Fifteen

Ⅰ Gradients and level curves

Ⅱ Gradients and tangent planes of level surfaces

Ⅲ Equations for tangent planes of level surfaces

Ⅳ Tangent planes of graphs

Ⅴ Curvature of graphs

Ⅵ Concavity

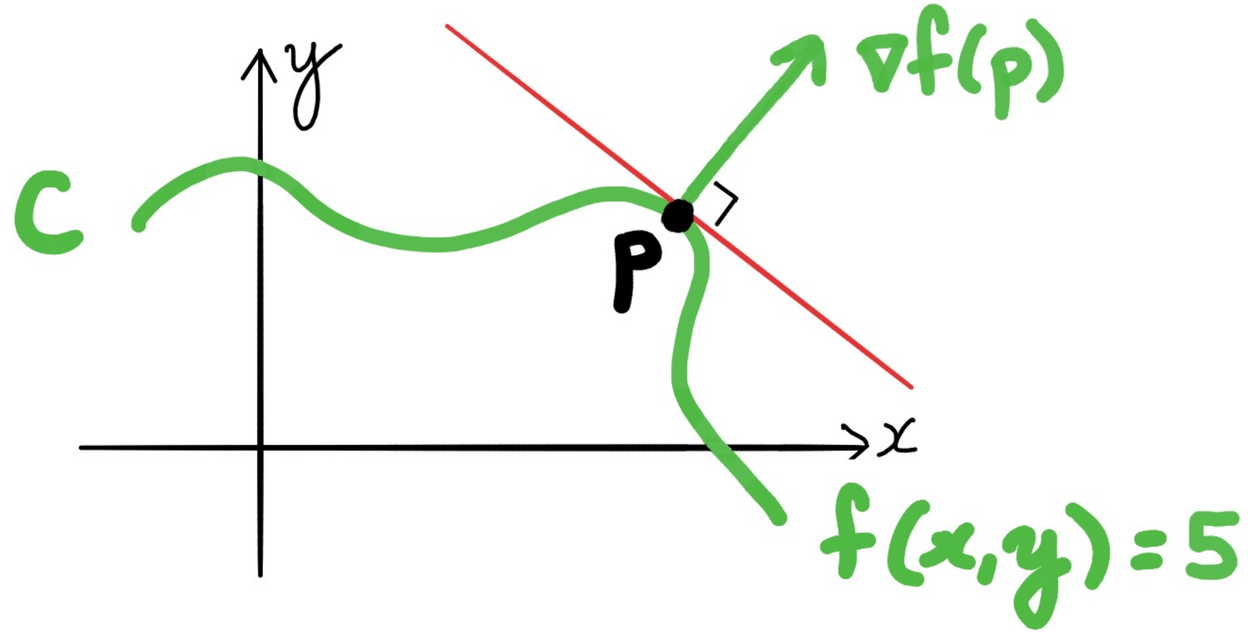
Ⓘ Gradients and level curves

Proposition:

If $p \in \mathbb{R}^2$ is on a level curve C for $f(x,y)$, then either

(a) $\nabla f(p) = \vec{0}$, or

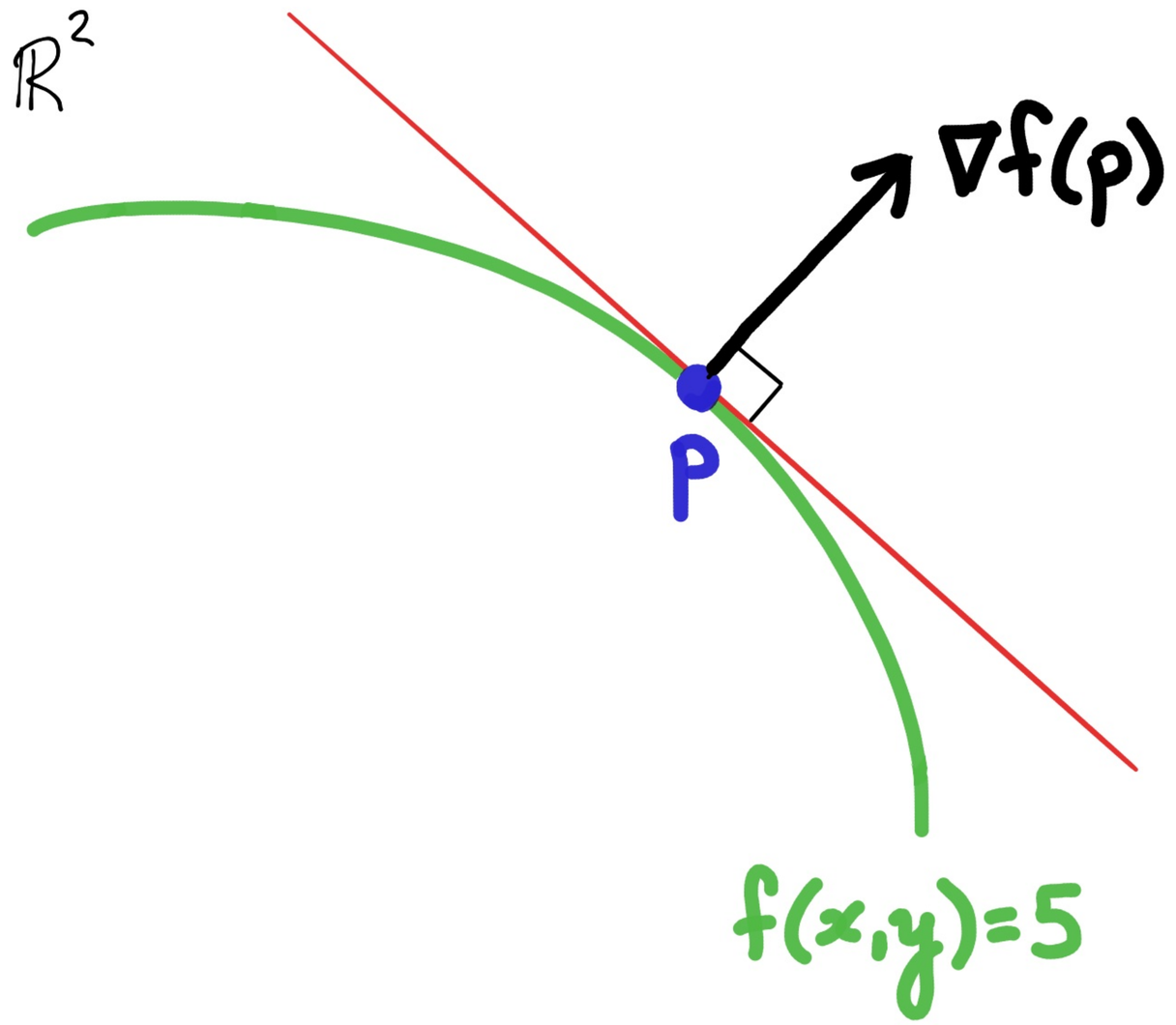
(b) $\nabla f(p)$ is orthogonal to the tangent line of C at p.



proof:

setup:

- $f : \mathbb{R}^2 \to \mathbb{R}$

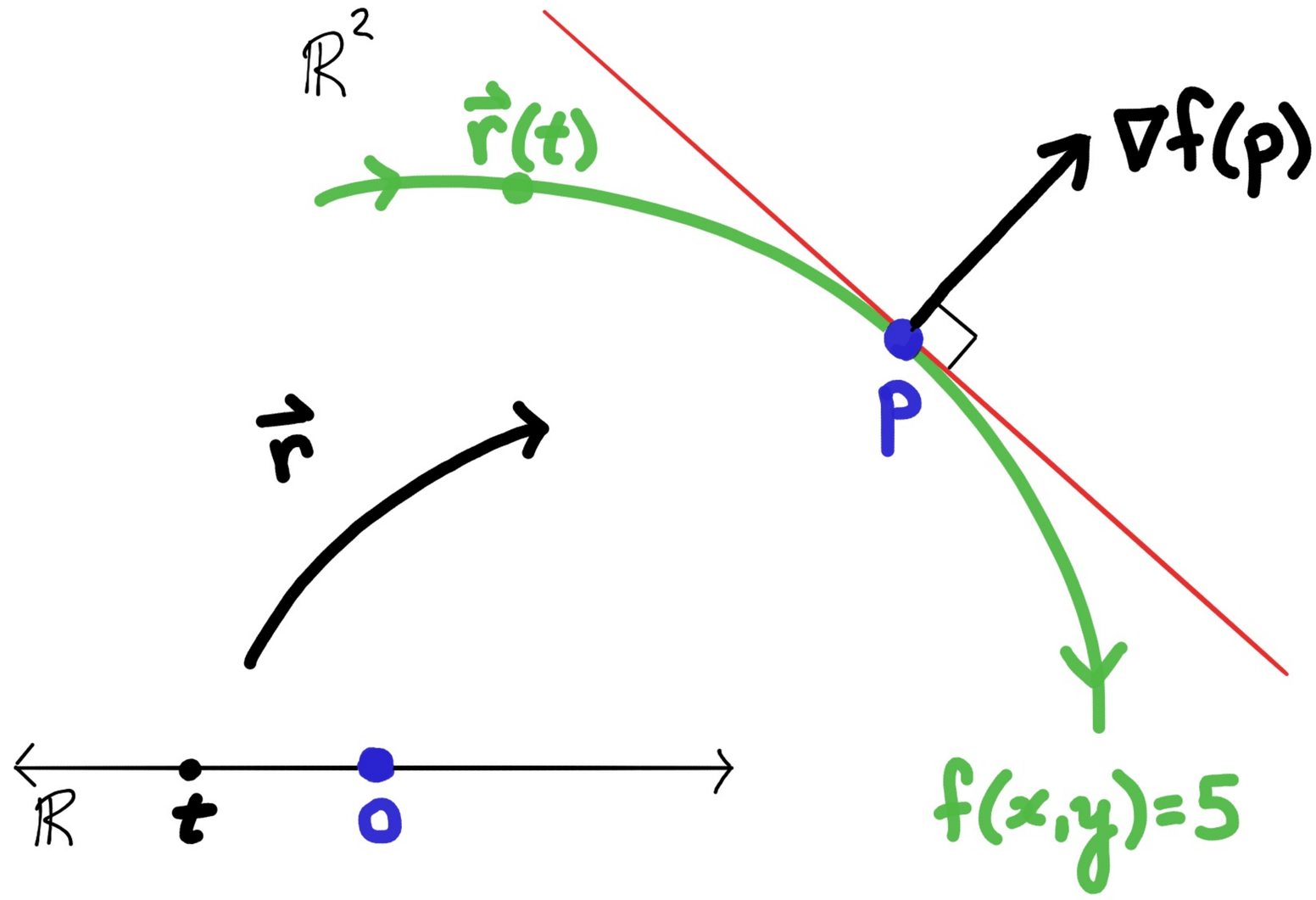


proof:

setup:

- $f : \mathbb{R}^2 \to \mathbb{R}$
- $\vec{r} : \mathbb{R} \to \mathbb{R}^2$

  param. $f(x,y)=5$

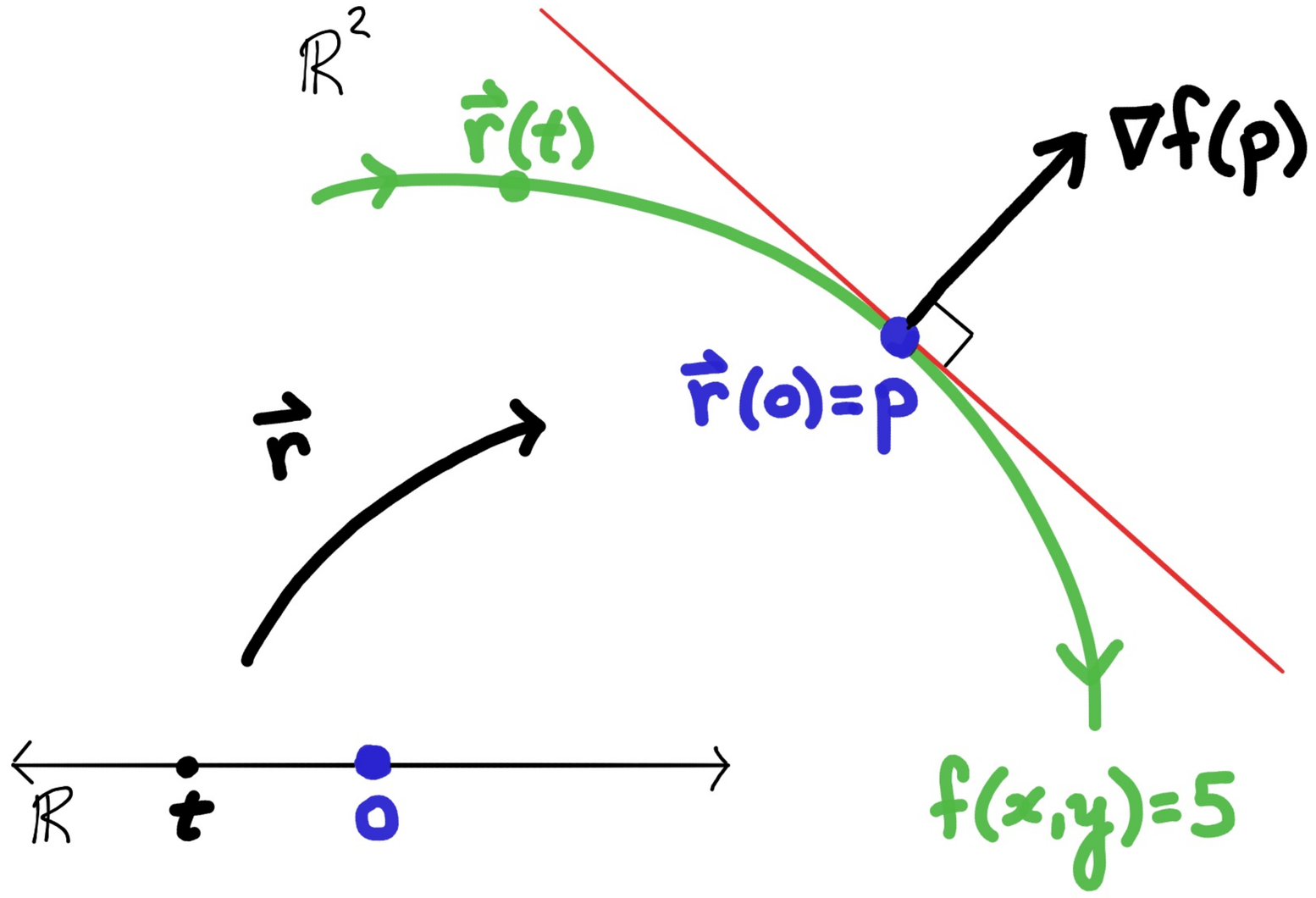


proof:

setup:

- $f : \mathbb{R}^2 \to \mathbb{R}$
- $\vec{r} : \mathbb{R} \to \mathbb{R}^2$

  parom. $f(x,y)=5$
- $\vec{r}(0) = p$



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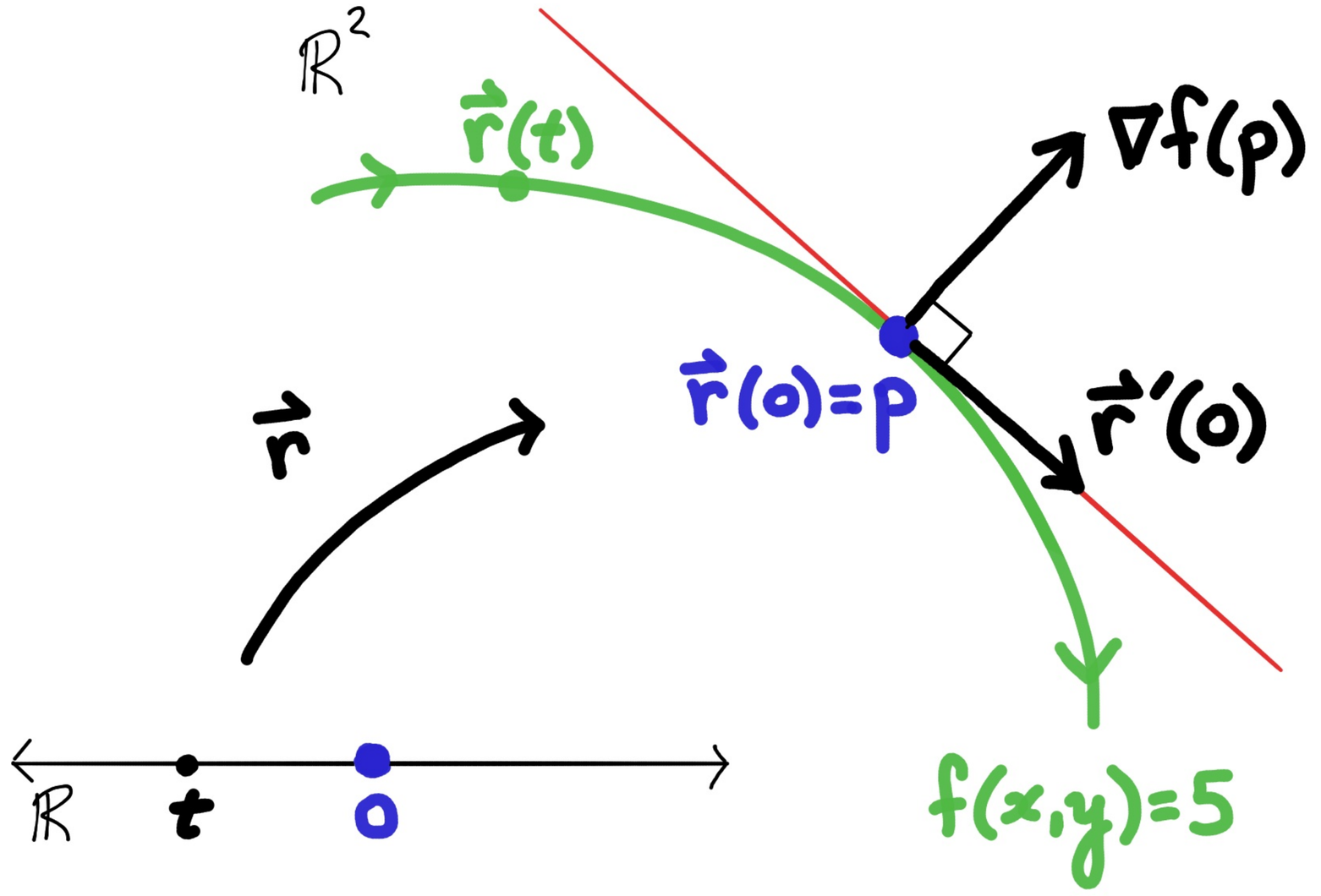

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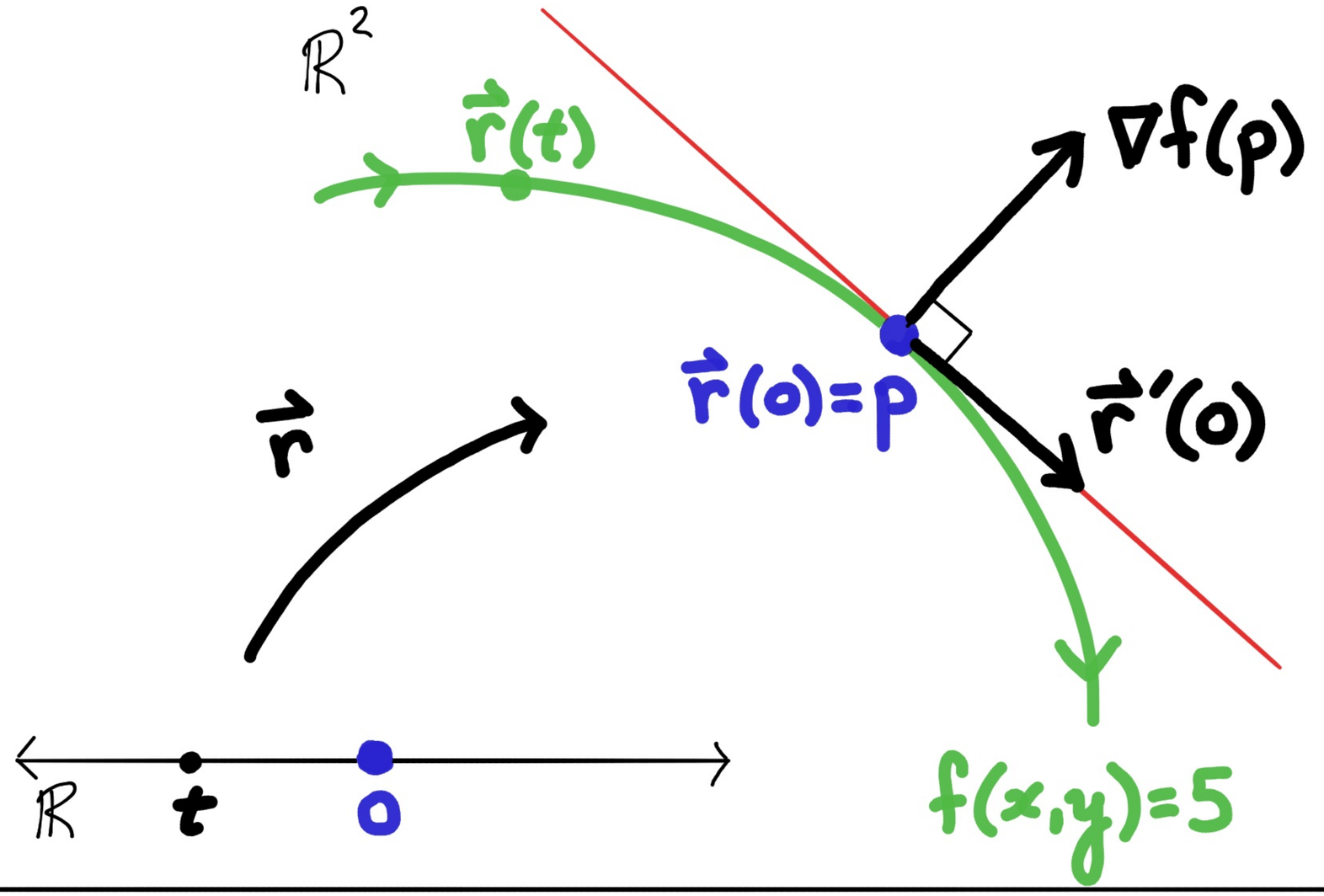

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conclusion:

- $f \circ \vec{r} : \mathbb{R} \to \mathbb{R}$, $f \circ \vec{r}(t) = f(\vec{r}(t))$

proof:

setup:

- $f : \mathbb{R}^2 \to \mathbb{R}$
- $\vec{r} : \mathbb{R} \to \mathbb{R}^2$
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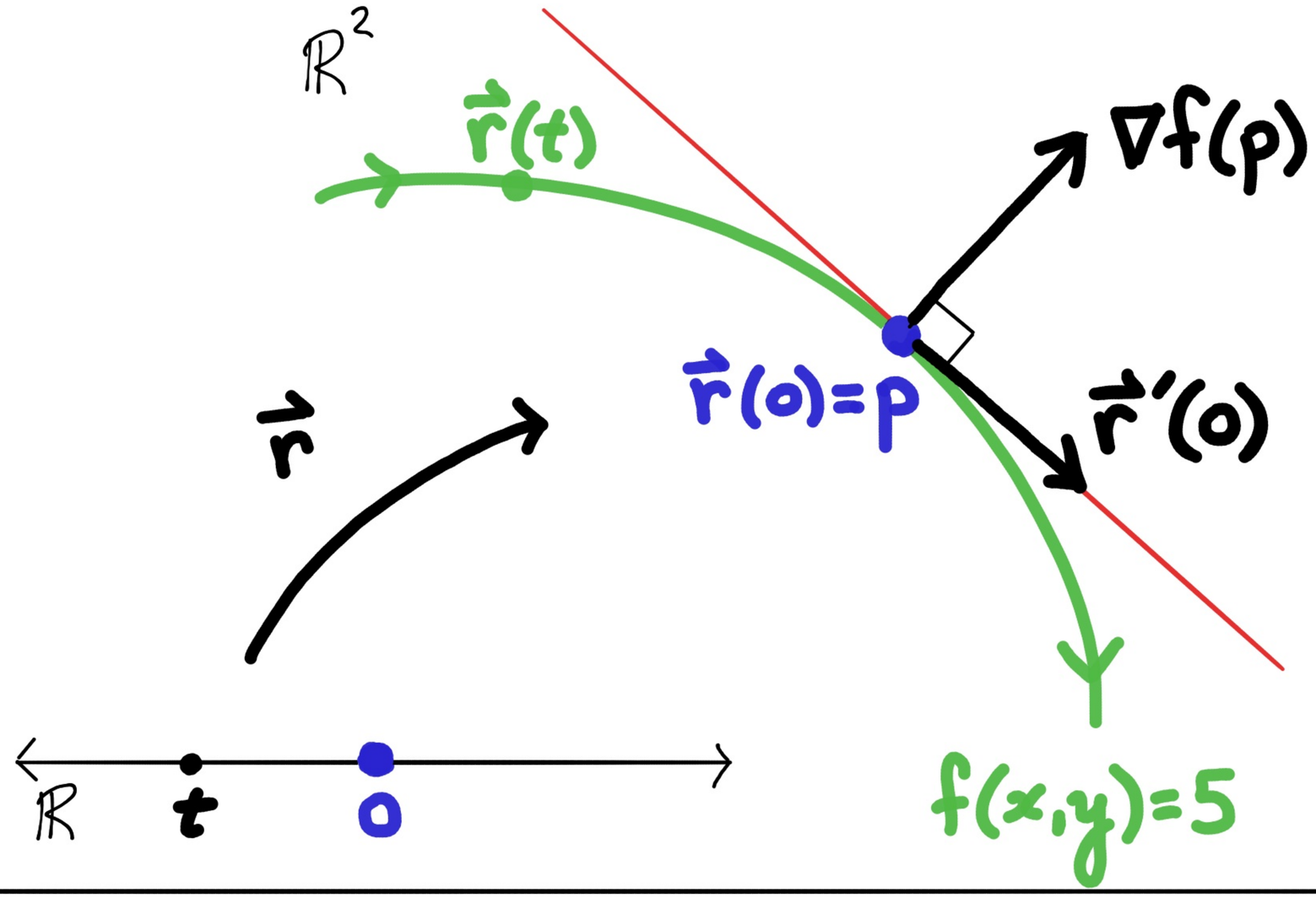

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conclusion:

- $f \circ \vec{r} : \mathbb{R} \to \mathbb{R}$, $f \circ \vec{r}(t) = f(\vec{r}(t)) = 5$

proof:

