Fen

Ⅰ Limits

Ⅱ Continuity

Ⅲ Boundary points

I Limits

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

If $p \in \mathbb{R}^n$, then $f(p) \in \mathbb{R}^m$.

Example:

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$f(x,y,z) = (2x-y, xyz)$

If $p \in \mathbb{R}^3$ is $p=(1,0,3)$, then

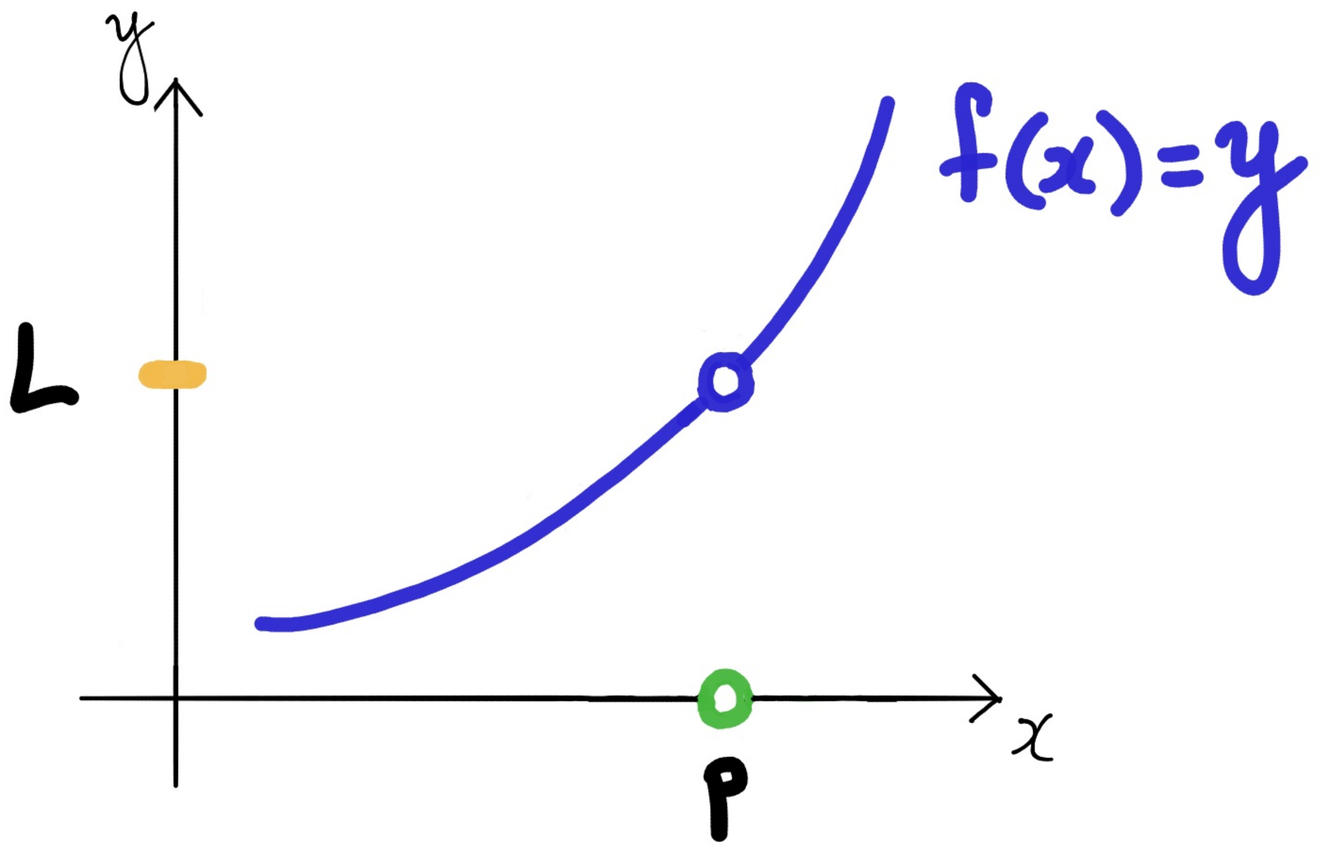
$f(p) \in \mathbb{R}^2$ is $f(p) = (2 \cdot 1 - 0, 1 \cdot 0 \cdot 3) = (2,0)$.

For $f: \mathbb{R}^n \to \mathbb{R}^m$ and $p \in \mathbb{R}^n$,

$f(p)$ could mean

- $f(x)$
- $f(x,y)$
- $f(x,y,z)$
- $f(x,y,z,w)$

Limits for $f: \mathbb{R} \to \mathbb{R}$



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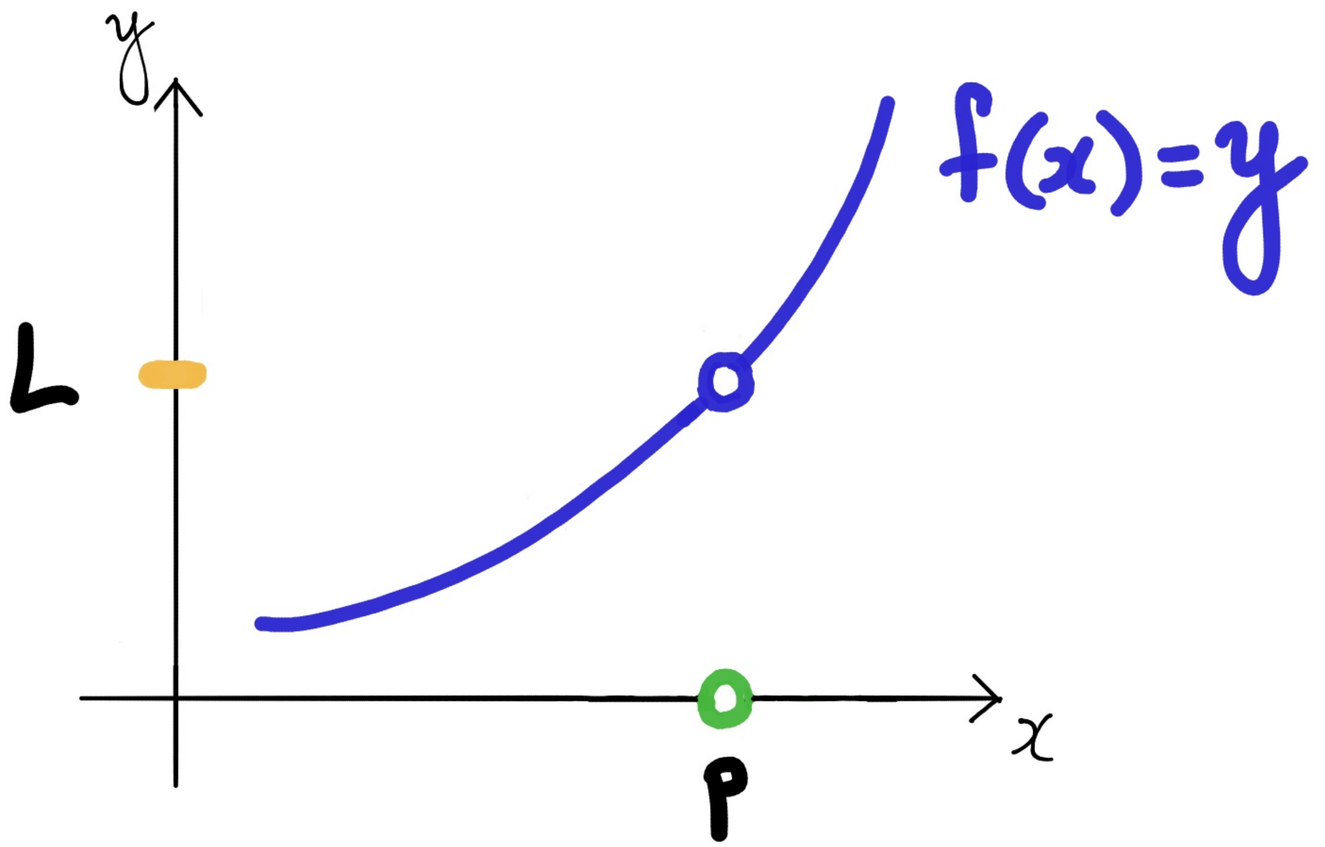

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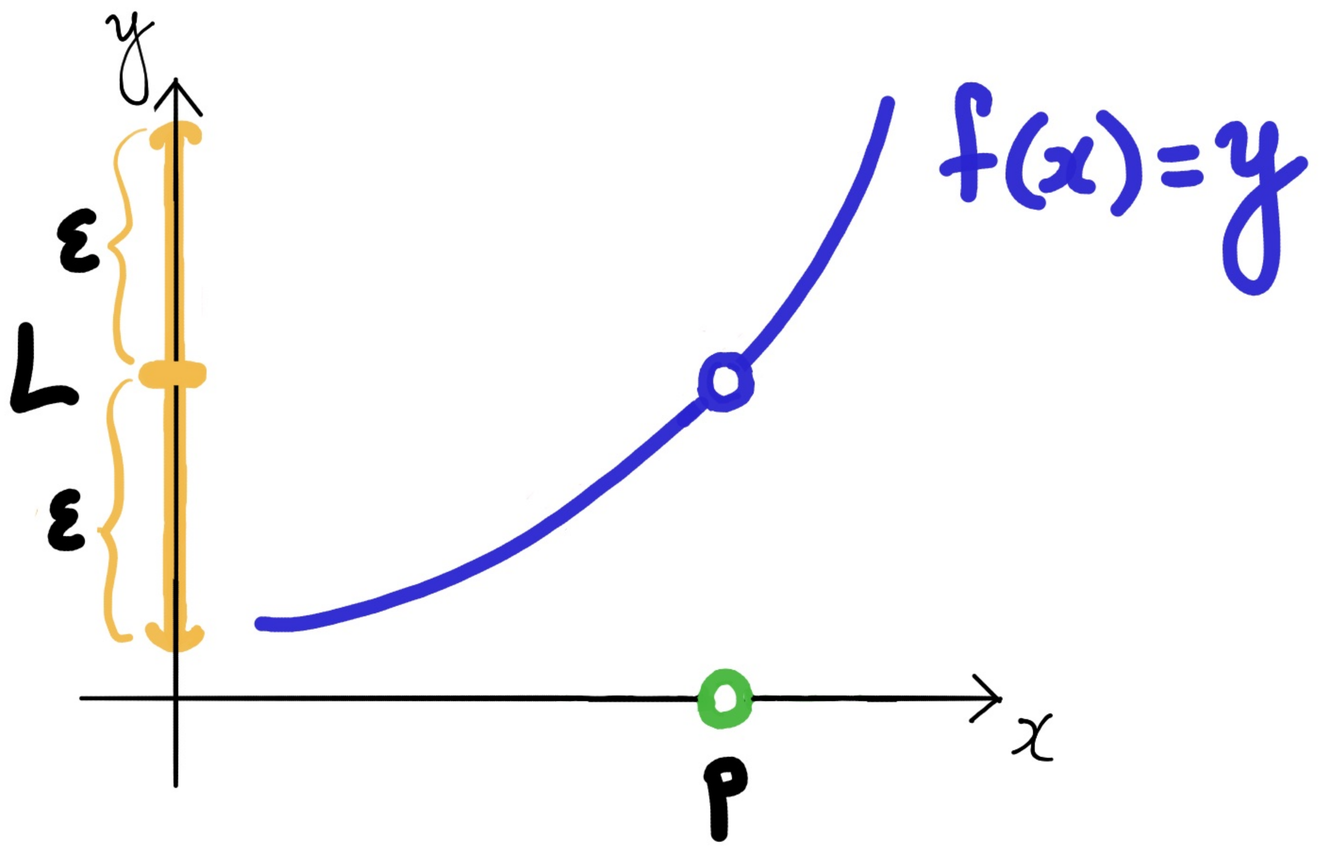

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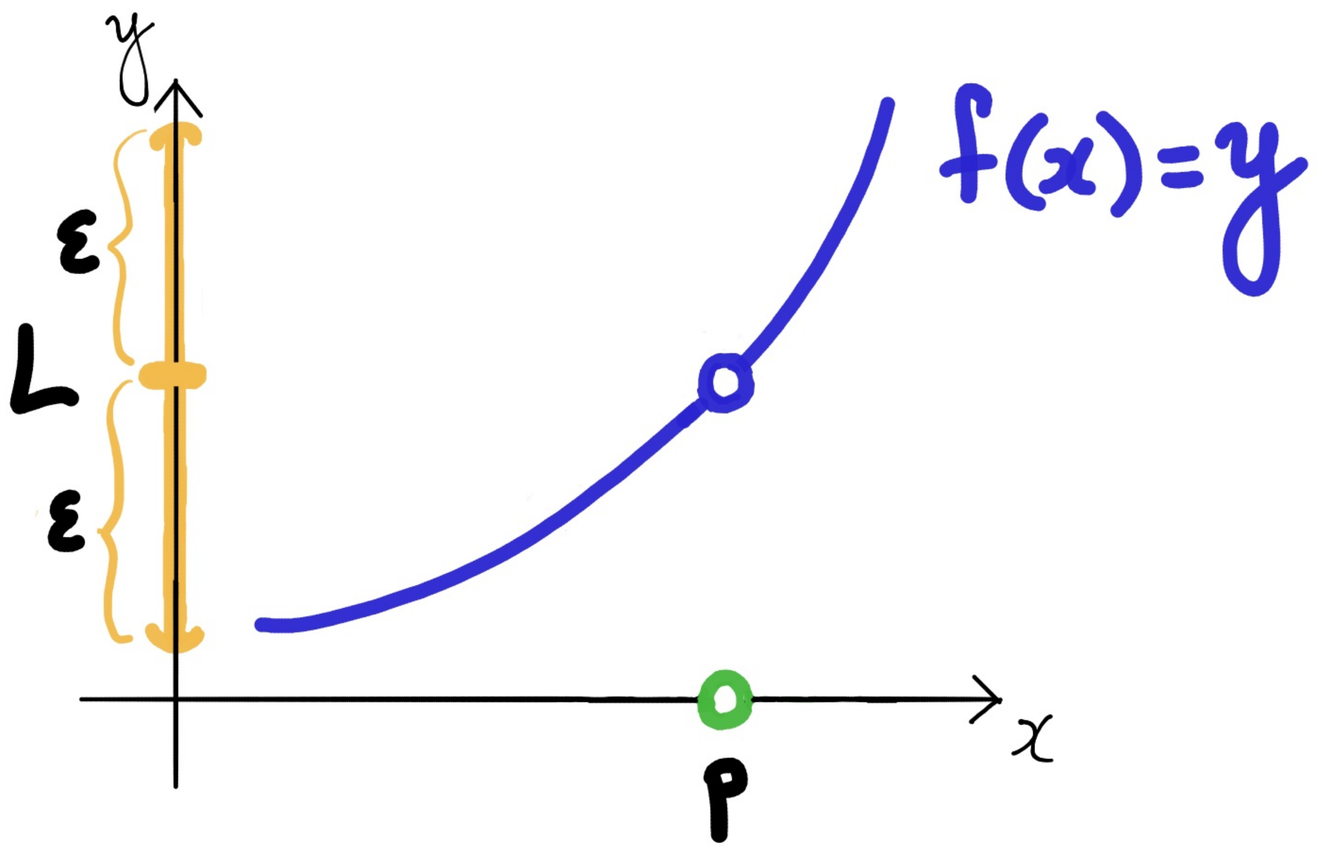

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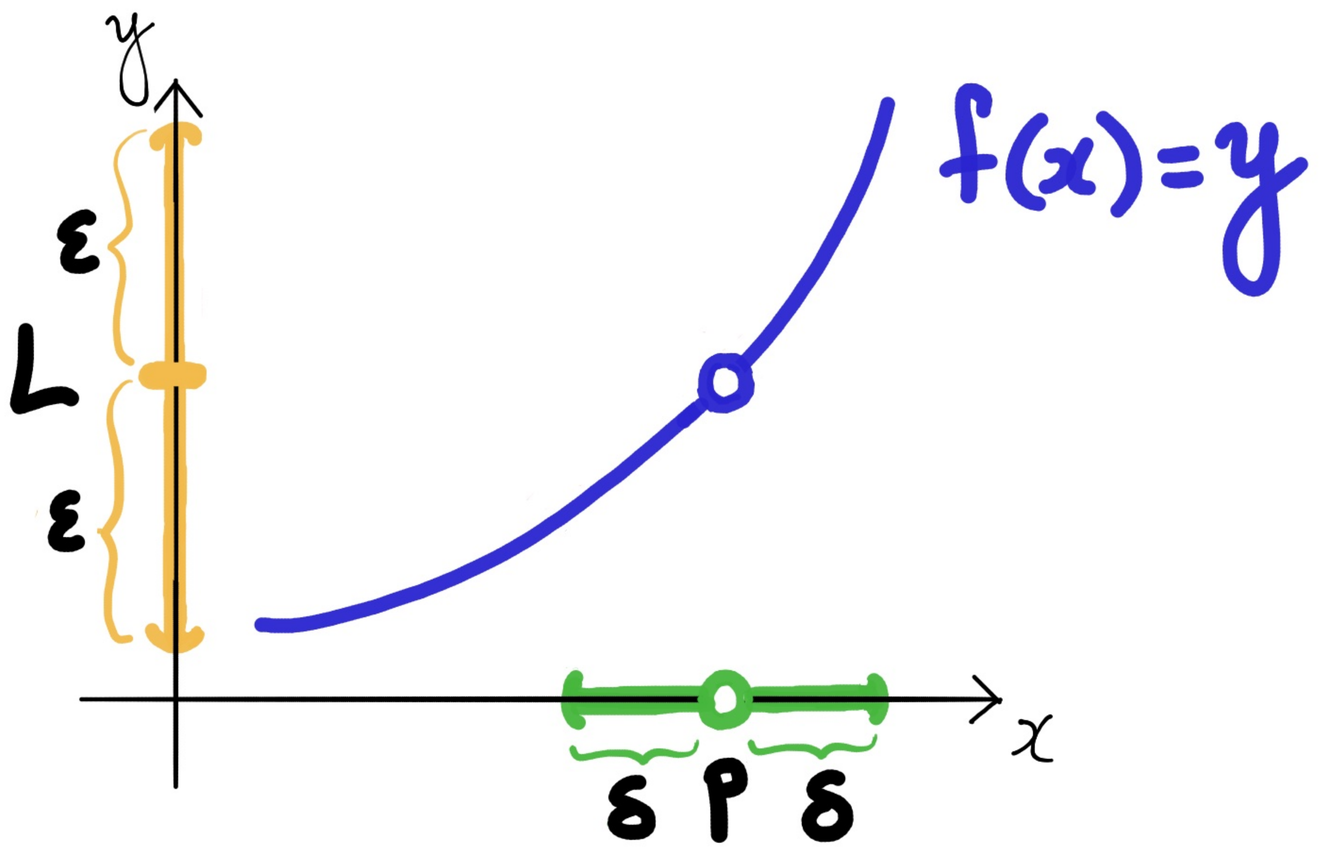

