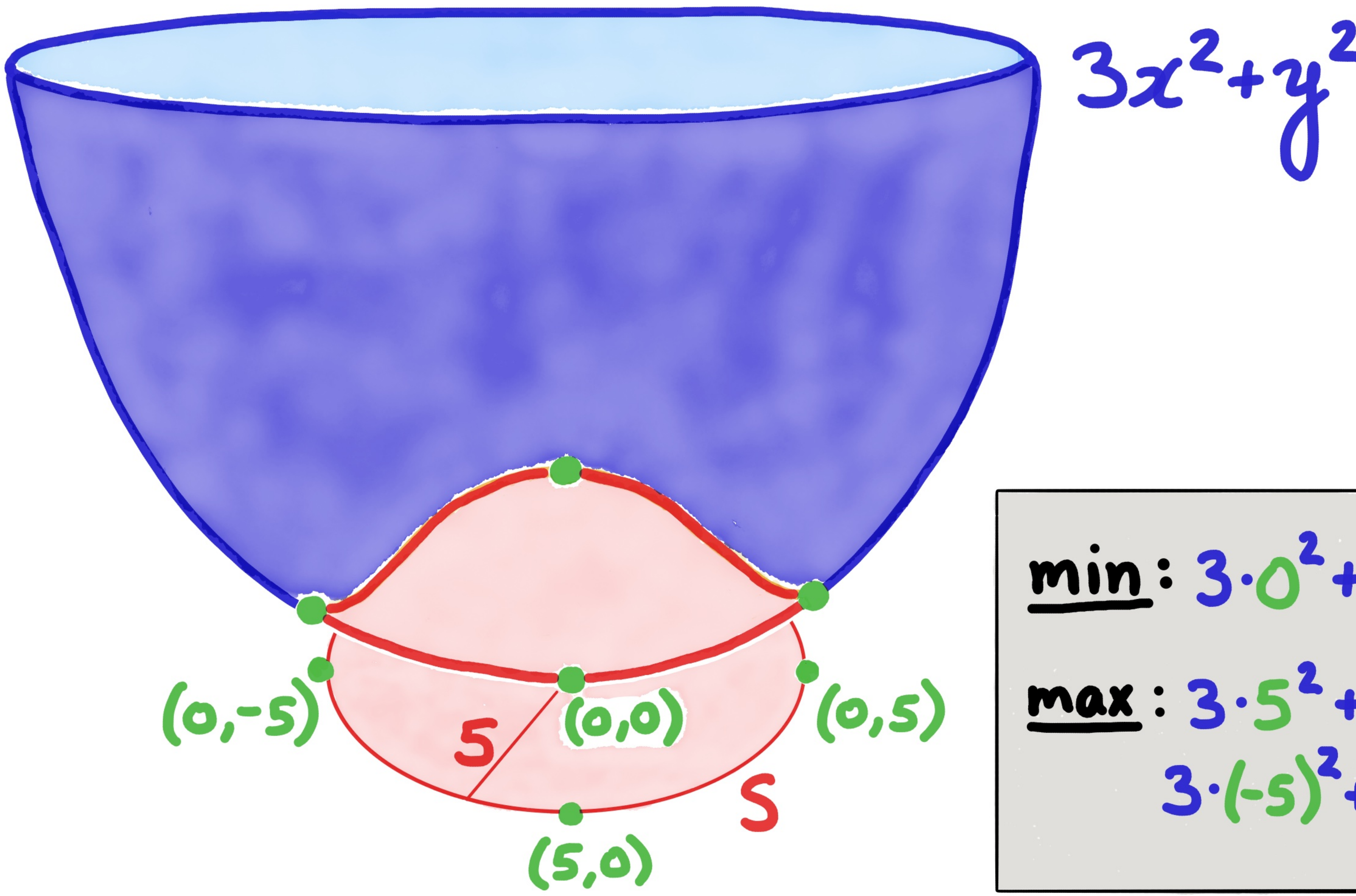
$(0,-5)$

Critical points from interior and boundary of S	Values of critical points under $3x^2+y^2$
$(0,0)$	$3\cdot 0^2 + 0^2 = 0$
$(5,0)$	$3\cdot 5^2 + 0^2 = 75$
$(-5,0)$	$3\cdot(-5)^2 + 0^2 = 75$
$(0,5)$	$3\cdot 0^2 + 5^2 = 25$
$(0,-5)$	$3\cdot 0^2 + (-5)^2 = 25$

Critical points from interior and boundary of S	Values of critical points under $3x^2+y^2$
$(0,0)$	$3\cdot 0^2+0^2=0$ ← min
$(5,0)$	$3\cdot 5^2+0^2=75$
$(-5,0)$	$3\cdot(-5)^2+0^2=75$
$(0,5)$	$3\cdot 0^2+5^2=25$
$(0,-5)$	$3\cdot 0^2+(-5)^2=25$

max

③ Find min and max values for $f(x,y) = 3x^2 + y^2$ on the domain $S = \{(x,y) : x^2 + y^2 \leq 25\}$.



min: $3 \cdot 0^2 + 0^2 = 0$

max: $3 \cdot 5^2 + 0^2 = 75$

$3 \cdot (-5)^2 + 0^2 = 75$

$3x^2+y^2$

x^2+y^2