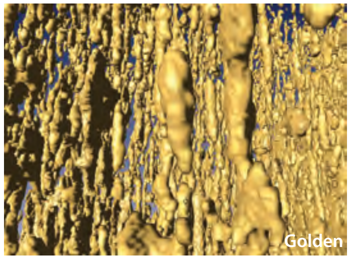
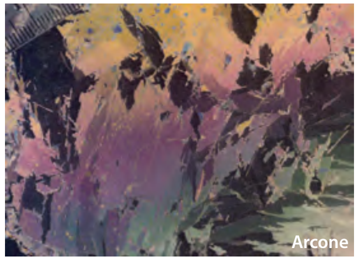
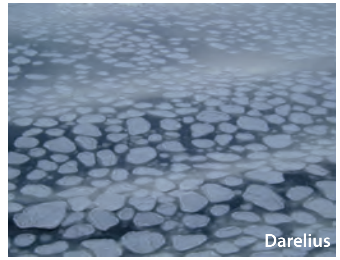
millimeters



centimeters

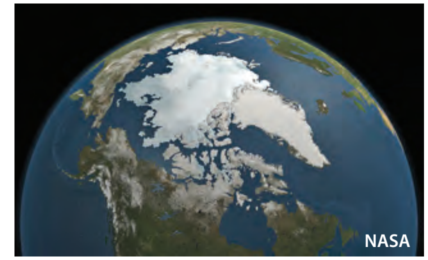


meters



kilometers

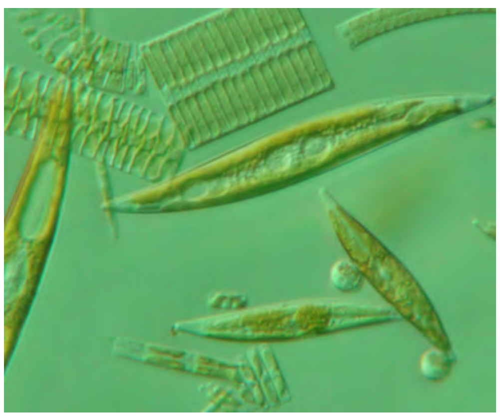
10^3 kilometers

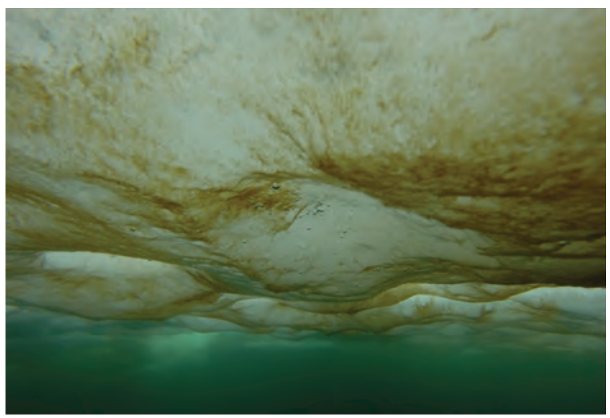


Modeling sea ice algal blooms using dynamical systems with random parameters

Ken Golden, University of Utah

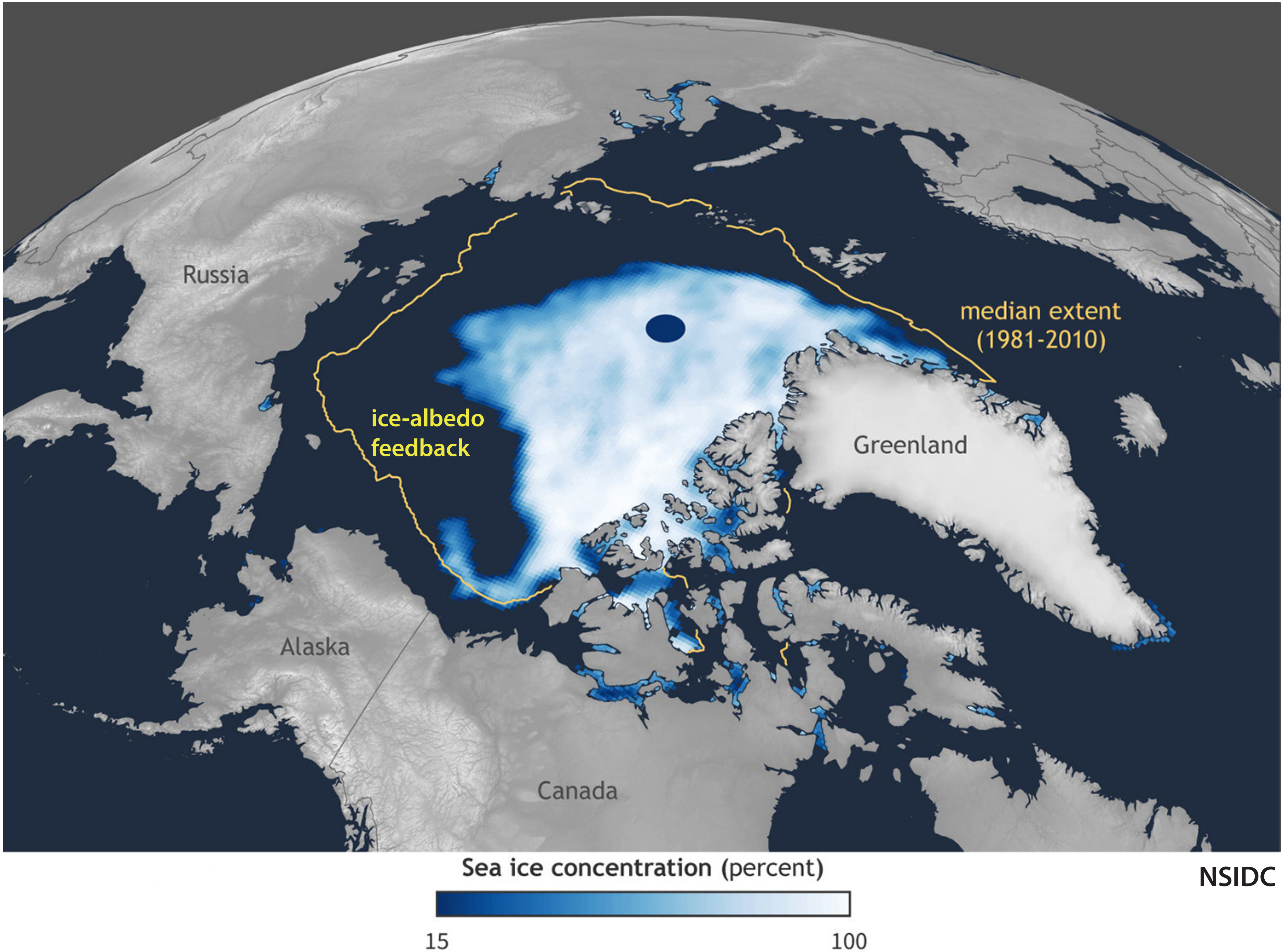
From micro to macro in the polar marine ecosystem





AMS Fall Southeastern Sectional Meeting
Ergodic Theory and Dynamical Systems
October 15, 2023

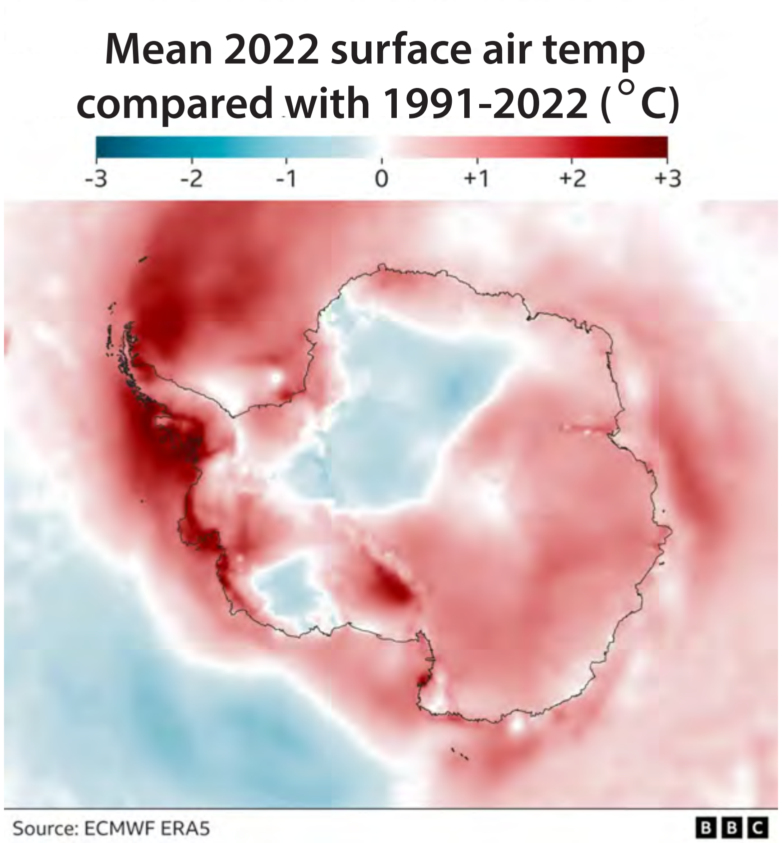
Arctic sea ice extent September 15, 2020



