Twelve

Ⅰ Definition of directional derivative

Ⅱ The practice of computing directional derivatives

Ⅲ Geometry of directional derivatives

Ⅳ Jacobian matrices

Ⓘ Definition of directional derivative

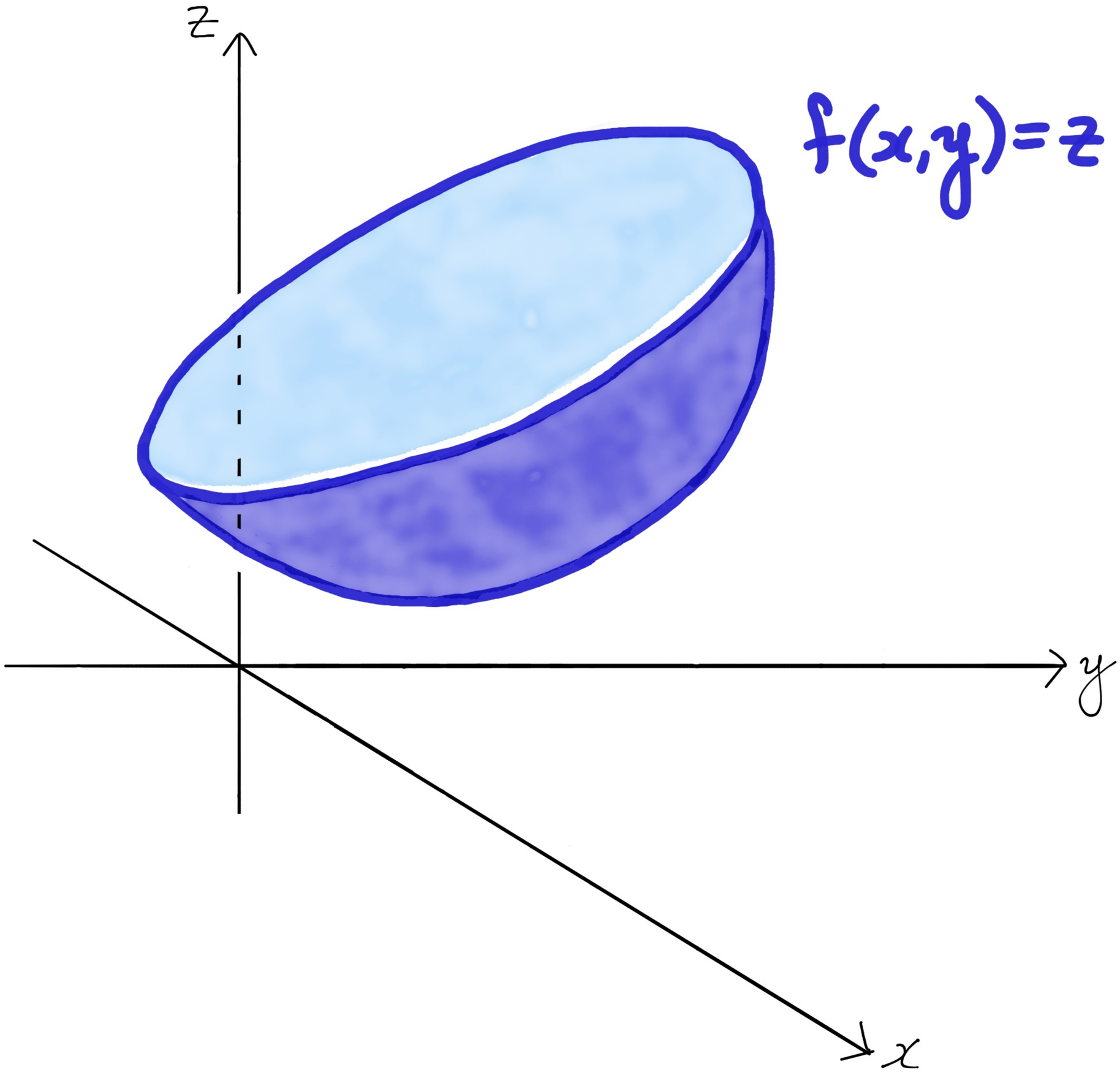
Word problem:

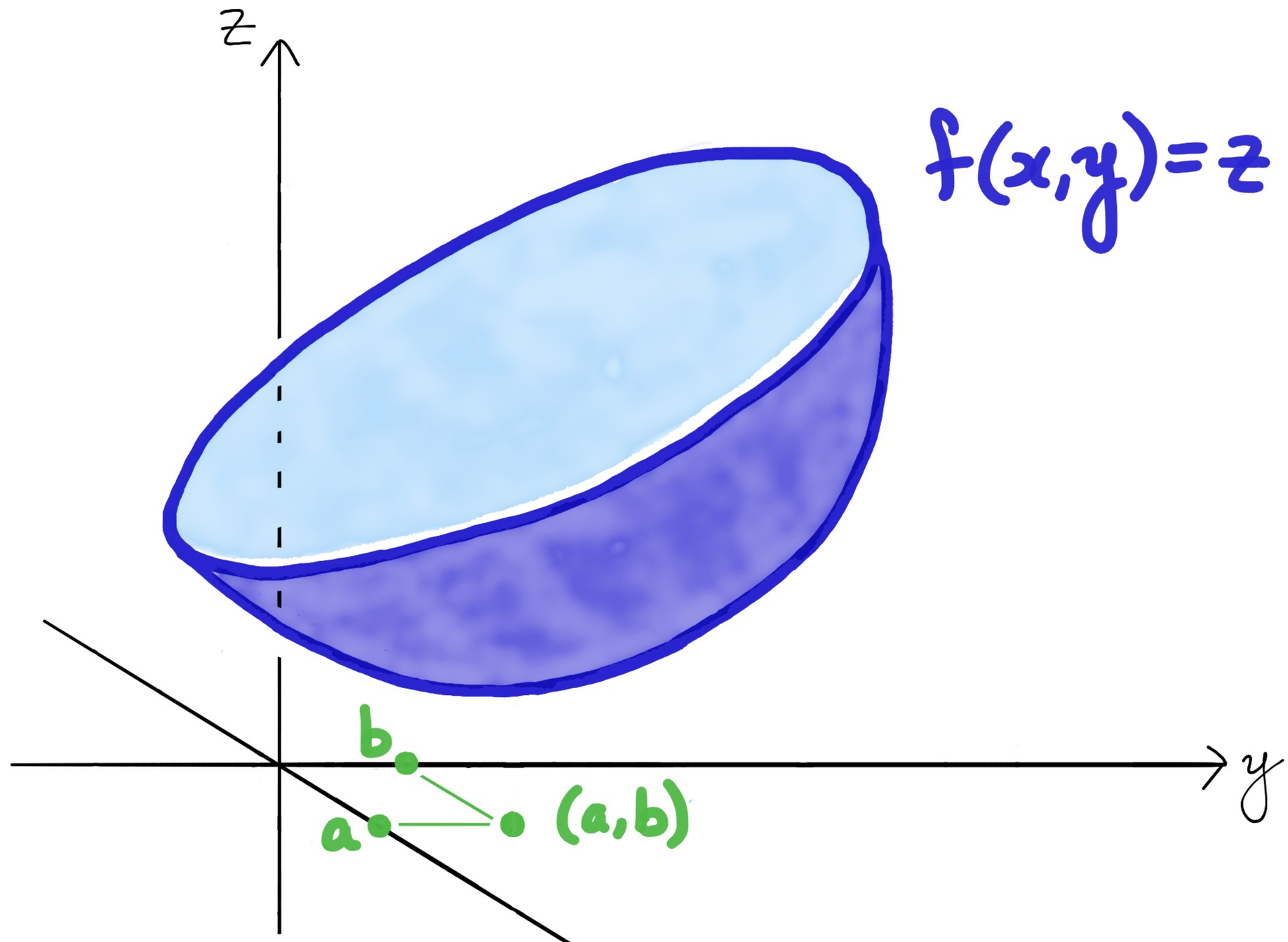
W - water

G - gasoline

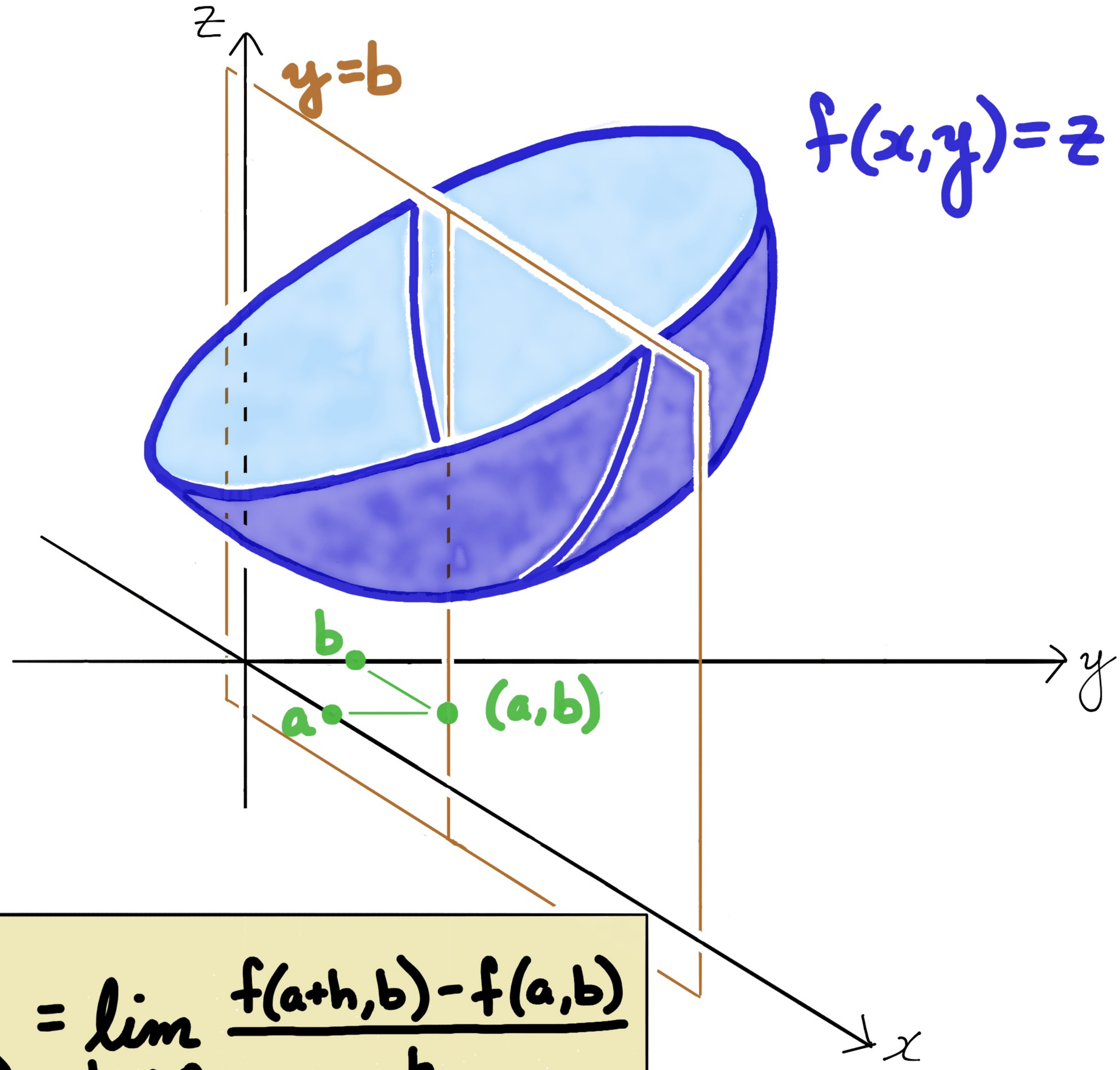
$F(W,G)$ - size of fire

$$\frac{\partial F}{\partial W}(W,G)<0, \quad \frac{\partial F}{\partial G}(W,G)>0$$

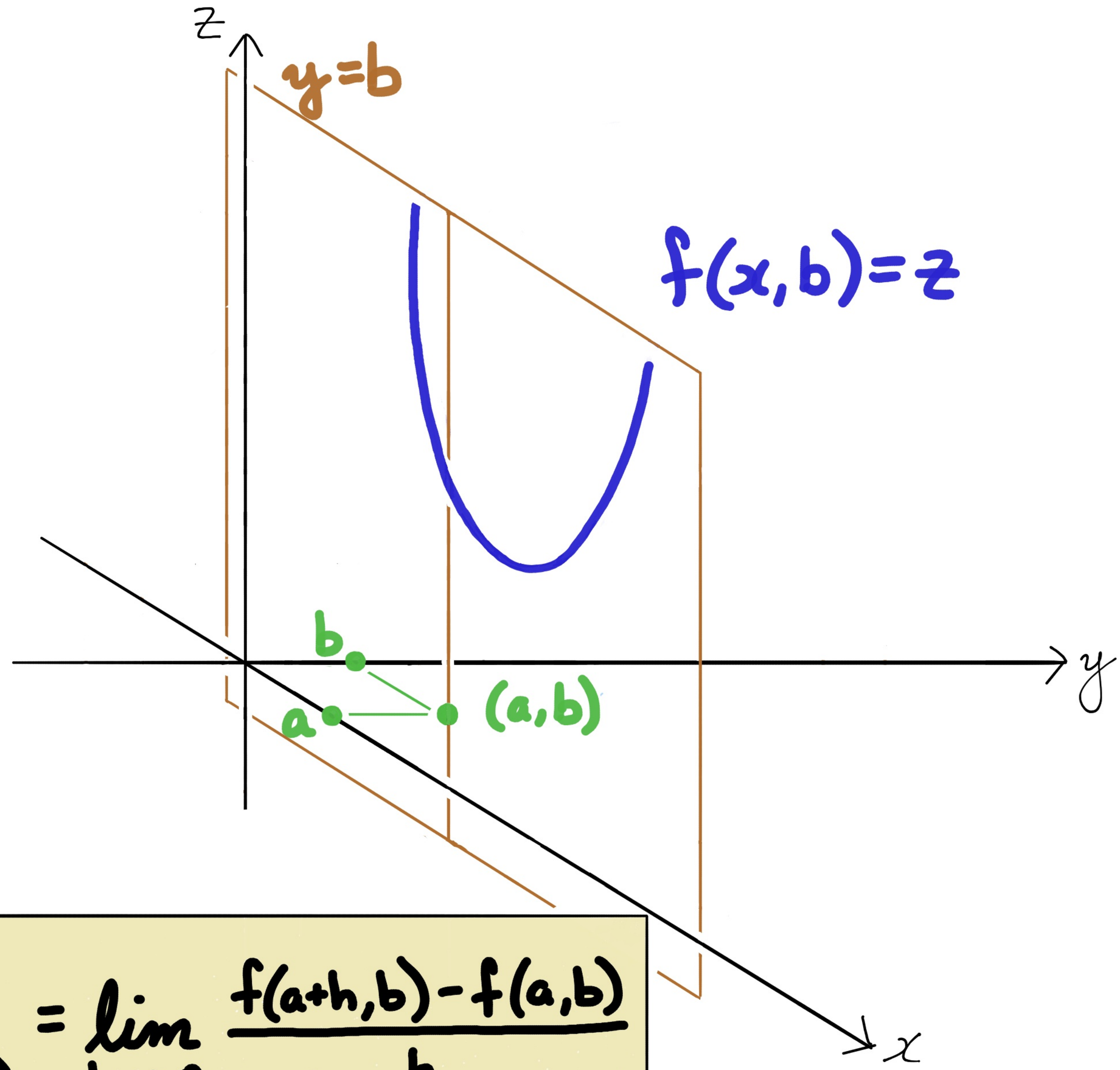

z
y
x
f(x,y)=z



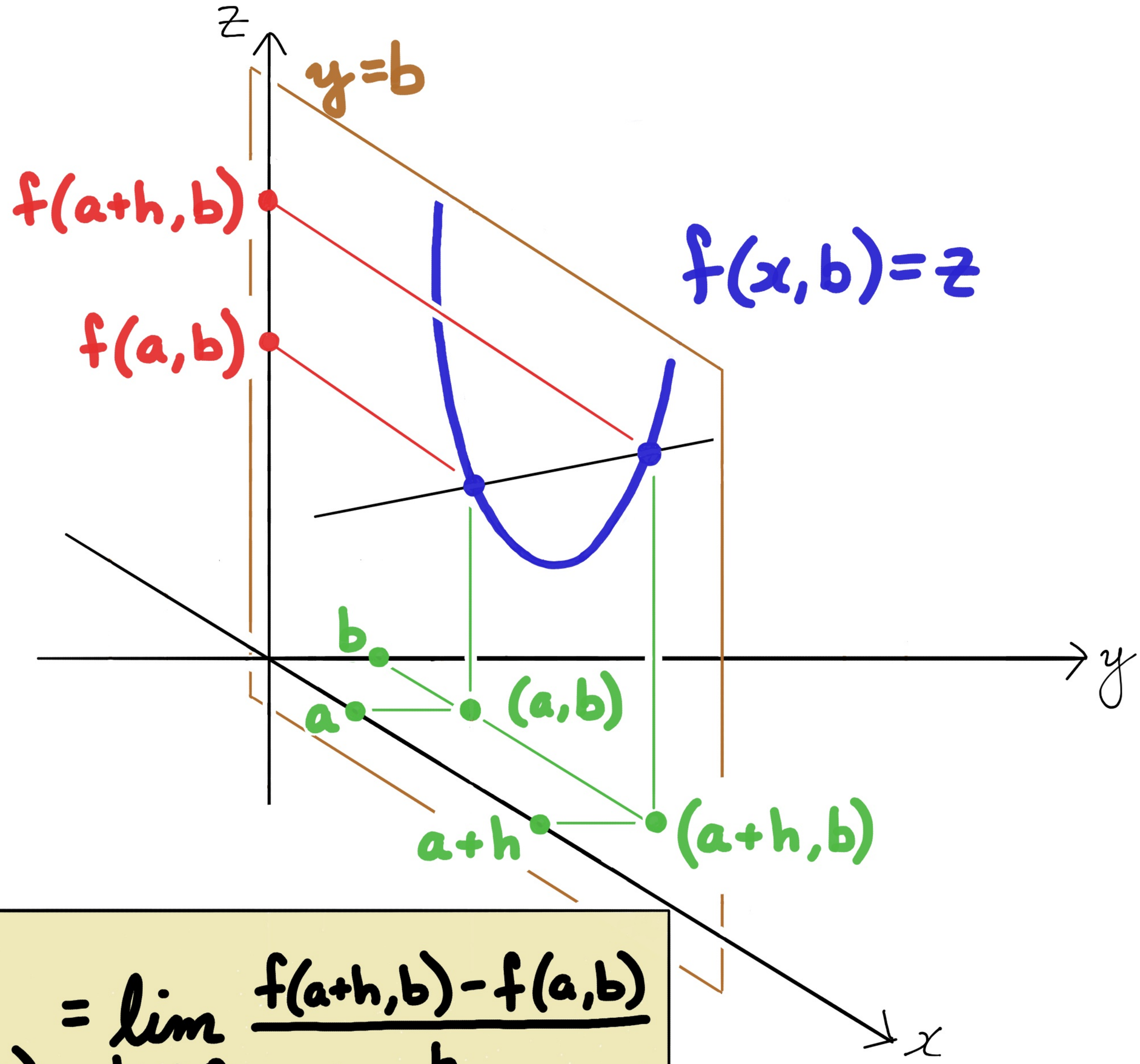
$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$