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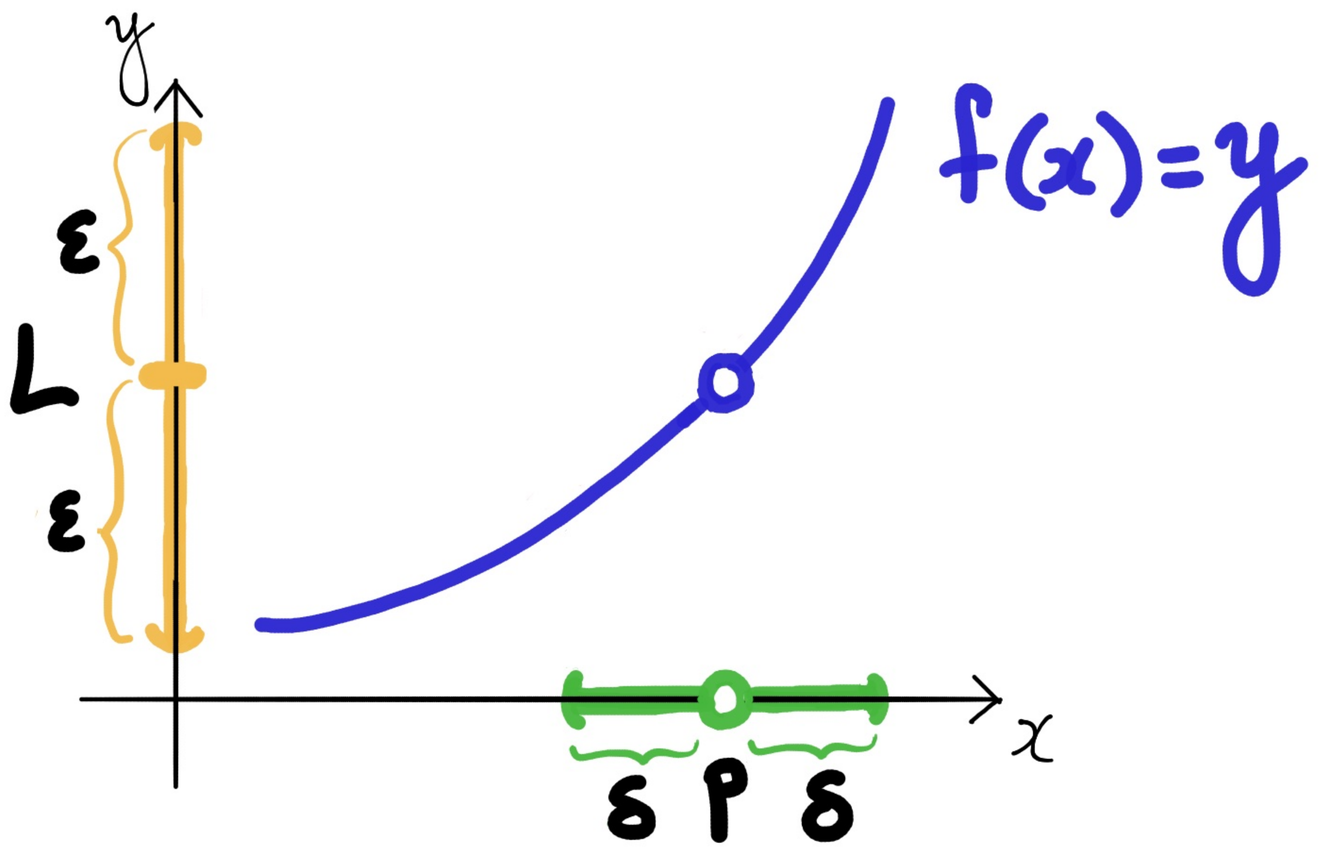

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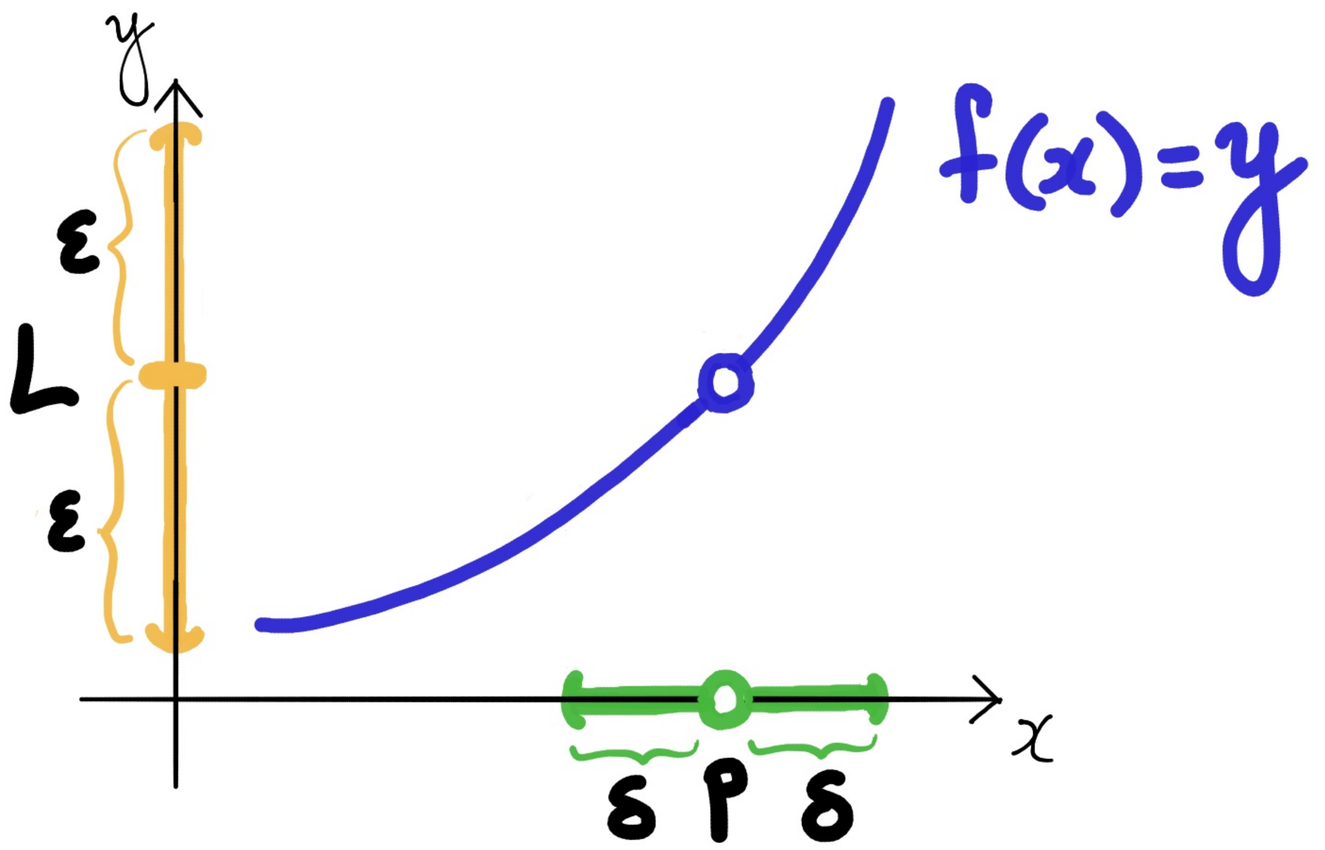
Limits for $f:\mathbb{R}\to\mathbb{R}$



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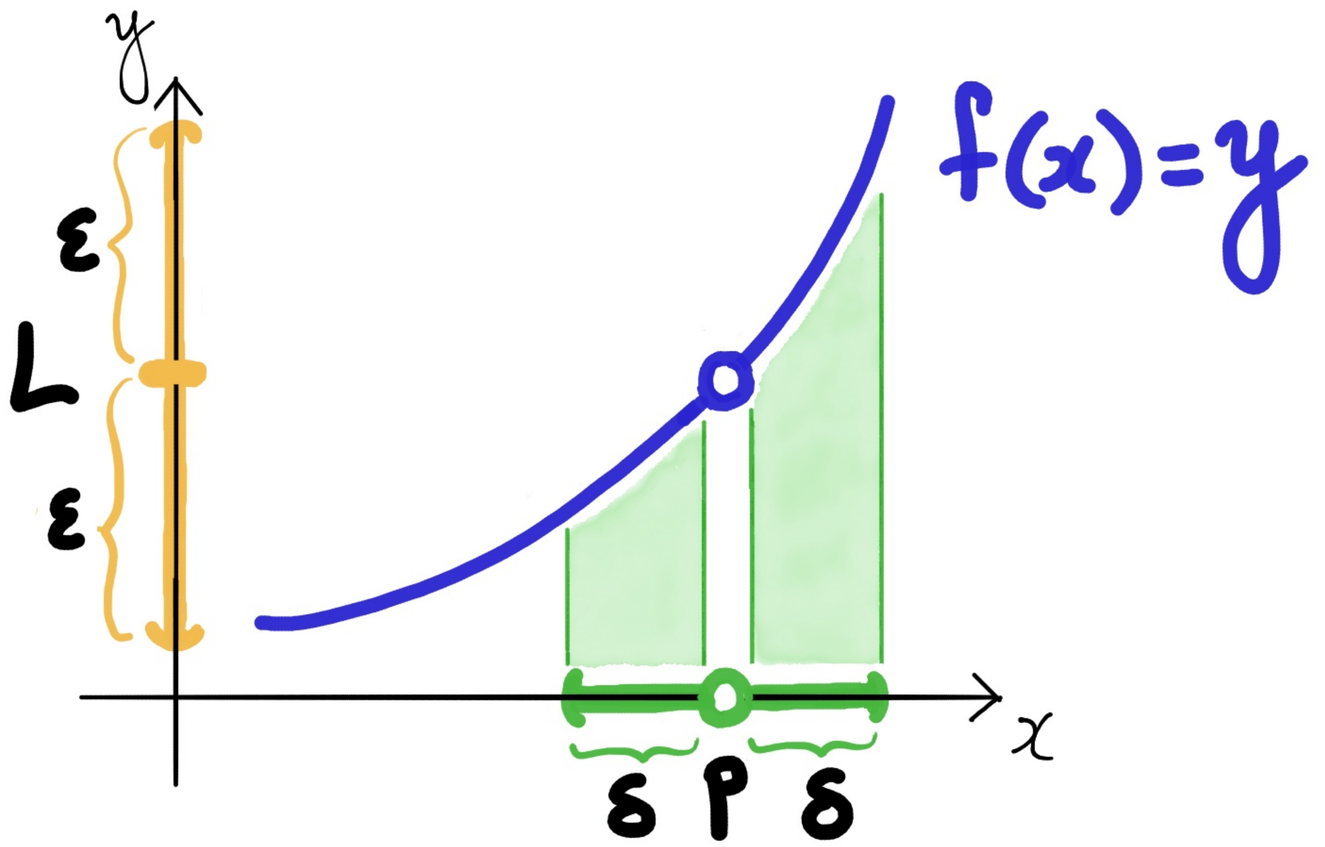

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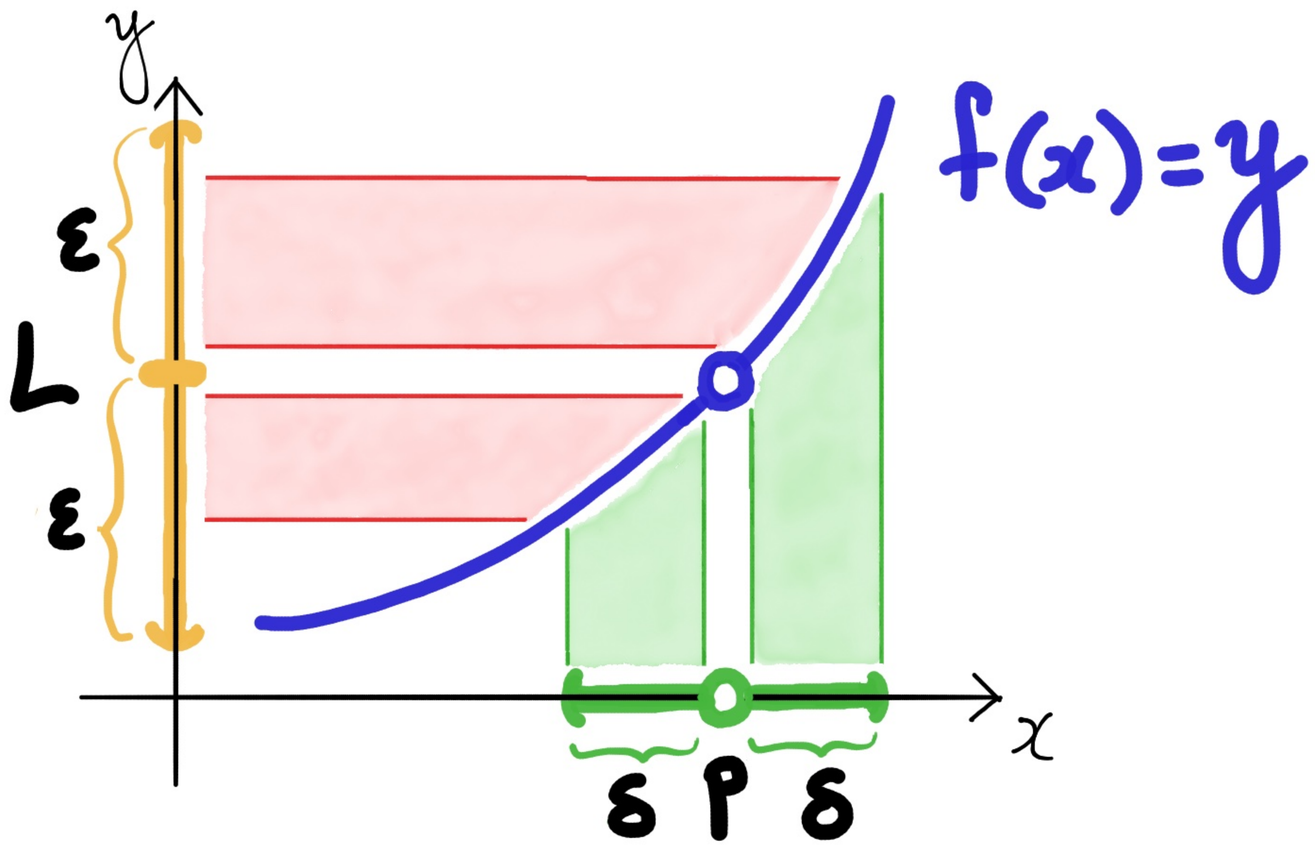

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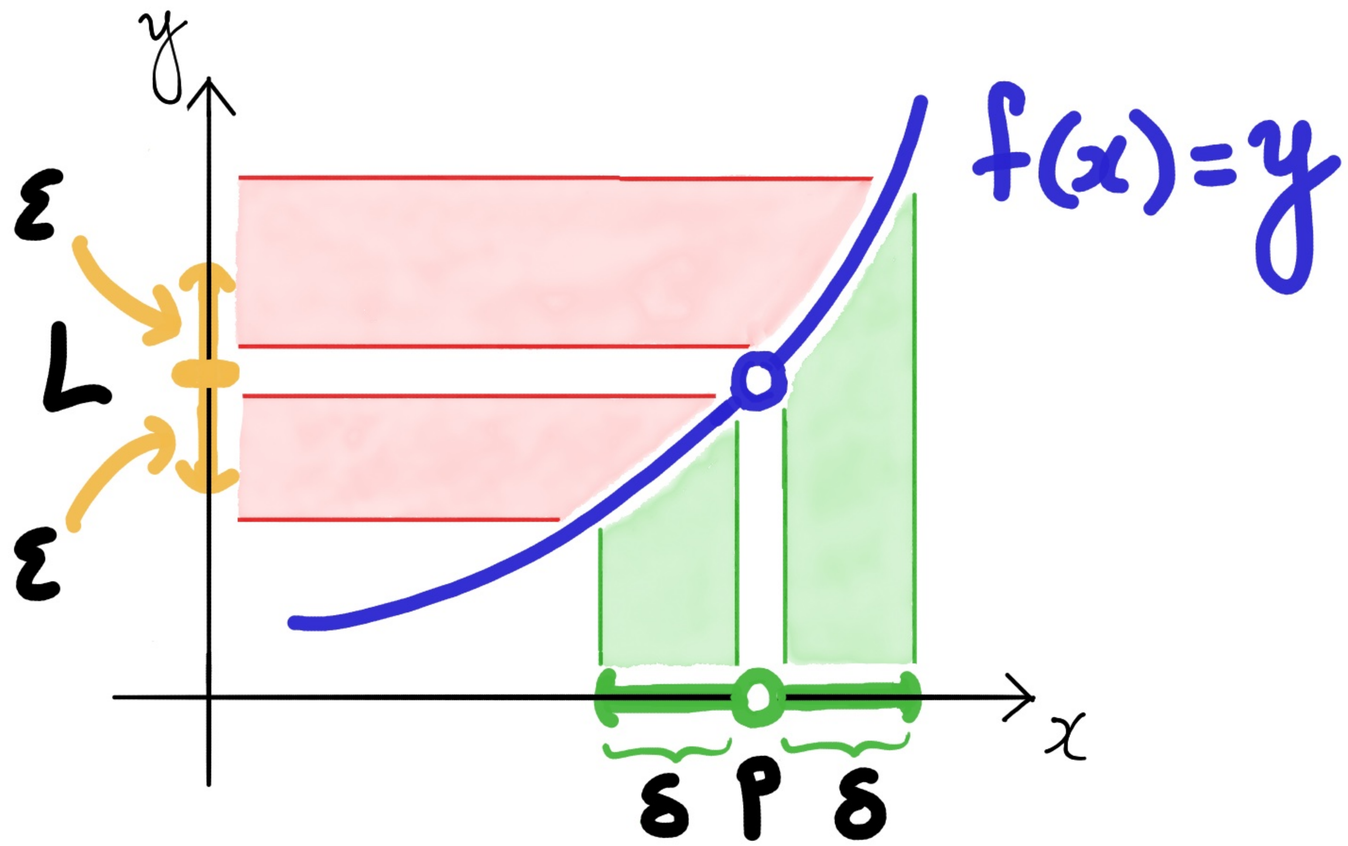

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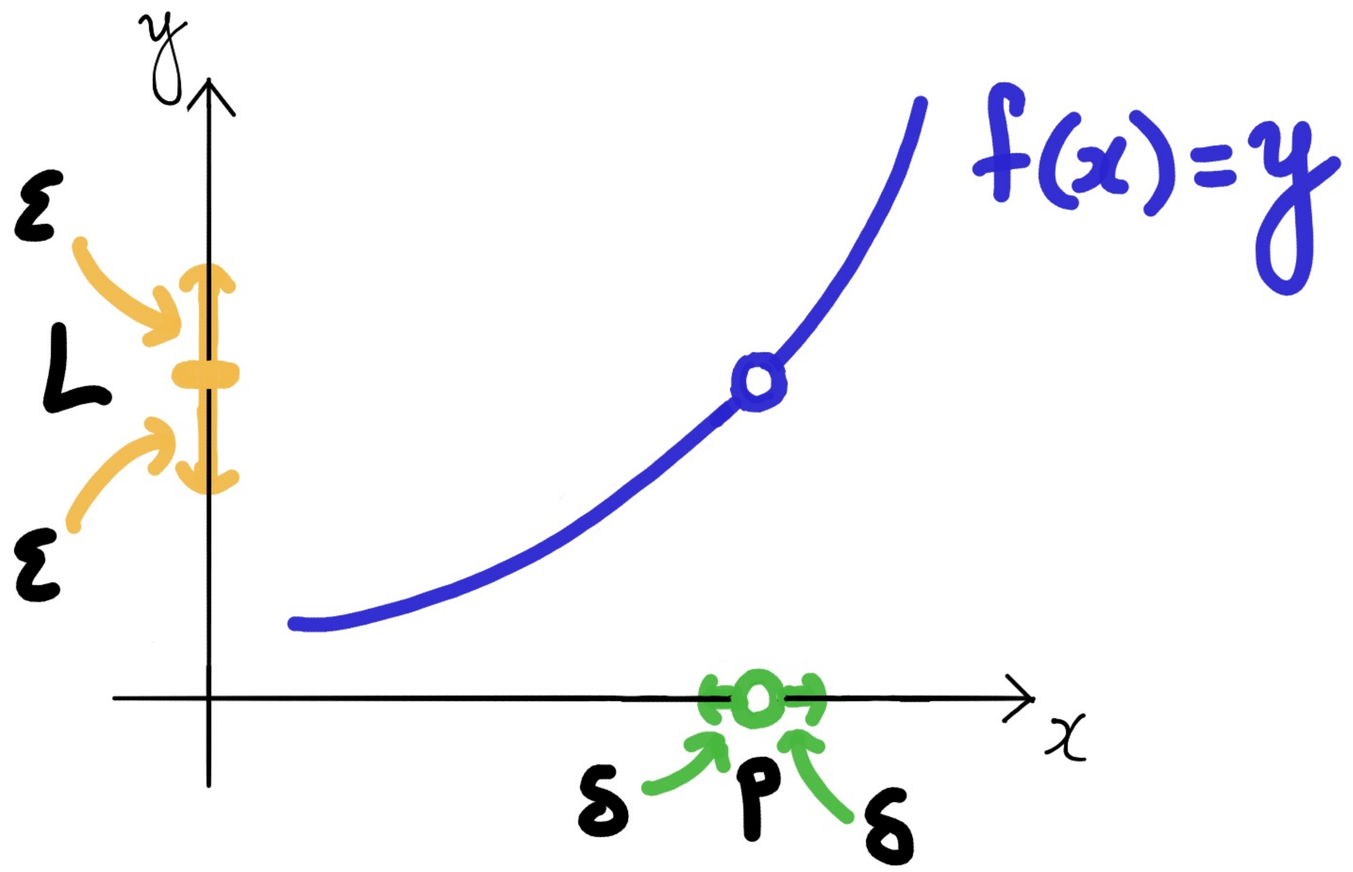

