Eleven

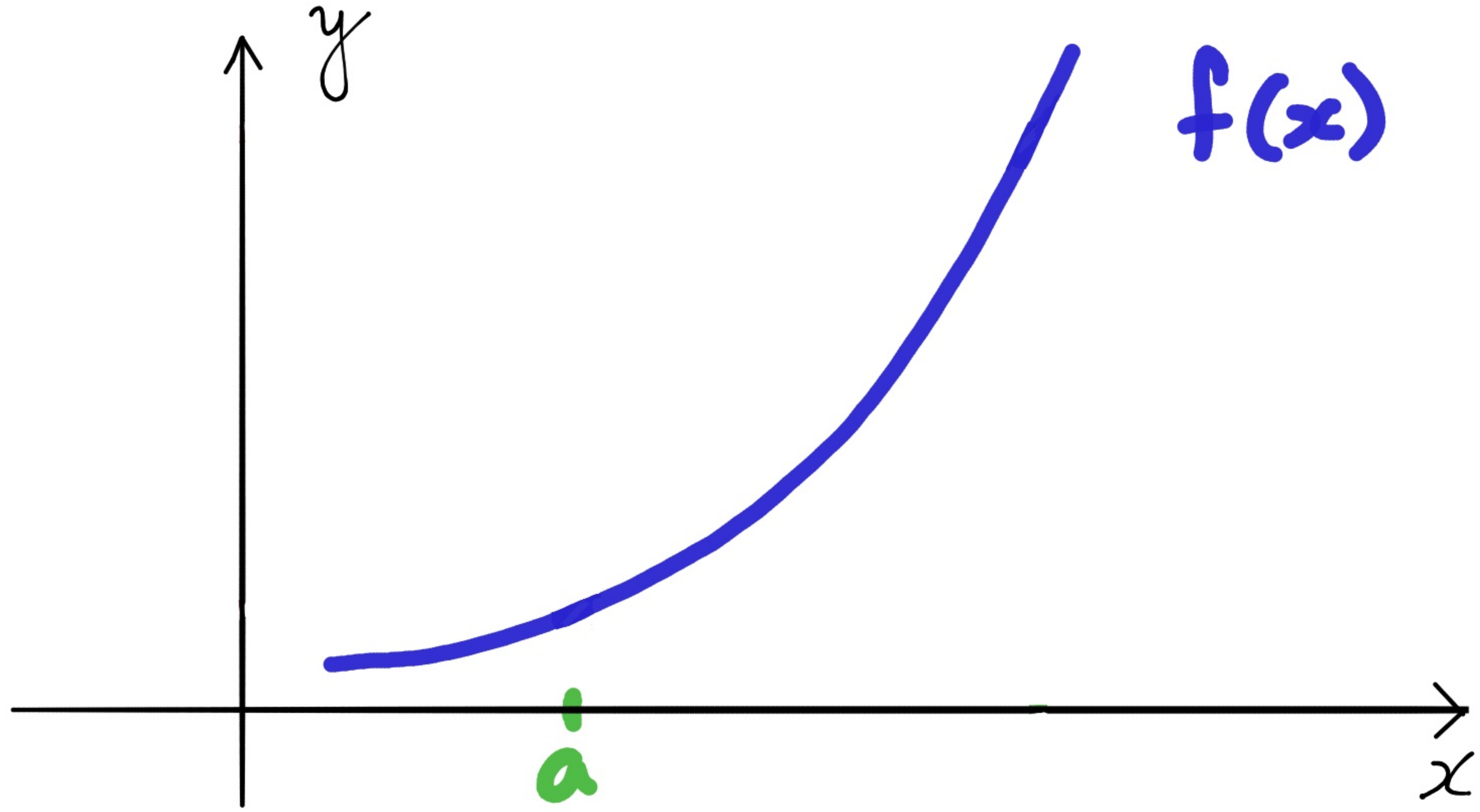
I First partial derivatives

II Second partial derivatives

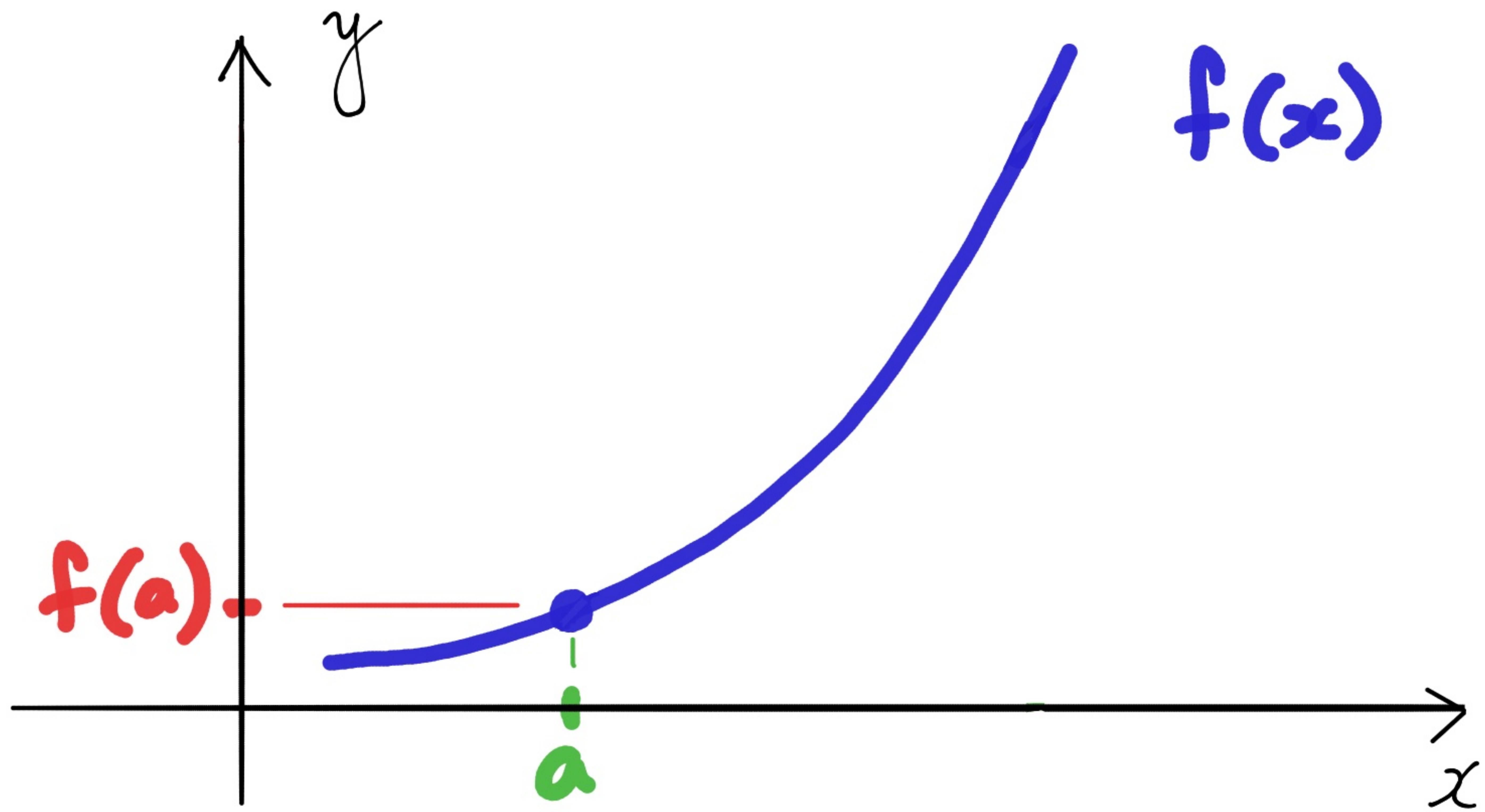
III C^1 and C^2 functions

I First partial derivatives

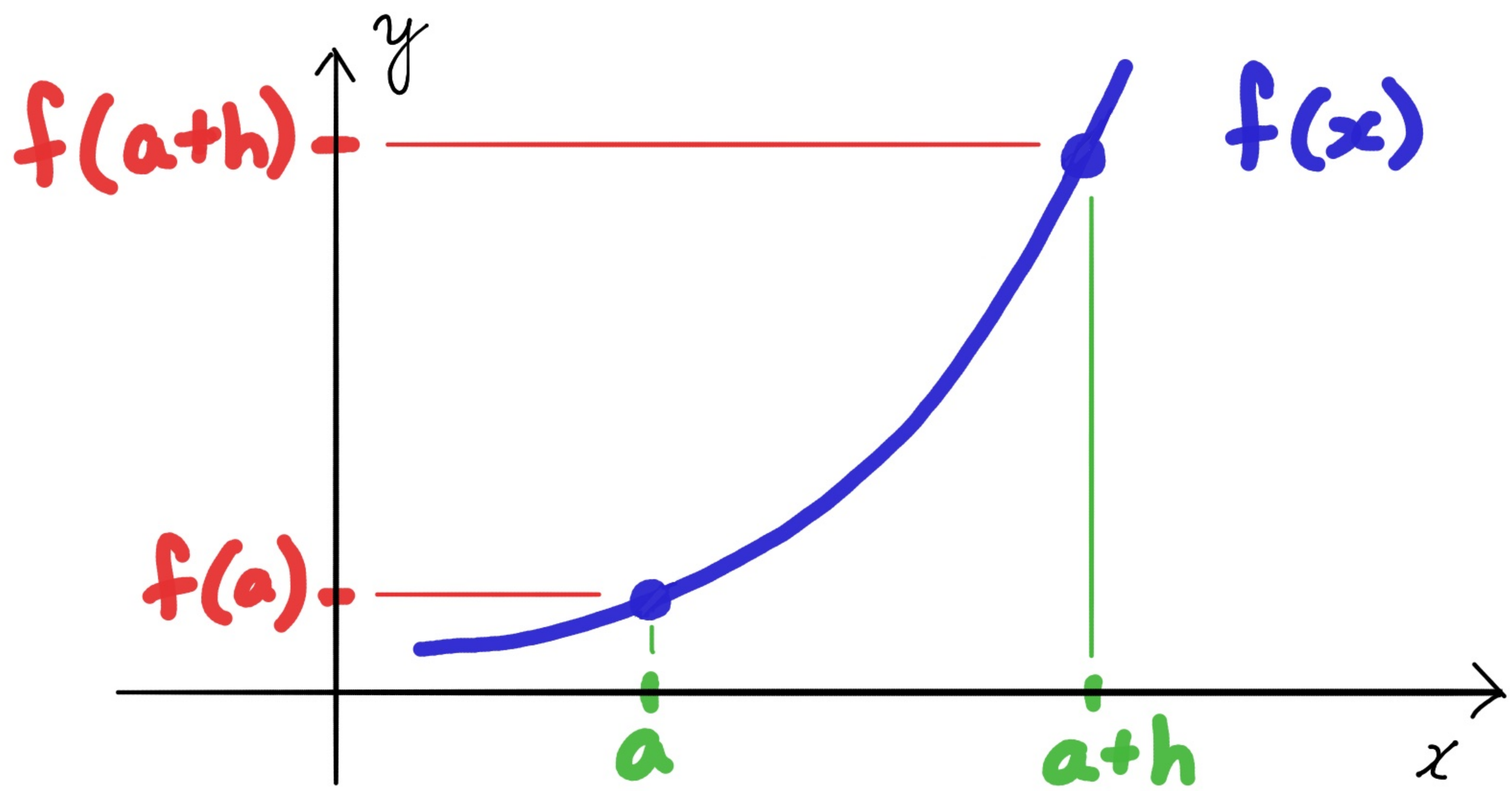
Recall, for $f : \mathbb{R} \to \mathbb{R}$