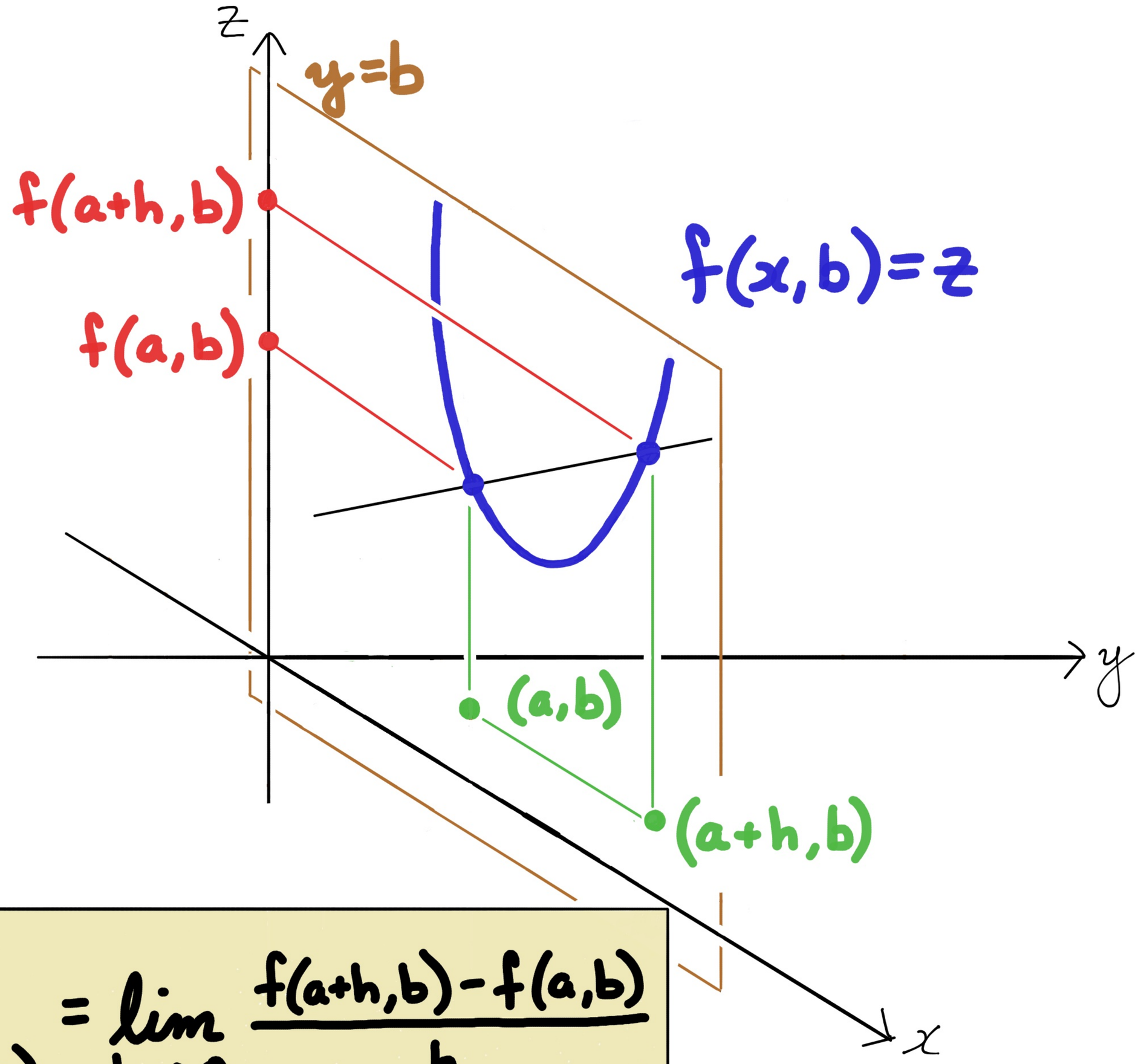
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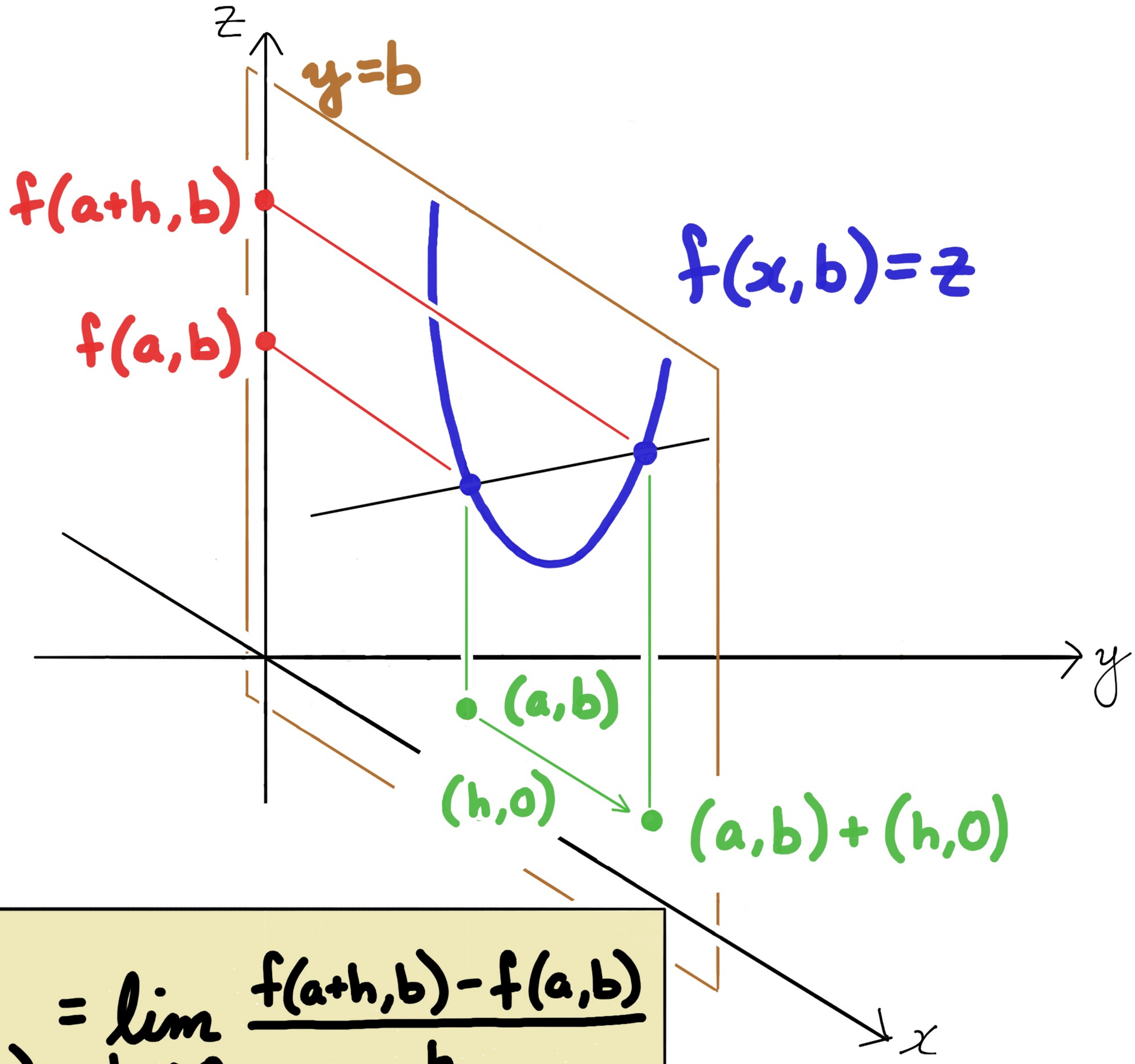
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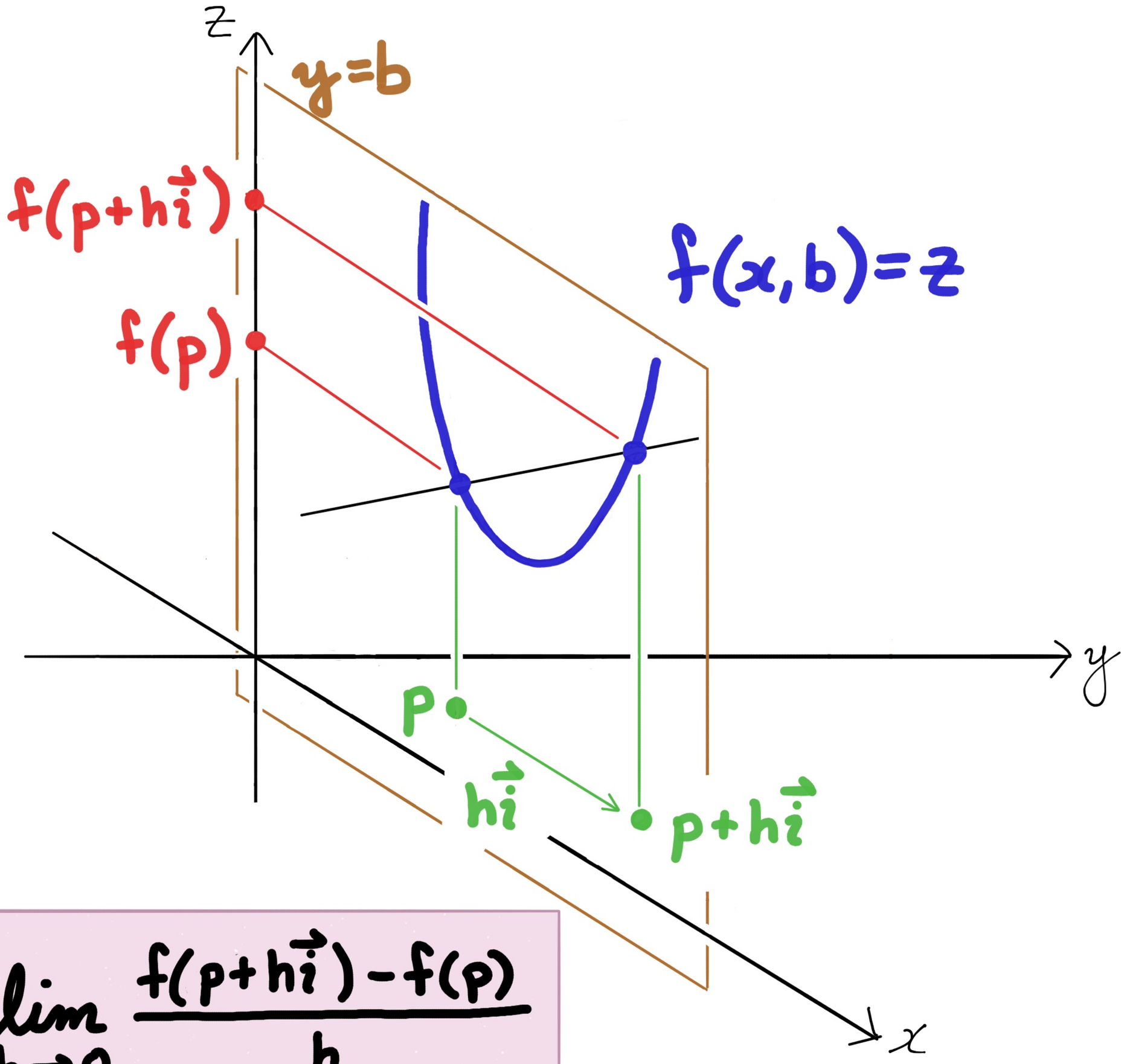
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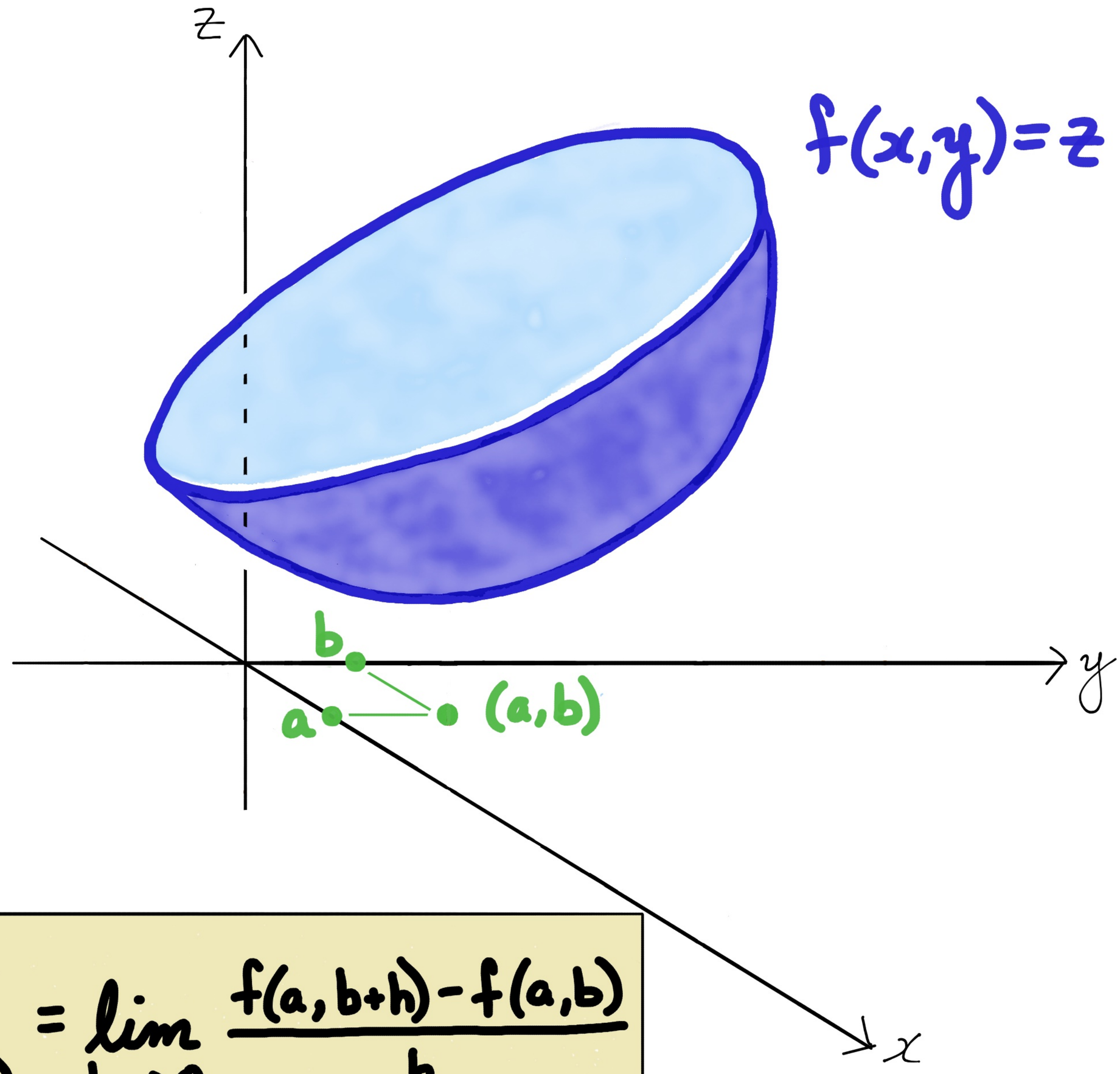
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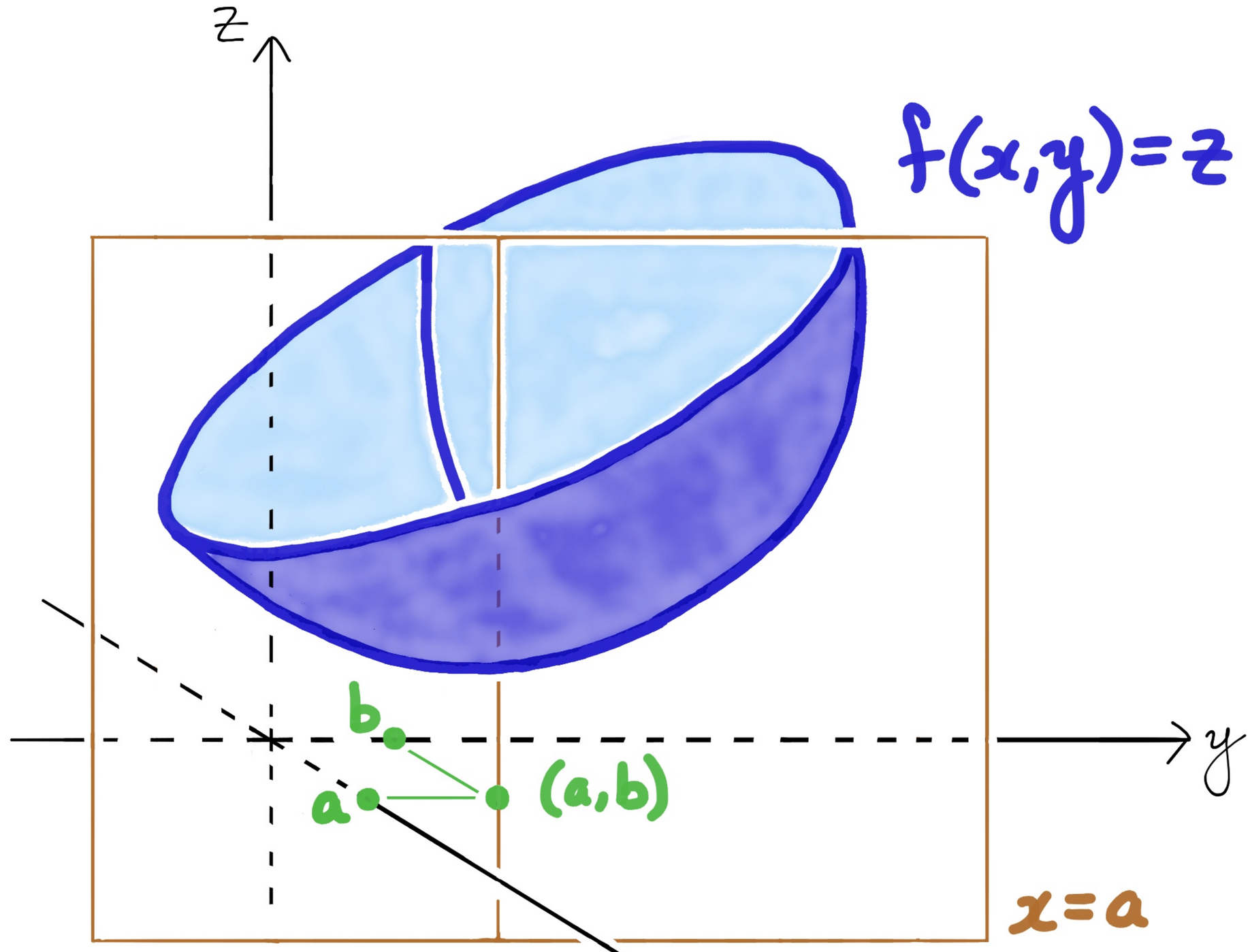
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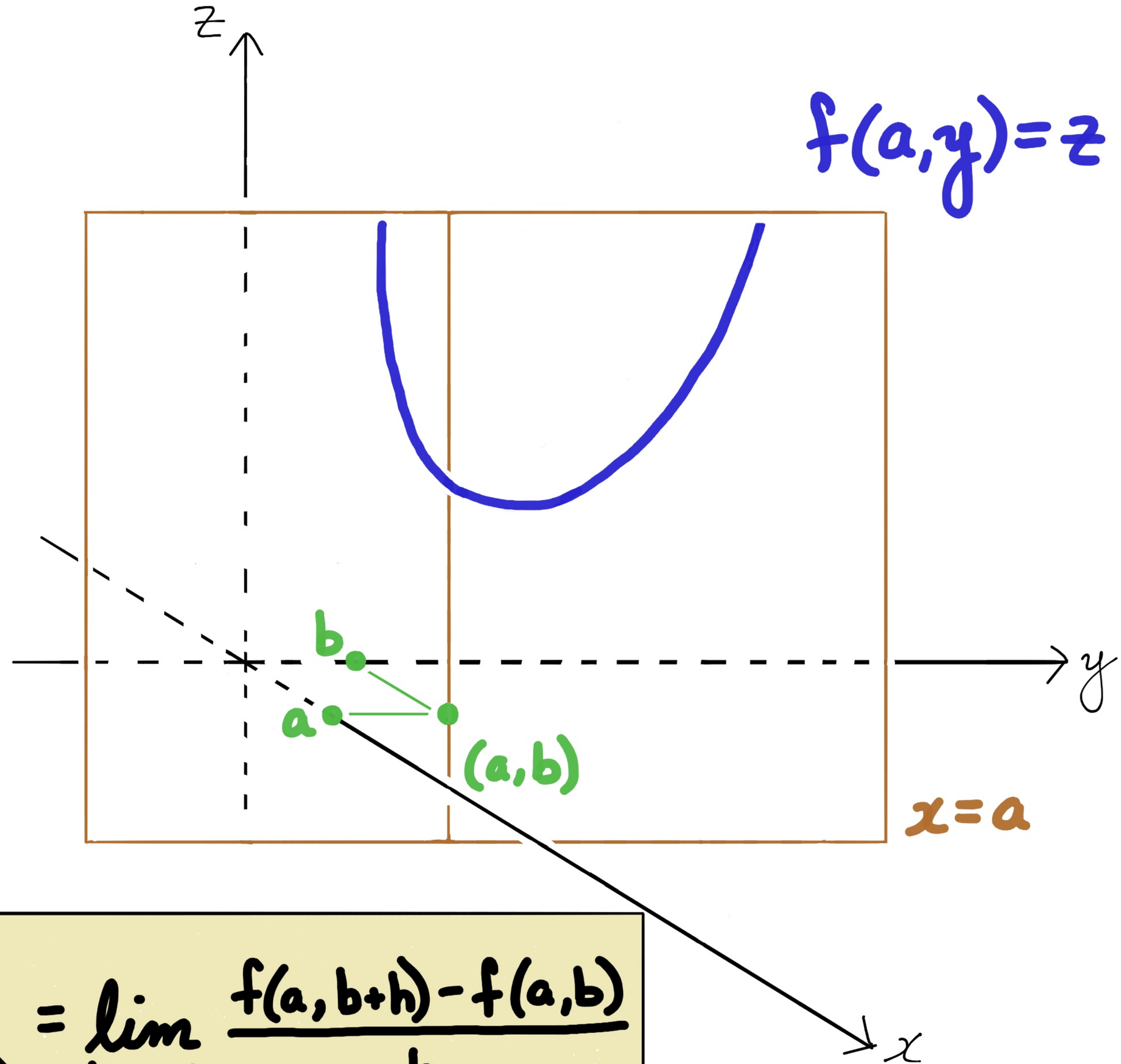
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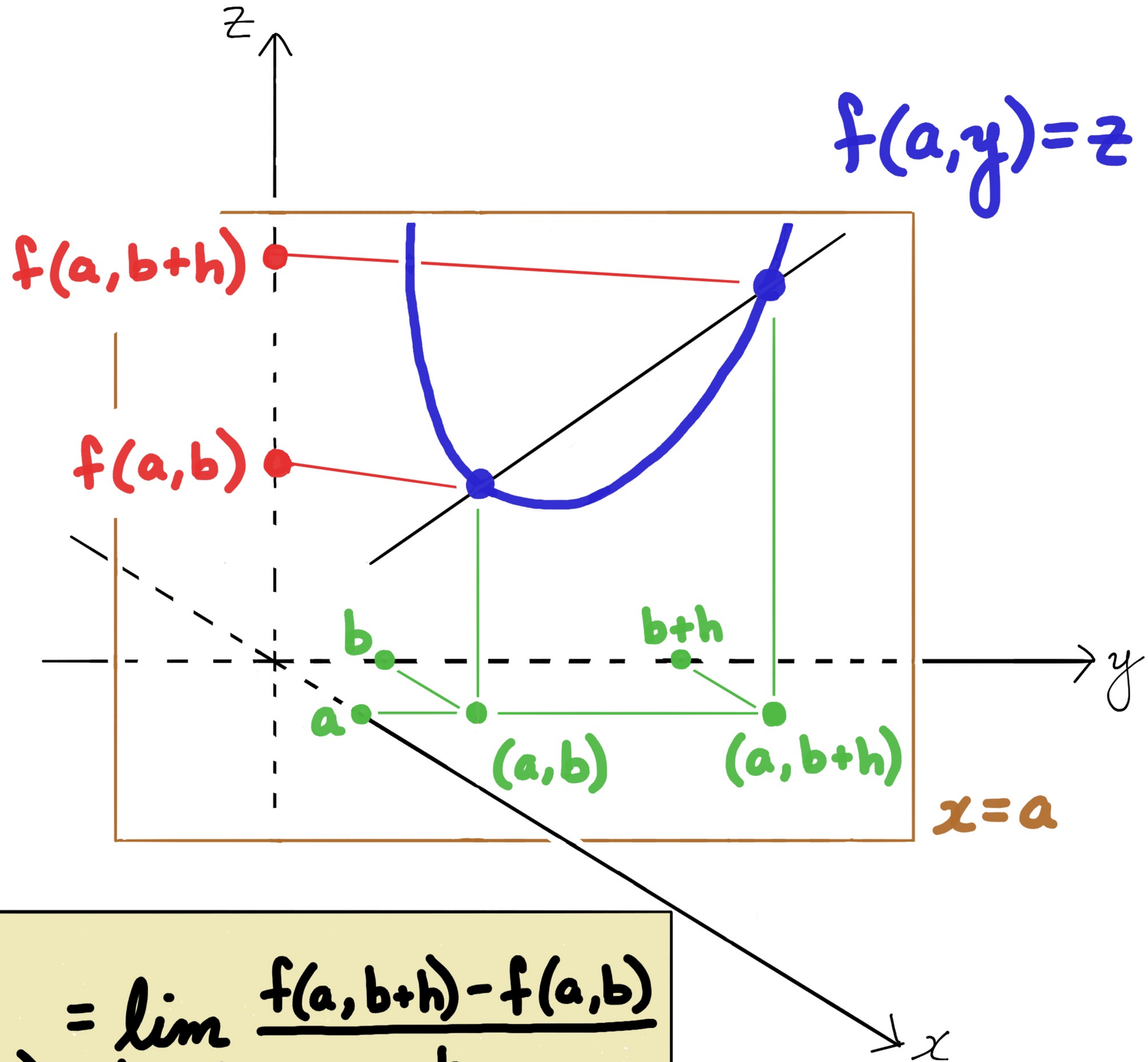
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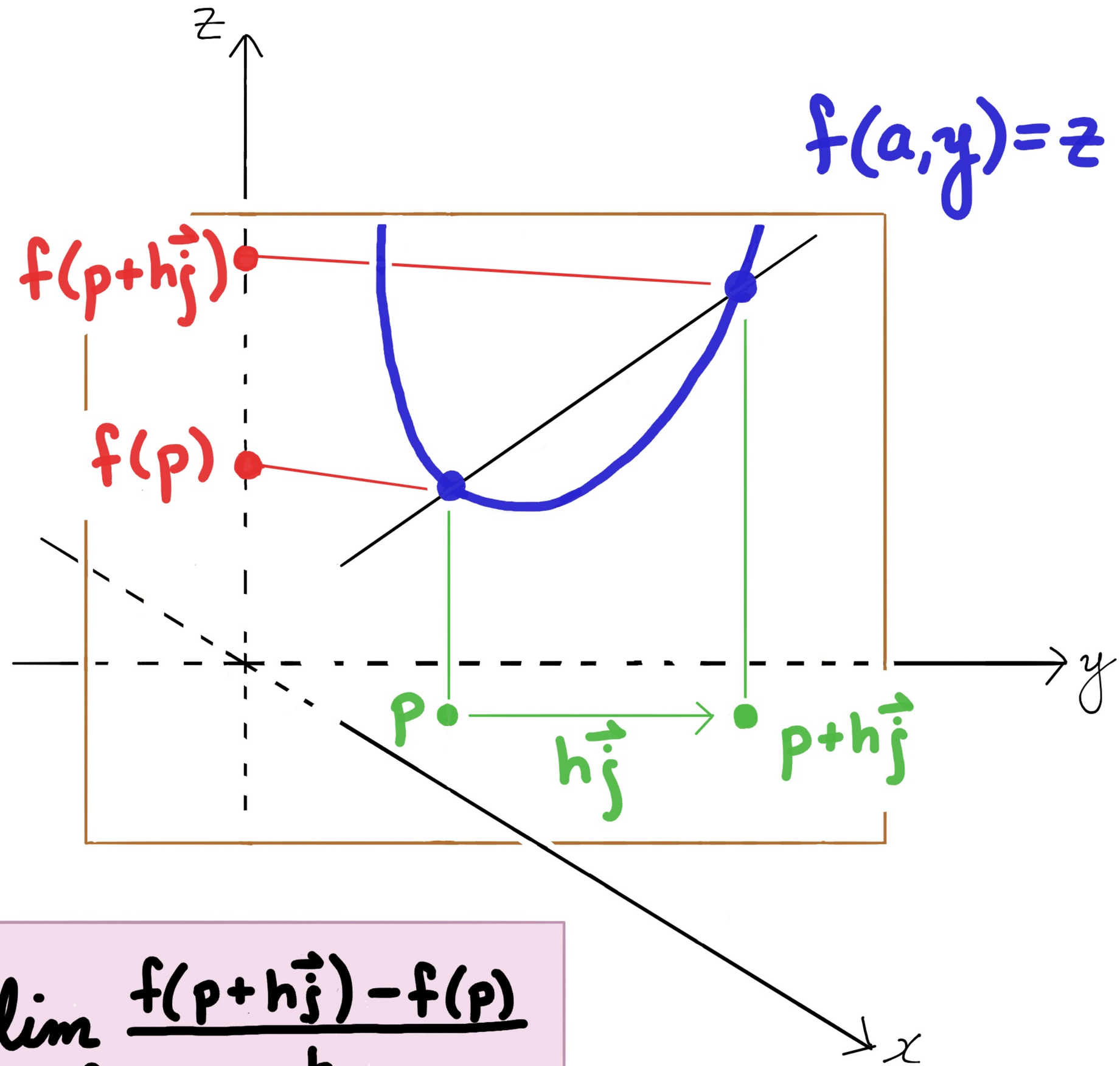
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Note: $\left.\frac{\partial f}{\partial x}\right|_p = D_p f(\vec{i})$ and $\left.\frac{\partial f}{\partial y}\right|_p = D_p f(\vec{j})$.

