Thirteen

I Definition of gradient

II Geometry of gradients

III Gradients, level curves, and level surfaces

Ⅰ Definition of gradient

For $f: \mathbb{R}^n \to \mathbb{R}$ and $p \in \mathbb{R}^n$,

$\nabla f(p) \in \mathbb{R}^n$ is a vector in the domain of f,

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For $f: \mathbb{R}^2 \to \mathbb{R}$

$$\nabla f(p) = \left(\frac{\partial f}{\partial x}\Big|_p, \frac{\partial f}{\partial y}\Big|_p \right)$$

For $f: \mathbb{R}^3 \to \mathbb{R}$

$$\nabla f(p) = \left(\frac{\partial f}{\partial x}\Big|_p, \frac{\partial f}{\partial y}\Big|_p, \frac{\partial f}{\partial z}\Big|_p \right)$$

Examples:

1.) Find $\nabla f(2,1)$ if $f(x,y) = xy$.

2.) Find $\nabla g(1,-1,2)$ if $g(x,y,z) = 2xyz^2 - 3x$

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$$\nabla g(1,-1,2) = \left((2yz^2-3)\big|_{(1,-1,2)}, 2xz^2\big|_{(1,-1,2)}, 4xyz\big|_{(1,-1,2)} \right)$$

$$= (-11, 8, -8)$$

$f(x,y)$

$$\left(\frac{\partial f}{\partial x}\Big|_p , \frac{\partial f}{\partial y}\Big|_p \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

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matrix

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$$\boxed{D_p f(\vec{v})}$$

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$$\boxed{\nabla f(p) \cdot \vec{v}}$$

Directional derivatives

$$D_p f(\vec{v}) = \nabla f(p) \cdot \vec{v} = \vec{v} \cdot \nabla f(p)$$

II Geometry of gradients

Proposition:

Suppose $f:\mathbb{R}^n \to \mathbb{R}$, $p \in \mathbb{R}^n$, and $\nabla f(p) \neq \vec{0}$.

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(a) At p, f increases most rapidly (has max. slope) in direction of $\nabla f(p)$.
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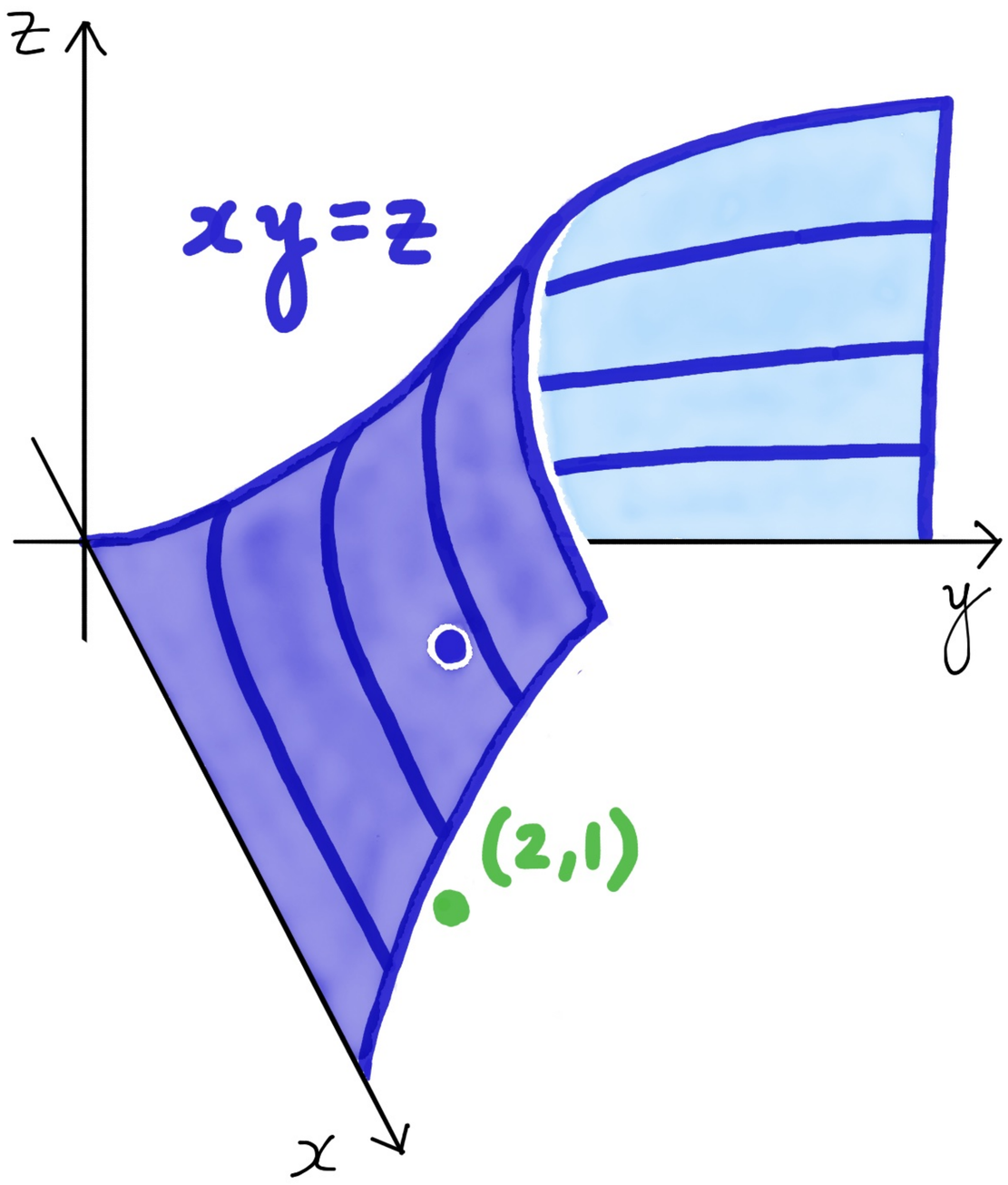
The slope in that direction is $-\|\nabla f(p)\|$.

Example: Steepest positive and negative slopes for $f(x,y) = xy$ at $(2,1)$.



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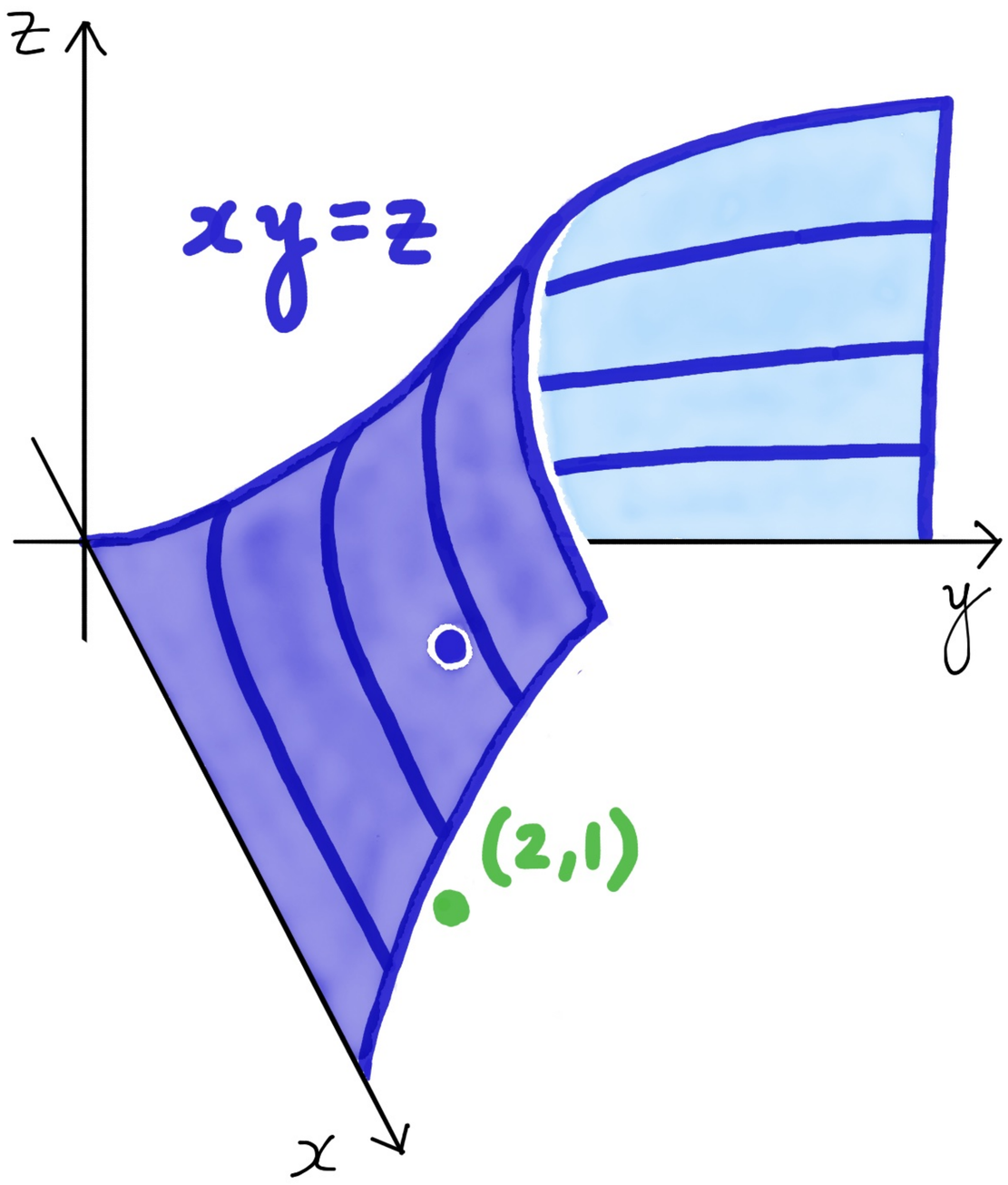

$$\nabla f(2,1) = (1,2)$$

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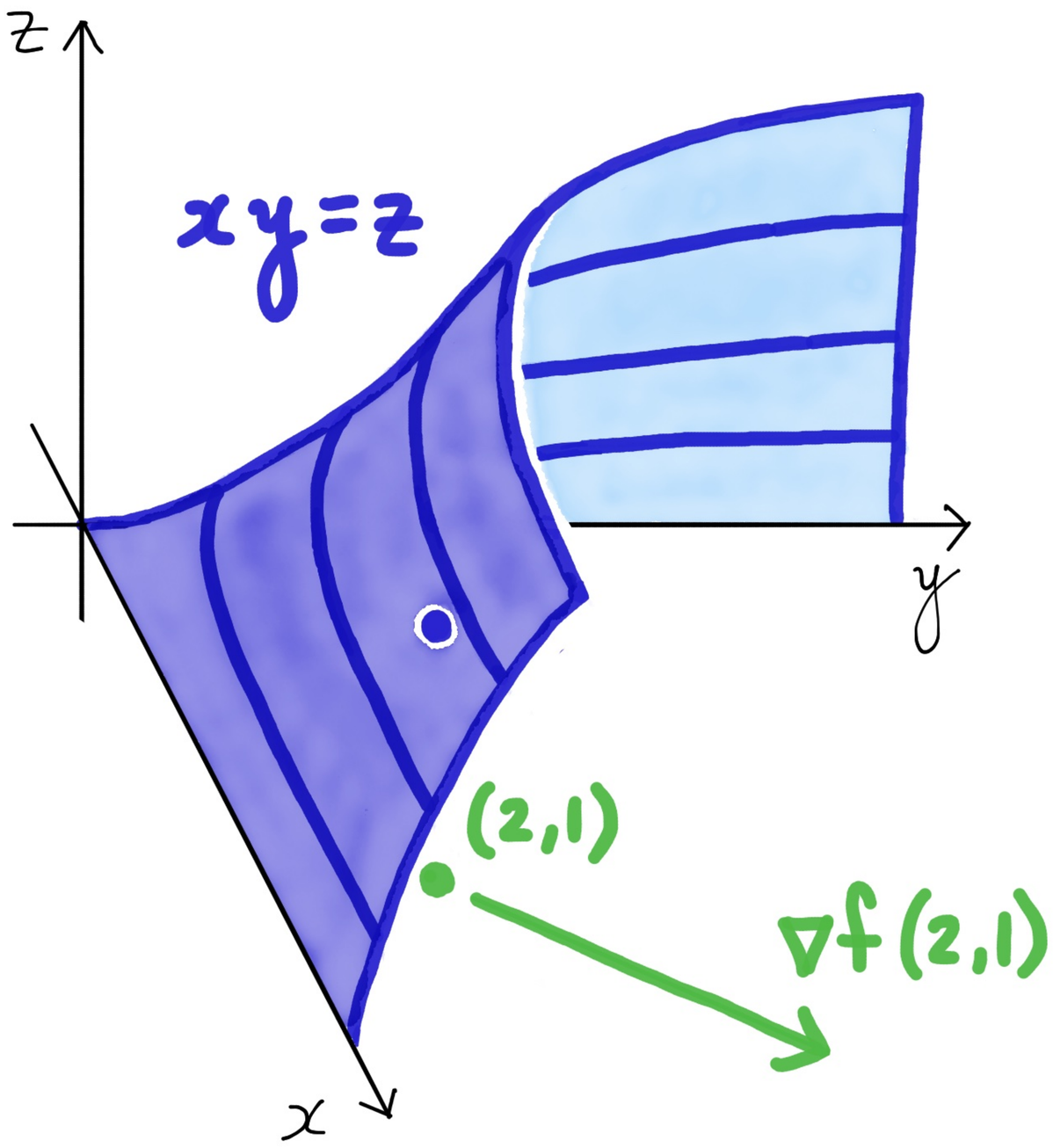