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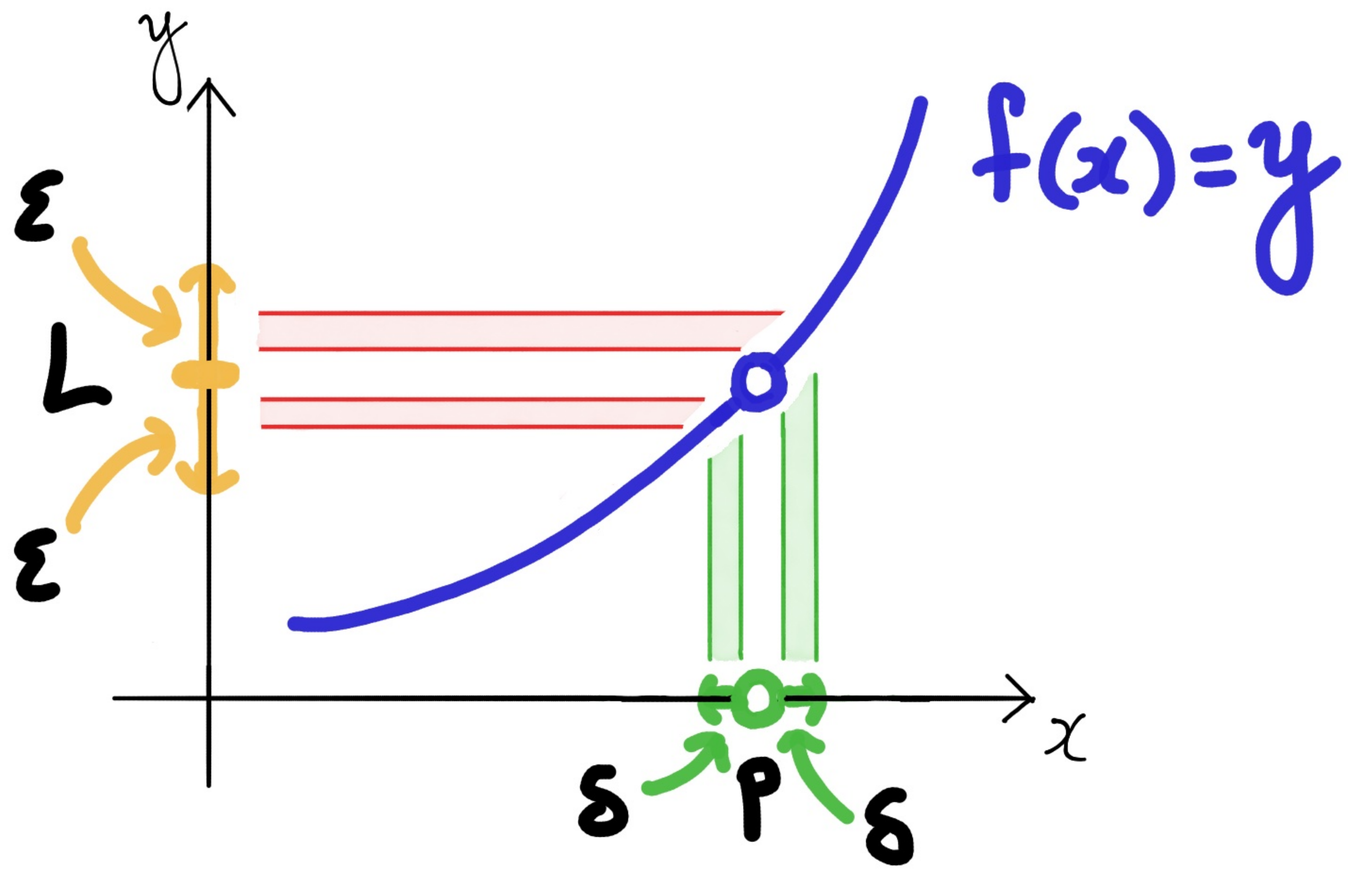

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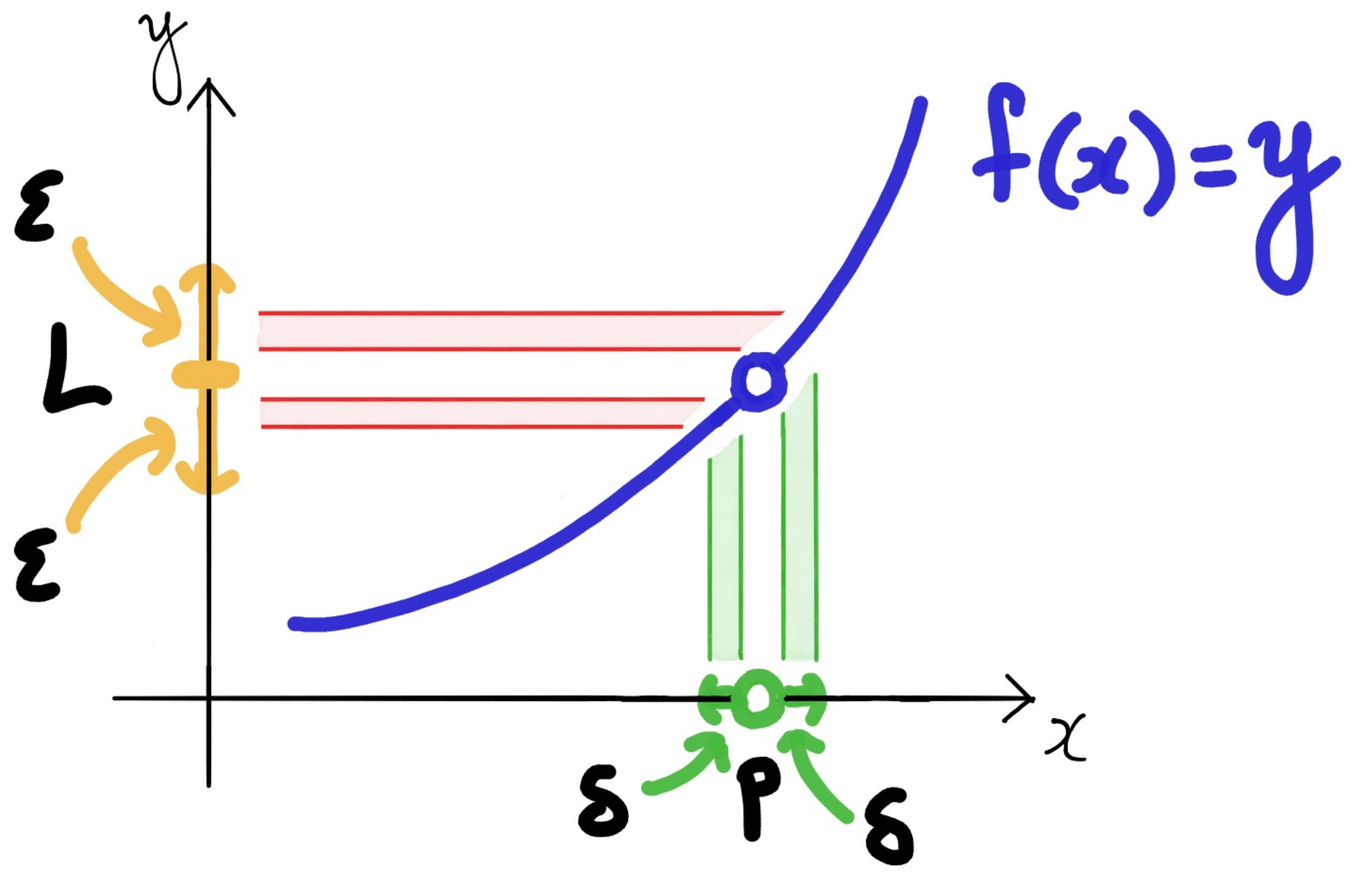

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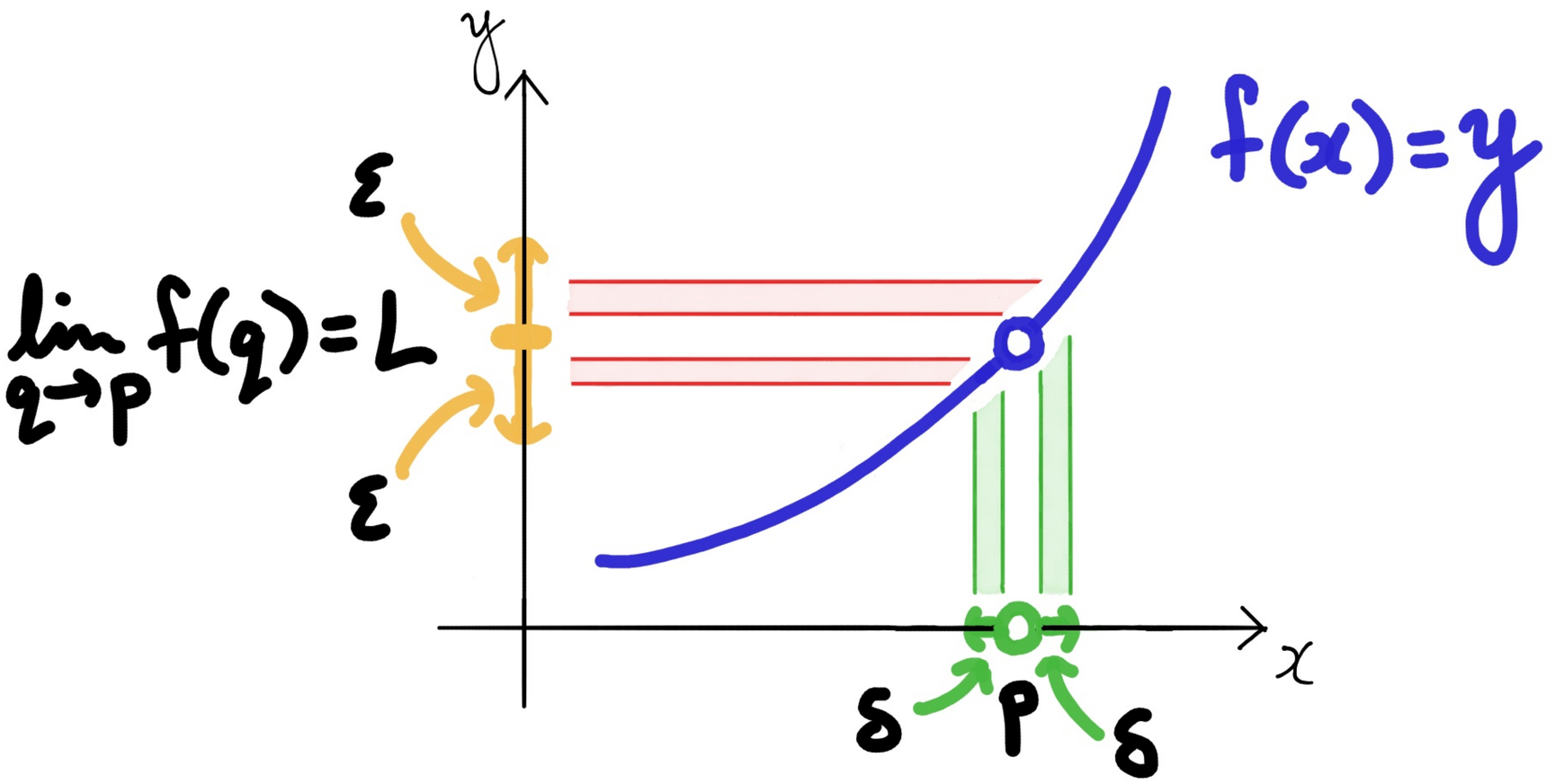

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$$\lim_{q \to p} f(q) = L$$

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

p, q ↑ $f(q), L$ ↑

For any $\varepsilon > 0$, there is some $\delta > 0$, such that
if $0 < |q - p| < \delta$, then $|f(q) - L| < \varepsilon$.

$$\lim_{q \to p} f(q) = L$$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

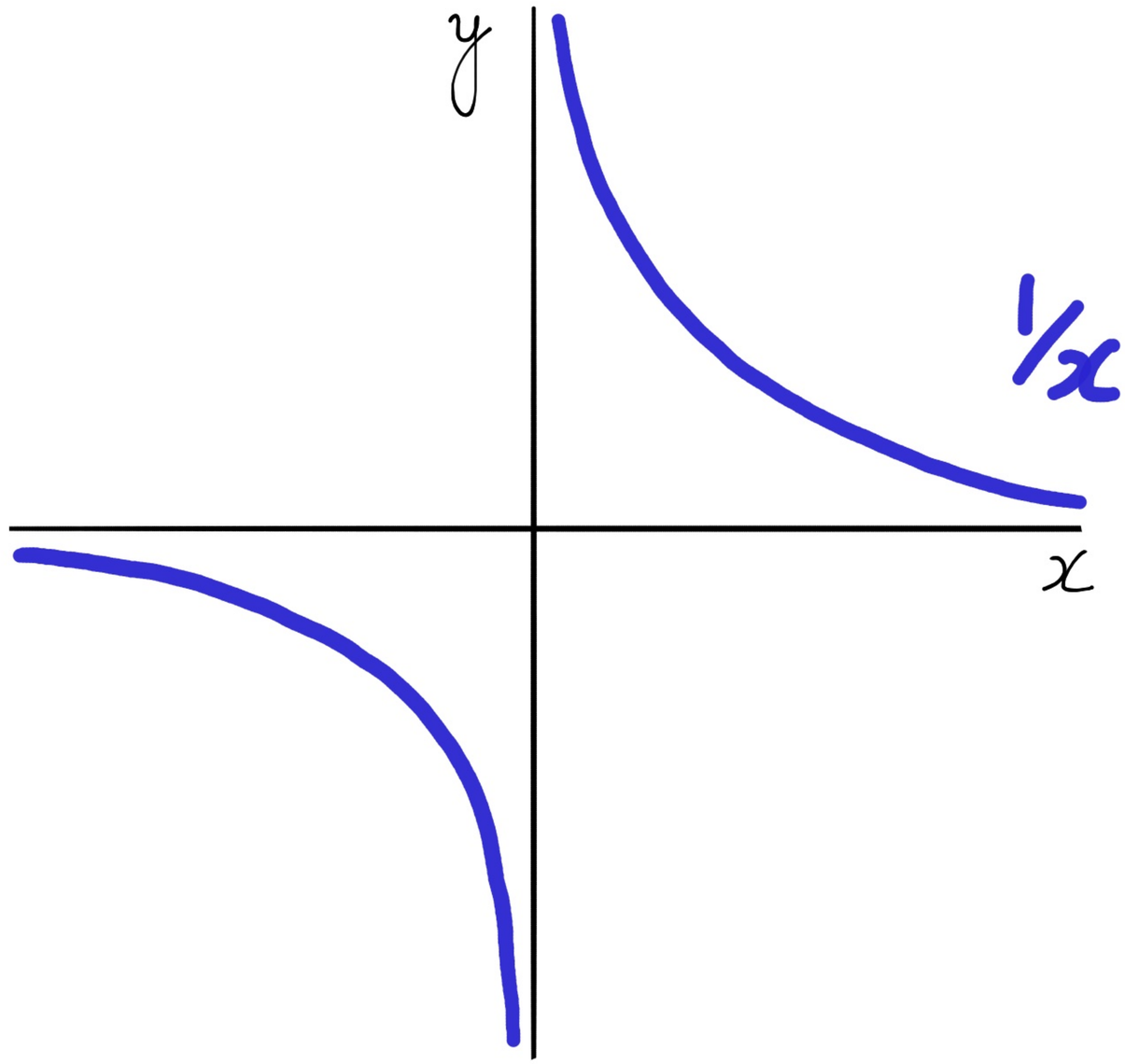
p, q ↑ $\mathbb{R}^n$

$\mathbb{R}^m$ ↑ $f(q), L$

For any $\varepsilon > 0$, there is some $\delta > 0$, such that if $0 < \|q - p\| < \delta$, then $\|f(q) - L\| < \varepsilon$.

If $f(x) = 1/x$,

then $\lim_{x \to 0} f(x)$ does not exist.



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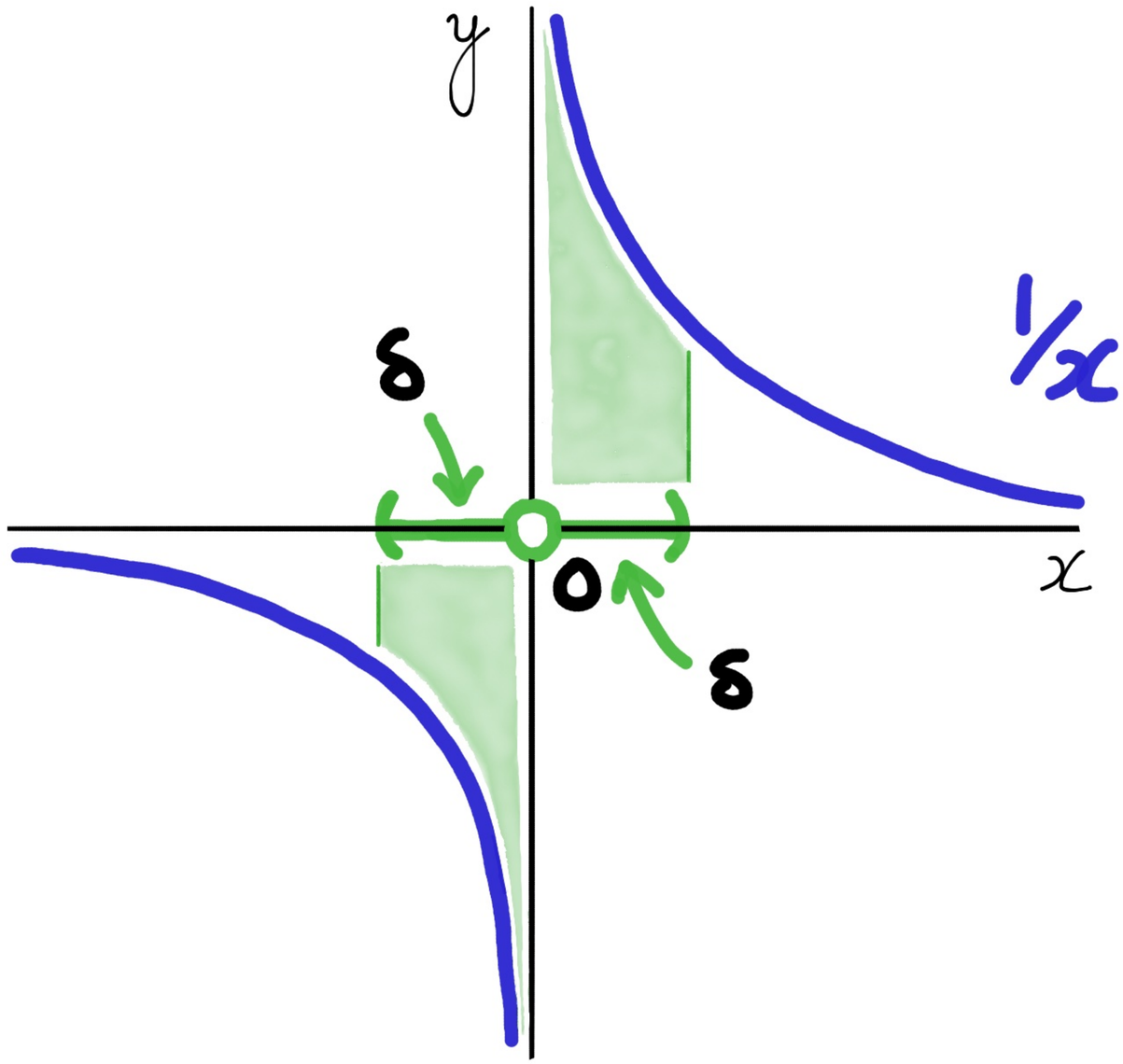