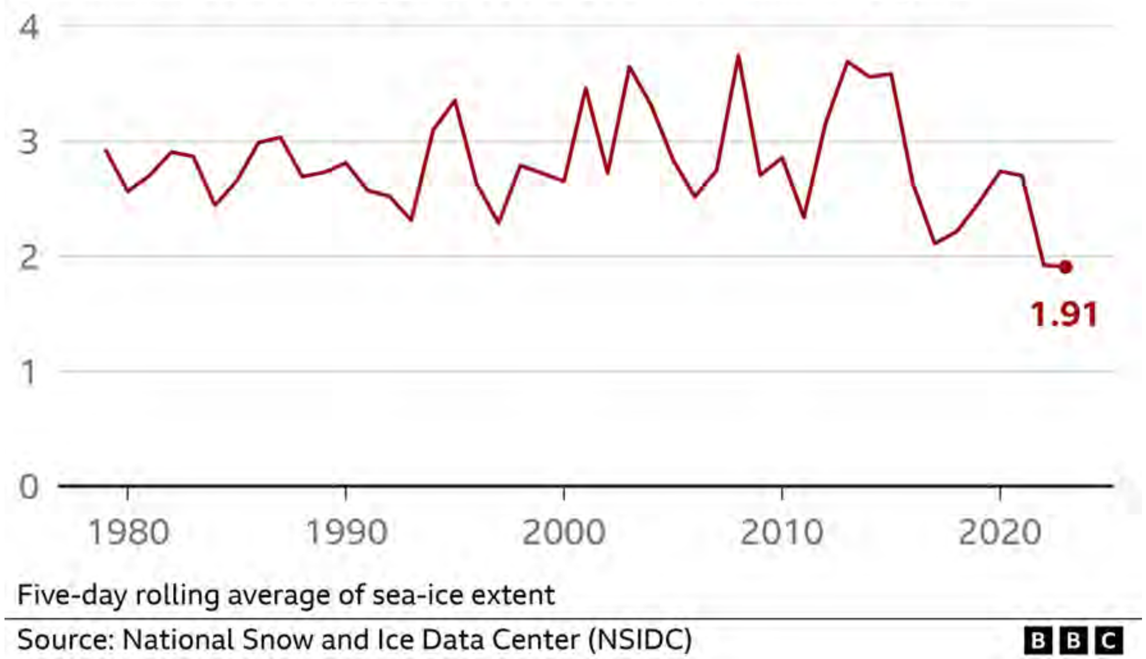
New Record Low for Antarctic Sea Ice

February 13, 2023

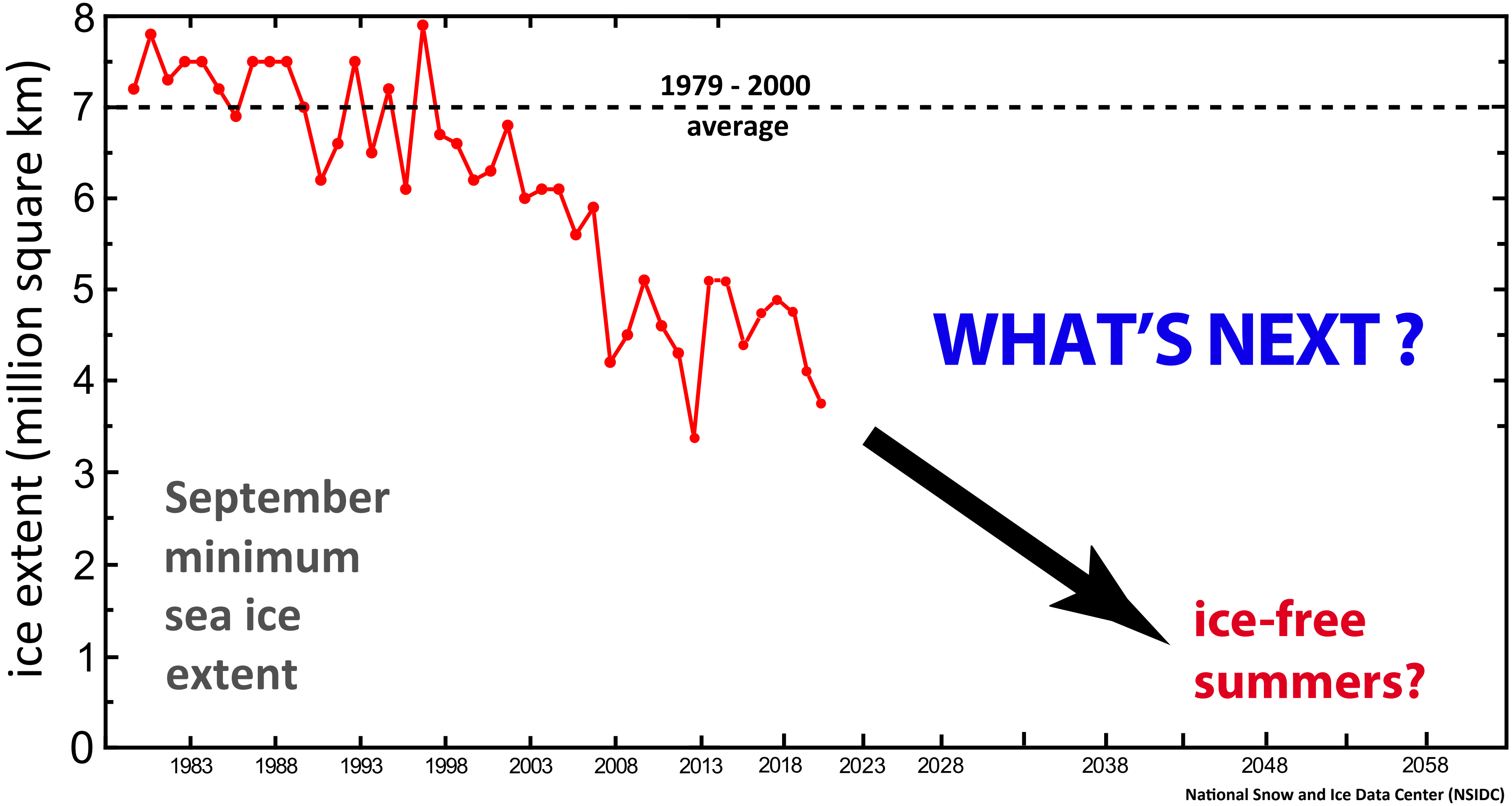
Much of Antarctica warmer than average



Minimum extent 1979-2023 (million sq km)



ARCTIC summer sea ice loss



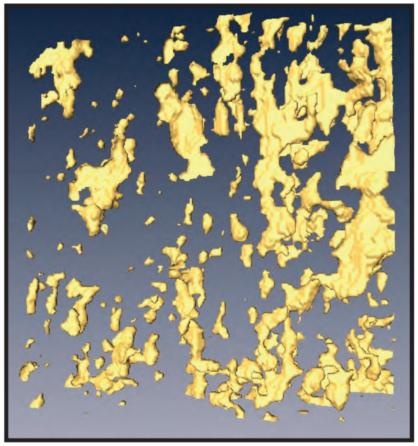
predictions require lots of math modeling

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

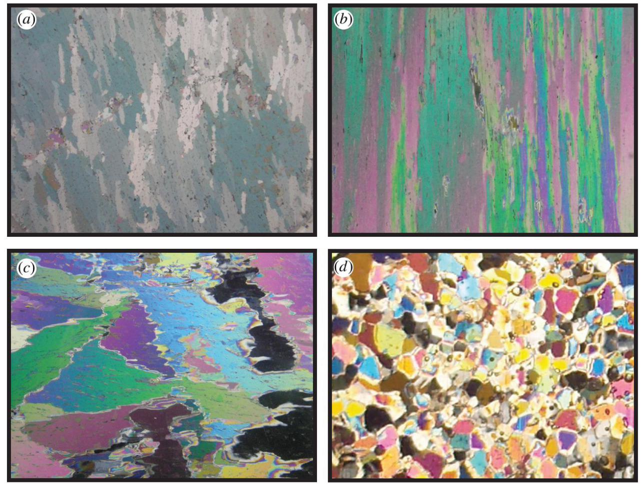
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

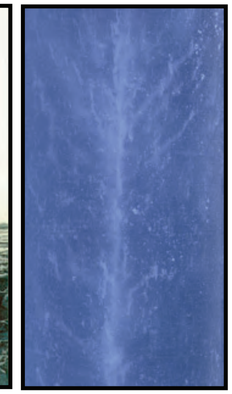
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

centimeters

---

mesoscale

Arctic melt ponds

K. Frey

Antarctic pressure ridges

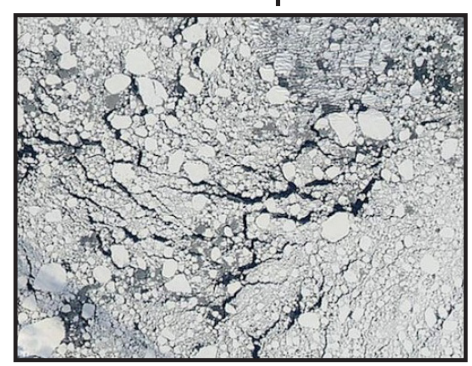
K. Golden

macroscale

sea ice floes

J. Weller

sea ice pack



NASA

meters

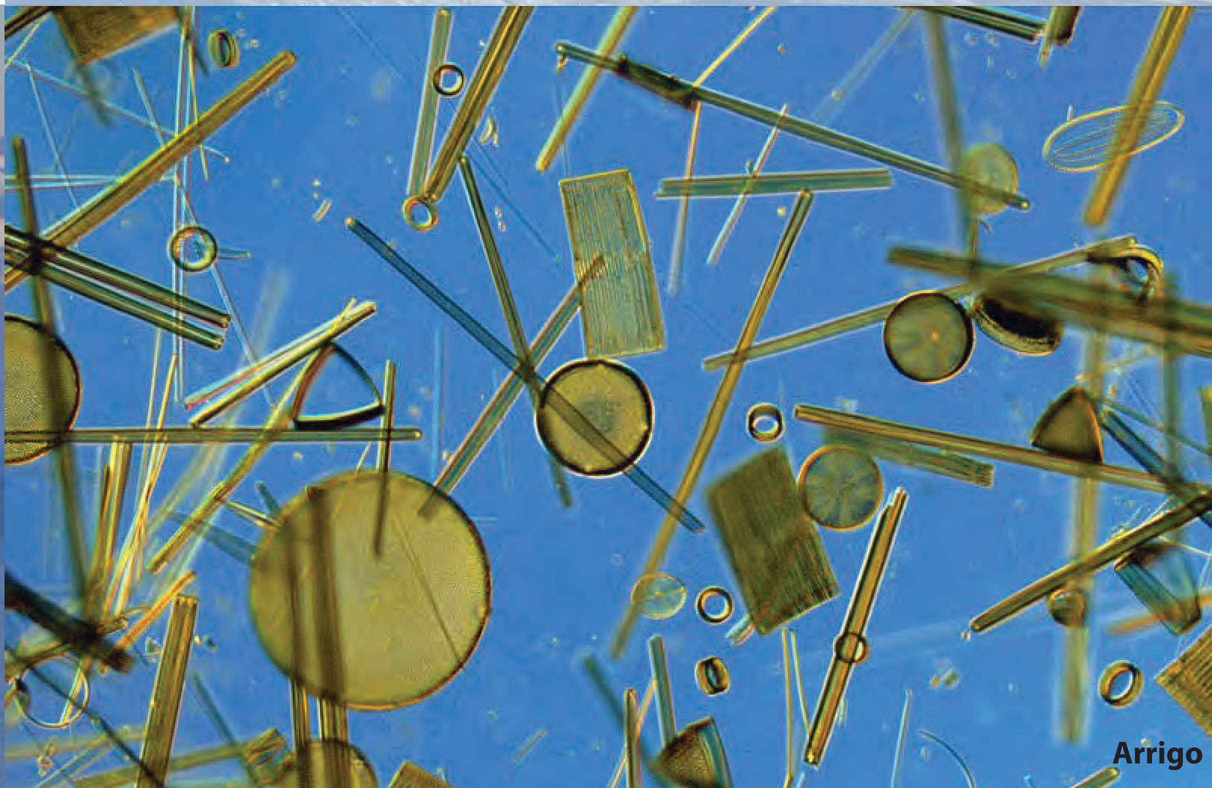
kilometers

microbes, megafauna, and the physics of sea ice

How do sea ice properties affect the life it hosts?

How does life in and on sea ice affect its physical properties?

Golden



Arrigo

What is this talk about?

A brief tour of recent results on multiscale modeling of physical and ecological processes in the sea ice system.

Focus:

1. **Physical processes regulating algal dynamics**
2. **Dynamical systems model with random parameters**

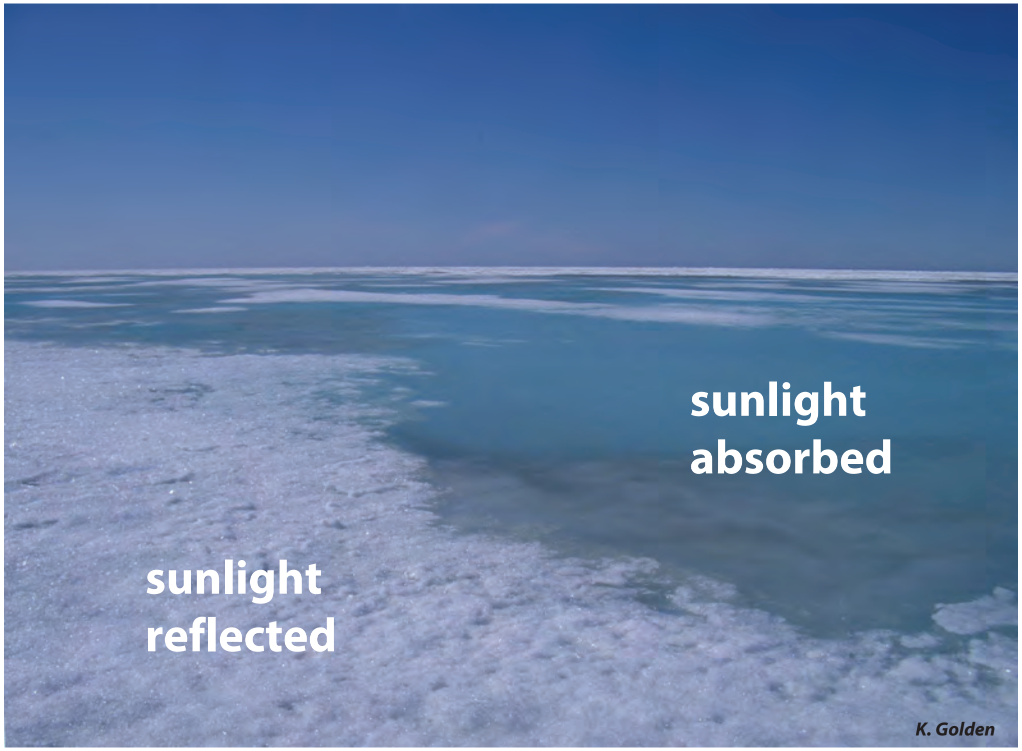
microscale, mesoscale, macroscale

(through the lens of fractal geometry)

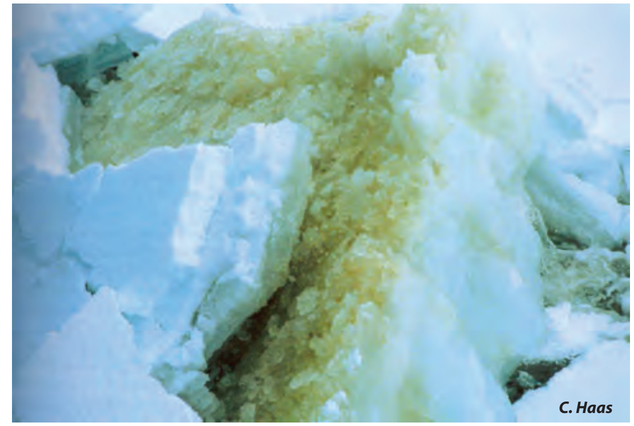
microscale

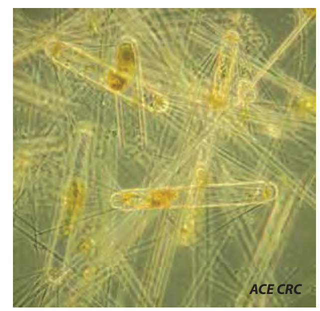
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