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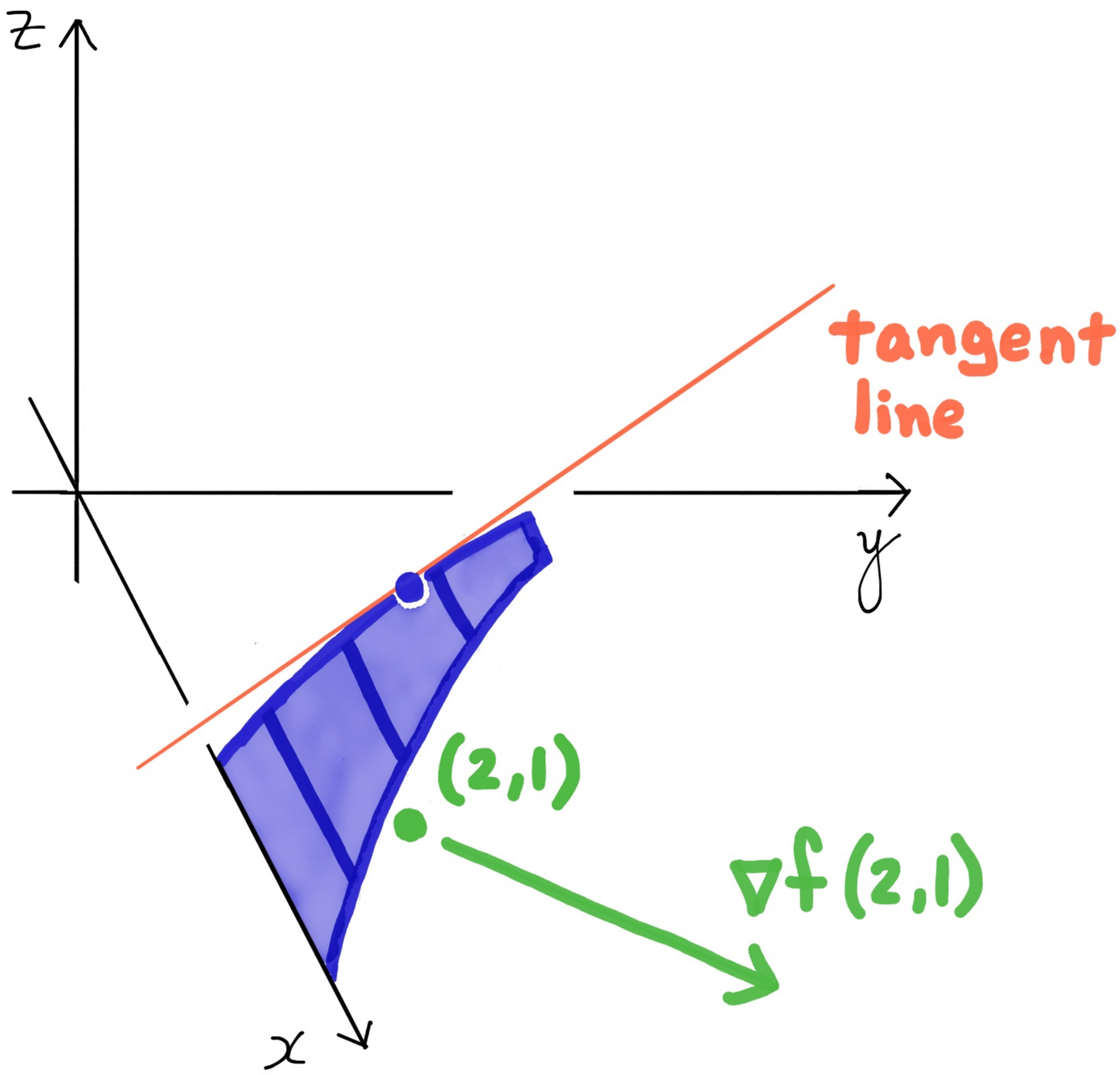

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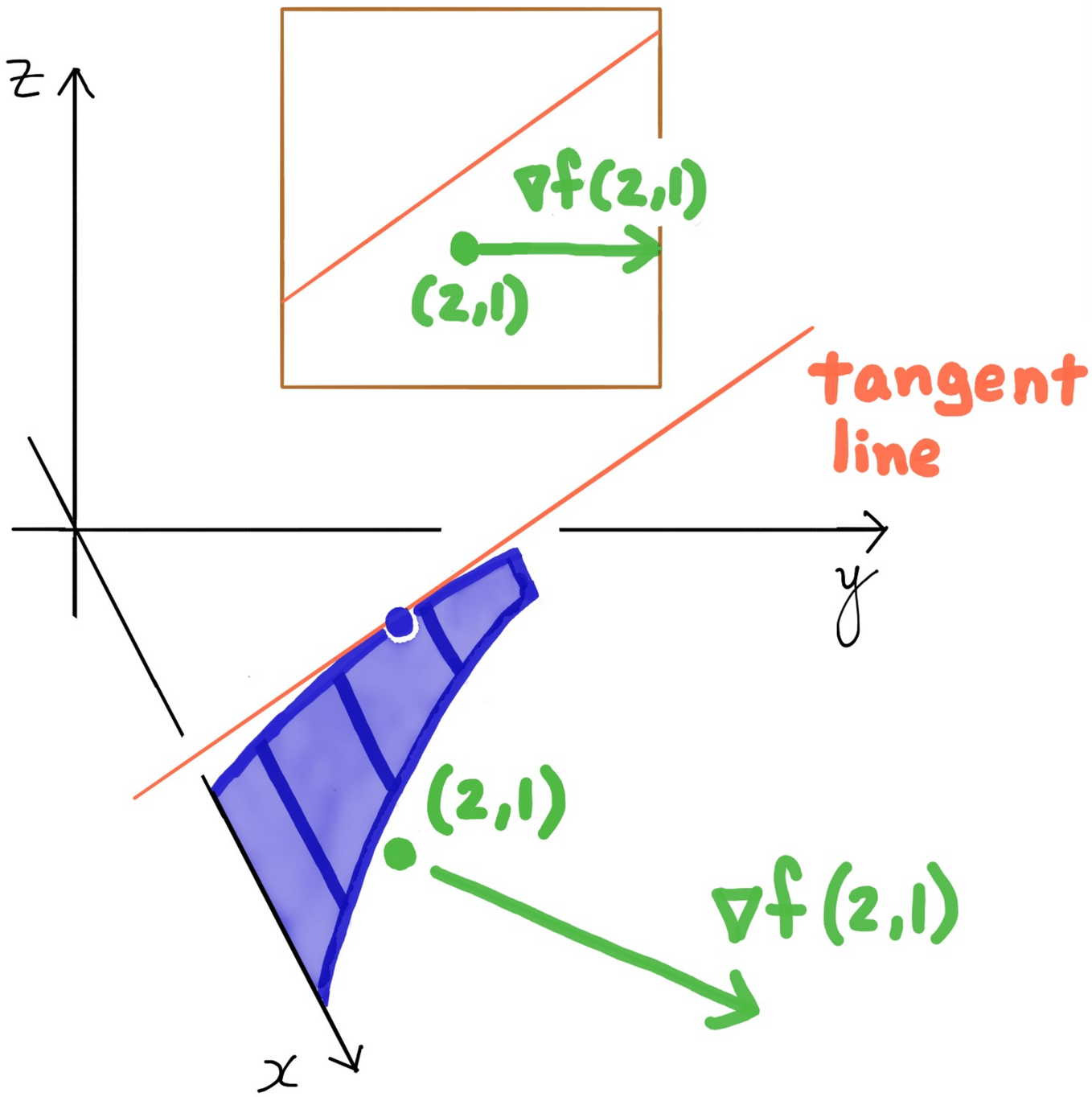

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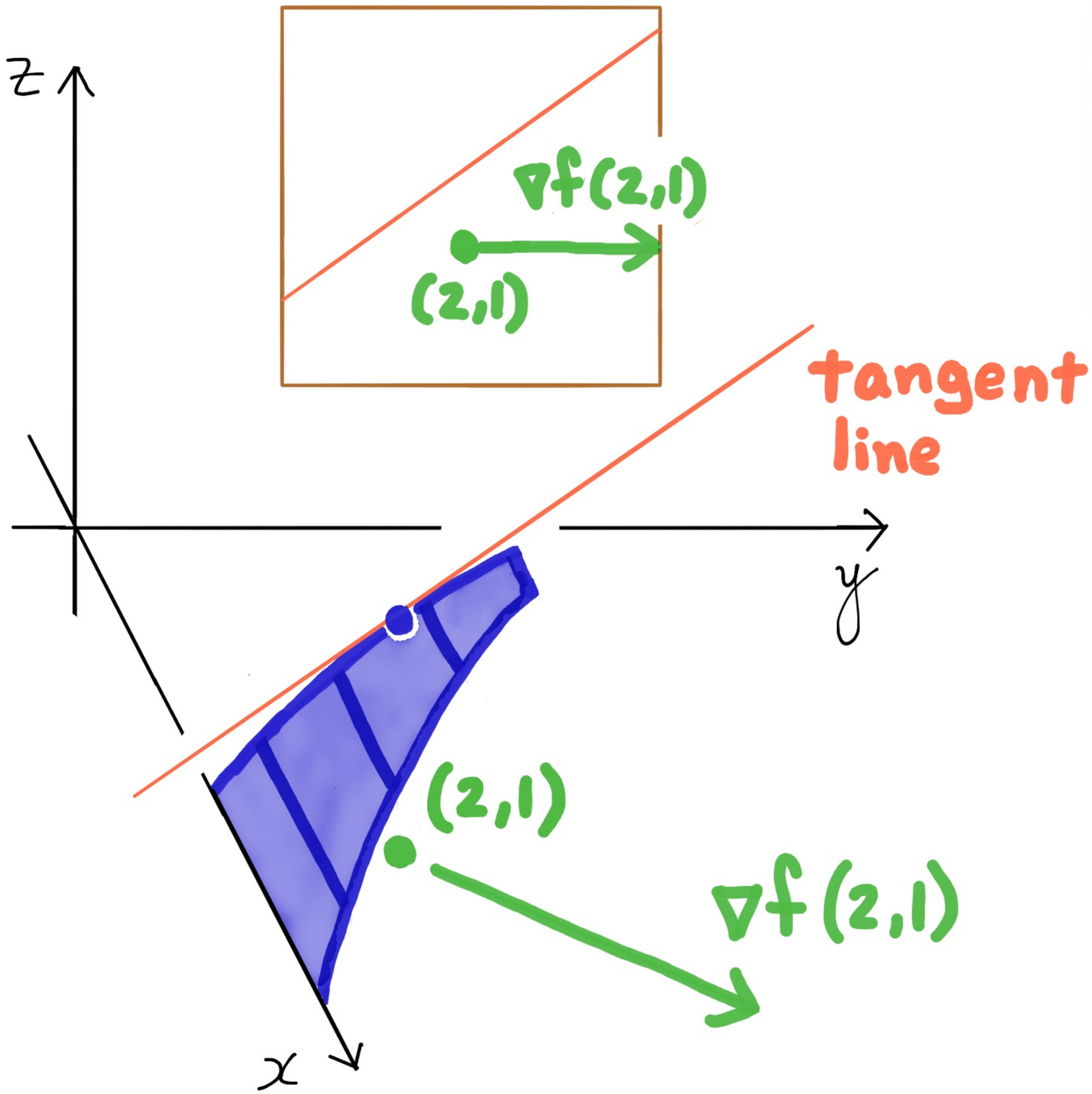

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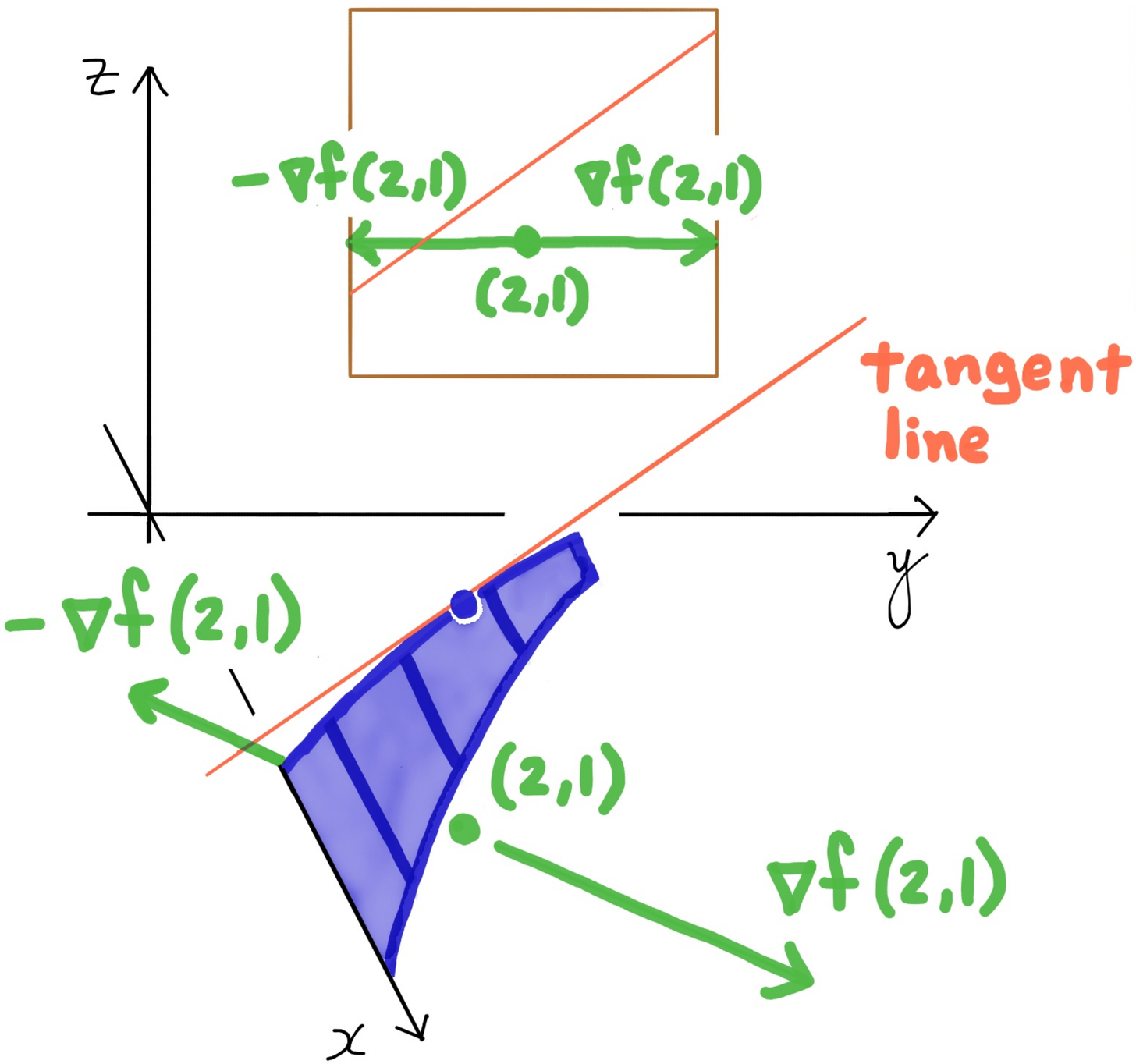