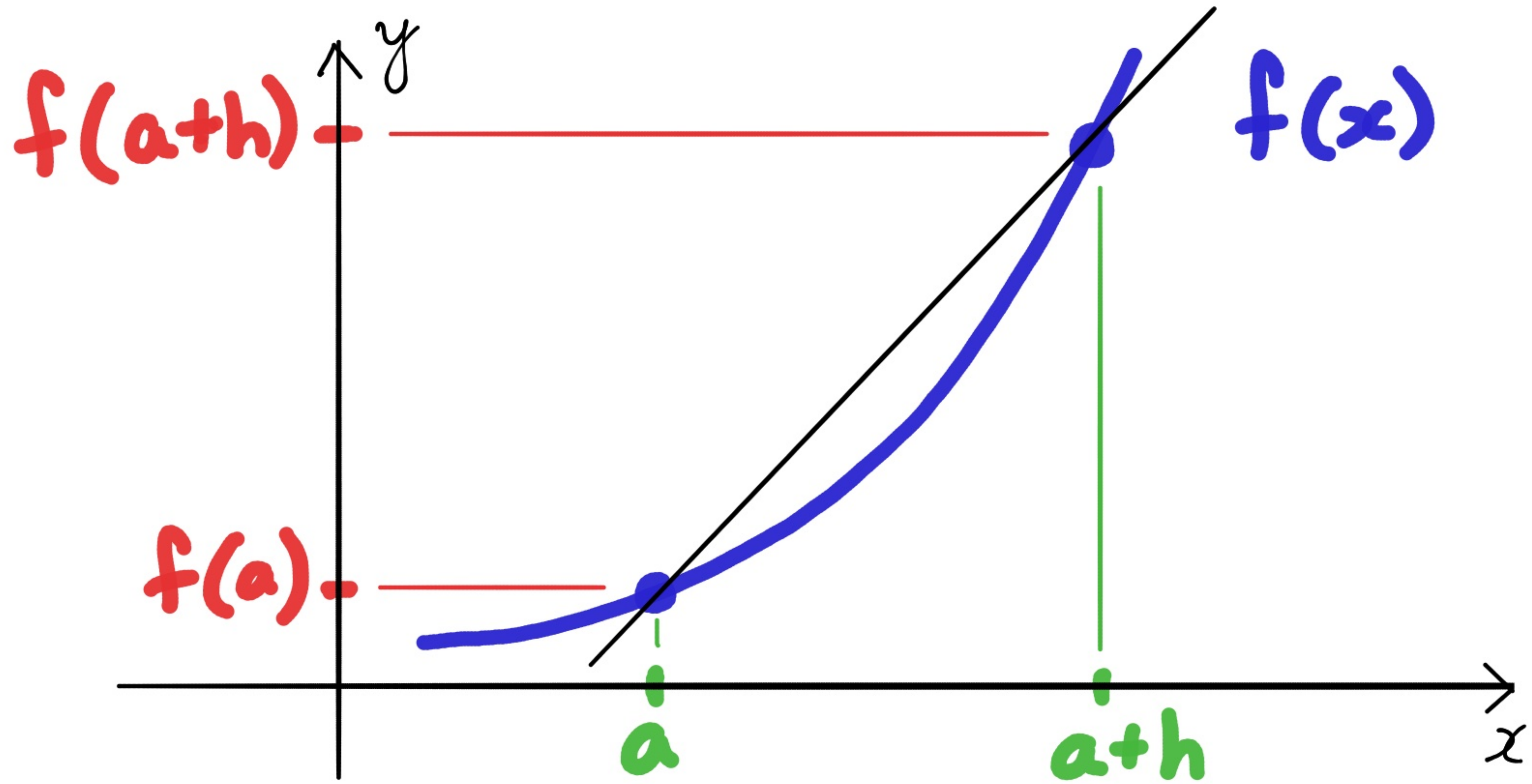
Recall, for $f: \mathbb{R} \to \mathbb{R}$



Recall, for $f:\mathbb{R}\to\mathbb{R}$



Recall, for $f: \mathbb{R} \to \mathbb{R}$



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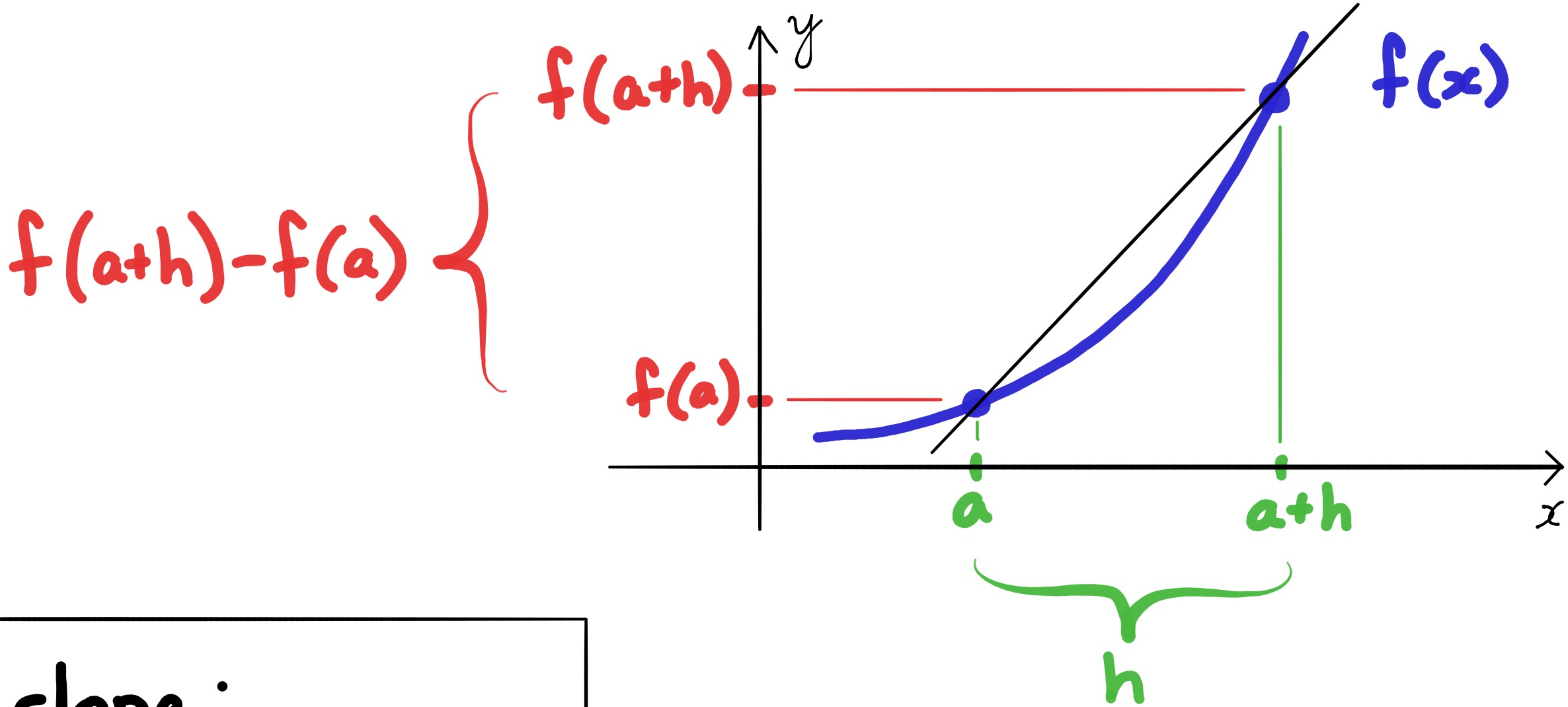