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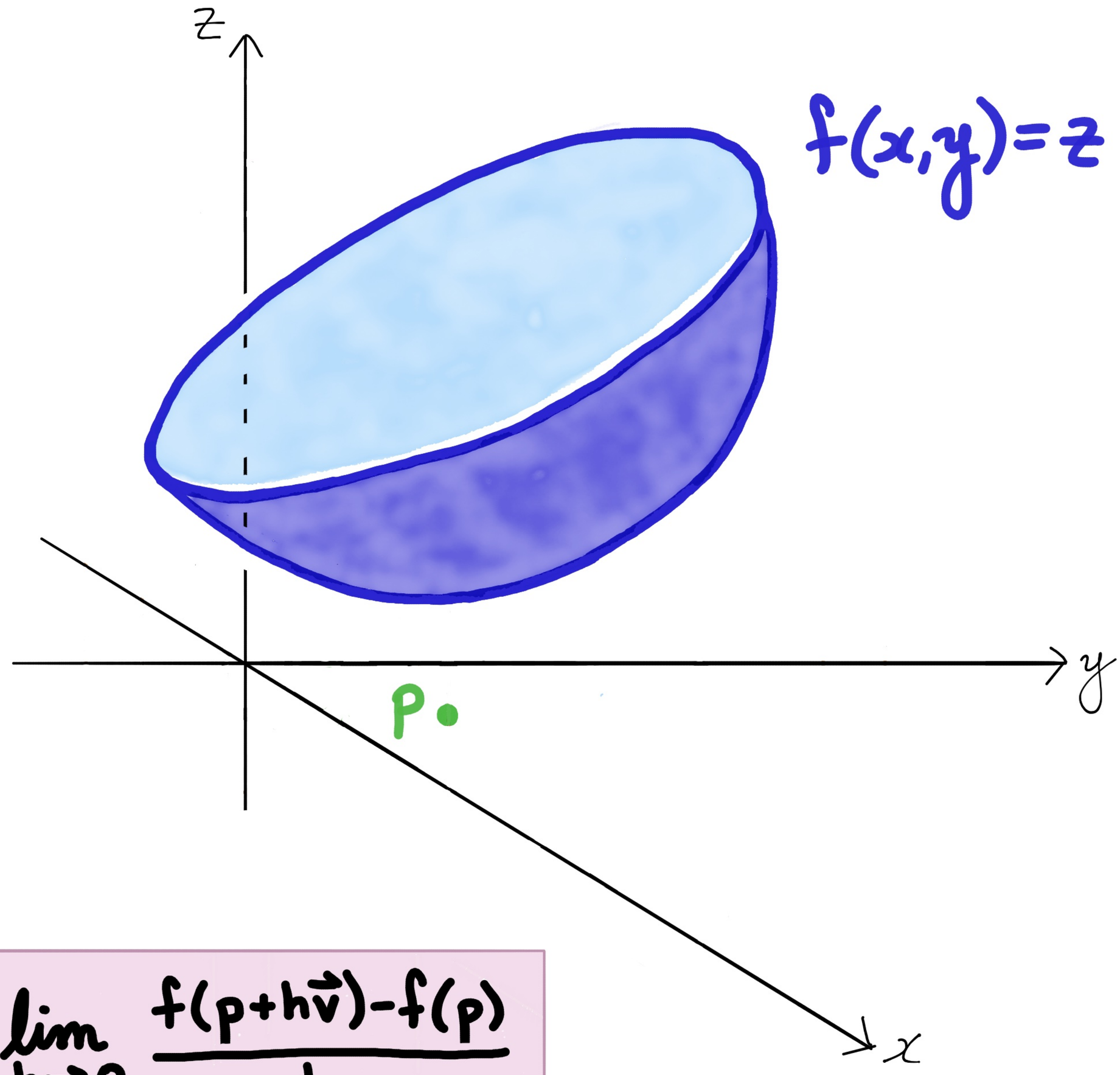
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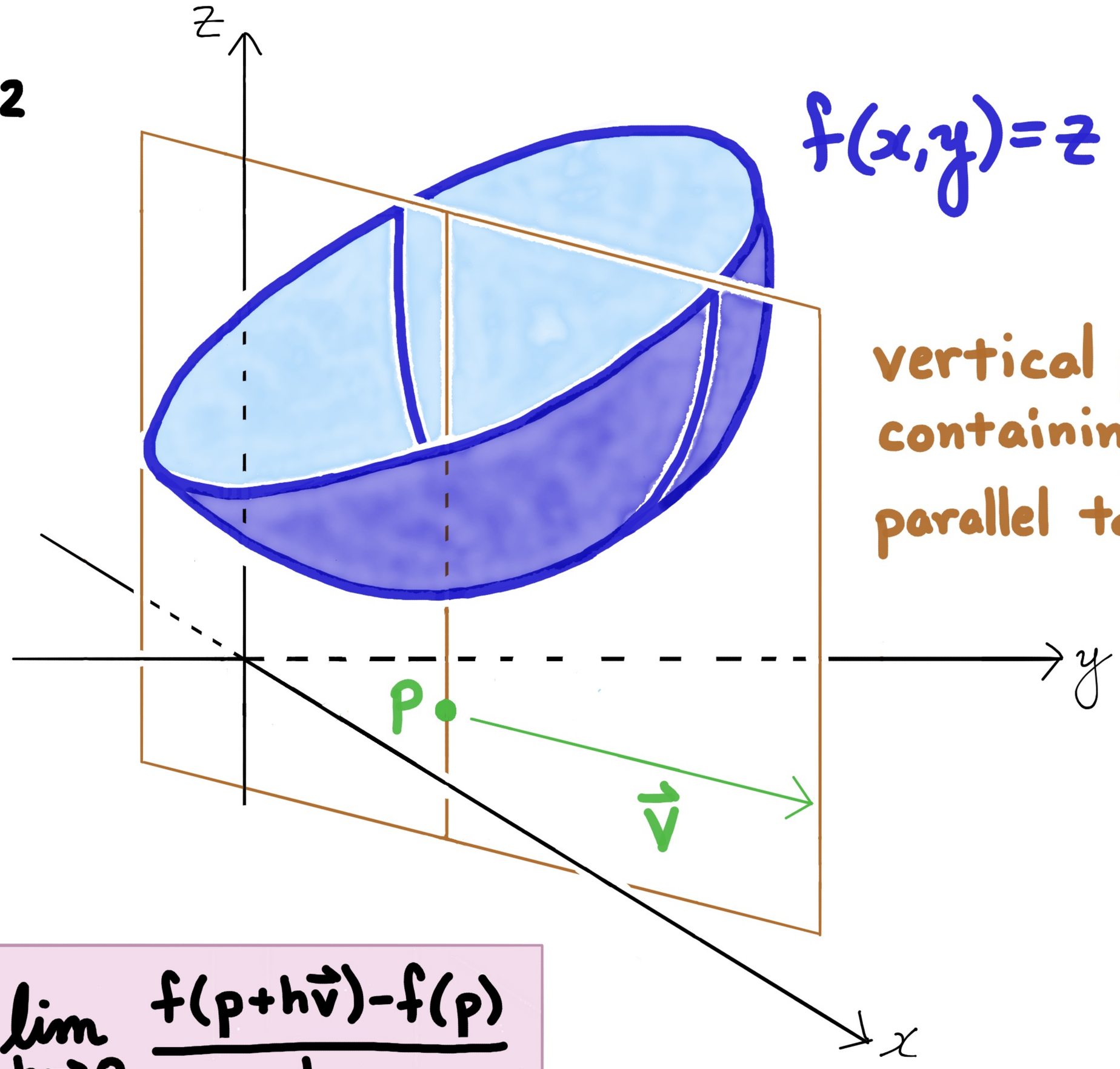
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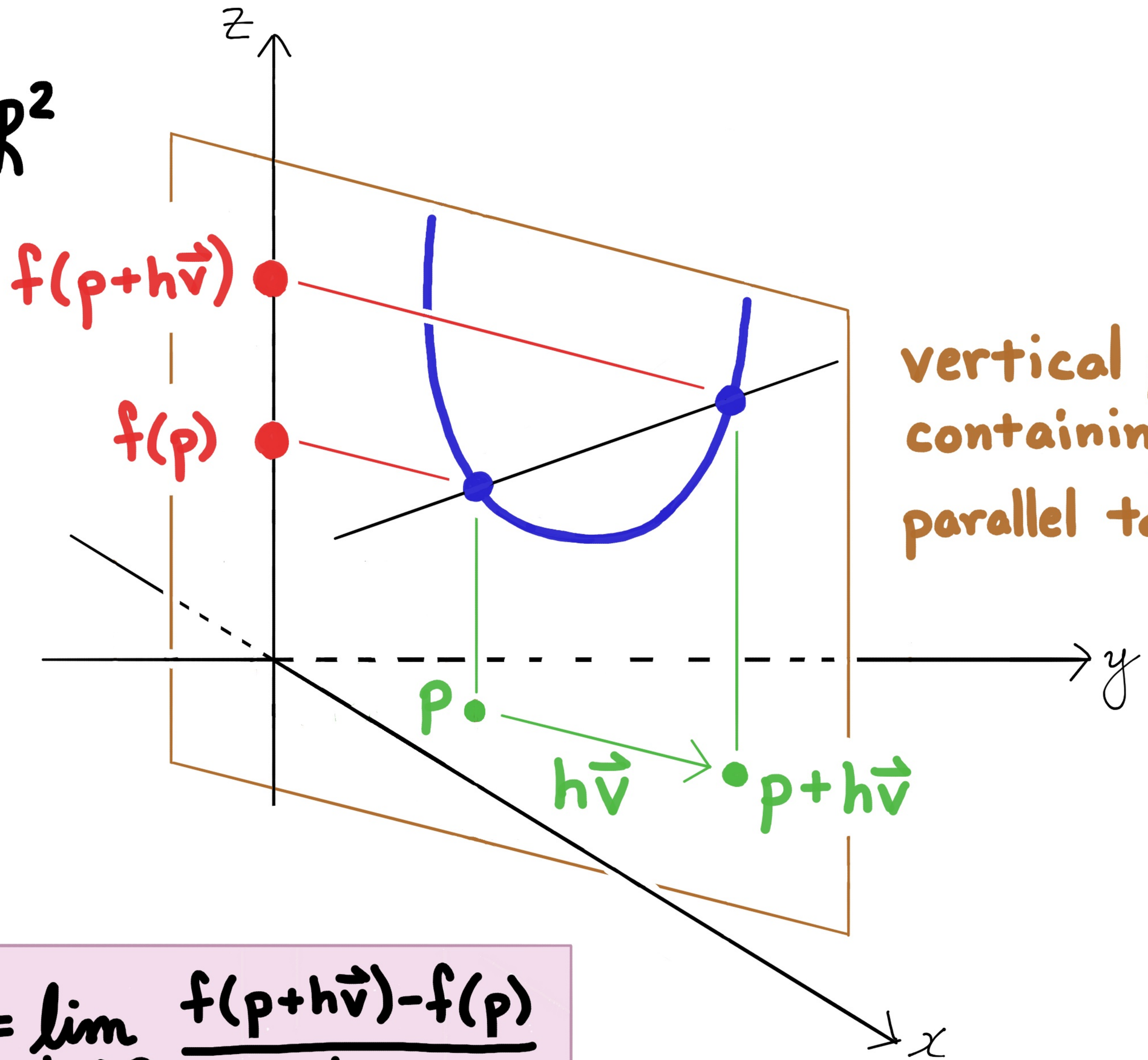
$f(x,y) = z$



