Fourteen

Chain Rule:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Example:

$$\frac{d}{dx} \cos(x^2) = \left(-\sin(x^2)\right) \cdot (2x)$$

Ⅰ Matrices

Ⅱ Jacobian matrices

Ⅲ The purpose of Jacobian matrices

Ⅳ Chain rule

Ⅴ The chain rule in practice

Ⓘ Matrices

A **kxn matrix** is a matrix with k rows and n columns.

1x1: 8

1x2: $(4,6)$

1x3: $(3,7,8)$

2x1: $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

2x2: $\begin{pmatrix} 3 & 4 \\ 7 & 2 \end{pmatrix}$

2x3: $\begin{pmatrix} 6 & 7 & 1 \\ 3 & 4 & 2 \end{pmatrix}$

3x1: $\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$

3x2: $\begin{pmatrix} 5 & 2 \\ 4 & 7 \\ 3 & 1 \end{pmatrix}$

3x3: $\begin{pmatrix} 1 & 3 & 4 \\ 5 & 7 & 2 \\ 3 & 4 & 1 \end{pmatrix}$

Matrices are functions

$$\begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 4x+7y \end{pmatrix}$$

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$$\begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 4x+7y \end{pmatrix}$$

$$(4,3,5) : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(4,3,5) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4x+3y+5z$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 7 & 8 \end{pmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 7 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y+5z \\ 4x+7y+8z \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 4 & 8 \\ 9 & 2 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} 1 & 3 \\ 4 & 8 \\ 9 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ 4x+8y \\ 9x+2y \end{pmatrix}$$

Matrices are functions

2x2

$$\begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 4x+7y \end{pmatrix}$$

1x3

$$(4,3,5) : \mathbb{R}^3 \longrightarrow \mathbb{R}^1$$

$$(4,3,5)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4x+3y+5z$$

2x3

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 7 & 8 \end{pmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 4 & 7 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y+5z \\ 4x+7y+8z \end{pmatrix}$$

3x2

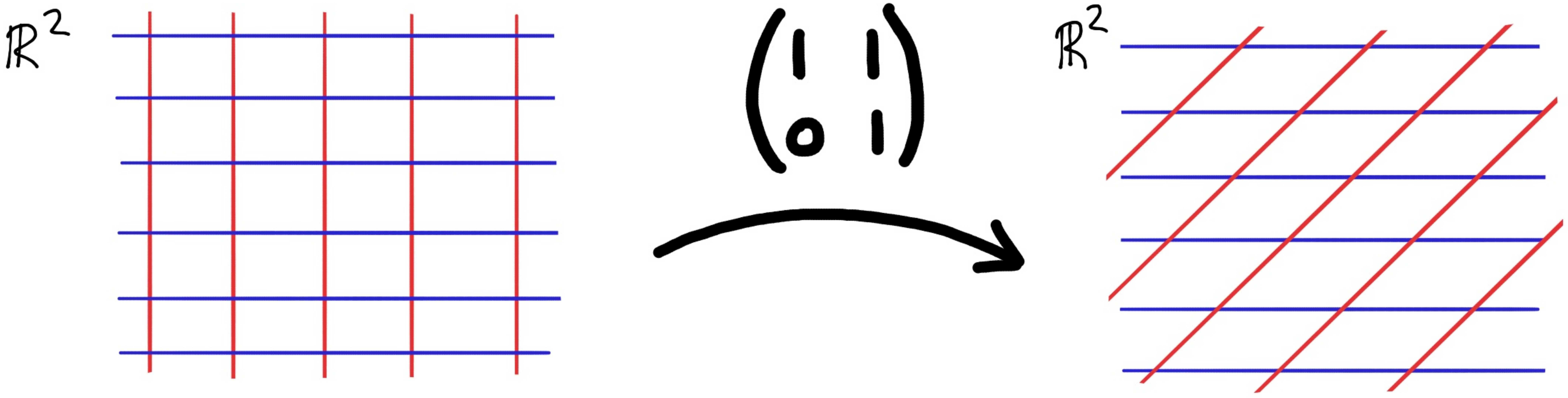
$$\begin{pmatrix} 1 & 3 \\ 4 & 8 \\ 9 & 2 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} 1 & 3 \\ 4 & 8 \\ 9 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ 4x+8y \\ 9x+2y \end{pmatrix}$$

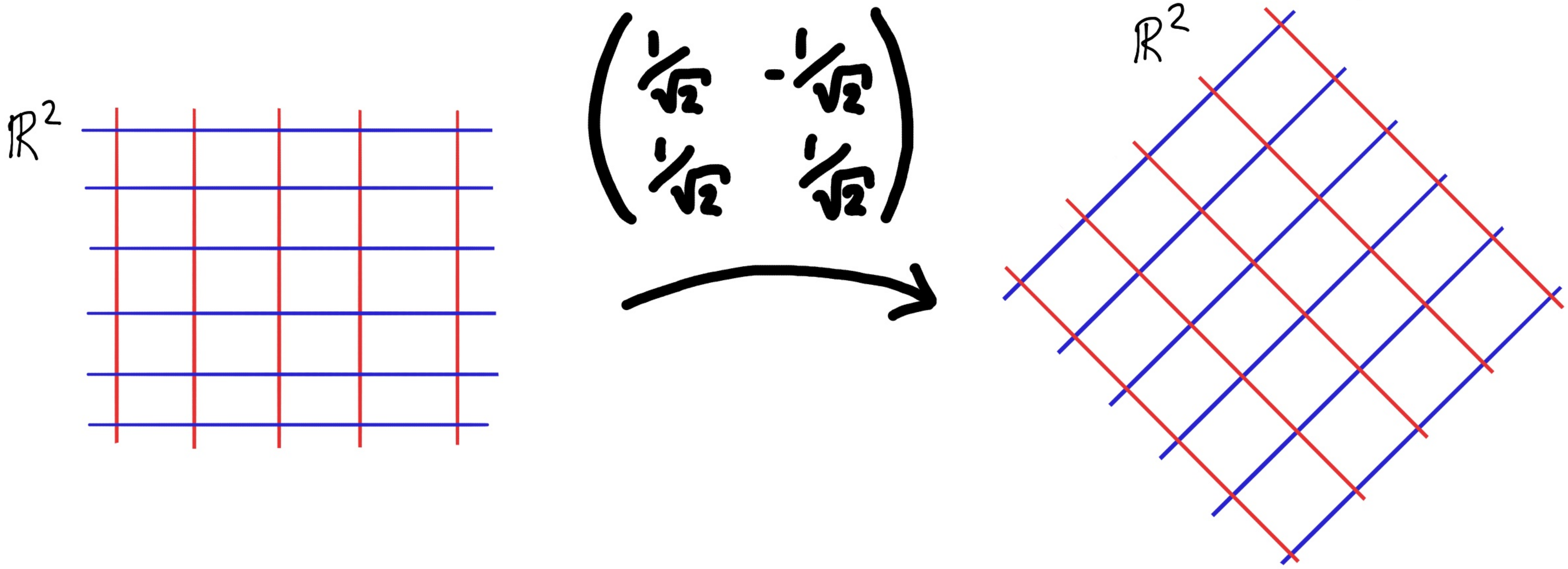
A k×n matrix is a function $\mathbb{R}^n \to \mathbb{R}^k$