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- $f : \mathbb{R}^2 \to \mathbb{R}$
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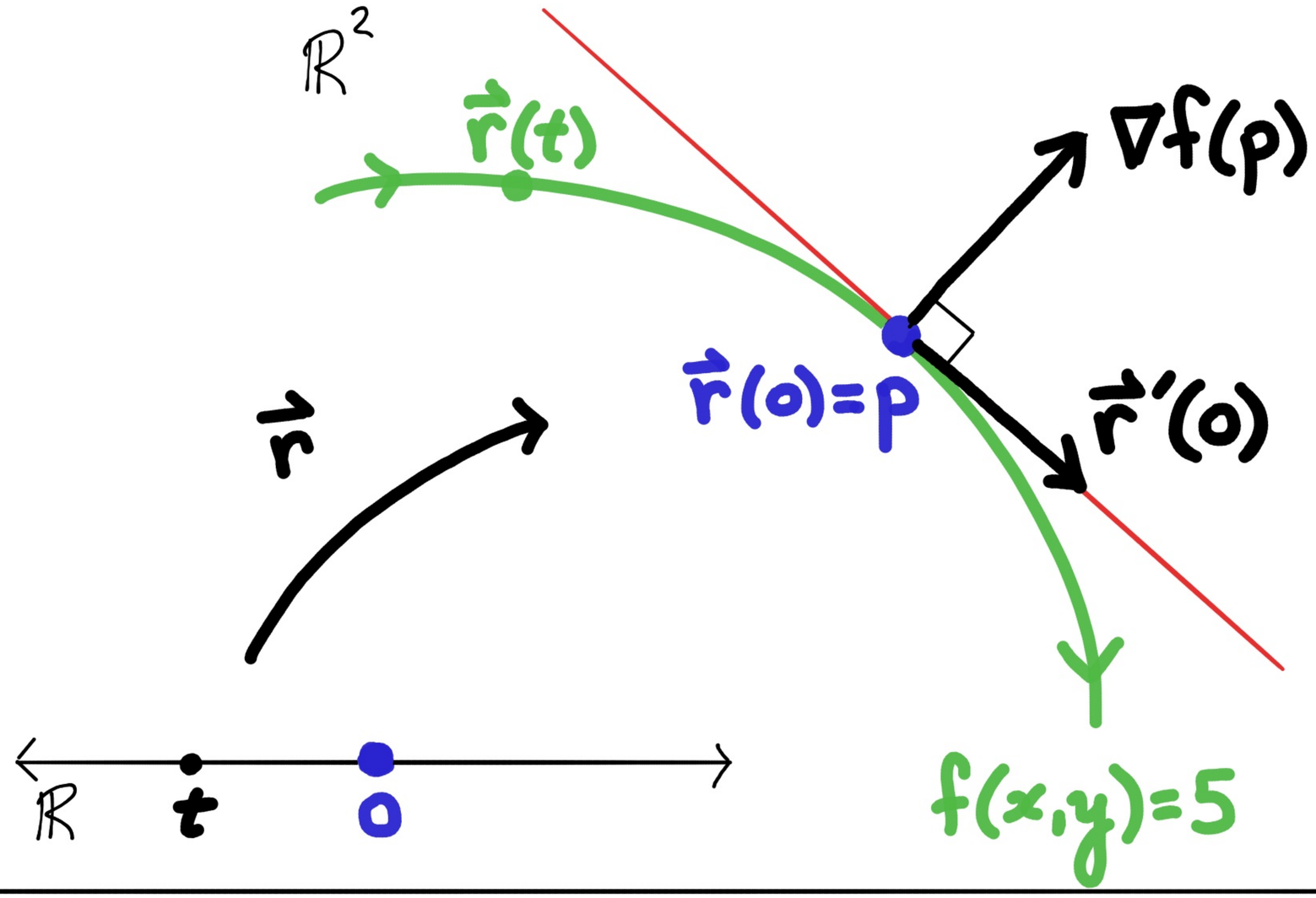

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conclusion:

- $f \circ \vec{r} : \mathbb{R} \to \mathbb{R}$, $f \circ \vec{r}(t) = f(\vec{r}(t)) = 5$
- $\left.\frac{d}{dt}(5)\right|_{t=0}$

proof:

setup:

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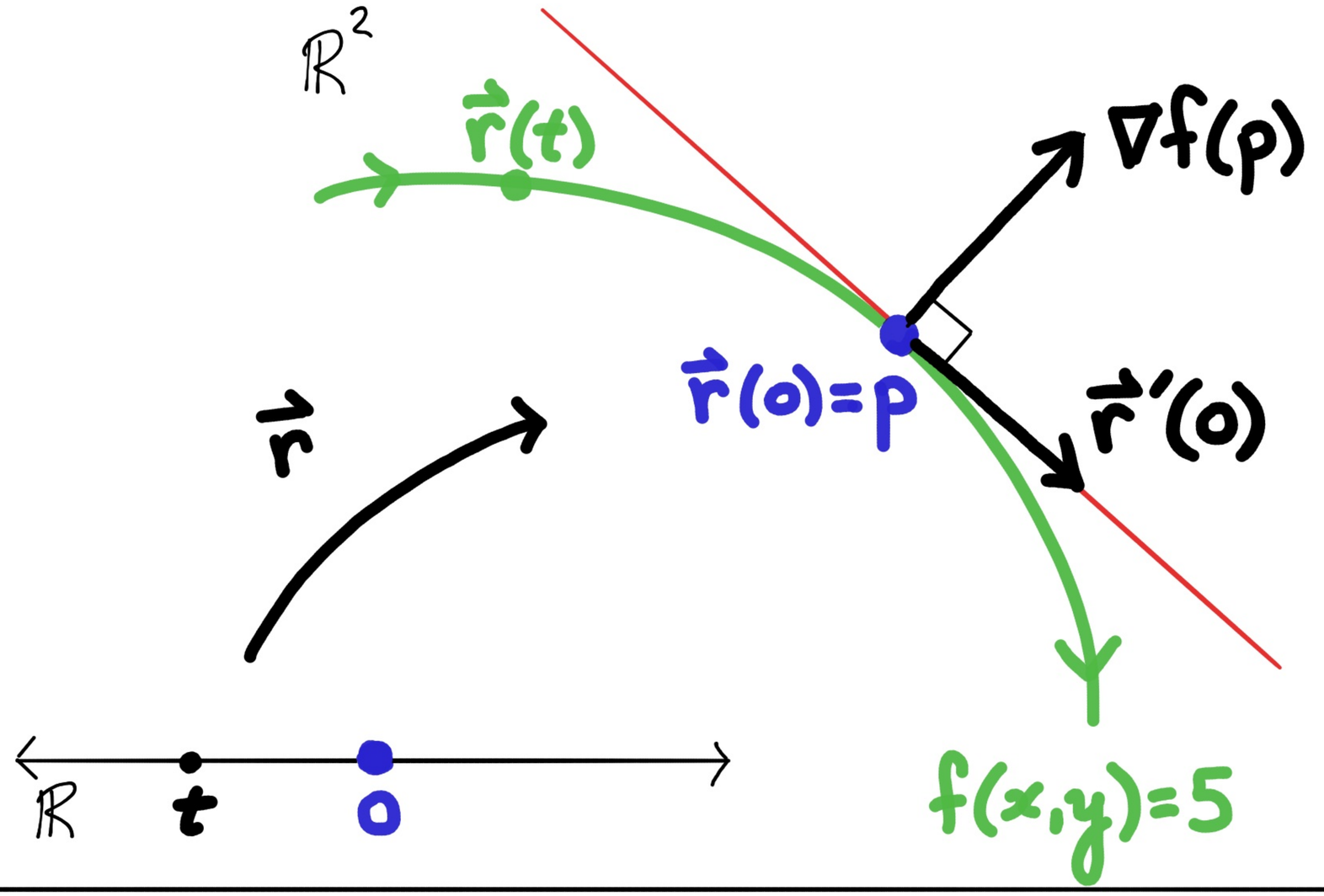

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conclusion:

- $f \circ \vec{r} : \mathbb{R} \to \mathbb{R}$, $f \circ \vec{r}(t) = f(\vec{r}(t)) = 5$
- $0 = \frac{d}{dt}(5)\Big|_{t=0}$

proof:

setup:

- $f : \mathbb{R}^2 \to \mathbb{R}$
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  param. $f(x,y)=5$
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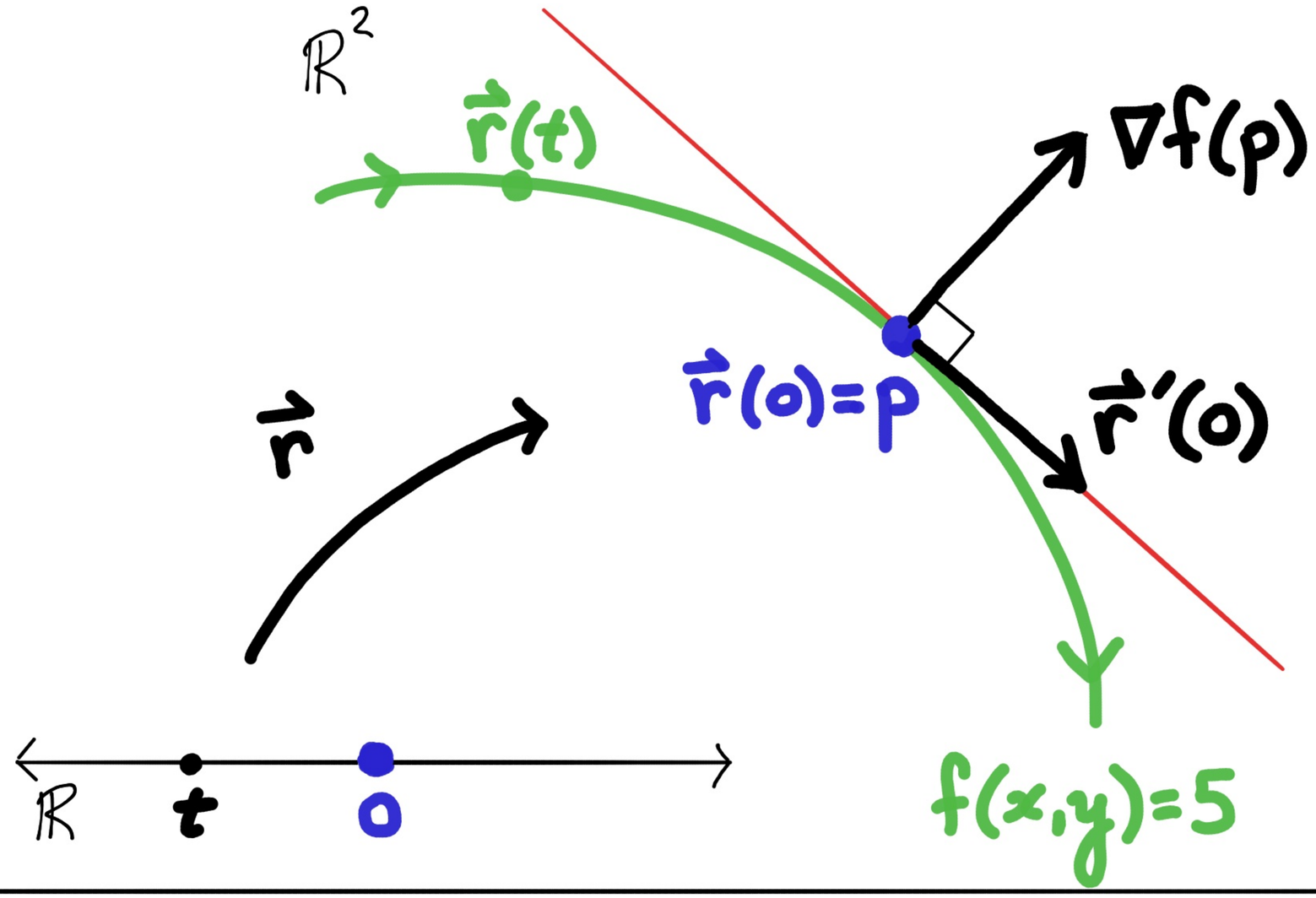

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conclusion:

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- $0 = \frac{d}{dt}(5)\Big|_{t=0} = \frac{d}{dt}(f \circ \vec{r})\Big|_{t=0}$

proof:

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- $f : \mathbb{R}^2 \to \mathbb{R}$
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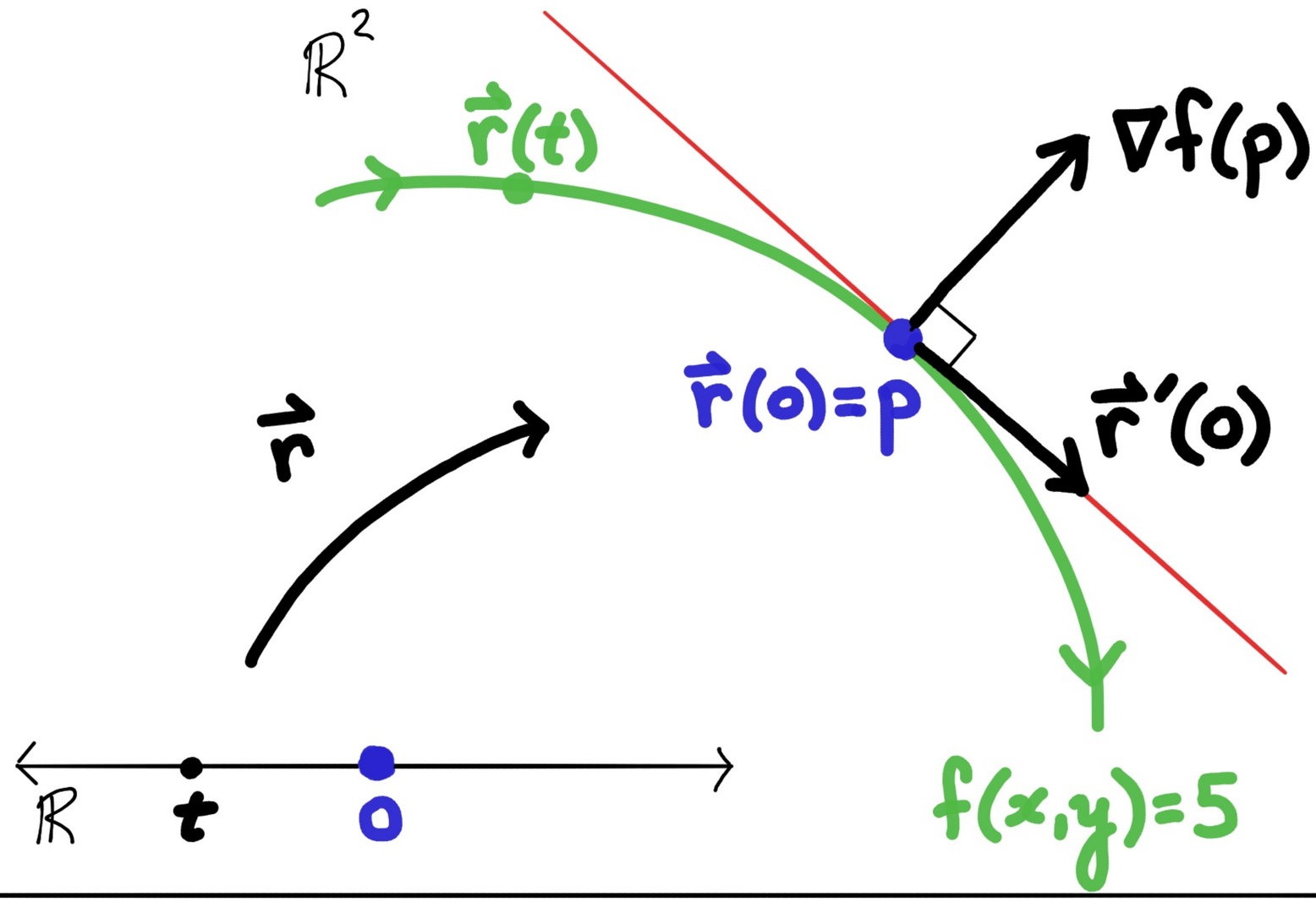

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conclusion:

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- $0 = \frac{d}{dt}(5)\Big|_{t=0} = \frac{d}{dt}(f \circ \vec{r})\Big|_{t=0} = D_0(f \circ \vec{r})$

proof:

setup:

- $f : \mathbb{R}^2 \to \mathbb{R}$
- $\vec{r} : \mathbb{R} \to \mathbb{R}^2$
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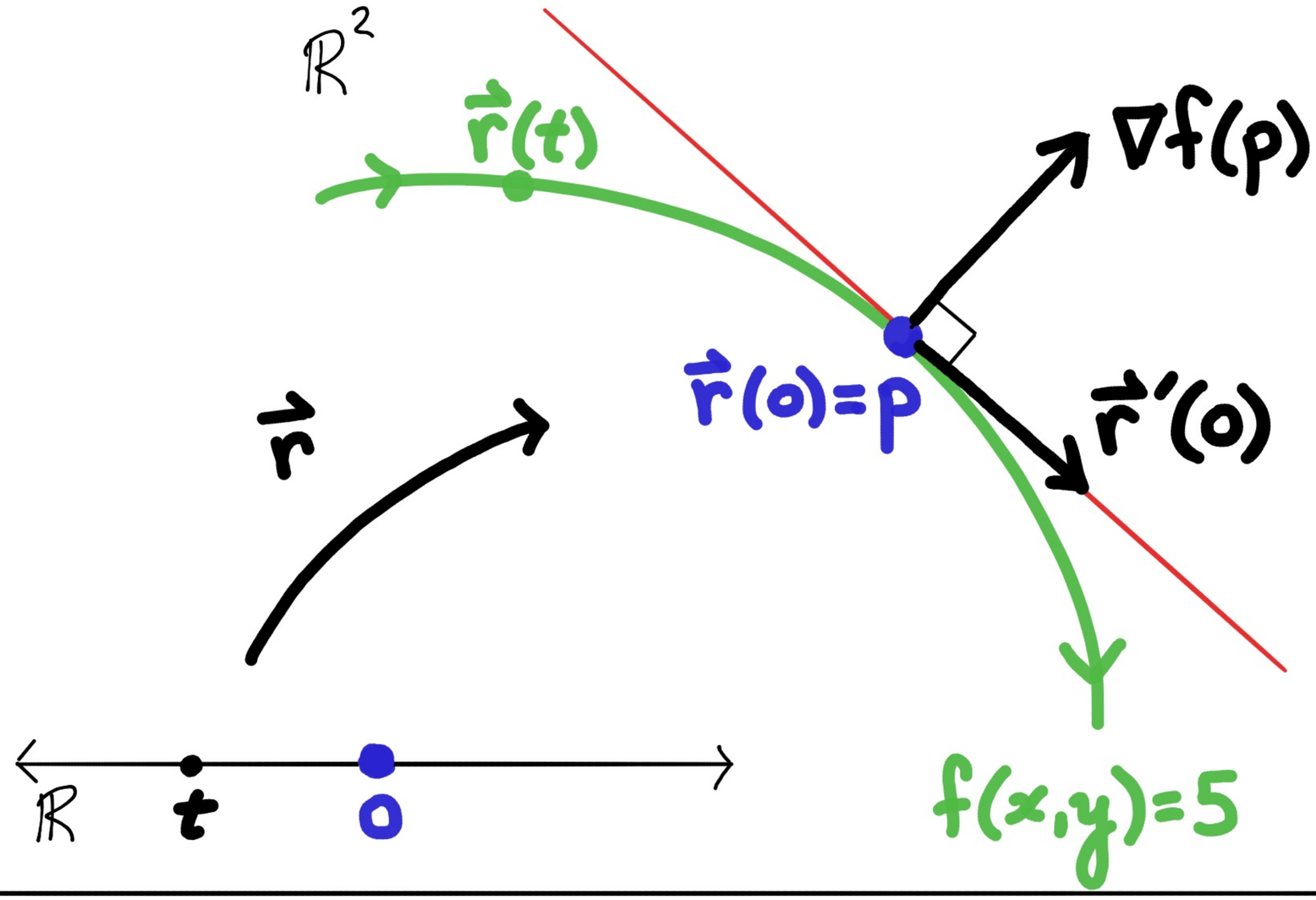

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- $0 = \frac{d}{dt}(5)\Big|_{t=0} = \frac{d}{dt}(f \circ \vec{r})\Big|_{t=0} = D_0(f \circ \vec{r}) = (D_{\vec{r}(0)} f)(D_0 \vec{r})$

proof:

setup:

- $f : \mathbb{R}^2 \to \mathbb{R}$
- $\vec{r} : \mathbb{R} \to \mathbb{R}^2$
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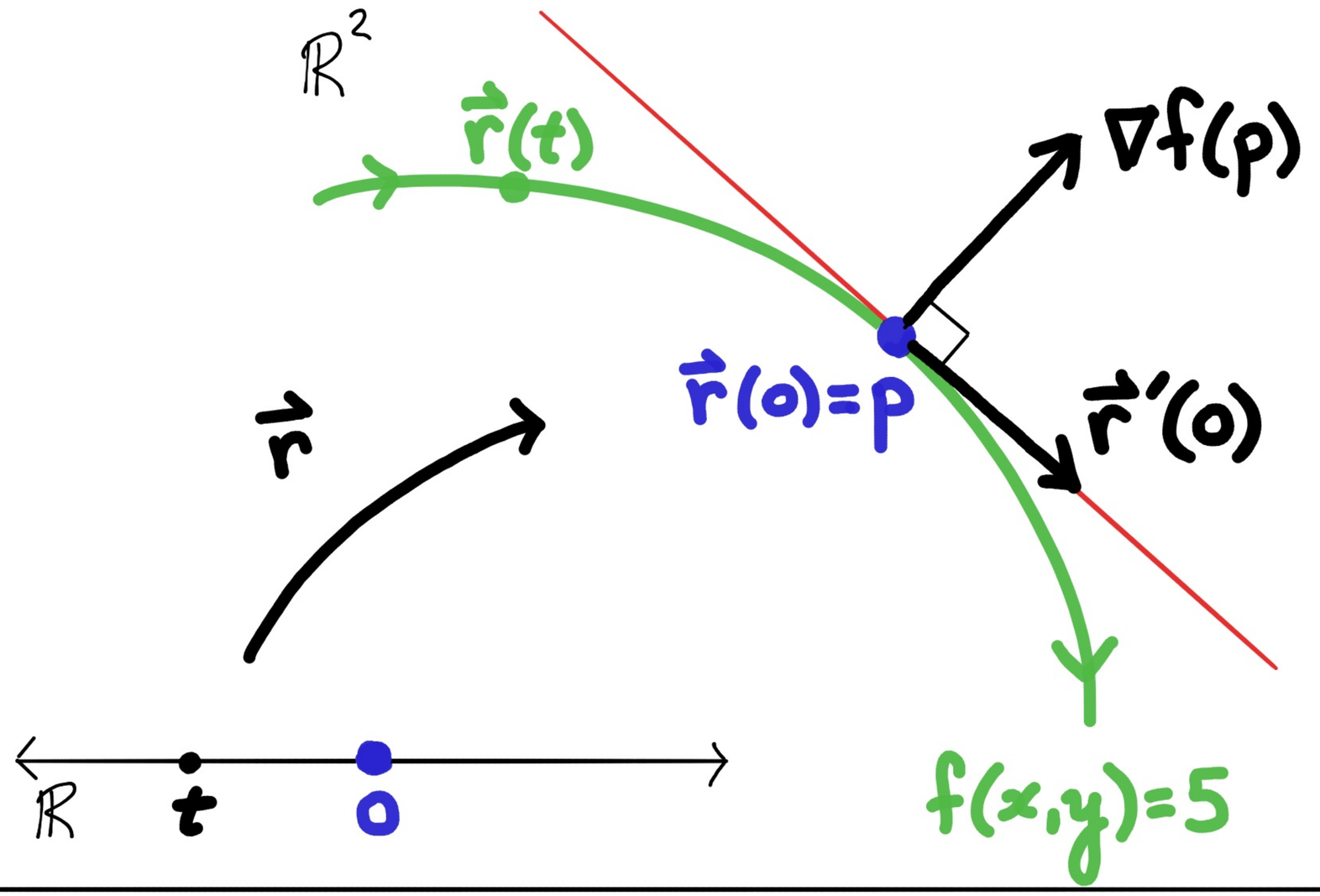

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$$= (D_p f)(D_0 \vec{r})$$

proof:

setup:

- $f : \mathbb{R}^2 \to \mathbb{R}$
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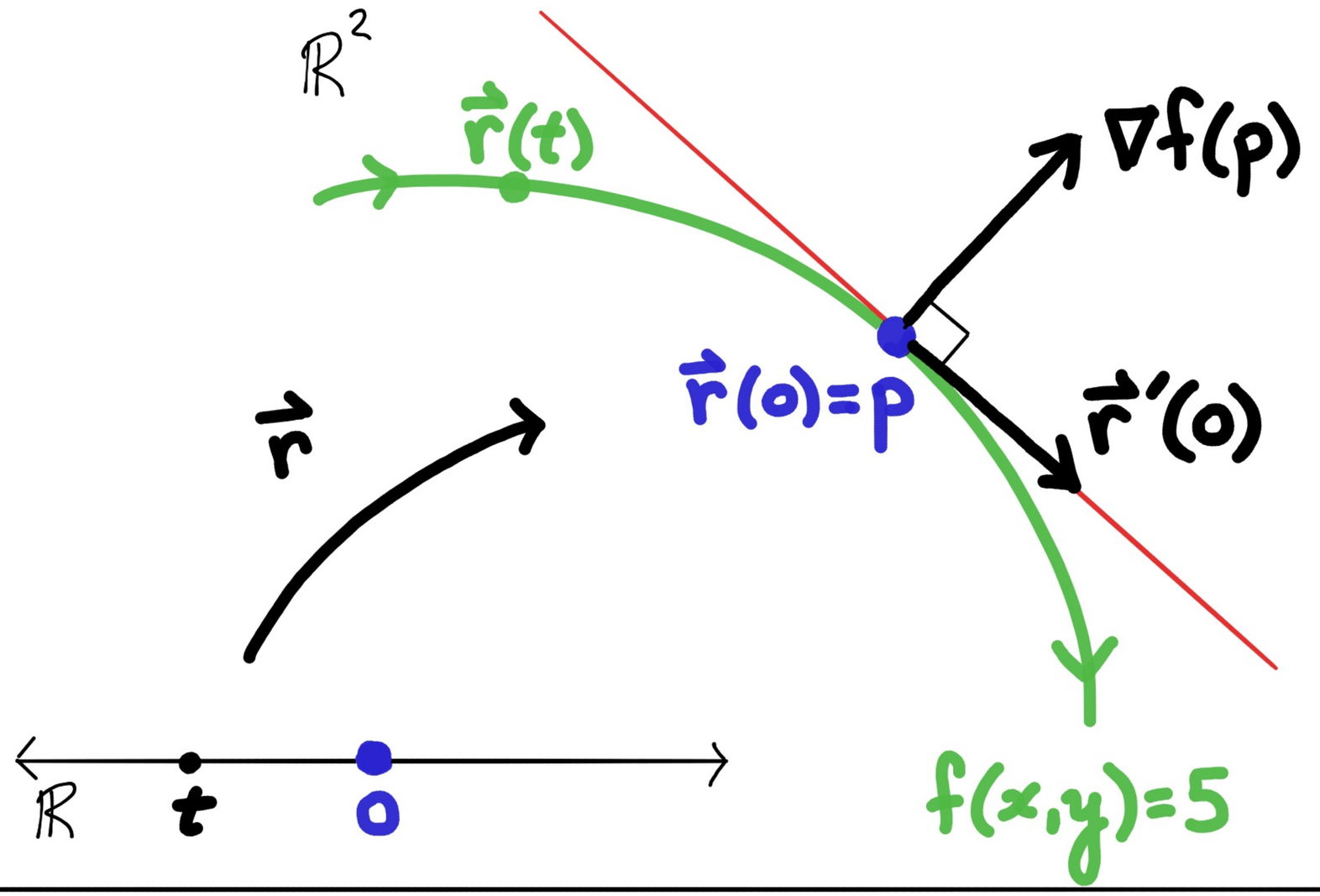

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$$= (D_p f)(D_0\vec{r}) = \left(\frac{\partial f}{\partial x}\Big|_p, \frac{\partial f}{\partial y}\Big|_p\right)\begin{pmatrix} \frac{d\vec{r}_1}{dt}\Big|_0 \\ \frac{d\vec{r}_2}{dt}\Big|_0 \end{pmatrix}$$

proof:

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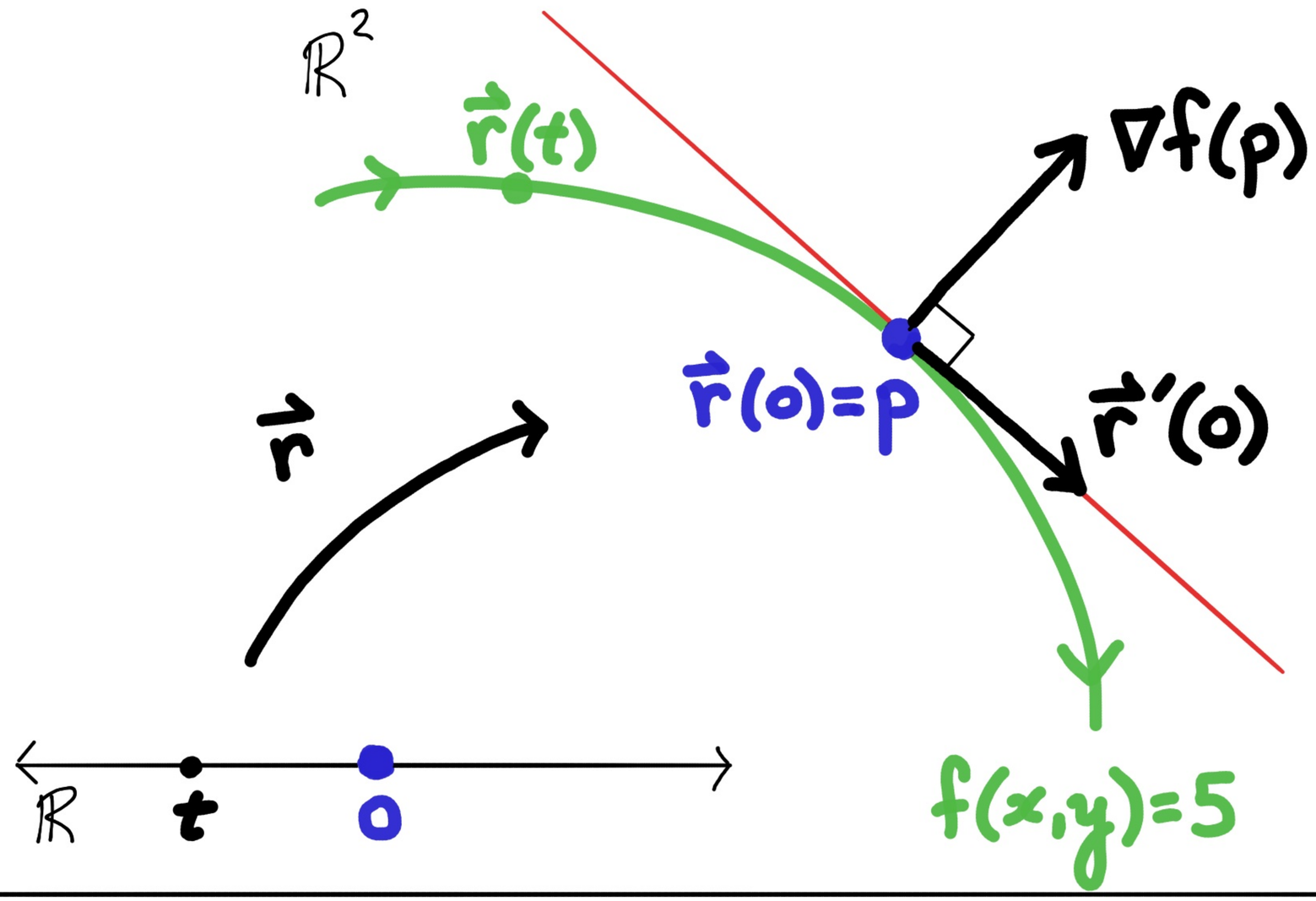

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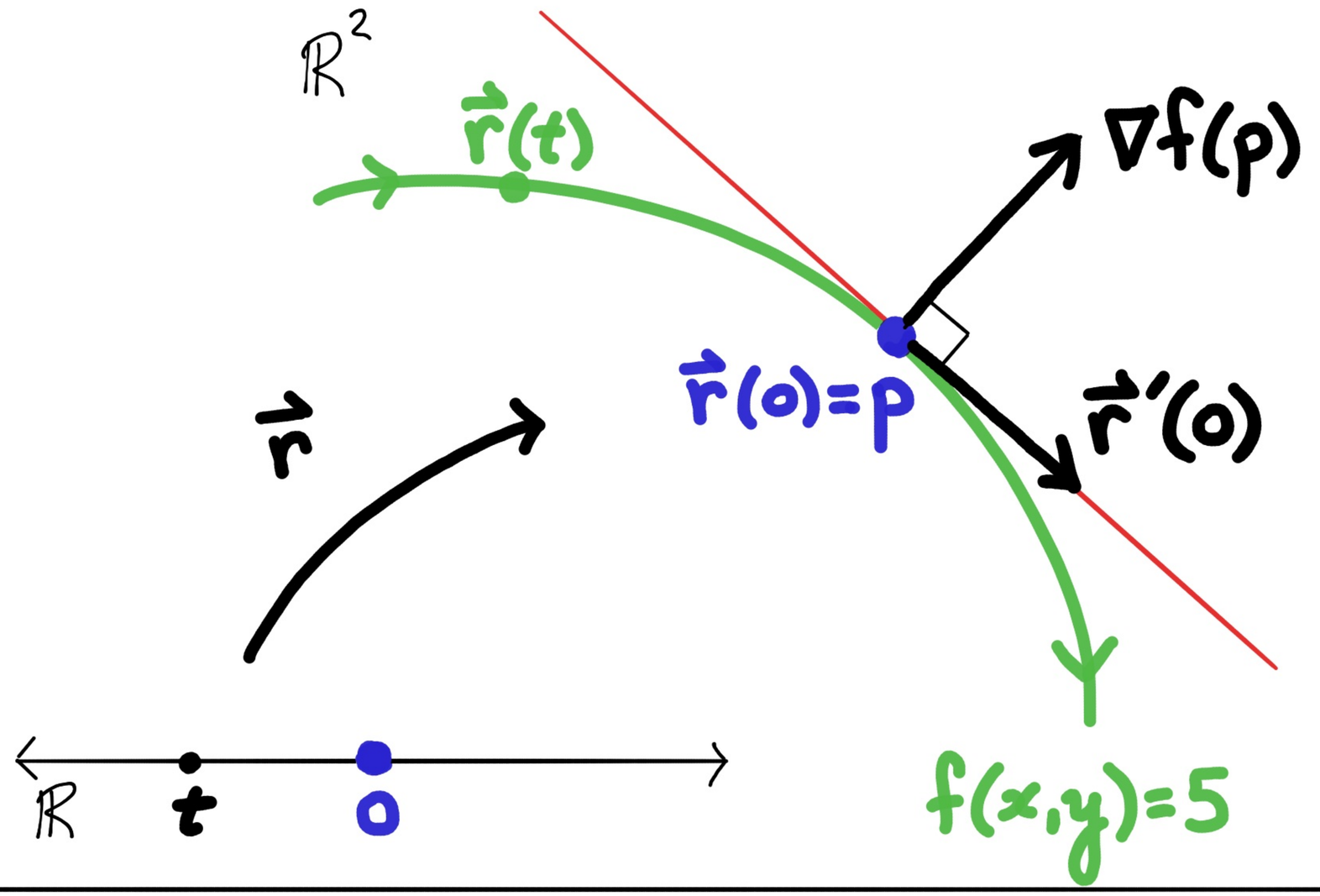

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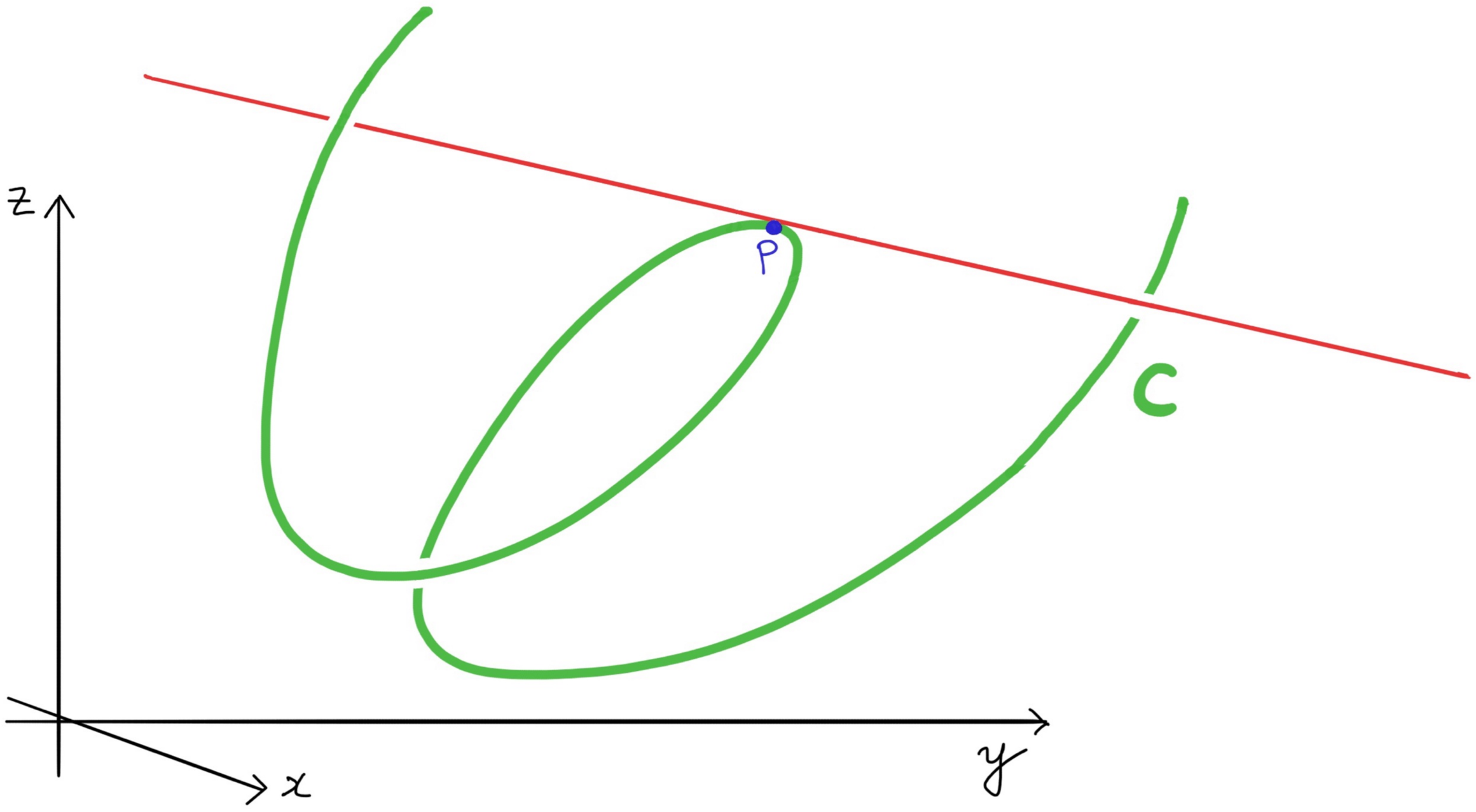
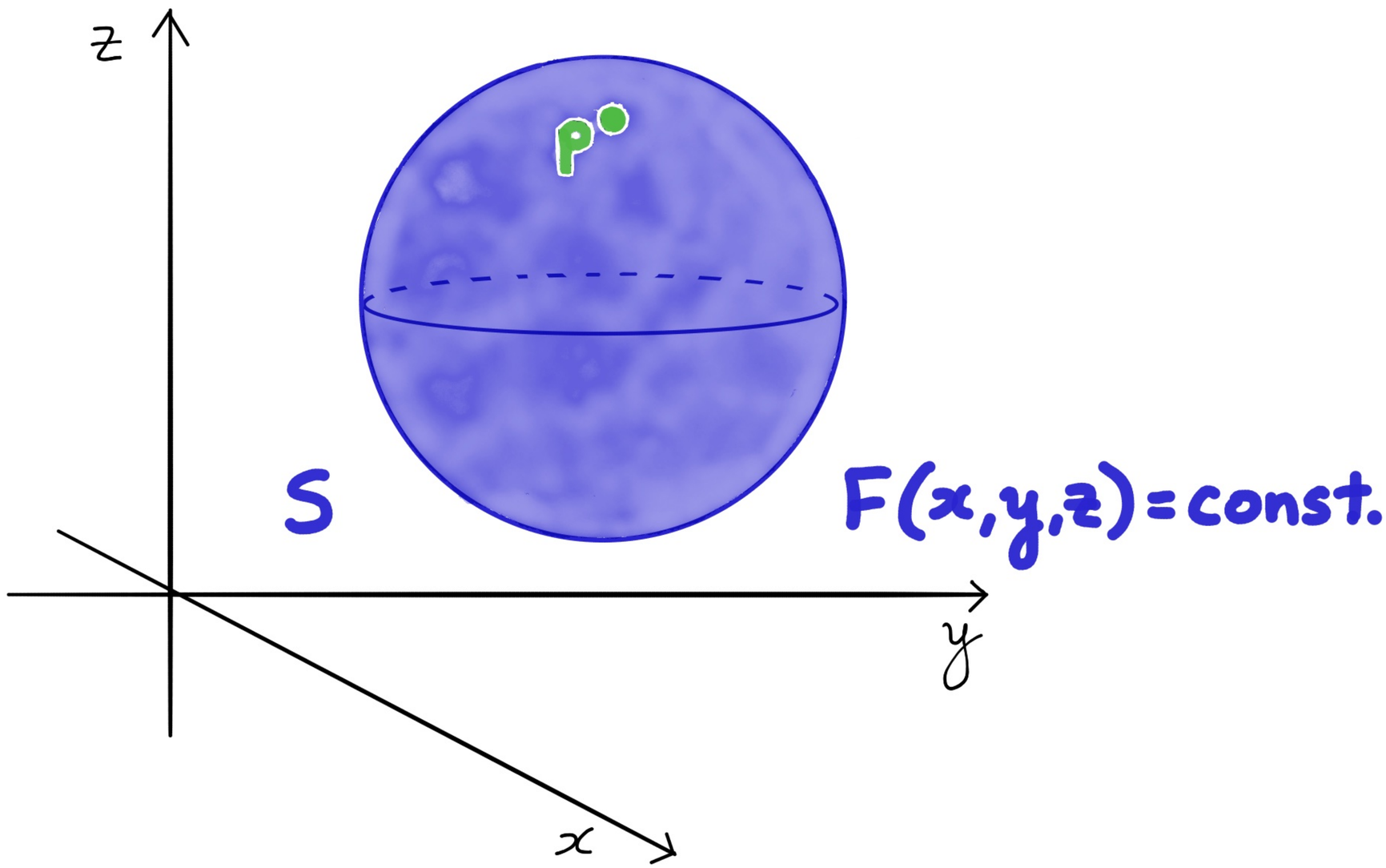
END

II Gradients and tangent planes of level surfaces

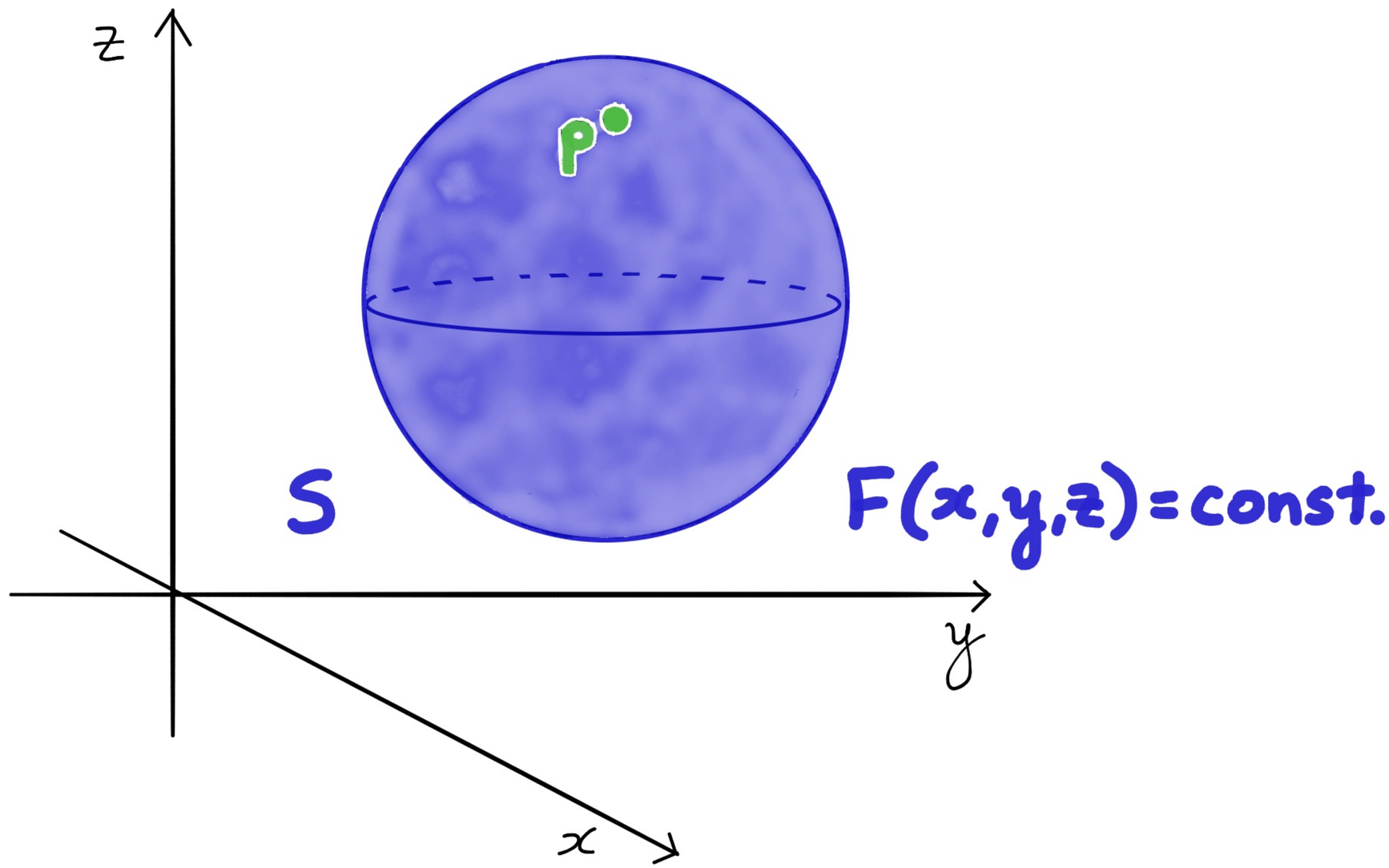
The tangent line of C at P is the best approximation possible of C near P by a line.



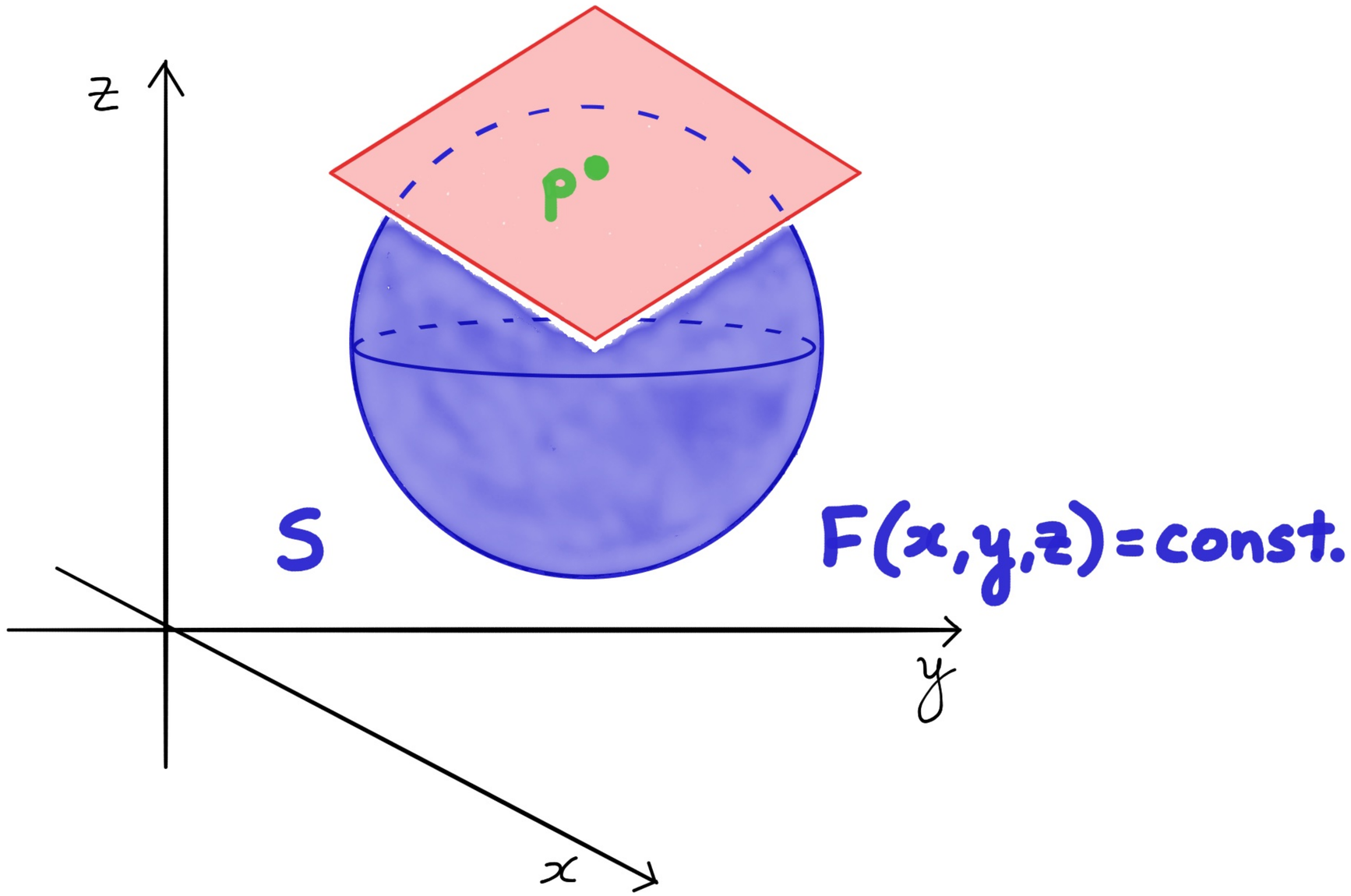
- $F: \mathbb{R}^3 \to \mathbb{R}$
- S a level surface
- $p \in S$



The tangent plane of S at $p \in S$ is the best approximation possible of S near p by a plane.



The tangent plane of S at p ∈ S is the best approximation possible of S near p by a plane.



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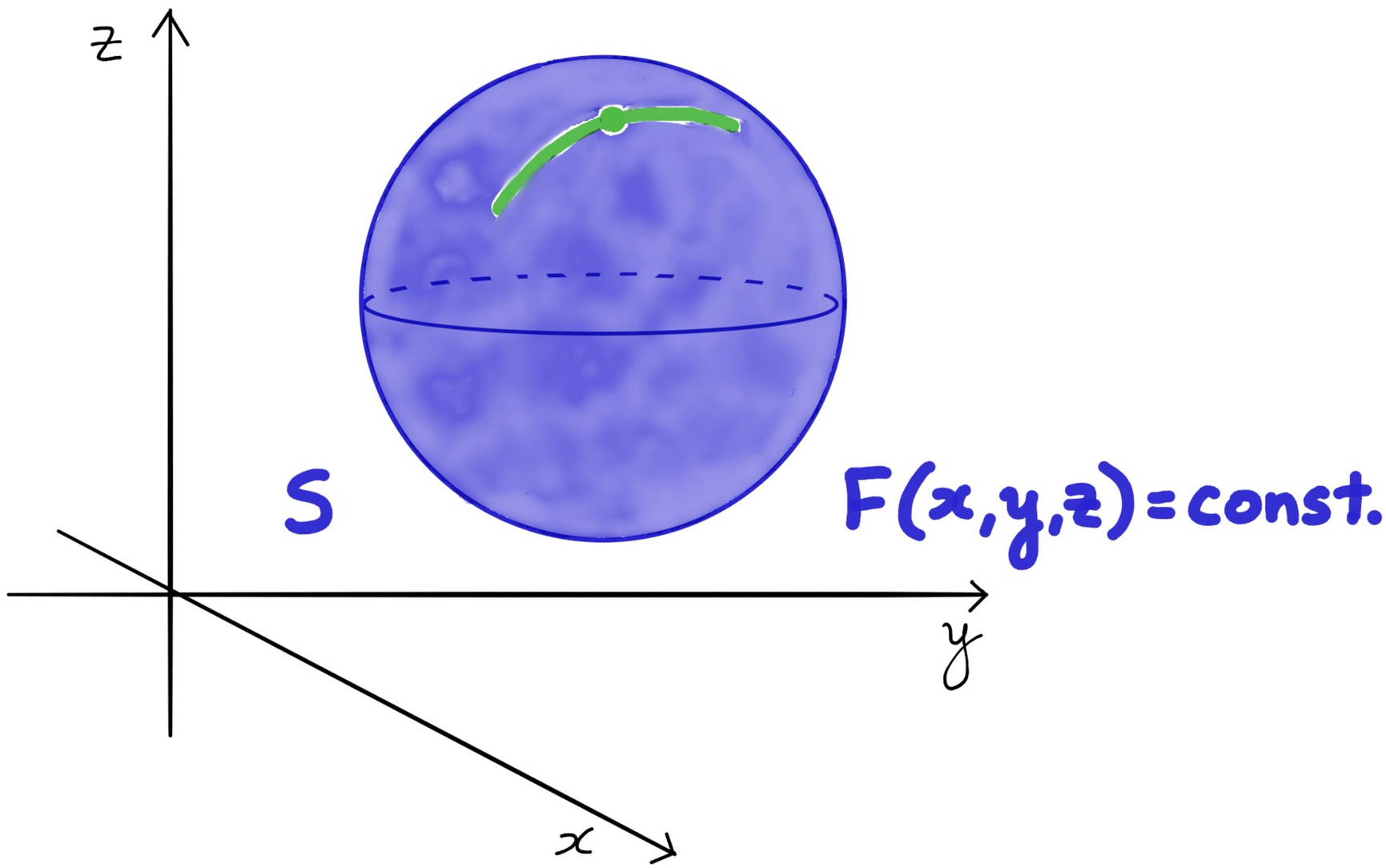

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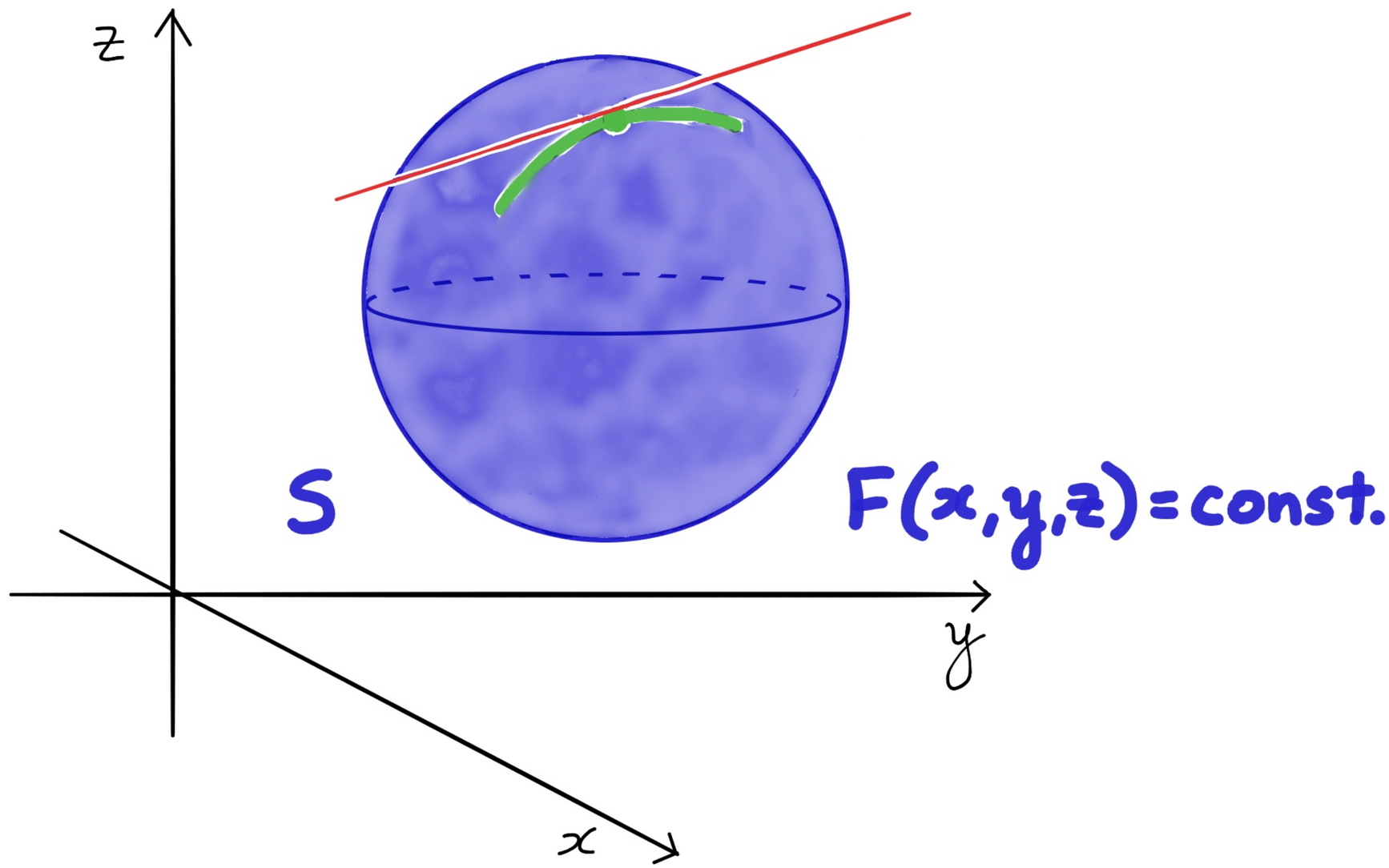