vertical plane containing p, parallel to $\vec{v}$.

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Ⅱ The practice of computing directional derivatives

Theorem: Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is C^1, that is f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are continuous. Then for any $p, \vec{v} \in \mathbb{R}^2$,

$$D_p f(\vec{v}) = v_1 \frac{\partial f}{\partial x}\Big|_p + v_2 \frac{\partial f}{\partial y}\Big|_p$$

where $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

$$p, \vec{v} \in \mathbb{R}^2, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad f: \mathbb{R}^2 \to \mathbb{R}$$

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Example:

$f(x,y) = xy$, $p = (2,1)$.

Find $\frac{\partial f}{\partial x}\Big|_p$ and $\frac{\partial f}{\partial y}\Big|_p$.

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$$\frac{\partial f}{\partial x}\Big|_p = y\Big|_{(2,1)} = 1$$

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Find

$D_p f\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

$D_p f\begin{pmatrix} -3 \\ -3 \end{pmatrix}$

$D_p f\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$D_p f\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

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$$D_p f\begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5 \cdot 1 + 4 \cdot 2 = 13$$

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Lemma: For any $\Delta \in \mathbb{R}$,

$$D_p f(\Delta \vec{v}) = \Delta D_p f(\vec{v})$$

proof:

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$$= D_p f \begin{pmatrix} \Delta v_1 \\ \Delta v_2 \end{pmatrix}$$

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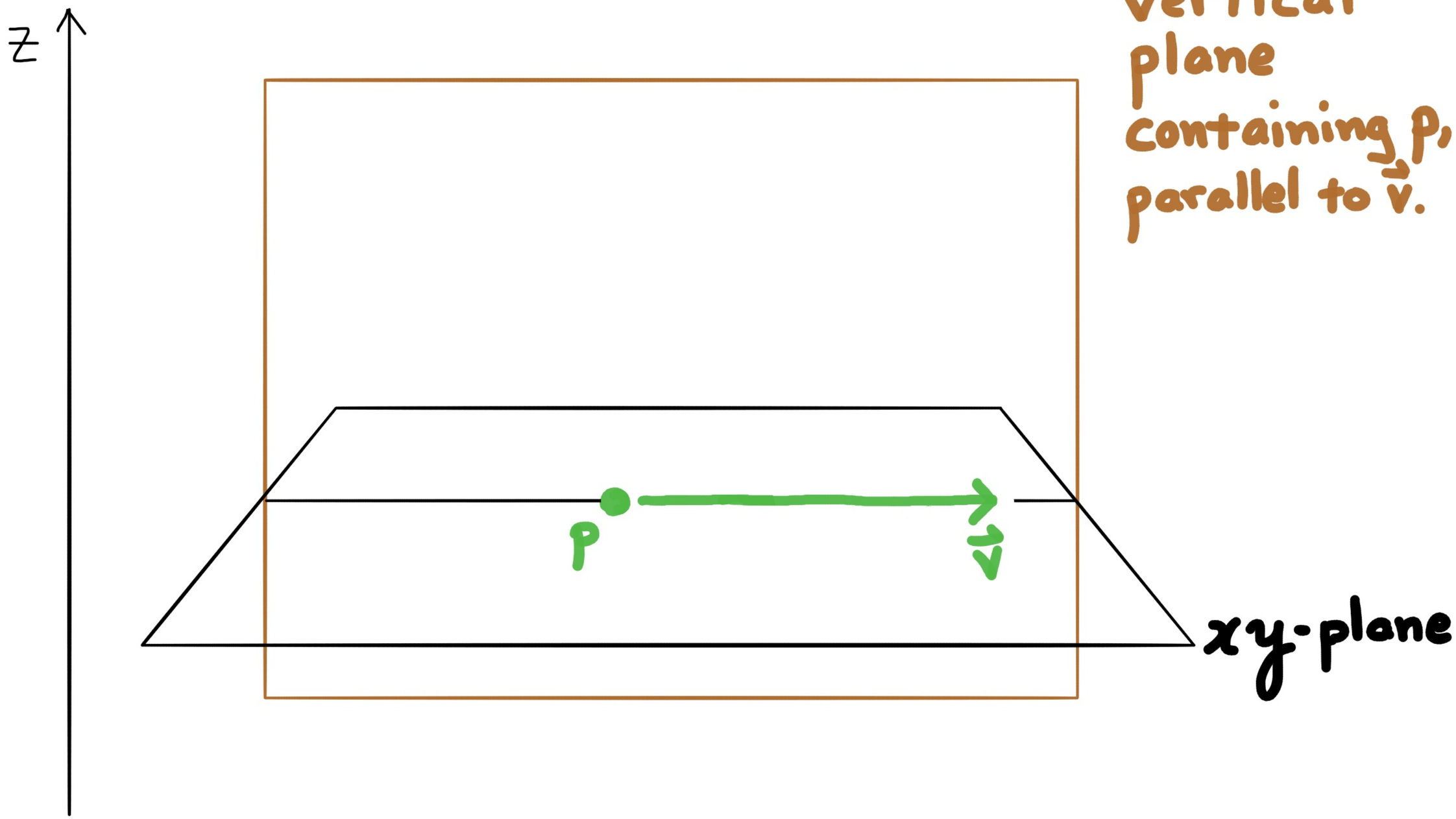
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$$= a\, D_p f(\vec{v}).$$

END

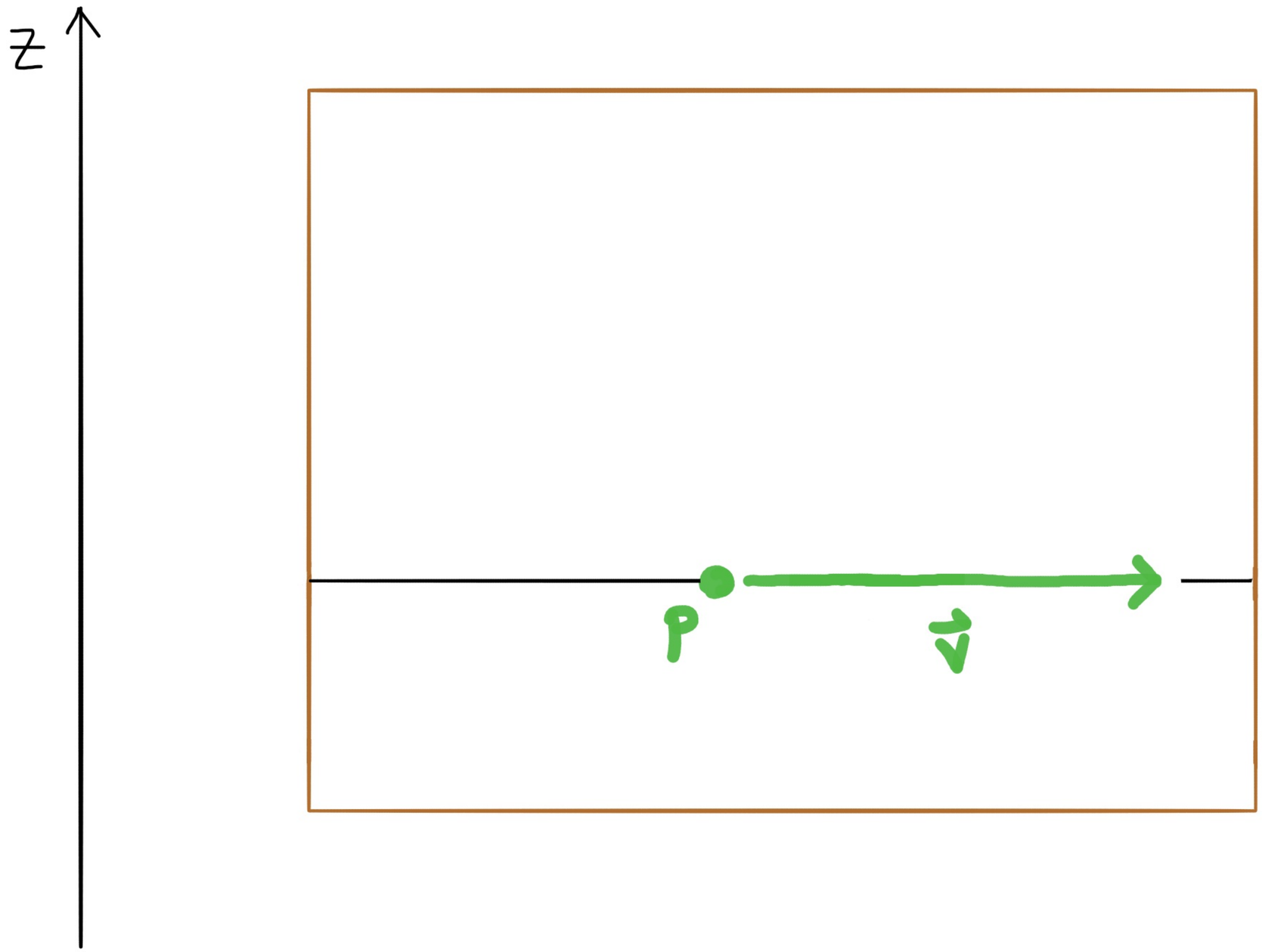
III Geometry of directional derivatives

$f: \mathbb{R}^2 \to \mathbb{R}$, $p, \vec{v} \in \mathbb{R}^2$, $\|\vec{v}\| = 1$.