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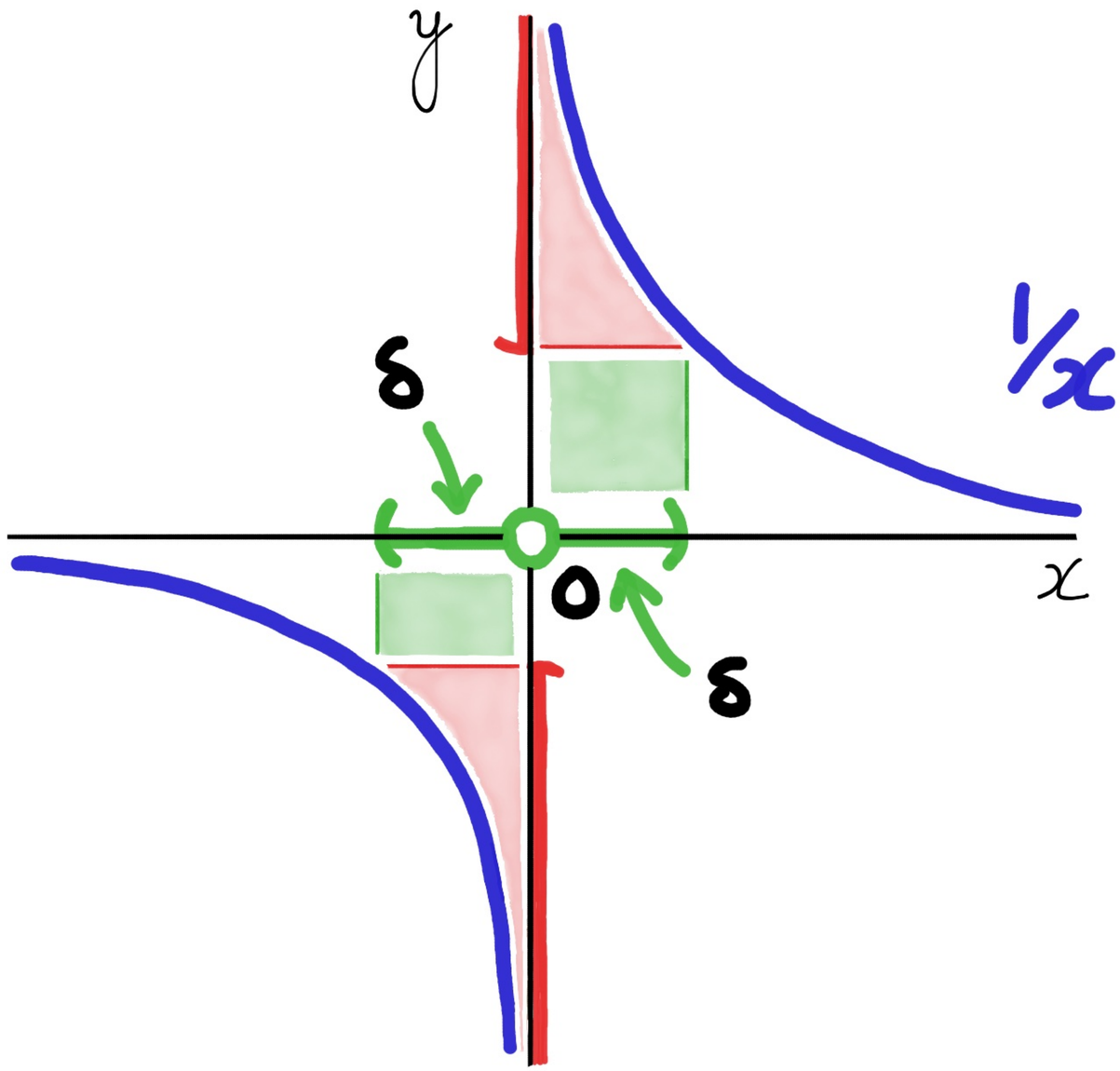

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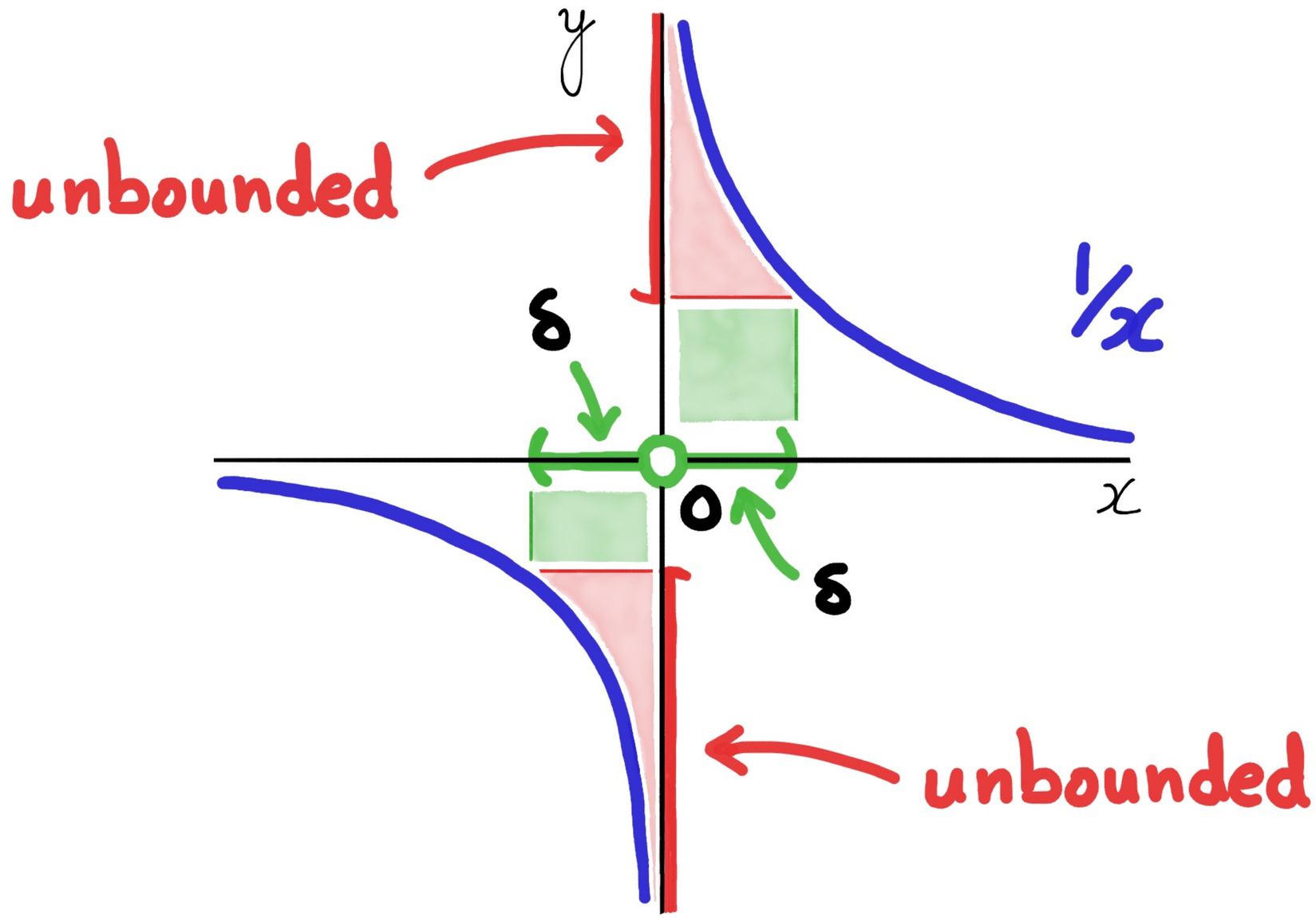


Generally, if

- $f: \mathbb{R}^n \longrightarrow \mathbb{R}$
- $g: \mathbb{R}^n \longrightarrow \mathbb{R}$
- $p \in \mathbb{R}^n$ with $g(p) = 0$

then

$$\lim_{q \to p} \frac{f(q)}{g(q)}$$ does not exist.

Generally, if

- $f: \mathbb{R}^n \longrightarrow \mathbb{R}$
- $g: \mathbb{R}^n \longrightarrow \mathbb{R}$
- $p \in \mathbb{R}^n$ with $g(p) = 0$

then

$$\lim_{q \to p} \frac{f(q)}{g(q)}$$ does not exist.

Examples:

$$\lim_{x \to 0} \frac{1}{x}$$

$$\lim_{x \to 3} \frac{2x + e^x}{x - 3}$$

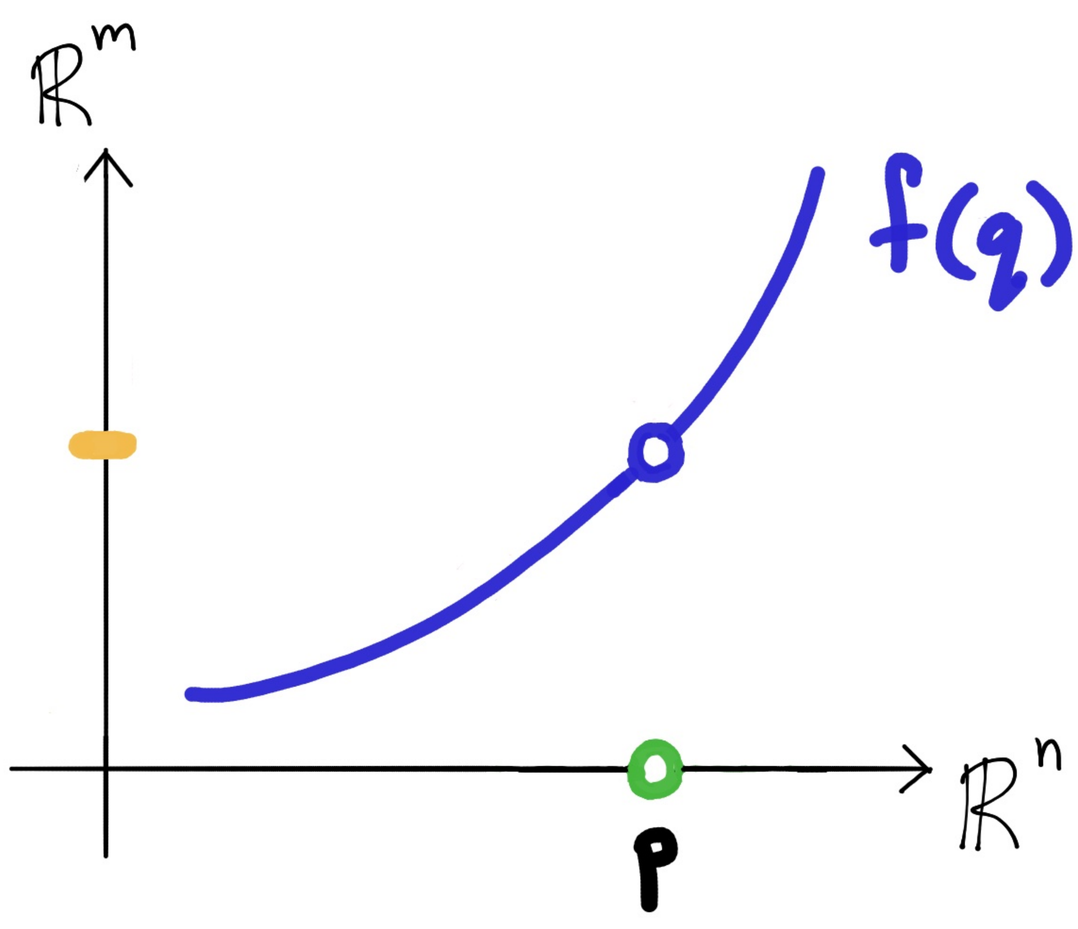
$$\lim_{(x,y) \to (2,4)} \frac{3xy}{2x - y}$$

$$\lim_{(x,y,z) \to (1,0,-1)} \frac{xz}{x + y + z}$$

Ⅱ Continuity

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is **continuous** if for every $p \in \mathbb{R}^n$, $\lim_{q \to p} f(q) = f(p)$.

$\lim_{q \to p} f(q)$



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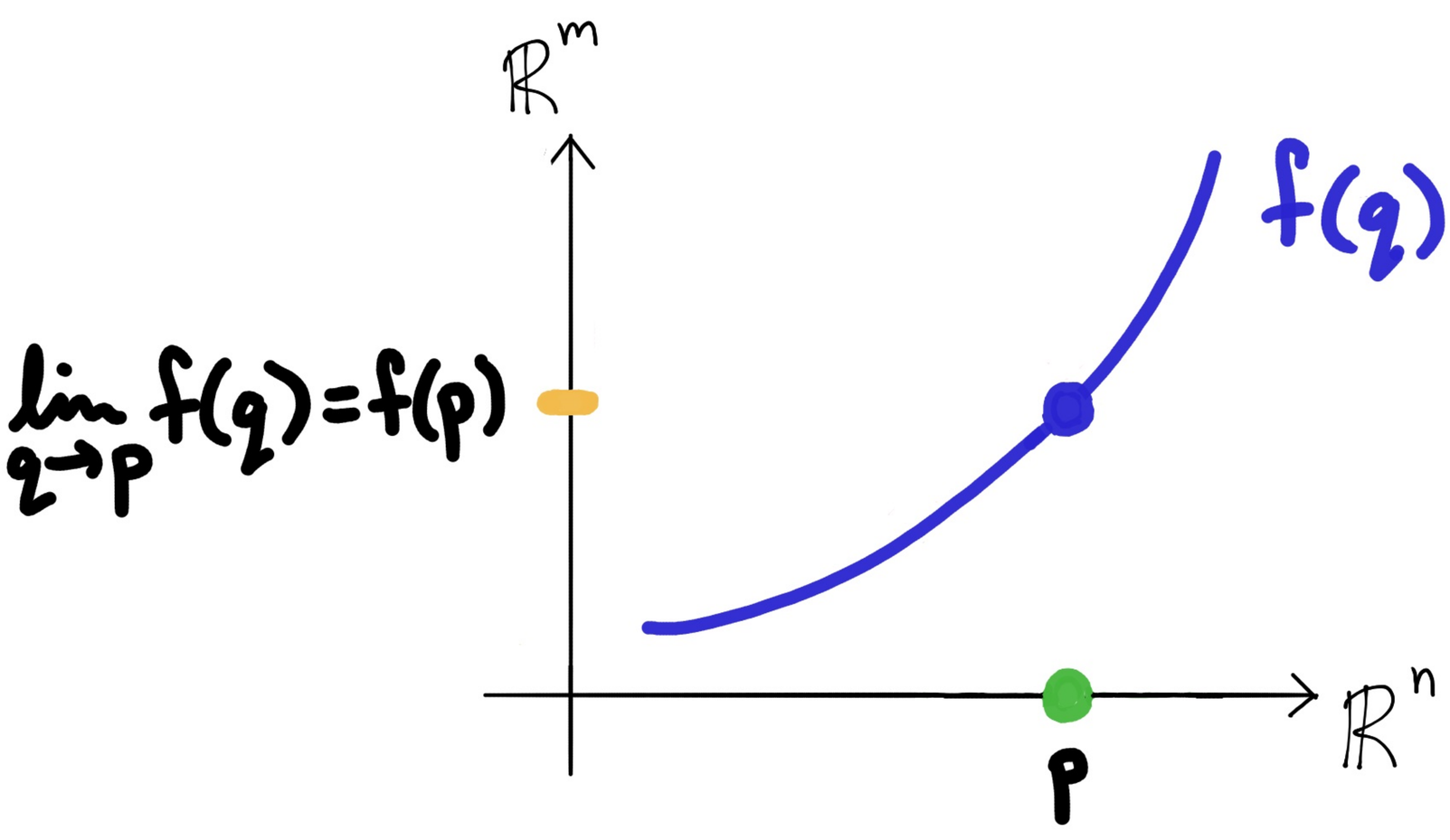
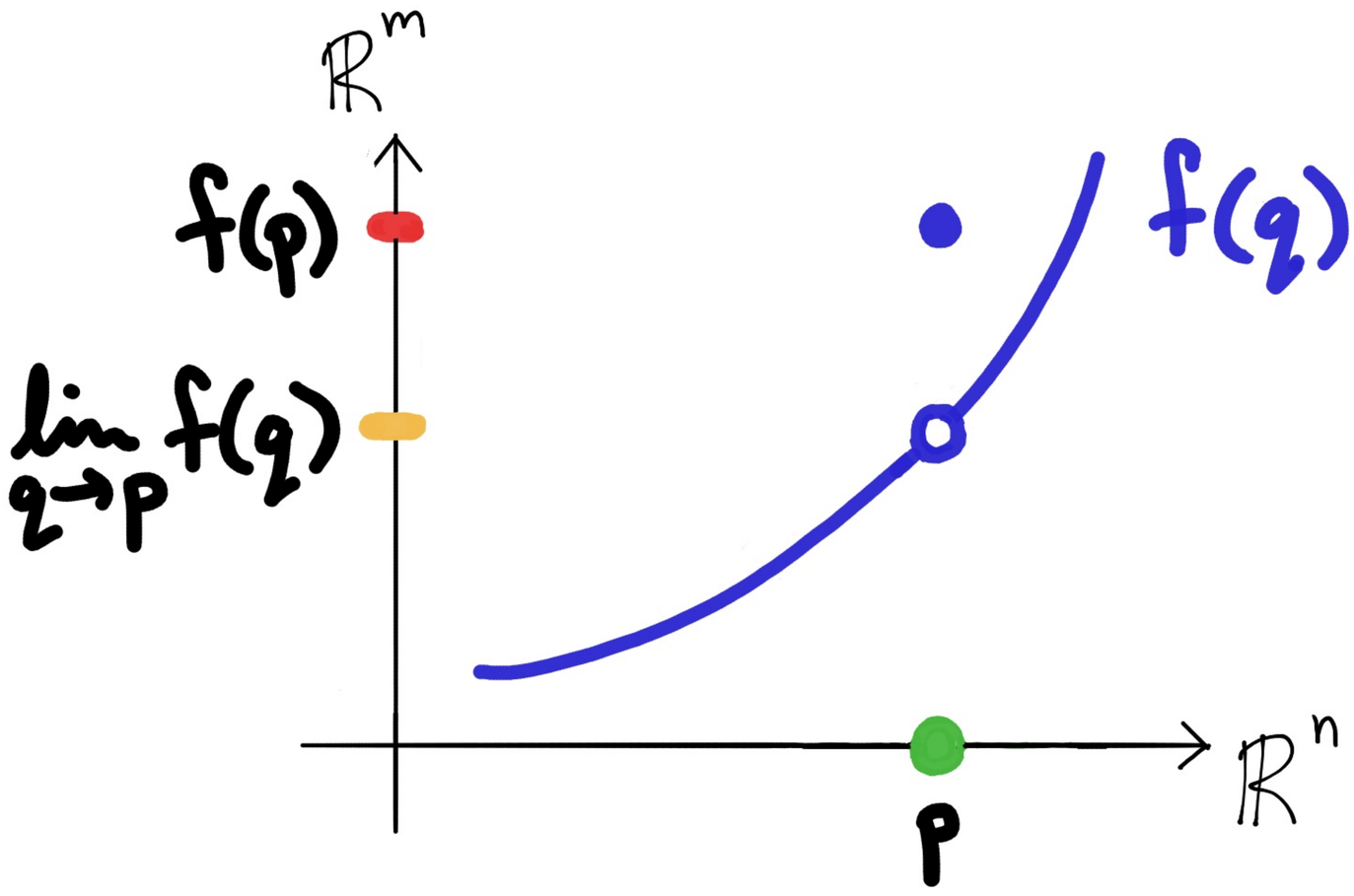