$\boxed{\underline{\text{slope:}} \quad \dfrac{f(a+h)-f(a)}{h}}$

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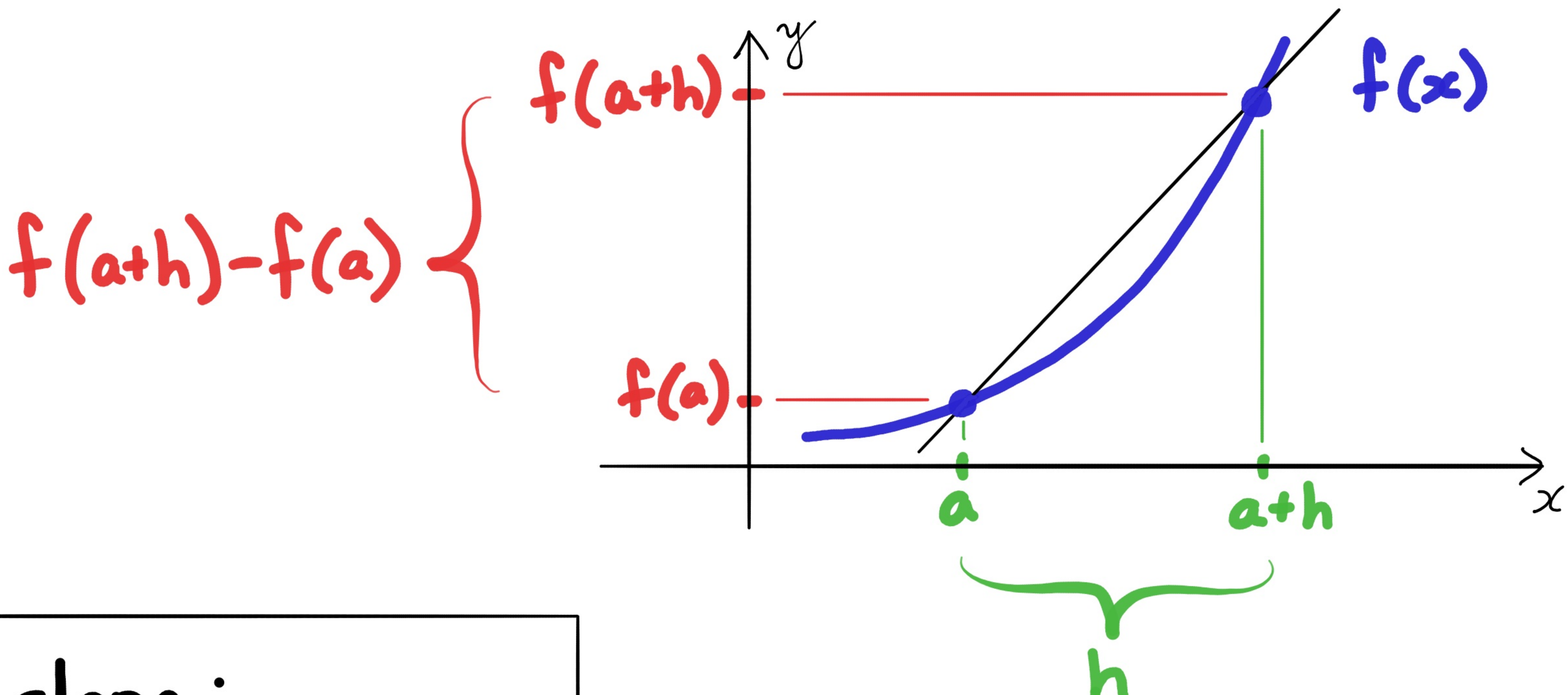
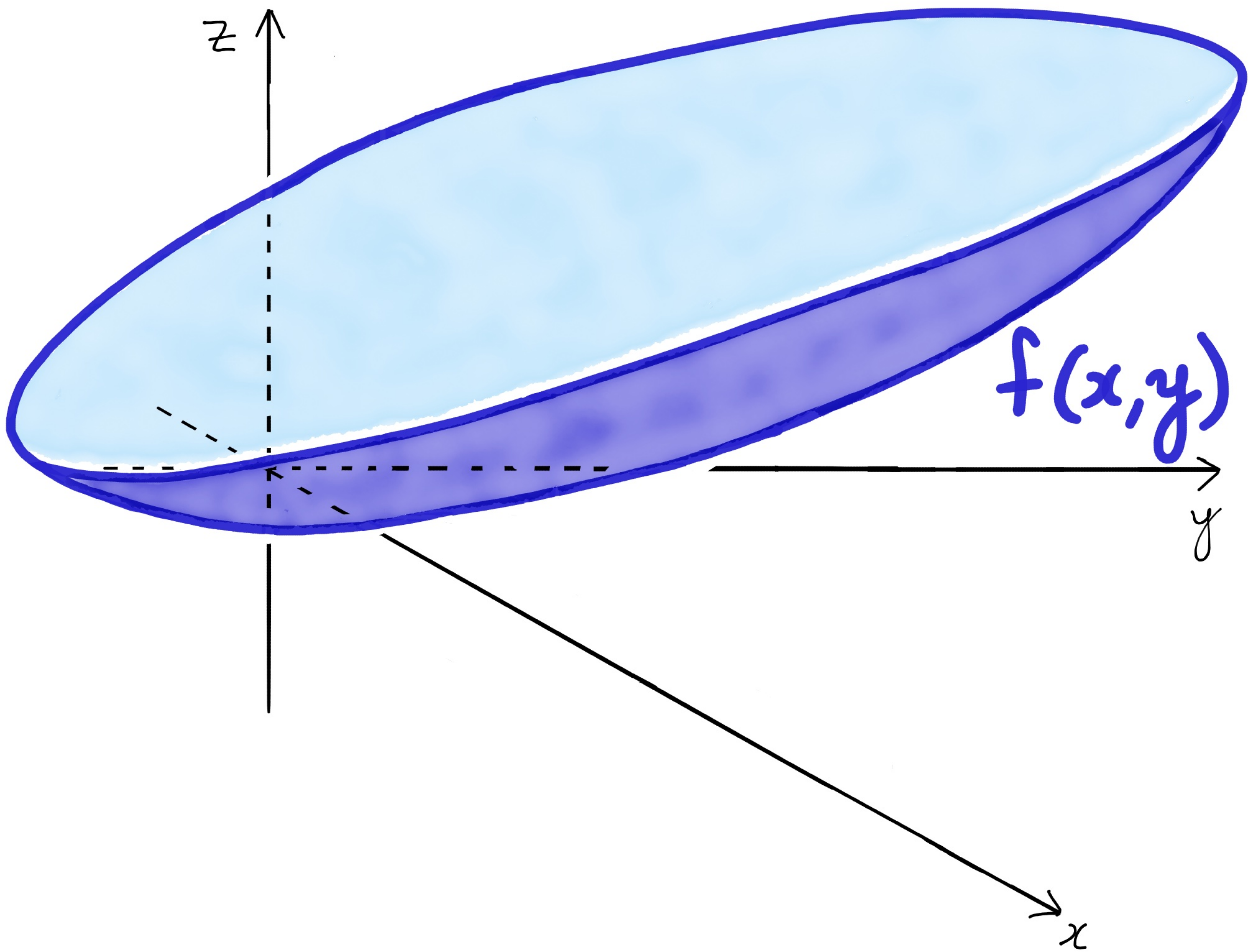


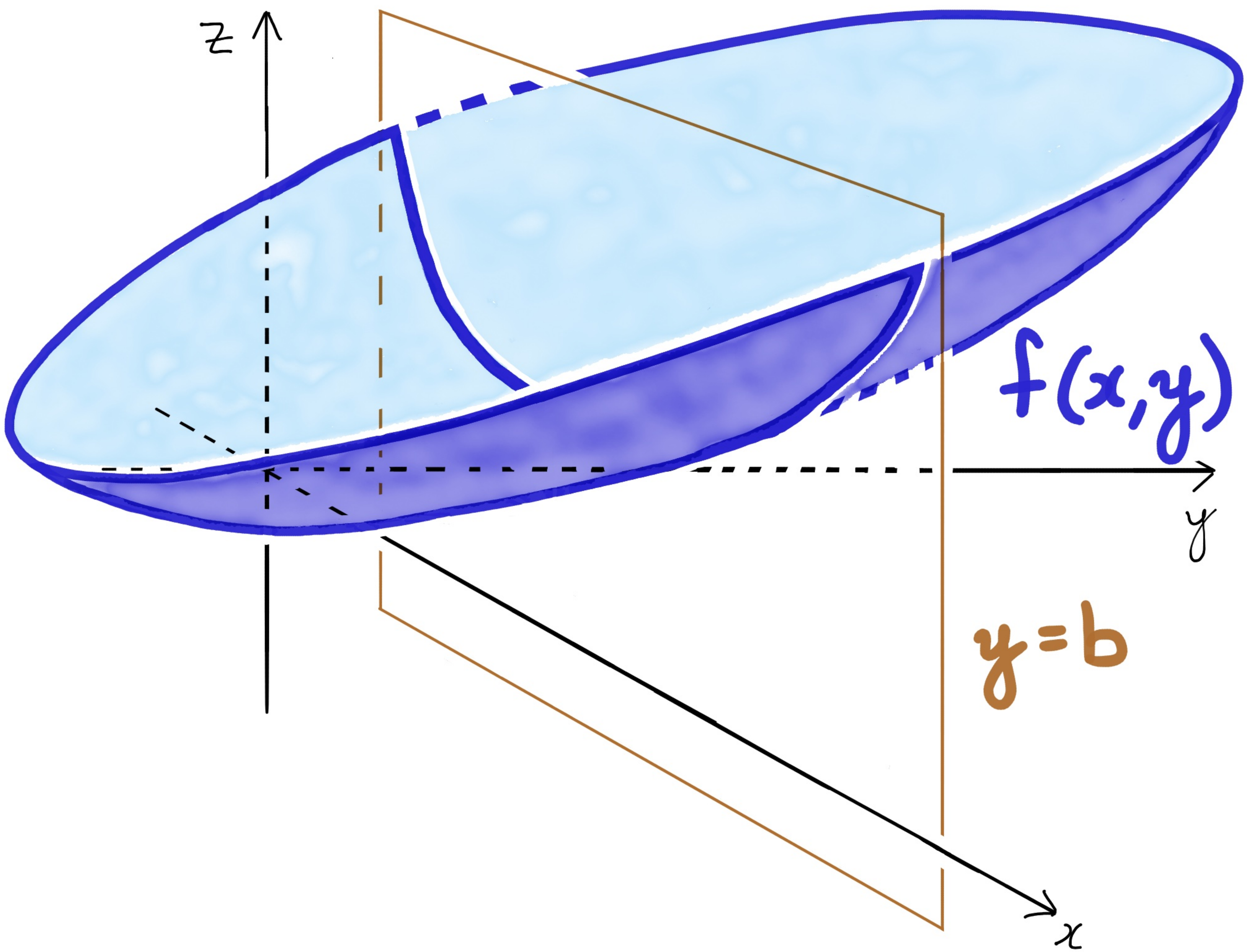
$$\boxed{\begin{array}{l}\underline{\text{slope}:}\\ \dfrac{f(a+h)-f(a)}{h}\end{array}}$$

$$\boxed{\frac{df}{dx}(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad \frac{\partial f}{\partial x}(a,b)$$