A $k \times n$ matrix is a "straight" function $\mathbb{R}^n \to \mathbb{R}^k$

A k×n matrix is a "straight" function $\mathbb{R}^n \to \mathbb{R}^k$



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Matrix multiplication is function composition

$$\begin{tikzcd}\mathbb{R}^2 \arrow[r, bend left, "\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}"] & \mathbb{R}^2 \arrow[r, bend left, "\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}"] & \mathbb{R}^2\end{tikzcd}$$

Matrix multiplication is function composition

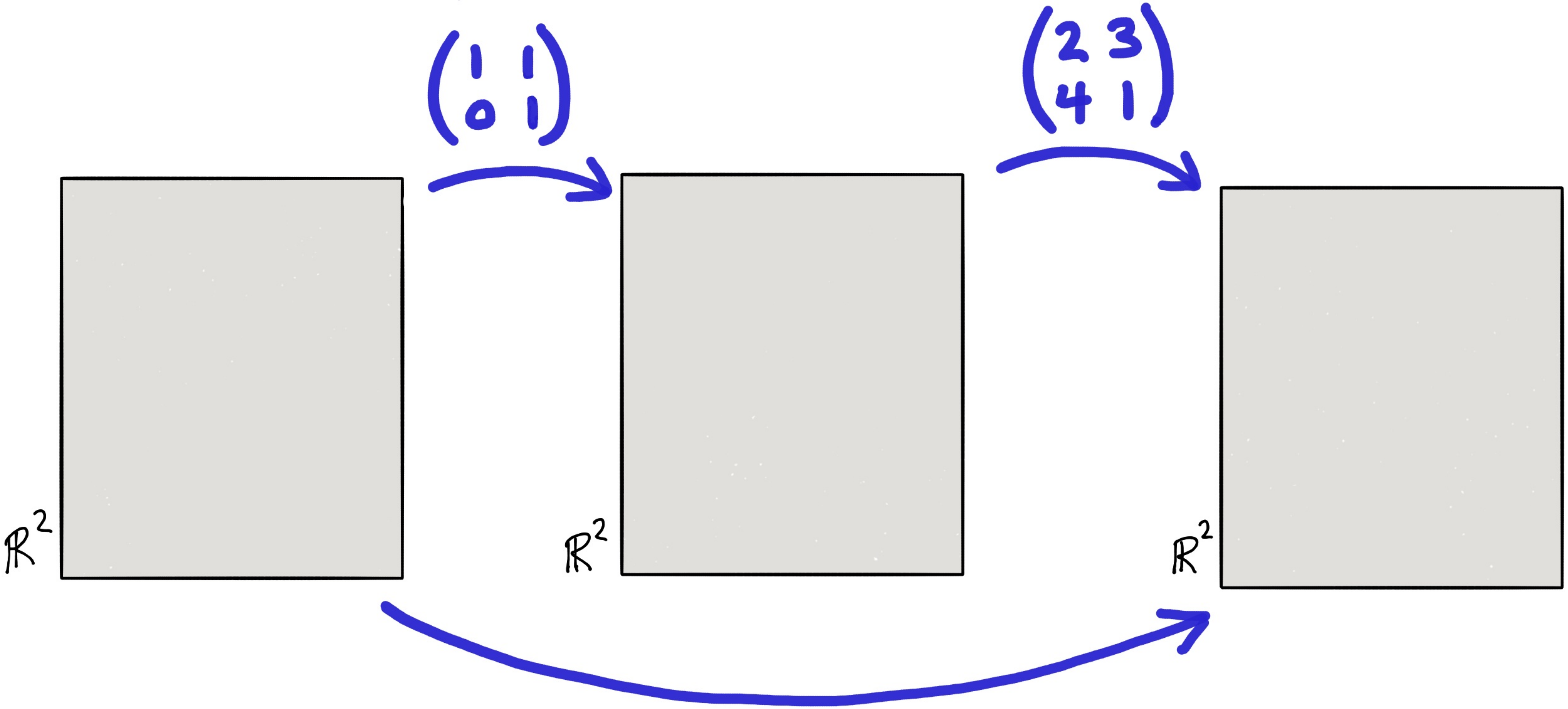
$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\begin{pmatrix}1&1\\0&1\end{pmatrix}"] \arrow[rr, bend right, "\begin{pmatrix}2&3\\4&1\end{pmatrix}\circ\begin{pmatrix}1&1\\0&1\end{pmatrix}"'] & \mathbb{R}^2 \arrow[r, "\begin{pmatrix}2&3\\4&1\end{pmatrix}"] & \mathbb{R}^2
\end{tikzcd}$$

Matrix multiplication is function composition

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}"] \arrow[rr, bend right] & \mathbb{R}^2 \arrow[r, "\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}"] & \mathbb{R}^2
\end{tikzcd}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Matrix multiplication is function composition



$$\begin{pmatrix}2&3\\4&1\end{pmatrix}\circ\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}2&3\\4&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}2&5\\4&5\end{pmatrix}$$

Ⅱ Jacobian matrices

$$f : \underset{x\ y\ z}{\mathbb{R}^3} \longrightarrow \underset{w}{\mathbb{R}} \qquad f(x, y, z) = w$$

Jacobian matrix:

$$D_p f = \left(\frac{\partial w}{\partial x}\bigg|_p ,\ \frac{\partial w}{\partial y}\bigg|_p ,\ \frac{\partial w}{\partial z}\bigg|_p \right)$$

$$f : \underset{x\,y\,z}{\mathbb{R}^3} \longrightarrow \underset{w}{\mathbb{R}} \qquad f(x, y, z) = w$$

Jacobian matrix:

output variable

↓

$$D_p f = \begin{array}{cccc} w & \left(\frac{\partial w}{\partial x}\Big|_p , \right. & \frac{\partial w}{\partial y}\Big|_p , & \left. \frac{\partial w}{\partial z}\Big|_p \right) \\ & x & y & z \end{array}$$

input variables ⟶

$$\underset{x\ y\ z}{f:\mathbb{R}^3} \longrightarrow \underset{w}{\mathbb{R}} \qquad f(x, y, z) = w$$

Jacobian matrix:

output variable

$$D_p f = \begin{array}{cccc} w & \bigg(\frac{\partial w}{\partial x}\Big|_p, & \frac{\partial w}{\partial y}\Big|_p, & \frac{\partial w}{\partial z}\Big|_p\bigg) \\ & x & y & z \end{array}$$

input variables ⟶

$f:\underset{t}{\mathbb{R}}\longrightarrow\underset{x\;y\;z}{\mathbb{R}^3}$ $\qquad$ $f(t)=(x,y,z)$