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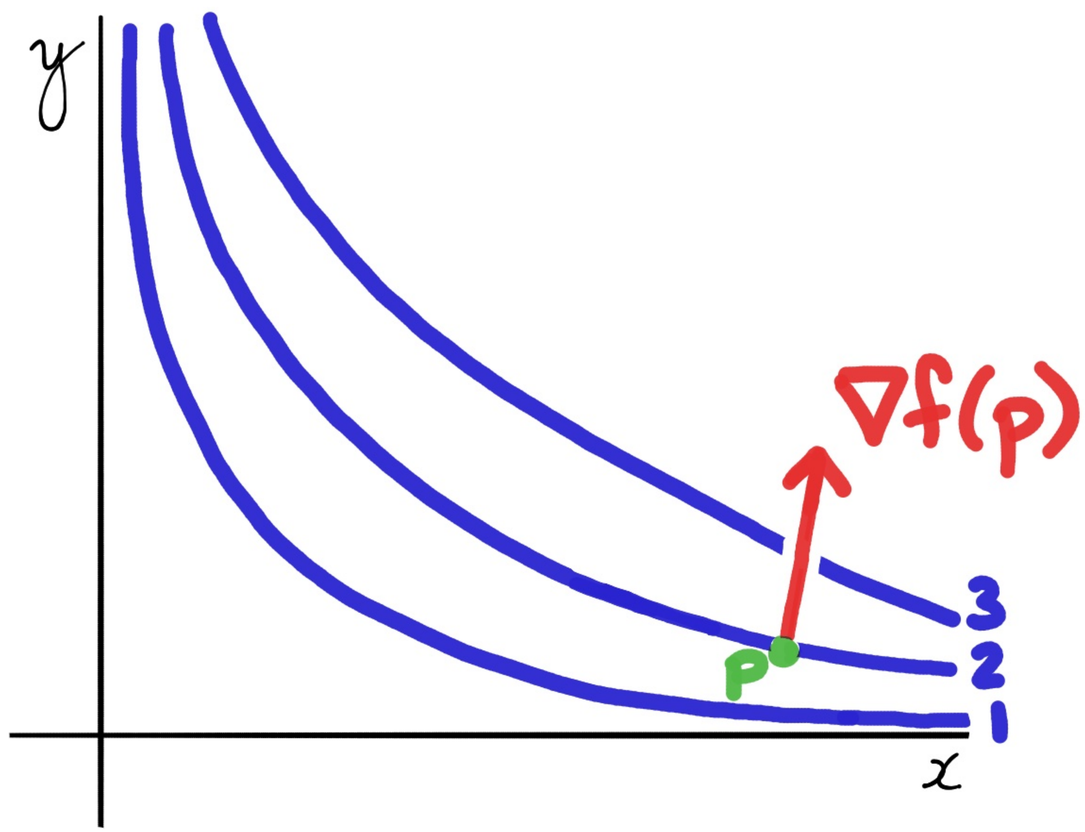
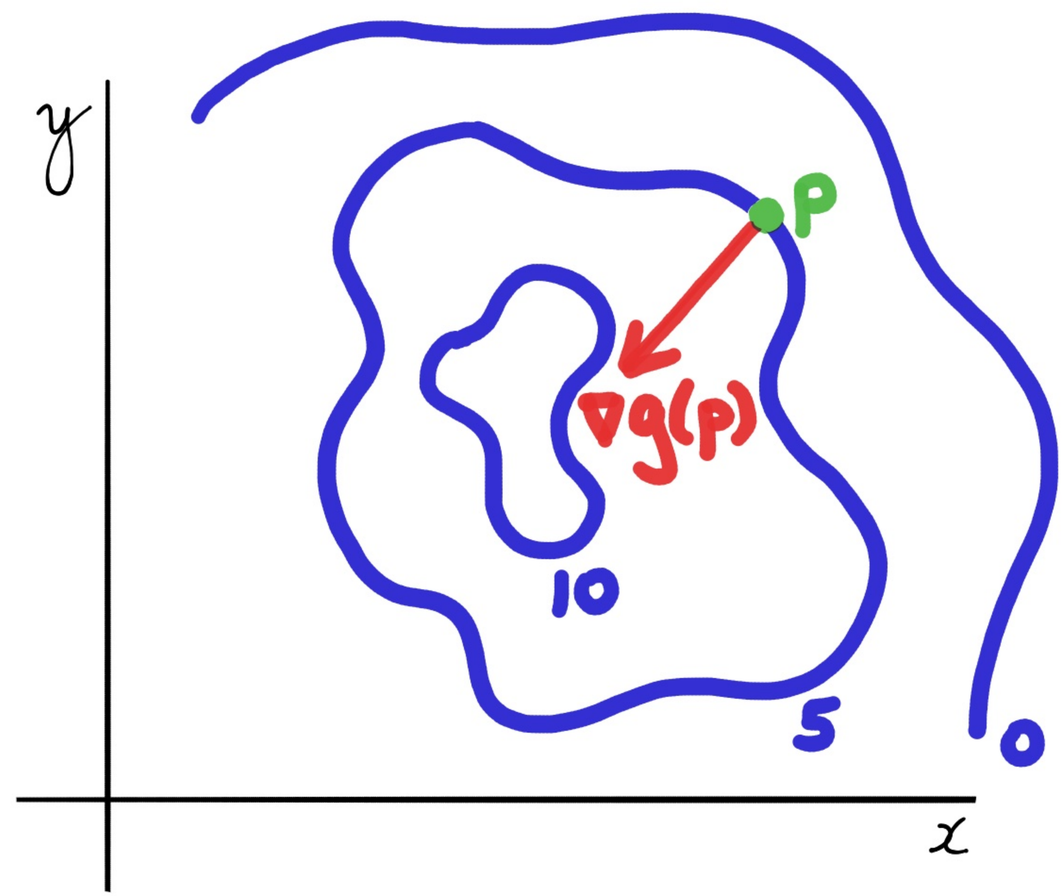
slope in direction of $-\nabla f(2,1)$: $-\sqrt{5}$

gradient points in direction of increasing values

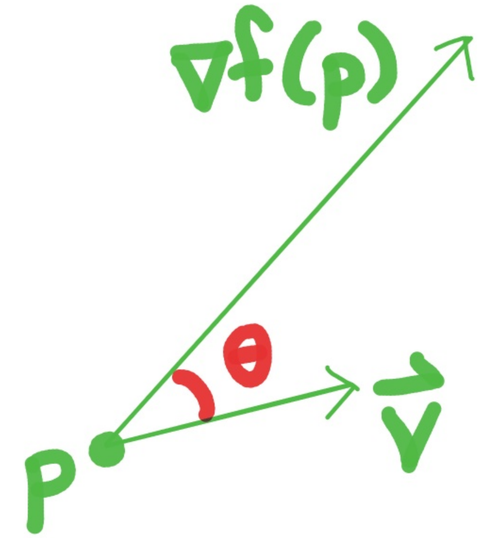


level curves for $f(x,y)=xy$



level curves for $g(x,y)$

Max slope of graph of f at p is in direction of $\nabla f(p)$.



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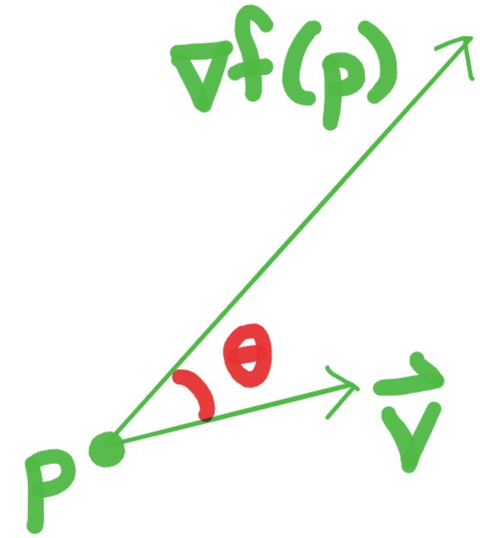

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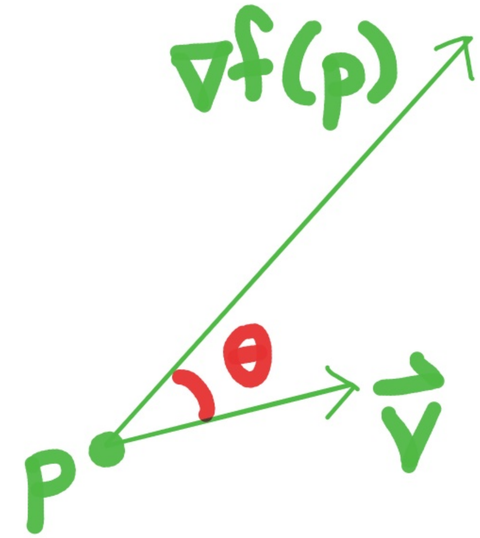

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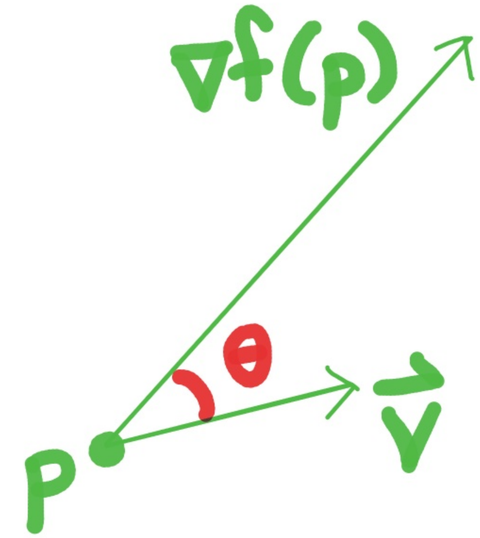

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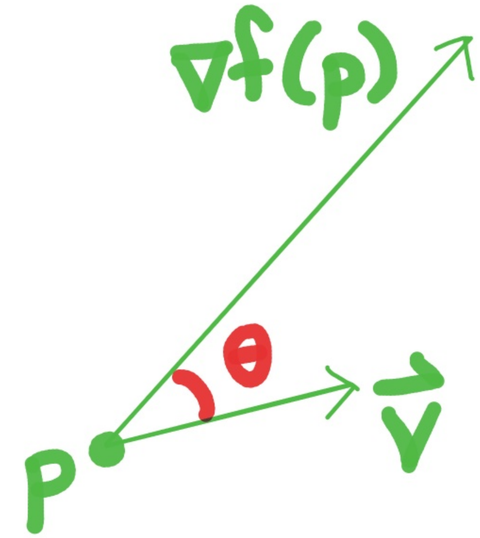

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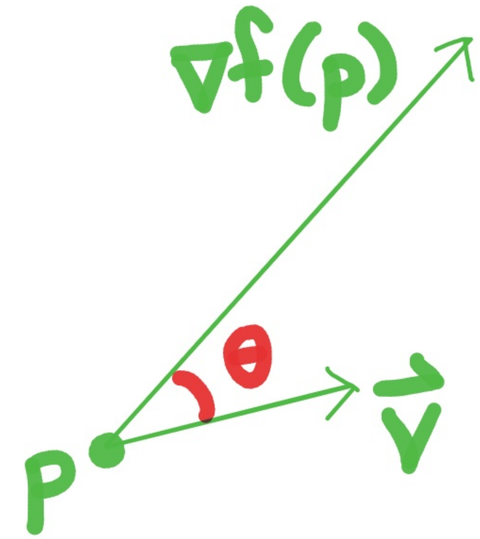

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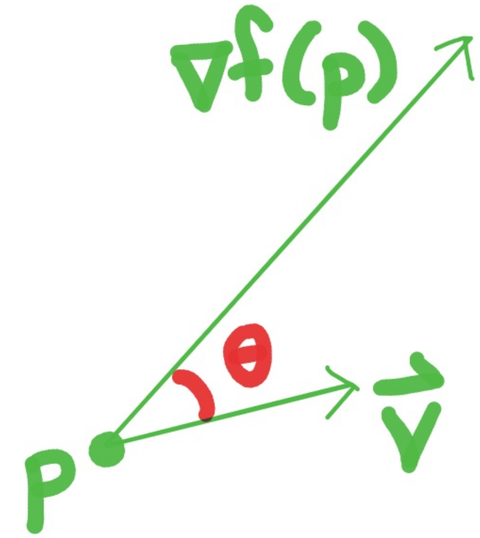

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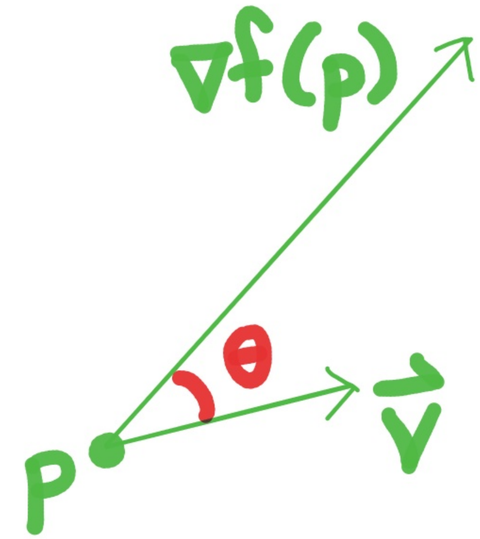

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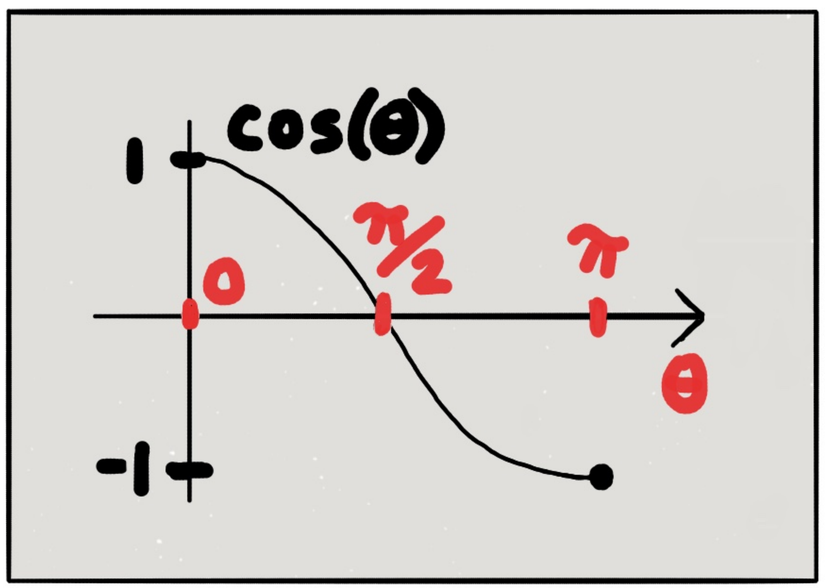