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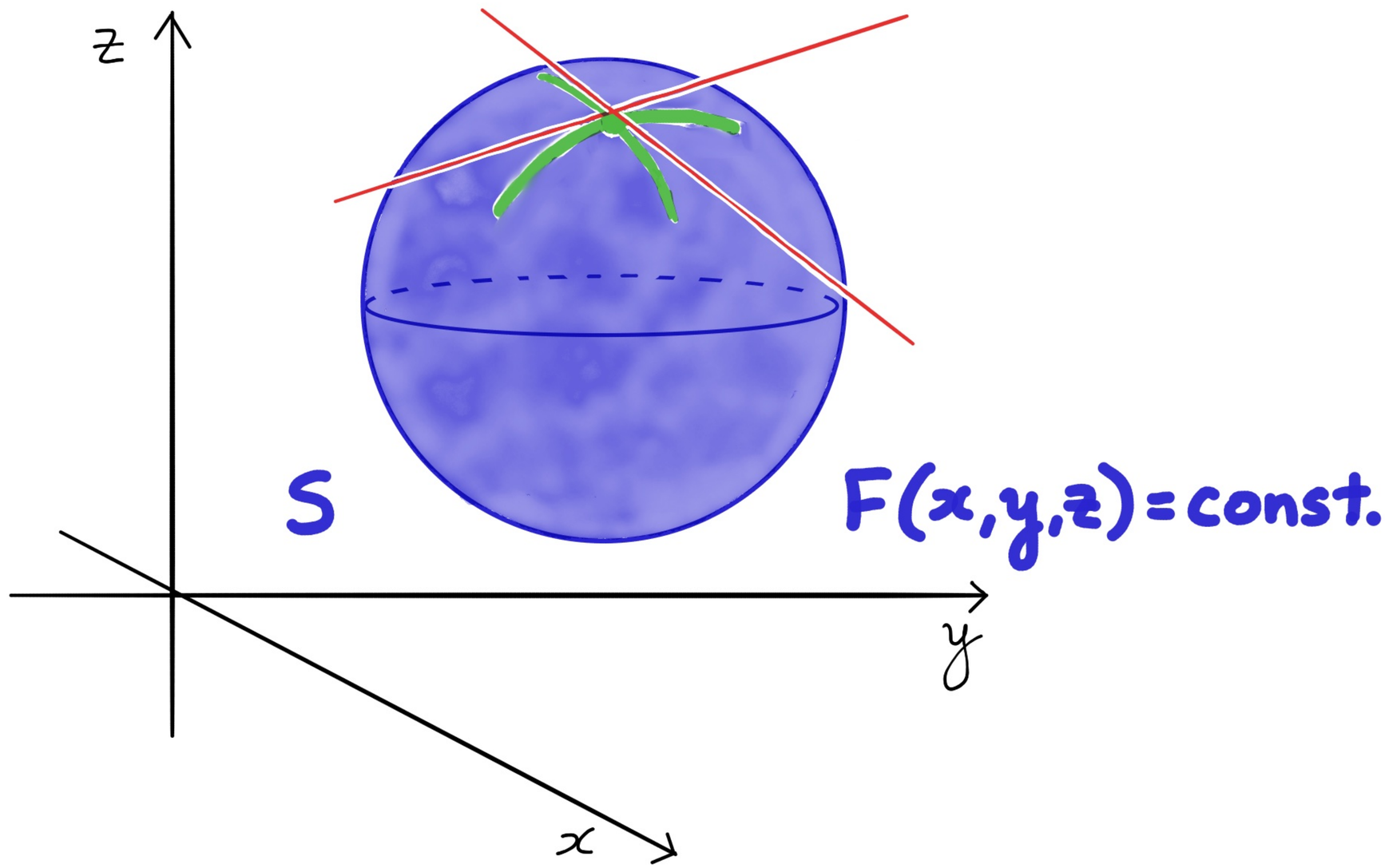

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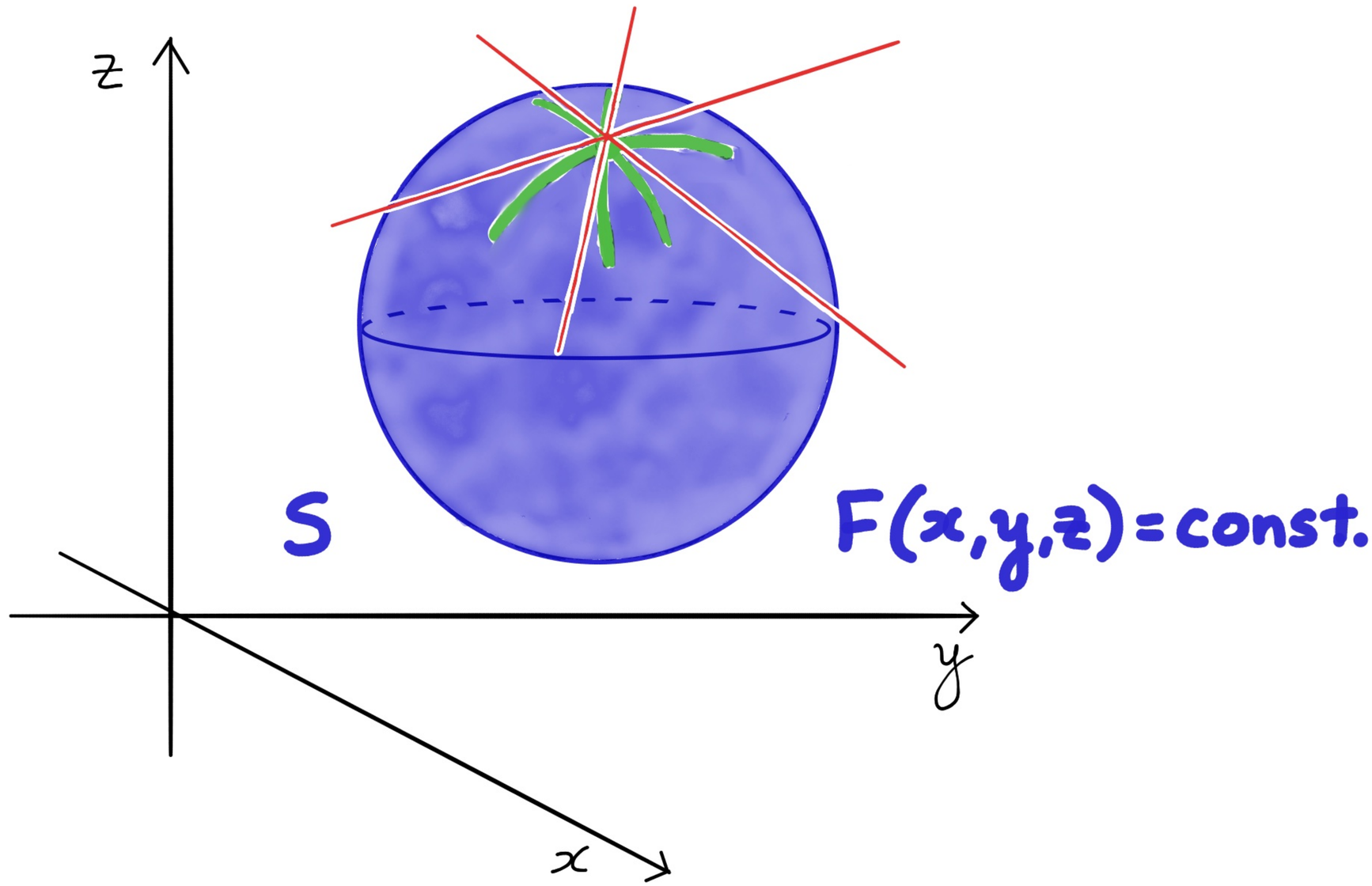

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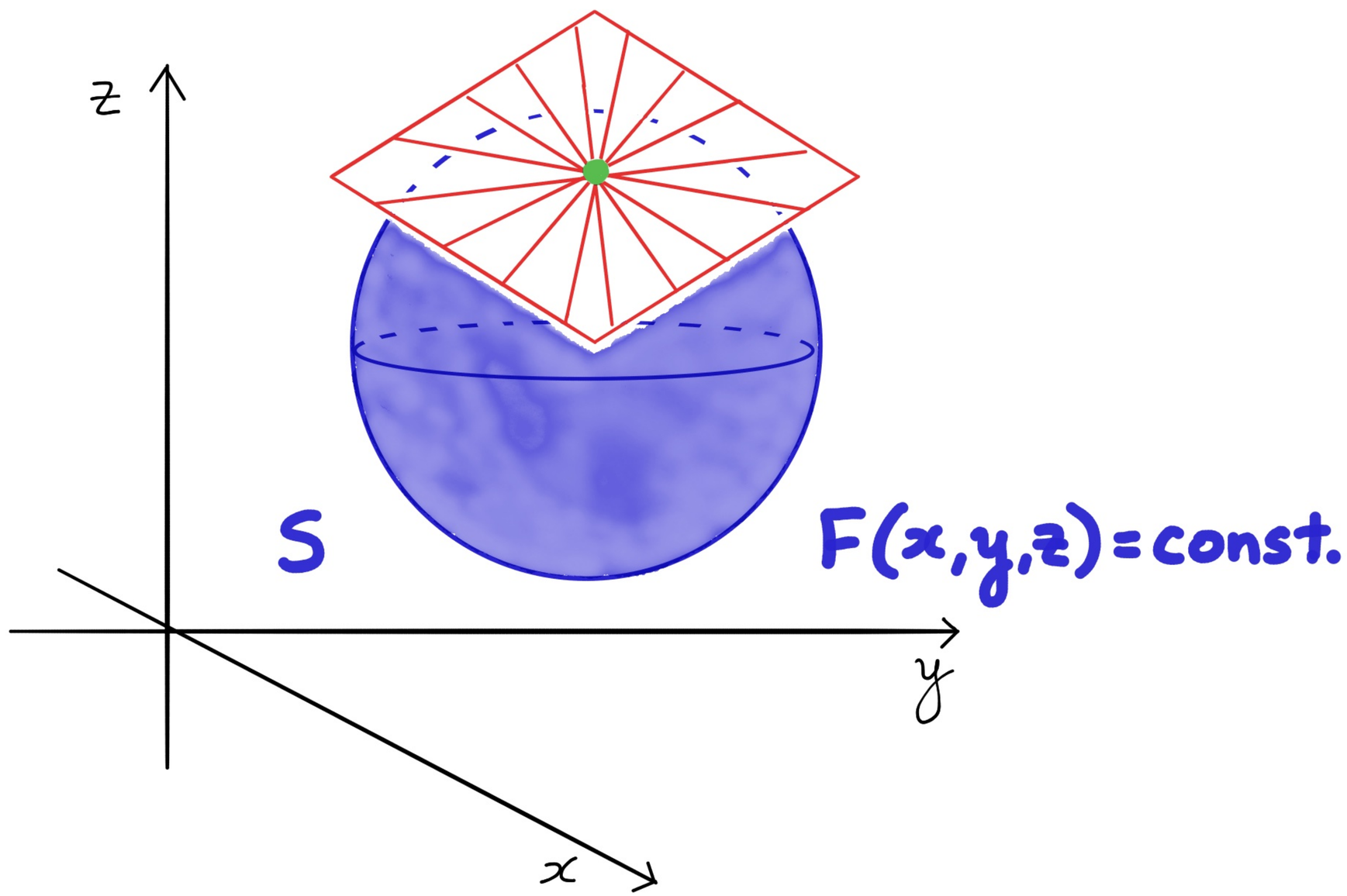

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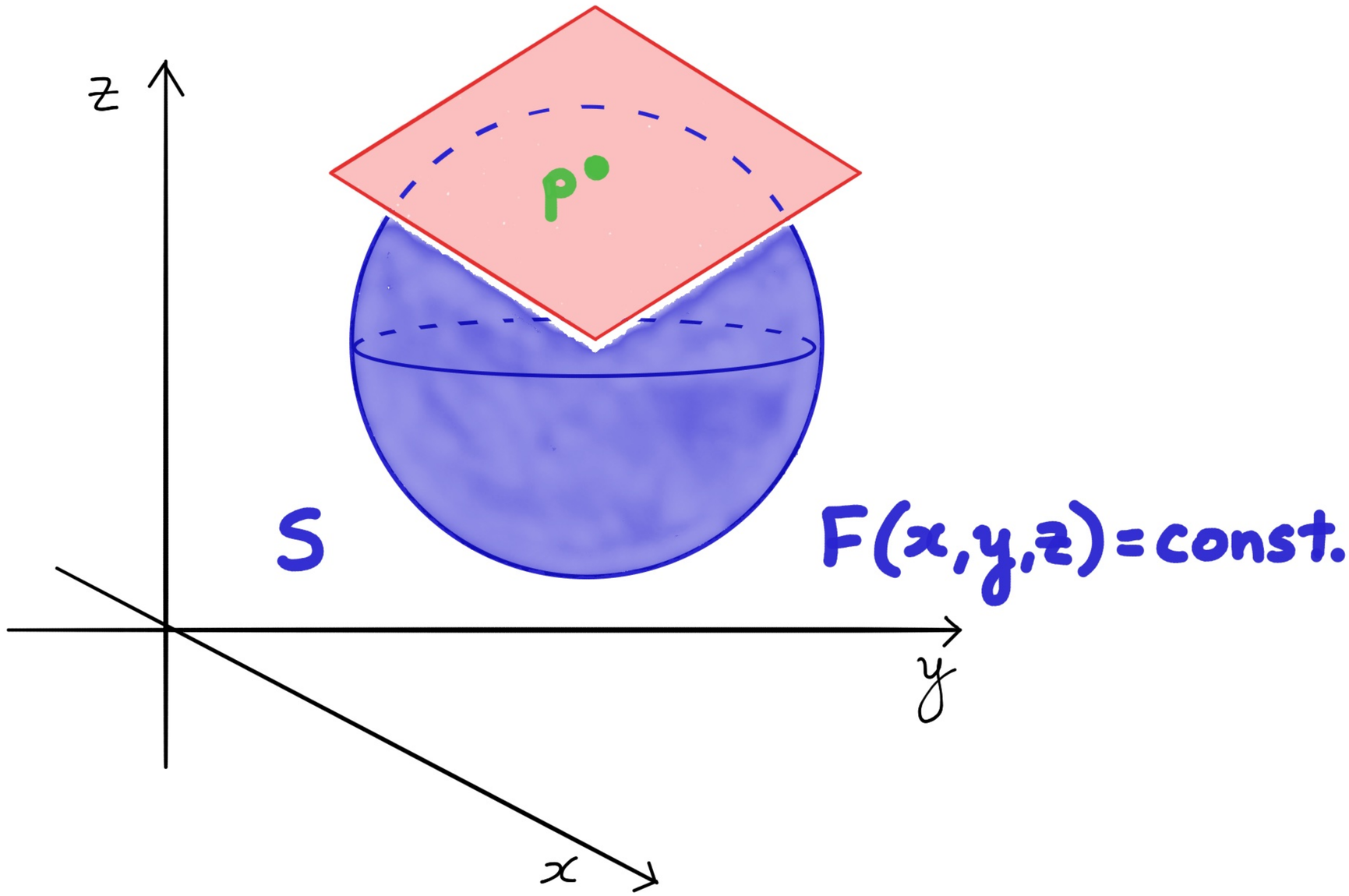
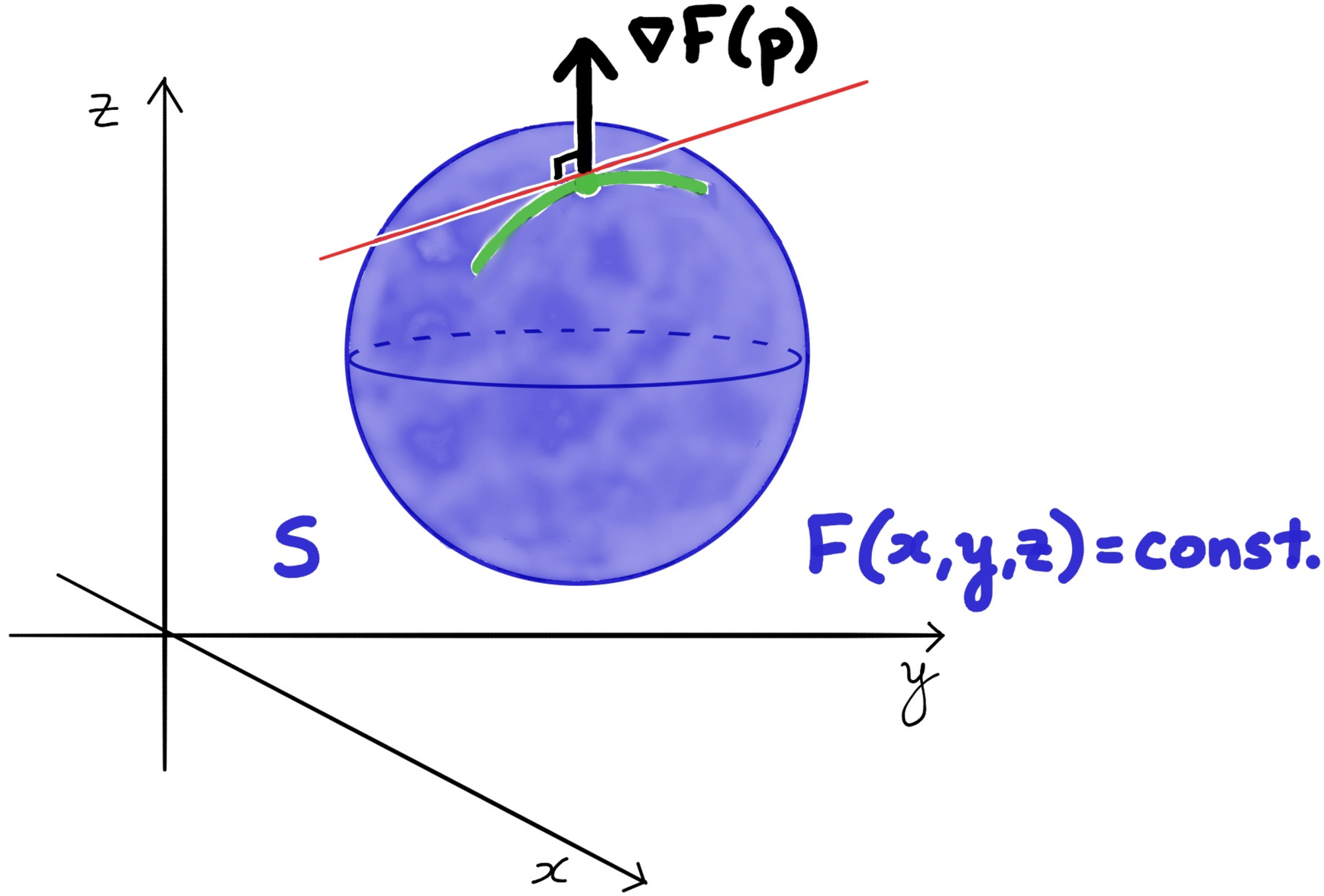


The tangent vectors of S at $p \in S$ are orthogonal to $\nabla F(p)$.



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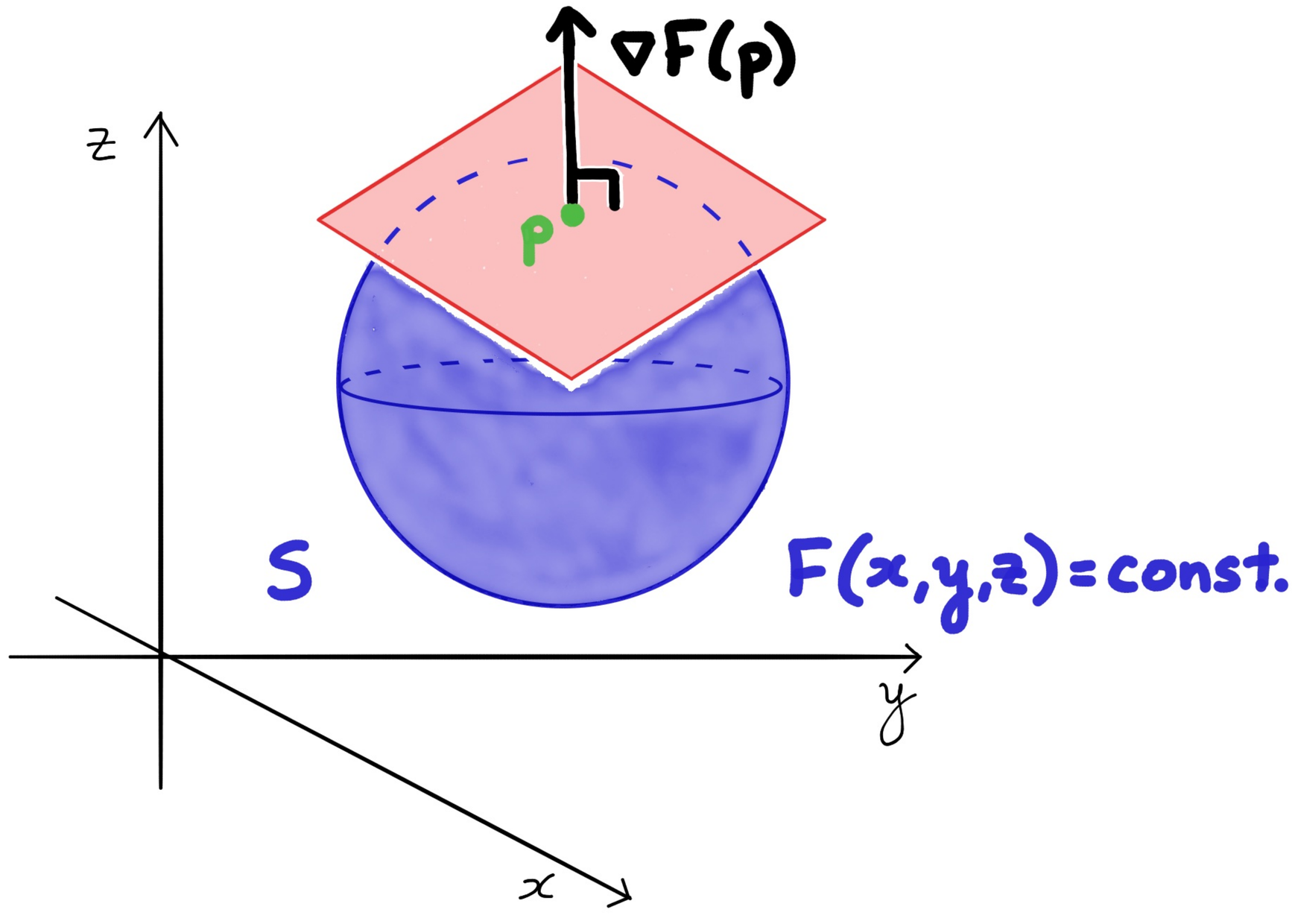
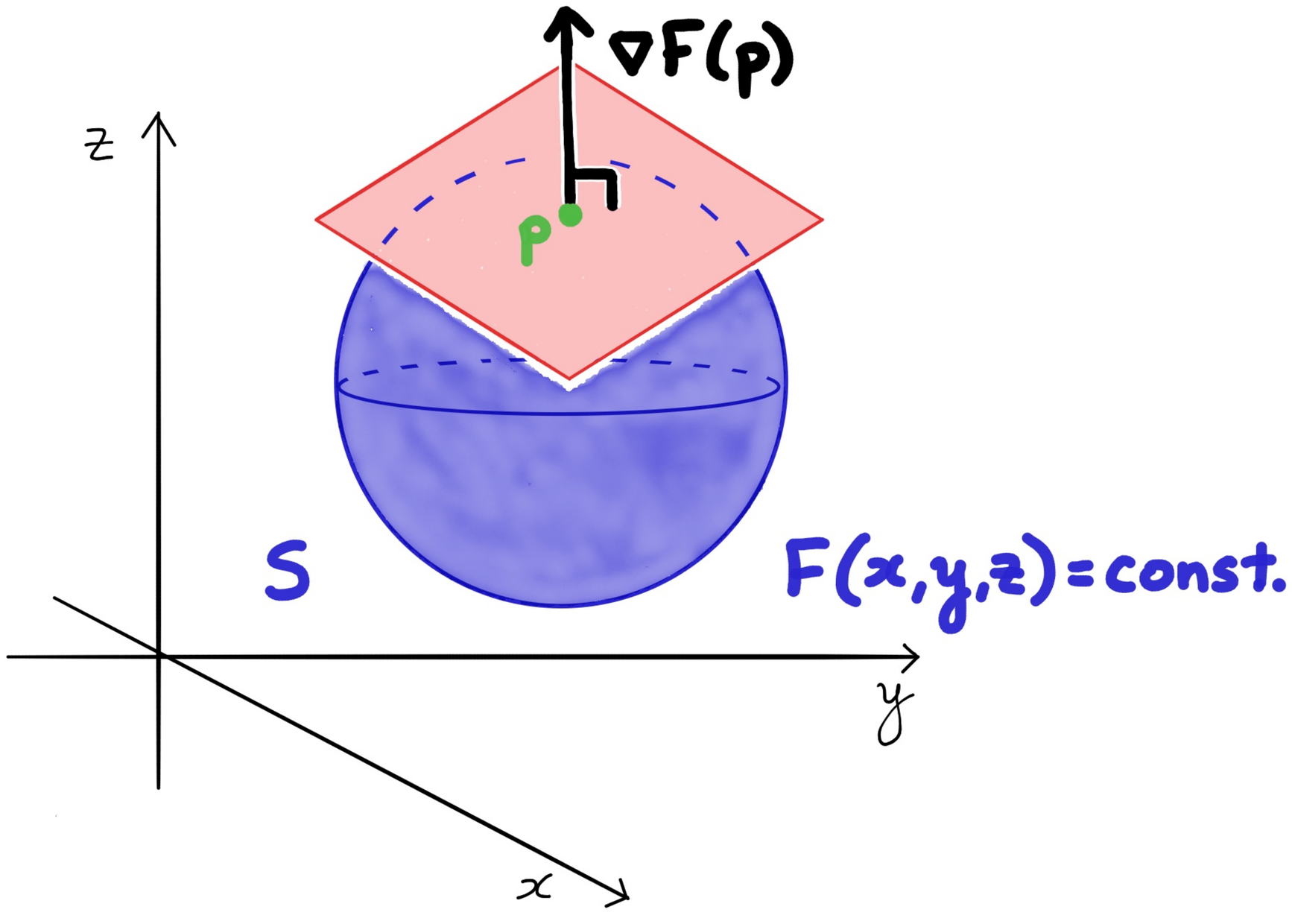


III Equations for tangent planes of level surfaces

The tangent plane of S at $p \in S$ contains p and has normal vector

$$\nabla F(p) = \left(\frac{\partial F}{\partial x}\Big|_p, \frac{\partial F}{\partial y}\Big|_p, \frac{\partial F}{\partial z}\Big|_p \right).$$



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$$\nabla F(p) = \left(\frac{\partial F}{\partial x}\Big|_p, \frac{\partial F}{\partial y}\Big|_p, \frac{\partial F}{\partial z}\Big|_p \right).$$

The tangent plane of S at $p \in S$ contains p and has normal vector

$$\nabla F(p) = \left(\frac{\partial F}{\partial x}\Big|_p, \frac{\partial F}{\partial y}\Big|_p, \frac{\partial F}{\partial z}\Big|_p \right).$$

The tangent plane of S at $p \in S$ for $p=(x_0,y_0,z_0)$ has equation

$$\frac{\partial F}{\partial x}\Big|_p (x-x_0) + \frac{\partial F}{\partial y}\Big|_p (y-y_0) + \frac{\partial F}{\partial z}\Big|_p (z-z_0) = 0$$

Tangent plane of $F(x,y,z)=\text{const.}$ at $p=(x_0,y_0,z_0)$:

$$\left.\frac{\partial F}{\partial x}\right|_p (x-x_0) + \left.\frac{\partial F}{\partial y}\right|_p (y-y_0) + \left.\frac{\partial F}{\partial z}\right|_p (z-z_0) = 0$$

Example:

Find the equation for the tangent plane of $(x-2)^2+(y-2)^2 = 1-(z-2)^2$ at $p=\left(\frac{5}{2}, 2, \frac{4+\sqrt{3}}{2}\right)$.