Jacobian matrix:

output variables ↓

$$D_pf = \begin{array}{cc} \begin{array}{c} x\\[4ex] y\\[4ex] z\end{array} & \begin{pmatrix} \left.\dfrac{\partial x}{\partial t}\right|_p \\[2ex] \left.\dfrac{\partial y}{\partial t}\right|_p \\[2ex] \left.\dfrac{\partial z}{\partial t}\right|_p \end{pmatrix} \\ & t \end{array}$$

input variable →

$$\underset{t}{f:\mathbb{R}} \longrightarrow \underset{x\ y\ z}{\mathbb{R}^3} \qquad f(t) = (x, y, z)$$

Jacobian matrix:

output variables ↓

$$D_p f = \begin{array}{cc} x & \\ y & \\ z & \\ & t \end{array} \!\!\!\! \begin{array}{c} \\ \end{array}$$

$$D_p f = \begin{array}{c c} x & \left. \frac{\partial x}{\partial t} \right|_p \\ y & \left. \frac{\partial y}{\partial t} \right|_p \\ z & \left. \frac{\partial z}{\partial t} \right|_p \\ & t \end{array}$$

input variable →

$$f:\underset{t}{\mathbb{R}} \longrightarrow \underset{x\;y\;z}{\mathbb{R}^3} \qquad f(t)=(x,y,z)$$

Jacobian matrix:

output variables ↓

$$D_pf = \begin{array}{c} x \\ \\ y \\ \\ z \\ \\ \end{array} \begin{pmatrix} \left.\frac{dx}{dt}\right|_p \\ \left.\frac{dy}{dt}\right|_p \\ \left.\frac{dz}{dt}\right|_p \end{pmatrix}$$
$$\qquad\qquad\qquad\quad t$$

input variable ⟶

$$f : \underset{s\ \ t}{\mathbb{R}^2} \longrightarrow \underset{x\ \ y}{\mathbb{R}^2} \qquad f(s,t) = (x,y)$$

Jacobian matrix:

output variables ↓

$$D_p f = \begin{array}{cc} \begin{array}{c} x \\[2em] y \end{array} & \begin{pmatrix} \left.\dfrac{\partial}{\partial}\right|_p & \left.\dfrac{\partial}{\partial}\right|_p \\[2em] \left.\dfrac{\partial}{\partial}\right|_p & \left.\dfrac{\partial}{\partial}\right|_p \end{pmatrix} \\ & \begin{array}{cc} s & \qquad t \end{array} \end{array}$$

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Jacobian matrix:

output variables ↓

$$D_p f = \begin{array}{cc} & \begin{array}{cc} \quad s \quad & \quad t \quad \end{array} \\ \begin{array}{c} x \\[2em] y \end{array} & \begin{pmatrix} \left.\dfrac{\partial x}{\partial s}\right|_p & \left.\dfrac{\partial x}{\partial t}\right|_p \\[1.5em] \left.\dfrac{\partial y}{\partial s}\right|_p & \left.\dfrac{\partial y}{\partial t}\right|_p \end{pmatrix} \end{array}$$

input variables ⟶

$$f: \underset{s\ \ t}{\mathbb{R}^2} \longrightarrow \underset{x\ \ y}{\mathbb{R}^2} \qquad f(s,t)=(x,y)$$

Jacobian matrix:

output variables ↓

$$D_p f = \begin{array}{c@{\;}c} \begin{array}{c} x \\[5ex] y \end{array} & \begin{pmatrix} \left.\dfrac{\partial x}{\partial s}\right|_p & \left.\dfrac{\partial x}{\partial t}\right|_p \\[4ex] \left.\dfrac{\partial y}{\partial s}\right|_p & \left.\dfrac{\partial y}{\partial t}\right|_p \end{pmatrix} \\ & \begin{array}{cc} s & \qquad t \end{array} \end{array}$$

input variables ⟶

Example: If $f(x,y) = (2xy, xy^3)$, then

$D_{(1,3)}f =$

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$$D_{(1,3)}f = \begin{pmatrix} \left.\frac{\partial\, 2xy}{\partial x}\right|_{(1,3)} & \left.\frac{\partial\, 2xy}{\partial y}\right|_{(1,3)} \\ \left.\frac{\partial\, xy^3}{\partial x}\right|_{(1,3)} & \left.\frac{\partial\, xy^3}{\partial y}\right|_{(1,3)} \end{pmatrix}$$

Example: If $f(x,y) = (2xy, xy^3)$, then

$$D_{(1,3)}f = \begin{pmatrix} \left.\frac{\partial 2xy}{\partial x}\right|_{(1,3)} & \left.\frac{\partial 2xy}{\partial y}\right|_{(1,3)} \\ \left.\frac{\partial xy^3}{\partial x}\right|_{(1,3)} & \left.\frac{\partial xy^3}{\partial y}\right|_{(1,3)} \end{pmatrix}$$

$$= \begin{pmatrix} 2y\big|_{(1,3)} & 2x\big|_{(1,3)} \\ y^3\big|_{(1,3)} & 3xy^2\big|_{(1,3)} \end{pmatrix}$$

Example: If $f(x,y) = (2xy, xy^3)$, then

$$D_{(1,3)} f = \begin{pmatrix} \left.\frac{\partial 2xy}{\partial x}\right|_{(1,3)} & \left.\frac{\partial 2xy}{\partial y}\right|_{(1,3)} \\ \left.\frac{\partial xy^3}{\partial x}\right|_{(1,3)} & \left.\frac{\partial xy^3}{\partial y}\right|_{(1,3)} \end{pmatrix}$$

$$= \begin{pmatrix} 2y|_{(1,3)} & 2x|_{(1,3)} \\ y^3|_{(1,3)} & 3xy^2|_{(1,3)} \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 2 \\ 27 & 27 \end{pmatrix}$$

$$f:\underset{x\;y\;z}{\mathbb{R}^3}\longrightarrow\underset{u\;w}{\mathbb{R}^2} \qquad f(x,y,z)=(u,w)$$

Jacobian matrix:

output variables ↓

$$D_p f = \begin{array}{cc} \begin{array}{c} u \\[3em] w \end{array} & \begin{pmatrix} \left.\frac{\partial u}{\partial x}\right|_p & \left.\frac{\partial u}{\partial y}\right|_p & \left.\frac{\partial u}{\partial z}\right|_p \\[2em] \left.\frac{\partial w}{\partial x}\right|_p & \left.\frac{\partial w}{\partial y}\right|_p & \left.\frac{\partial w}{\partial z}\right|_p \end{pmatrix} \\ & \begin{array}{ccc} x & \qquad y & \qquad z \end{array} \end{array}$$

input variables →

$f : \underset{t}{\mathbb{R}} \to \underset{z}{\mathbb{R}}$ $\qquad f(t) = z$

Jacobian matrix:

output variable

↓

$$D_p f = \begin{array}{cc} z & \\ & \left.\dfrac{\partial z}{\partial t}\right|_p \\ & t \end{array}$$

input variable ⟶

$$f:\underset{t}{\mathbb{R}} \to \underset{z}{\mathbb{R}} \qquad f(t) = z$$

Jacobian matrix:

output variable

$$D_p f = \begin{array}{cc} \downarrow z & \left.\dfrac{\partial z}{\partial t}\right|_p \\ & t \end{array} = \left.\frac{dz}{dt}\right|_p = f'(p)$$