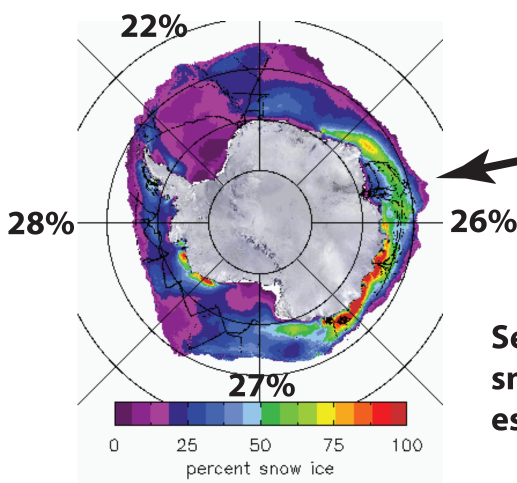
evolution of Arctic melt ponds and sea ice albedo



nutrient flux for algal communities







T. Maksym and T. Markus, 2008

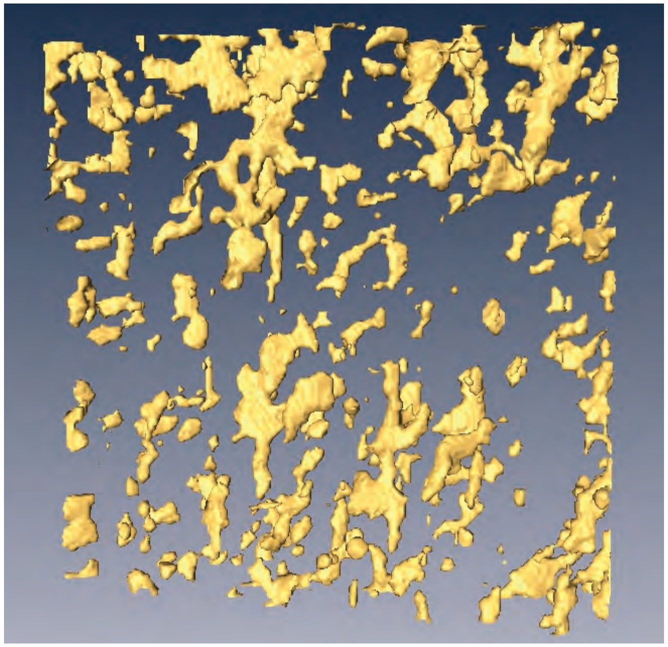
September snow-ice estimates

Antarctic surface flooding and snow-ice formation

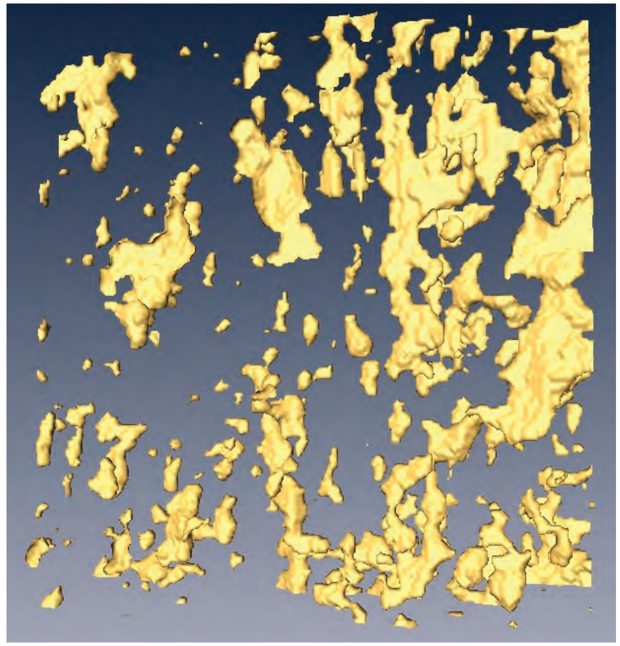
- evolution of salinity profiles

- ocean-ice-air exchanges of heat, CO_2

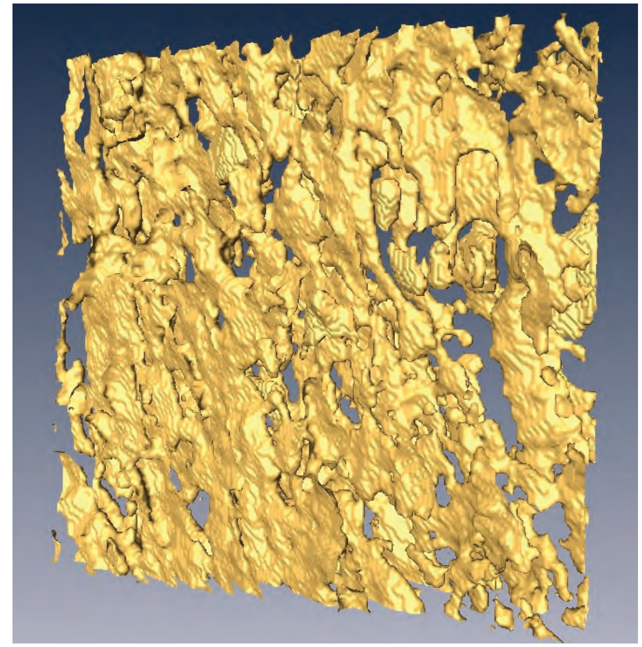
brine volume fraction and ***connectivity*** increase with temperature



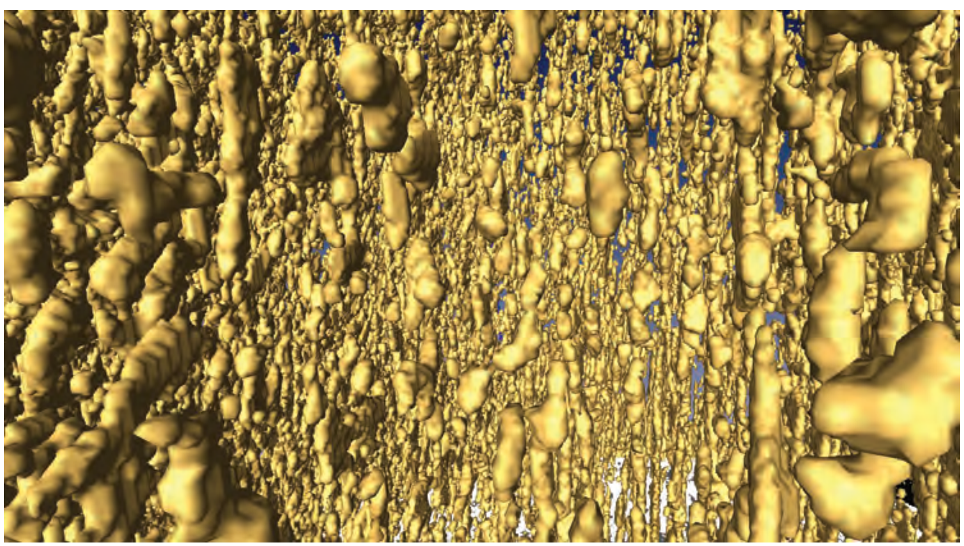
T = -15 °C, $\phi = 0.033$



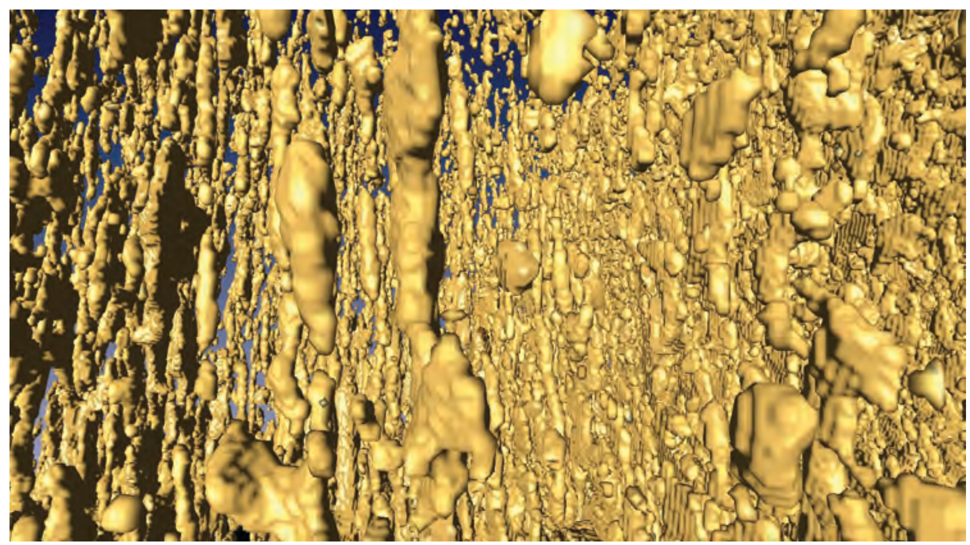
T = -6 °C, $\phi = 0.075$



T = -3 °C, $\phi = 0.143$



T = -8 °C, $\phi = 0.057$

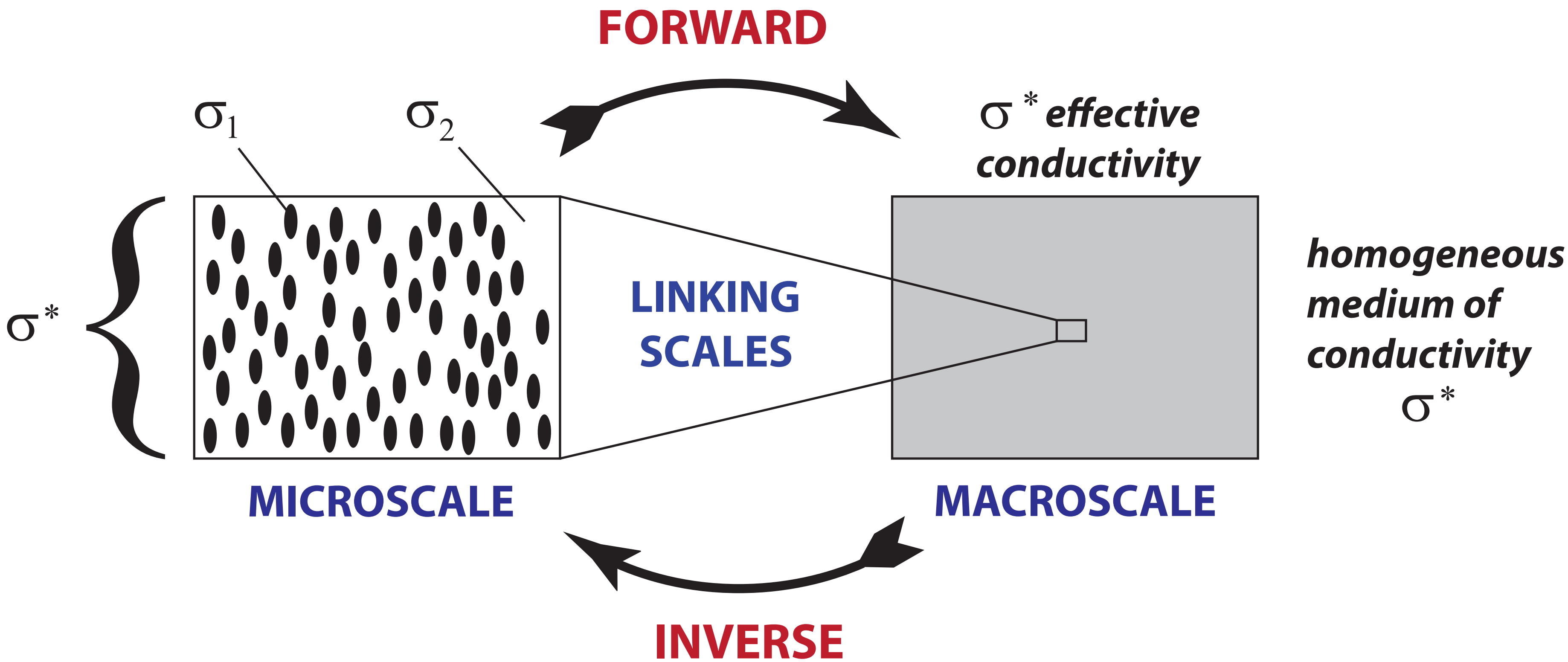


T = -4 °C, $\phi = 0.113$

X-ray tomography for brine in sea ice

Golden *et al.*, *Geophysical Research Letters*, 2007

HOMOGENIZATION for Composite Materials



Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*
*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

