# §14

## Jacobian matrices

For $f(x,y)=z$ , $D_pf = \left( \frac{\partial z}{\partial x}\Big|_p , \frac{\partial z}{\partial y}\Big|_p \right)$

For $f(x,y,z)=w$, $D_pf = \left( \frac{\partial w}{\partial x}\Big|_p , \frac{\partial w}{\partial y}\Big|_p , \frac{\partial w}{\partial z}\Big|_p \right)$

For $f(x,y) = (z,w)$, $D_pf = \begin{pmatrix} \frac{\partial z}{\partial x}\Big|_p & \frac{\partial z}{\partial y}\Big|_p \\ \frac{\partial w}{\partial x}\Big|_p & \frac{\partial w}{\partial y}\Big|_p \end{pmatrix}$

For $f(x,y,z)=(u,v,w)$,

$$D_pf = \begin{pmatrix} \frac{\partial u}{\partial x}\Big|_p & \frac{\partial u}{\partial y}\Big|_p & \frac{\partial u}{\partial z}\Big|_p \\ \frac{\partial v}{\partial x}\Big|_p & \frac{\partial v}{\partial y}\Big|_p & \frac{\partial v}{\partial z}\Big|_p \\ \frac{\partial w}{\partial x}\Big|_p & \frac{\partial w}{\partial y}\Big|_p & \frac{\partial w}{\partial z}\Big|_p \end{pmatrix}$$

$$\boxed{\begin{array}{l} \text{For } f: \mathbb{R}^n \to \mathbb{R}^k \text{ and } p \in \mathbb{R}^n, \\ \quad D_pf : \mathbb{R}^n \to \mathbb{R}^k \text{ approximates } f \text{ near } p. \end{array}}$$

## Chain rule:

For $f : \mathbb{R}^n \longrightarrow \mathbb{R}^k$,

$g : \mathbb{R}^k \longrightarrow \mathbb{R}^m$,

and $p \in \mathbb{R}^n$,

$$D_p(g \circ f) = \left(D_{f(p)}g\right)\left(D_p f\right)$$

## Particular instances of the chain rule:

① If $x = x(t)$, $y = y(t)$, and $z = z(x,y)$,

then $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

② If $x = x(t)$, $y = y(t)$, $z = z(t)$, and $w = w(x,y,z)$,

then $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

③ If $x = x(s,t)$, $y = y(s,t)$, and $z = z(x,y)$,

then $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$

and $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$