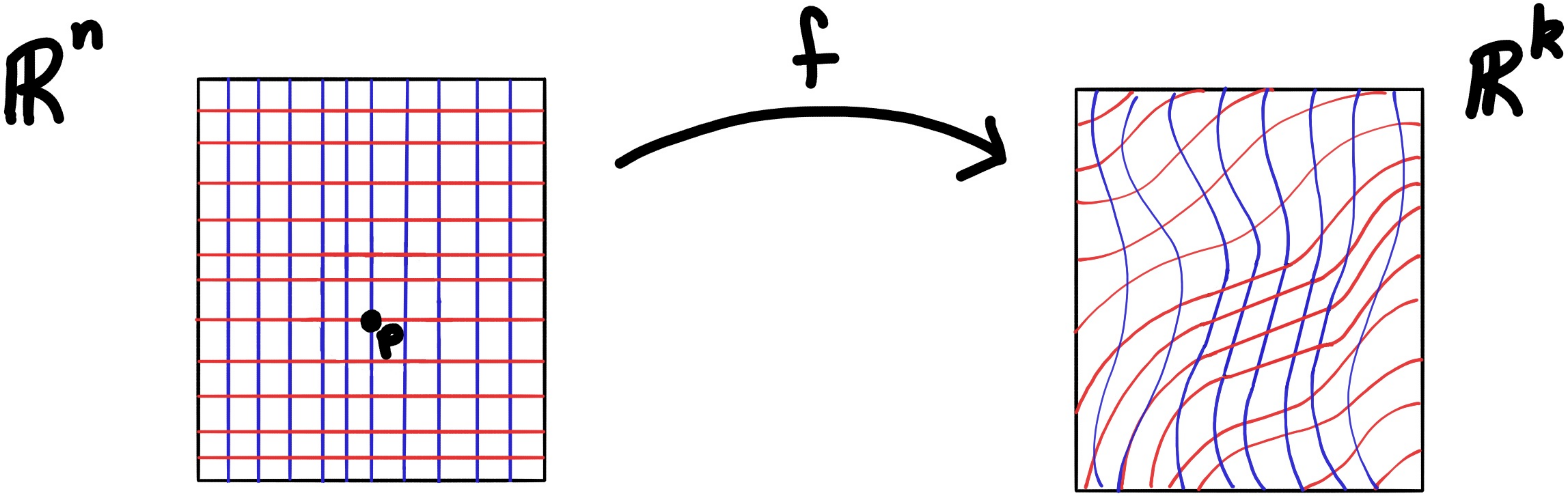
input variable →

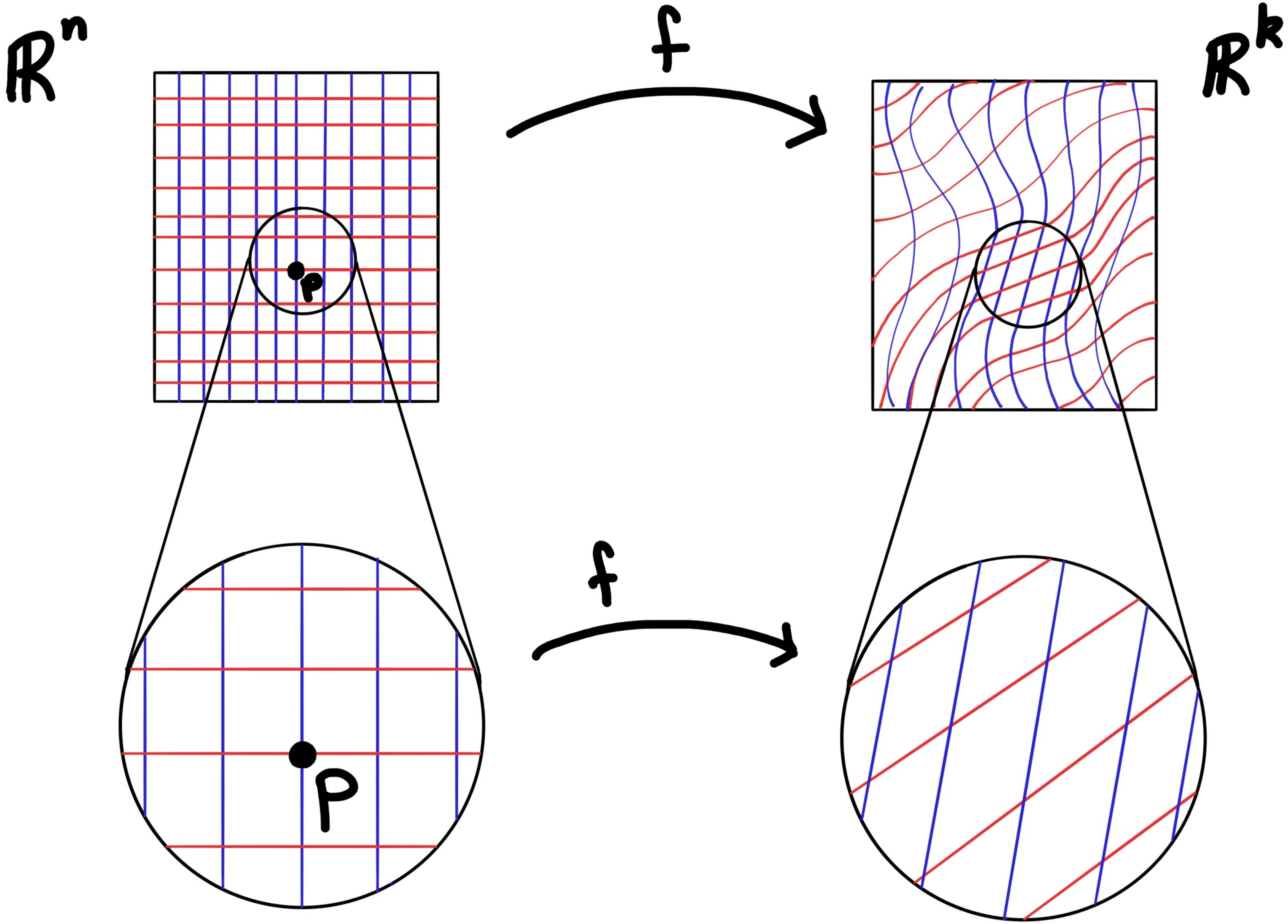
If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^k$,