Critical behavior of fluid transport in sea ice

impermeable permeable

"on - off" switch for bulk fluid flow

Arctic field data

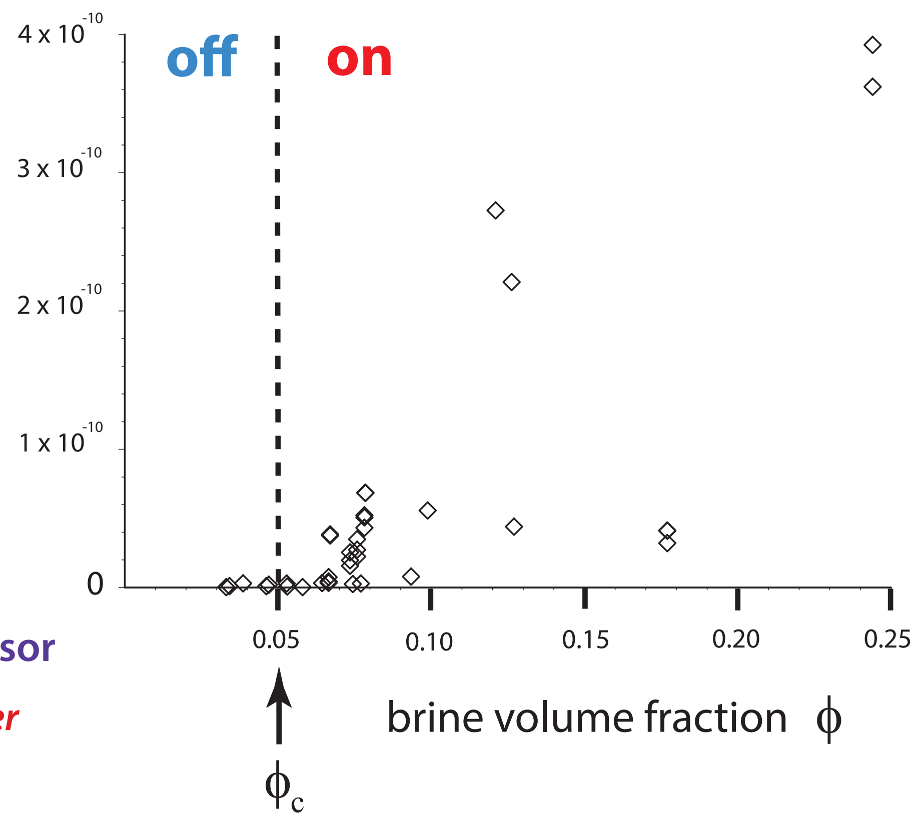
vertical fluid permeability k (m^2)

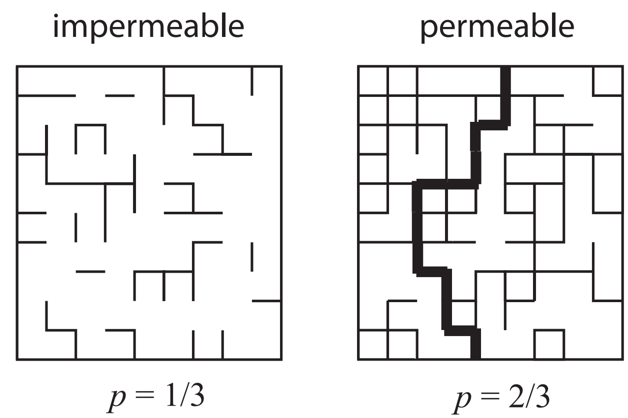
Darcy's Law

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\nabla p$$

$\mathbf{k}$ = fluid permeability tensor

homogenized parameter





lattice percolation

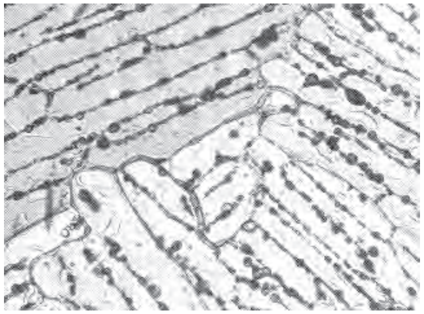
FRACTAL percolation clusters

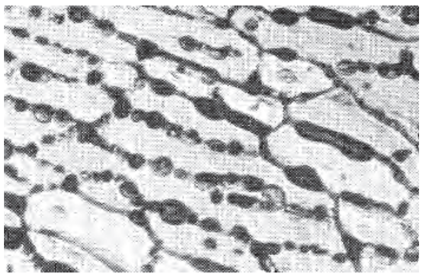
PERCOLATION THRESHOLD $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ \mathrm{C},\ S \approx 5$ ppt

RULE OF FIVES

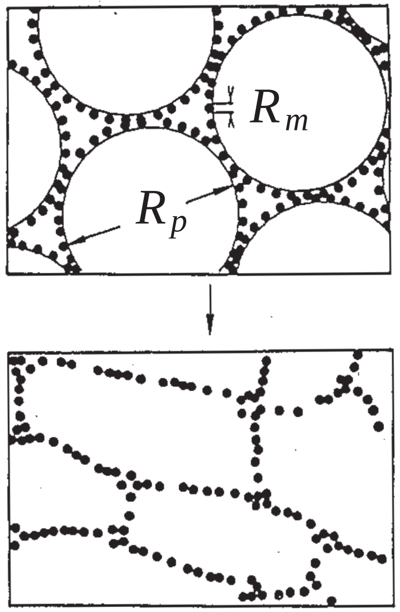
Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

cross pollination

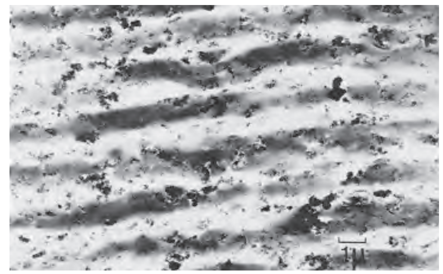




sea ice



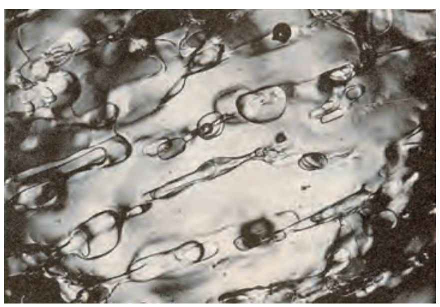
compressed powder



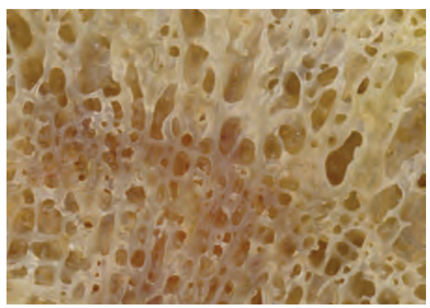
radar absorbing composite

use stealth technology to predict 5%

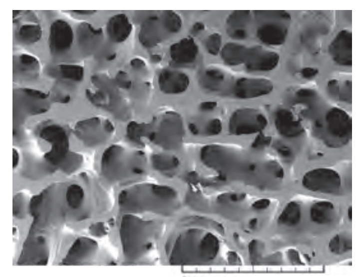
sea ice is a radar absorbing composite!



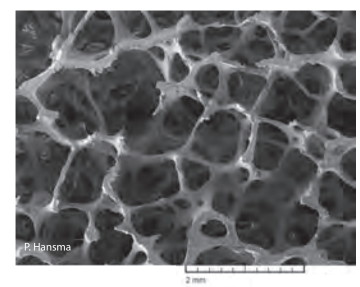
sea ice



human bone



young healthy trabecular bone



old osteoporotic trabecular bone

the math doesn't care if it's sea ice or bone!

new method of monitoring osteoporosis from sea ice

Golden, Murphy, Cherkaev, J. Biomech. 2011

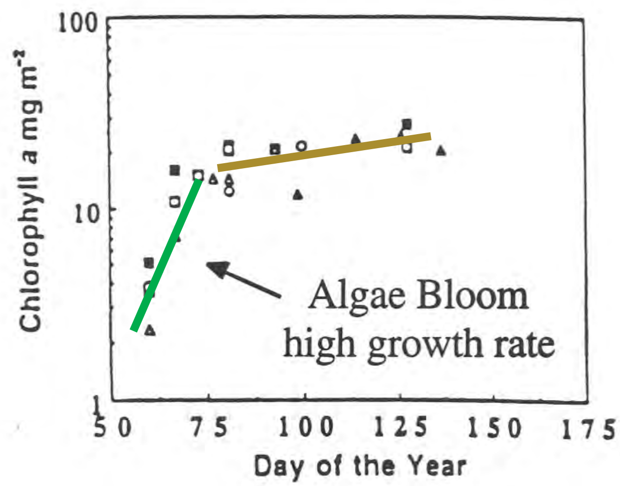
sea ice algal communities

D. Thomas 2004

nutrient replenishment controlled by ice permeability

biological activity turns on or off according to ***rule of fives***

critical behavior of microbial activity



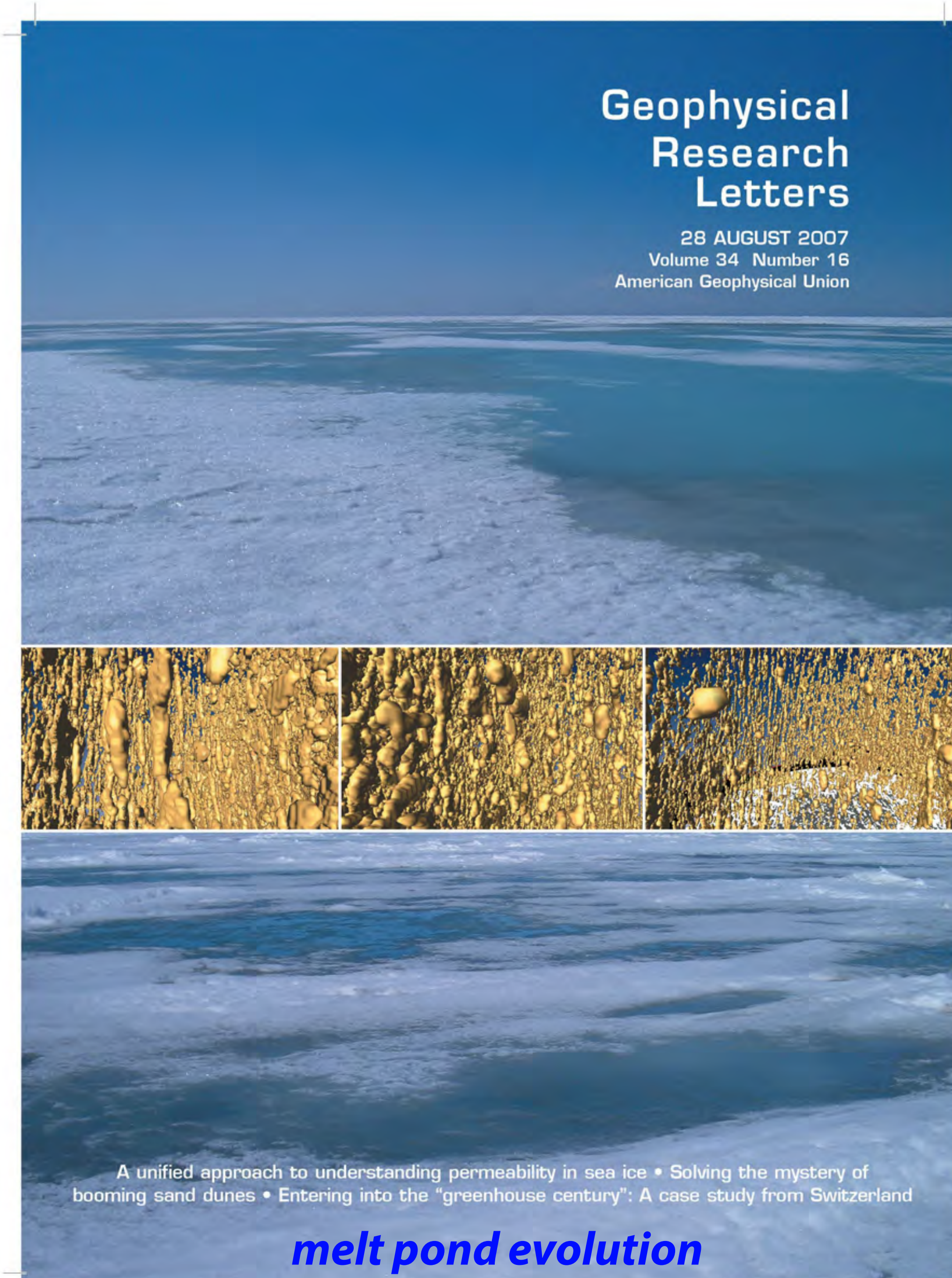
Convection-fueled algae bloom
Ice Station Weddell

Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophysical Research Letters 2007*



microscale

governs

mesoscale processes

melt pond evolution

percolation theory for fluid permeability

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8}\ \mathrm{m}^2$$

from critical path analysis in hopping conduction

hierarchical model

rock physics

network model

rigorous bounds

X-ray tomography for brine inclusions

confirms rule of fives

brine percolation threshold of $\phi = 5\%$ for bulk fluid flow

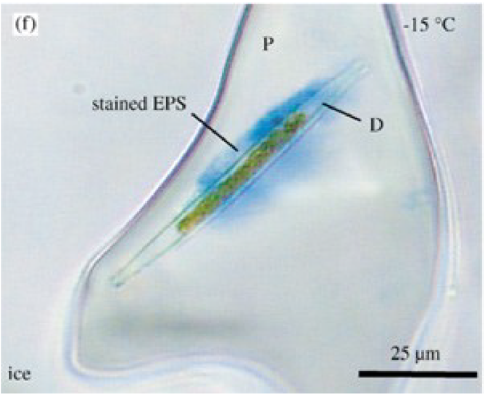
Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

theories agree closely with field data

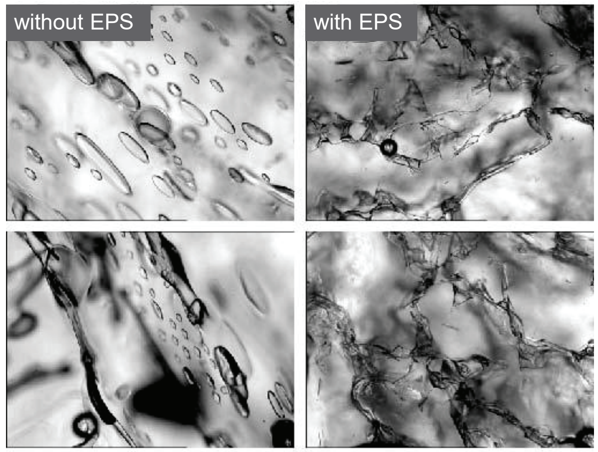
Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

How does EPS affect fluid transport? **How does the biology affect the physics?**

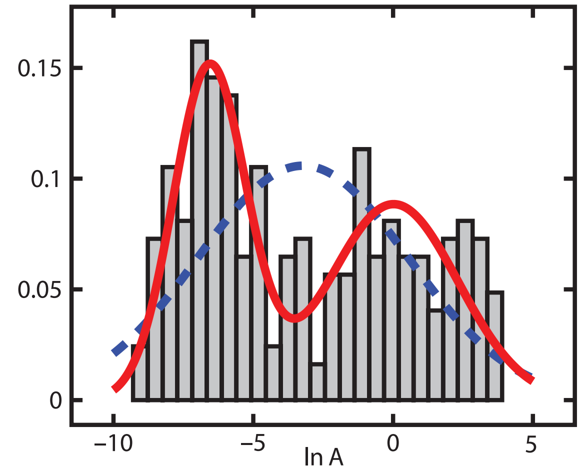
FRACTAL



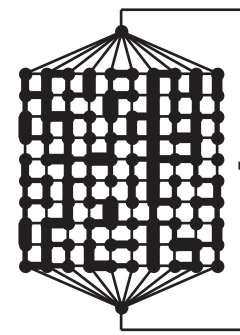
Krembs

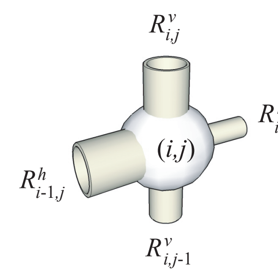


Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL





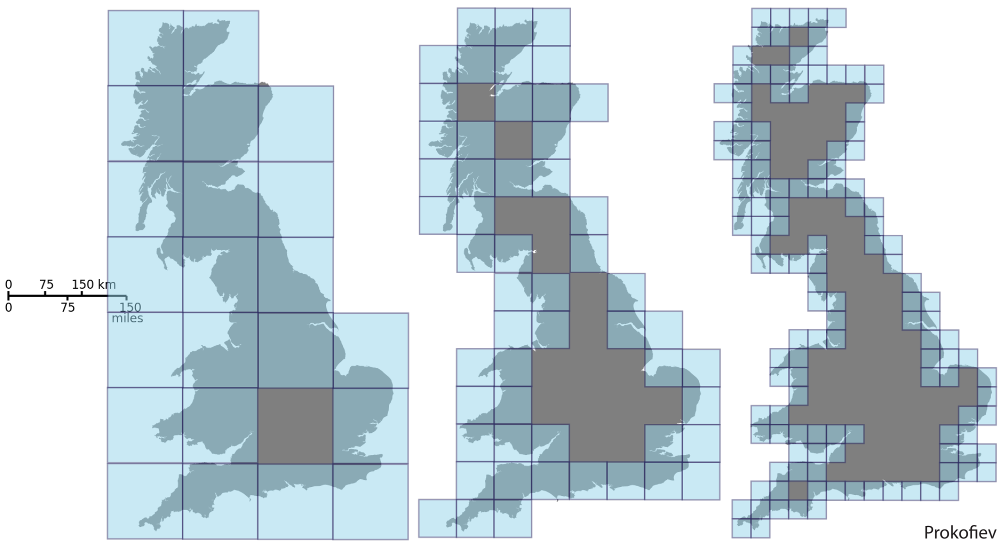
- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability *k;* results predict observed drop in *k*

Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
Multiscale Modeling and Simulation, 2018

Zhu, Jabini, Golden, Eicken, Morris
Ann. Glac. 2006

Thermal Evolution of Brine Fractal Geometry in Sea Ice

Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O'Sadnick, Elena Cherkaev and Kenneth Golden, 2023

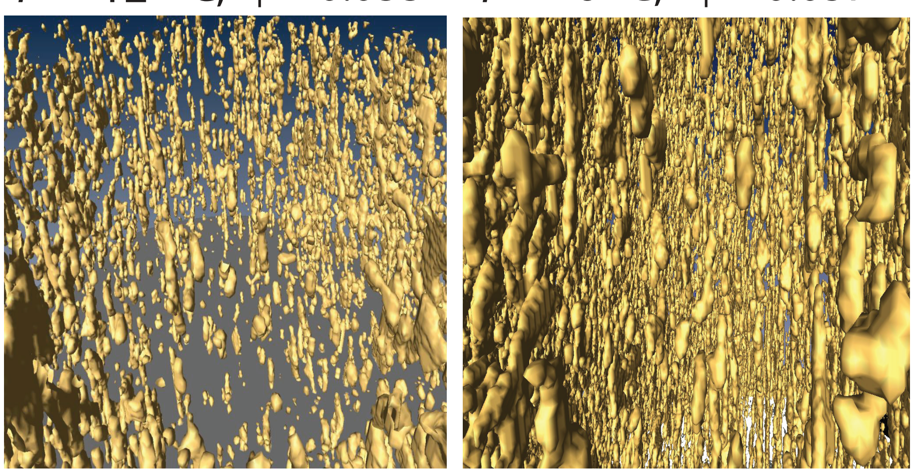


fractal dimension of the coastline of Great Britain by box counting

$$N(\epsilon) \sim \epsilon^{-D}$$

brine channels and inclusions "look" like fractals (from 30 yrs ago)

T = -12° C, ϕ = 0.033 T = -8° C, ϕ = 0.057

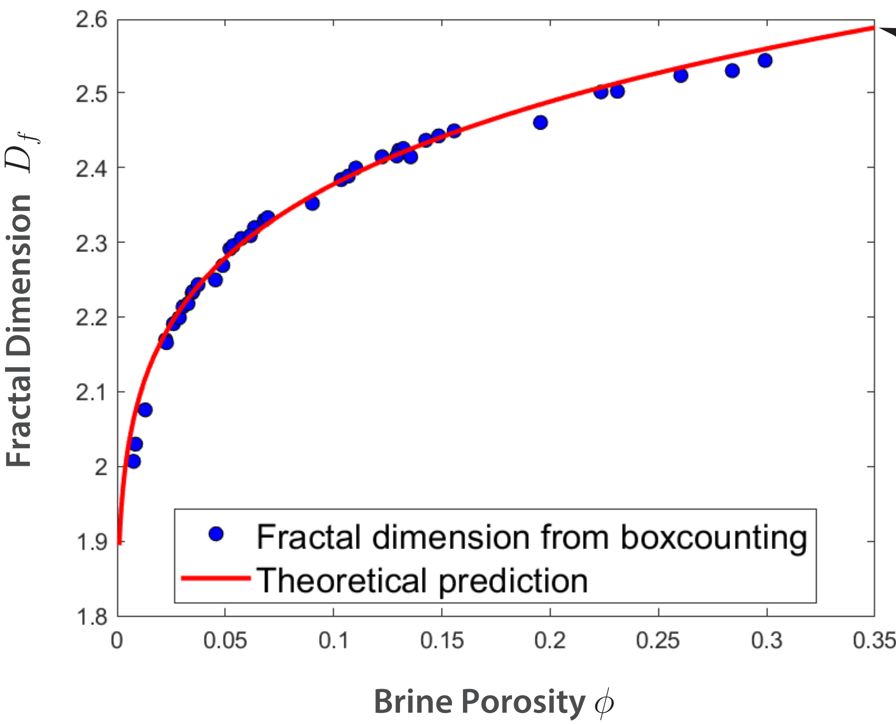


Golden, Eicken, et al. *GRL*, 2007

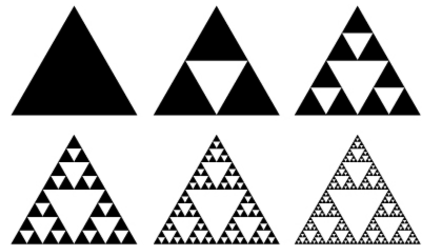
X-ray computed tomography of brine in sea ice

columnar and granular

The first comprehensive, quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.

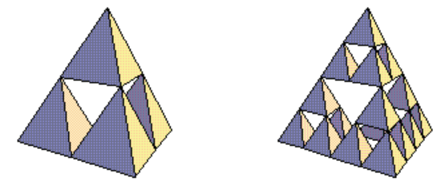


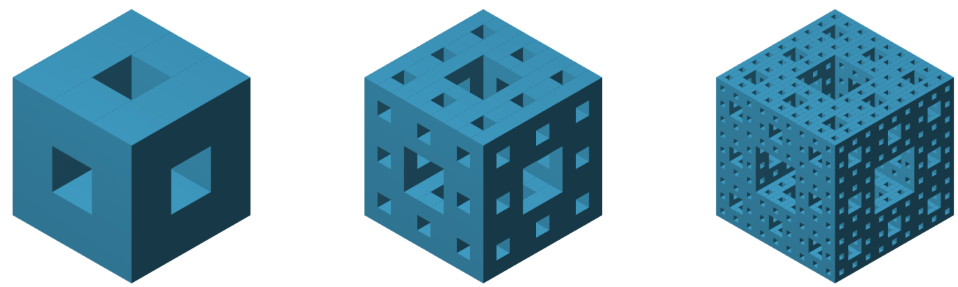
$$D_f = 3 - \frac{\ln \phi}{\ln(\lambda_{min}/\lambda_{max})}$$

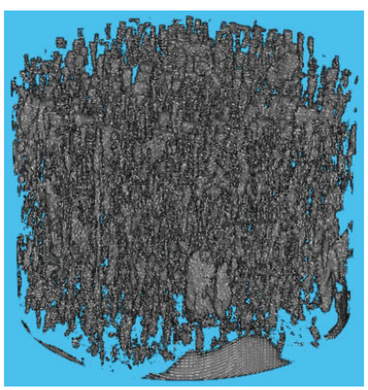


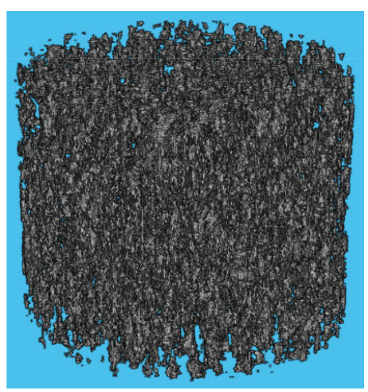
The red curve is exact for the Sierpinski pyramid (an exactly self-similar geometry); discovered for sandstones - statistically self-similar porous media like sea ice.