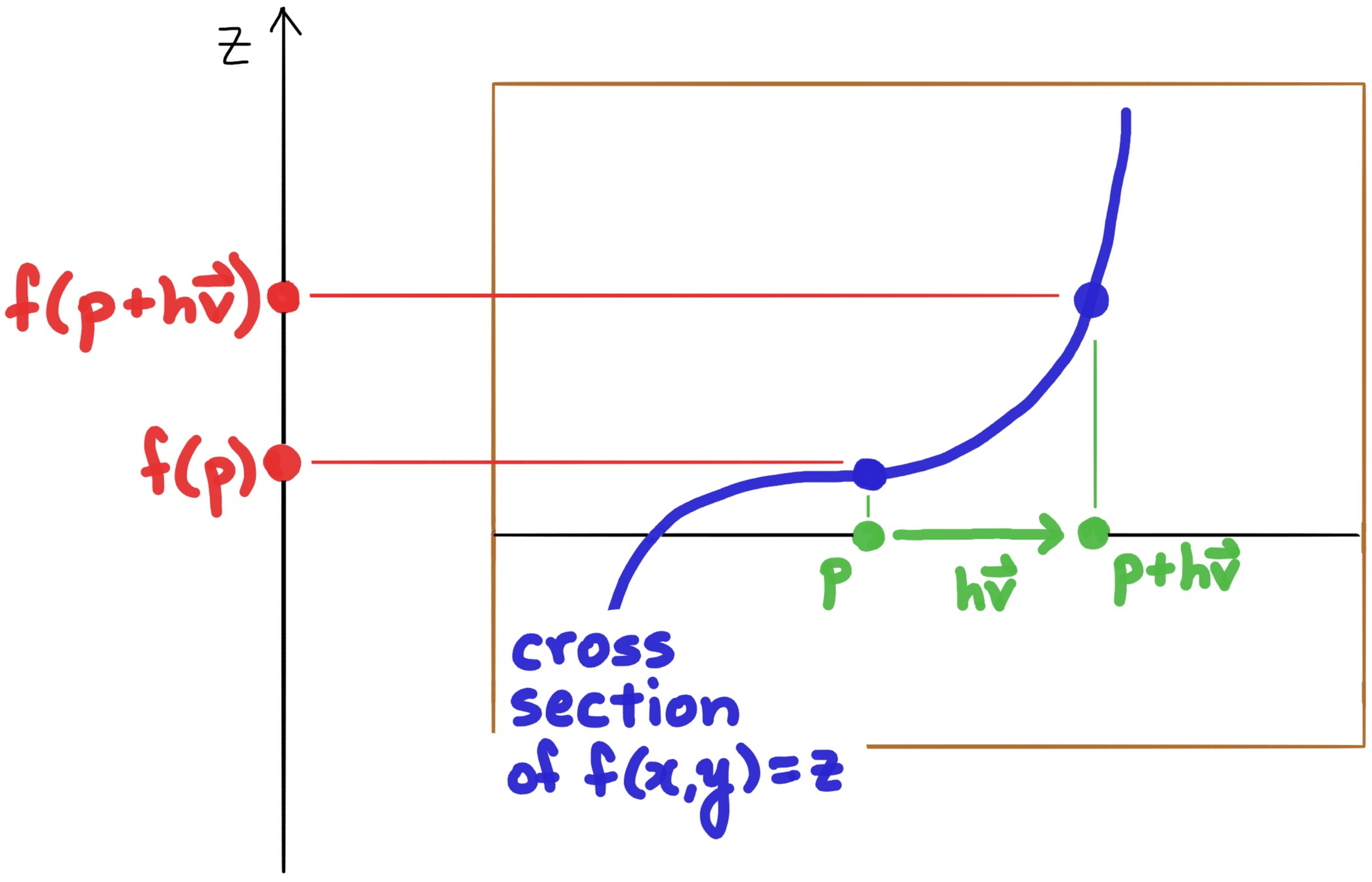
vertical plane containing p, parallel to $\vec{v}$.

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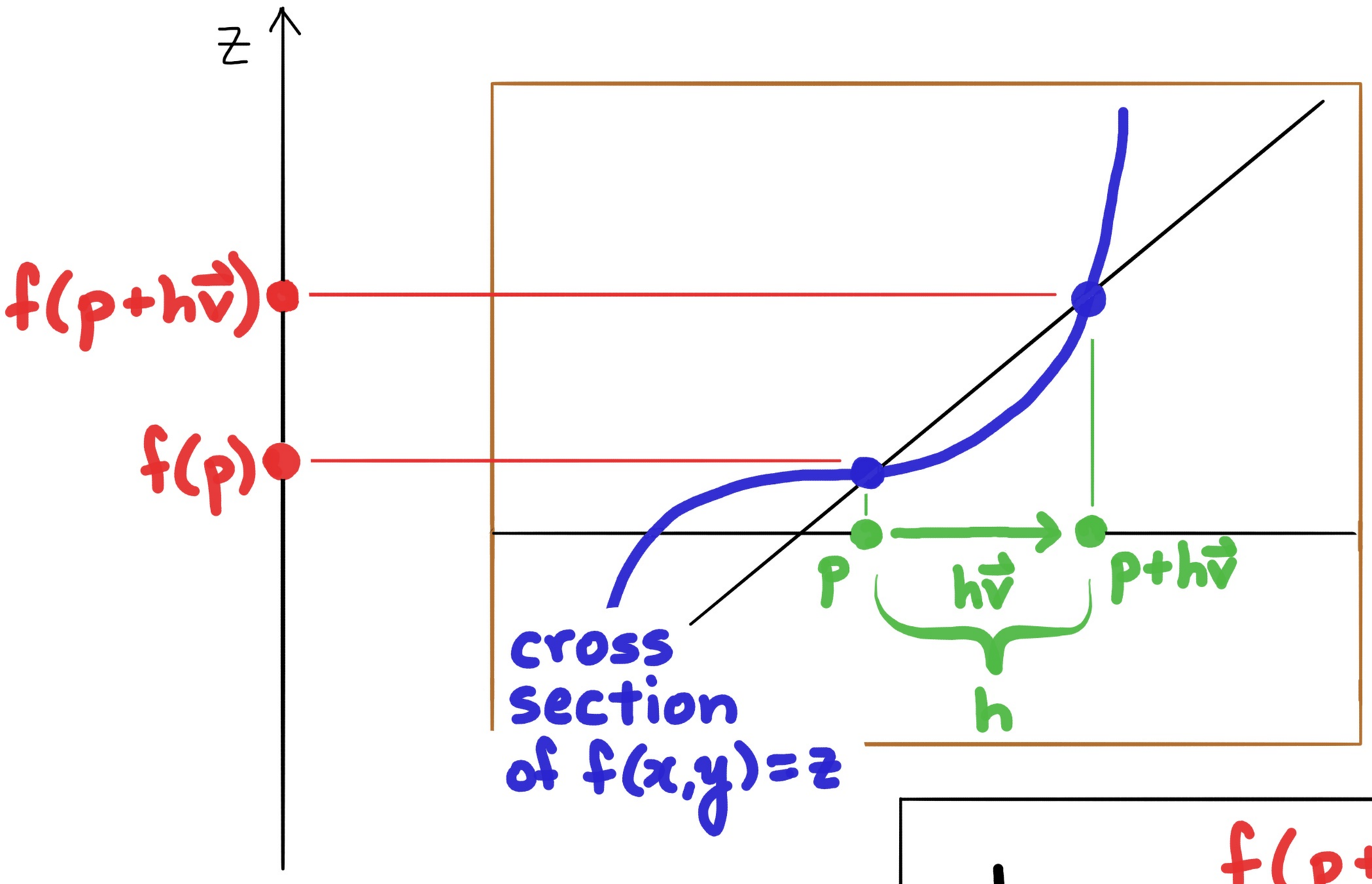


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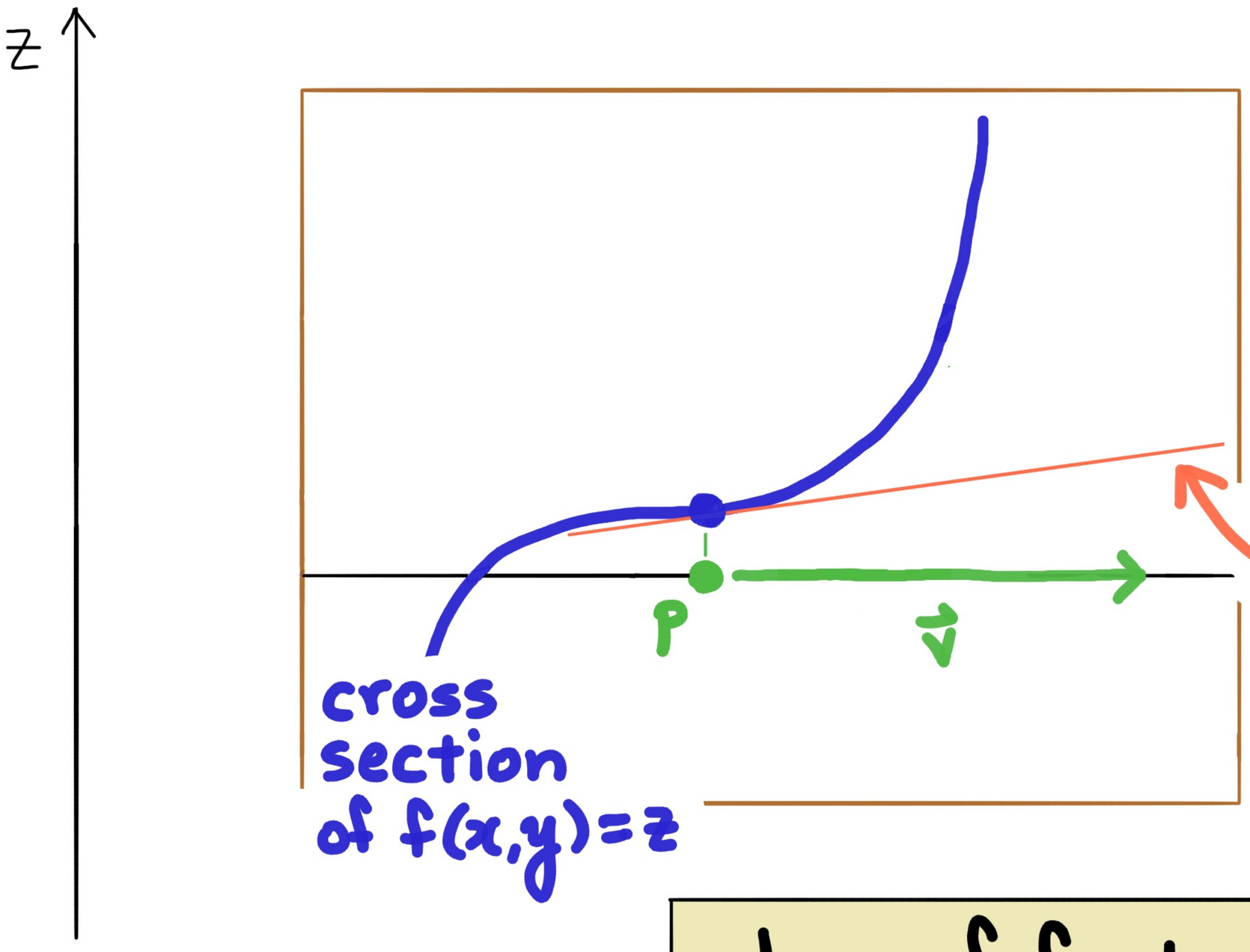
$f: \mathbb{R}^2 \to \mathbb{R}$, $p, \vec{v} \in \mathbb{R}^2$, $\|\vec{v}\| = 1$.



vertical plane containing p, parallel to $\vec{v}$.

slope: $\dfrac{f(p+h\vec{v}) - f(p)}{h}$

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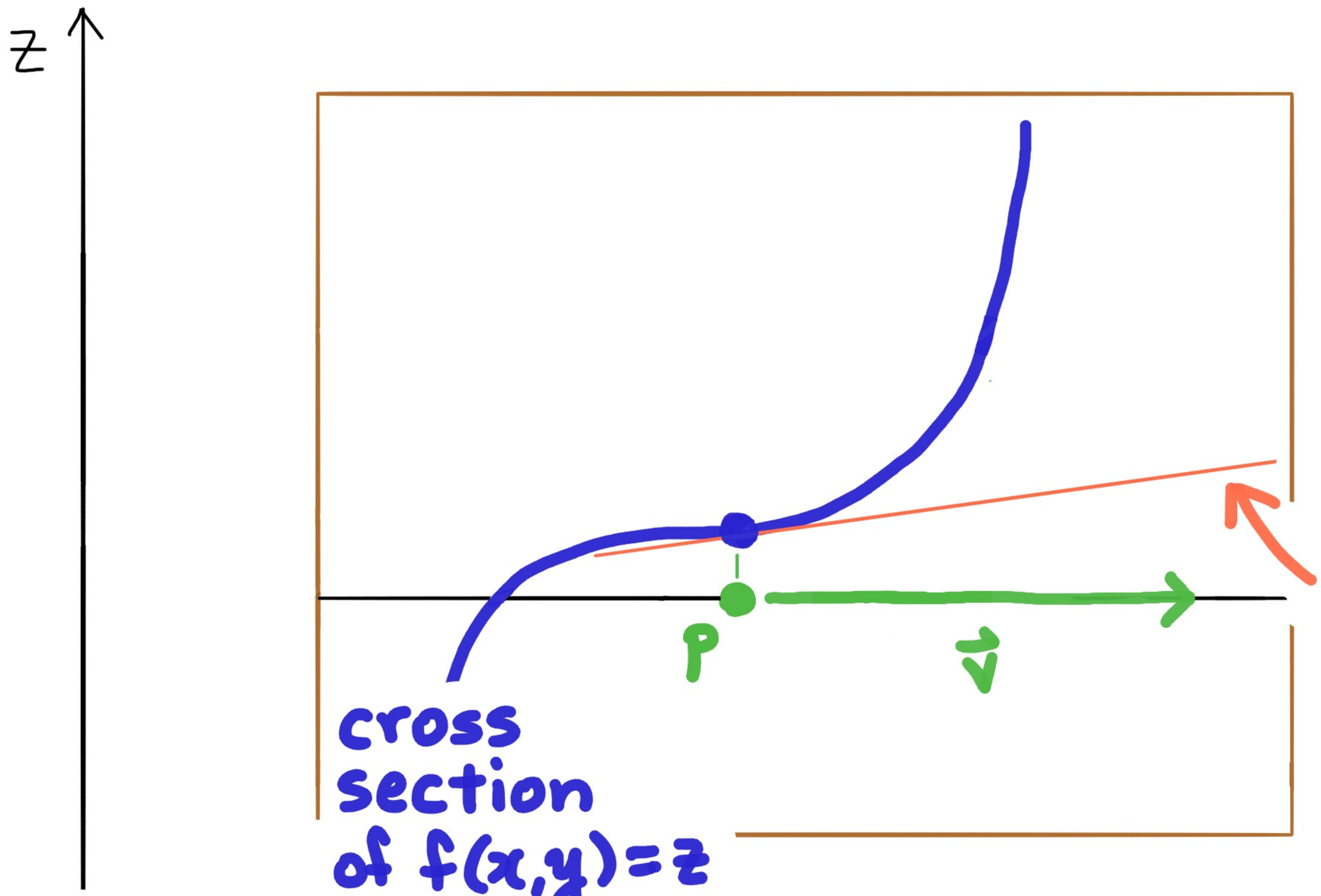
vertical plane containing p, parallel to $\vec{v}$.

tangent line of cross section

cross section of $f(x,y)=z$

slope of f at p in direction of $\vec{v}$: $D_p f(\vec{v})$

$f: \mathbb{R}^2 \to \mathbb{R}$, $p, \vec{v} \in \mathbb{R}^2$, $\|\vec{v}\| = 1$.



vertical plane containing p, parallel to $\vec{v}$.

slope is "rate of change"

slope of f at p in direction of $\vec{v}$: $D_p f(\vec{v})$

Summary: $f: \mathbb{R}^2 \to \mathbb{R}$, $p, \vec{v} \in \mathbb{R}^2$, $\|\vec{v}\| = 1$.