$f: \mathbb{R}^2 \to \mathbb{R}, \quad \frac{\partial f}{\partial x}(a,b)$



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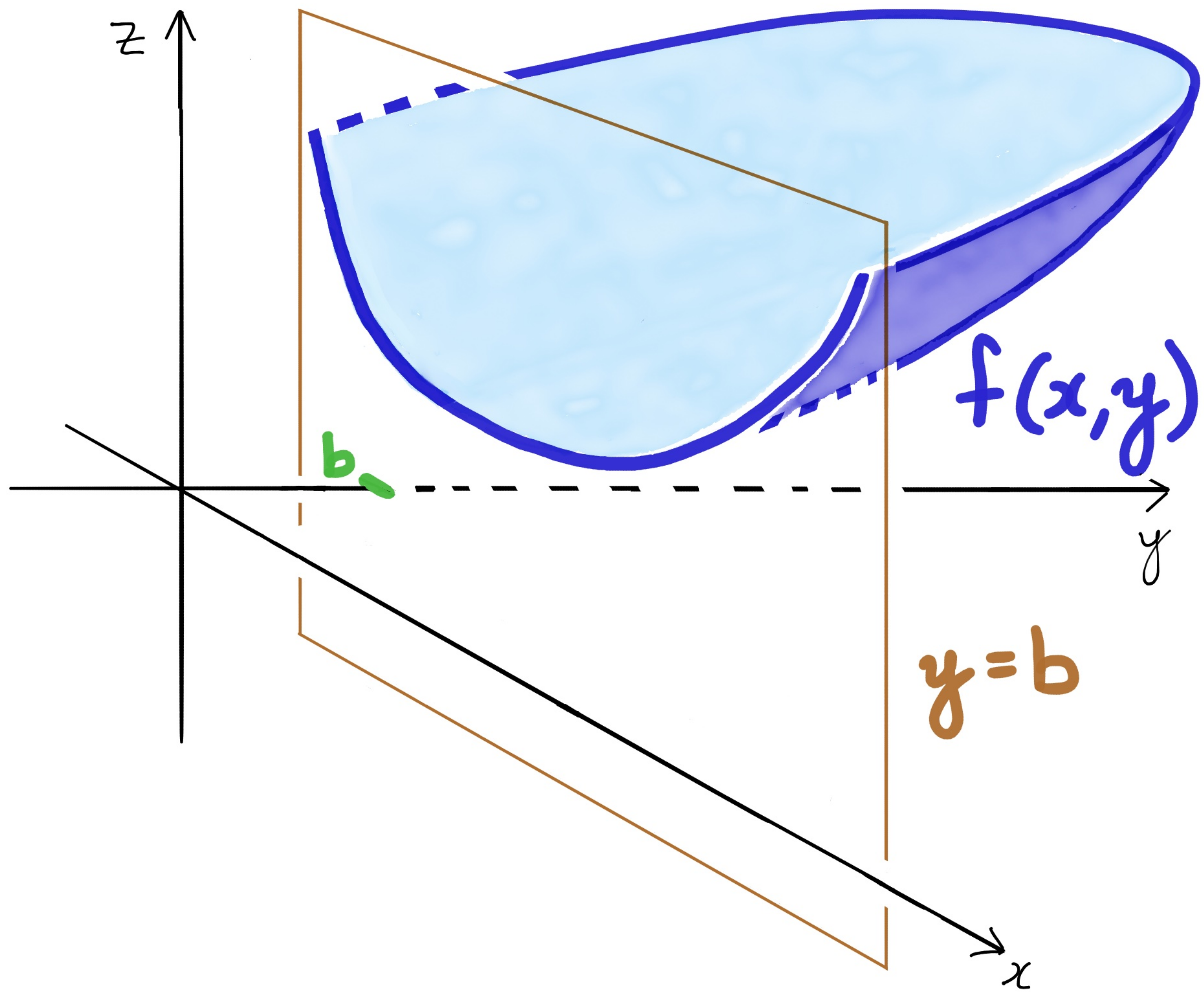

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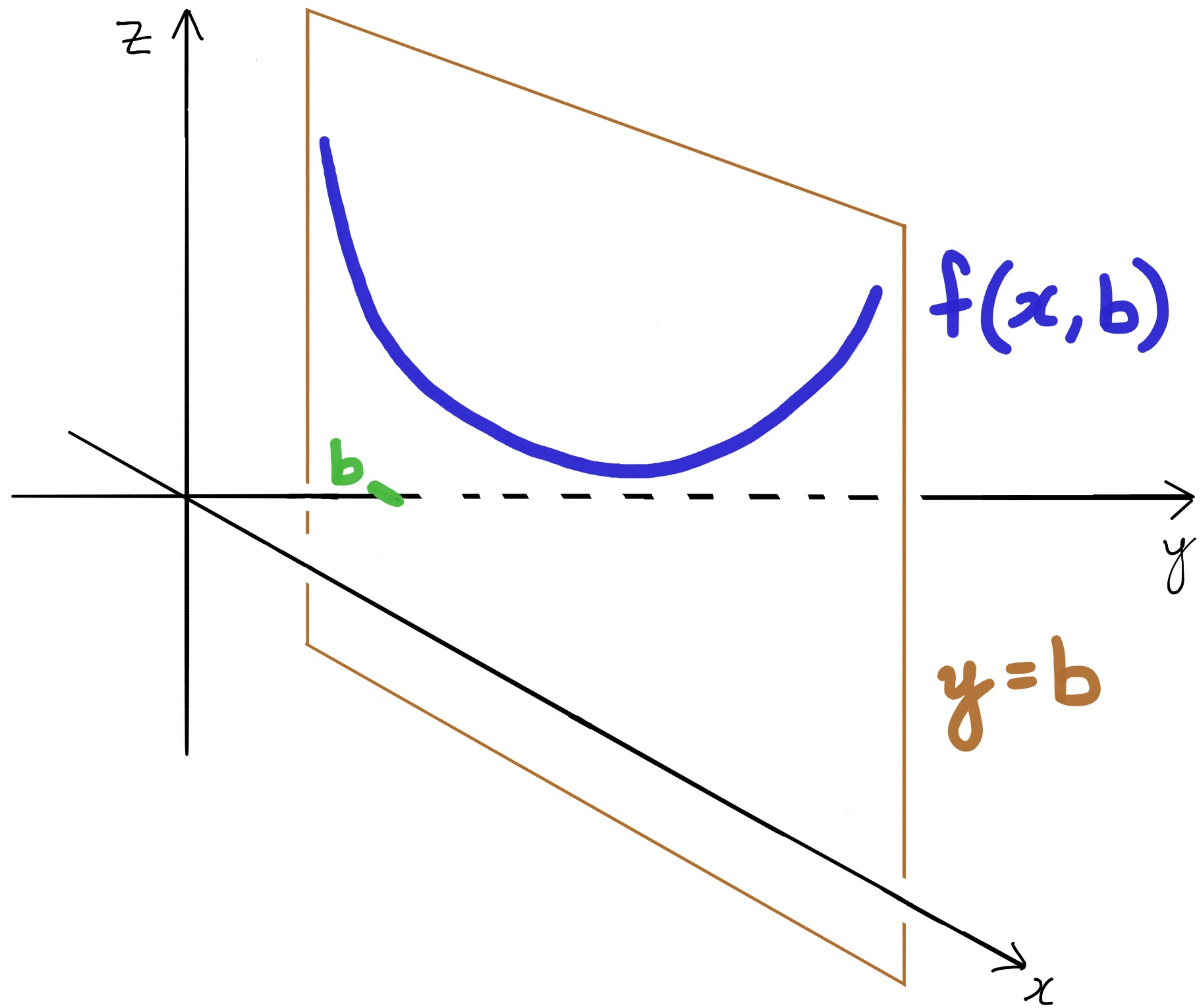

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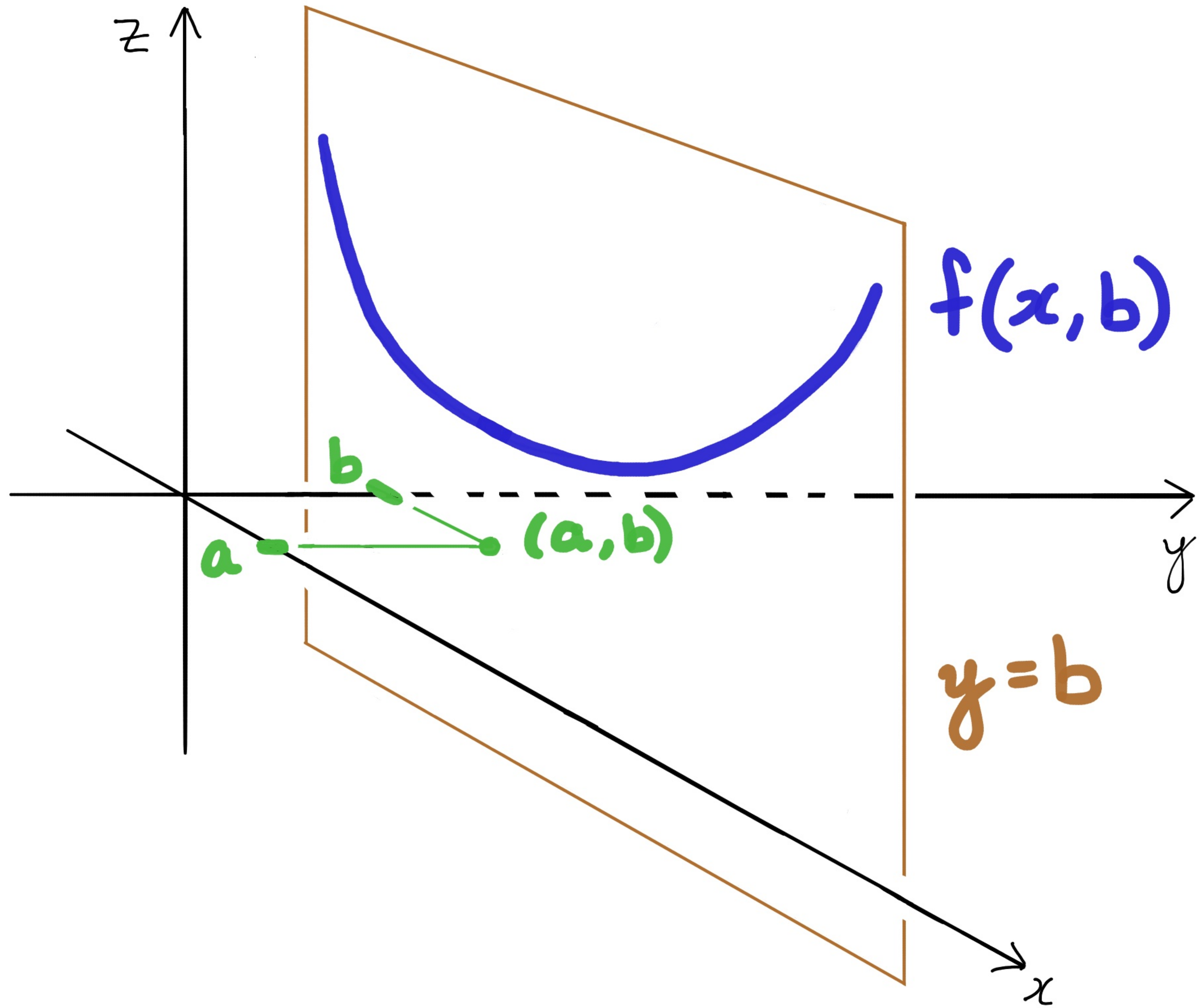