Tangent plane of $F(x,y,z)=\text{const.}$ at $p=(x_0,y_0,z_0)$:

$$\left.\frac{\partial F}{\partial x}\right|_p (x-x_0)+\left.\frac{\partial F}{\partial y}\right|_p (y-y_0)+\left.\frac{\partial F}{\partial z}\right|_p (z-z_0)=0$$

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Solution:

① $(x-2)^2+(y-2)^2+(z-2)^2=1$, $F(x,y,z)=(x-2)^2+(y-2)^2+(z-2)^2$

Tangent plane of $F(x,y,z)=\text{const.}$ at $p=(x_0,y_0,z_0)$:

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Find the equation for the tangent plane of $(x-2)^2+(y-2)^2=1-(z-2)^2$ at $p=\left(\frac{5}{2},2,\frac{4+\sqrt{3}}{2}\right)$.

Solution:

① $(x-2)^2+(y-2)^2+(z-2)^2=1$, $F(x,y,z)=(x-2)^2+(y-2)^2+(z-2)^2$

② $\left.\frac{\partial F}{\partial x}\right|_p=2\left(\frac{5}{2}-2\right)=1$, $\left.\frac{\partial F}{\partial y}\right|_p=2(2-2)=0$, $\left.\frac{\partial F}{\partial z}\right|_p=2\left(\frac{4+\sqrt{3}}{2}-2\right)=\sqrt{3}$

Tangent plane of $F(x,y,z)=\text{const.}$ at $p=(x_0,y_0,z_0)$:

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Example:

Find the equation for the tangent plane of $(x-2)^2+(y-2)^2=1-(z-2)^2$ at $p=\left(\frac{5}{2},2,\frac{4+\sqrt{3}}{2}\right)$.

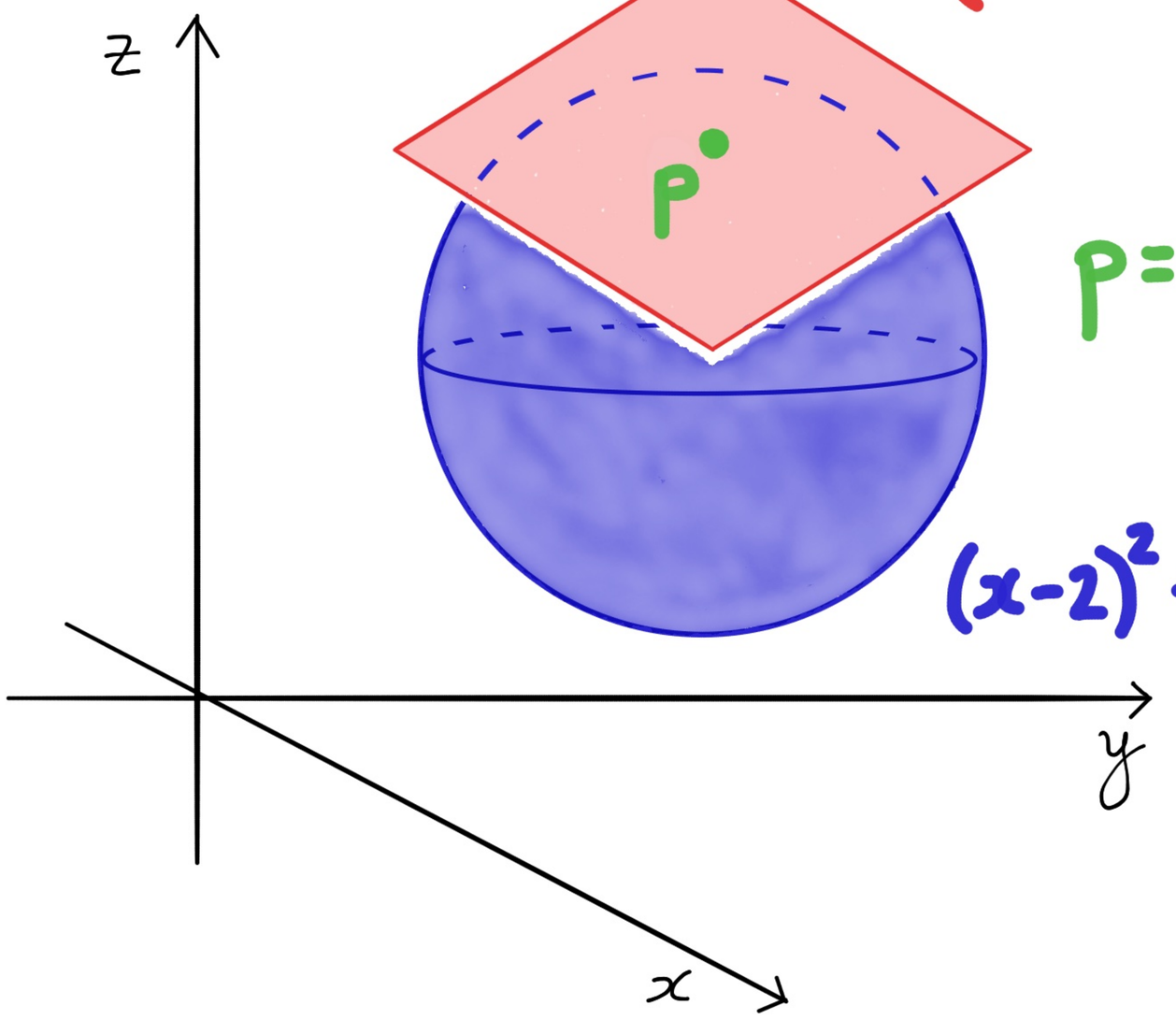
Solution:

① $(x-2)^2+(y-2)^2+(z-2)^2=1$, $F(x,y,z)=(x-2)^2+(y-2)^2+(z-2)^2$

② $\left.\frac{\partial F}{\partial x}\right|_p=2\left(\frac{5}{2}-2\right)=1$, $\left.\frac{\partial F}{\partial y}\right|_p=2(2-2)=0$, $\left.\frac{\partial F}{\partial z}\right|_p=2\left(\frac{4+\sqrt{3}}{2}-2\right)=\sqrt{3}$

③ $(x-\frac{5}{2})+\sqrt{3}\left(z-\frac{4+\sqrt{3}}{2}\right)=0$

$$\left(x-\tfrac{5}{2}\right)+\sqrt{3}\left(z-\frac{4+\sqrt{3}}{2}\right)=0$$

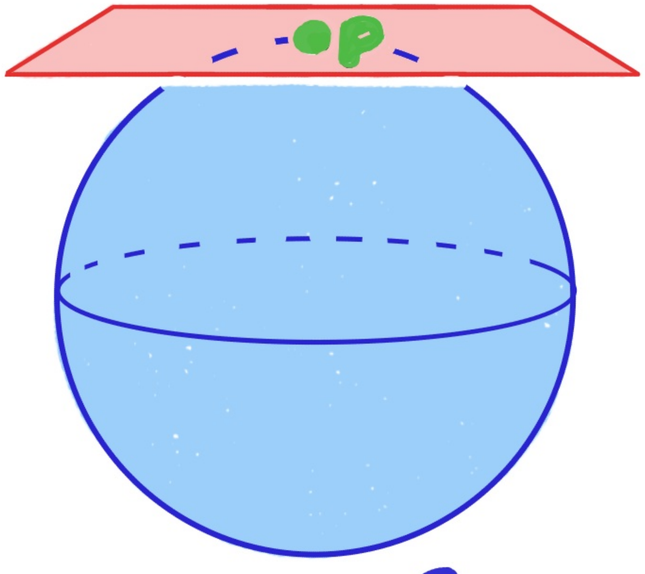


$$P=\left(\tfrac{5}{2},2,\frac{4+\sqrt{3}}{2}\right)$$

$$(x-2)^2+(y-2)^2+(z-2)^2=1$$

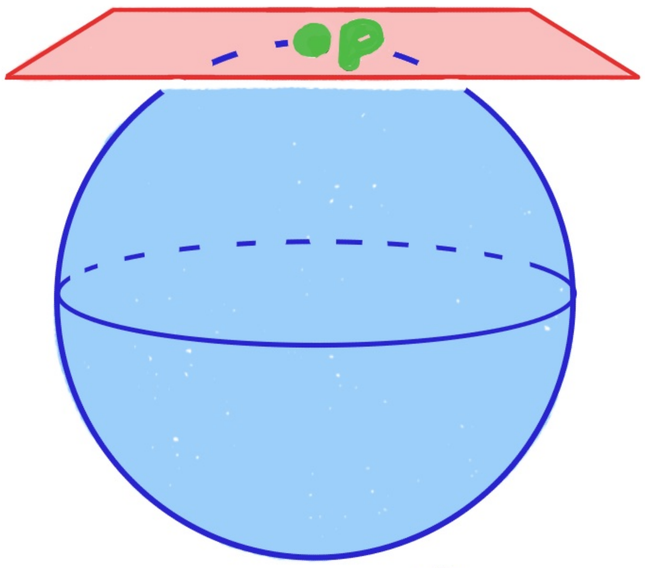
Example:

Find the equation for the tangent plane of $x^2+y^2+z^2=1$ at the point $p=(0,0,1)$.



Example:

Find the equation for the tangent plane of $x^2+y^2+z^2=1$ at the point $p=(0,0,1)$.



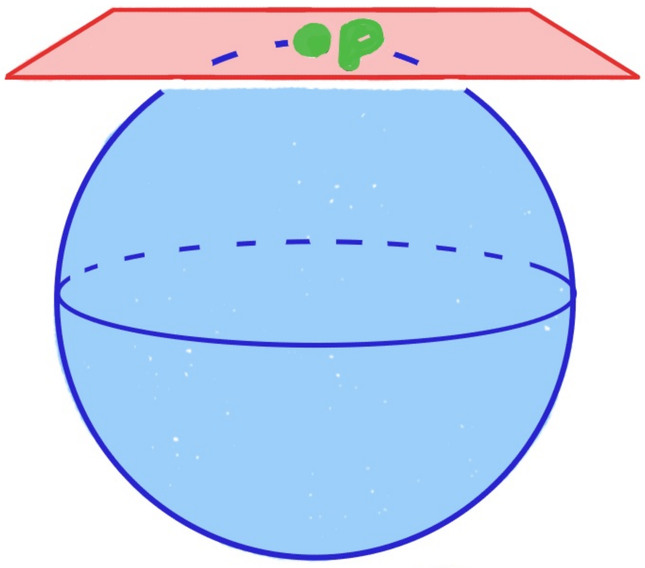
$x^2+y^2+z^2=1$

Solution:

① $F(x,y,z)=x^2+y^2+z^2$

Example:

Find the equation for the tangent plane of $x^2+y^2+z^2=1$ at the point $p=(0,0,1)$.



$x^2+y^2+z^2=1$

Solution:

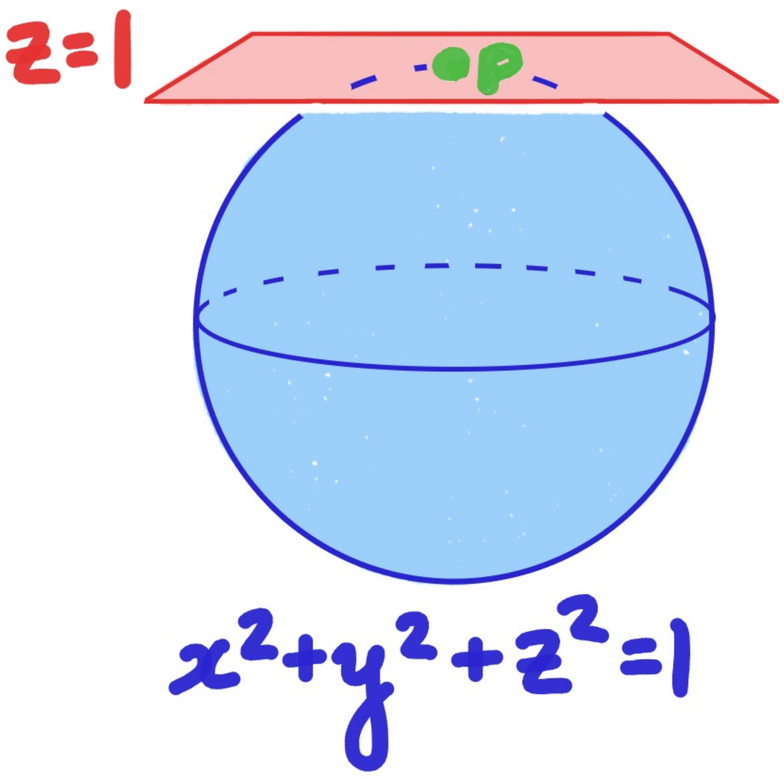
① $F(x,y,z)=x^2+y^2+z^2$

② $\left.\frac{\partial F}{\partial x}\right|_p = \left.\frac{\partial F}{\partial y}\right|_p = 2\cdot 0 = 0$

$\left.\frac{\partial F}{\partial z}\right|_p = 2\cdot 1 = 2$

Example:

Find the equation for the tangent plane of $x^2+y^2+z^2=1$ at the point $p=(0,0,1)$.



Solution:

① $F(x,y,z)=x^2+y^2+z^2$

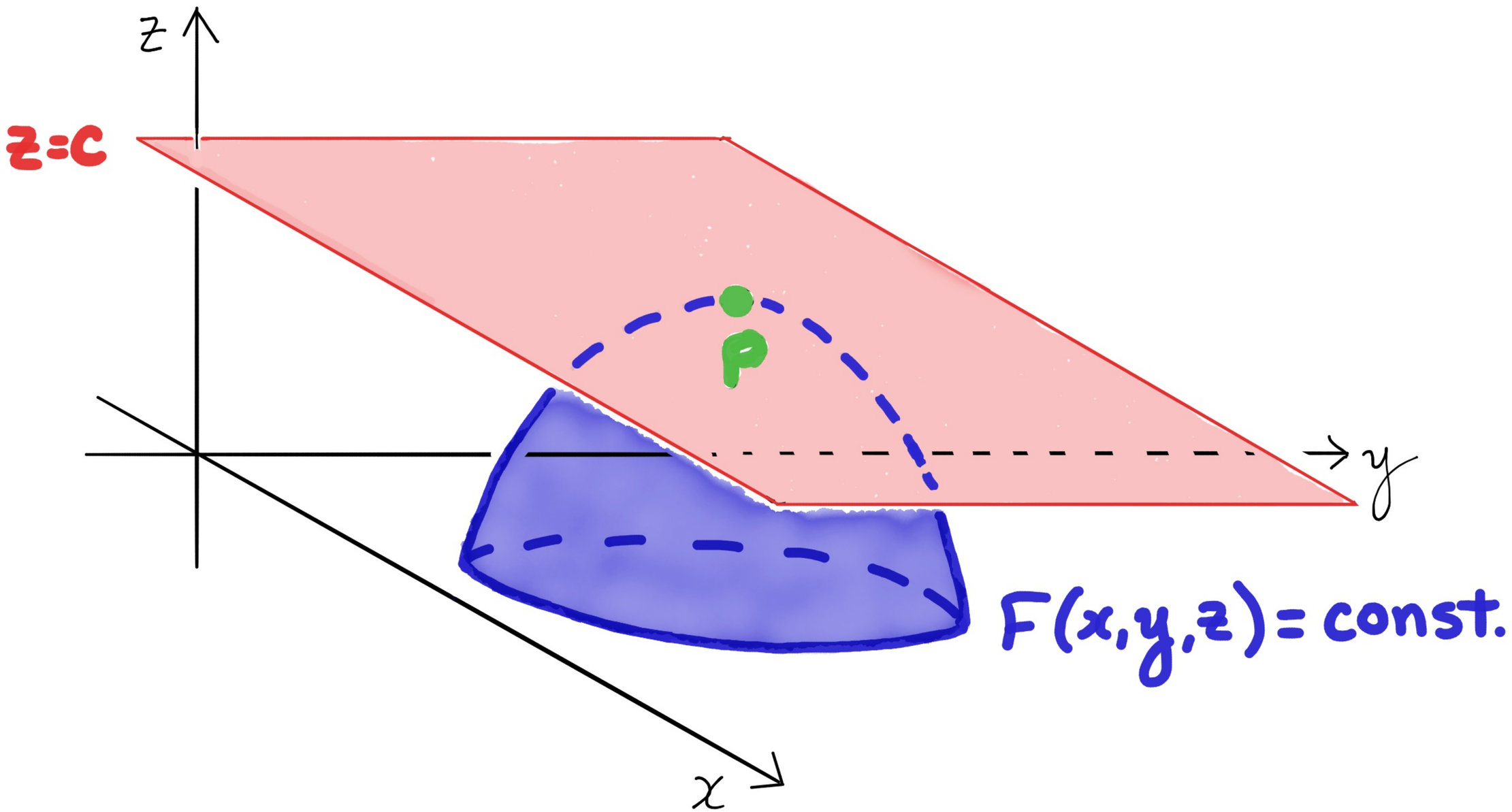
② $\left.\frac{\partial F}{\partial x}\right|_p = \left.\frac{\partial F}{\partial y}\right|_p = 2\cdot 0 = 0$

$\left.\frac{\partial F}{\partial z}\right|_p = 2\cdot 1 = 2$

③ $2(z-1)=0$, or $z=1$

Horizontal tangent plane

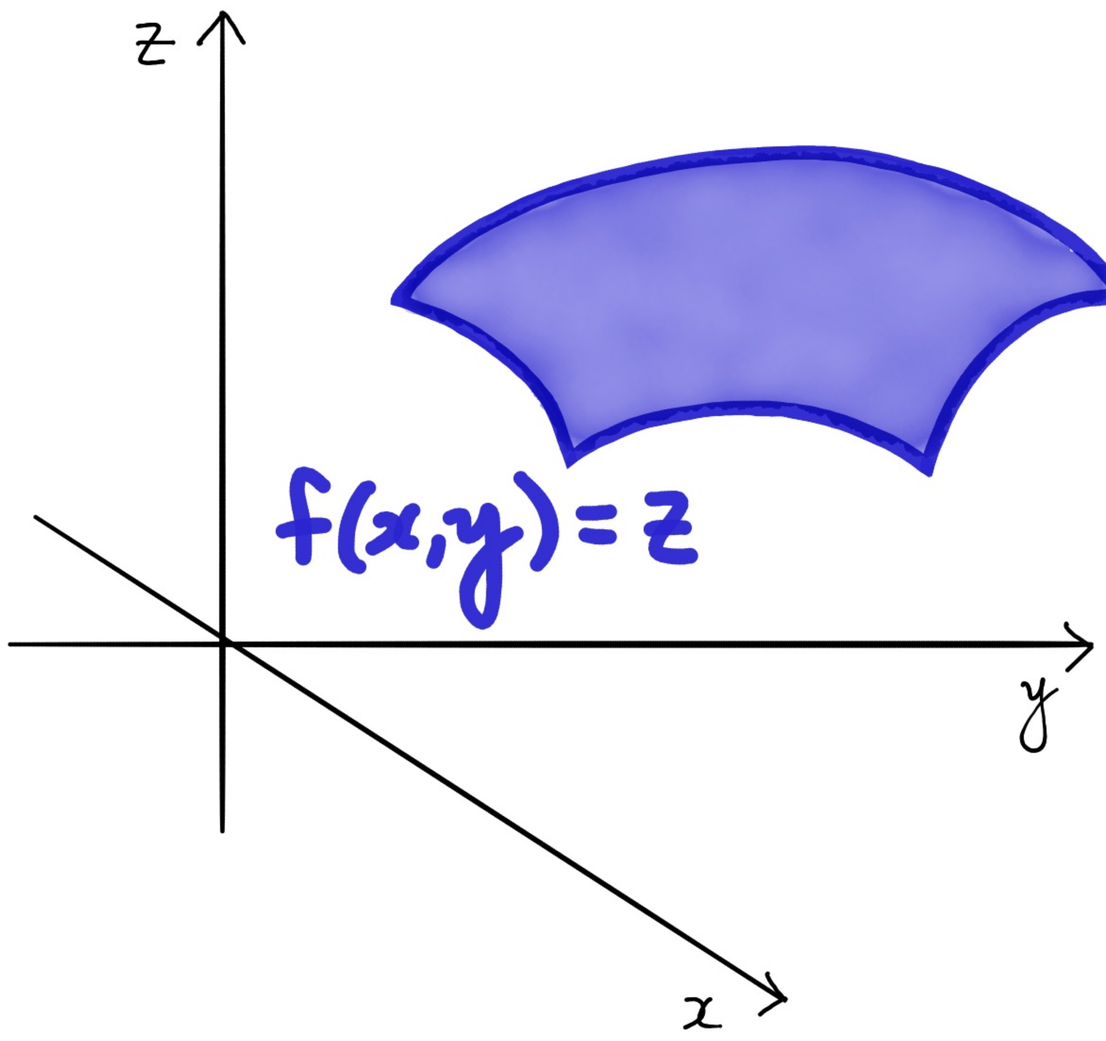
$$\left.\frac{\partial F}{\partial x}\right|_P = \left.\frac{\partial F}{\partial y}\right|_P = 0$$



