§29

Below, C is a curve in $\mathbb{R}^n$, parametrized by $a \le t \le b$ and the coordinate functions $x_1 = x_1(t),\ x_2 = x_2(t),\ \ldots,\ x_n = x_n(t)$.

Furthermore, any function written with an f – be it $f, f_1, f_2, \ldots, f_n$ – is of the form $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

$$\int_C f(x_1,\ldots,x_n)\,ds = \int_a^b f(x_1(t),\ldots,x_n(t))\,\|(x_1'(t),\ldots,x_n'(t))\|\,dt$$

$$\int_C f(x_1,\ldots,x_n)\,dx_i = \int_a^b f(x_1(t),\ldots,x_n(t))\,x_i'(t)\,dt$$

$$\int_C \sum_{i=1}^n f_i(x_1,\ldots,x_n)\,dx_i = \sum_{i=1}^n \int_C f_i(x_1,\ldots,x_n)\,dx_i \qquad \text{equals}$$

$$\int_a^b \left(\sum_{i=1}^n f_i(x_1(t),\ldots,x_n(t))\,x_i'(t) \right) dt$$

$$\int_{C_1 \cup C_2} f(x_1,\dots,x_n)\,ds = \int_{C_1} f(x_1,\dots,x_n)\,ds + \int_{C_2} f(x_1,\dots,x_n)\,ds$$

$$\int_{C_1 \cup C_2} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i = \int_{C_1} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i + \int_{C_2} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i$$

$$\int_{-C} f(x_1,\dots,x_n)\,ds = \int_{C} f(x_1,\dots,x_n)\,ds$$

$$\int_{-C} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i = -\int_{C} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i$$

$\int_{C} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i$ is called <u>independent of path</u>

if whenever C_1 and C_2 are curves with the same initial points and the same terminal points, then

$$\int_{C_1} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i = \int_{C_2} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i$$

In this case, when C is a curve with initial point $\vec{a} \in \mathbb{R}^n$ and terminal point $\vec{b}$, we write

$\int_{\vec{a}}^{\vec{b}} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i$ for $\int_{C} \sum_{i=1}^{n} f_i(x_1,\dots,x_n)\,dx_i$.