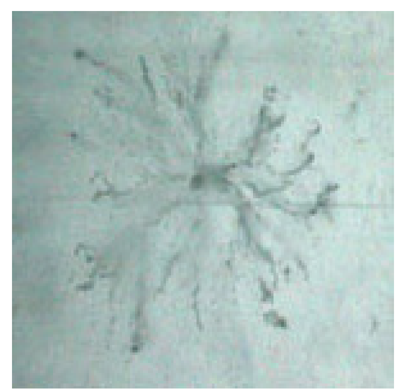
Katz and Thompson, 1985; Yu and Li, 2001

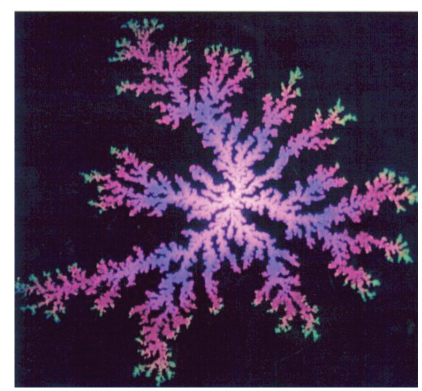












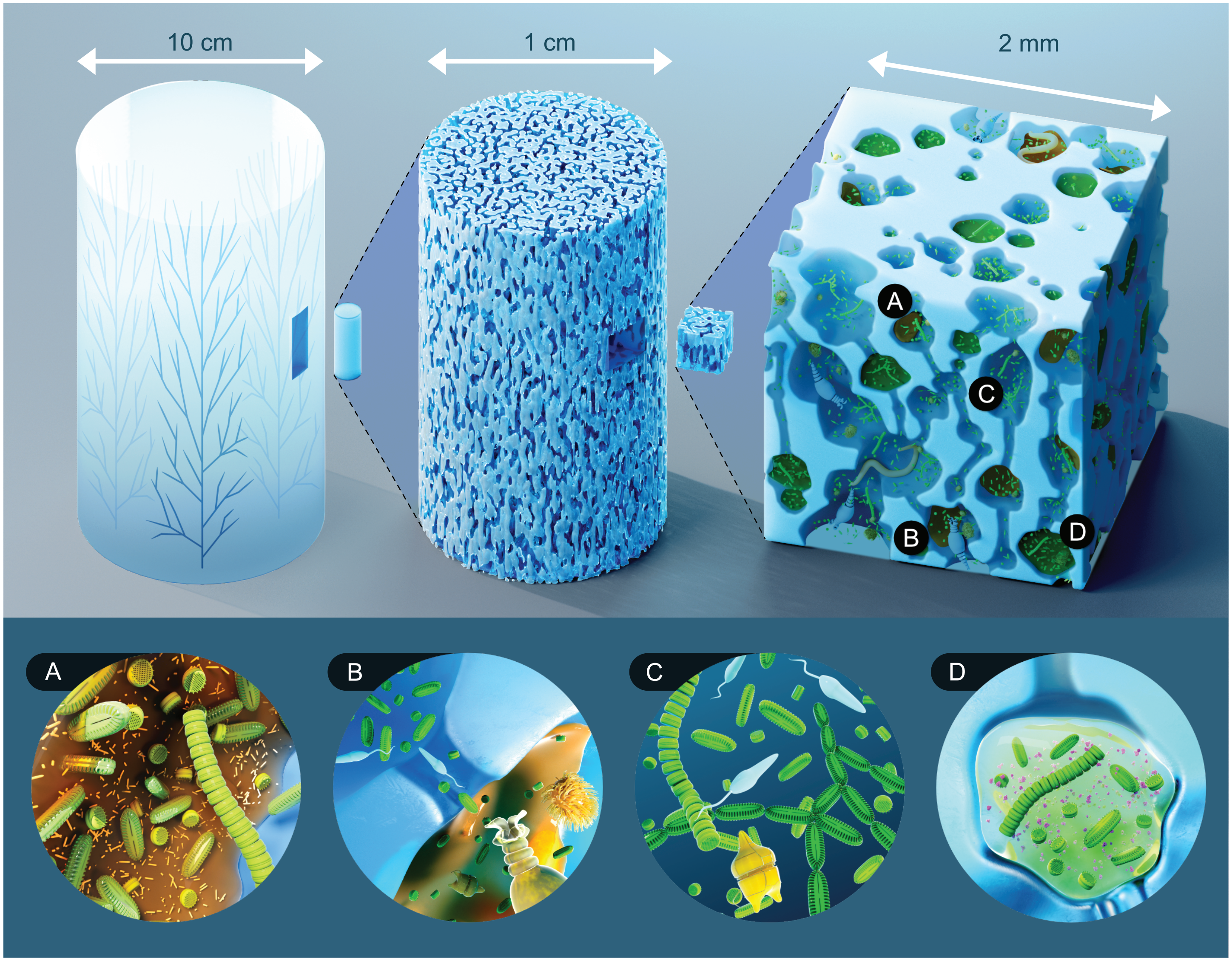
X-ray tomography

DLA model

brine channel in sea ice

diffusion limited aggregation

Implications of brine fractal geometry on sea ice ecology and biogeochemistry



Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

(A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.

(B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).

(D) Ice algae secrete extracellular polymeric substances (EPS) which alter incusion geometry and may further increase the fractal dimension.

Remote sensing of sea ice

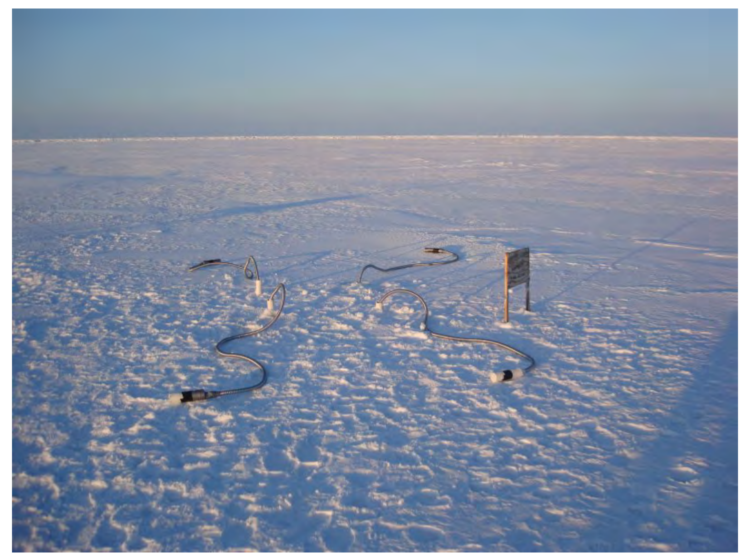
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

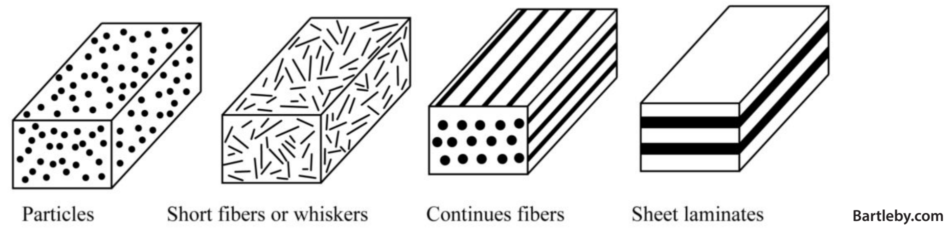
$$\varepsilon^*$$

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

complexities of mixture geometry



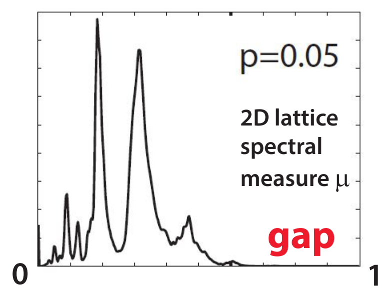
distilled

distilled

Analytic Continuation Method

Stieltjes Integral Representations
for Homogenized Parameters

Bergman 1978, Milton 1979
Golden & Papanicolaou 1983



spectral properties of operator (matrix)
~ quantum states, energy levels for atoms

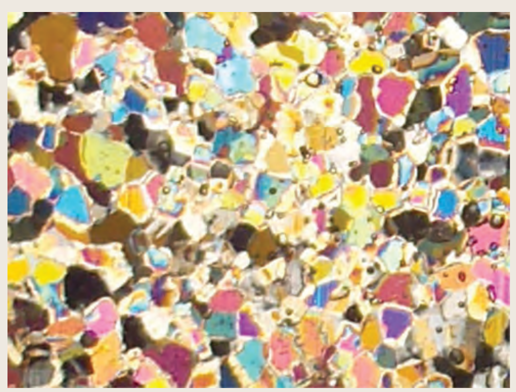
eigenvectors eigenvalues

EXTEND to: polycrystals, advection diffusion, waves through ice pack

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
  orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



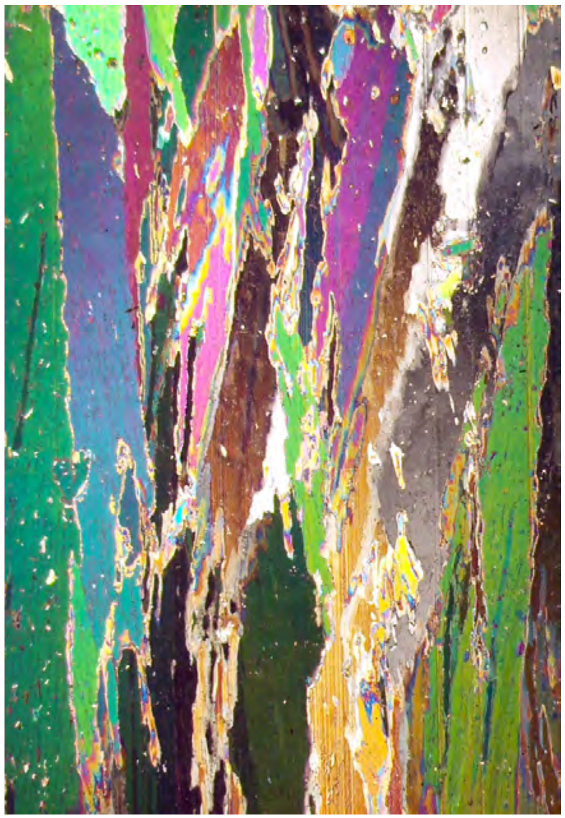
Proc. Roy. Soc. A 8 Feb 2015

higher threshold for fluid flow in granular sea ice

microscale details impact "mesoscale" processes

nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar



5%

granular

10%

Golden, Sampson, Gully, Lubbers, Tison 2023

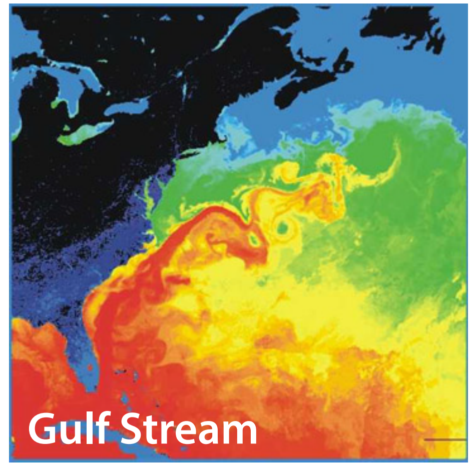
electromagnetically distinguish ice types
inverse homogenization for polycrystals