Ⅳ Tangent planes of graphs

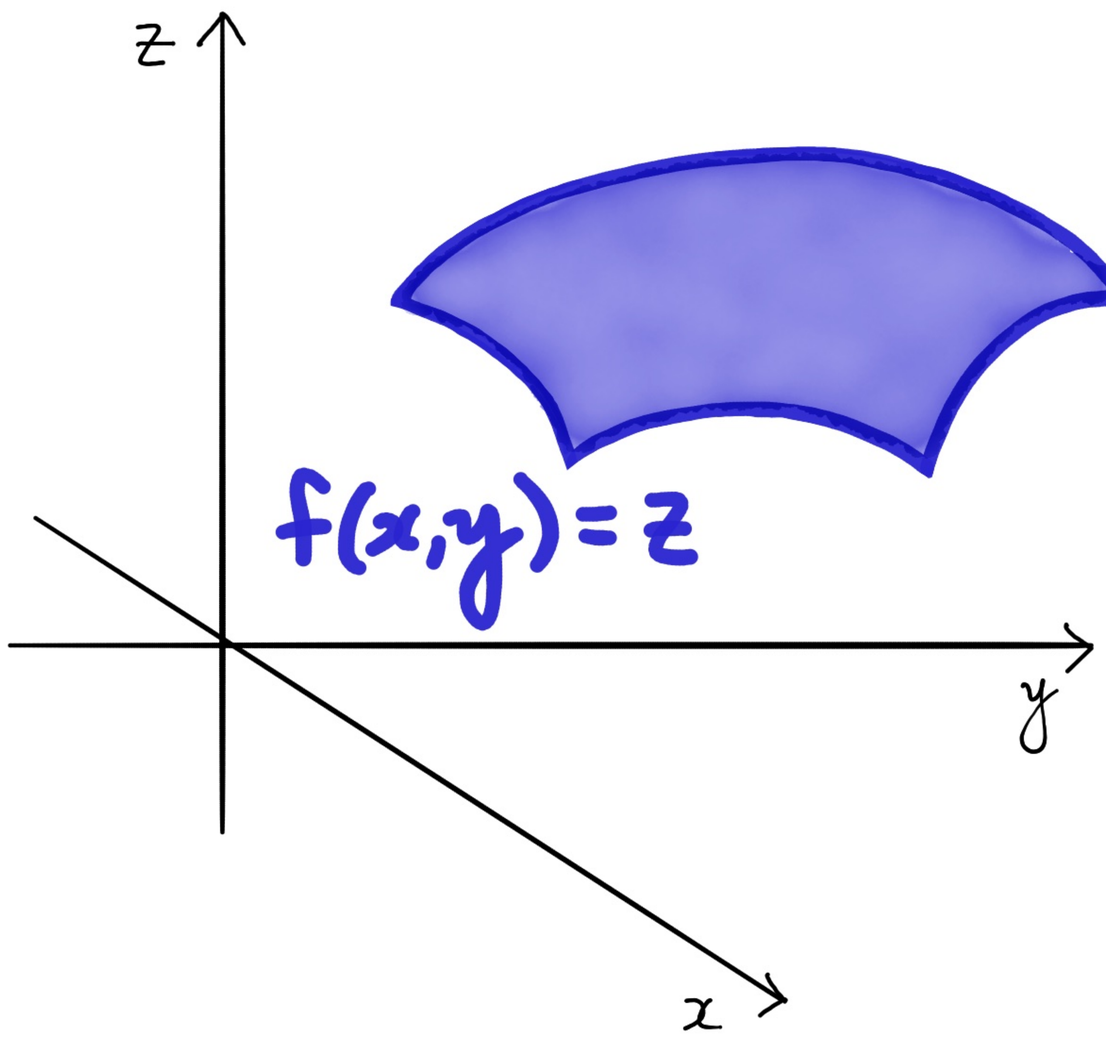
Graphs as level surfaces

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



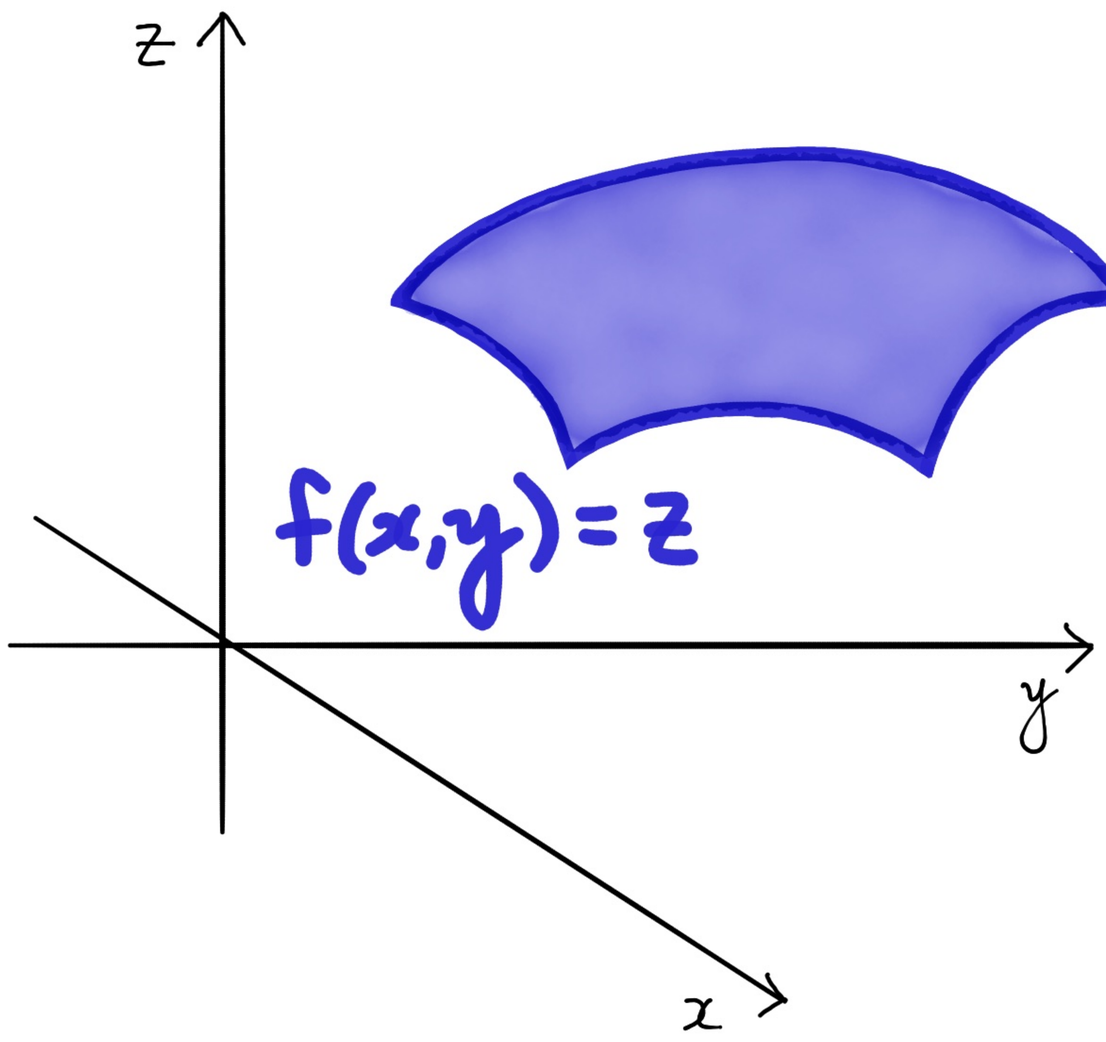
Graphs as level surfaces

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



Graphs as level surfaces

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



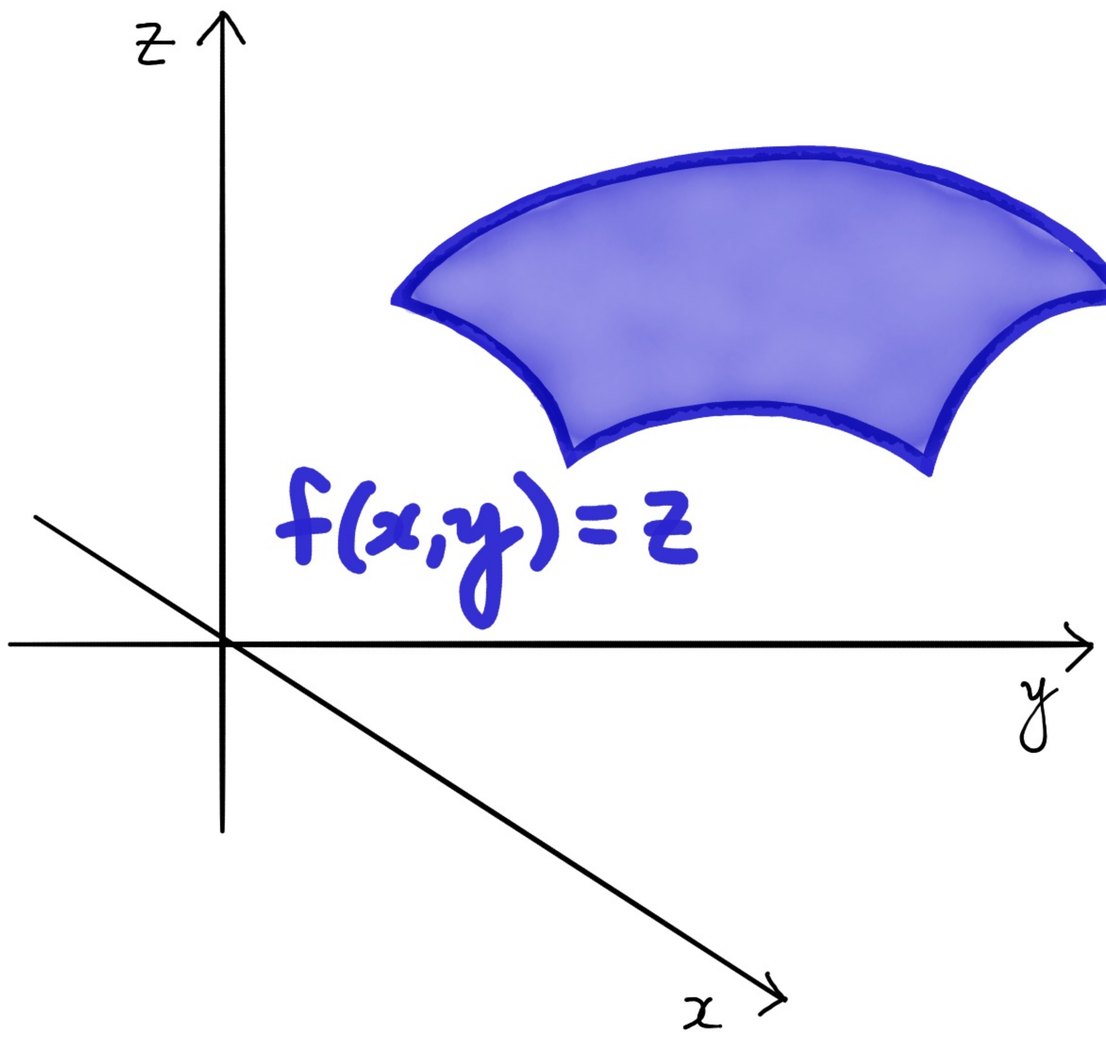
Define:

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$F(x,y,z) = f(x,y) - z$

Graphs as level surfaces

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



$f(x,y) - z = 0$

$F(x,y,z) = 0$

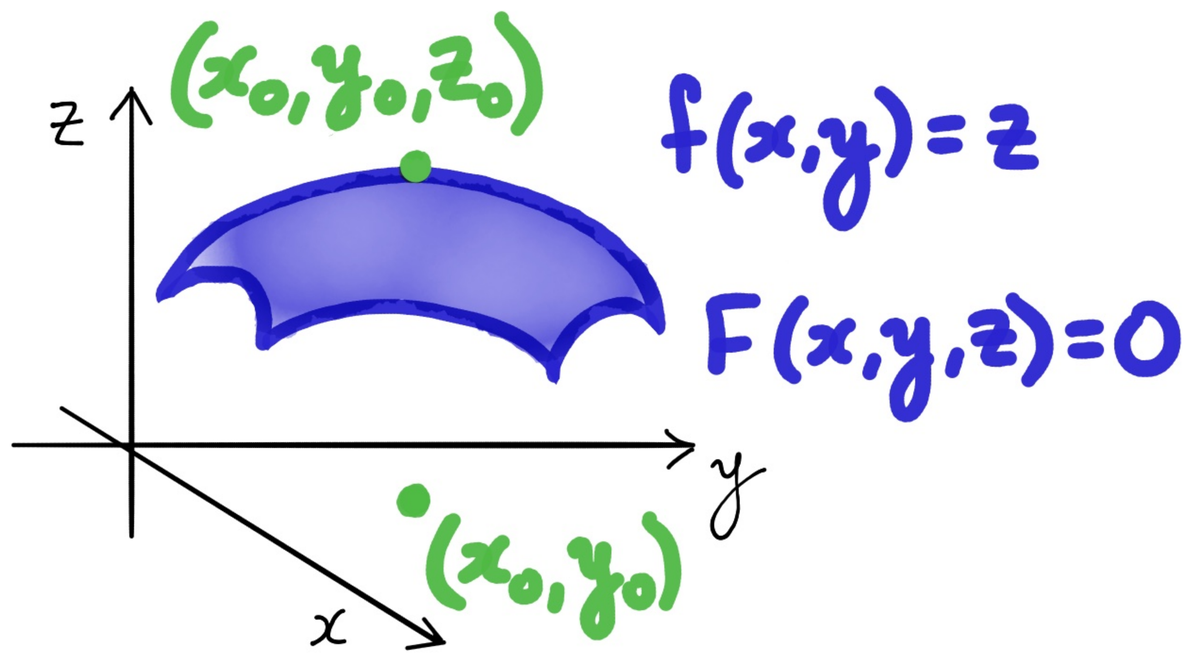
Define:

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$F(x,y,z) = f(x,y) - z$

Horizontal tangent planes for graphs

$$F(x,y,z) = f(x,y) - z$$

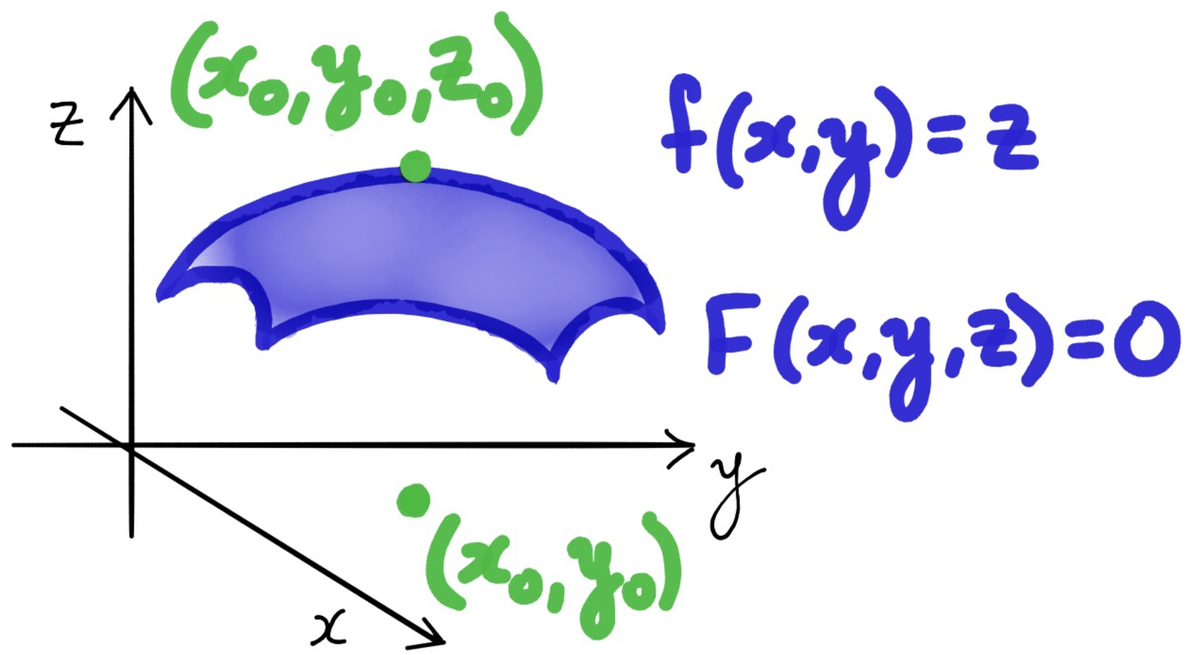


$$0 = \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0,z_0)}$$

$$0 = \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0,z_0)}$$

Horizontal tangent planes for graphs

$$F(x,y,z) = f(x,y) - z$$

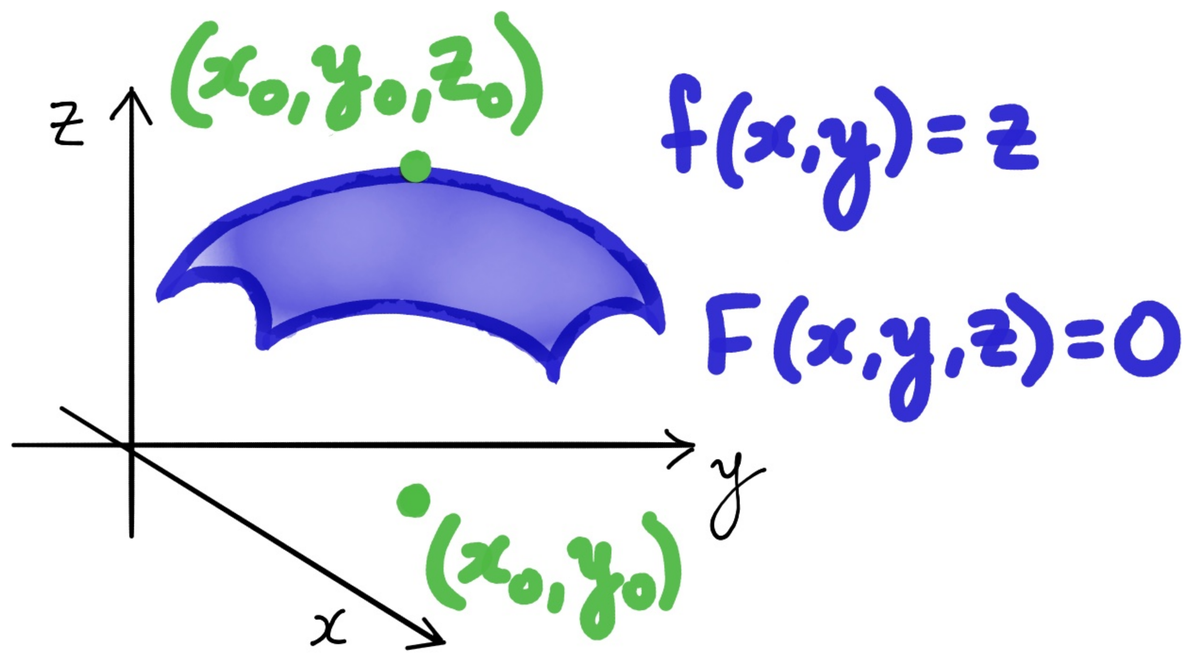


$$0 = \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0,z_0)} = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)}$$

$$0 = \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0,z_0)}$$

Horizontal tangent planes for graphs

$$F(x,y,z) = f(x,y) - z$$

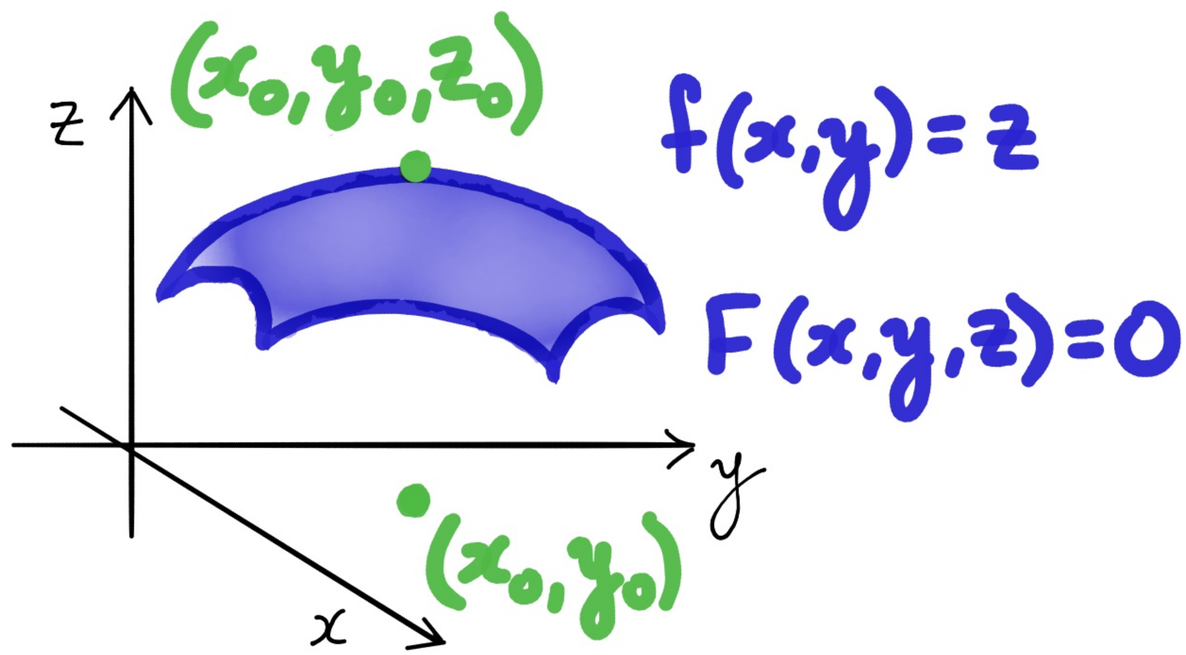


$$0 = \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0,z_0)} = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)} = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$$

$$0 = \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0,z_0)}$$

Horizontal tangent planes for graphs

$$F(x,y,z) = f(x,y) - z$$



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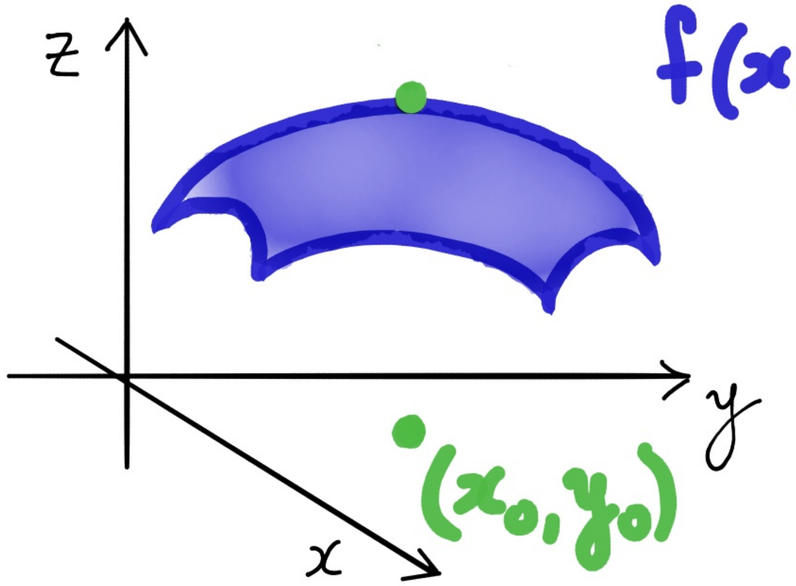
Horizontal tangent planes for graphs

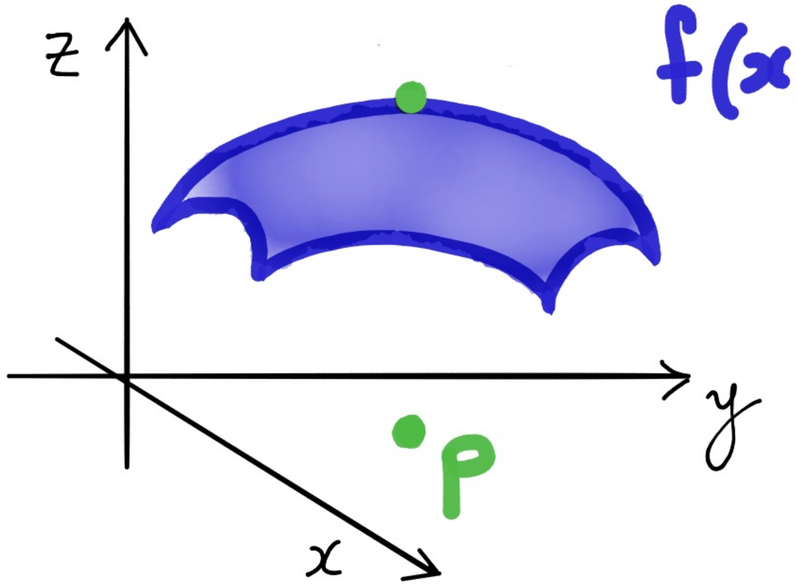


$f(x,y) = z$

Tangent plane for graph of $f: \mathbb{R}^2 \to \mathbb{R}$ is horizontal at $(x_0, y_0) \in \mathbb{R}^2$ when

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = 0$$

Horizontal tangent planes for graphs

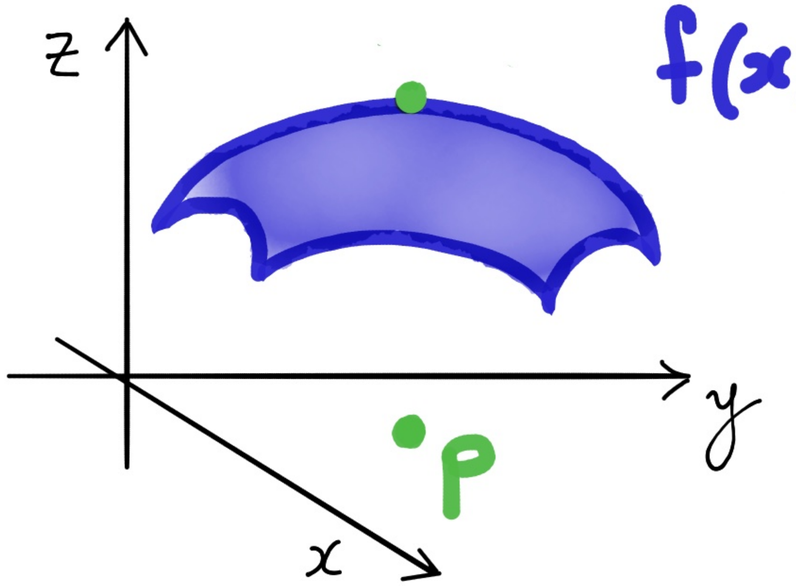


$f(x,y) = z$

Tangent plane for graph of $f: \mathbb{R}^2 \to \mathbb{R}$ is horizontal at $p \in \mathbb{R}^2$ when

$$\left.\frac{\partial f}{\partial x}\right|_p = \left.\frac{\partial f}{\partial y}\right|_p = 0$$

Horizontal tangent planes for graphs

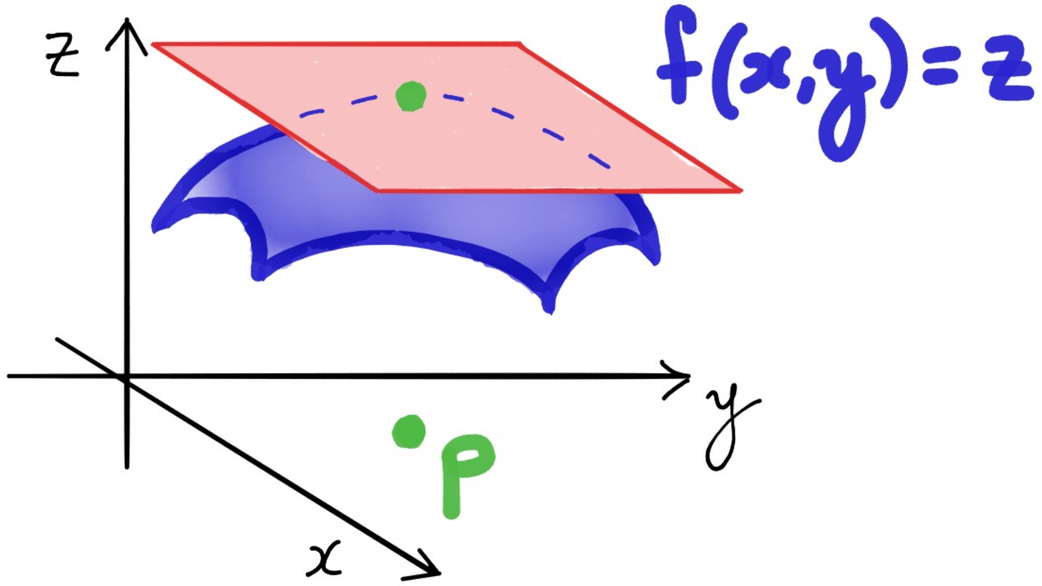


$f(x,y)=z$

Tangent plane for graph of $f:\mathbb{R}^2\to\mathbb{R}$ is horizontal at $p\in\mathbb{R}^2$ when

$$\nabla f(p)=\vec{0}.$$

Horizontal tangent planes for graphs

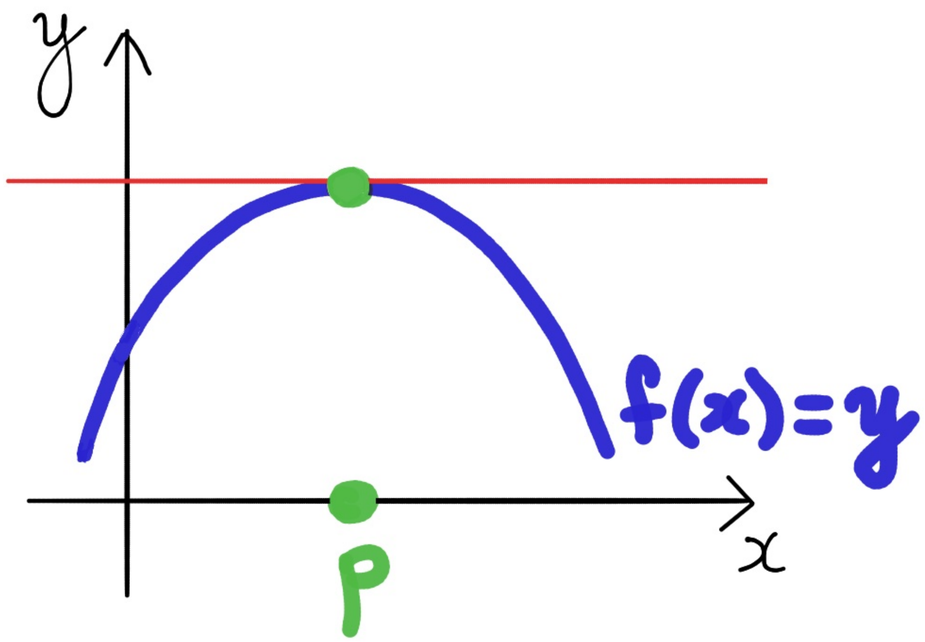


Tangent plane for graph of $f: \mathbb{R}^2 \to \mathbb{R}$ is horizontal at $p \in \mathbb{R}^2$ when

$$\nabla f(p) = \vec{0}.$$

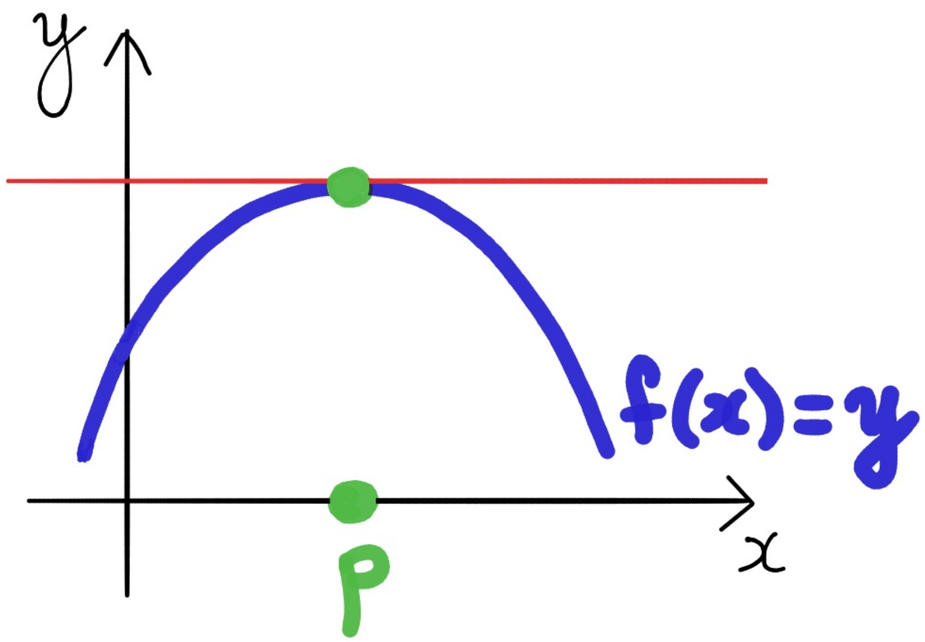
$f: \mathbb{R} \to \mathbb{R}$ has horizontal tangent line at $p \in \mathbb{R}$ when