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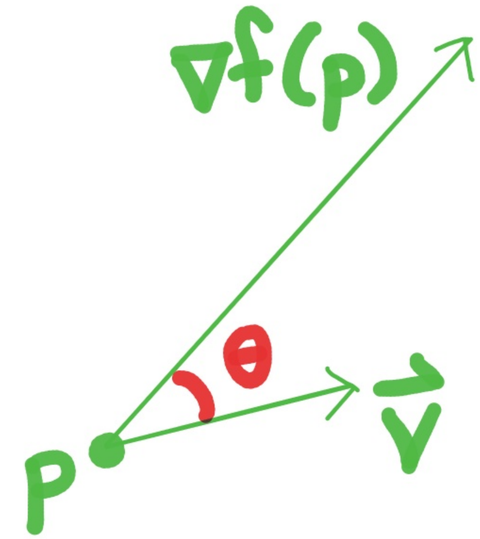

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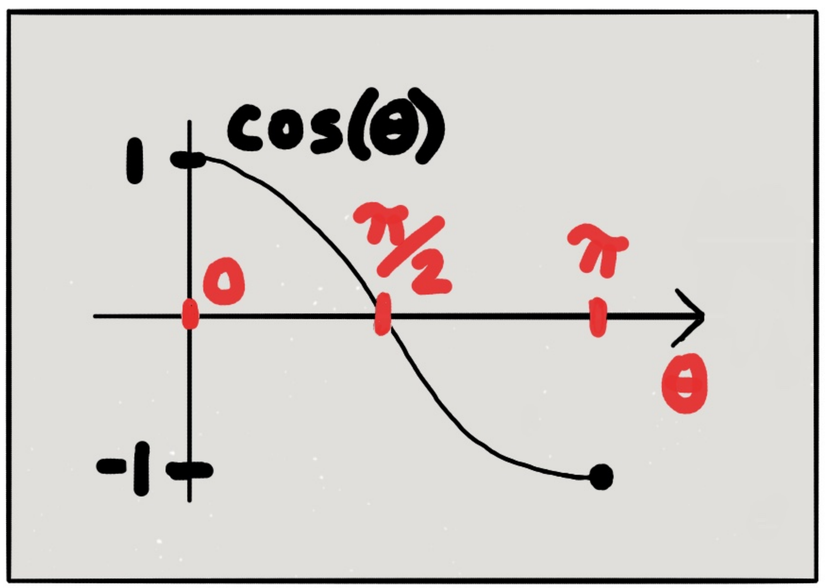

<u>Max slope:</u> $\theta = 0$

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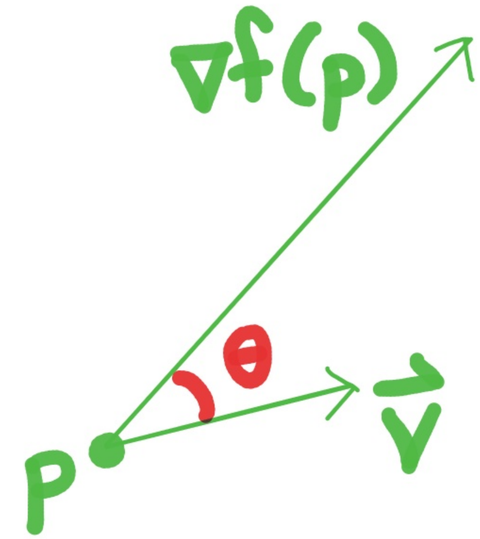

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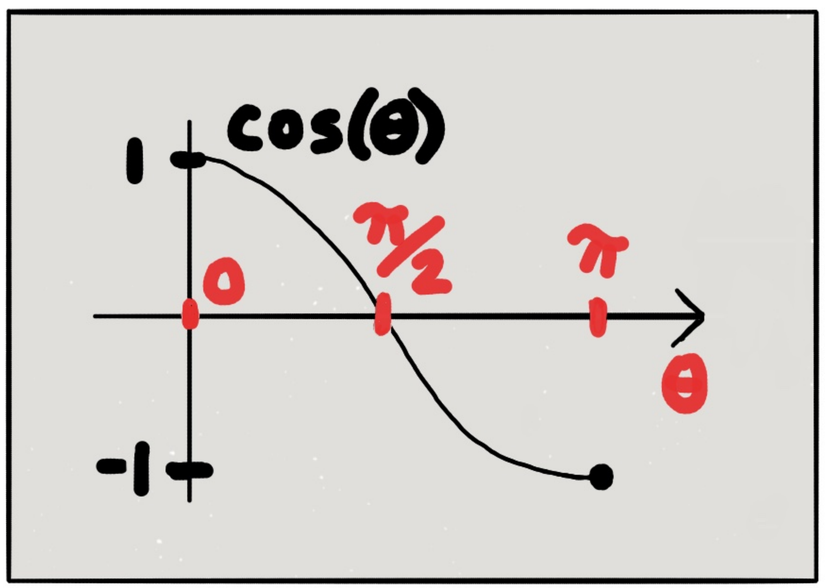

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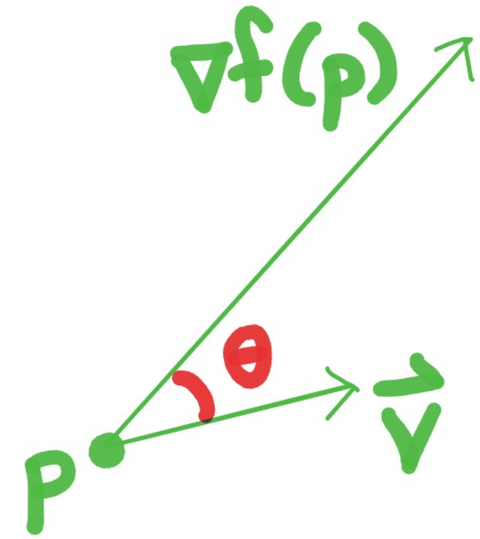

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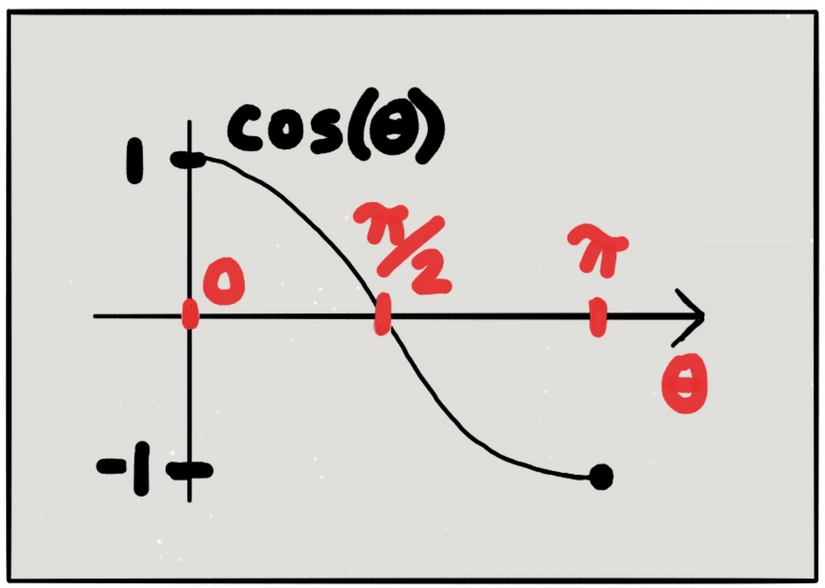

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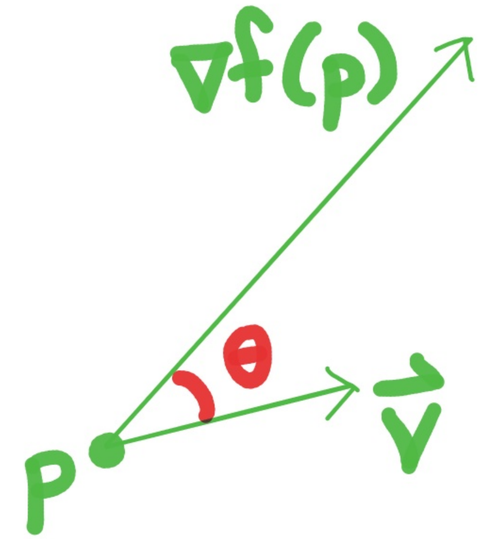

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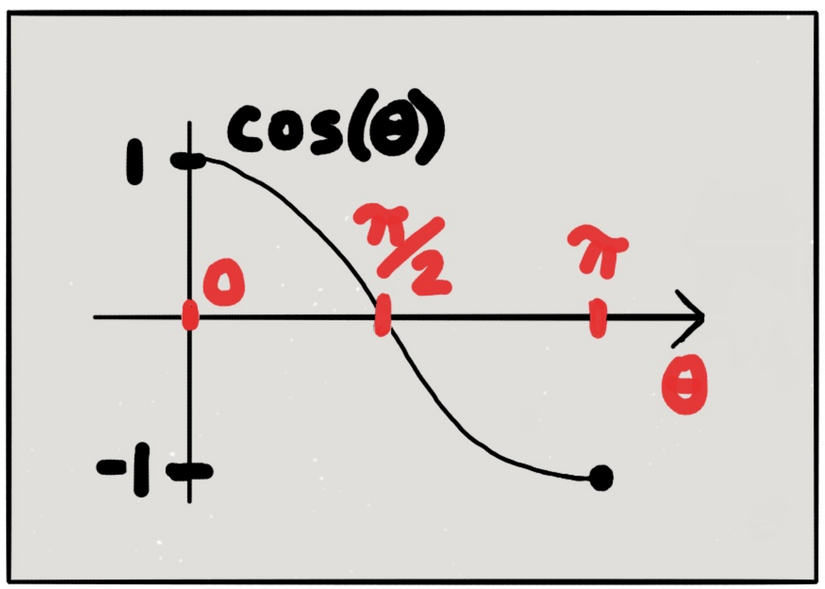

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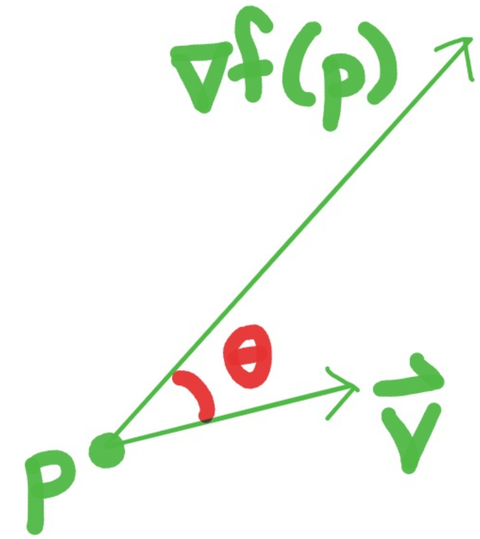
Min slope of graph of f at p is in direction of $\nabla f(p)$.



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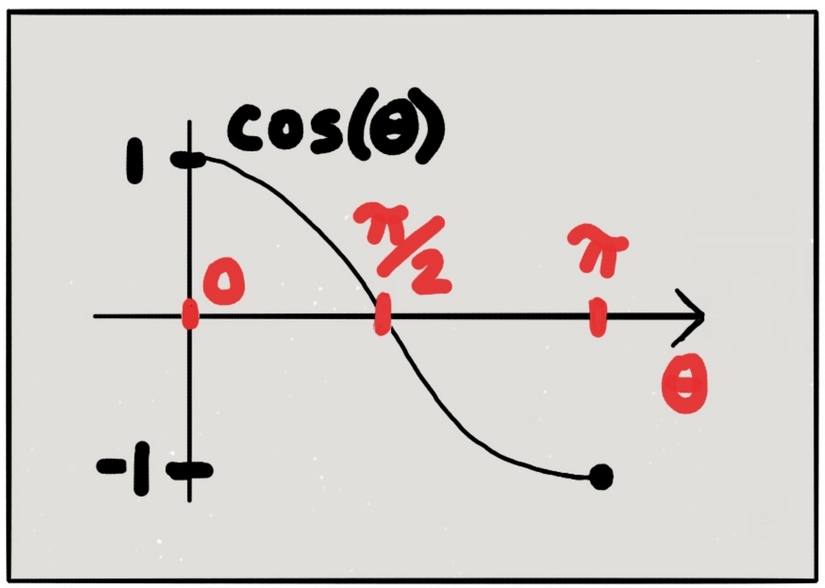

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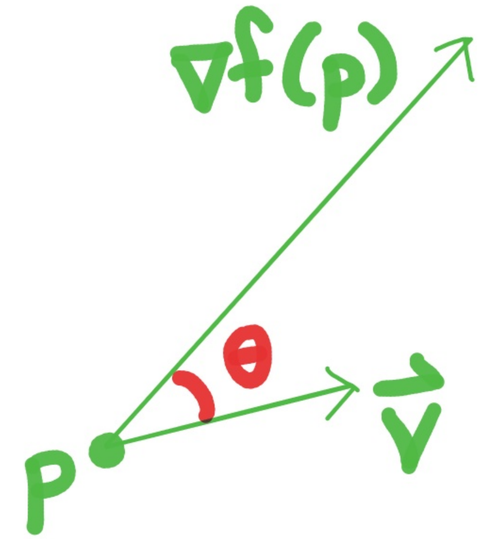