$D_p f(\vec{v})$ is the slope of the graph of f at p in direction of $\vec{v}$.

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Generally, slope of graph of f at p in direction of $\vec{v}$ is

$$D_p f\left(\frac{\vec{v}}{\|\vec{v}\|}\right)$$

since $\left\|\frac{\vec{v}}{\|\vec{v}\|}\right\| = 1$.

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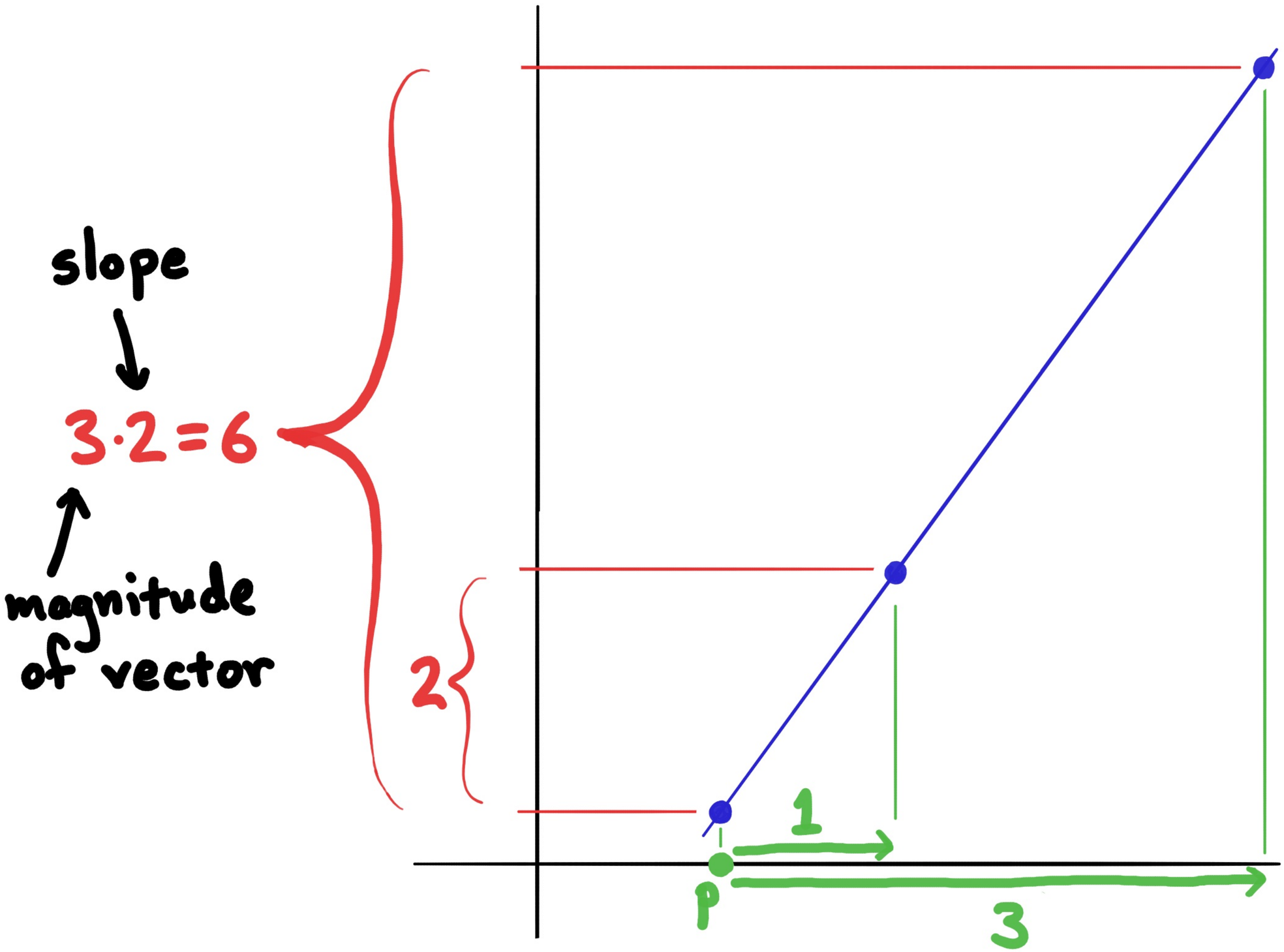
$$D_p f(\vec{v}) = \|\vec{v}\| \frac{D_p f(\vec{v})}{\|\vec{v}\|}$$