$$\left.\frac{df}{dx}\right|_p = 0$$



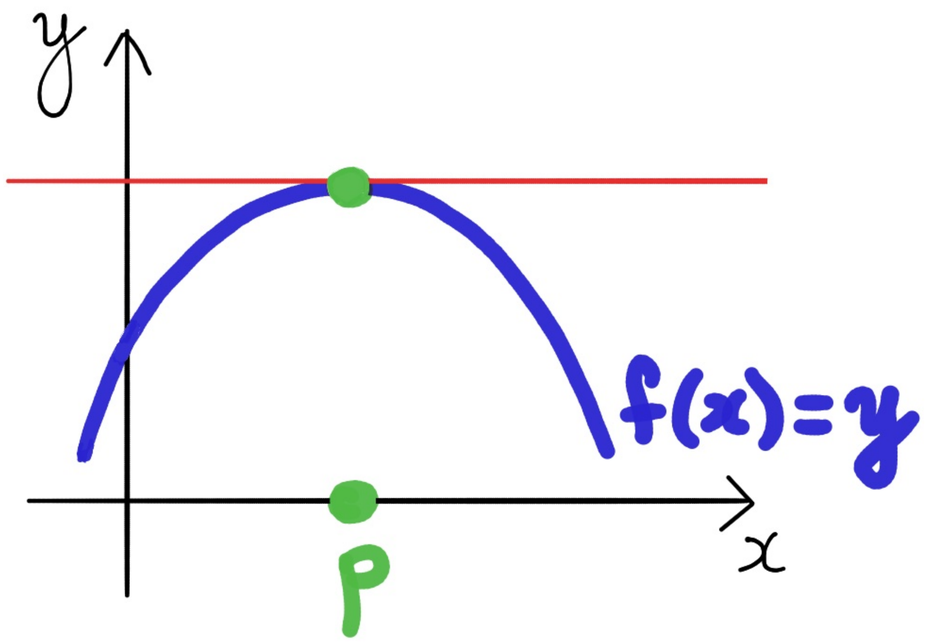
$f: \mathbb{R} \to \mathbb{R}$ has horizontal tangent line at $p \in \mathbb{R}$ when

$$\nabla f(p) = \left.\frac{df}{dx}\right|_p = 0$$



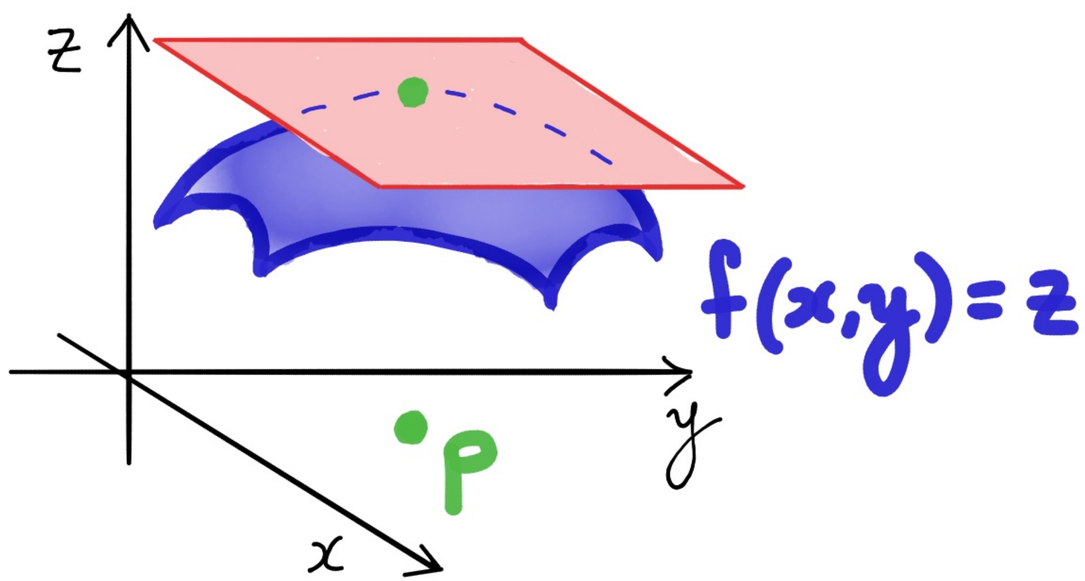
$f:\mathbb{R}\to\mathbb{R}$ has horizontal tangent line at $p\in\mathbb{R}$ when

$$\nabla f(p)=\left.\frac{df}{dx}\right|_p=0$$



$f:\mathbb{R}^2\to\mathbb{R}$ has horizontal tangent plane at $p\in\mathbb{R}^2$ when

$$\nabla f(p)=\vec{0}$$

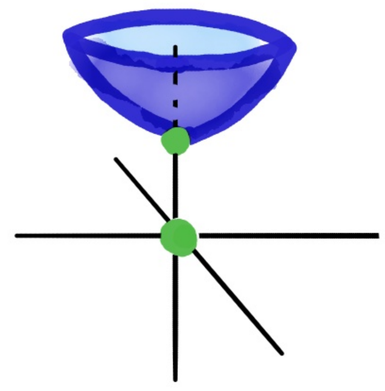


Examples:

Find $\nabla f(p)$ for $f: \mathbb{R}^2 \to \mathbb{R}$ and $p \in \mathbb{R}^2$ below.

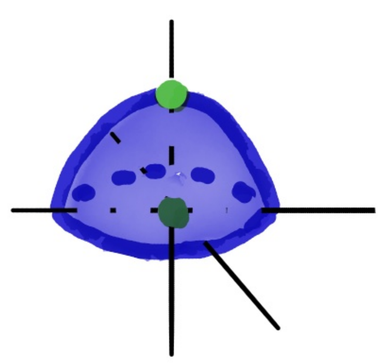
① $f(x,y) = x^2 + y^2 + 1$

$p = (0,0)$



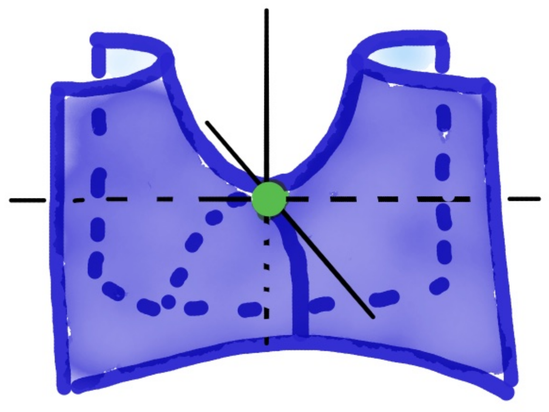
② $f(x,y) = -x^2 - y^2 + 1$

$p = (0,0)$



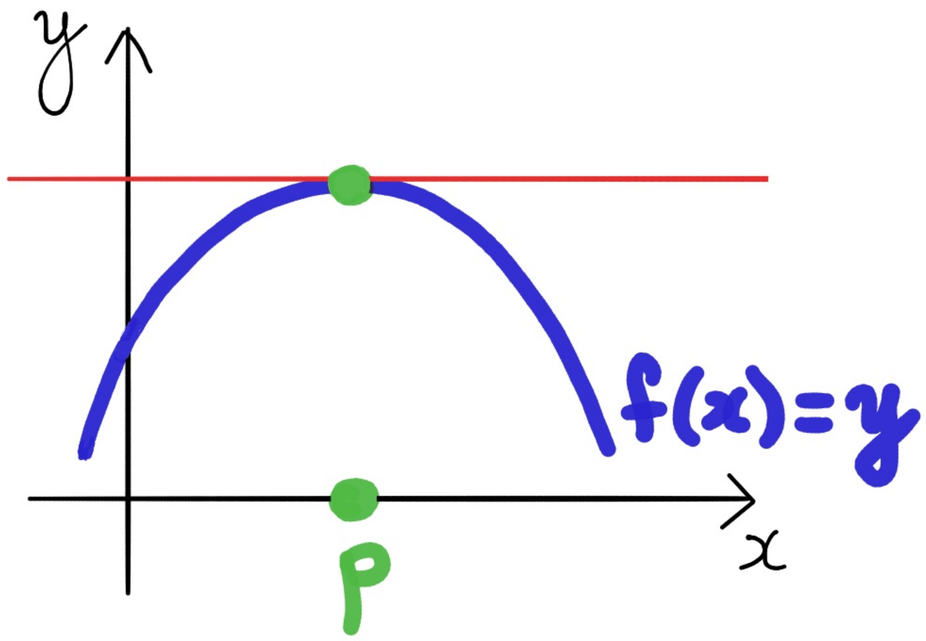
③ $f(x,y) = y^2 - x^2$

$p = (0,0)$



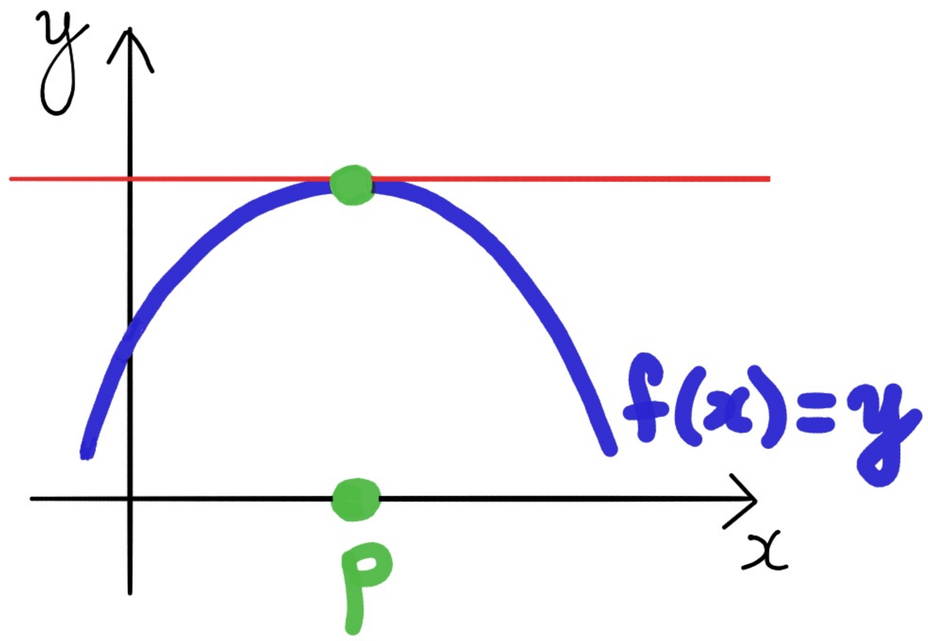
Ⓥ Curvature of graphs

$f : \mathbb{R} \longrightarrow \mathbb{R}$



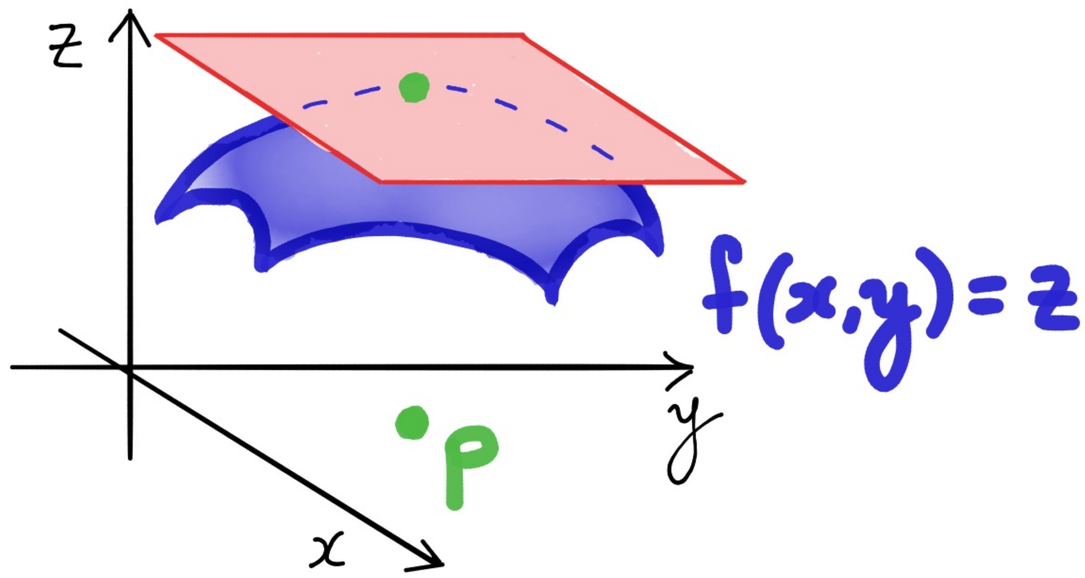
If $\nabla f(p) = 0$, curvature of graph at $p \in \mathbb{R}$ is $\left| f''(p) \right|$.

$f : \mathbb{R} \longrightarrow \mathbb{R}$



If $\nabla f(p) = 0$, curvature of graph at $p \in \mathbb{R}$ is $\left| \frac{d^2 f}{dx^2}\Big|_p \right|$.

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$



If $\nabla f(p) = \vec{0}$, curvature of graph at $p \in \mathbb{R}^2$ is

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2}\Big|_p & \frac{\partial^2 f}{\partial x \partial y}\Big|_p \\ \frac{\partial^2 f}{\partial x \partial y}\Big|_p & \frac{\partial^2 f}{\partial y^2}\Big|_p \end{vmatrix}.$$

Examples:

Find the curvature for the graphs of $f: \mathbb{R}^2 \to \mathbb{R}$ at $p \in \mathbb{R}^2$ below.

① $f(x,y) = 3x^2 + y^2$

$p = (0,0)$

② $f(x,y) = -3x^2 - y^2$

$p = (0,0)$

③ $f(x,y) = y^2 - x^2$

$p = (0,0)$

Examples:

Find the curvature for the graphs of $f: \mathbb{R}^2 \to \mathbb{R}$ at $p \in \mathbb{R}^2$ below.

① $f(x,y) = 3x^2 + y^2$

$p = (0,0)$

$\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12$

② $f(x,y) = -3x^2 - y^2$

$p = (0,0)$

$\begin{vmatrix} -6 & 0 \\ 0 & -2 \end{vmatrix} = 12$

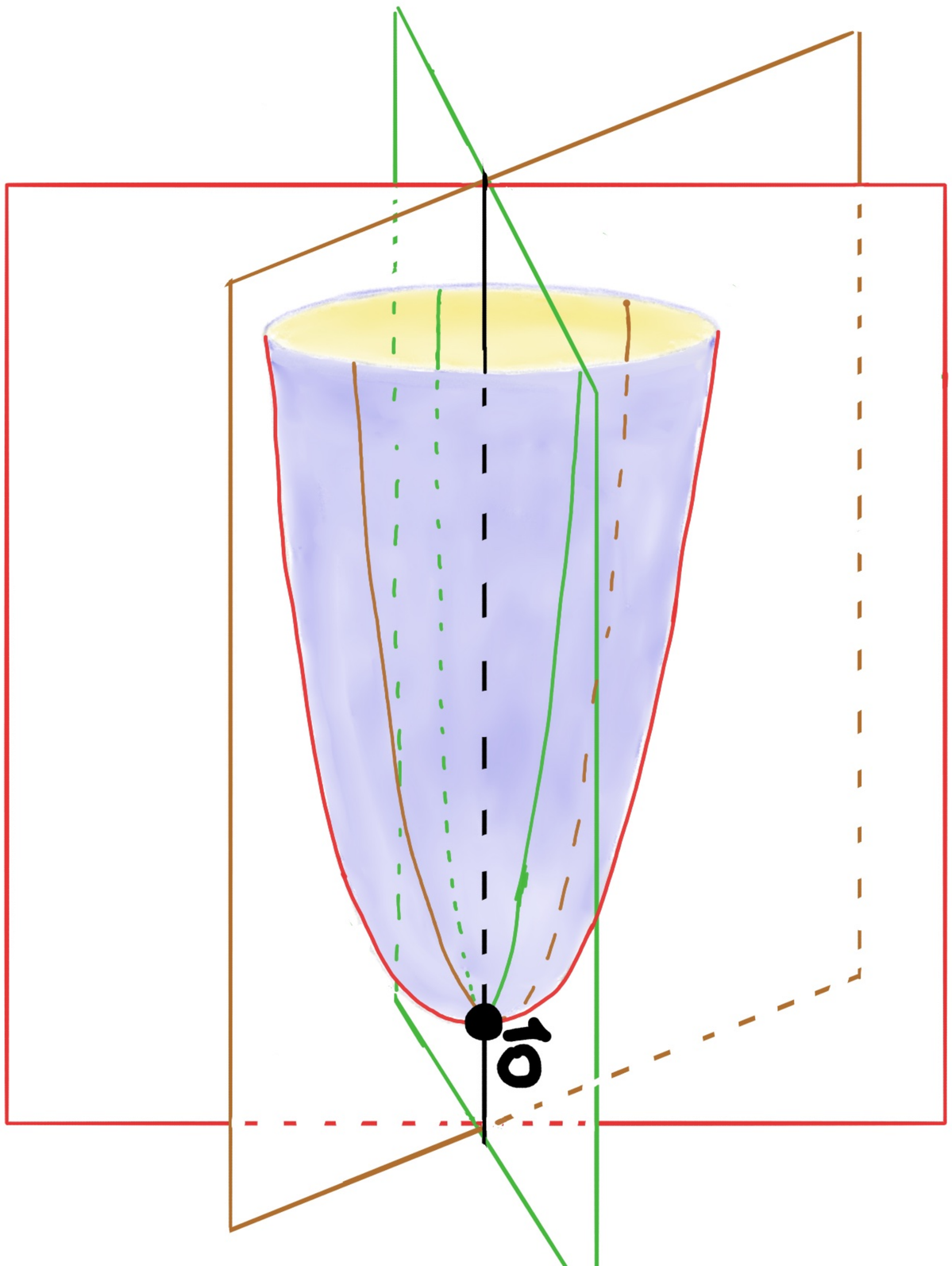
③ $f(x,y) = y^2 - x^2$

$p = (0,0)$

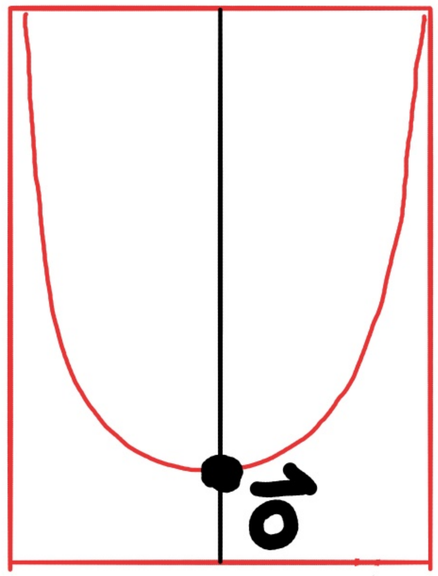
$\begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4$

