§30

Suppose C is a curve in $\mathbb{R}^n$ with initial point $\vec{a} \in \mathbb{R}^n$ and terminal point $\vec{b} \in \mathbb{R}^n$.

If $f : \mathbb{R}^n \to \mathbb{R}$, then $\int_C \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i = [f]_{\vec{a}}^{\vec{b}}$.

We often write this integral as $\int_{\vec{a}}^{\vec{b}} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$.

Equivalent to the above paragraph: If $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field with $\vec{F} = (F_1, \ldots, F_n)$ and $\vec{F}$ is conservative with potential function f (meaning $\vec{F} = \nabla f$), then $\int_C \sum_{i=1}^{n} F_i \, dx_i = [f]_{\vec{a}}^{\vec{b}}$. We often write this integral as $\int_{\vec{a}}^{\vec{b}} \sum_{i=1}^{n} F_i \, dx_i$.

If C is a simple closed curve, then we often write $\oint_C \sum_{i=1}^{n} F_i \, dx_i$ instead of $\int_C \sum_{i=1}^{n} F_i \, dx_i$.

Theorem:

For a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, $\vec{F} = (F_1, \dots, F_n)$ the following are equivalent:

1. $\vec{F} = \nabla f$ for some $f : \mathbb{R}^n \to \mathbb{R}$.
2. $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ for all $1 \le i, j \le n$.
3. $\displaystyle\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$ for any closed simple curve C in $\mathbb{R}^n$.
4. $\displaystyle\int_C \sum_{i=1}^{n} F_i \, dx_i$ is independent of path.