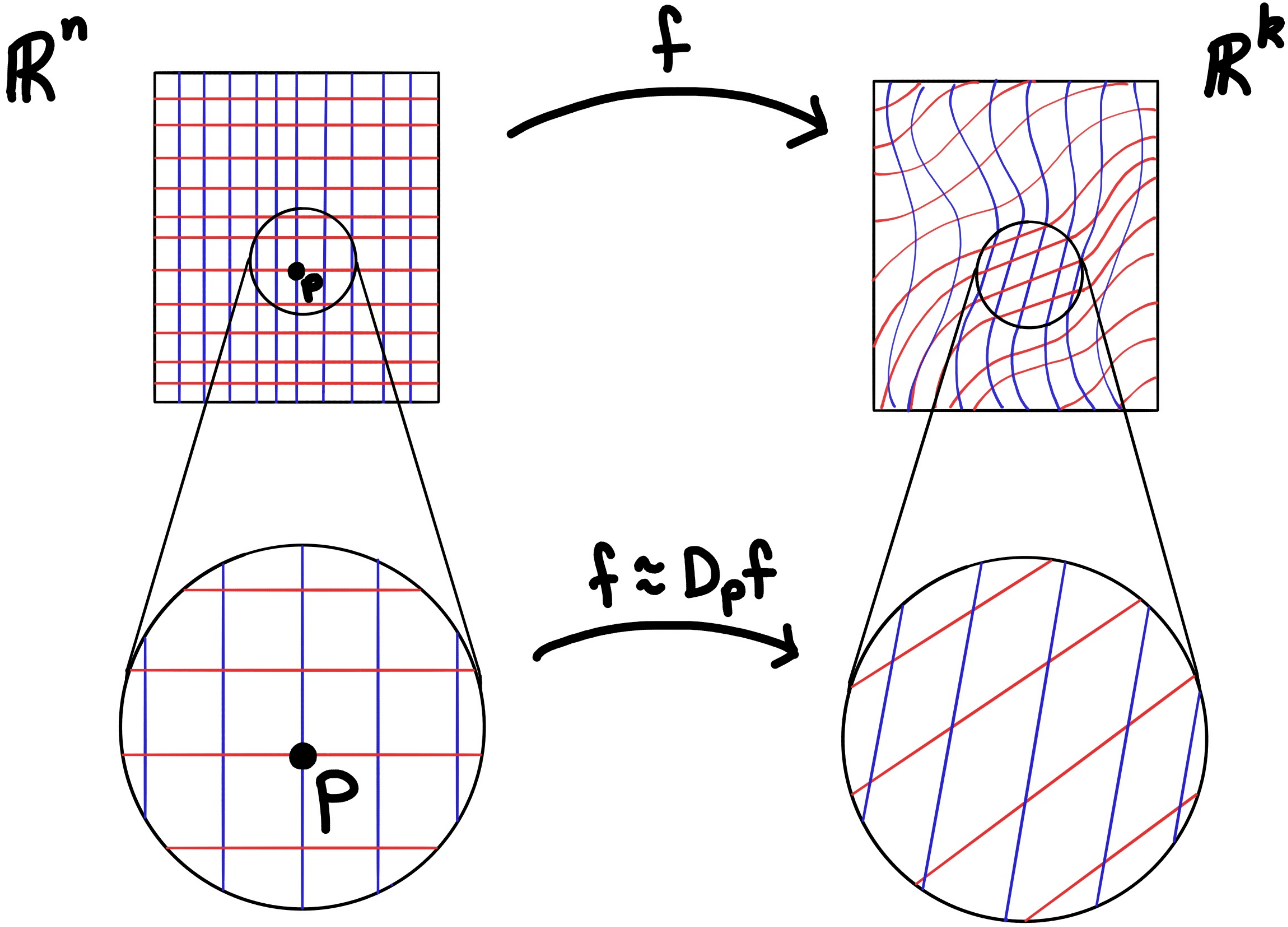
and $p \in \mathbb{R}^n$,

then $D_p f: \mathbb{R}^n \longrightarrow \mathbb{R}^k$

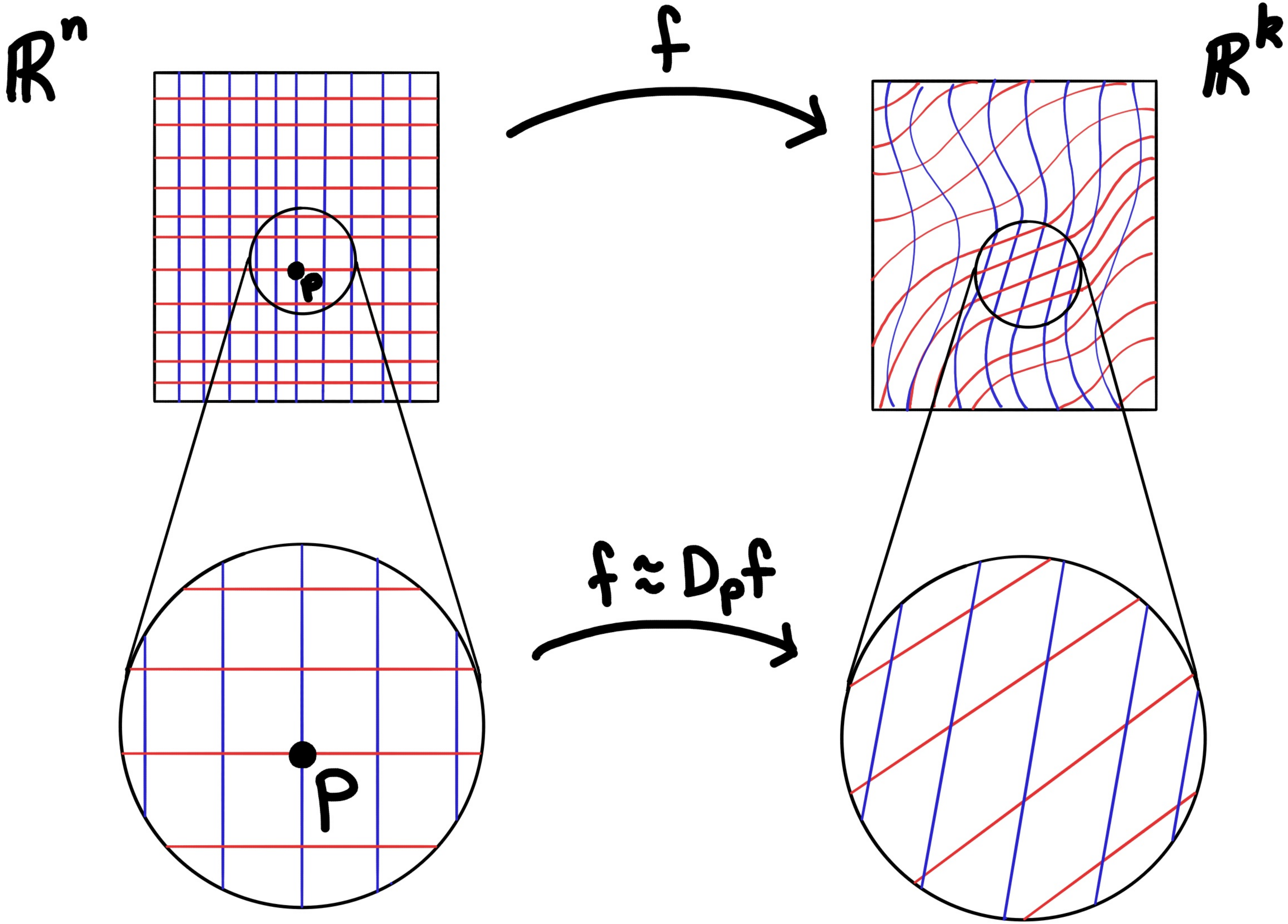
Ⅲ The purpose of Jacobian matrices


$\mathbb{R}^n$
p
f
$\mathbb{R}^k$

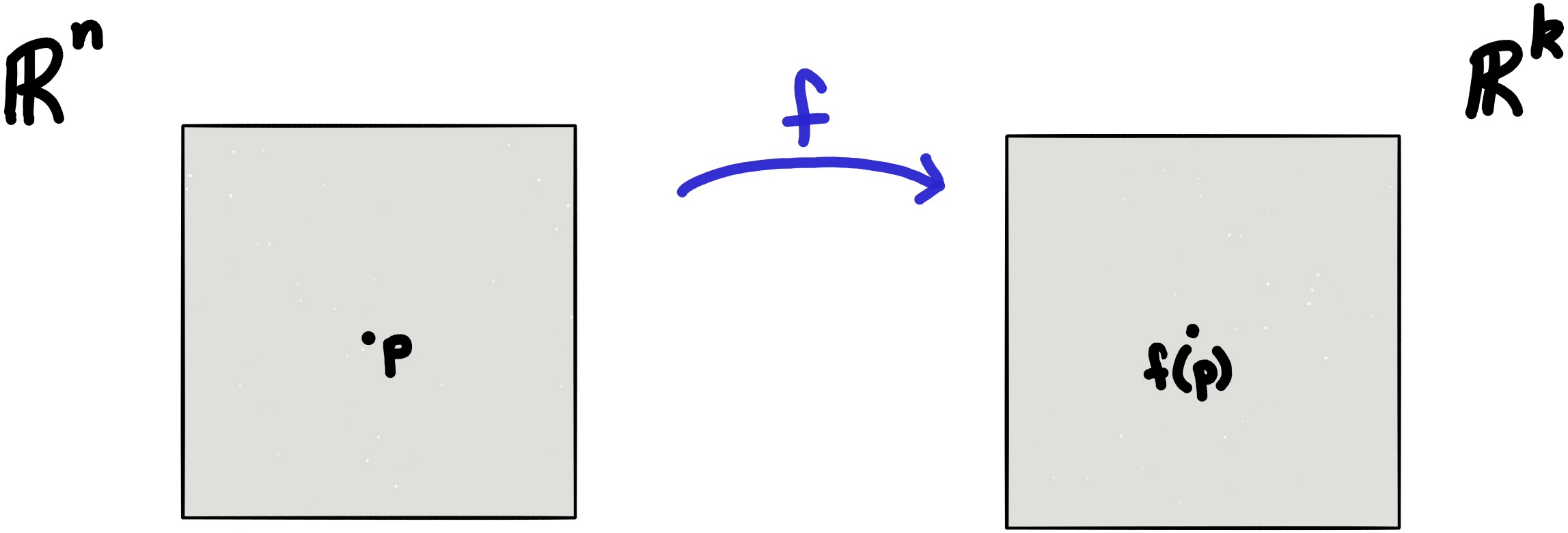