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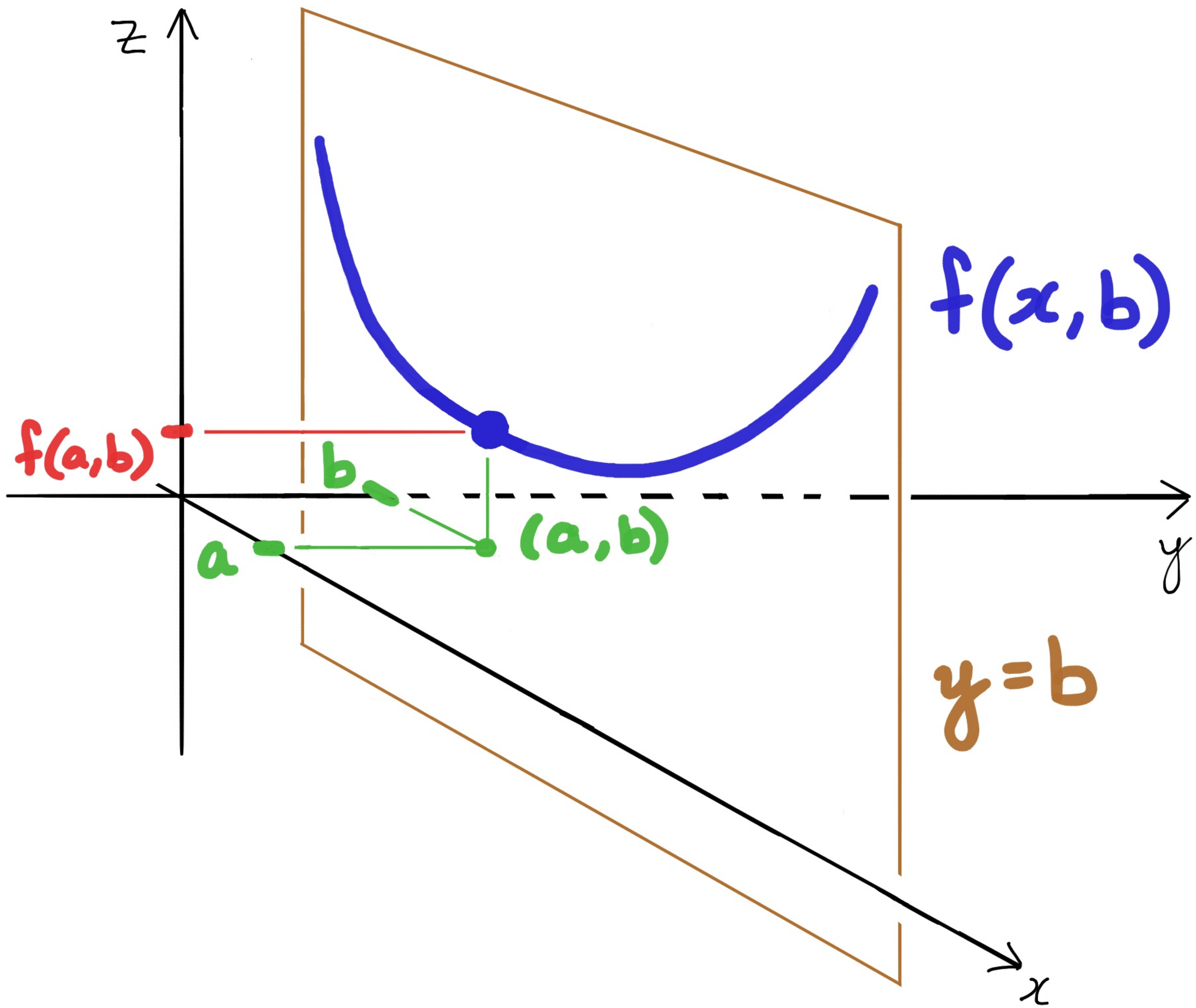

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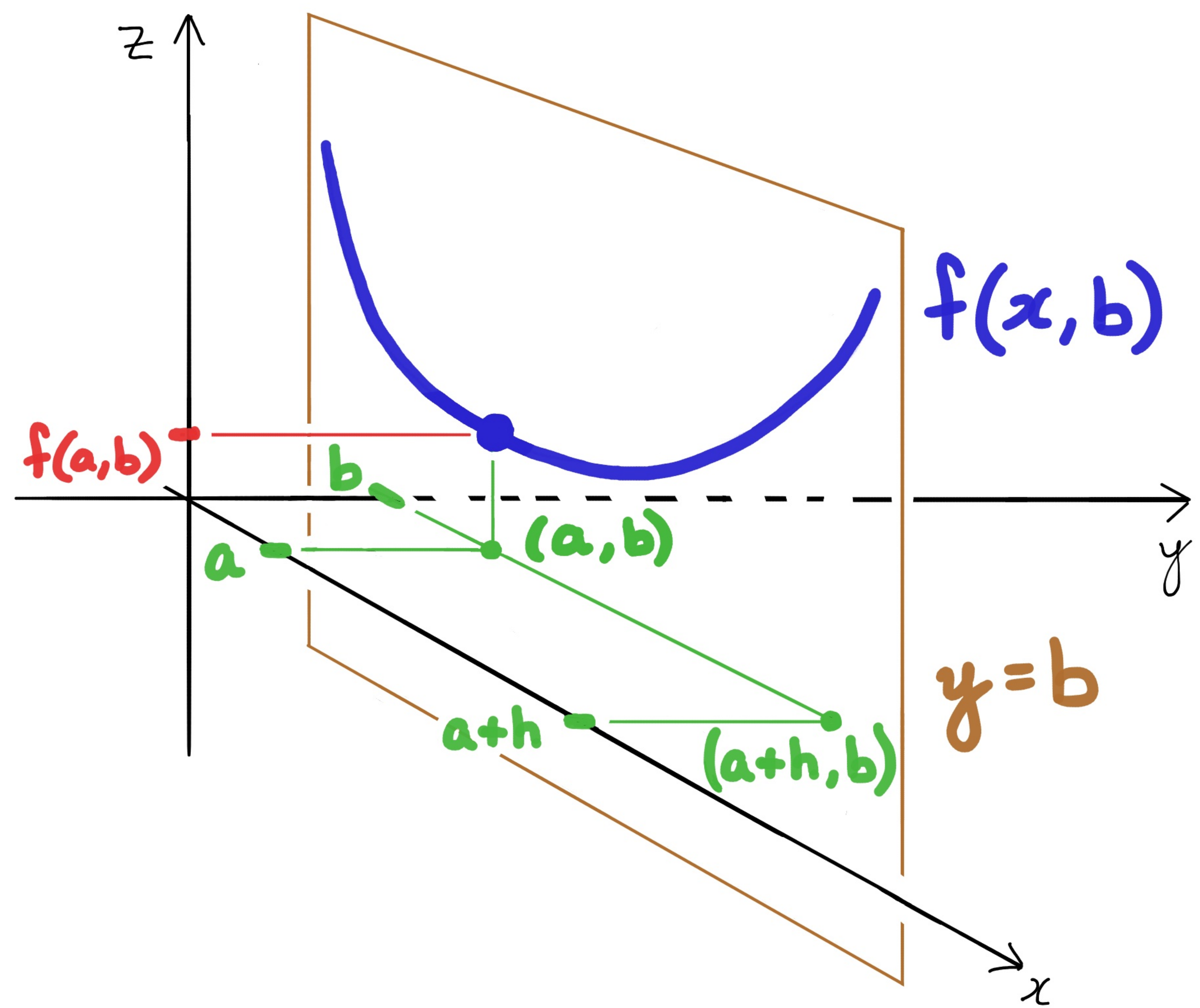

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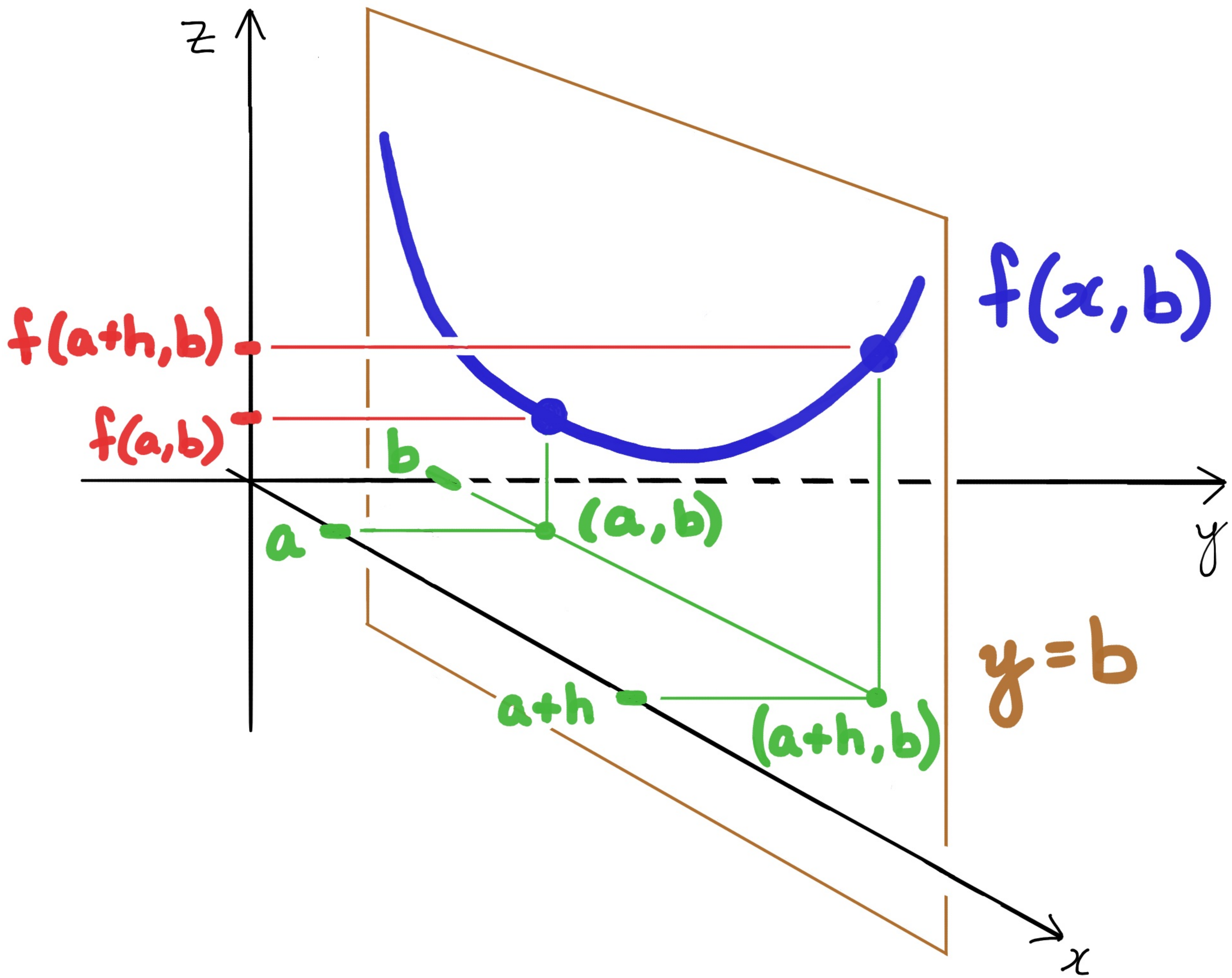