↑ derivative of f at p with respect to $\vec{v}$

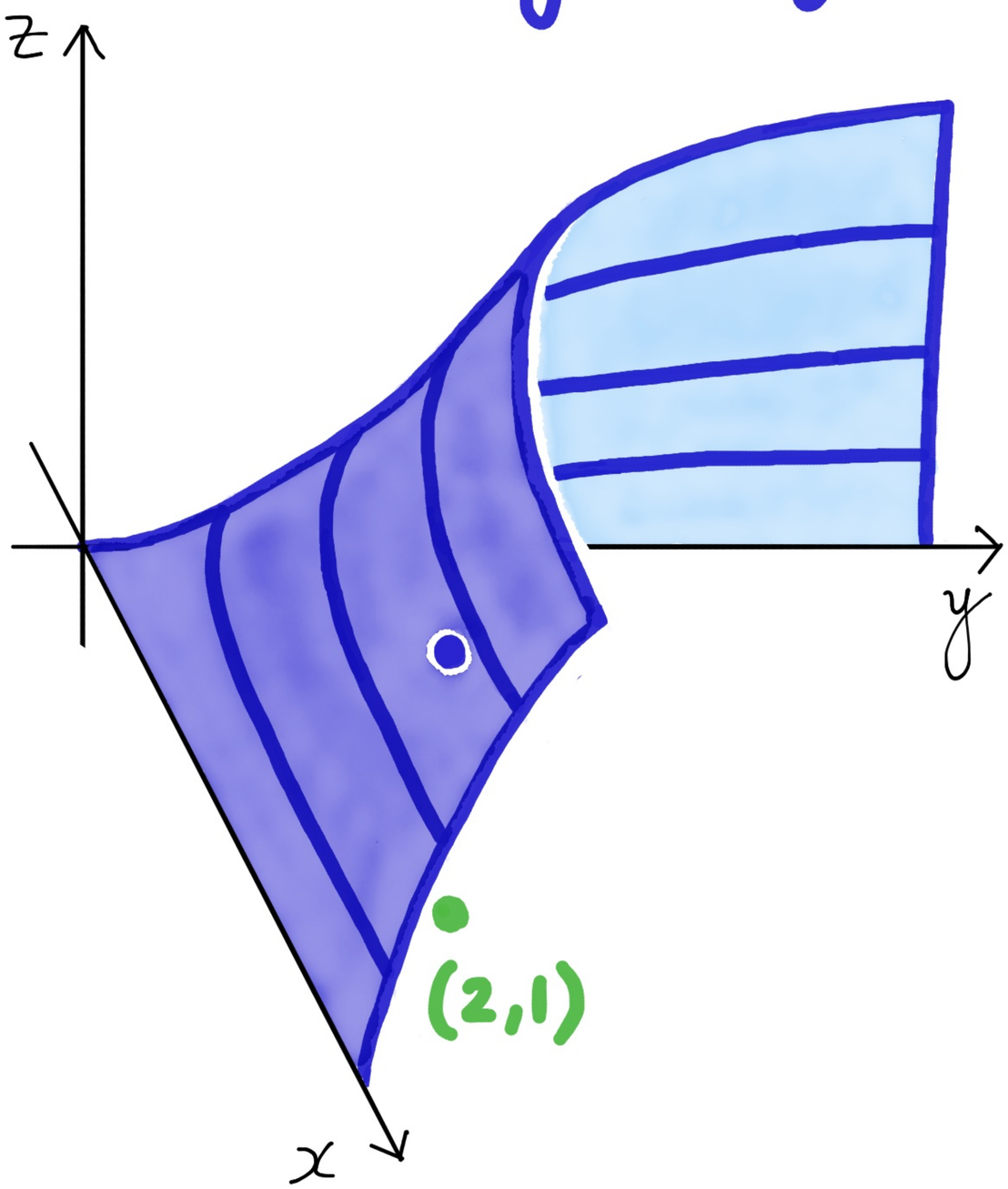
↑ magnitude of $\vec{v}$

↑ slope of graph of f at p in direction of $\vec{v}$

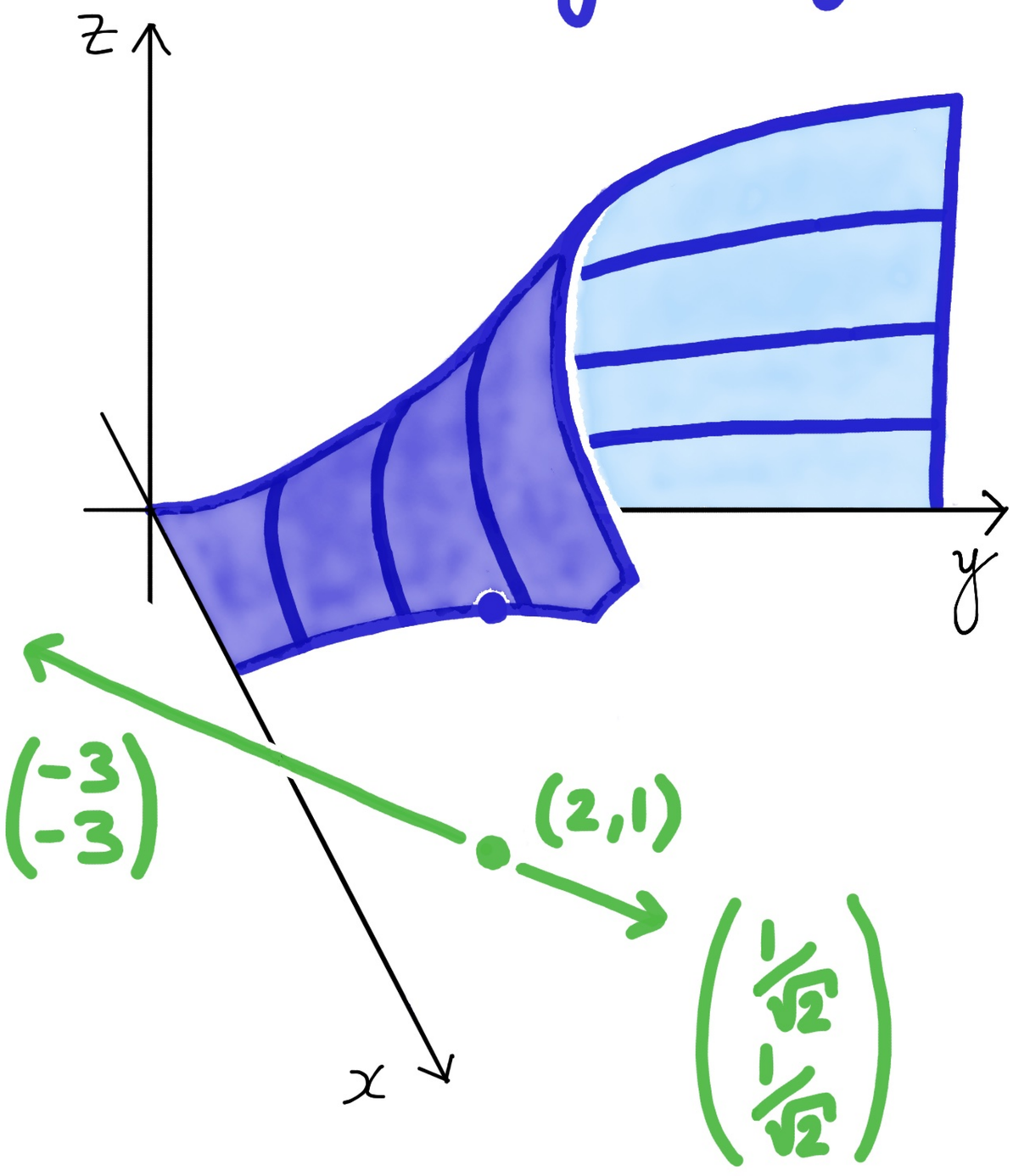
line of slope 2



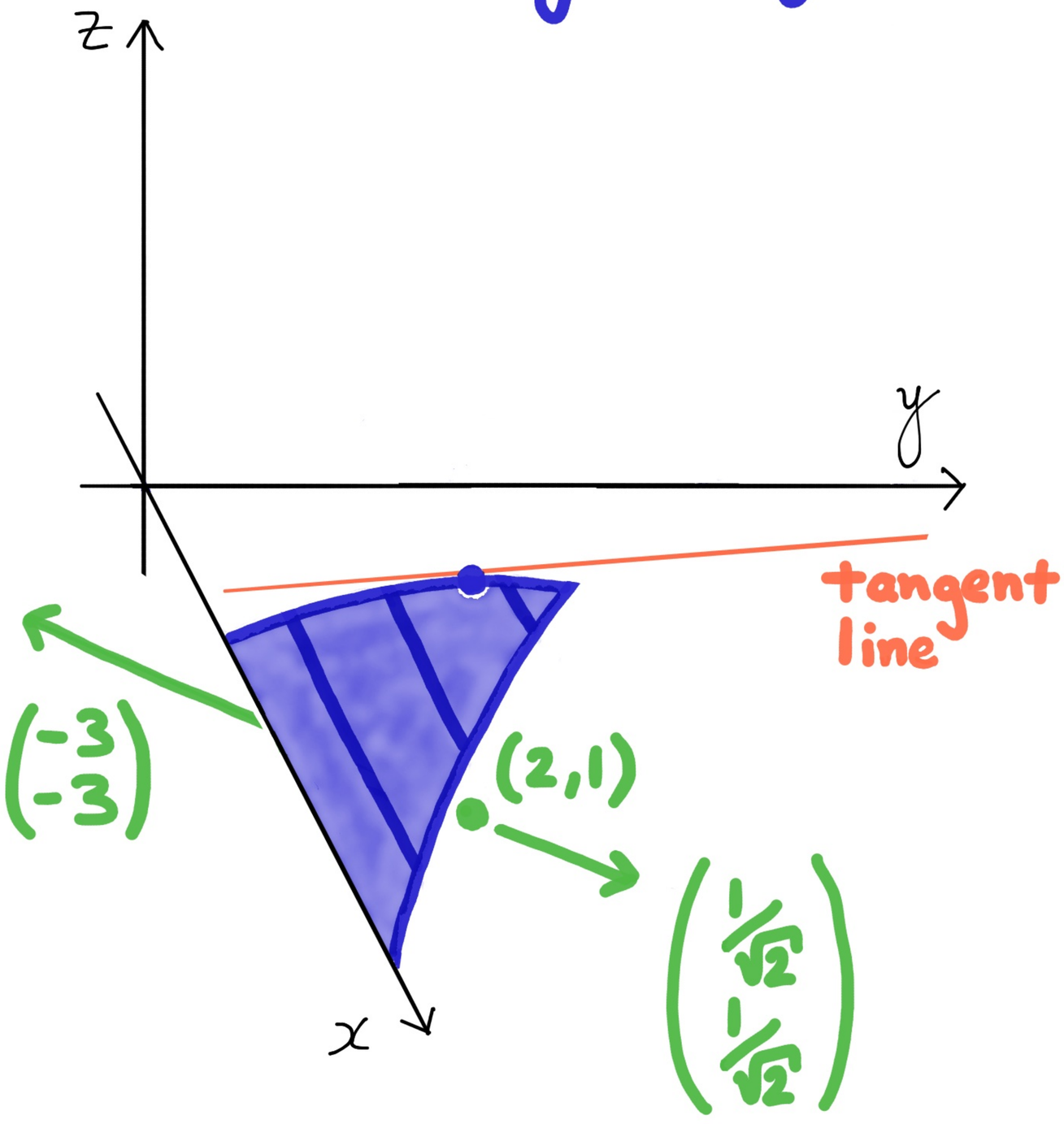
$$f(x,y)=xy$$



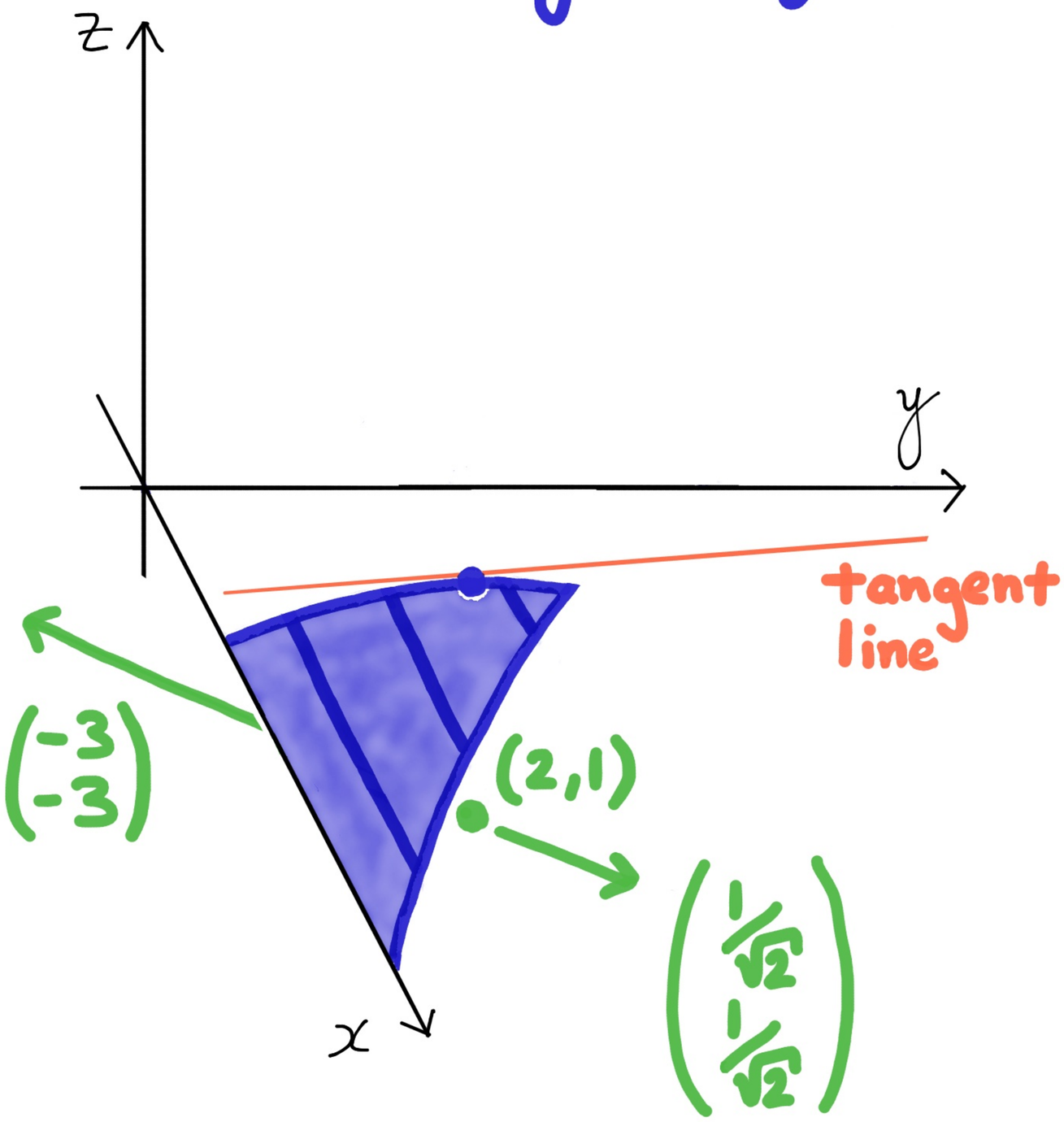
$$f(x,y) = xy$$

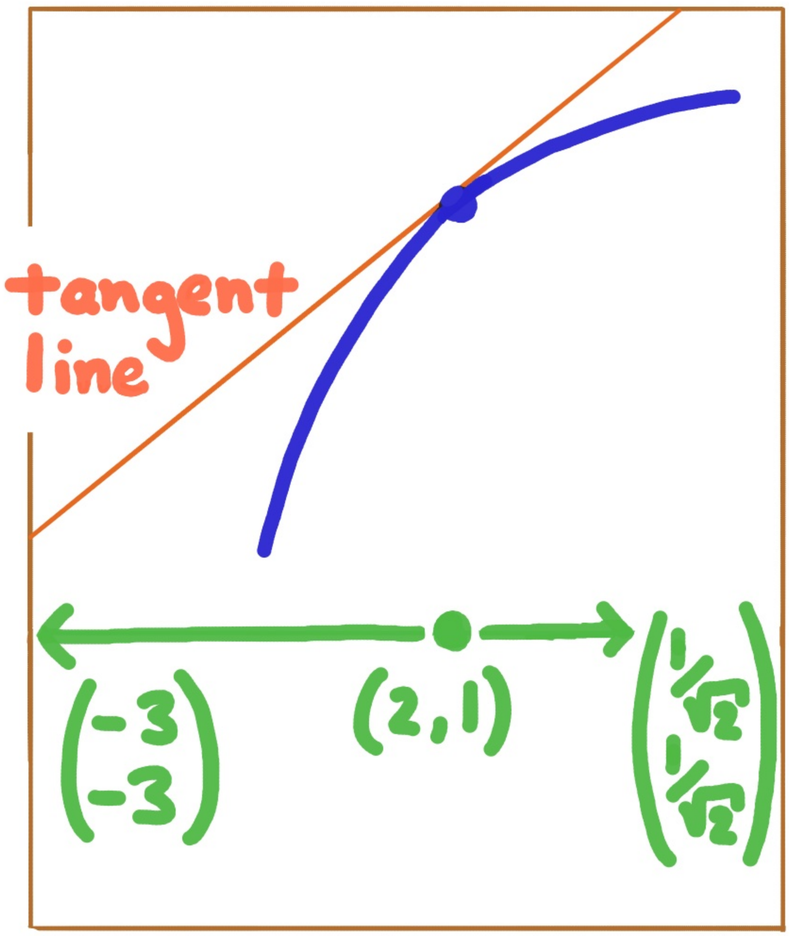

z
y
x
$\begin{pmatrix} -3 \\ -3 \end{pmatrix}$
(2,1)
$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$f(x,y) = xy$$

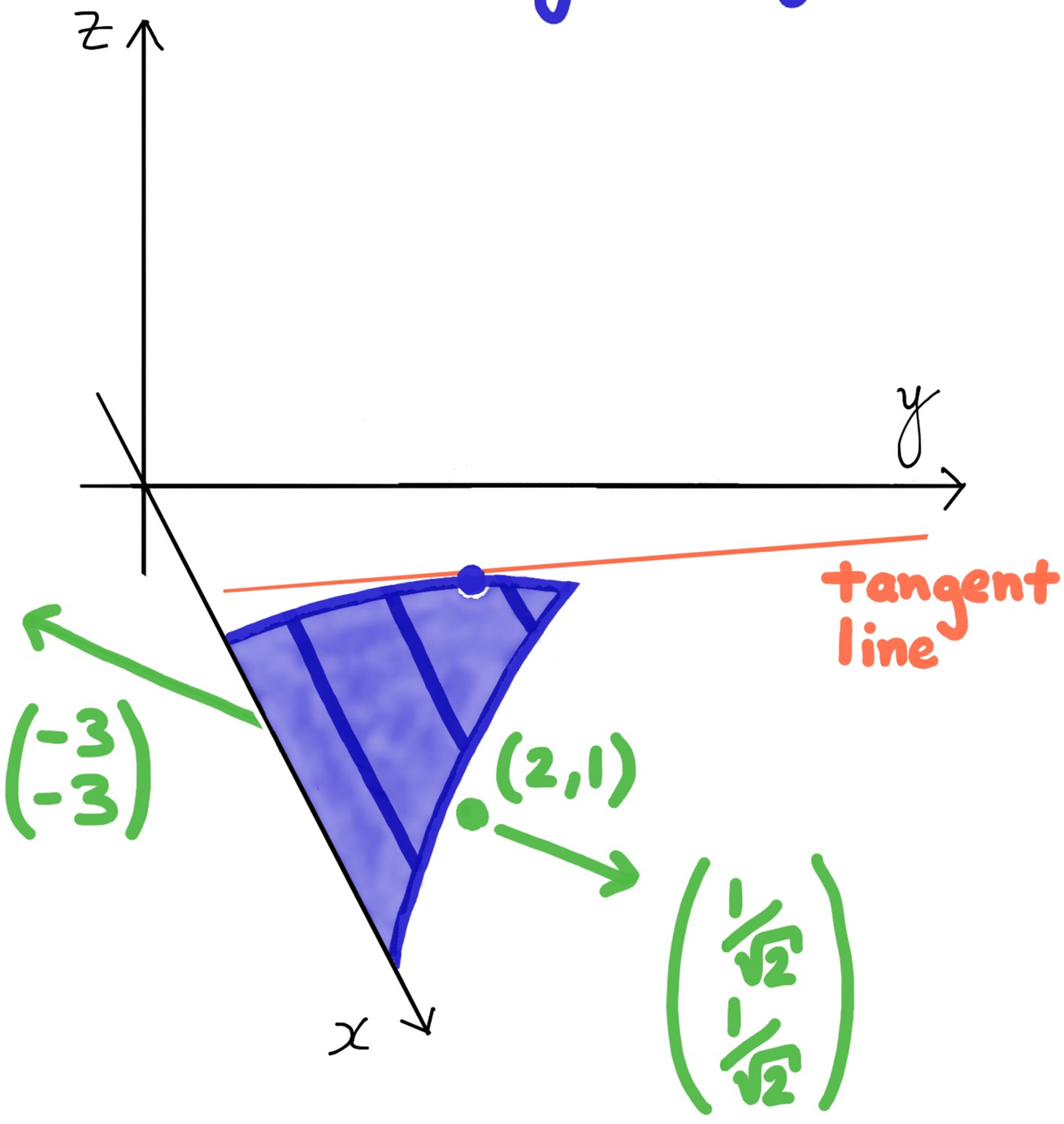

z
y
x
tangent line
$\begin{pmatrix} -3 \\ -3 \end{pmatrix}$
(2,1)
$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$f(x,y) = xy$$





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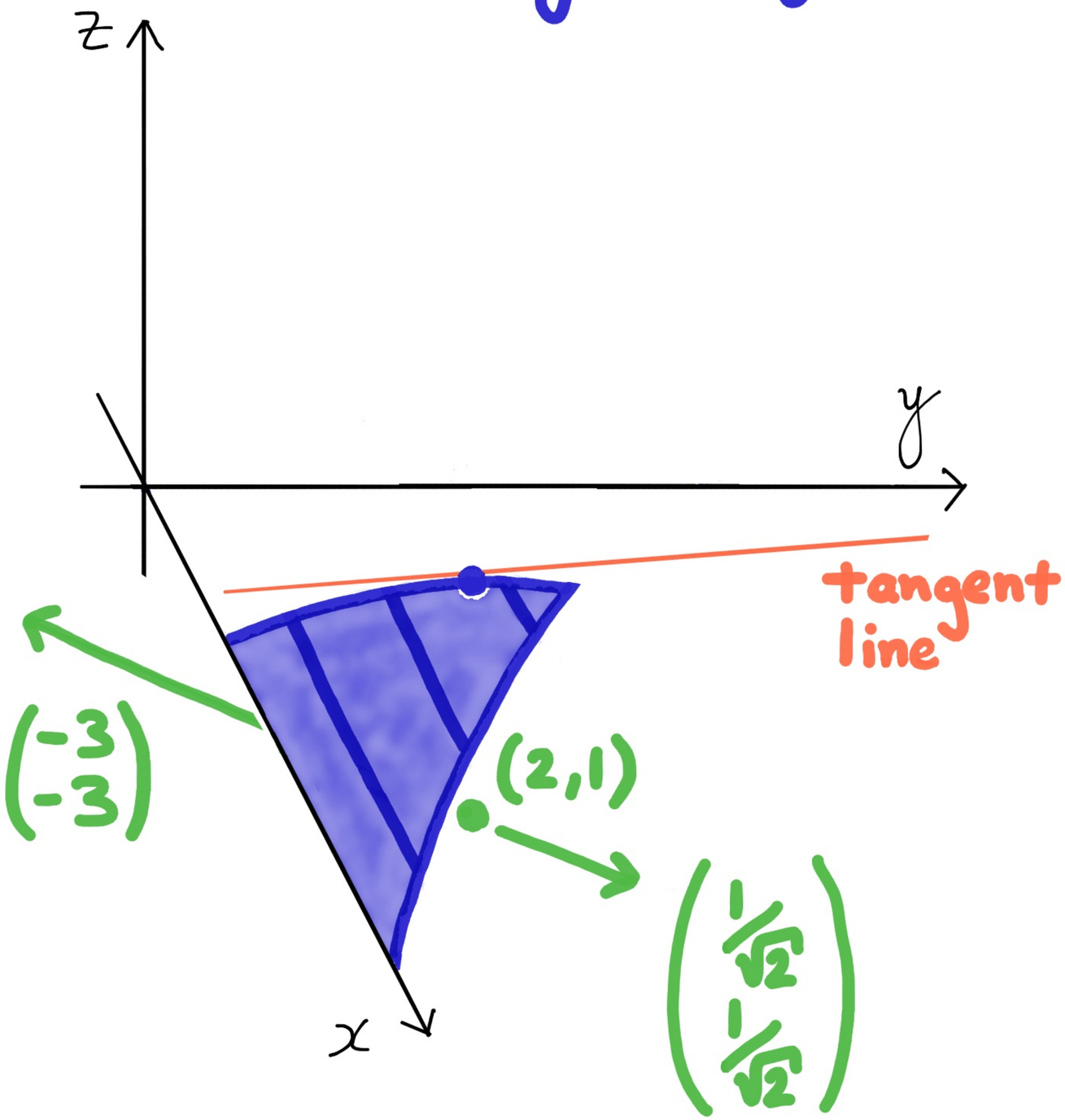


slope in direction of $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$:

$$D_{(2,1)}f\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \Big/ \left\|\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\|$$

$$= \frac{3}{\sqrt{2}} \Big/ 1 = \frac{3}{\sqrt{2}}$$

$$f(x,y) = xy$$



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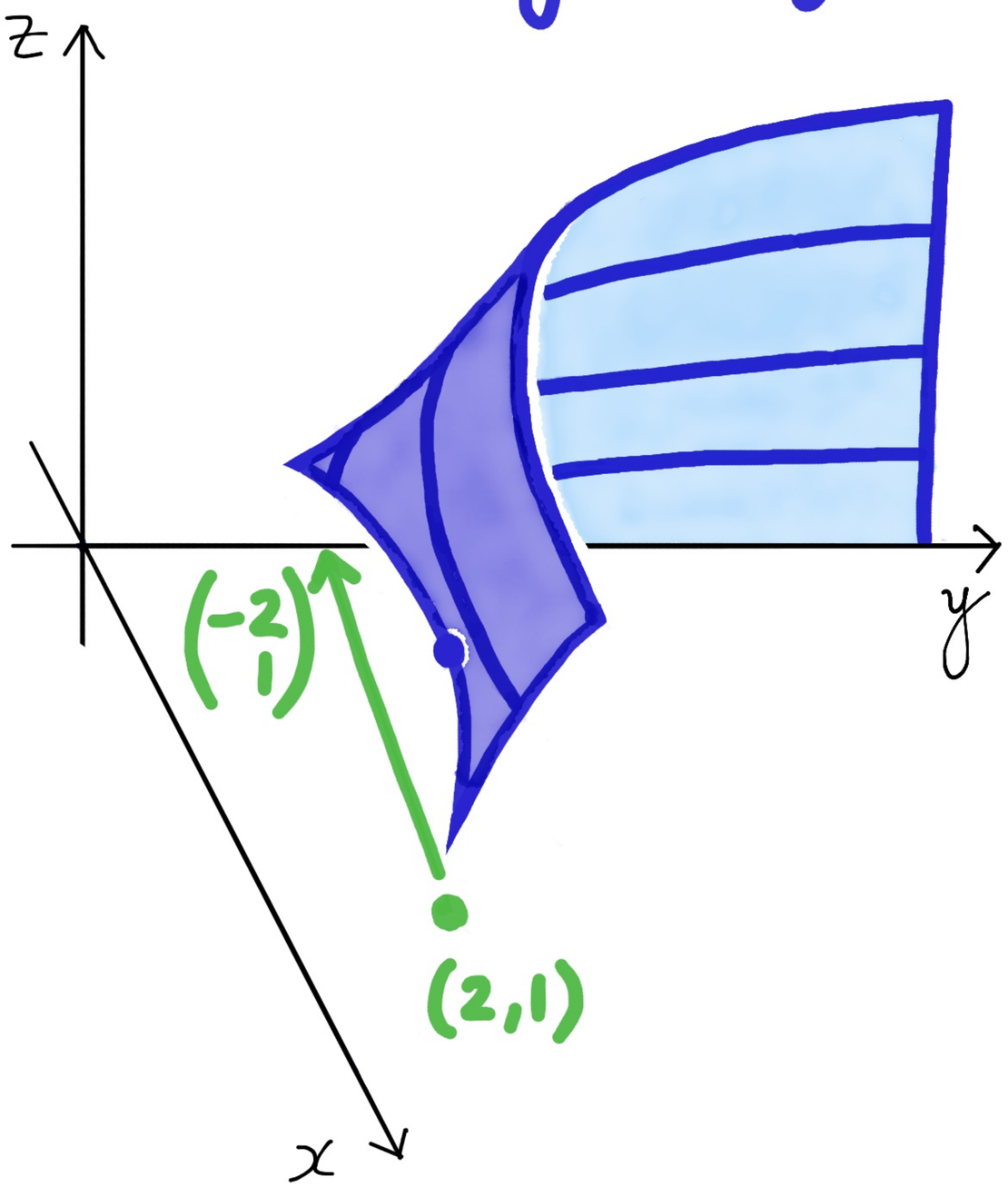
$$= \frac{3/\sqrt{2}}{1} = \frac{3}{\sqrt{2}}$$

slope in direction of $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$:

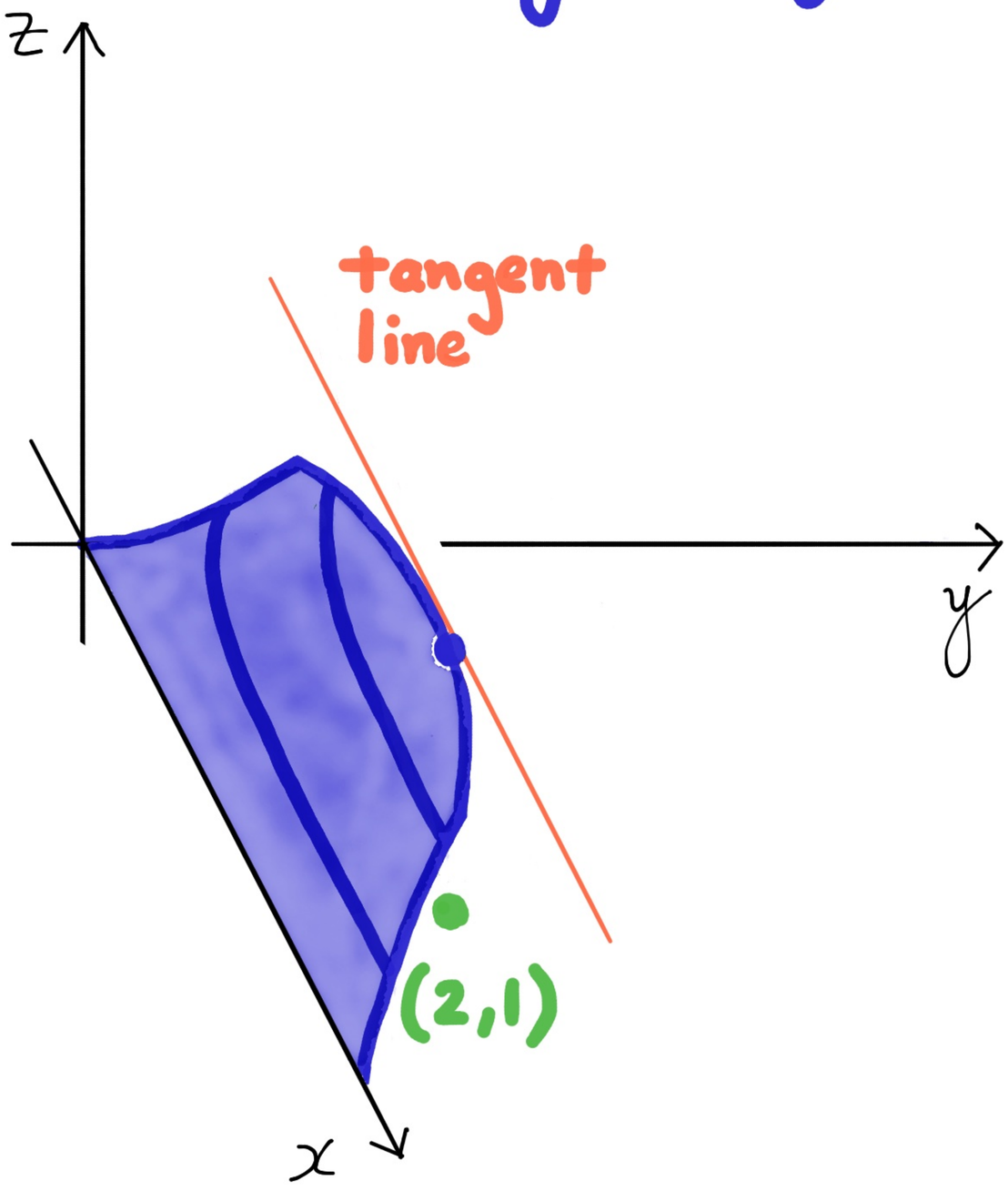
$$D_{(2,1)}f\begin{pmatrix} -3 \\ -3 \end{pmatrix} \Big/ \left\|\begin{pmatrix} -3 \\ -3 \end{pmatrix}\right\|$$

$$= \frac{-9}{\sqrt{18}} = -\frac{3}{\sqrt{2}}$$

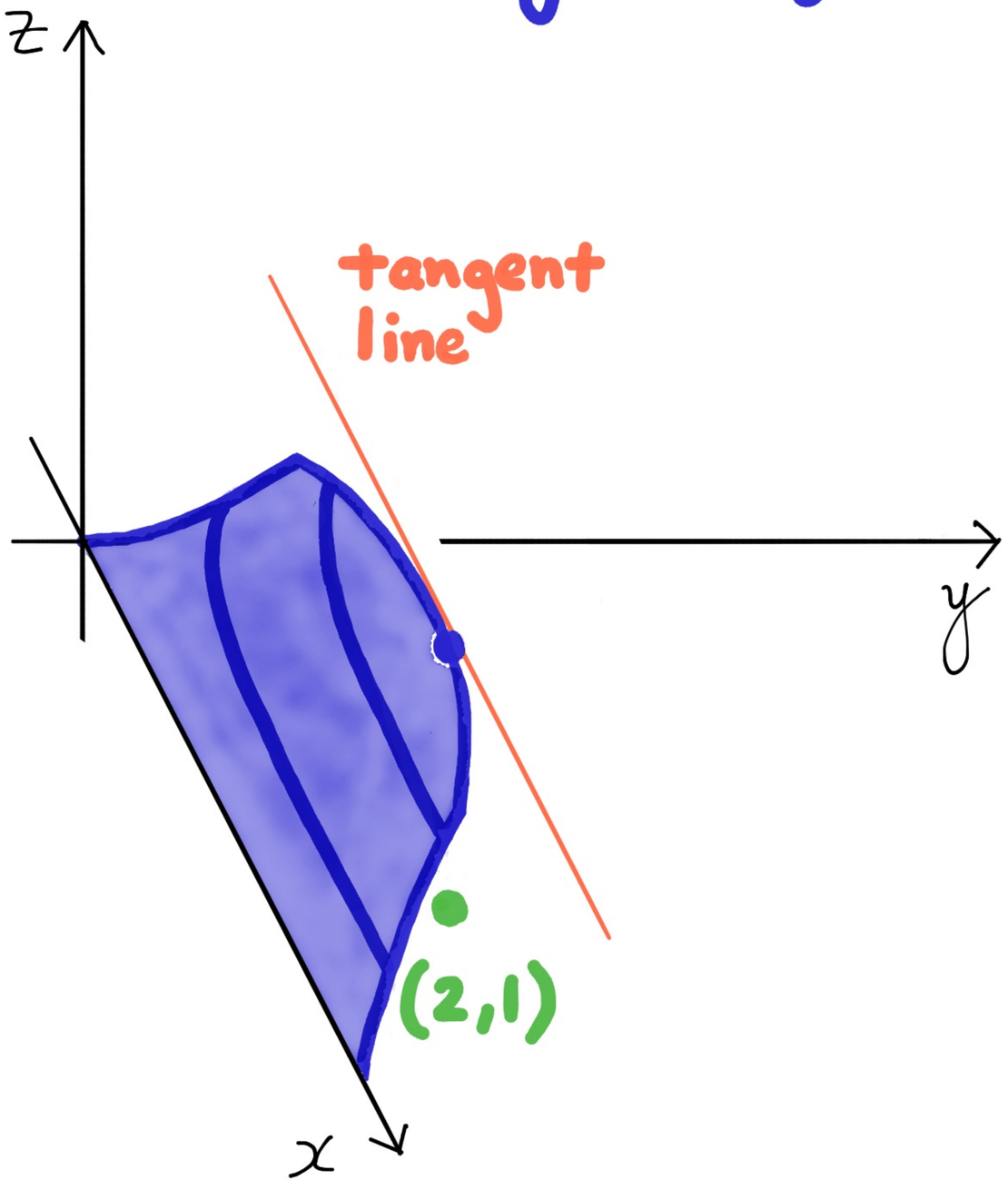
$$f(x,y) = xy$$

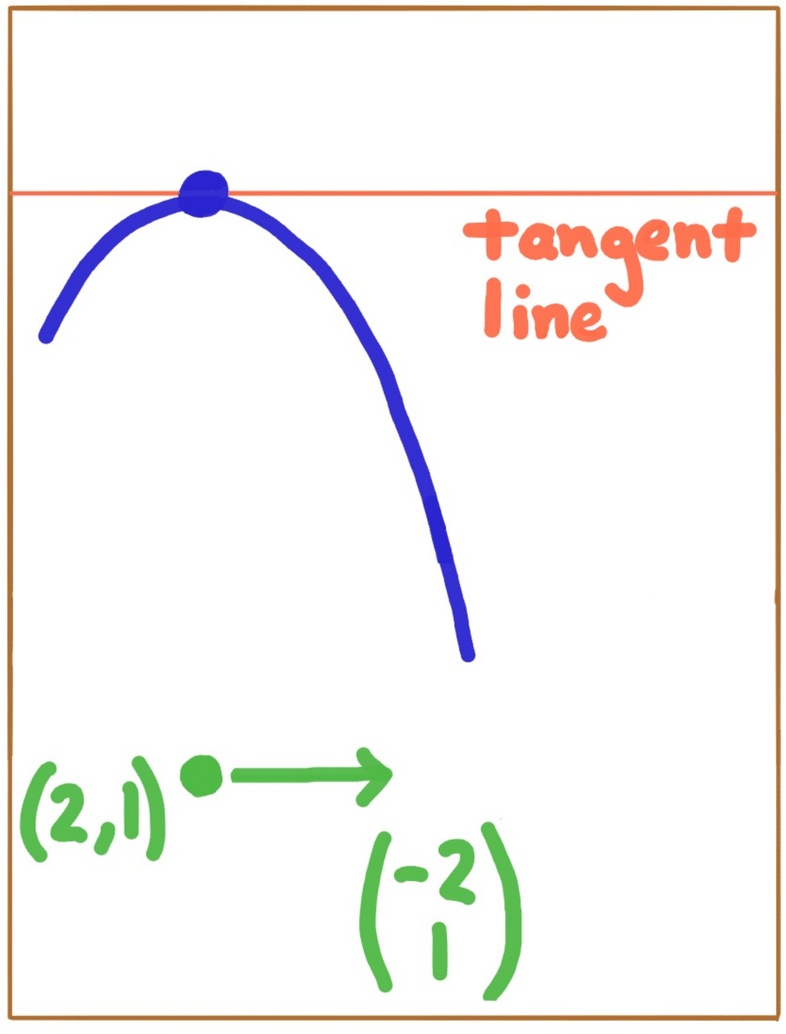


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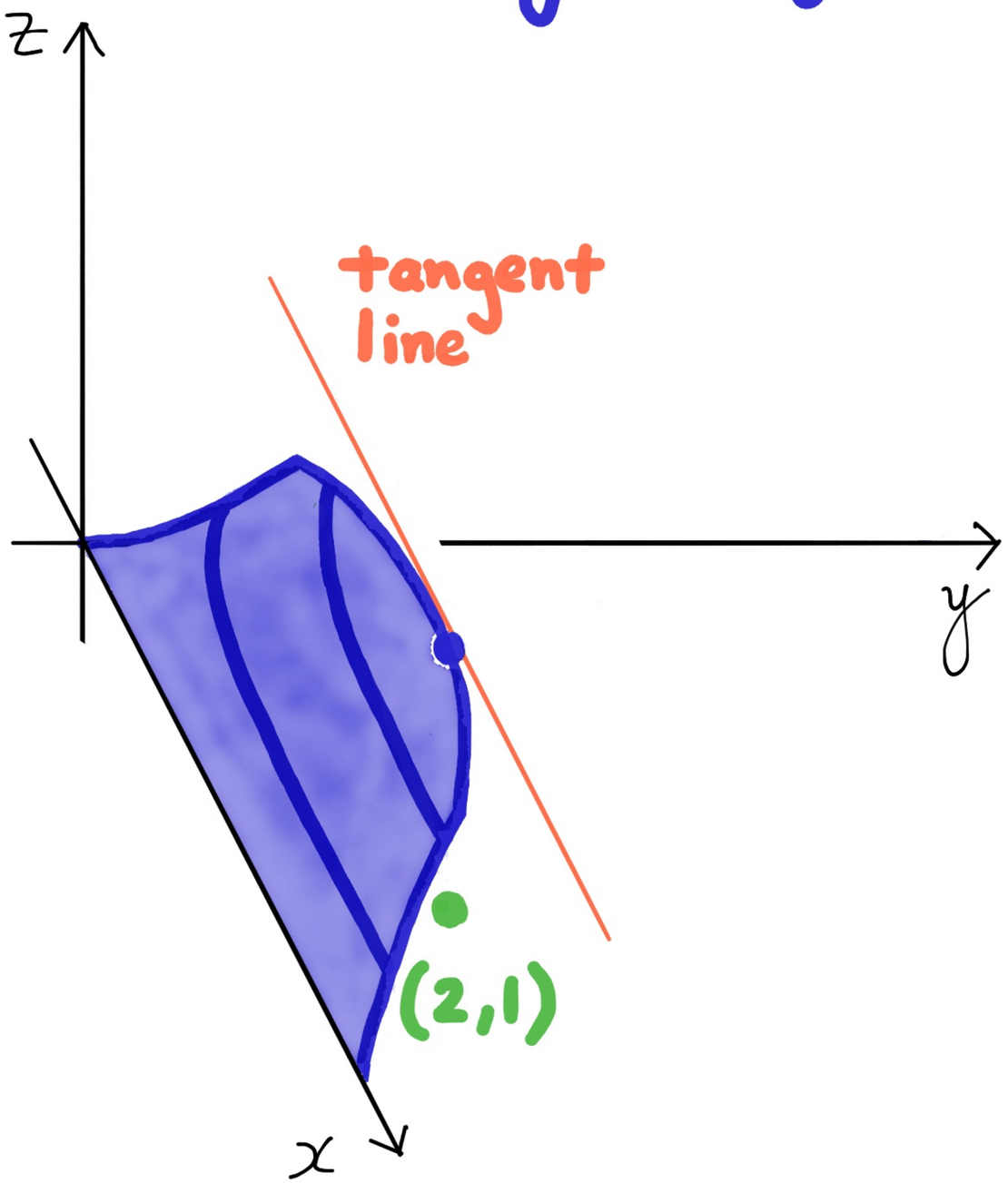


$$f(x,y) = xy$$





$$f(x,y) = xy$$



slope in direction of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$:

$$\frac{D_{(2,1)}f\begin{pmatrix} -2 \\ 1 \end{pmatrix}}{\left\|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\|}$$

$$= \frac{0}{\left\|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\|}$$

$$= 0$$

Ⅳ Jacobian matrices

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$D_p f(\vec{v}) = v_1 \left.\frac{\partial f}{\partial x}\right|_p + v_2 \left.\frac{\partial f}{\partial y}\right|_p$$

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$D_p f = \left(\frac{\partial f}{\partial x}\Big|_p , \frac{\partial f}{\partial y}\Big|_p \right)$ is the Jacobian matrix of f at p.

Example:

$f(x,y) = 2xy^2 - 3x$. $D_p f = \left(\frac{\partial f}{\partial x}\Big|_p, \frac{\partial f}{\partial y}\Big|_p \right)$.

Find the Jacobian matrix of f at $(3,-1)$.

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Find $D_{(3,-1)}f\begin{pmatrix}4\\3\end{pmatrix}$.

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Find the slope of the graph of f at $(3,-1)$ in the direction of $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

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The slope of the graph of f at $(3,-1)$ in the direction of $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is

$$D_{(3,-1)}f \begin{pmatrix} 4 \\ 3 \end{pmatrix} \Big/ \left\| \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right\| = -40 \Big/ \sqrt{4^2+3^2} = -8.$$

Example:

$g(x,y,z) = xyz^2$. $D_p g = \left(\frac{\partial g}{\partial x}\Big|_p, \frac{\partial g}{\partial y}\Big|_p, \frac{\partial g}{\partial z}\Big|_p \right)$.

Find the slope of the graph of g at $(2,1,3)$ in the direction of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Example:

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$$\text{slope: } \frac{D_{(2,1,3)} g \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\|} = \frac{19}{\sqrt{1^2+2^2+2^2}} = \frac{19}{3}$$

C^1 functions are approximated by matrices

$$\lim_{h \to 0} \frac{f(p + h\vec{v}) - f(p)}{h} = D_p f(\vec{v}) \,,$$

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> If $h\vec{v} \approx \vec{0}$, then $f(p+h\vec{v})-f(p) \approx D_p f(h\vec{v})$.

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> If $\vec{w} \approx \vec{0}$, then $f(p+\vec{w})-f(p) \approx D_p f(\vec{w})$.

C^1 functions are approximated by matrices

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Rephrased:

> If $\vec{v} \approx \vec{0}$, then $f(p+\vec{v})-f(p) \approx D_p f(\vec{v})$.

If $\vec{v} \approx \vec{0}$, then $f(p+\vec{v}) - f(p) \approx D_p f(\vec{v})$.