Not continuous



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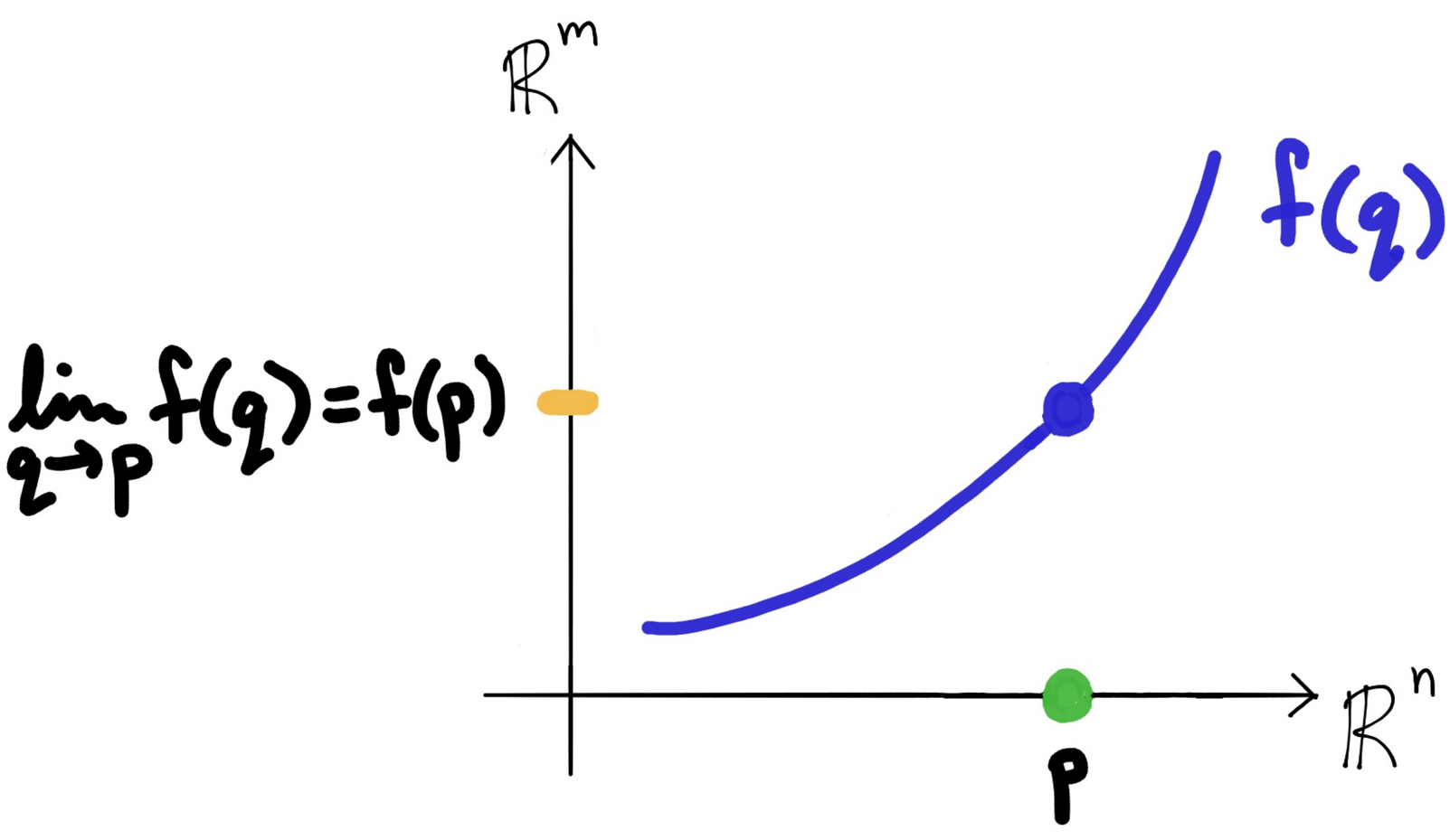

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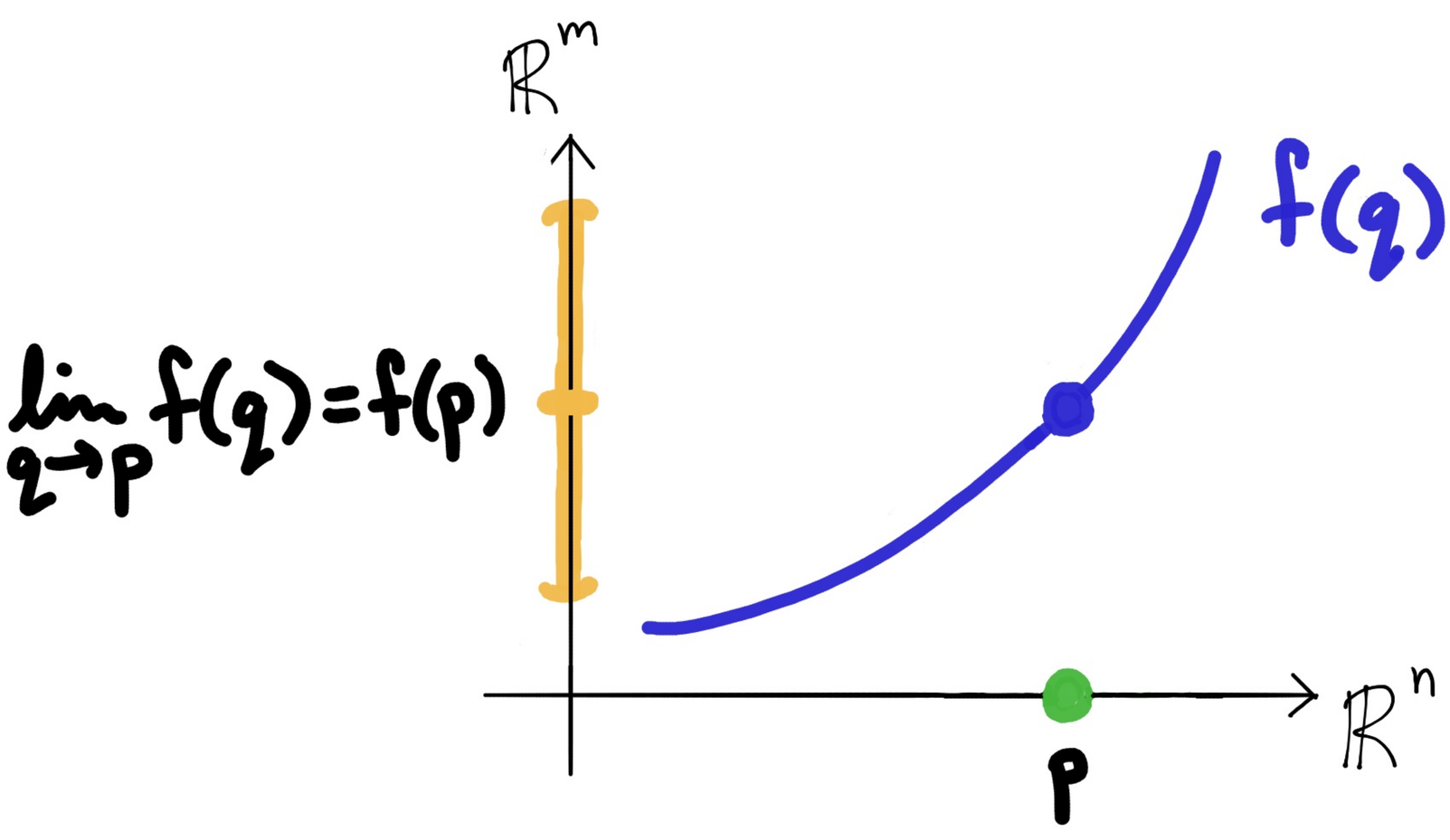

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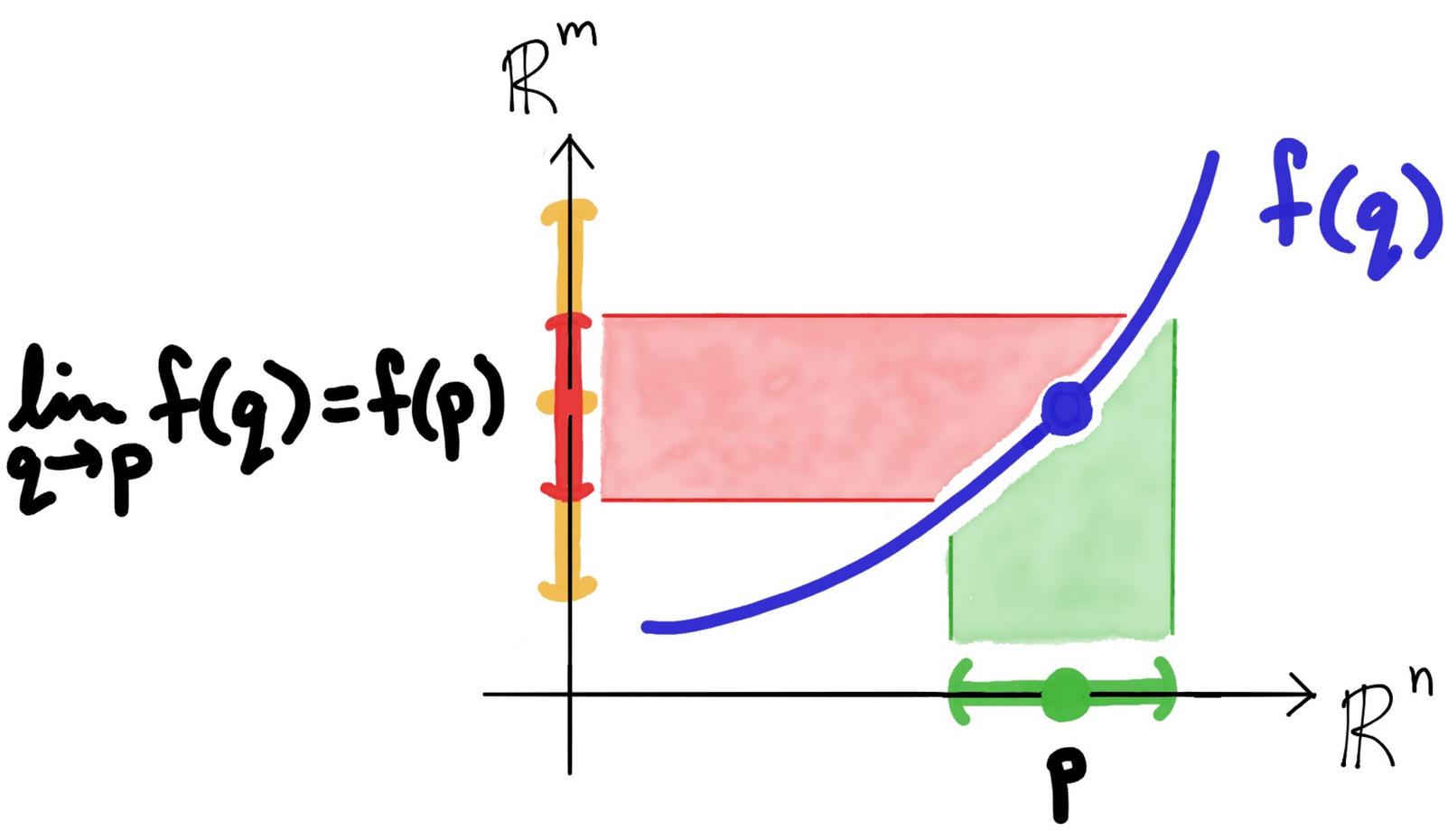
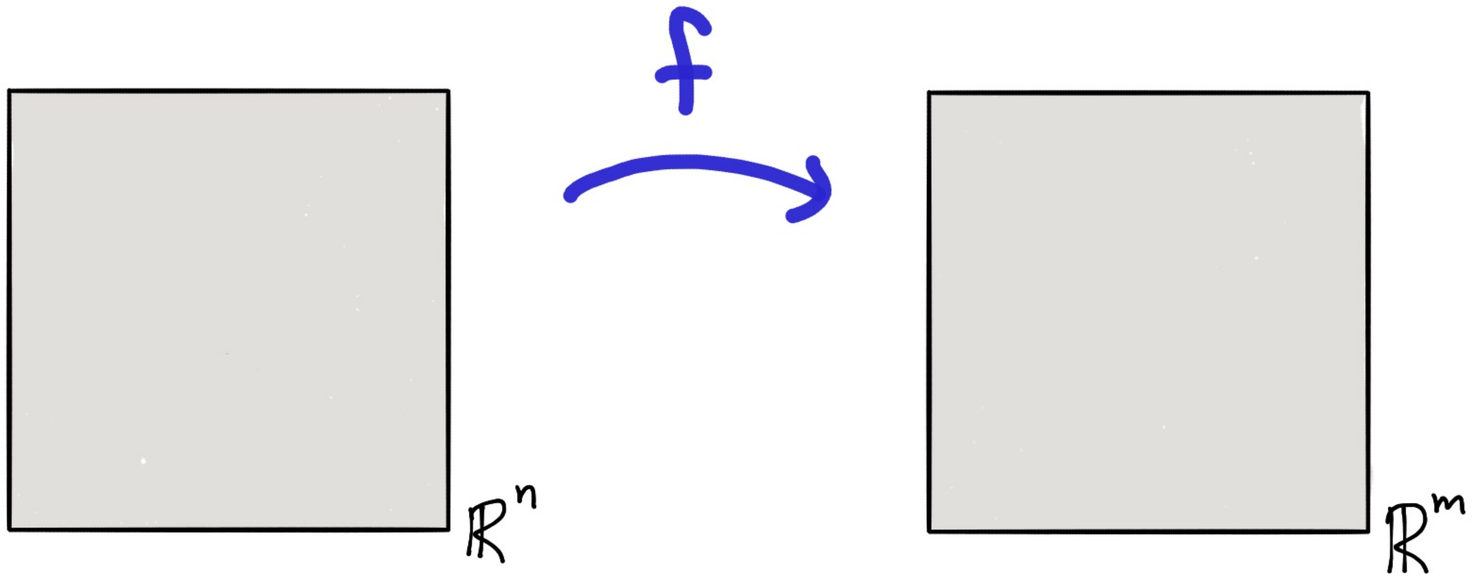


Continuous



Continuous

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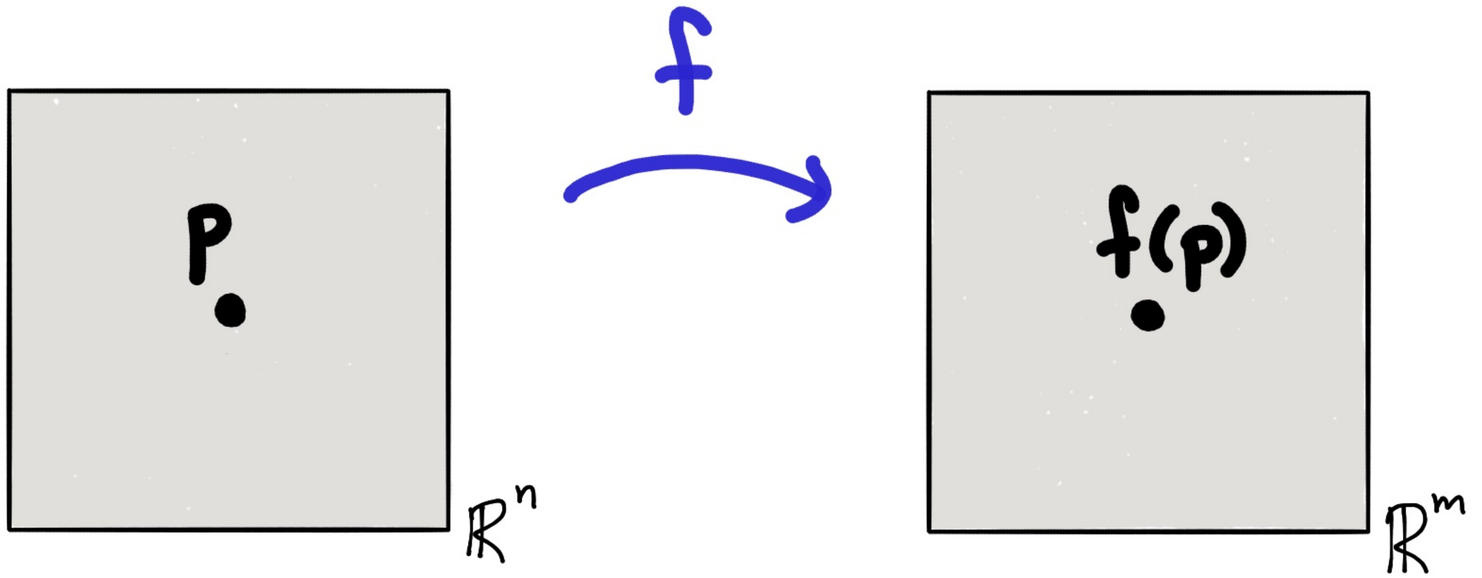

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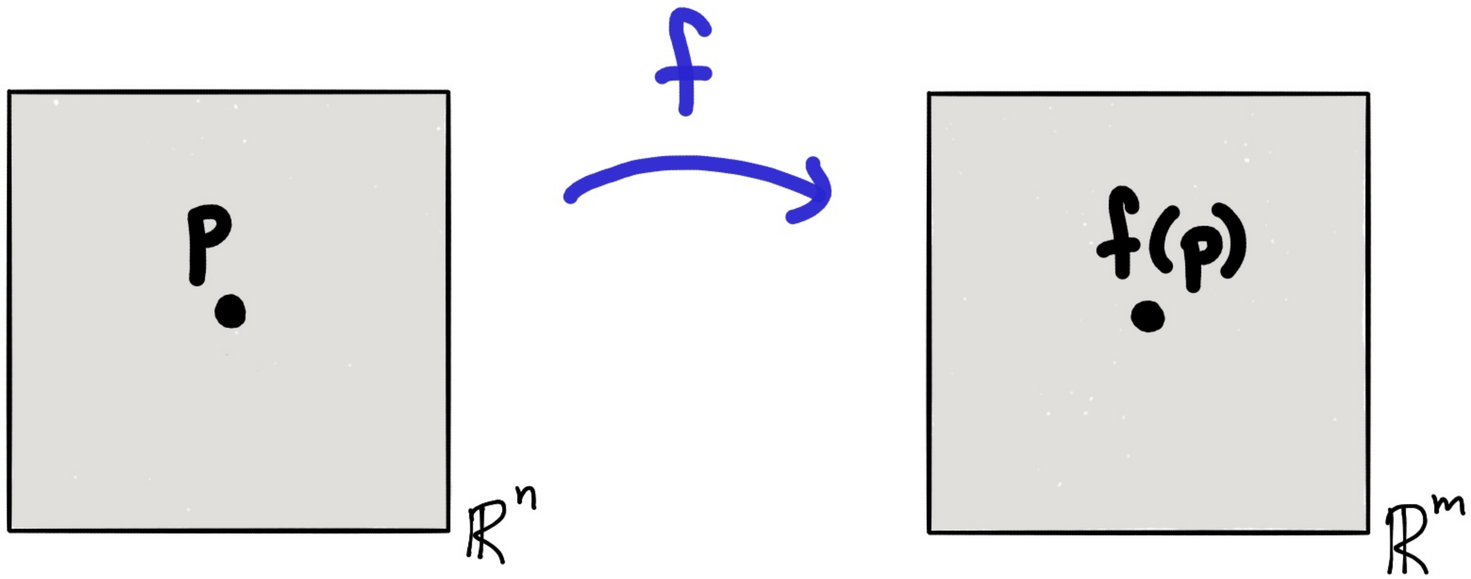
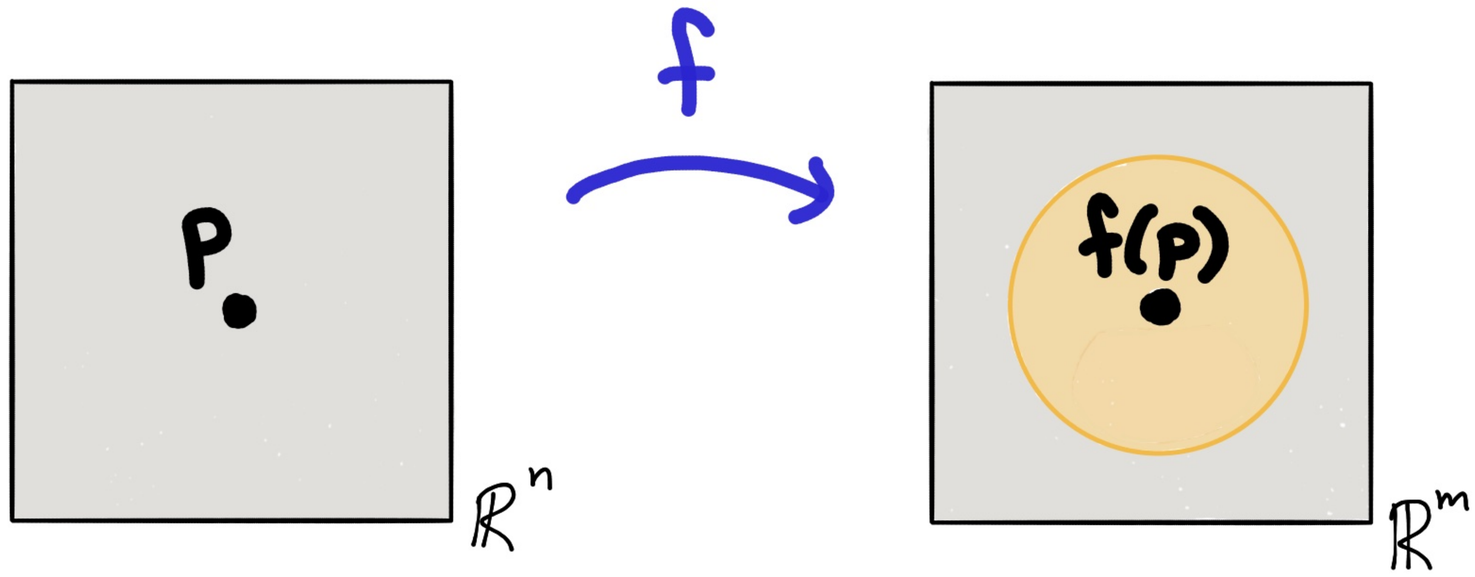