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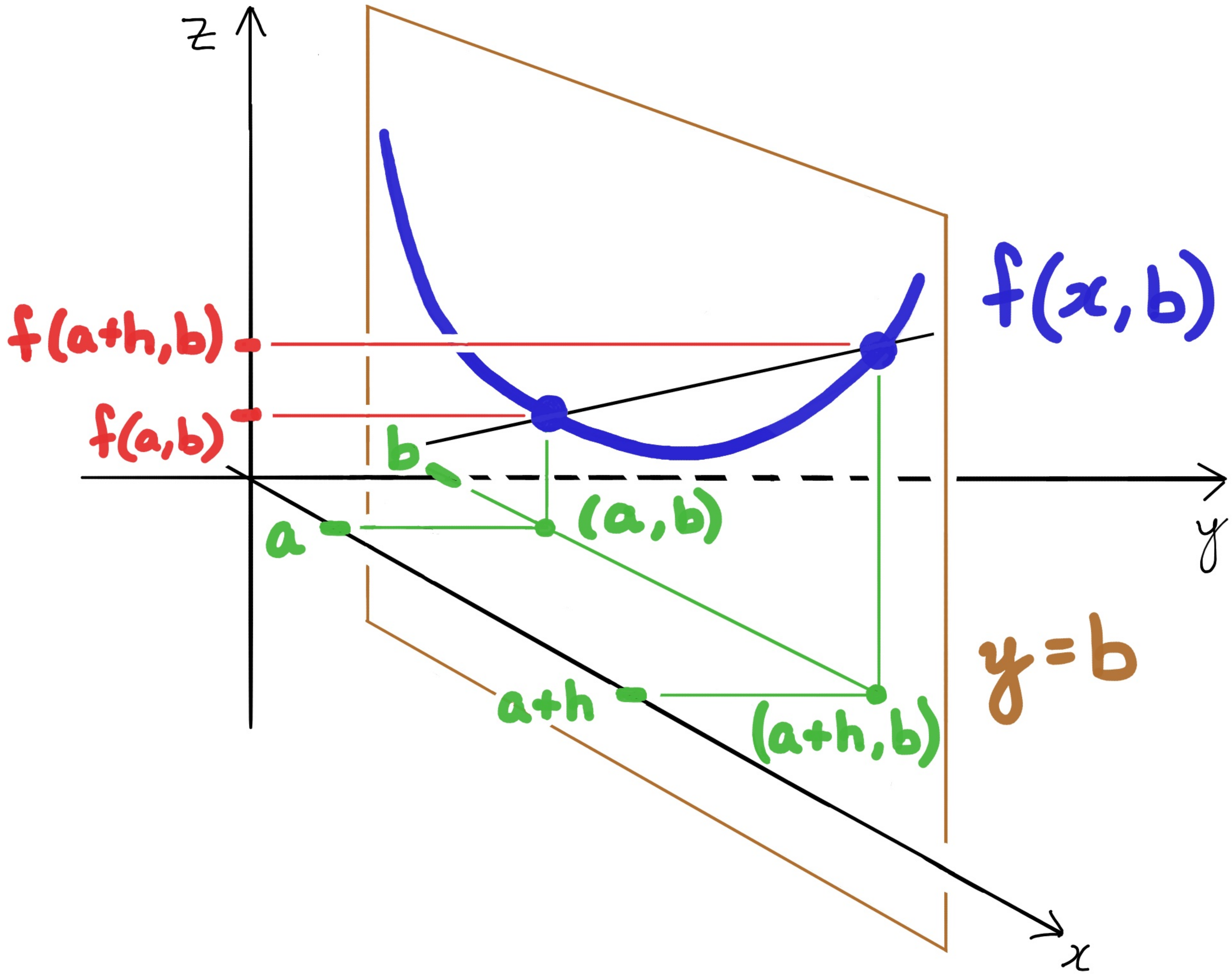

$$f:\mathbb{R}^2 \to \mathbb{R}, \qquad \frac{\partial f}{\partial x}(a,b)$$

slope:

$$\frac{f(a+h,b)-f(a,b)}{h}$$

$f : \mathbb{R}^2 \to \mathbb{R}$,

$$\boxed{\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}}$$

slope:

$$\frac{f(a+h,b) - f(a,b)}{h}$$

To find $\frac{\partial f}{\partial x}$, take derivative of f with respect to x. Treat all other variables as constants.

Examples:

1. $\frac{\partial}{\partial x}(3x) =$
2. $\frac{\partial}{\partial x}(yx) =$
3. $\frac{\partial}{\partial x}(yx^2 + yx + 4) =$
4. $\frac{\partial}{\partial x}\left(\cos(x)\sin y + wux^2 + \frac{x}{t+\Delta}\right) =$

To find $\frac{\partial f}{\partial x}$, take derivative of f with respect to x. Treat all other variables as constants.

Examples:

1. $\frac{\partial}{\partial x}(3x) = 3$
2. $\frac{\partial}{\partial x}(yx) =$
3. $\frac{\partial}{\partial x}(yx^2 + yx + 4) =$
4. $\frac{\partial}{\partial x}\left(\cos(x)\sin y + wux^2 + \frac{x}{t+\Delta}\right) =$

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Examples:

① $\frac{\partial}{\partial x}(3x) = 3$

② $\frac{\partial}{\partial x}(yx) = y$

③ $\frac{\partial}{\partial x}(yx^2 + yx + 4) =$

④ $\frac{\partial}{\partial x}\left(\cos(x)\sin y + wux^2 + \frac{x}{t+\Delta}\right) =$

To find $\frac{\partial f}{\partial x}$, take derivative of f with respect to x. Treat all other variables as constants.

Examples:

1. $\frac{\partial}{\partial x}(3x) = 3$
2. $\frac{\partial}{\partial x}(yx) = y$
3. $\frac{\partial}{\partial x}(yx^2 + yx + 4) = 2yx + y$
4. $\frac{\partial}{\partial x}\left(\cos(x)\sin y + wux^2 + \frac{x}{t+A}\right) =$

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$$= -\sin(x)\sin y + 2wux + \frac{1}{t+\Delta}$$

To find $\frac{\partial f}{\partial x}$, take derivative of f with respect to x. Treat all other variables as constants.

Example:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x,y) = 3xy^2 - 8x^2 + 7y^3$$

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$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x,y) = 3xy^2 - 8x^2 + 7y^3$$

$$\frac{\partial f}{\partial x}(x,y) = 3y^2 - 16x$$

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$$\frac{\partial f}{\partial x}(4,2) = 3 \cdot 2^2 - 16 \cdot 4 = 12 - 64 = -52$$