$\mathbb{R}^n$
f
$\mathbb{R}^k$
p
f
p


$\mathbb{R}^n$
f
$\mathbb{R}^k$
p
$f \approx D_p f$
p

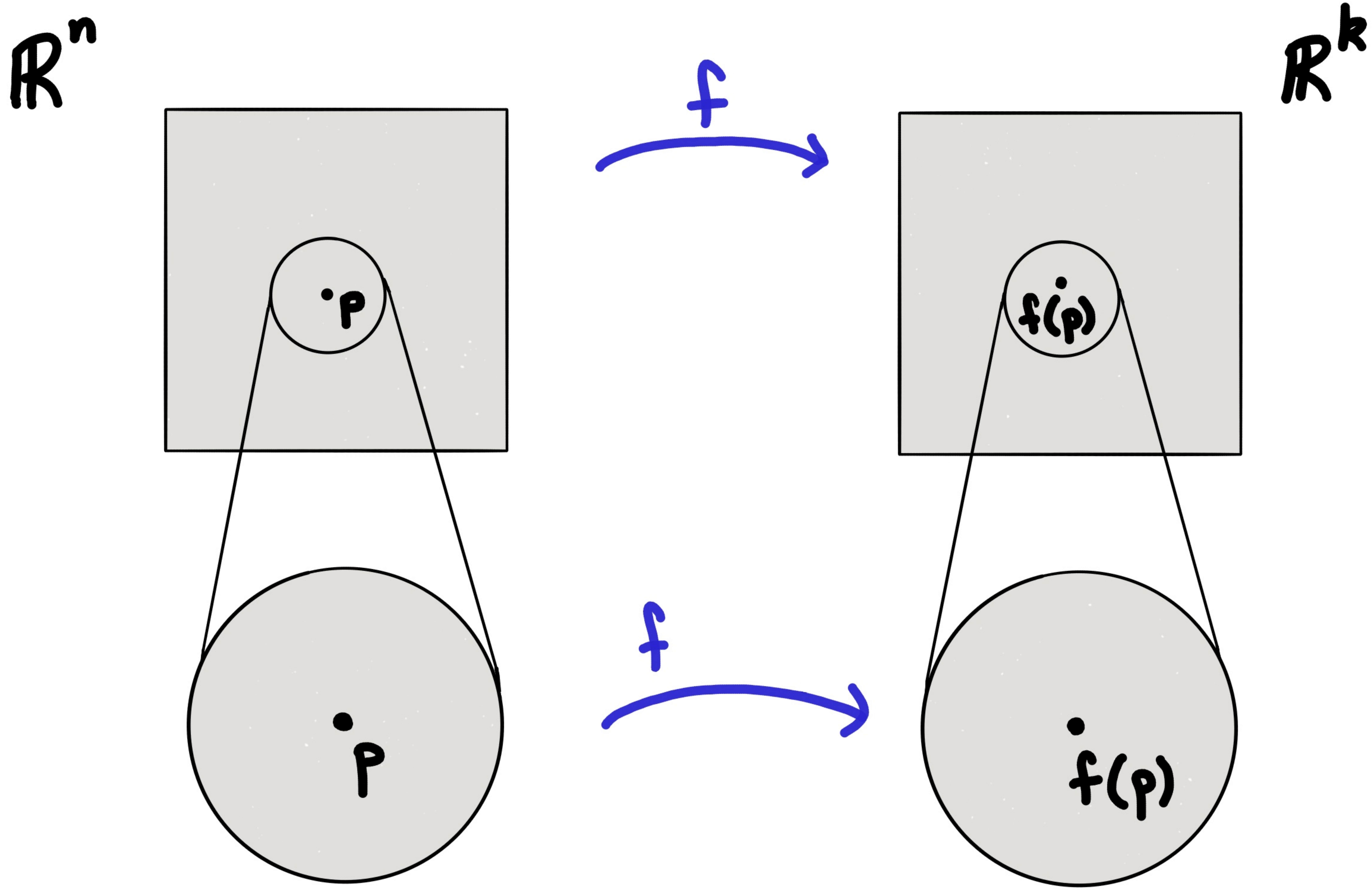
Near p, D_pf approximates f



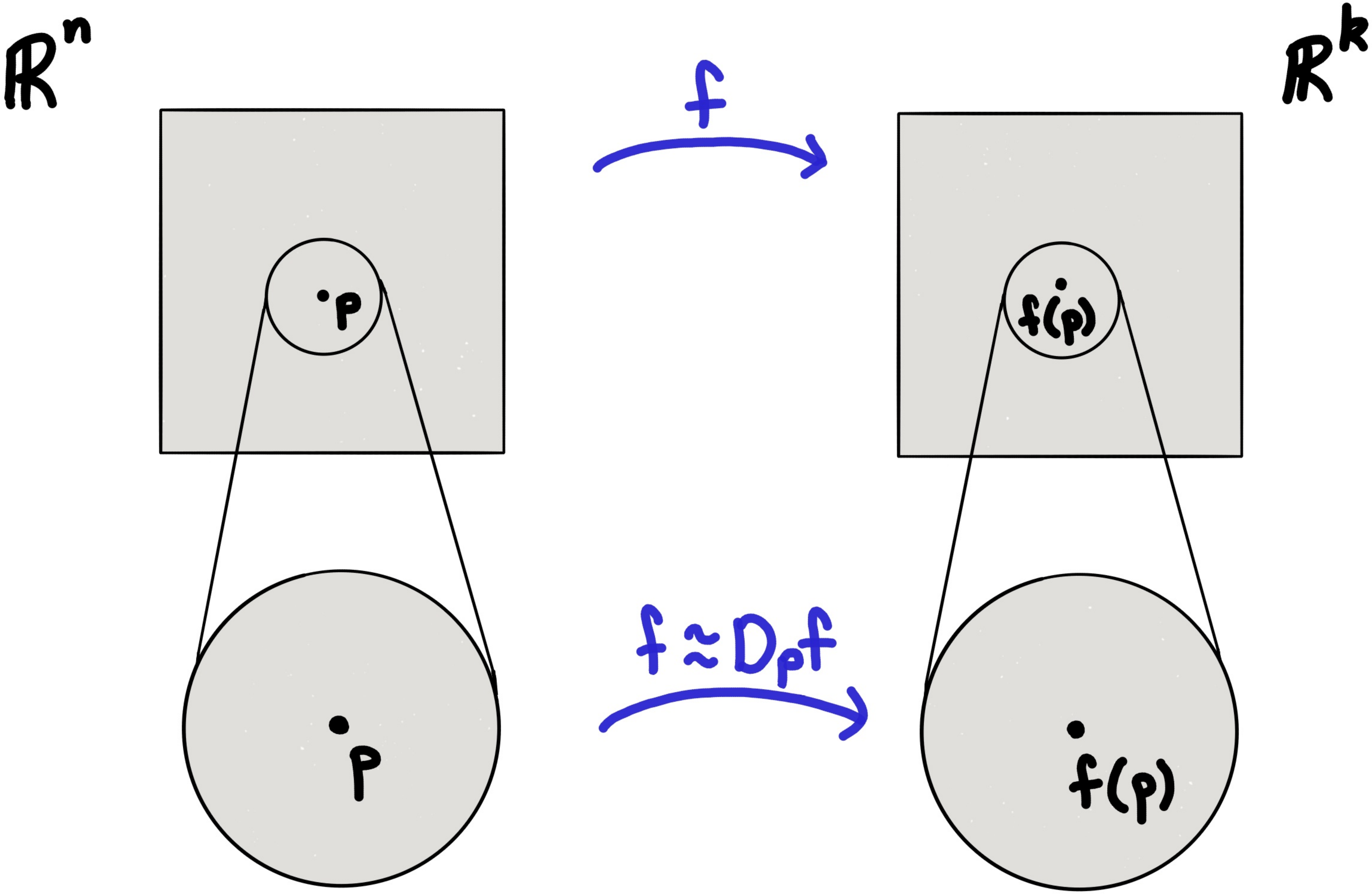
Near p, D_pf approximates f



Near p, D_pf approximates f