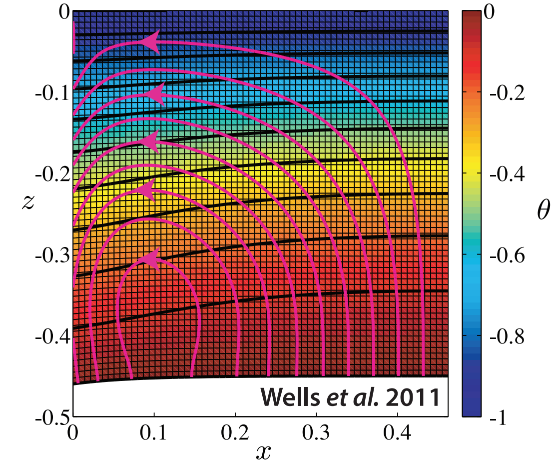
mesoscale

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

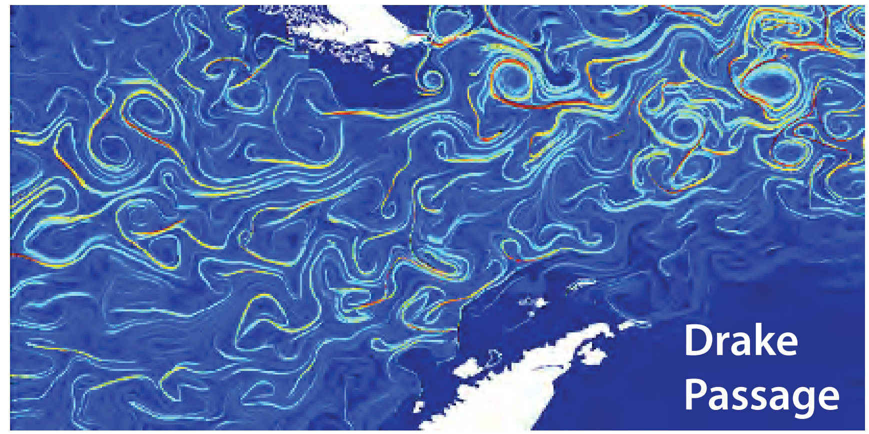
homogenize

$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

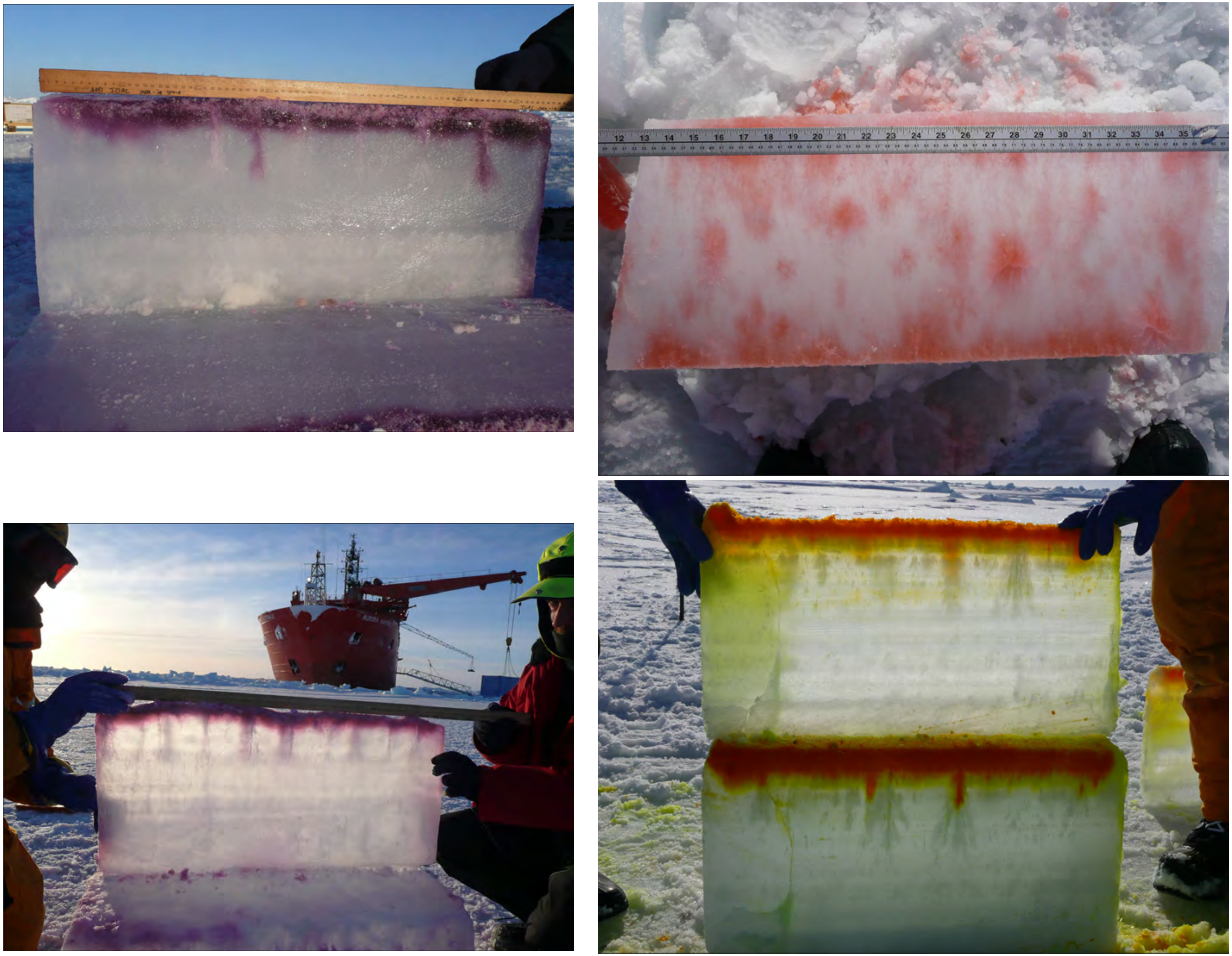
Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91



Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

tracers flowing through inverted sea ice blocks



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

$$\kappa^* = \kappa\left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2+\tau^2}\right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2+\tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H\Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H\Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

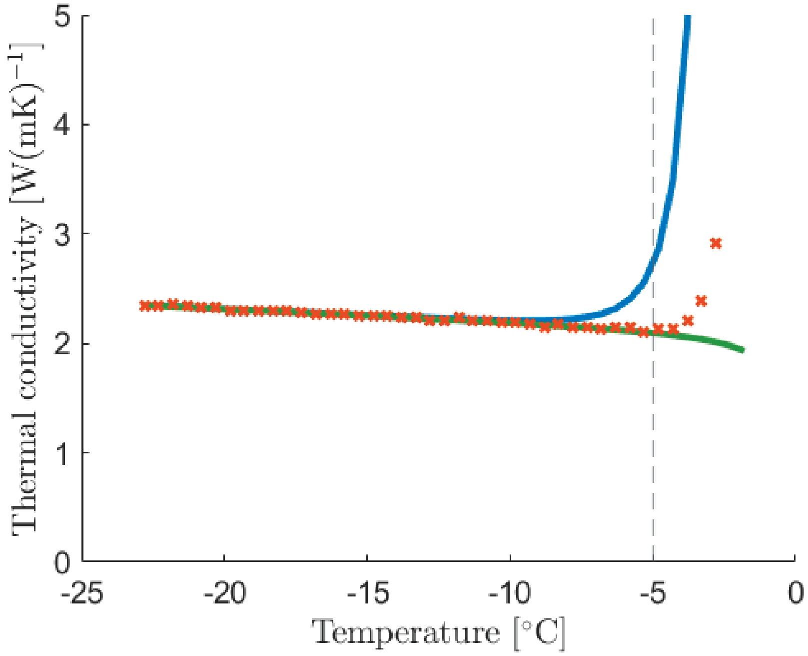
rigorous framework for numerical computations of spectral measures and effective diffusivity for model flows

new integral representations, theory of moment calculations

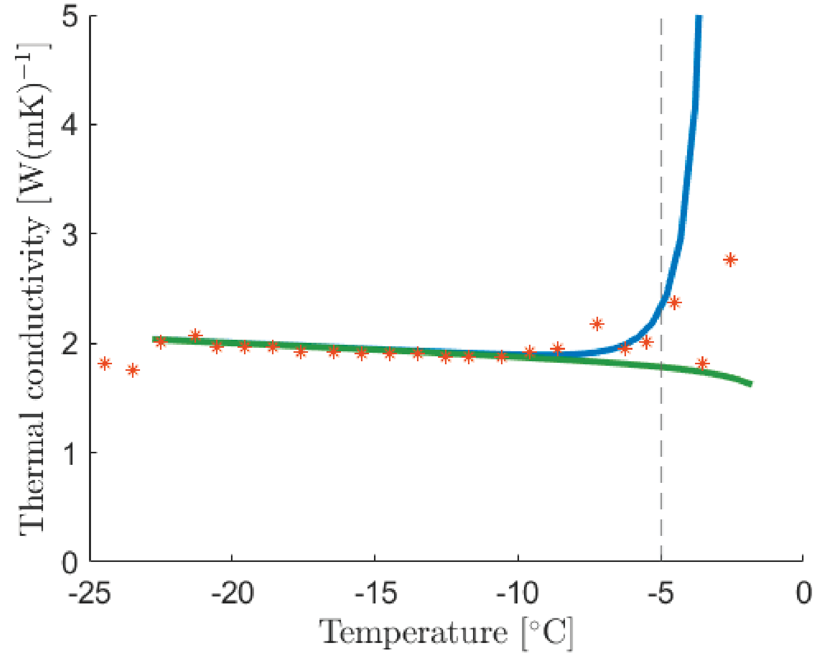
separation of material properties and flow field

Bounds on Convection Enhanced Thermal Transport

simulations



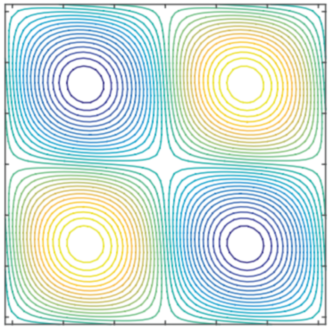
data [Trodahl et al., 2001]



Monte-Carlo simulations of SDE with temperature dependent Péclet number P

strength of advection $B = \kappa P/2\pi$

Euler-Maruyama and subsampling methods for SDE



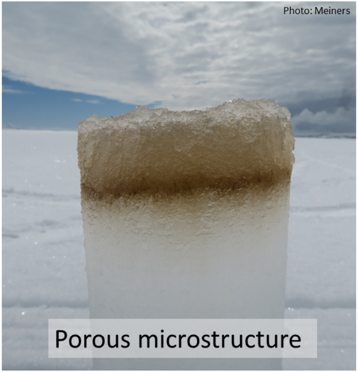
cat's eye flow model for brine convective flow

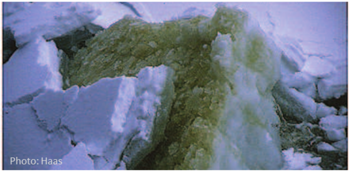
Rigorous Padé approximant bounds in terms of P using Stieltjes integral + analytic continuation method for the measure

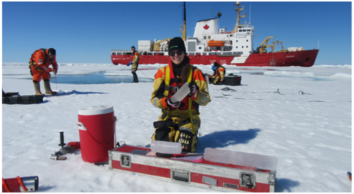
Darcy velocity v = 0.5 [m/s]

Kraitzman, Hardenbrook, Dinh, Murphy, Cherkaev, Zhu & Golden, 2023

SEA ICE ALGAE







Can we improve agreement between algae models and data?

80% of polar bear diet can be traced to ice algae*.

* Brown TA, et al. (2018). *PloS one*, 13(1), e0191631

ALGAL BLOOM MODEL*

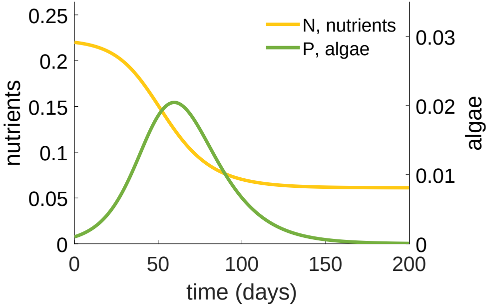
$$\text{nutrients:} \qquad \frac{dN}{dt} = \underbrace{\alpha}_{\text{input}} - \underbrace{\beta NP}_{\text{uptake}} - \underbrace{\eta N}_{\text{loss}}$$

$$\text{algae:} \qquad \frac{dP}{dt} = \underbrace{\gamma\beta NP}_{\text{growth}} - \underbrace{\delta P}_{\text{death}},$$

$$N(0) = n_0, \qquad P(0) = p_0$$

* Huppert, A., et al. (2002). *American Naturalist*, 159(2), 156-171

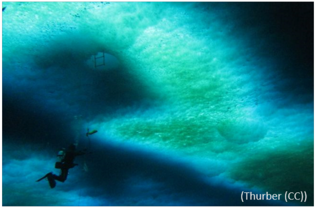
Algal bloom model

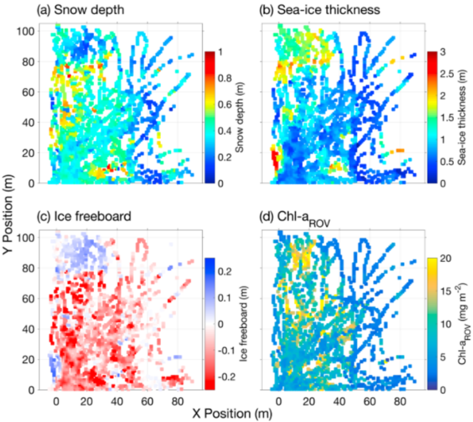


- poor agreement with data
- poor agreement between models

Steinacher, M., et al. (2010). *Biogeosciences*, 7(3), 979-1005

HETEROGENEITY





Meiners, K.M., et al. (2017). *Geophysical Research Letters*, 44(14), 7382-7390

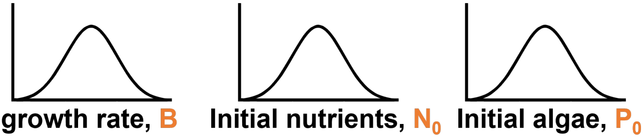
Heterogeneity in initial conditions

At each location within a larger region, we could consider

Nutrients $$\frac{dN}{dt} = \alpha - BNP - \eta N$$

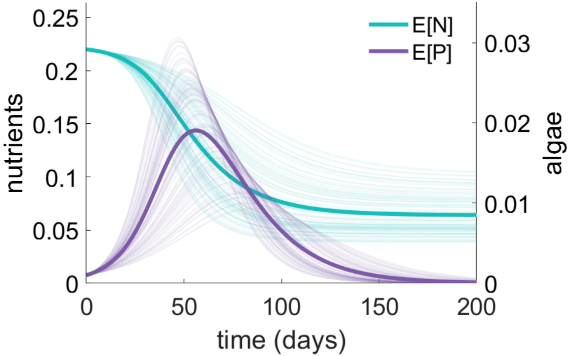
Algae $$\frac{dP}{dt} = \gamma BNP - \delta P$$

$$N(0) = N_0, \qquad P(0) = P_0$$



HOW DO WE ANALYZE THIS MODEL?