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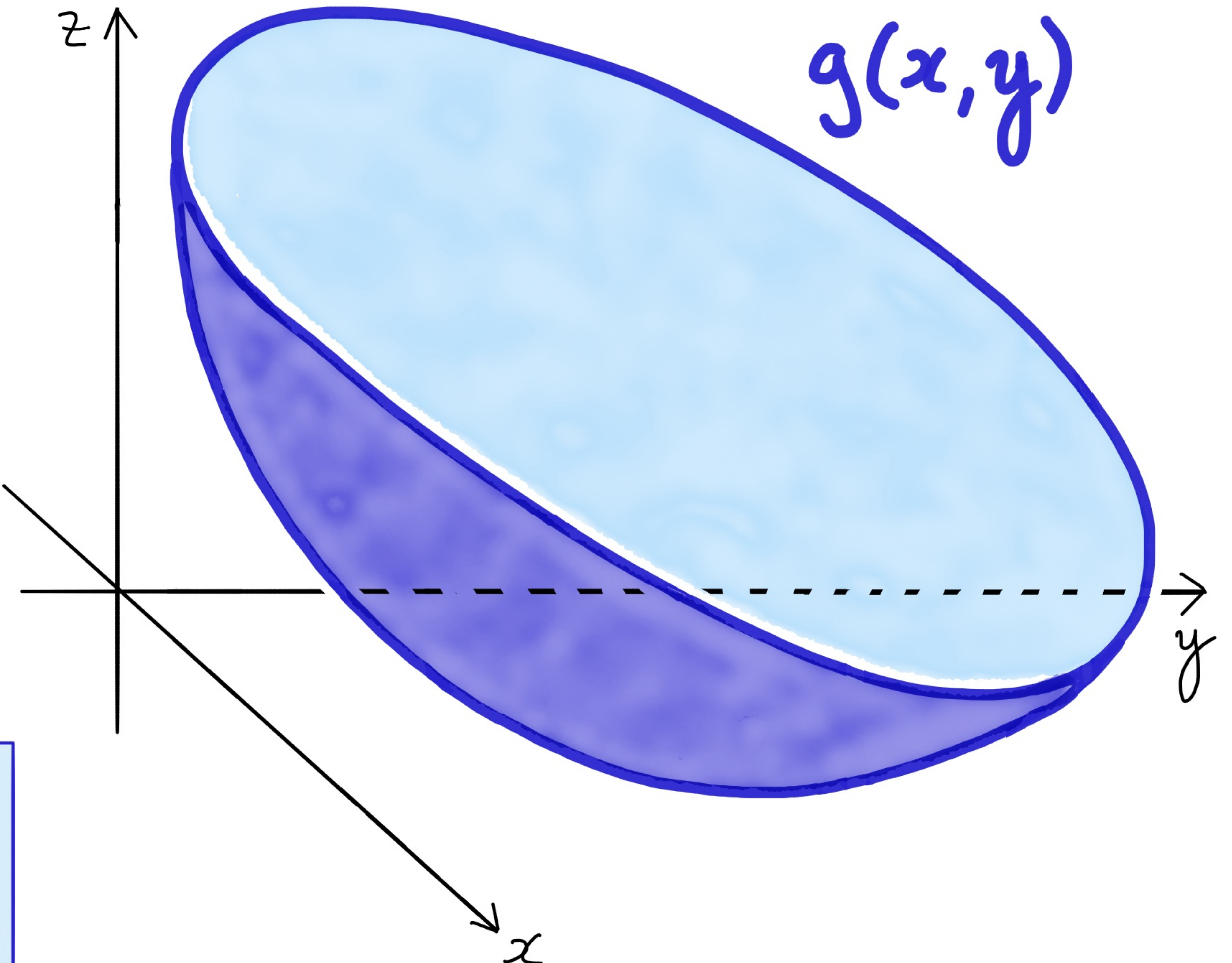
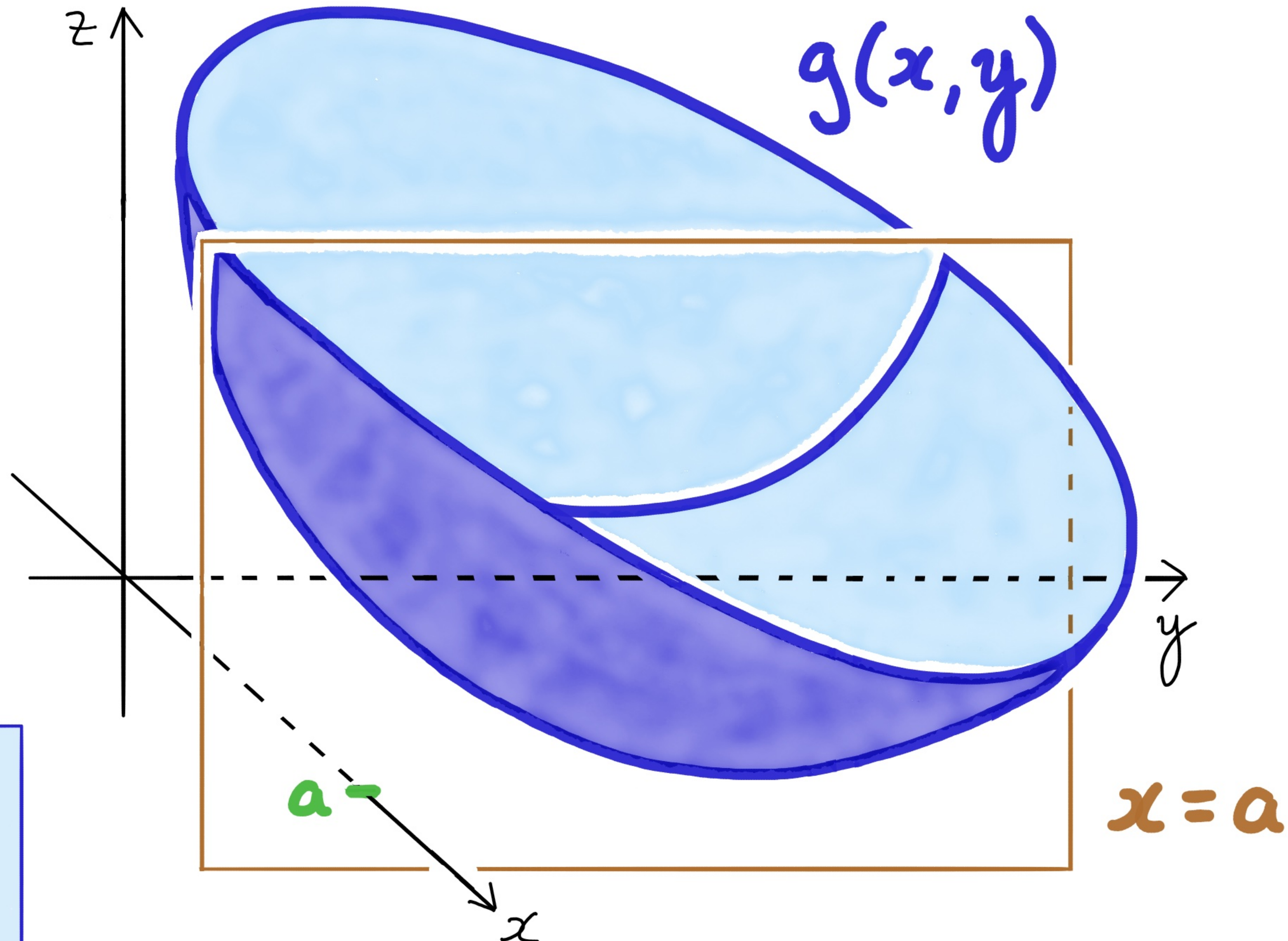
$$\boxed{\frac{\partial f}{\partial x} : \mathbb{R}^2 \longrightarrow \mathbb{R}}$$



$$\frac{\partial g}{\partial y}(a,b)$$



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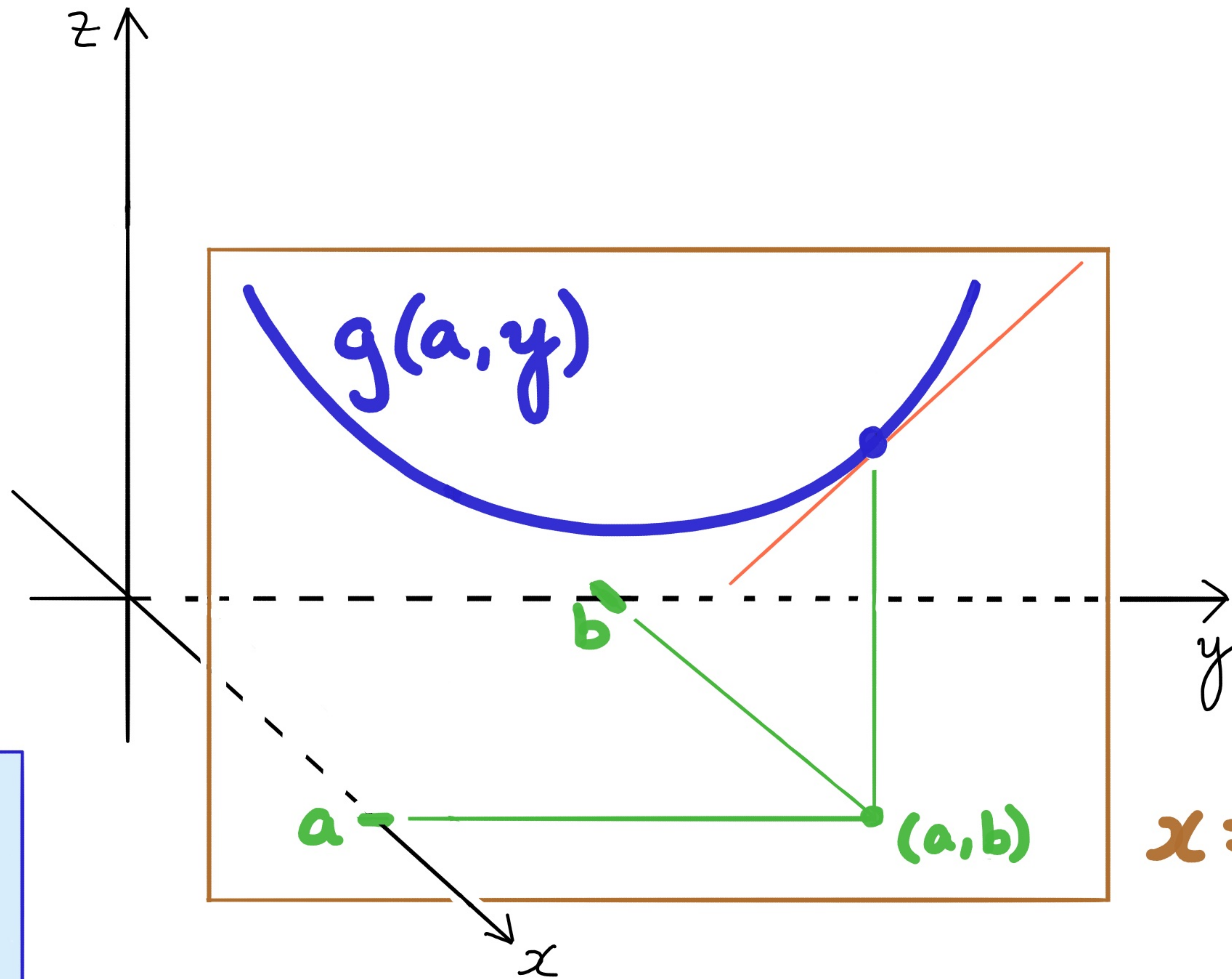

$$\boxed{\frac{\partial g}{\partial y}(a,b)}$$

measures slope of $x=a$ cross section of graph of $g(x,y)$ at $(x,y)=(a,b)$.

Examples:

① $\frac{\partial}{\partial y}\left(y^2\right) =$

② $\frac{\partial}{\partial y}\left(xwy^2 + v\right) =$

③ $\frac{\partial}{\partial w}\left(xwy^2 + v\right) =$

④ $\frac{\partial}{\partial v}\left(xwy^2 + v\right) =$

Examples:

① $\frac{\partial}{\partial y}\left(y^2\right) = 2y$

② $\frac{\partial}{\partial y}\left(xwy^2 + v\right) =$

③ $\frac{\partial}{\partial w}\left(xwy^2 + v\right) =$

④ $\frac{\partial}{\partial v}\left(xwy^2 + v\right) =$

Examples:

1. $\frac{\partial}{\partial y}\left(y^2\right) = 2y$

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① $\frac{\partial}{\partial y}(y^2) = 2y$

② $\frac{\partial}{\partial y}(xwy^2 + v) = 2xwy$

③ $\frac{\partial}{\partial w}(xwy^2 + v) = xy^2$

④ $\frac{\partial}{\partial v}(xwy^2 + v) = 1$

Word problem:

W - water

G - gasoline

$F(W, G)$ - size of fire

$$\frac{\partial F}{\partial W}(W,G) < 0, \quad \frac{\partial F}{\partial G}(W,G) > 0$$

Alternate expressions

$$\frac{\partial f}{\partial x}(x,y) = \left.\frac{\partial f}{\partial x}\right|_{(x,y)} = f_x(x,y)$$

$$\frac{\partial f}{\partial y}(x,y) = \left.\frac{\partial f}{\partial y}\right|_{(x,y)} = f_y(x,y)$$