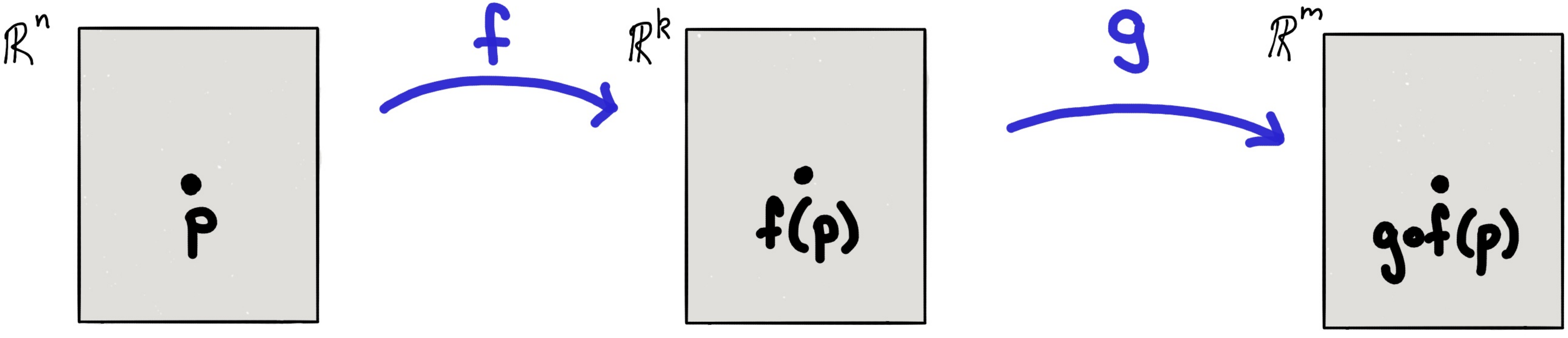


Near p, $D_p f$ approximates f



Ⓘ Chain rule

Chain Rule



Chain Rule

Chain Rule

Chain Rule


$\mathbb{R}^n$
f
$\mathbb{R}^k$
g
$\mathbb{R}^m$
p
$f(p)$
$g \circ f(p)$
p
$f \approx D_p f$
$f(p)$
g
$g \circ f(p)$
$g \circ f$