2. Pick any distance from f(p).



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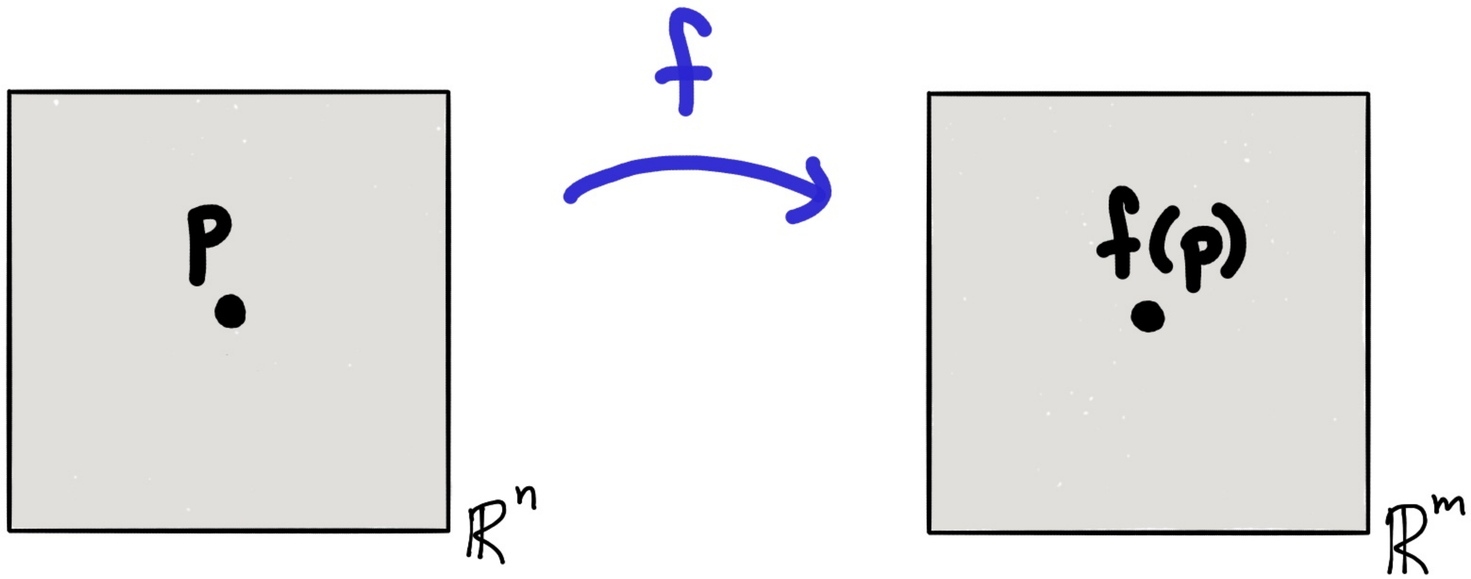

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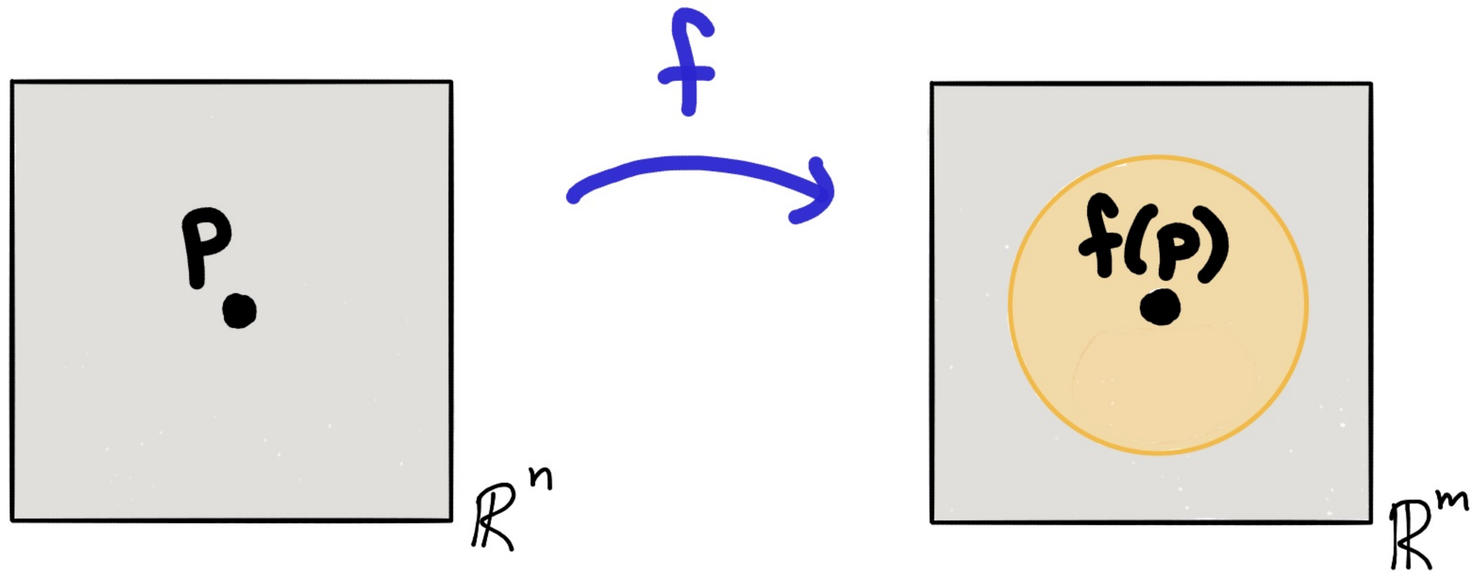
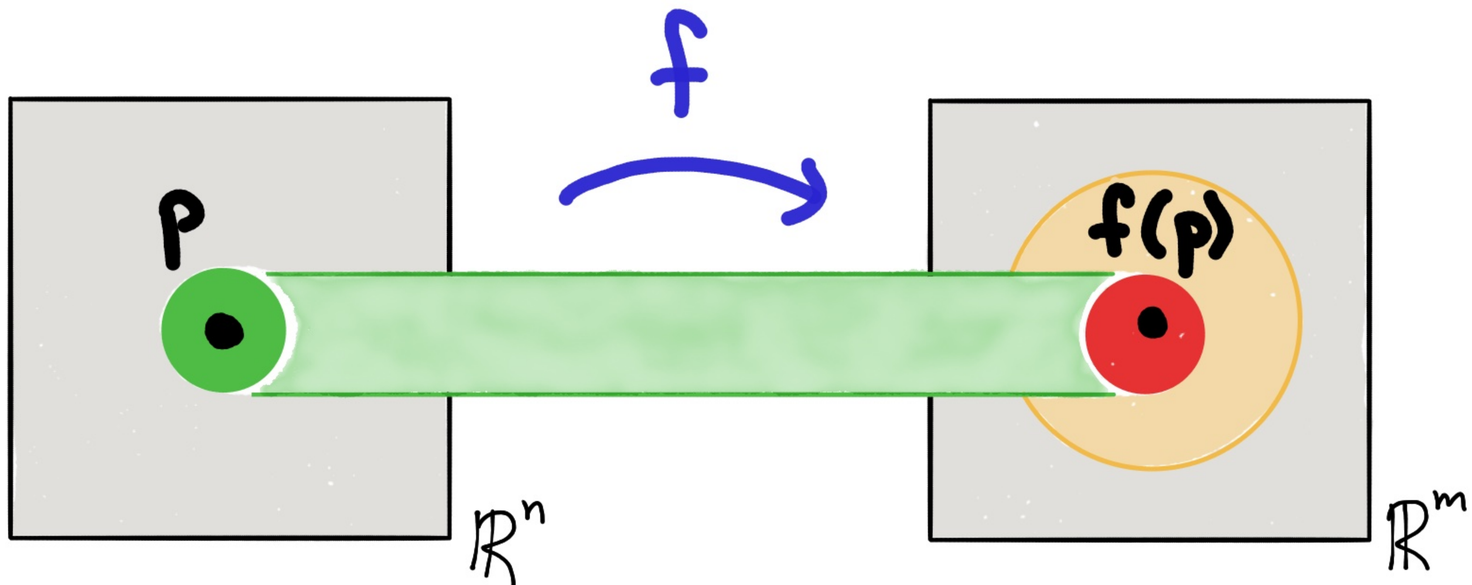
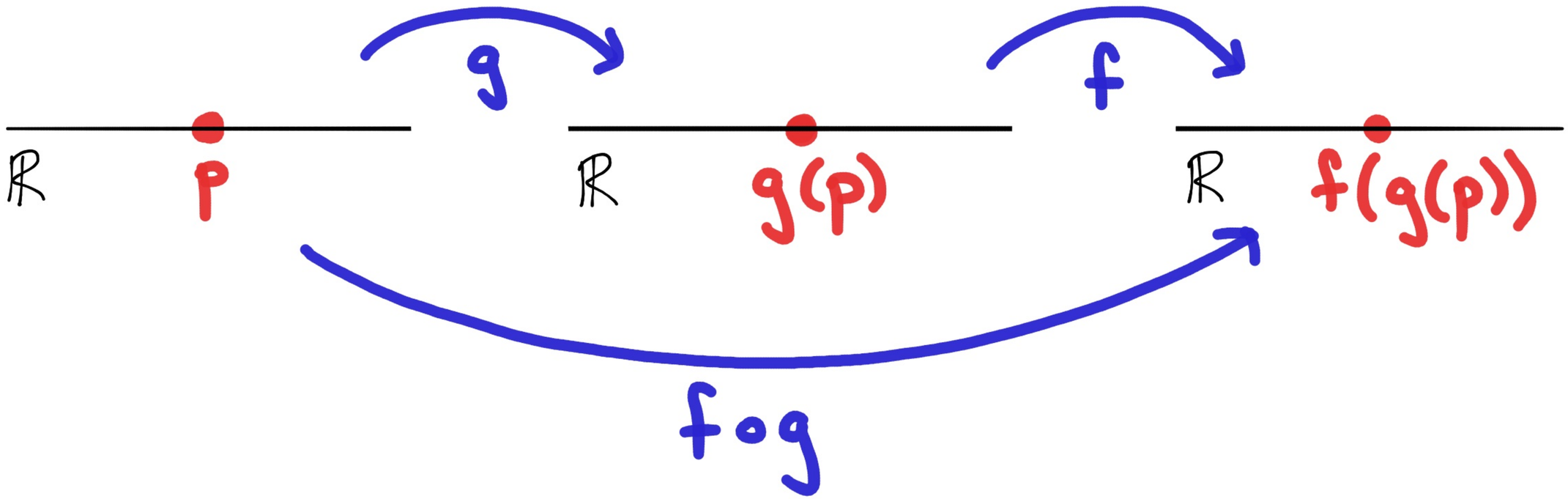


3. Find a distance from p that f puts near f(p).



function composition



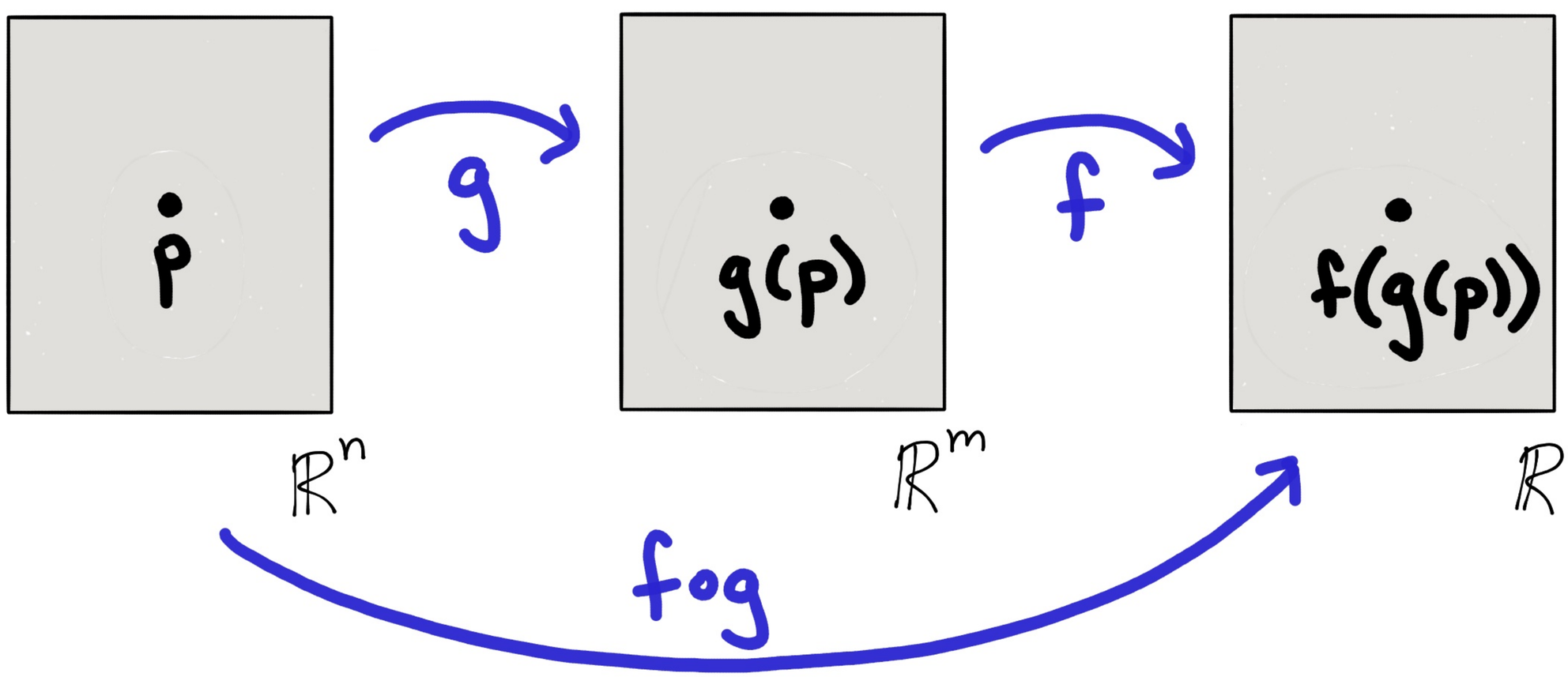
Example

$f(x) = \sin x$

$g(x) = \log x$

$f \circ g(x) = \sin(\log x)$

function composition



Example:

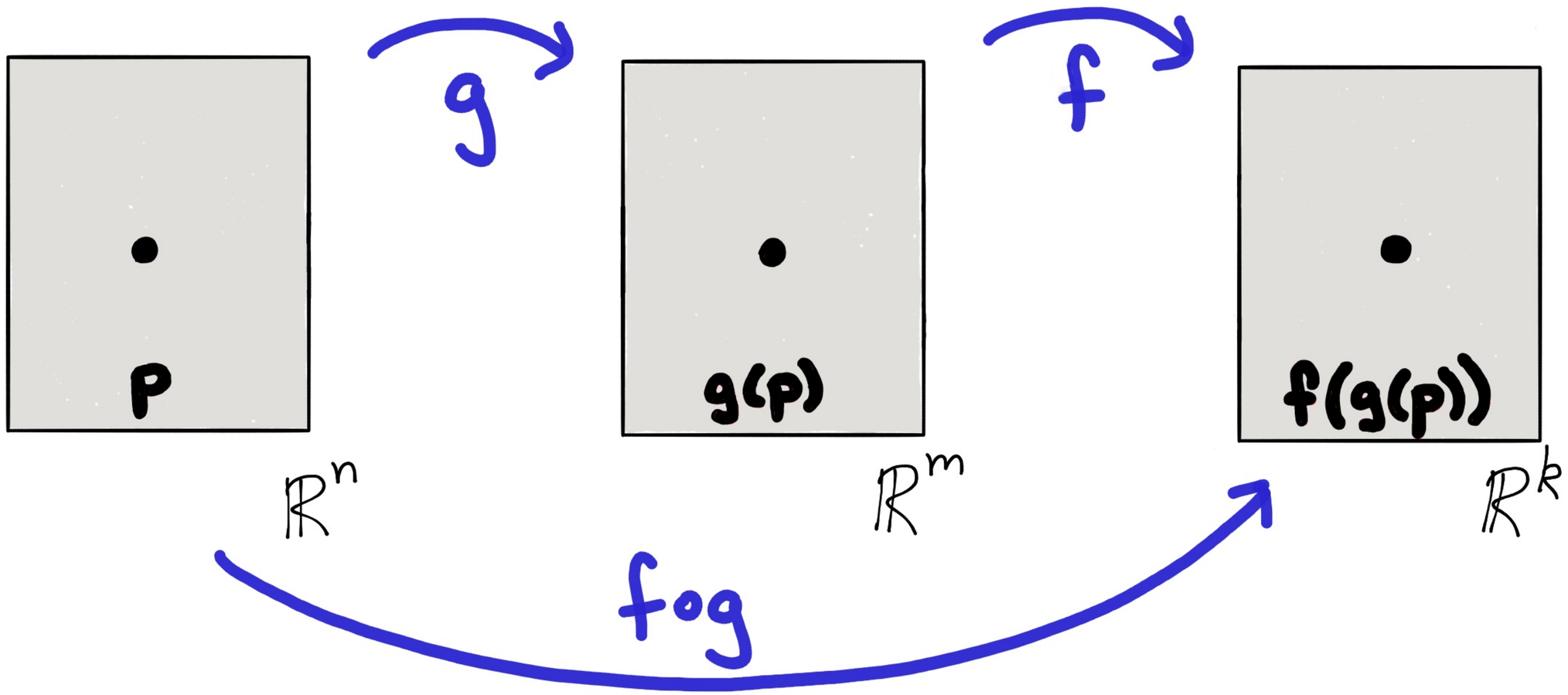
$f(x) = \log x$

$g(x,y) = xy$

$f \circ g(x,y) = \log(xy)$

Proposition: If f and g are continuous, so is $f \circ g$.

proof:



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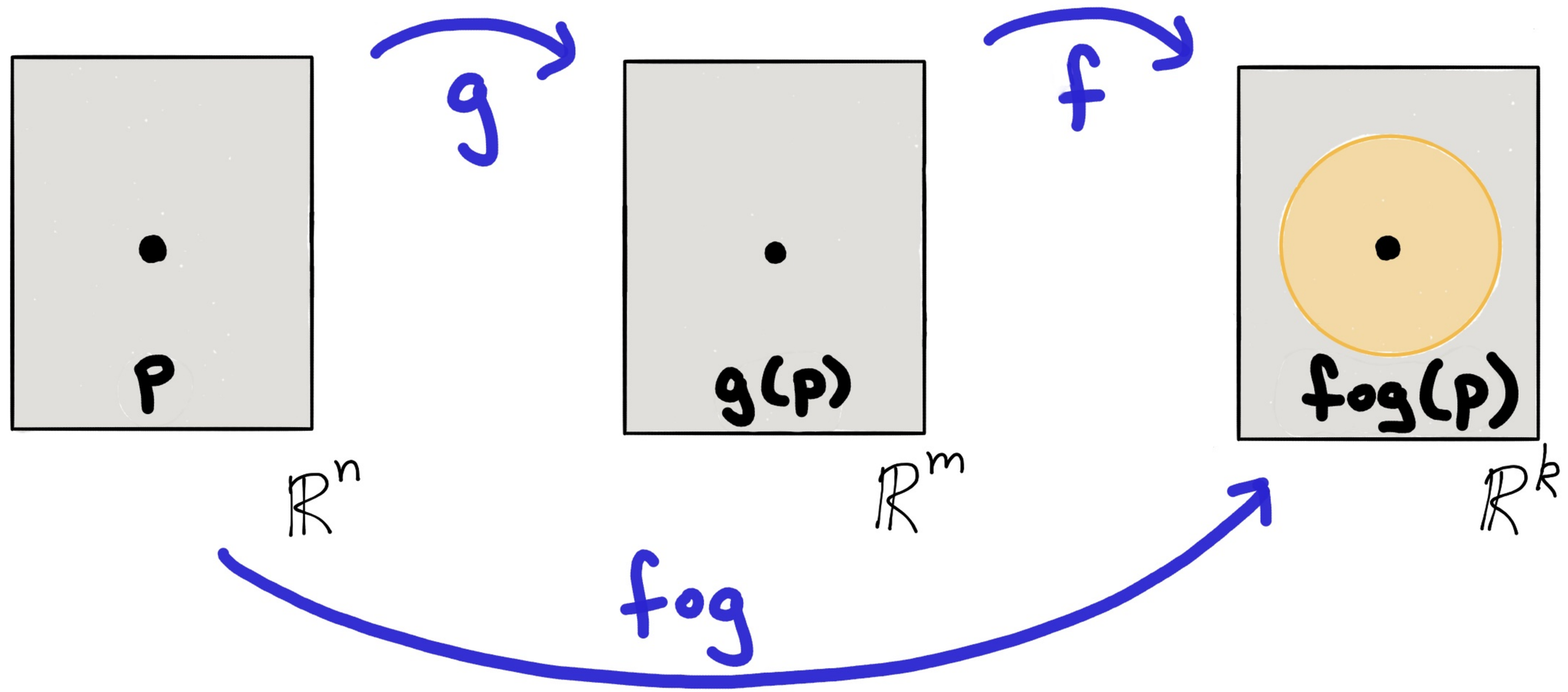