$3x^2+y^2=z$

Curvature
$2 \cdot 6 = 12$



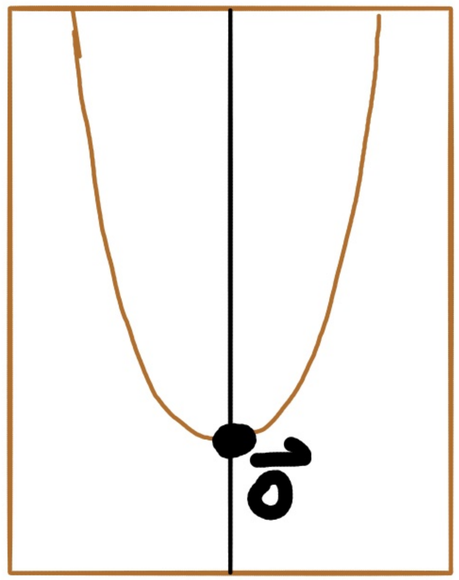
$x=0$



$y^2=z$

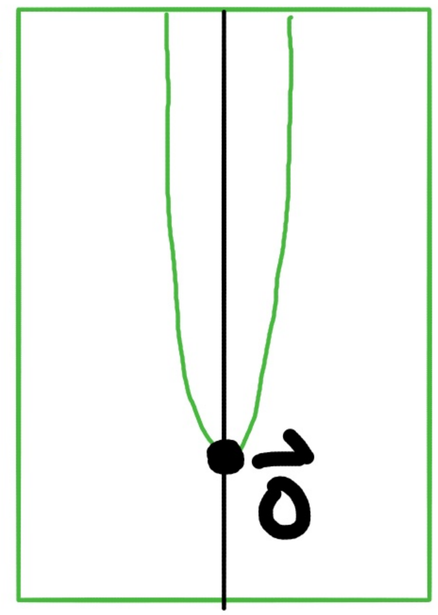
$\hat{K}(\vec{0})=2$

In between



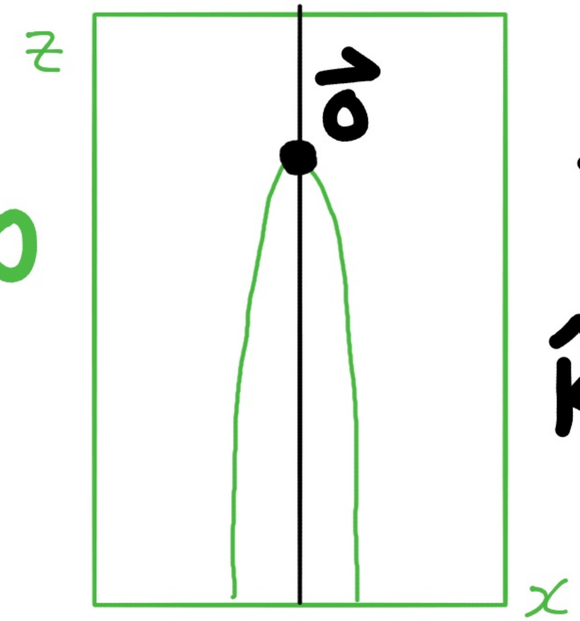
$2 \le \hat{K}(\vec{0}) \le 6$

$y=0$



$3x^2=z$

$\hat{K}(\vec{0})=6$

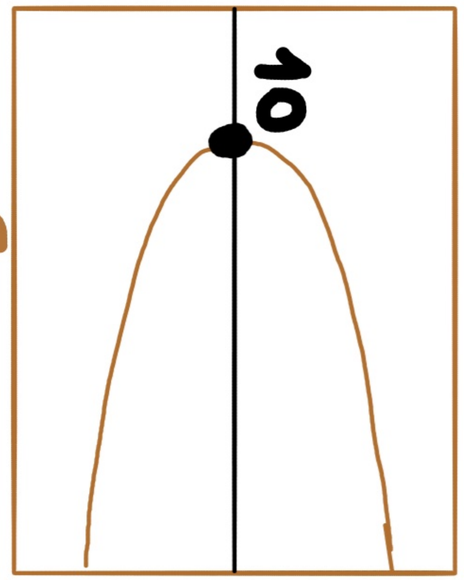


$y=0$

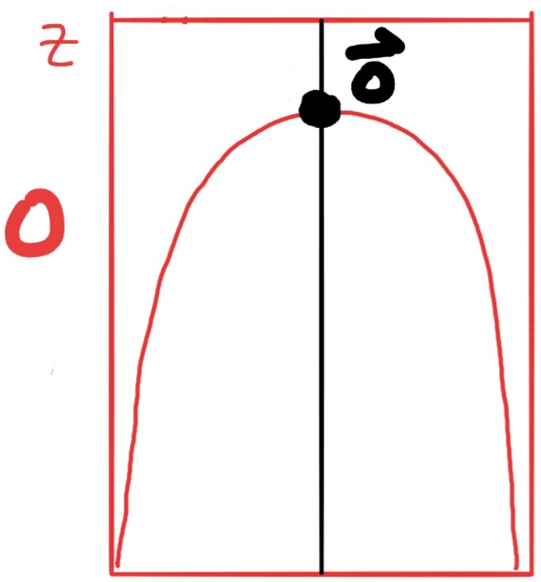
$-3x^2=z$

$\hat{K}(\vec{0})=-6$

In between



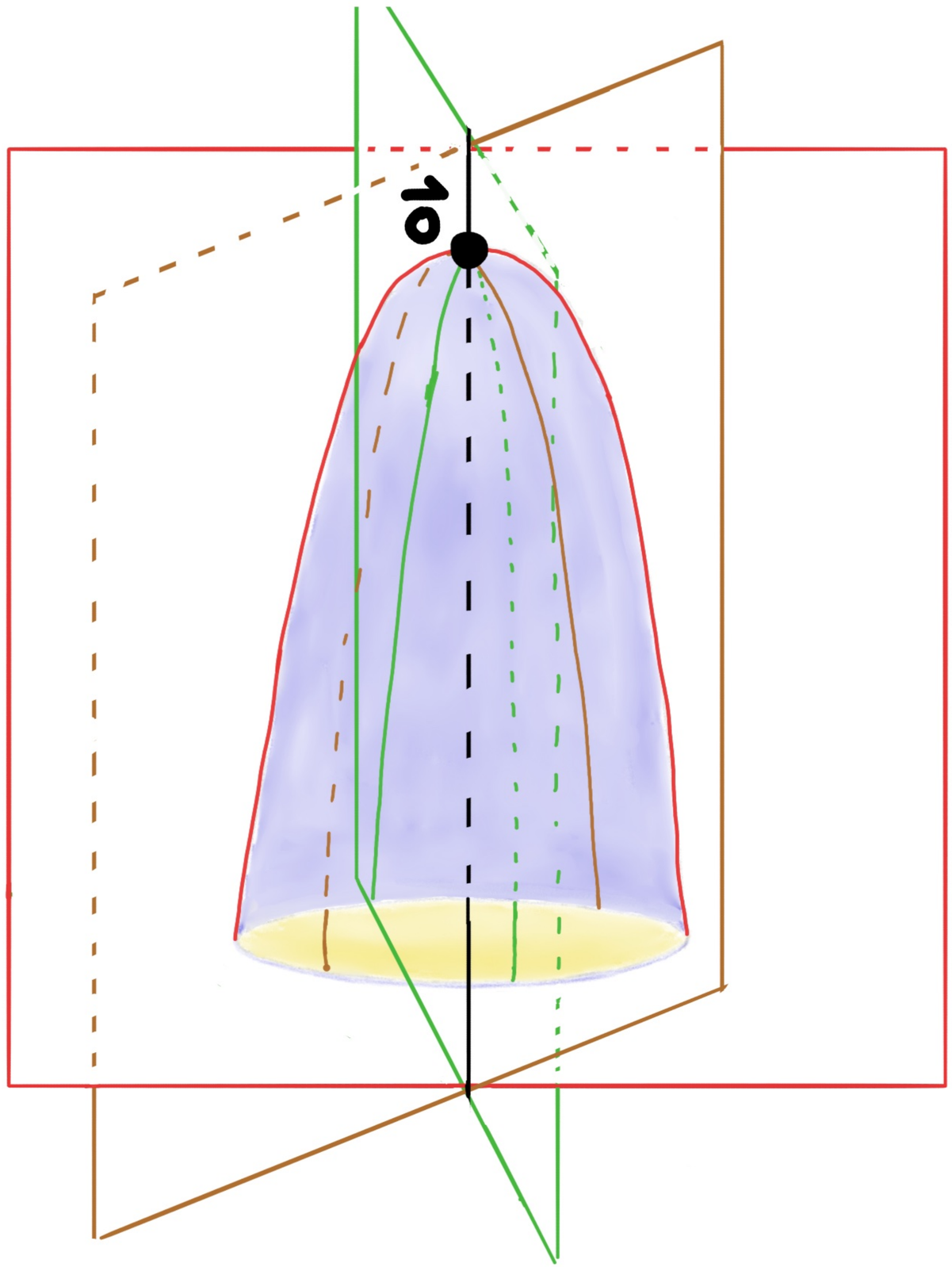
$-6 \le \hat{K}(\vec{0}) \le -2$



$x=0$

$-y^2=z$

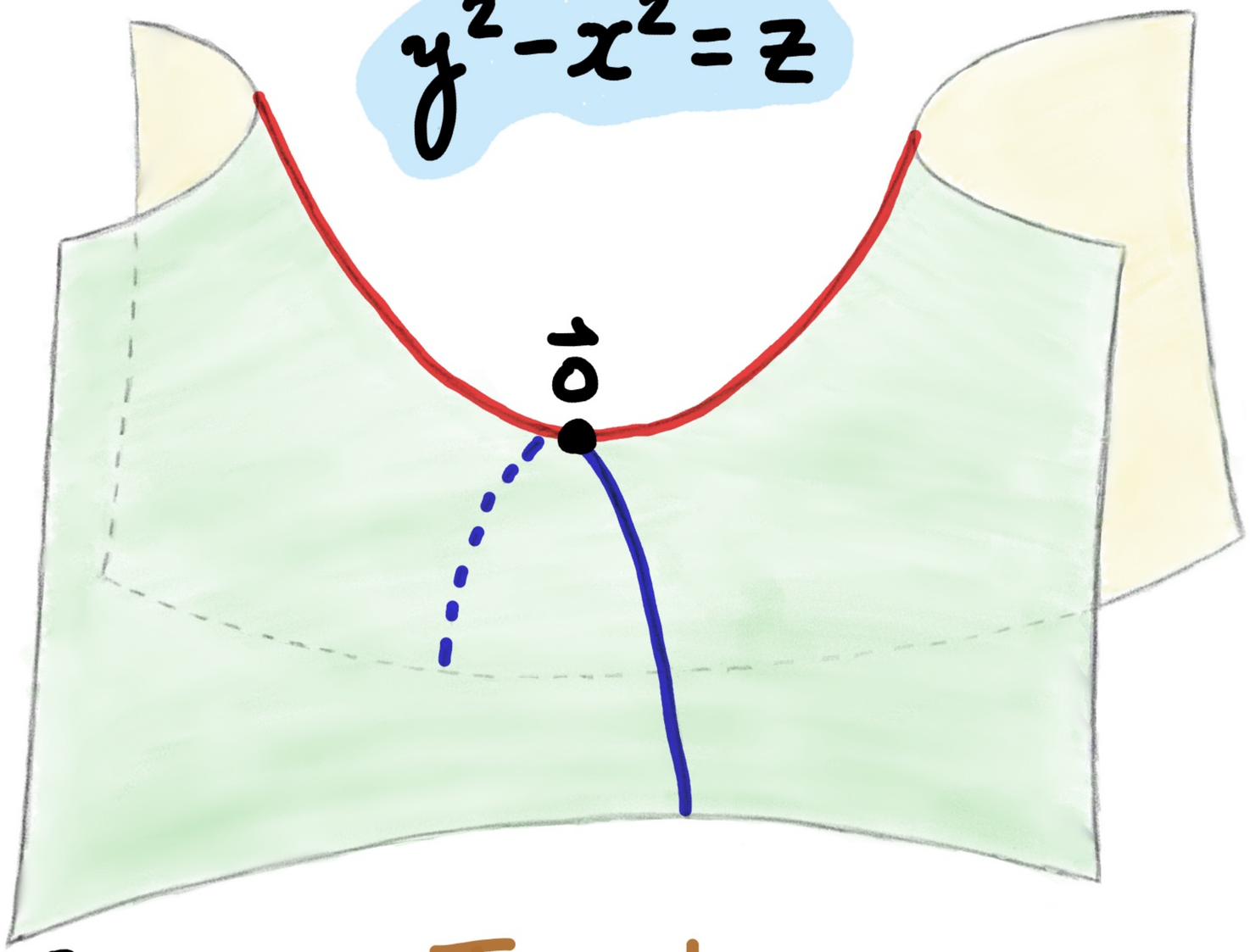
$\hat{K}(\vec{0})=-2$



Curvature
$(-2)(-6)=12$

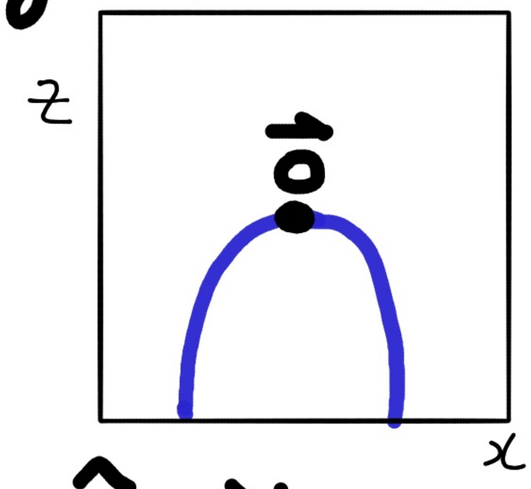
$-3x^2-y^2=z$

$y^2 - x^2 = z$



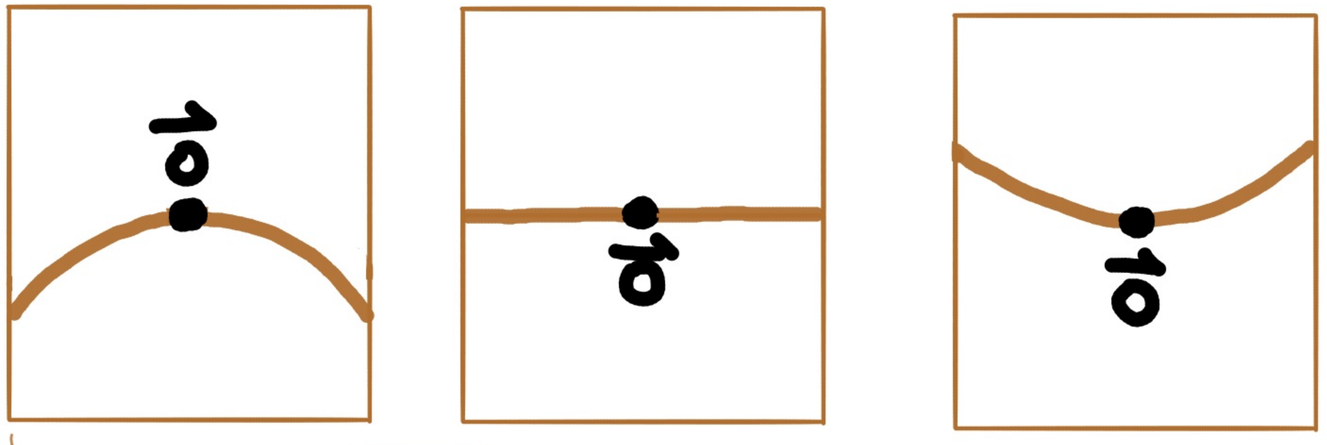
Curvature
$-2 \cdot 2 = -4$

$y=0,\ -x^2=z$



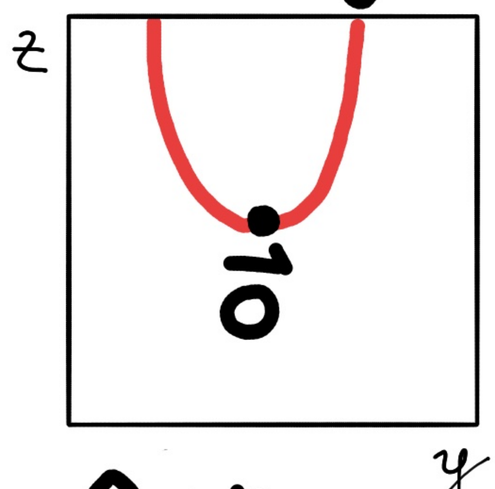
$\hat{K}(\vec{0}) = -2$

In between



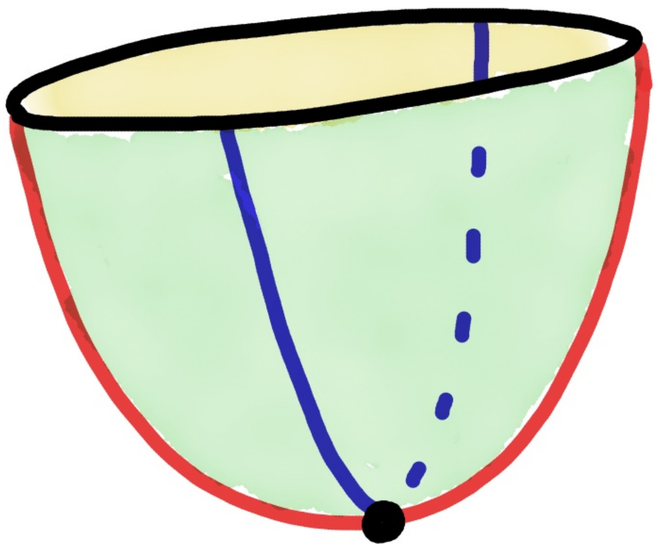
$-2 \le \hat{K}(\vec{0}) \le 2$

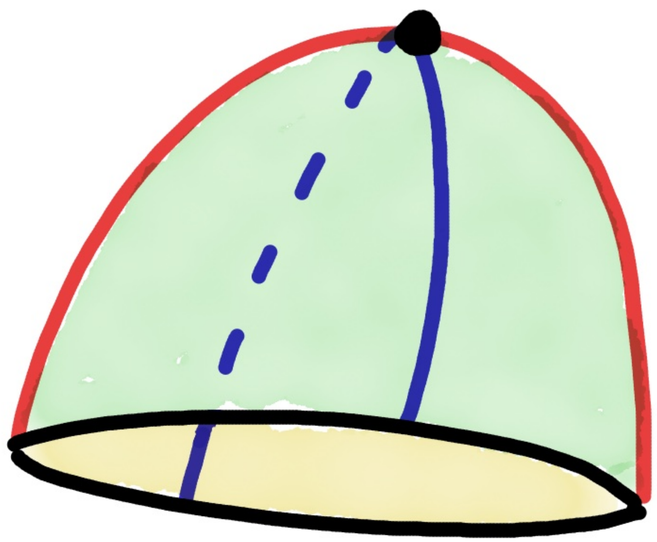
$x=0,\ y^2=z$



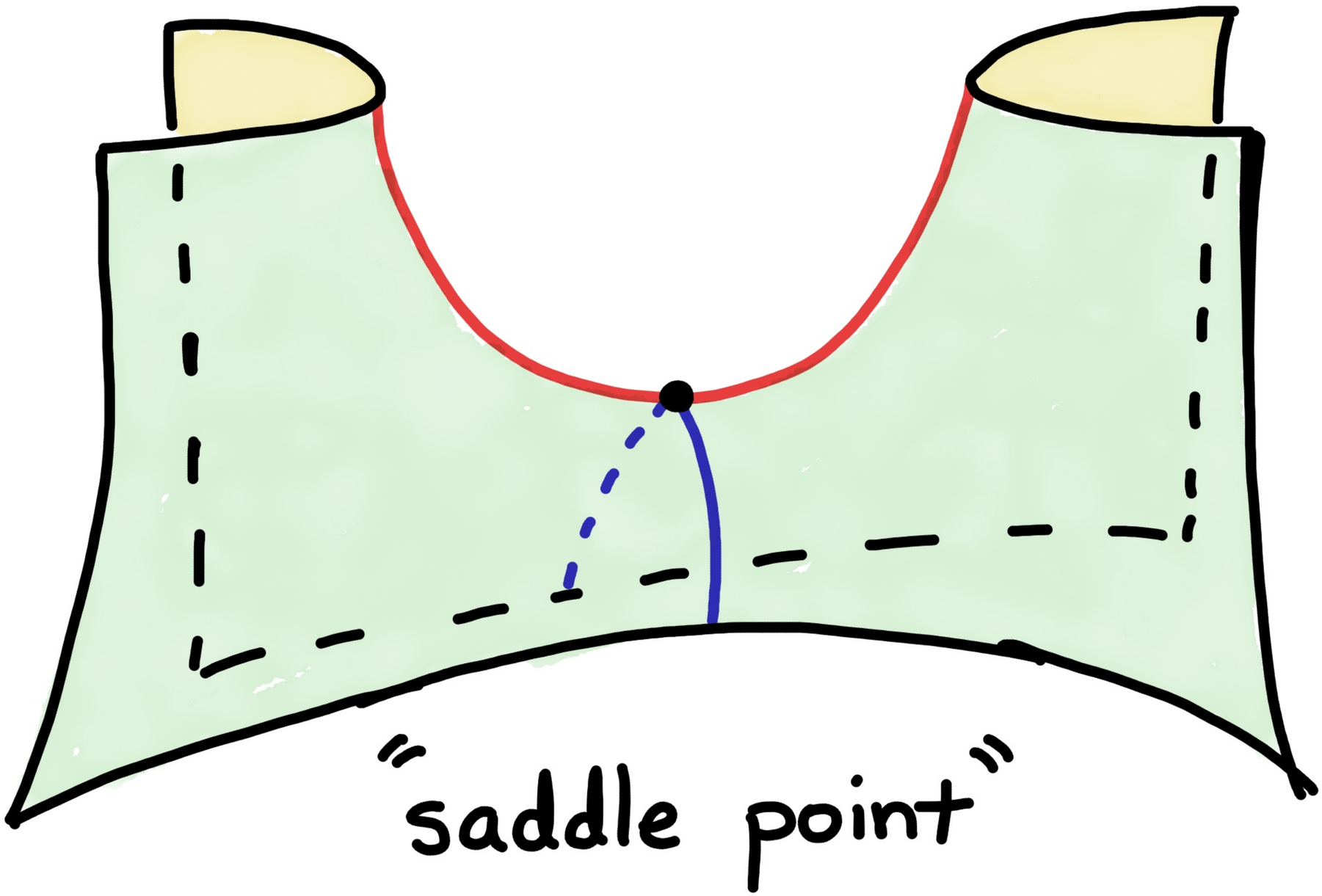
$\hat{K}(\vec{0}) = 2$

positive curvature



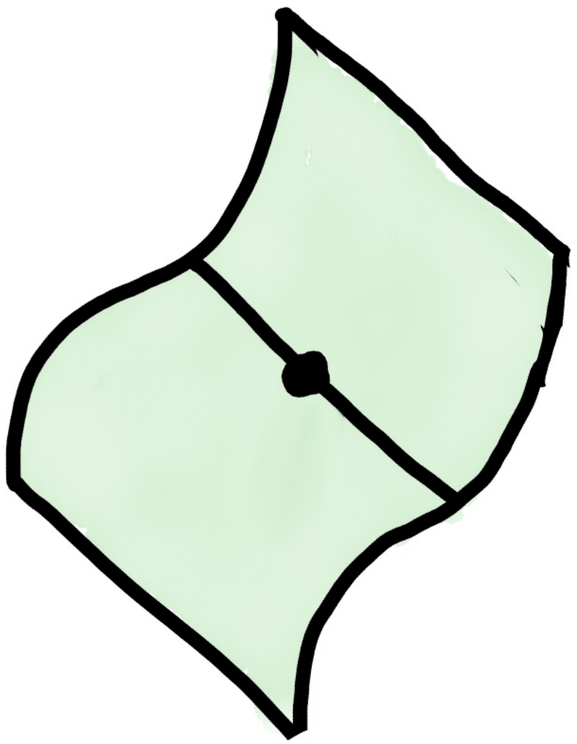


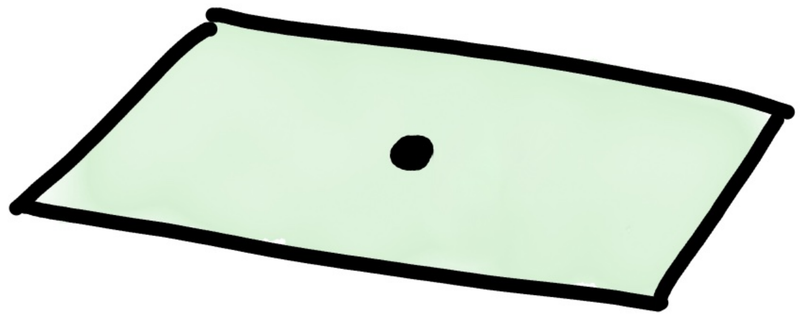
negative curvature

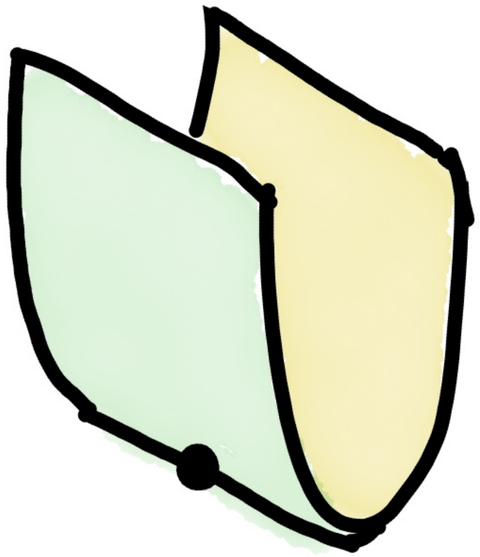


"saddle point"

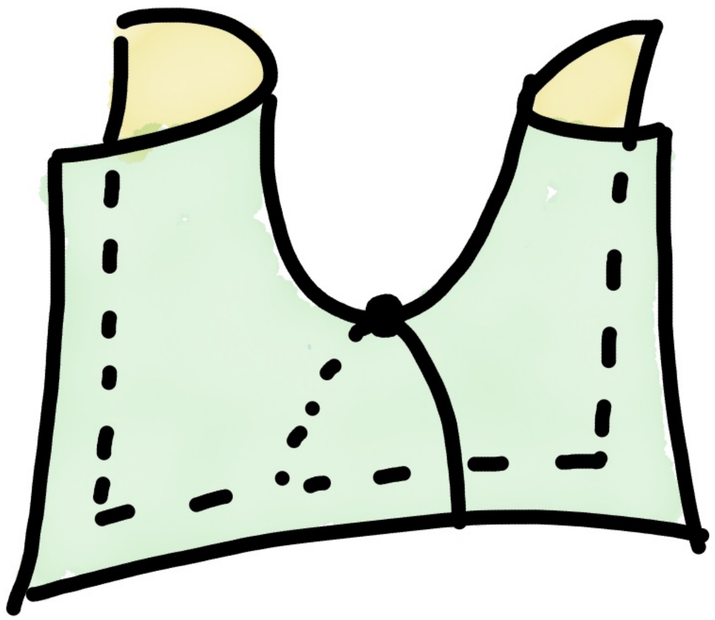
Zero curvature







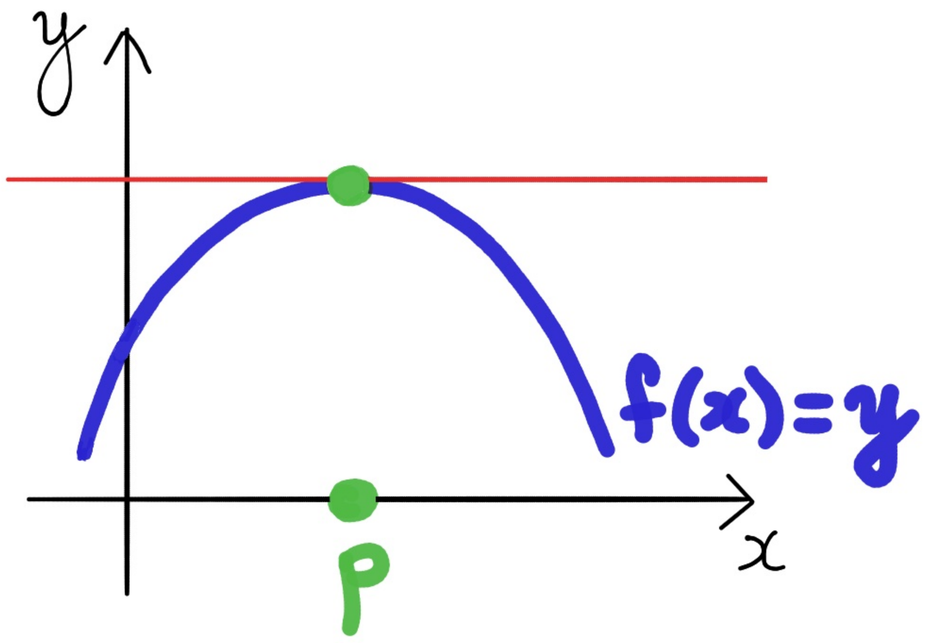
$x^4+y^4=z$



$y^4-x^4=z$

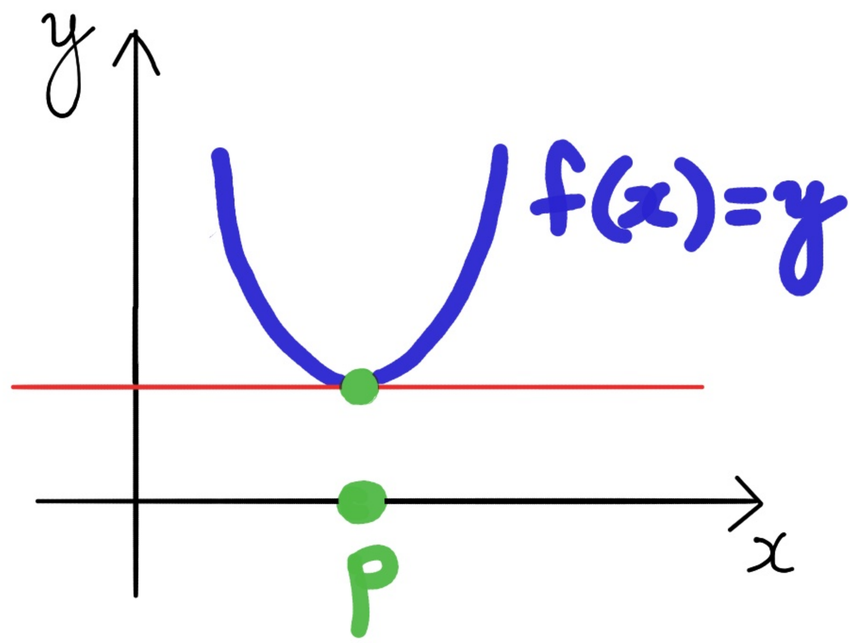
VI Concavity

$$f:\mathbb{R}\to\mathbb{R}, \quad \nabla f(p)=0$$



Concave down

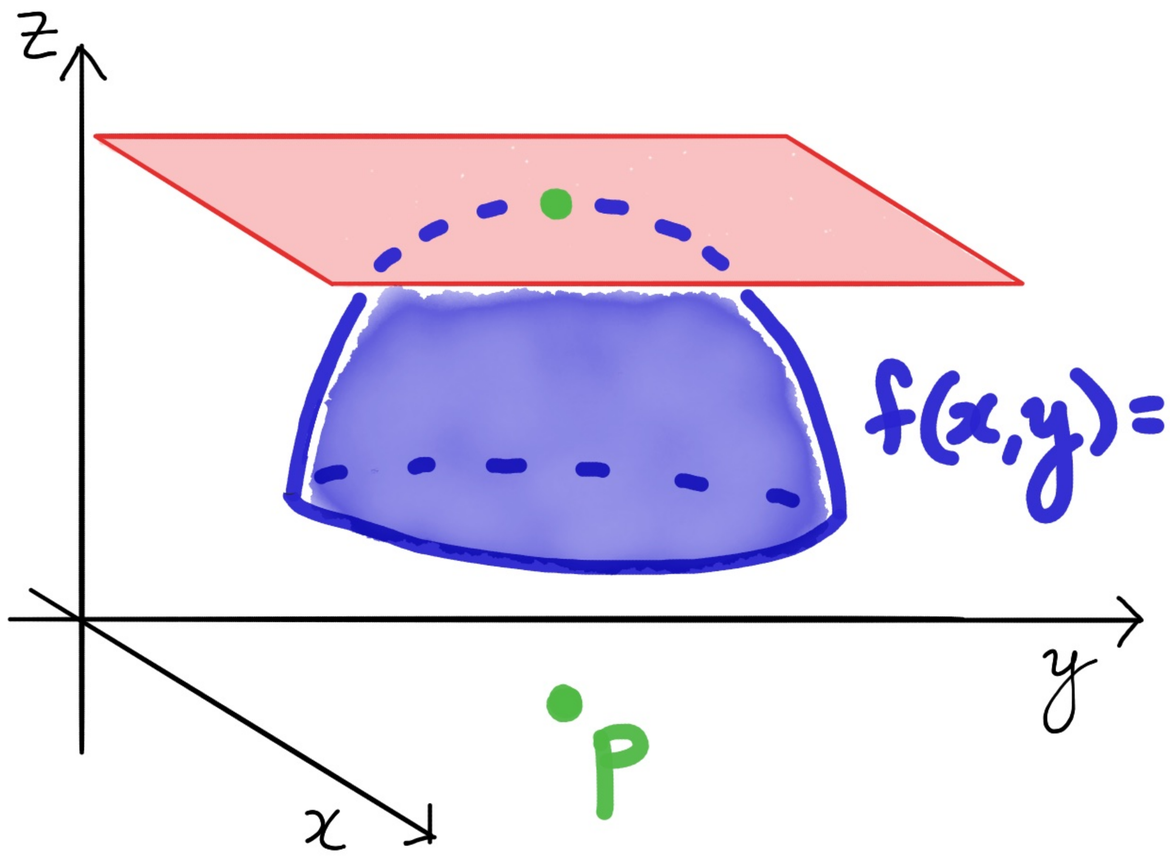
$$\left.\frac{d^2f}{dx^2}\right|_p < 0$$



Concave up

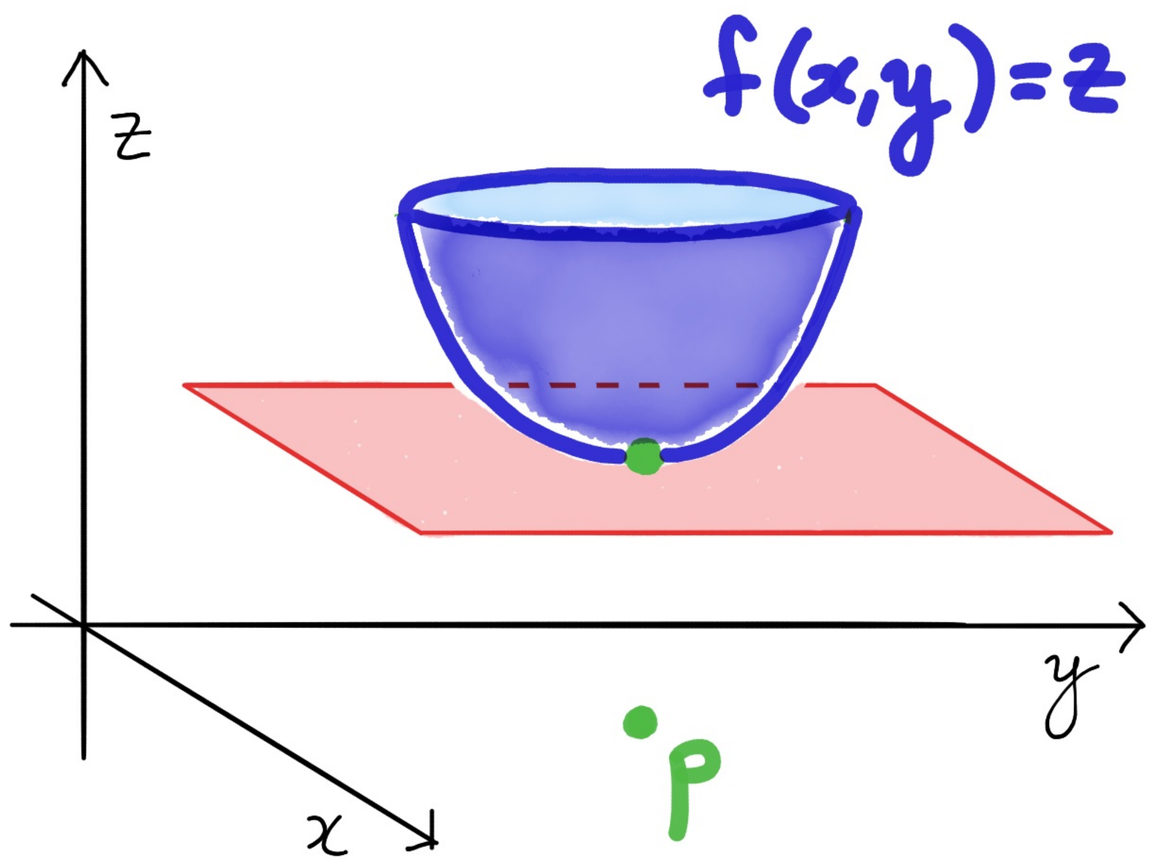
$$\left.\frac{d^2f}{dx^2}\right|_p > 0$$

$$f:\mathbb{R}^2\to\mathbb{R}, \qquad \nabla f(p)=\vec{0}$$



Concave down

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_p < 0$$



Concave up

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_p > 0$$