Chain Rule


$\mathbb{R}^n$
f
$\mathbb{R}^k$
g
$\mathbb{R}^m$
p
f(p)
g∘f(p)
$f \approx D_p f$
$g \approx D_{f(p)} g$
p
f(p)
g∘f(p)
g∘f

Chain Rule


$\mathbb{R}^n$
f
$\mathbb{R}^k$
g
$\mathbb{R}^m$
p
$f(p)$
$g \circ f(p)$
p
$f \approx D_p f$
$f(p)$
$g \approx D_{f(p)} g$
$g \circ f(p)$
$D_p(g \circ f) \approx g \circ f$

Chain Rule

$$D_p(g\circ f) \approx g\circ f \approx$$

Chain Rule

$D_p(g \circ f) \approx g \circ f \approx$

Chain Rule

$$D_p(g\circ f) \approx g\circ f \approx (D_{f(p)}g)\circ(D_pf)$$

Chain Rule

$$D_p(g\circ f) \approx g\circ f \approx (D_{f(p)}g)\circ(D_p f)$$

Chain Rule

$$D_p(g\circ f) \approx g\circ f \approx (D_{f(p)}g)\circ(D_p f) = (D_{f(p)}g)(D_p f)$$

Chain Rule

$$D_p(g\circ f) \approx g\circ f \approx (D_{f(p)}g)\circ(D_p f) = (D_{f(p)}g)(D_p f)$$

Chain Rule

If $f:\mathbb{R}^n \to \mathbb{R}^k$, $g:\mathbb{R}^k \to \mathbb{R}^m$, and $p \in \mathbb{R}^n$,

then

$$\boxed{D_p(g\circ f) = \left(D_{f(p)}g\right)\left(D_p f\right)}$$

Example:

$$(g\circ f)'(p) = g'(f(p))f'(p)$$

Ⓥ The chain rule in practice

$$\begin{array}{ccc} \mathbb{R} & \mathbb{R}^2 & \mathbb{R} \\ t & (x,y) & z \end{array}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\substack{x=x(t)\\ y=y(t)}"] & \mathbb{R}^2 \arrow[r, "z=z(x,y)"] & \mathbb{R} \\
t & (x,y) & z
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\begin{array}{c} x=x(t) \\ y=y(t) \end{array}"] & \mathbb{R}^2 \arrow[r, "z=z(x,y)"] & \mathbb{R} \\
t \arrow[rr, bend right, "z=z(t)"'] & (x,y) & z
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "{\substack{x=x(t)\\ y=y(t)}}"] & \mathbb{R}^2 \arrow[r, "z=z(x,y)"] & \mathbb{R} \\
t \arrow[rr, bend right, "z=z(t)"'] & (x,y) & z
\end{tikzcd}$$

Chain rule:

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\substack{x=x(t)\\ y=y(t)}"] & \mathbb{R}^2 \arrow[r, "z=z(x,y)"] & \mathbb{R} \\
t \arrow[rr, bend right, "z=z(t)"'] & (x,y) & z
\end{tikzcd}$$

Chain rule:

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\begin{array}{ccc} \mathbb{R} & \mathbb{R}^3 & \mathbb{R} \\ t & (x,y,z) & w \end{array}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\substack{x=x(t)\\ y=y(t)\\ z=z(t)}"] & \mathbb{R}^3 \arrow[r, "w=w(x,y,z)"] & \mathbb{R} \\
t & (x,y,z) & w
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "{\begin{matrix} x=x(t) \\ y=y(t) \\ z=z(t) \end{matrix}}"] \arrow[rr, bend right, "{w=w(t)}"'] & \mathbb{R}^3 \arrow[r, "{w=w(x,y,z)}"] & \mathbb{R} \\
t & (x,y,z) & w
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\begin{array}{c} x=x(t) \\ y=y(t) \\ z=z(t) \end{array}"] \arrow[rr, bend right, "w=w(t)"'] & \mathbb{R}^3 \arrow[r, "{w=w(x,y,z)}"] & \mathbb{R} \\
t & (x,y,z) & w
\end{tikzcd}$$

Chain rule:

$$\frac{dw}{dt} = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}$$

$$\begin{tikzcd}
\underset{t}{\mathbb{R}} \arrow[r, "\substack{x=x(t)\\ y=y(t)\\ z=z(t)}"] \arrow[rr, bend right, "w=w(t)"'] & \underset{(x,y,z)}{\mathbb{R}^3} \arrow[r, "w=w(x,y,z)"] & \underset{w}{\mathbb{R}}
\end{tikzcd}$$

Chain rule:

$$\frac{dw}{dt}=\left(\frac{\partial w}{\partial x},\frac{\partial w}{\partial y},\frac{\partial w}{\partial z}\right)\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\\ \frac{dz}{dt}\end{pmatrix}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\begin{array}{ccc} \mathbb{R}^2 & \mathbb{R}^2 & \mathbb{R} \\ (s,t) & (x,y) & z \end{array}$$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\begin{array}{c} x=x(s,t) \\ y=y(s,t) \end{array}"] & \mathbb{R}^2 \arrow[r, "z=z(x,y)"] & \mathbb{R} \\
(s,t) & (x,y) & z
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{(s,t)}{\mathbb{R}^2} \arrow[r, "\begin{array}{c}x=x(s,t)\\ y=y(s,t)\end{array}"] \arrow[rr, bend right, "z=z(s,t)"'] & \underset{(x,y)}{\mathbb{R}^2} \arrow[r, "z=z(x,y)"] & \underset{z}{\mathbb{R}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\substack{x=x(s,t)\\ y=y(s,t)}"] & \mathbb{R}^2 \arrow[r, "z=z(x,y)"] & \mathbb{R} \\
(s,t) \arrow[rr, bend right, "z=z(s,t)"'] & (x,y) & z
\end{tikzcd}$$

Chain rule:

$$\left(\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}\right) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "{\begin{matrix} x=x(s,t) \\ y=y(s,t) \end{matrix}}"] & \mathbb{R}^2 \arrow[r, "{z=z(x,y)}"] & \mathbb{R} \\
(s,t) \arrow[rr, bend right, "{z=z(s,t)}"'] & (x,y) & z
\end{tikzcd}$$

Chain rule:

$$\left(\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}\right) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$

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Suppose $z = xy$, $x = 2t+1$, and $y = t^2$.

Find $\frac{dz}{dt}$ as a function of t using the chain rule.

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$$= 6t^2 + 2t$$

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Suppose $z = xy$, $x = t + s$, and $y = st^2$.

Find $\frac{\partial z}{\partial t}$ as a function of s and t using the chain rule.

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$$= 3st^2 + 2s^2t$$