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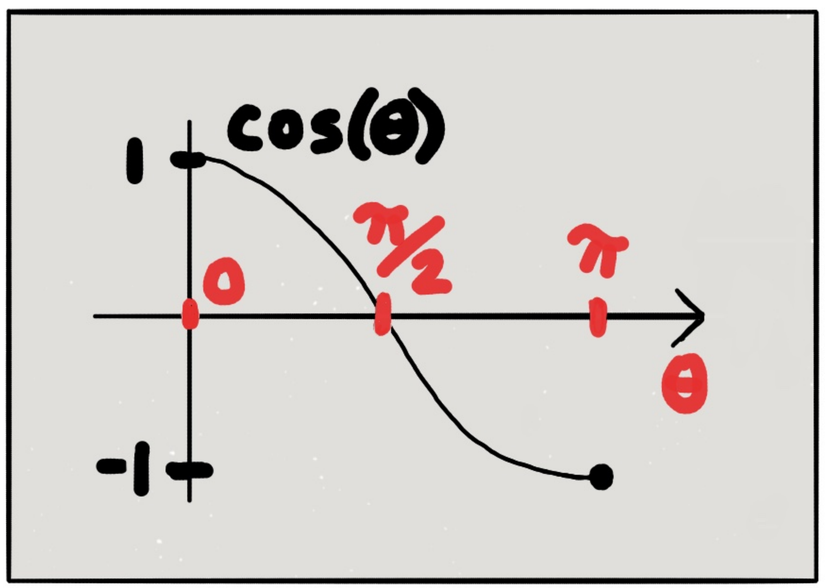

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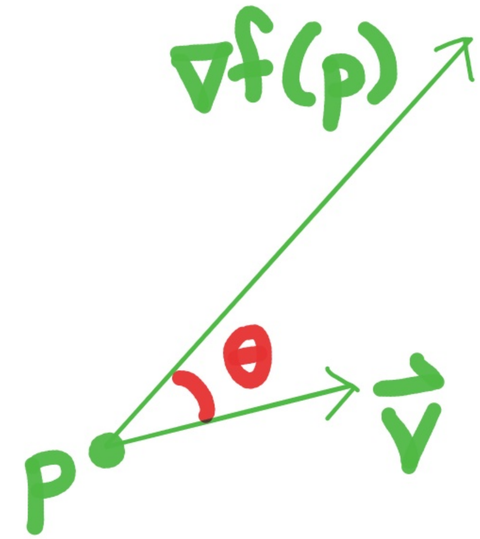

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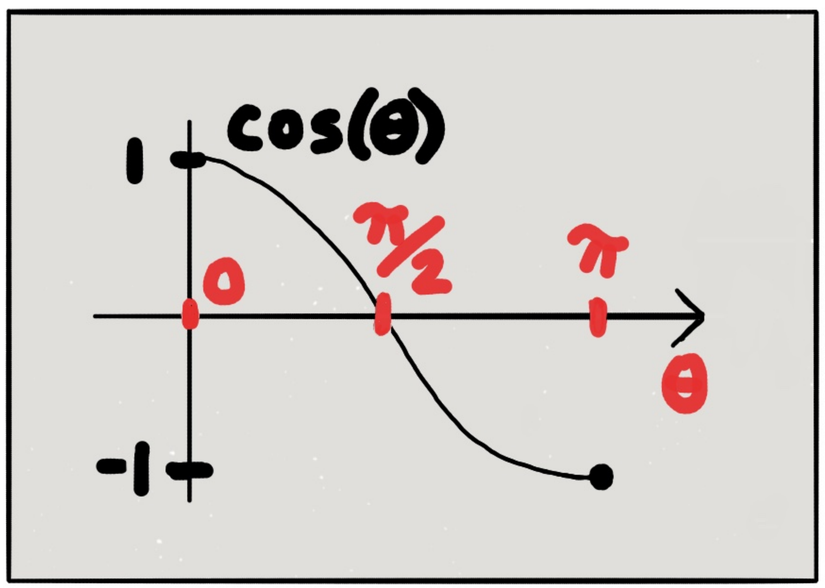

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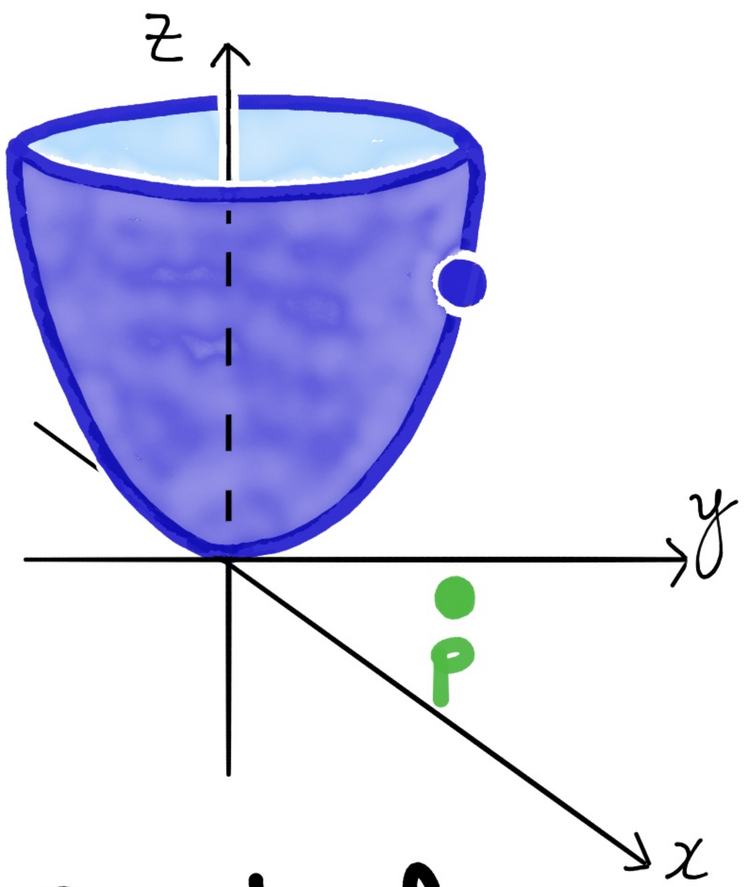
Slope is $D_p f(\vec{v}) = \|\nabla f(p)\| \cos\pi = -\|\nabla f(p)\|$.

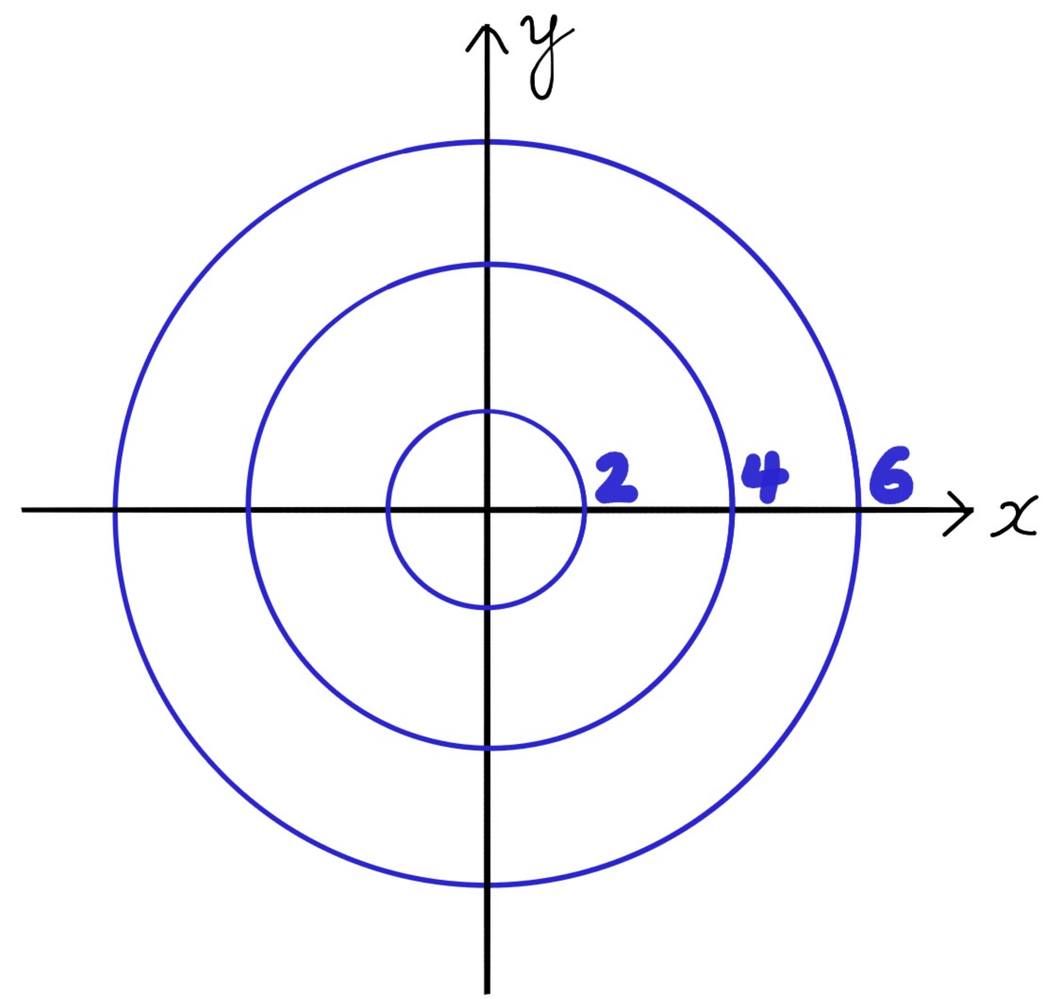
III Gradients, level curves, and level surfaces

Example: $f(x,y) = \|(x,y)\|$

"distance from $\vec{0}$ in the plane"



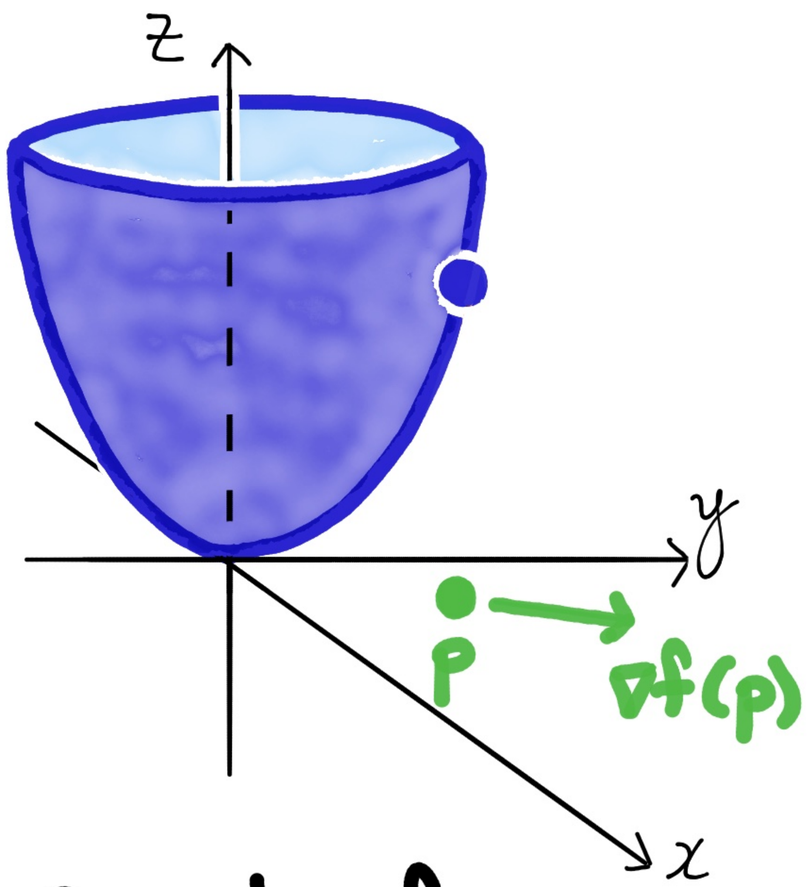
Graph of $f(x,y)$



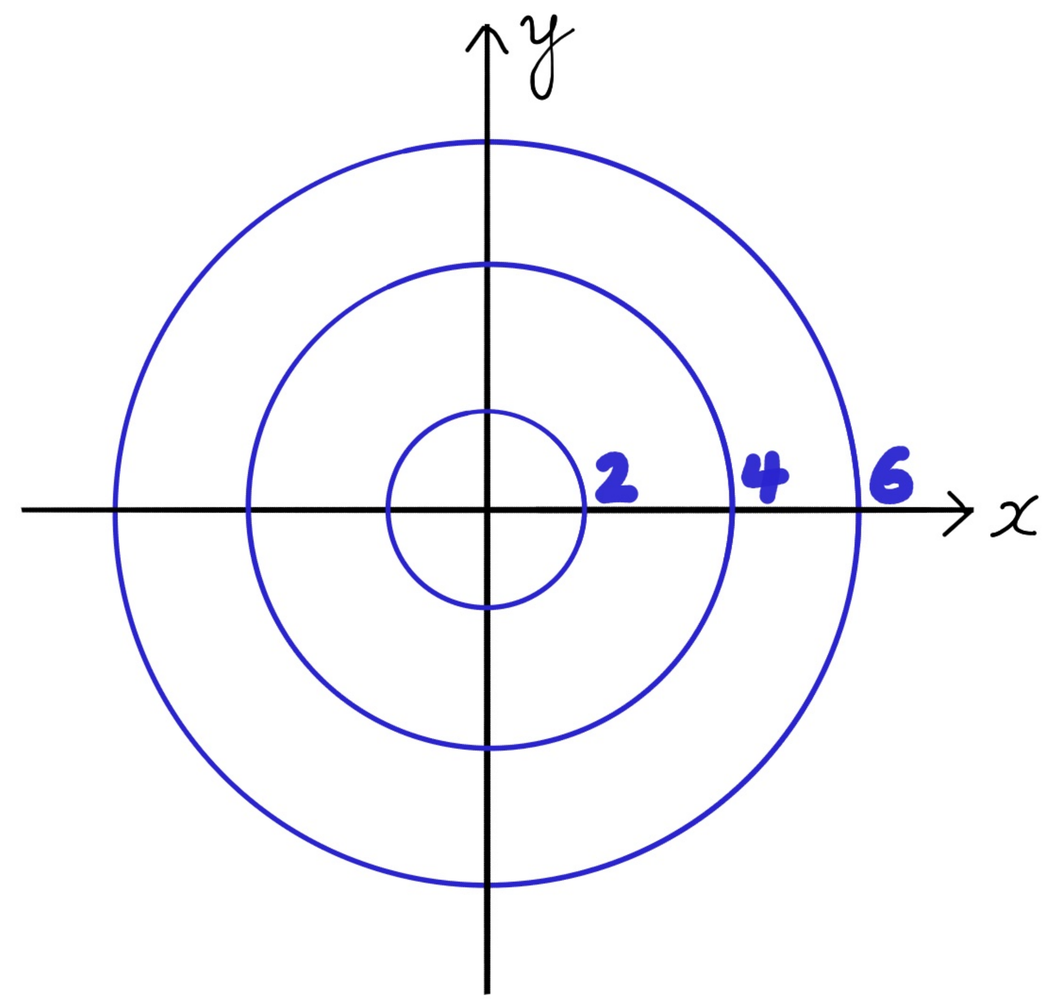
level curves and contour map for $f(x,y)$ in domain of f.