Alternate expressions

If $f : \mathbb{R}^n \to \mathbb{R}$, and $p \in \mathbb{R}^n$,

$$\frac{\partial f}{\partial x}(p) = \left.\frac{\partial f}{\partial x}\right|_p = f_x(p)$$

$$\frac{\partial f}{\partial y}(p) = \left.\frac{\partial f}{\partial y}\right|_p = f_y(p)$$

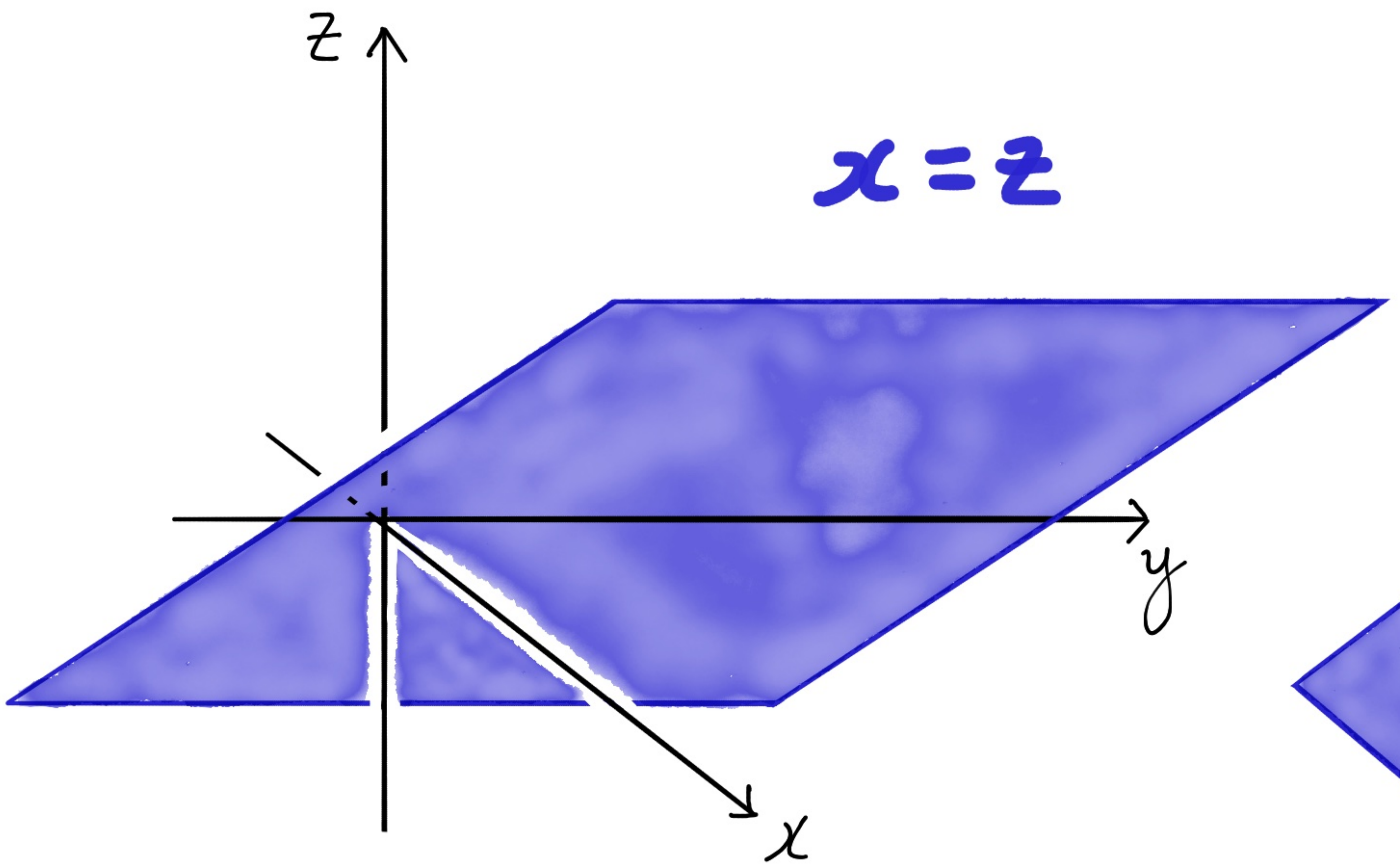
Graph of $f(x,y) = xy$

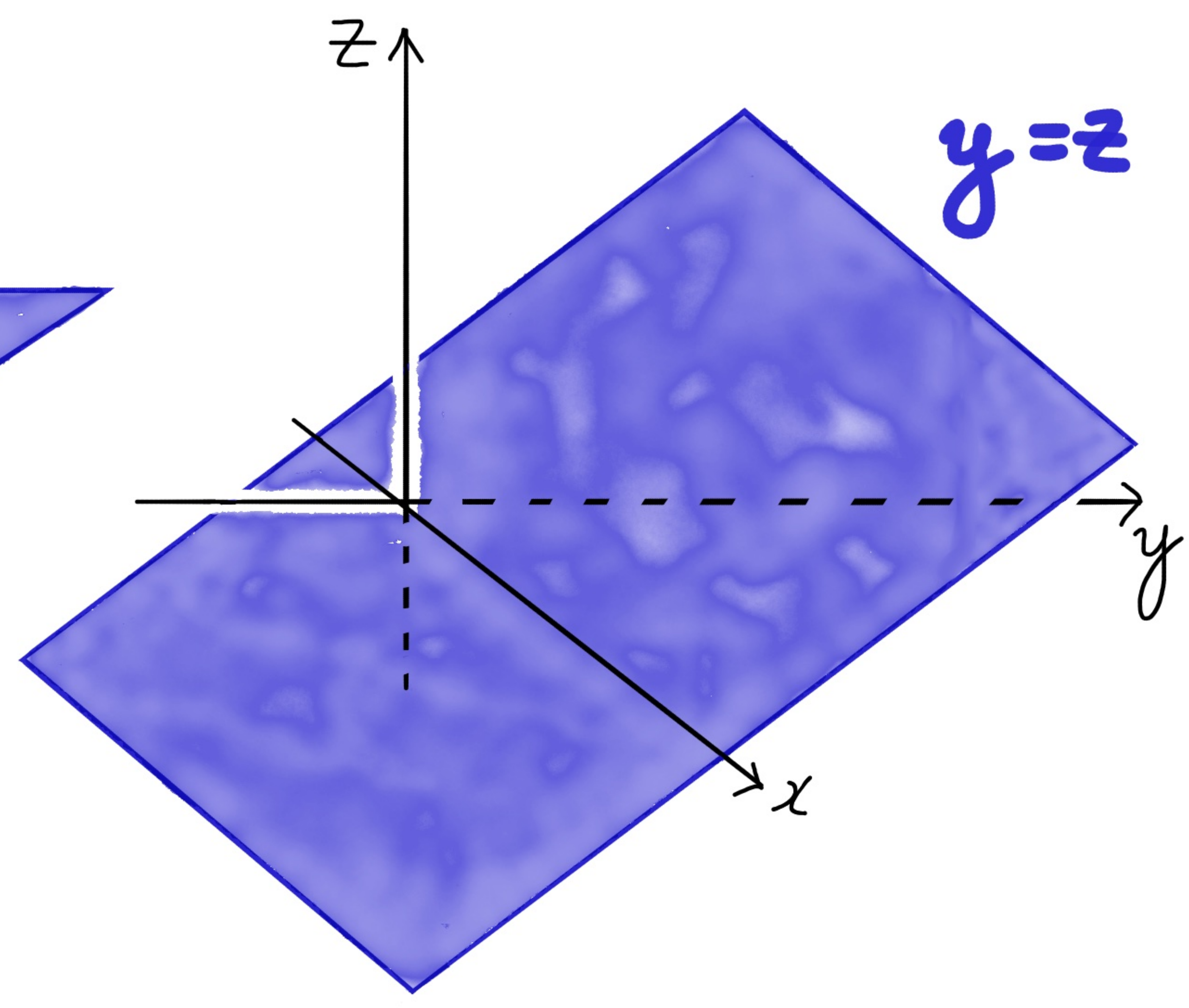
$$f_y(x,y) = x$$

$$f(x,y) = xy$$

$$f_y(x,y) = x$$

$$f_x(x,y) = y$$





II Second partial derivatives

If $f: \mathbb{R}^n \to \mathbb{R}$, then

$\frac{\partial f}{\partial x}: \mathbb{R}^n \to \mathbb{R}$ and $\frac{\partial f}{\partial y}: \mathbb{R}^n \to \mathbb{R}$.

If $f:\mathbb{R}^n \longrightarrow \mathbb{R}$, then

$\frac{\partial f}{\partial x}:\mathbb{R}^n \longrightarrow \mathbb{R}$ and $\frac{\partial f}{\partial y}:\mathbb{R}^n \longrightarrow \mathbb{R}$.

So,

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

are all functions $\mathbb{R}^n \longrightarrow \mathbb{R}$.

Writing conventions

- $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right)$ is written as

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad \frac{\partial^2}{\partial x \partial y} f \quad \text{or} \quad f_{yx}.$$

Writing conventions

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- $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f\right)$ is $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.

Examples:

1. Find all four second partial derivatives of $f(x,y) = 3xy^2 - 2x + 8 + y^2$.

Examples:

① Find all four second partial derivatives of $f(x,y) = 3xy^2 - 2x + 8 + y^2$.

$$\frac{\partial f}{\partial x} = 3y^2 - 2$$

$$\frac{\partial f}{\partial y} = 6xy + 2y$$

Examples:

① Find all four second partial derivatives of $f(x,y) = 3xy^2 - 2x + 8 + y^2$.

$$\frac{\partial f}{\partial x} = 3y^2 - 2 \quad \rightarrow \quad \frac{\partial^2 f}{\partial x^2} = 0, \qquad \frac{\partial^2 f}{\partial y \partial x} = 6y$$

$$\frac{\partial f}{\partial y} = 6xy + 2y \quad \rightarrow \quad \frac{\partial^2 f}{\partial x \partial y} = 6y, \qquad \frac{\partial^2 f}{\partial y^2} = 6x + 2$$

III C^1 and C^2 functions

$$\boxed{f : \mathbb{R}^2 \to \mathbb{R}}$$

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<u>f is C^1</u> if f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are continuous.

$$\boxed{f : \mathbb{R}^2 \to \mathbb{R}}$$

<u>f is C^1</u> if f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are continuous.

<u>f is C^2</u> if f is C^1 and $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous.

Theorem: If $f: \mathbb{R}^n \to \mathbb{R}$ is C^2, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$