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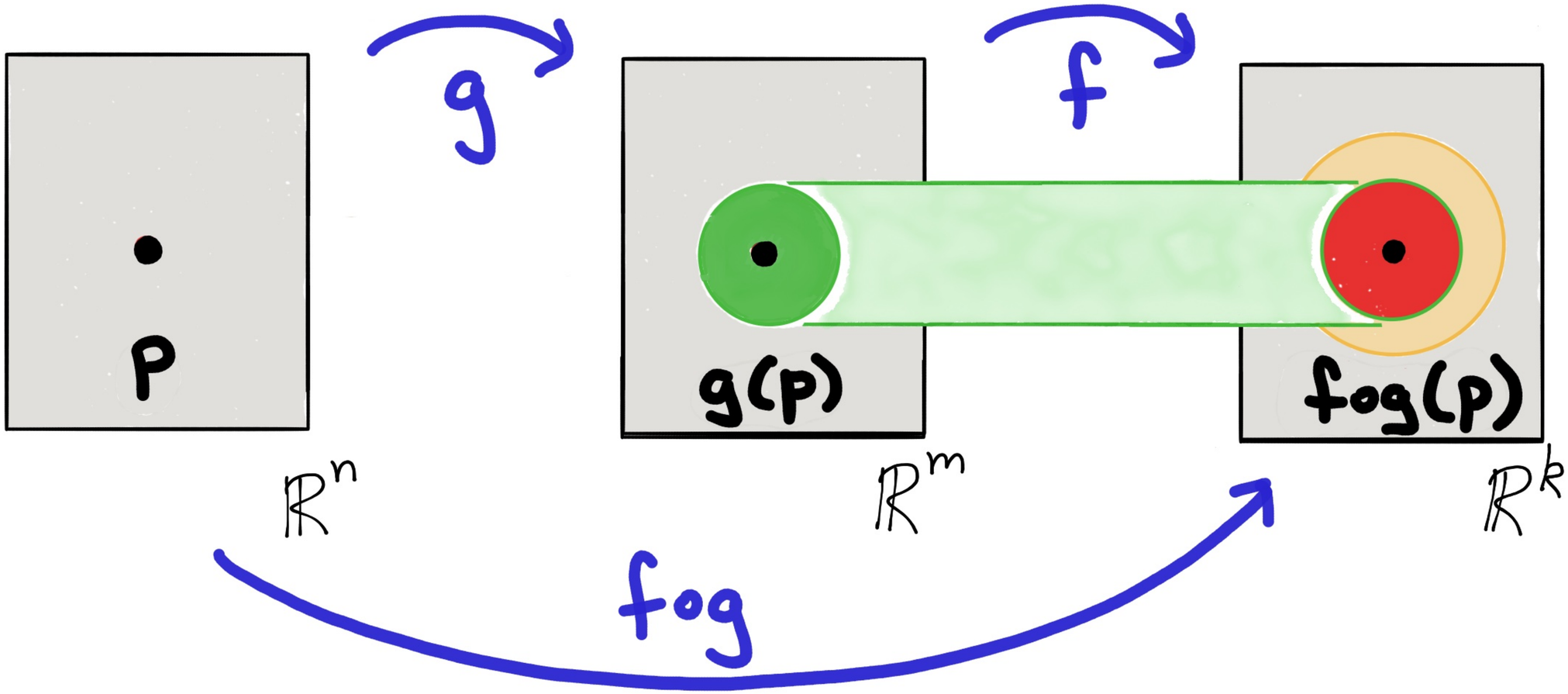

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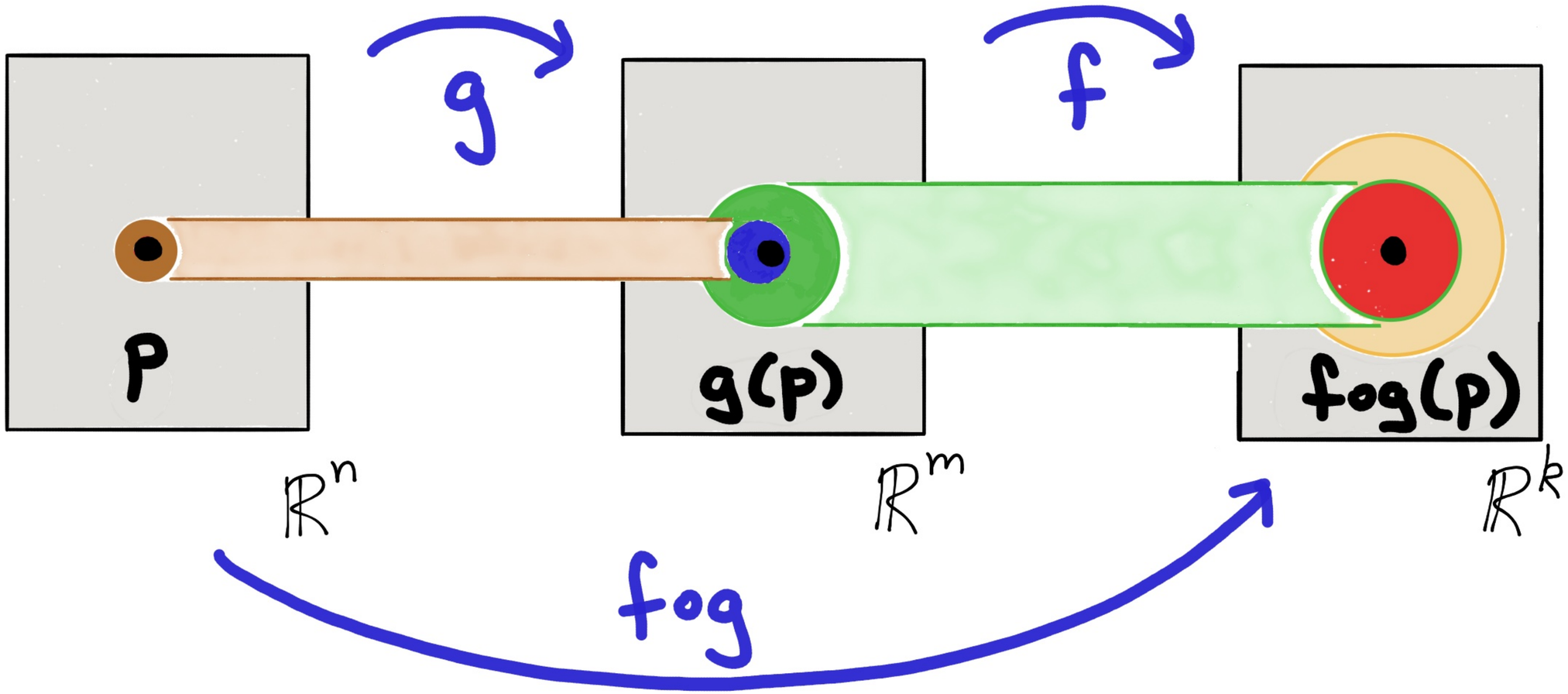

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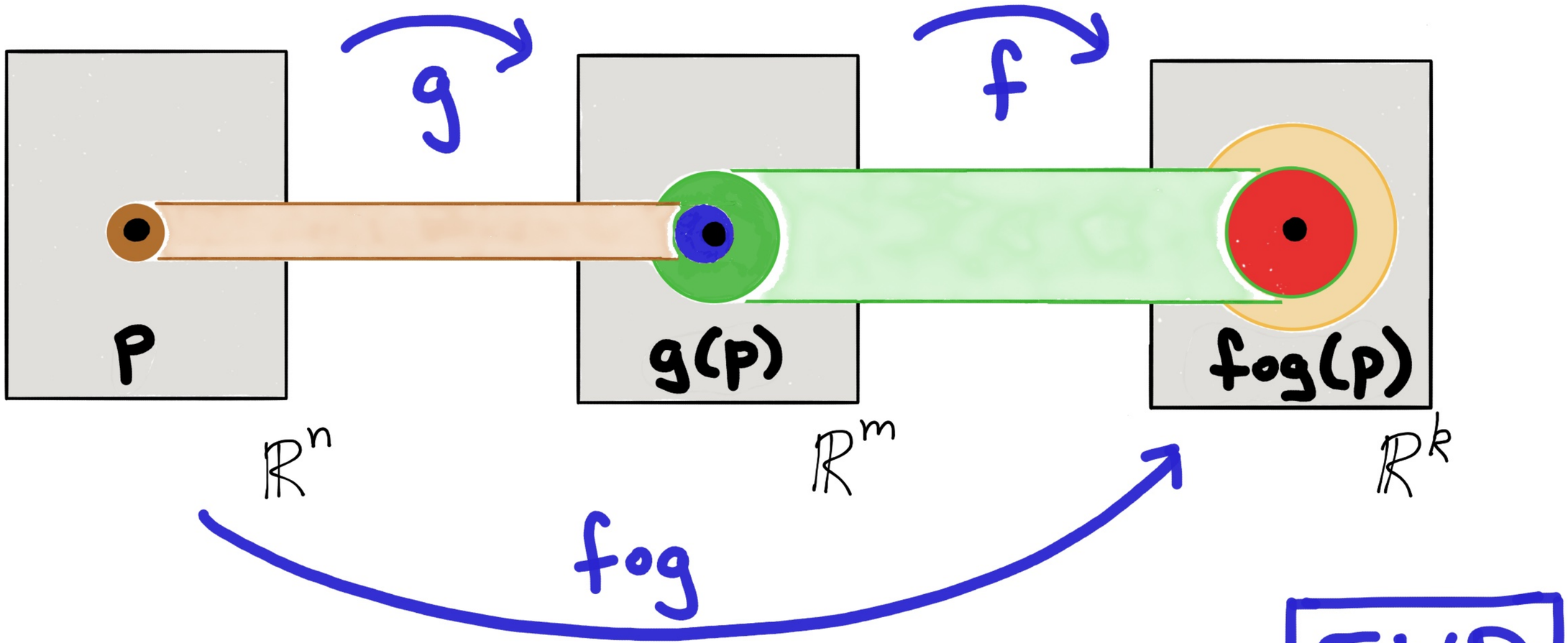

END

Continuous on their domain

$h_1(x,y,z) = x$

$h_2(x,y,z) = y$

$h_3(x,y,z) = z$

$h_4(x,y,z) = \text{const.}$

e^x

$\sin x$

$\cos x$

$\log x$

$\sqrt{x}$

$f \circ g$

$f + g$

$f - g$

$f g$

f / g

If f and g are

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fg

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If f and g are

Example:

$x, y, z, 5$

Continuous on their domain

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$f + g$

$f - g$

$f g$

f / g

If f and g are

Example:

$x, y, z, 5$

Continuous on their domain

$h_1(x,y,z) = x$

$h_2(x,y,z) = y$

$h_3(x,y,z) = z$

$h_4(x,y,z) = \text{const.}$

e^x

$\sin x$

$\cos x$

$\log x$

$\sqrt{x}$

$f \circ g$

$f + g$

$f - g$

$f \cdot g$

f / g

} If f and g are

Example:

$x, y, z, 5 \rightarrow 5x$

Continuous on their domain

$h_1(x,y,z) = x$

$h_2(x,y,z) = y$

$h_3(x,y,z) = z$

$h_4(x,y,z) = \text{const.}$

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If f and g are

Example:

$x, y, z, 5$ → $5xy$ → $\log(5xy)$

Continuous on their domain

$h_1(x,y,z) = x$

$h_2(x,y,z) = y$

$h_3(x,y,z) = z$

$h_4(x,y,z) = \text{const.}$

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$f + g$

$f - g$

$f \cdot g$

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Example:

$x, y, z, 5 \rightarrow 5xy \rightarrow \log(5xy)$

Continuous on their domain

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$\log x$

$\sqrt{x}$

$f \circ g$

$f + g$

$f - g$

$f\,g$

f/g

If f and g are

Example:

$x, y, z, 5 \rightarrow 5xy \rightarrow \log(5xy)$

$x, y, z, 5 \rightarrow x+y$

Continuous on their domain

$h_1(x,y,z) = x$

$h_2(x,y,z) = y$

$h_3(x,y,z) = z$

$h_4(x,y,z) = \text{const.}$

e^x

$\sin x$

$\cos x$

$\log x$

$\sqrt{x}$

$f \circ g$

$f + g$

$f - g$

$f \cdot g$

f / g

If f and g are

Example:

$x, y, z, 5 \rightarrow 5xy \rightarrow \log(5xy)$

$x, y, z, 5 \rightarrow x+y$

Continuous on their domain

$h_1(x,y,z) = x$

$h_2(x,y,z) = y$

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$f + g$

$f - g$

$f \cdot g$

f / g

If f and g are

Example:

$x, y, z, 5 \rightarrow 5xy \rightarrow \log(5xy)$

$x, y, z, 5 \rightarrow x+y \rightarrow (x+y)^2$

Continuous on their domain

$h_1(x,y,z) = x$

$h_2(x,y,z) = y$

$h_3(x,y,z) = z$

$h_4(x,y,z) = \text{const.}$

e^x

$\sin x$

$\cos x$

$\log x$

$\sqrt{x}$

$f \circ g$

$f + g$

$f - g$

$f\,g$

f/g

If f and g are

Example:

$x, y, z, 5 \rightarrow 5xy \rightarrow \log(5xy)$

$x, y, z, 5 \rightarrow x+y \rightarrow (x+y)^2$

Continuous on their domain

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$h_2(x,y,z) = y$

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If f and g are

Example:

$x, y, z, 5$ → $5xy$ → $\log(5xy)$

$x, y, z, 5$ → $x+y$ → $(x+y)^2$

$$\frac{\log(5xy)}{(x+y)^2}$$

Continuous on their domain

$h_1(x,y,z) = x$

$h_2(x,y,z) = y$

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$f - g$

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If f and g are

Example:

$x, y, z, 5$ → $5xy$ → $\log(5xy)$

$x, y, z, 5$ → $x+y$ → $(x+y)^2$

$$\frac{\log(5xy)}{(x+y)^2}$$

Examples:

1.) $\lim_{(x,y)\to(2,1)} \dfrac{3xy-2}{x+y}$

2.) $\lim_{(x,y)\to(1,-1)} \dfrac{3xy-2}{x+y}$

Examples:

1.) $\lim_{(x,y)\to(2,1)} \frac{3xy-2}{x+y} = \frac{3\cdot 2\cdot 1-2}{2+1} = \frac{4}{3}$

2.) $\lim_{(x,y)\to(1,-1)} \frac{3xy-2}{x+y}$

Examples:

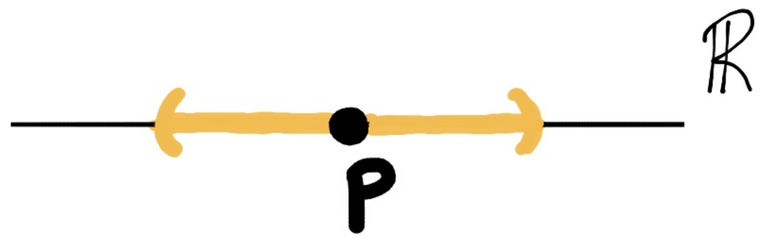
1.) $\lim_{(x,y)\to(2,1)} \frac{3xy-2}{x+y} = \frac{3\cdot 2\cdot 1-2}{2+1} = \frac{4}{3}$

2.) $\lim_{(x,y)\to(1,-1)} \frac{3xy-2}{x+y}$ does not exist.

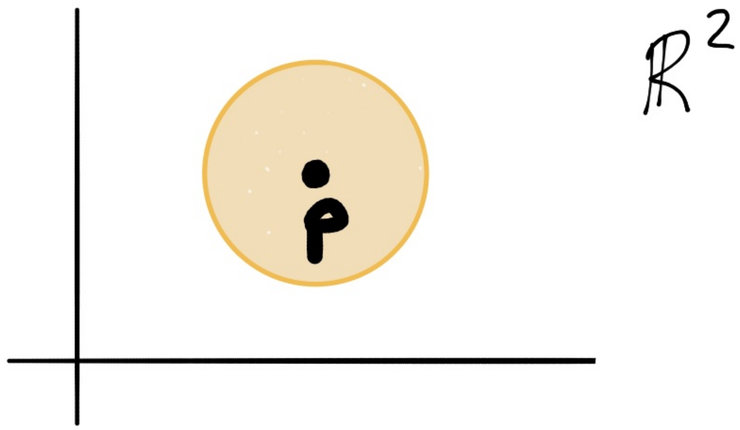
Ⅲ Boundary points

A neighborhood of $p \in \mathbb{R}^n$ is:

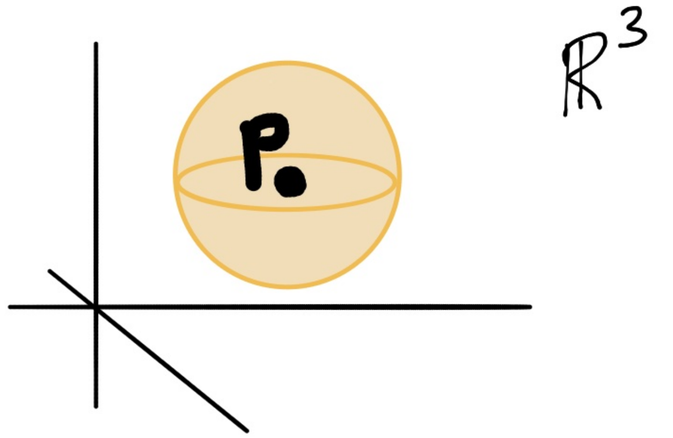
$p \in \mathbb{R}$: The interior of an interval centered at p.