Example: $f(x,y) = \|(x,y)\|$

"distance from $\vec{0}$ in the plane"



Graph of $f(x,y)$



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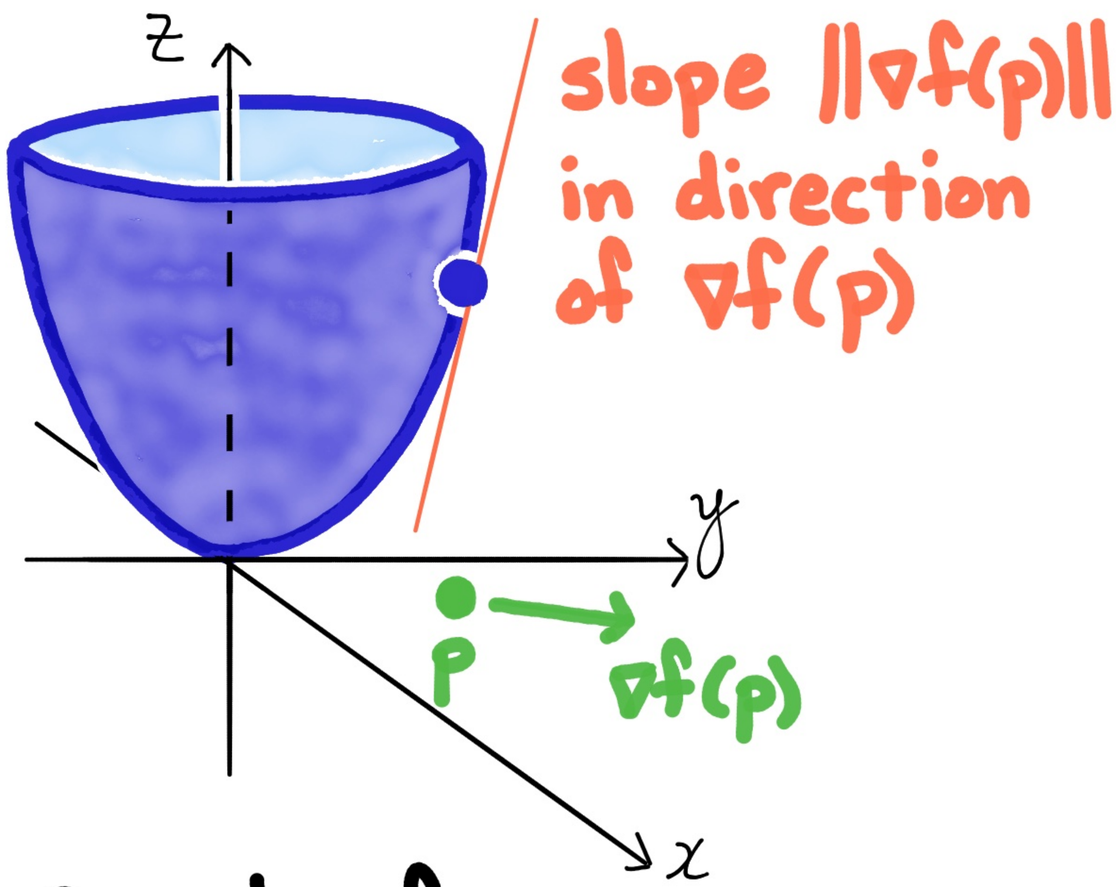
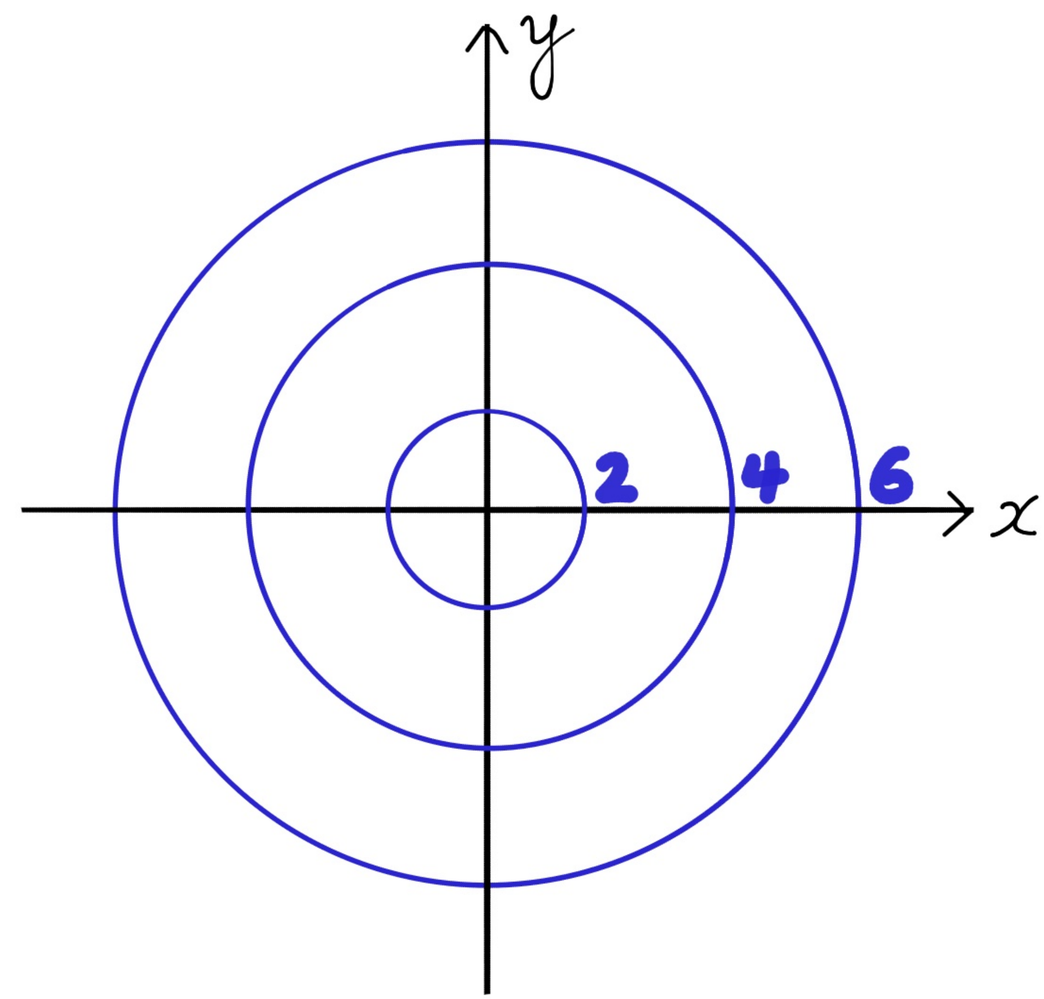


Graph of $f(x,y)$



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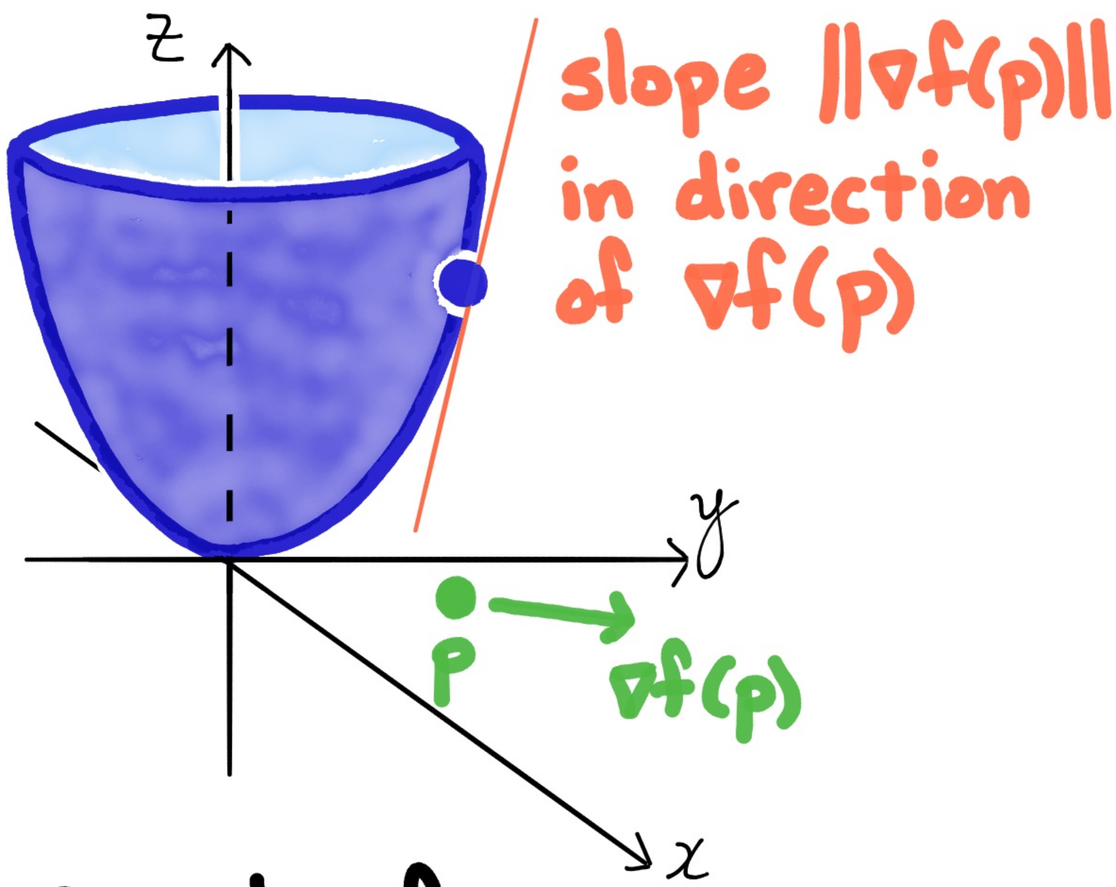
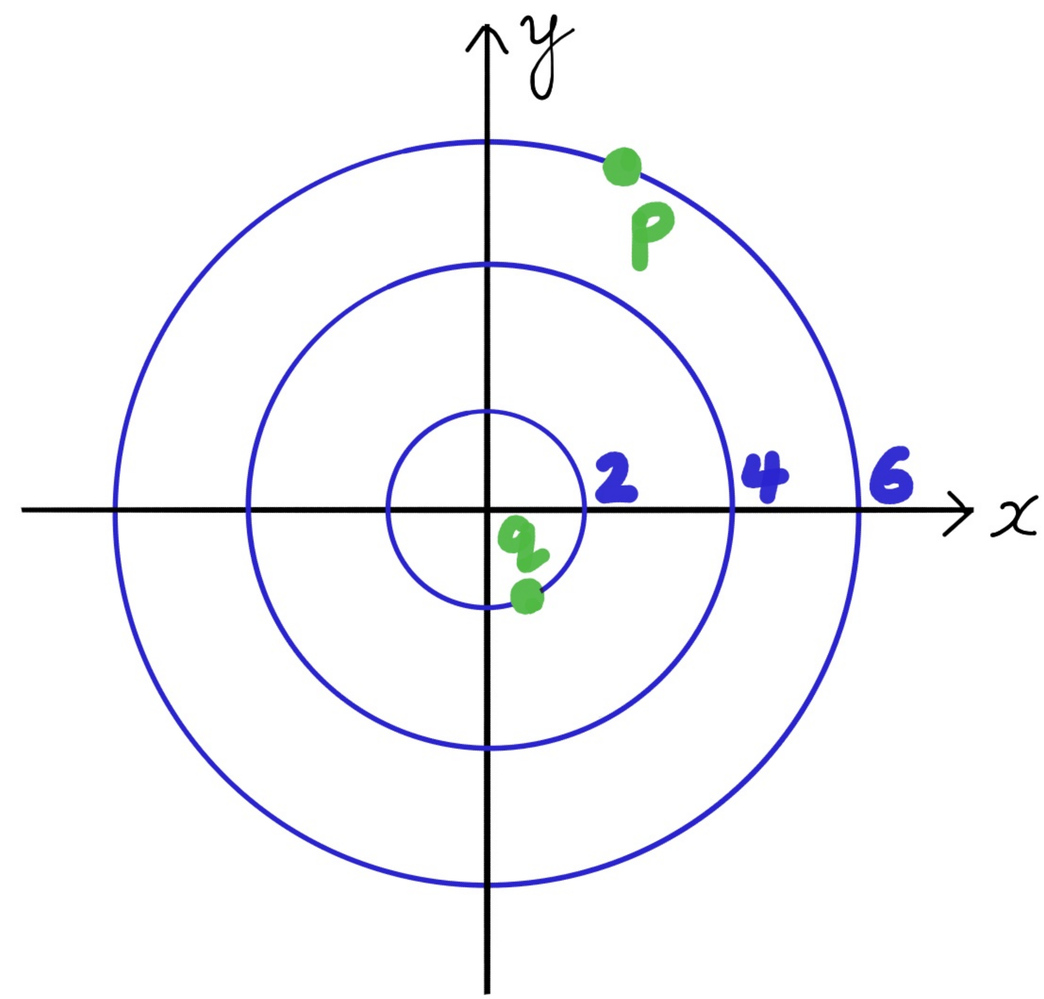


Graph of $f(x,y)$



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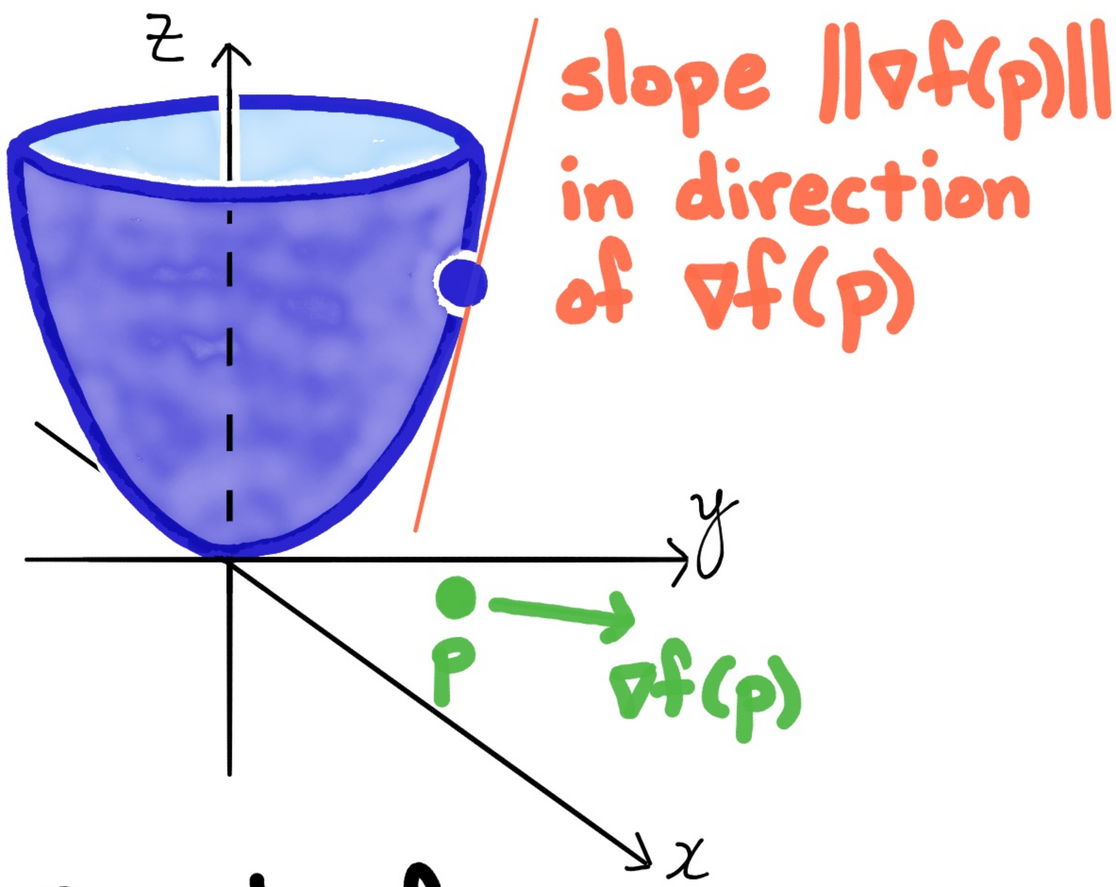

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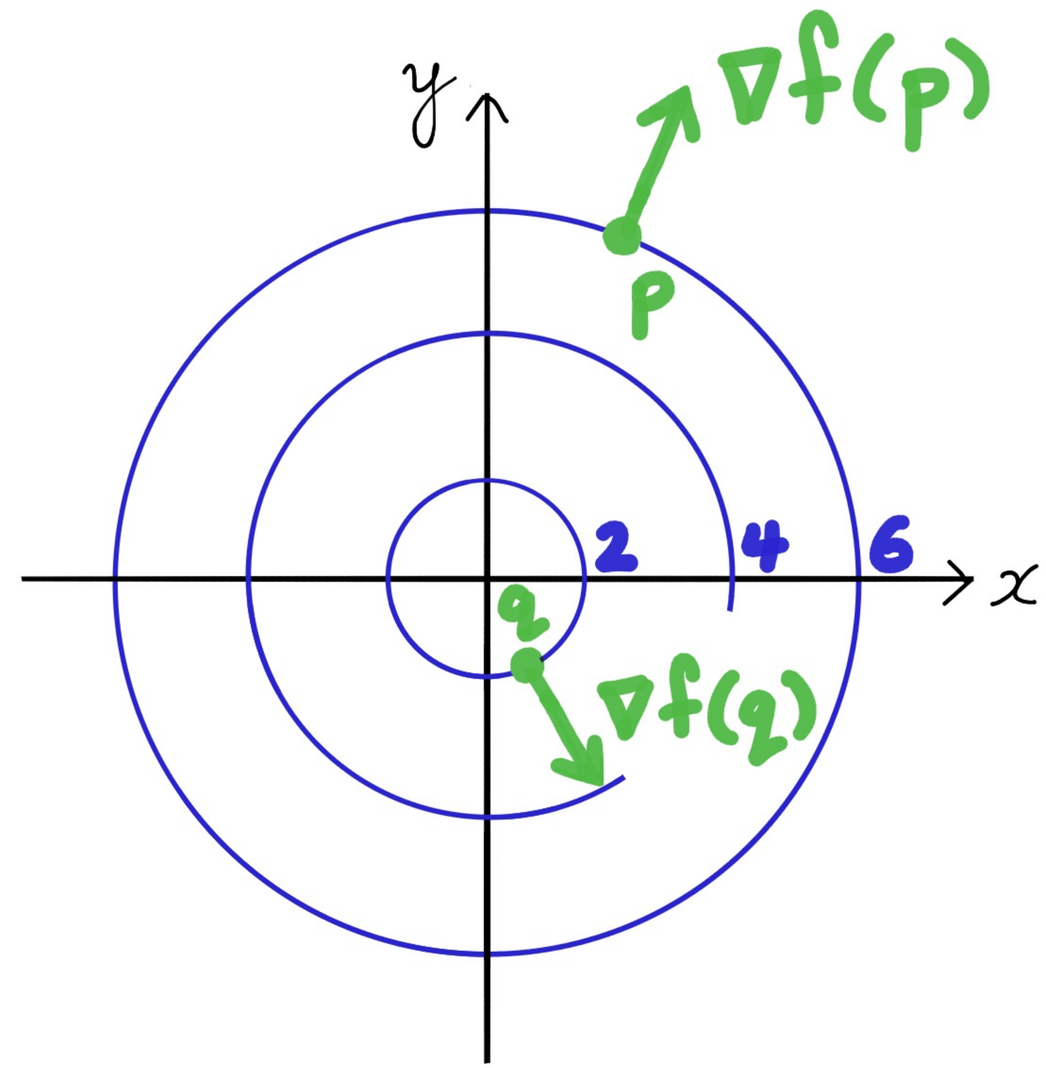

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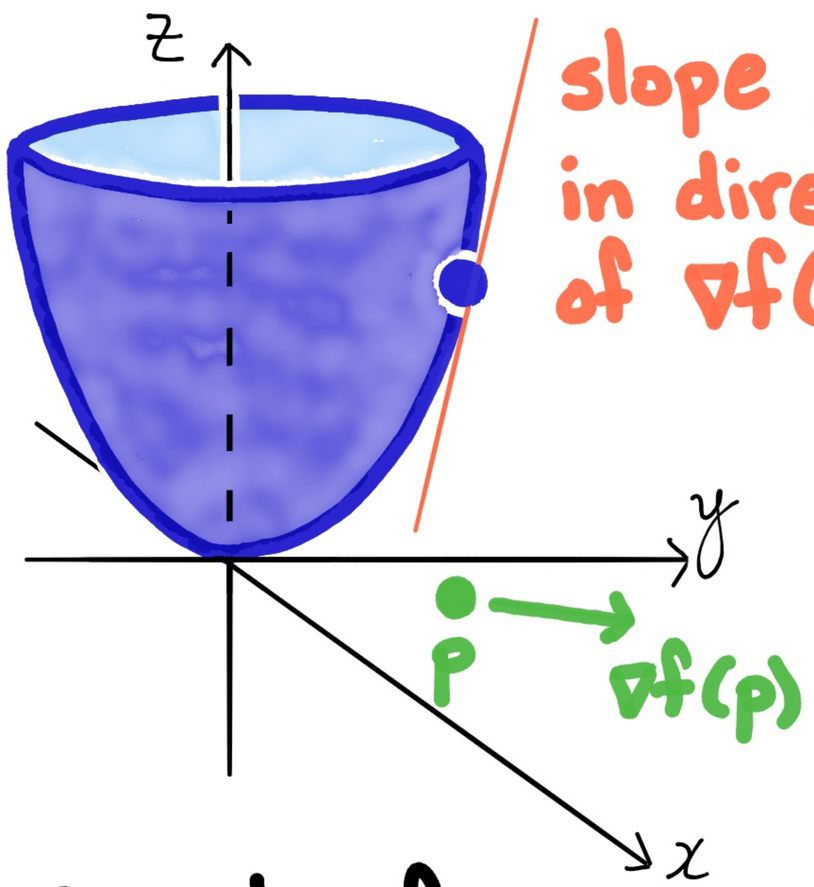
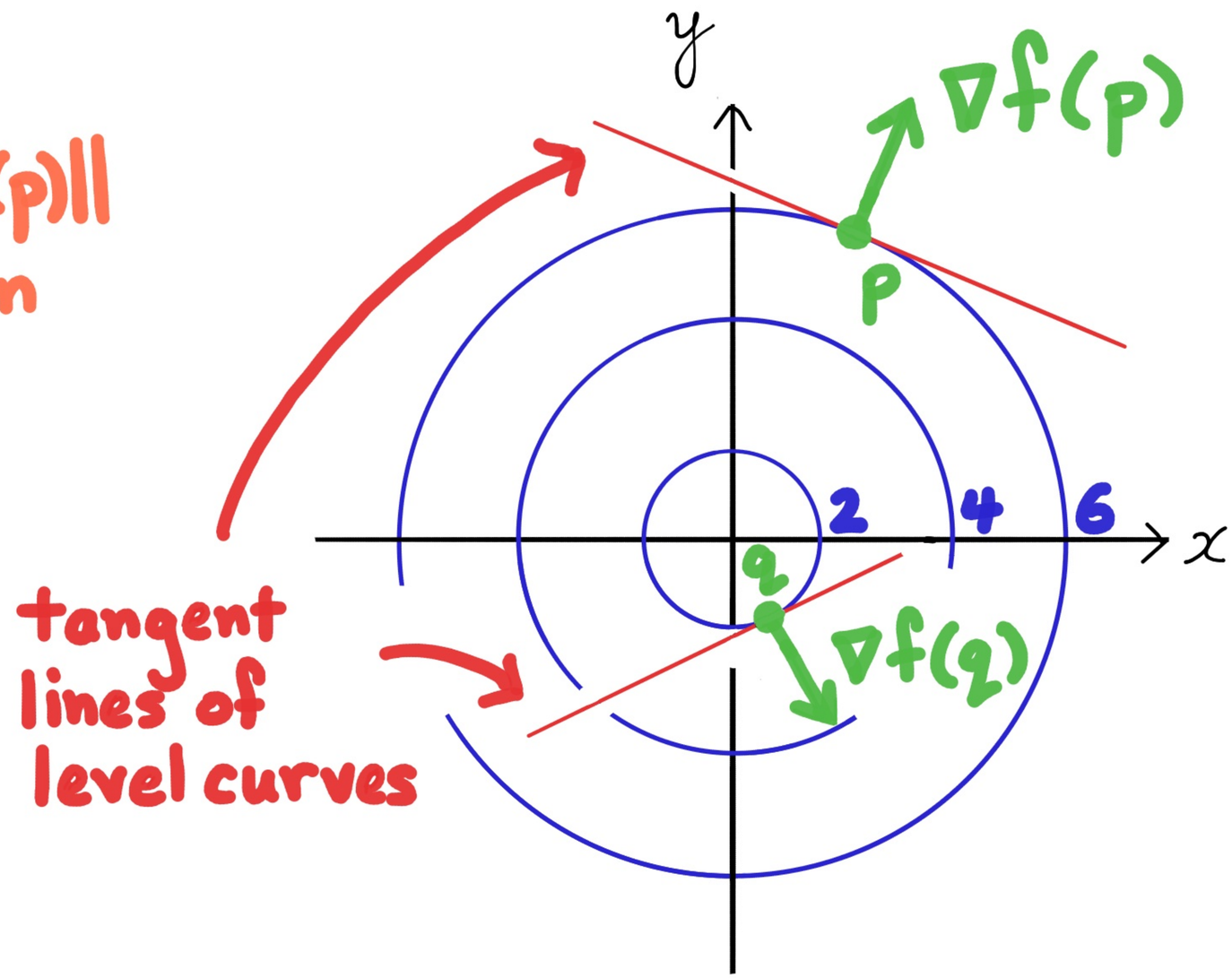


slope $\|\nabla f(p)\|$ in direction of $\nabla f(p)$

Graph of $f(x,y)$

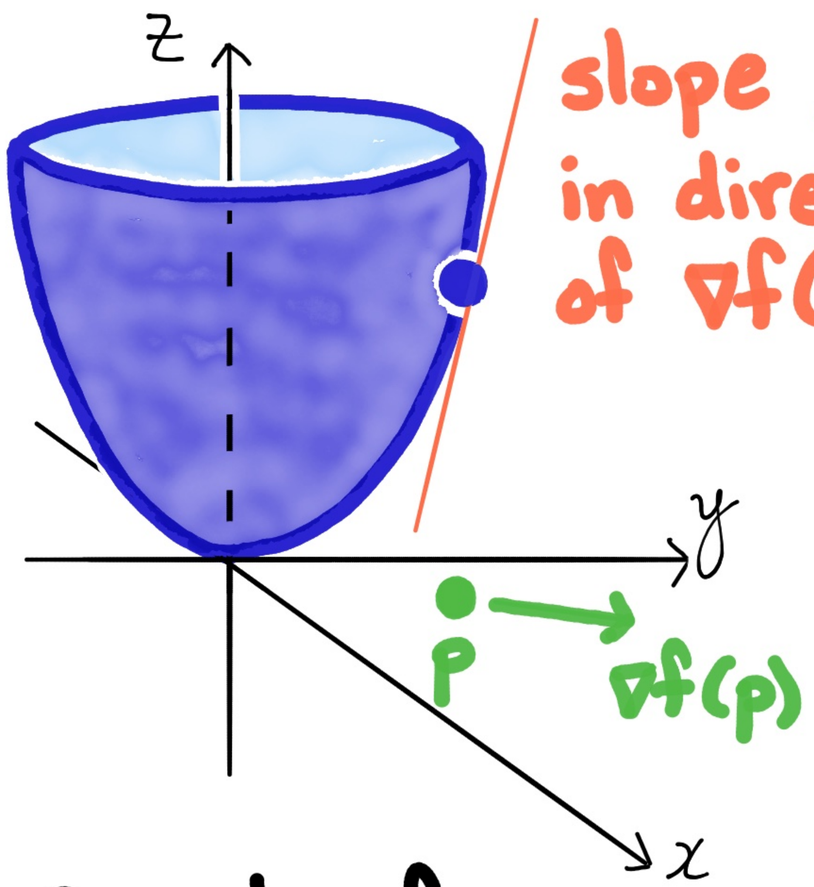


tangent lines of level curves

level curves and contour map for $f(x,y)$ in domain of f.