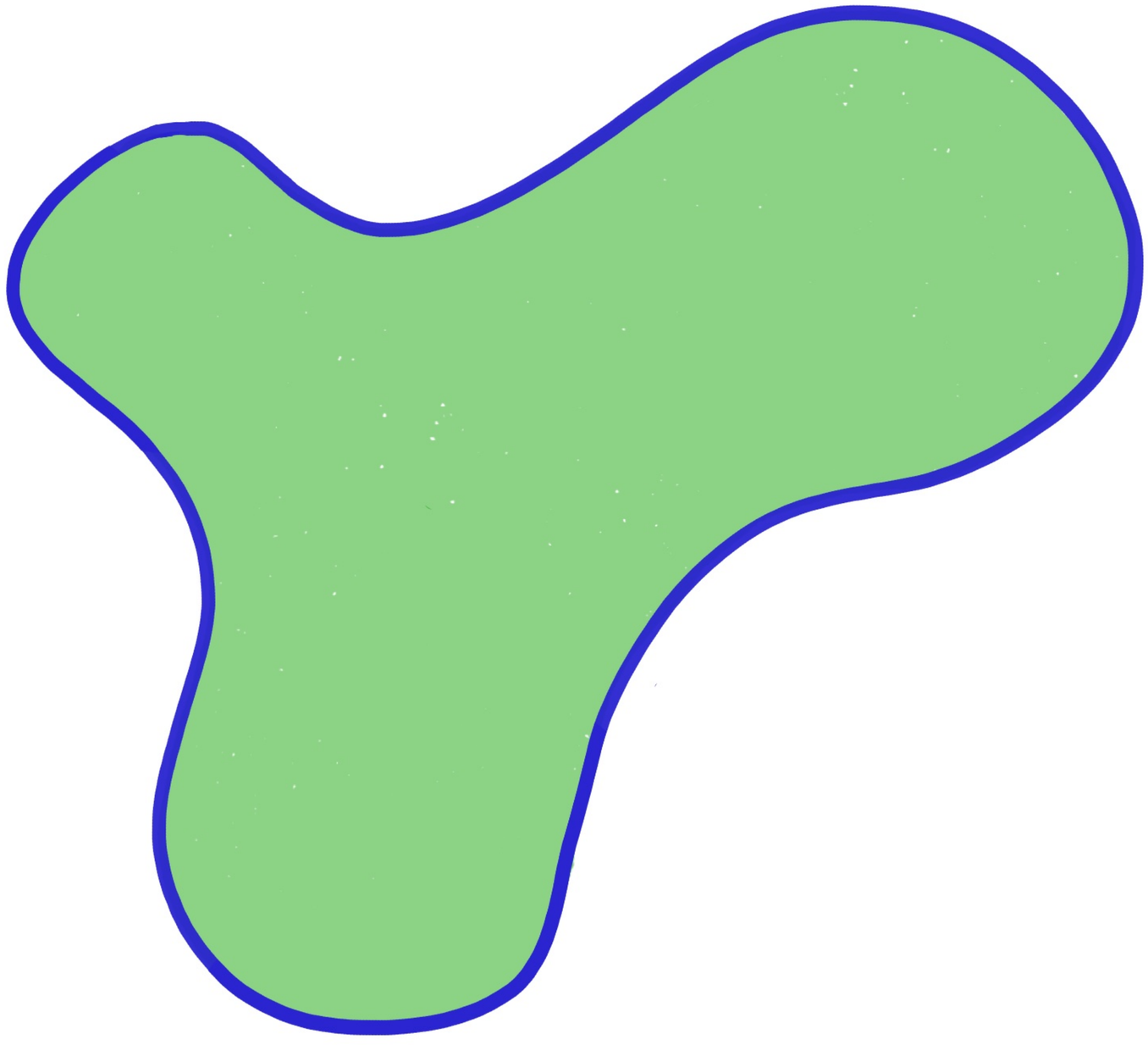
$p \in \mathbb{R}^2$: The interior of a circle centered at p.



$p \in \mathbb{R}^3$: The interior of a sphere centered at p.



S, a subset of $\mathbb{R}^n$



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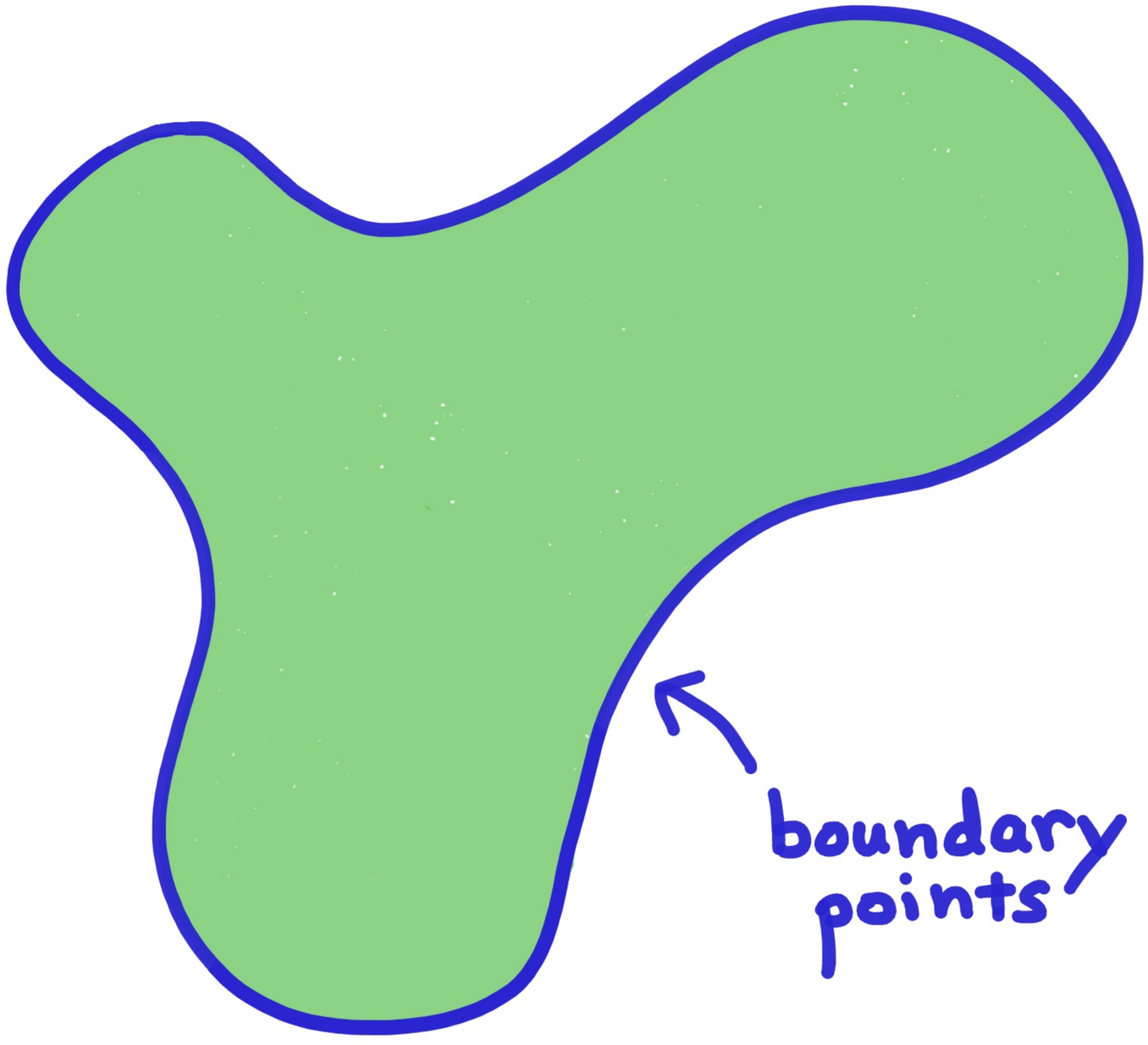

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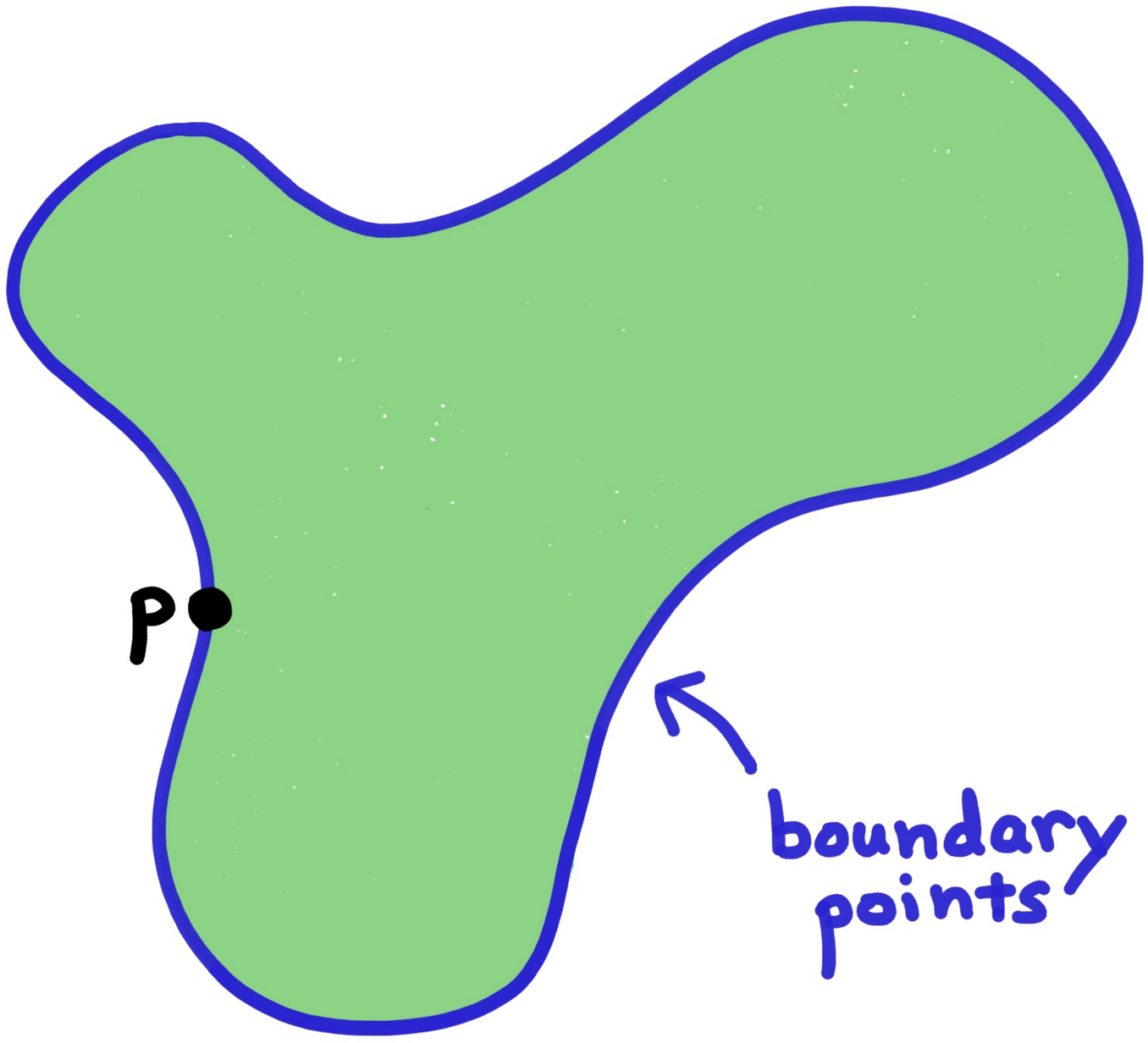

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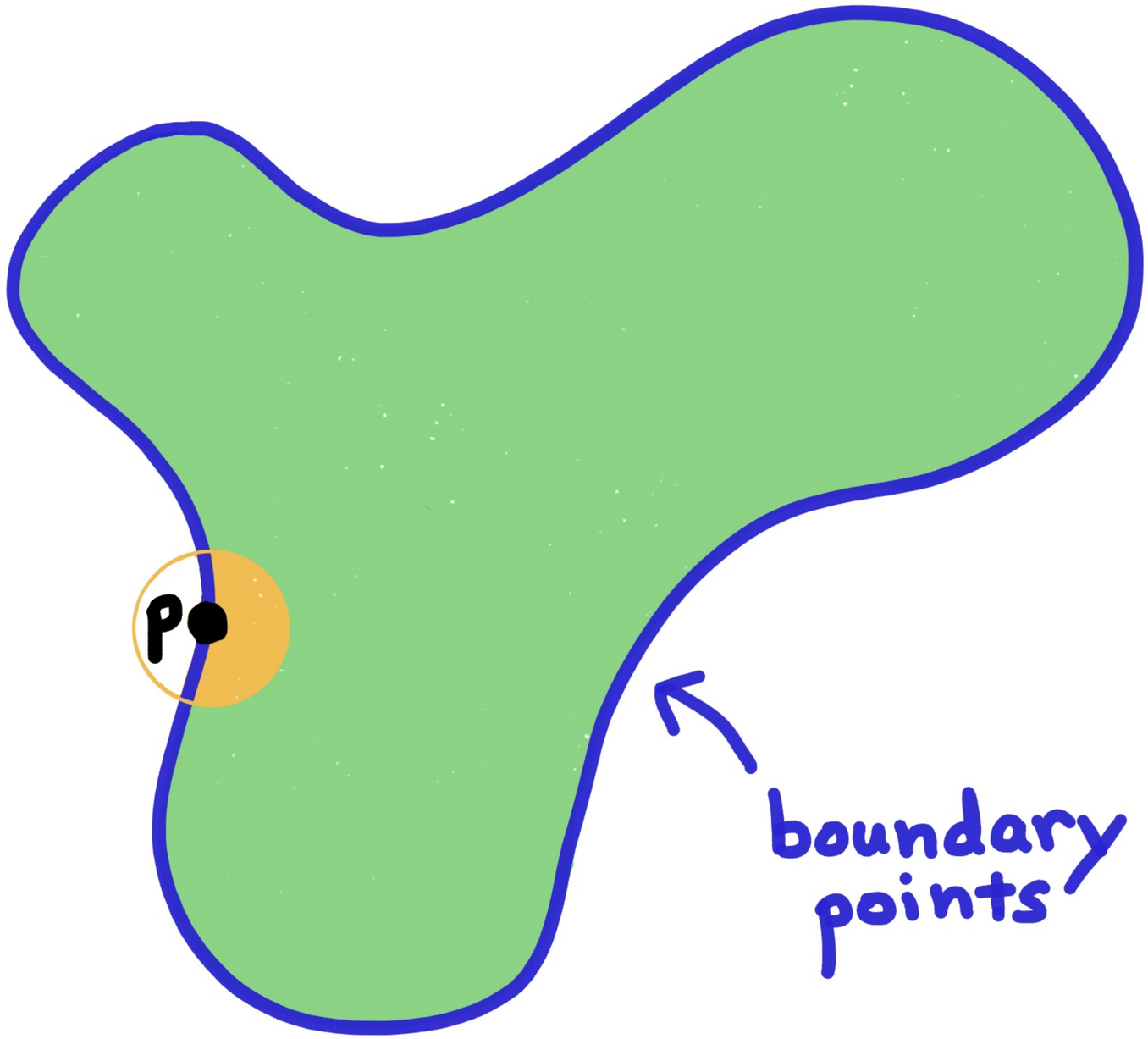
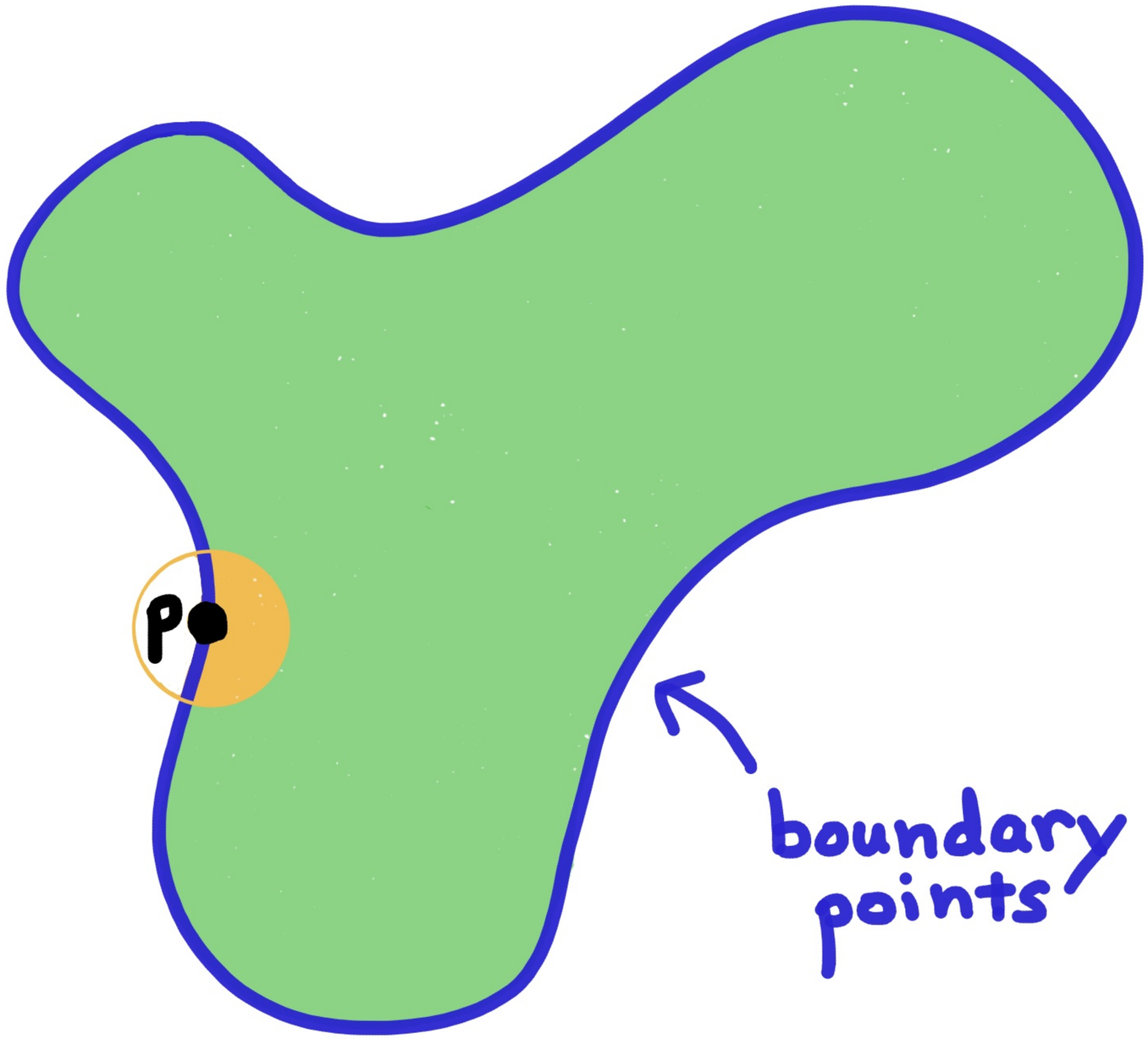


S, a subset of $\mathbb{R}^n$

$p \in \mathbb{R}^n$ is a **boundary point** of S if every neighborhood of p contains a point in S and a point not in S.

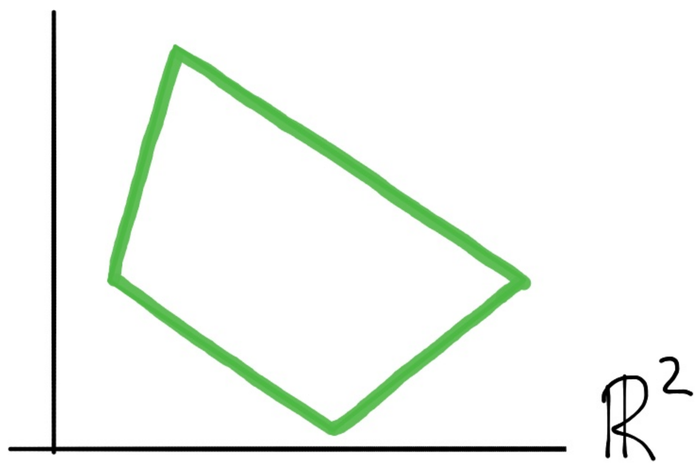


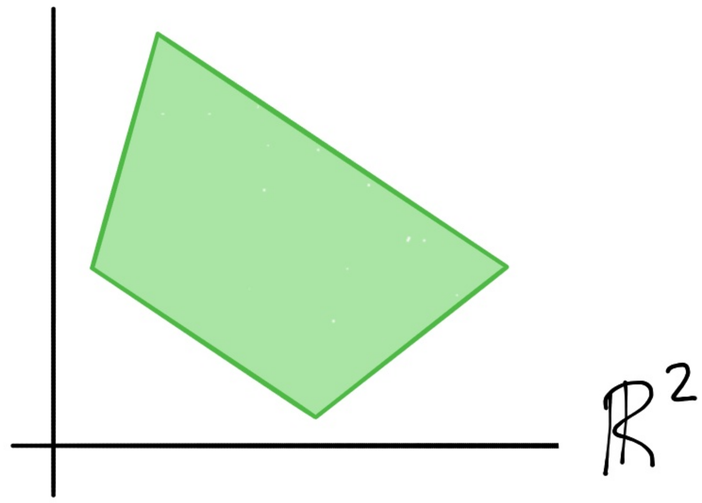
Examples:

1.) The boundary of the interval $2 < x \leq 5$ is 2 and 5.

2.) The boundary of

is





A set is **open** if it has none of its boundary points.

Examples:

- $\mathbb{R}^n$
- An interval $2<x<5$ in $\mathbb{R}$.
- Interior of a circle or polygon in $\mathbb{R}^2$.
- Interior of a sphere in $\mathbb{R}^3$.

A set is **closed** if it contains all of its boundary points.

Examples:

- $\mathbb{R}^n$
- An interval $2 \leq x \leq 5$ in $\mathbb{R}$.
- Perimeter and interior of a circle or polygon in $\mathbb{R}^2$.
- Surface of a sphere in $\mathbb{R}^3$.

Examples: Sets neither open nor closed.

1.) The interval $2 < x \leq 5$.

2.)