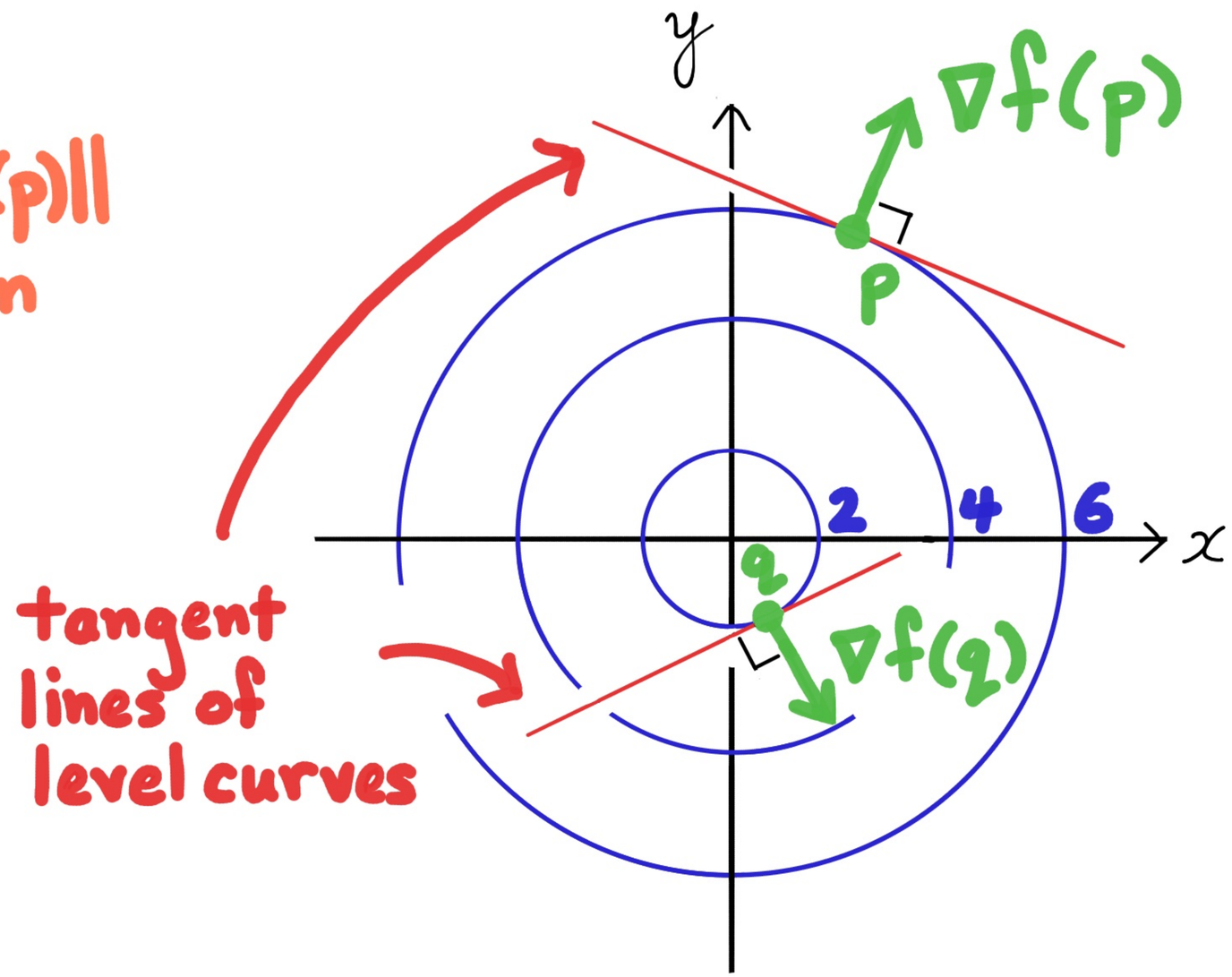
Example: $f(x,y) = \|(x,y)\|$

"distance from $\vec{0}$ in the plane"



slope $\|\nabla f(p)\|$ in direction of $\nabla f(p)$

Graph of $f(x,y)$



tangent lines of level curves

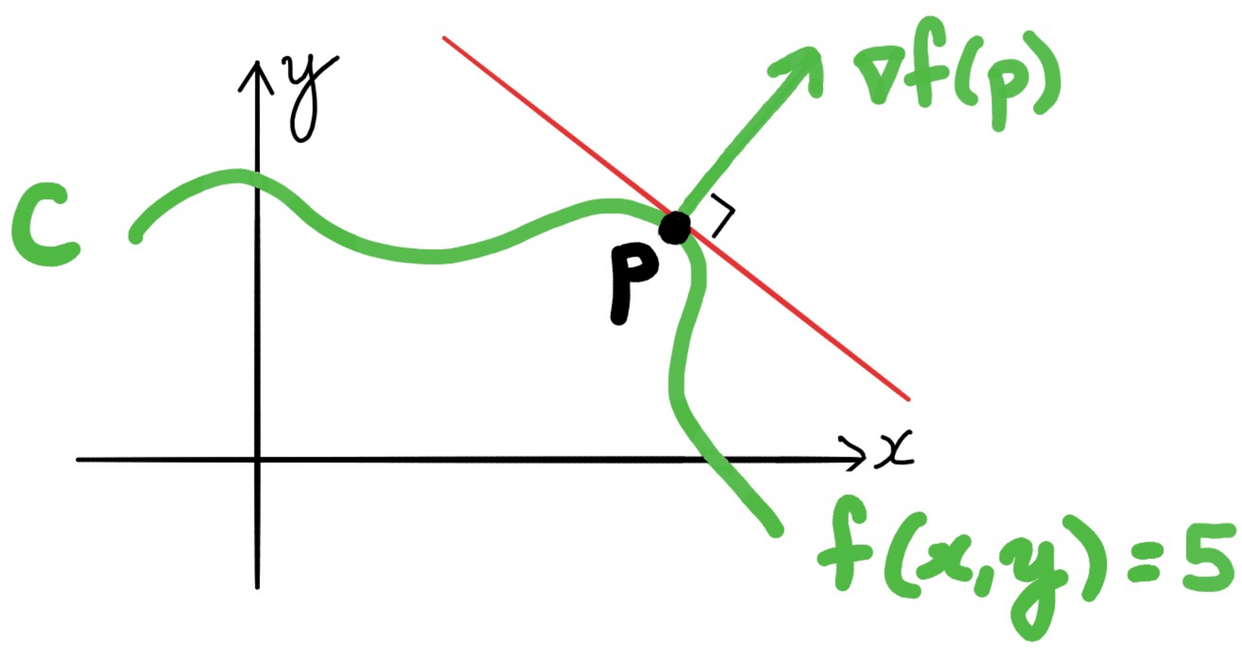
level curves and contour map for $f(x,y)$ in domain of f.

Proposition:

If $p \in \mathbb{R}^2$ is on a level curve C for $f(x,y)$, then either

(a) $\nabla f(p) = \vec{0}$, or

(b) $\nabla f(p)$ is orthogonal to the tangent line of C at p.

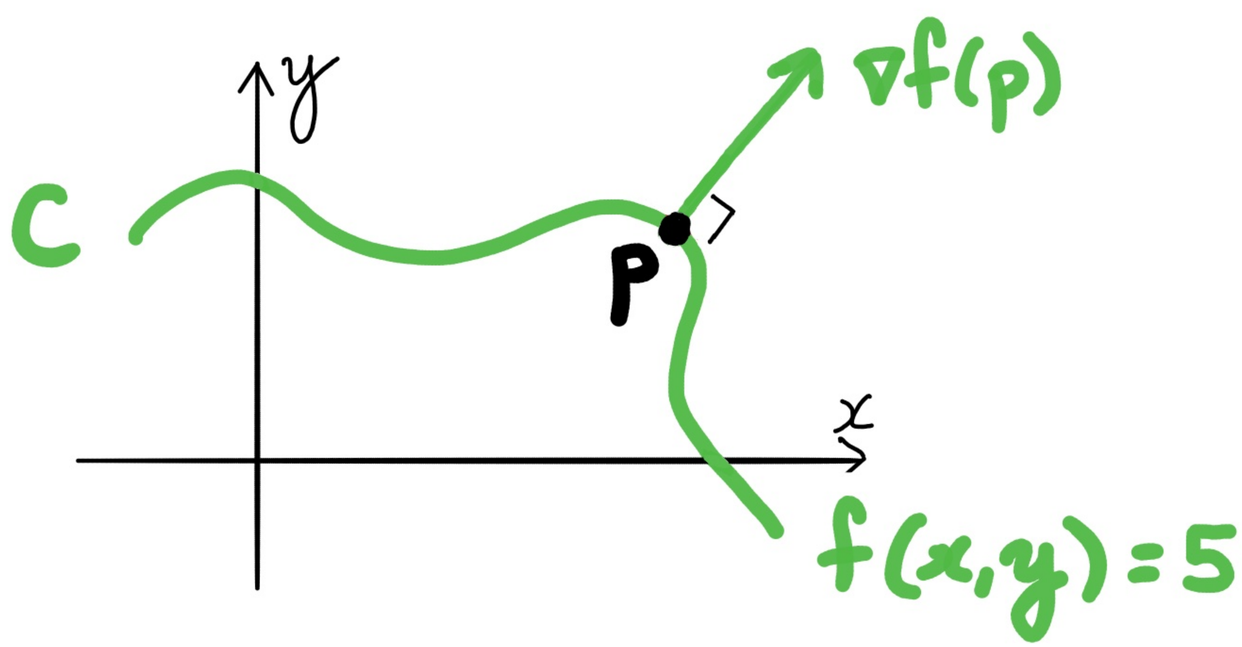


Proposition:

If $p \in \mathbb{R}^2$ is on a level curve C for $f(x,y)$, then either

(a) $\nabla f(p) = \vec{0}$, or

(b) $\nabla f(p)$ is orthogonal to C at p.



Proposition:

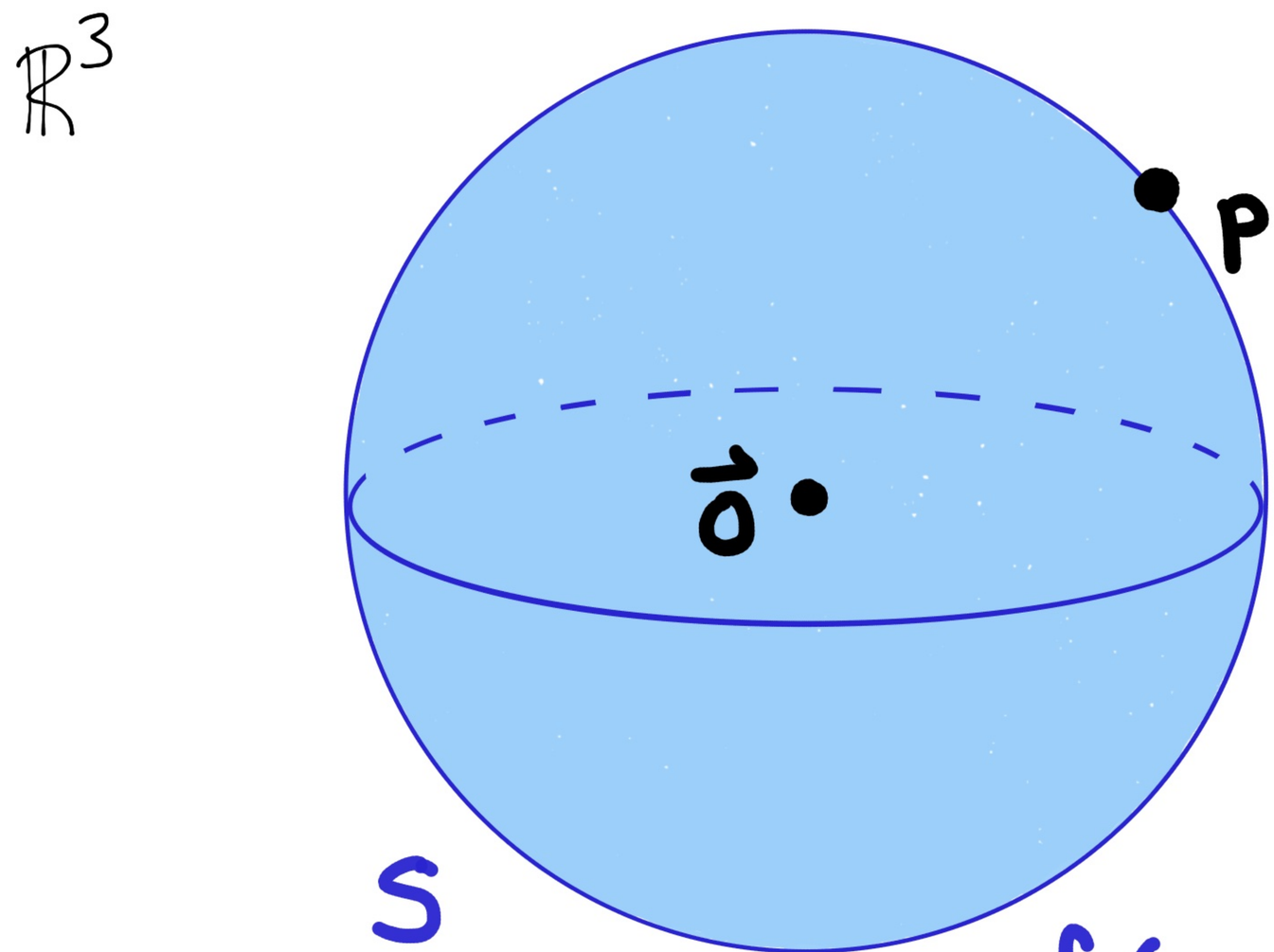
If $p \in \mathbb{R}^3$ is on a level surface S for $f(x,y,z)$, then either

(a) $\nabla f(p) = \vec{0}$, or

(b) $\nabla f(p)$ is orthogonal to S at p.

Example: $f(x,y,z) = \|(x,y,z)\|$

"distance from $\vec{0}$ in $\mathbb{R}^3$"



$f(x,y,z) = 7$

Example: $f(x,y,z) = \|(x,y,z)\|$

"distance from $\vec{0}$ in $\mathbb{R}^3$"