Monte Carlo simulations?



Too slow! Full algae model takes **8 hours** (cloud computing).

Received: 7 June 2022 | Revised: 1 August 2022 | Accepted: 1 August 2022
DOI: 10.1111/ele.14095

ECOLOGY LETTERS WILEY

METHOD

Uncertainty quantification for ecological models with random parameters

Jody R. Reimer[1,2] | Frederick R. Adler[1,2] | Kenneth M. Golden[1] | Akil Narayan[1,3]

[1]Department of Mathematics, University of Utah, Salt Lake City, Utah, USA
[2]School of Biological Sciences, University of Utah, Salt Lake City, Utah, USA
[3]Scientific Computing and Imaging Institute, University of Utah, Salt Lake City, Utah, USA

Correspondences
Jody R. Reimer, Department of Mathematics and School of Biological Sciences, University of Utah, Salt Lake City, Utah, USA.
Email: reimer@math.utah.edu

Abstract

There is often considerable uncertainty in parameters in ecological models. This uncertainty can be incorporated into models by treating parameters as random variables with distributions, rather than fixed quantities. Recent advances in uncertainty quantification methods, such as polynomial chaos approaches, allow for the analysis of models with random parameters. We introduce these methods with a motivating case study of sea ice algal blooms in heterogeneous environments. We compare Monte Carlo methods with polynomial chaos techniques to help understand the dynamics of an algal bloom model with random parameters.

Introduce polynomial chaos approach to widely used ecological ODE models, but with random parameters.

Polynomial chaos expansions

$$N(t;B,P_0,N_0) \approx N_V(t;B,P_0,N_0) := \sum_{j=1}^{n} \widetilde{N}_j(t)\phi_j(B,P_0,N_0),$$

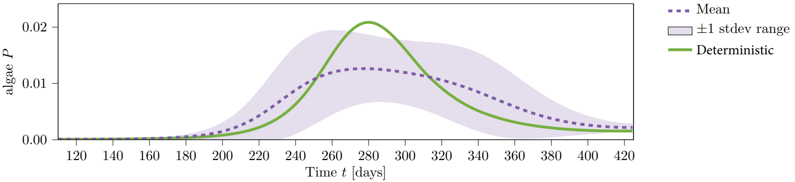
$$P(t;B,P_0,N_0) \approx P_V(t;B,P_0,N_0) := \sum_{j=1}^{n} \widetilde{P}_j(t)\phi_j(B,P_0,N_0),$$

where

- $V := \text{span}\{\phi_j\}_{j=1}^{n}$
- ϕ_j are orthogonal polynomials that form a basis for V
- $(\widetilde{N}_j, \widetilde{P}_j)$ need to be computed

Xiu, D. (2010). Numerical methods for stochastic computations. Princeton university press.

ECOLOGICAL INSIGHTS



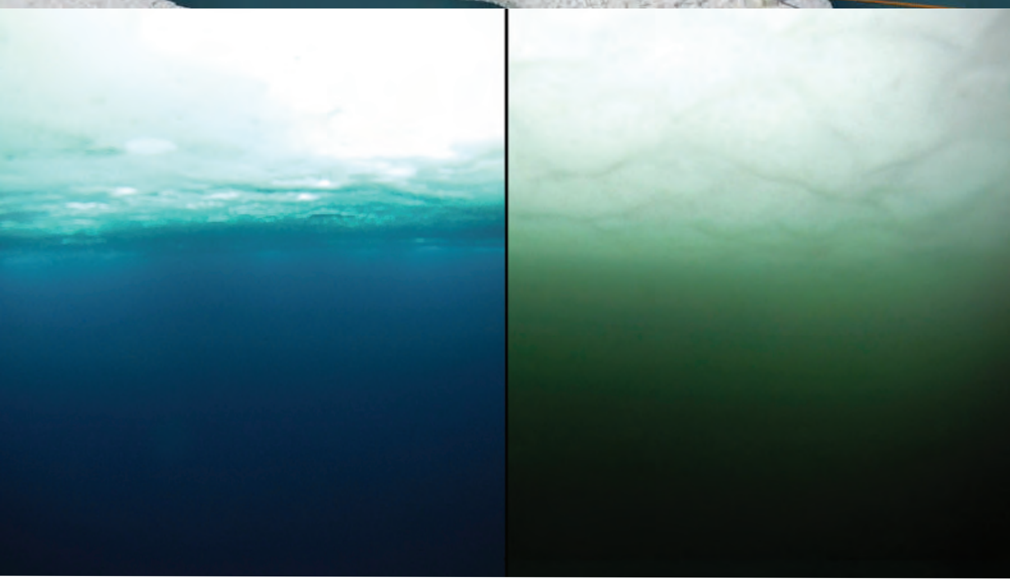
- lower peak bloom intensity
- longer bloom duration
- able to compare variance to data

macroscale

Perovich

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS



no bloom bloom

massive under-ice algal bloom

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

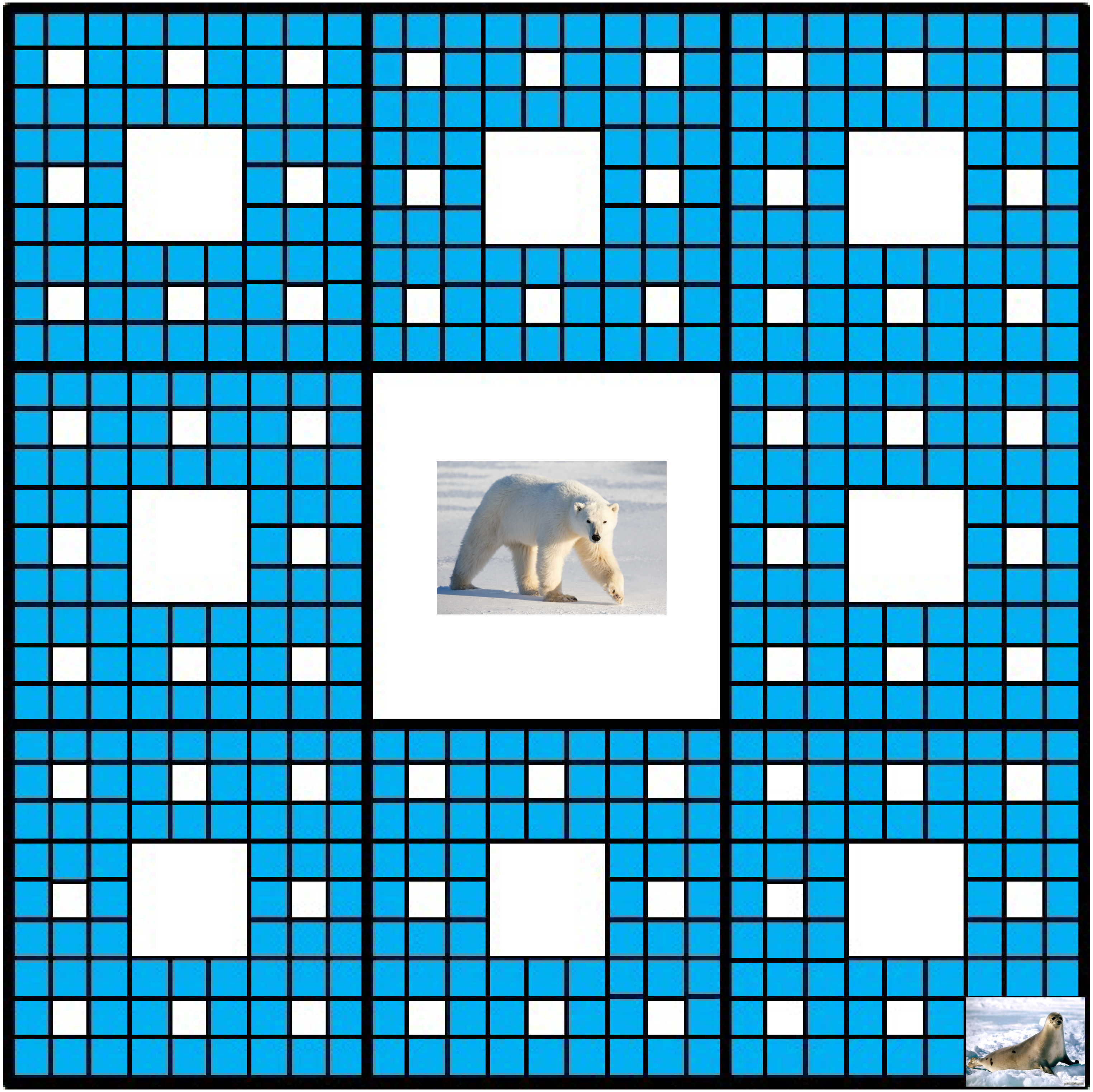
The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

Horvat, Rees Jones, Iams, Schroeder, Flocco, Feltham, ***Science Advances*** 2017

The effect of melt pond geometry on the distribution of solar energy under first year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden
Geophys. Res. Lett. 2019

(2015 AMS MRC)



polar bear foraging in a fractal icescape

Nicole Forrester
Jody Reimer
Ken Golden

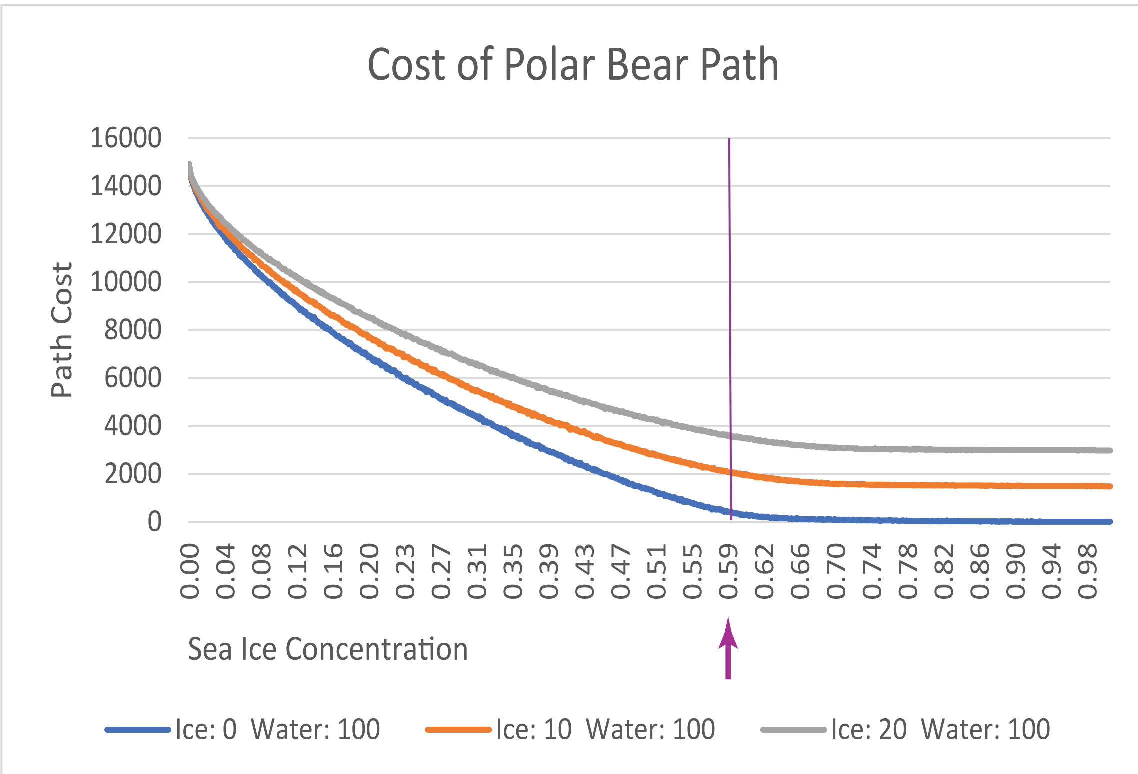
It costs the polar bear 5 times the energy to swim through water than to walk on sea ice.

What pathway to a seal minimizes energy spent?

Polar Bear Percolation

Optimal Movement of a Polar Bear in a Heterogenous Icescape

$C(p)$



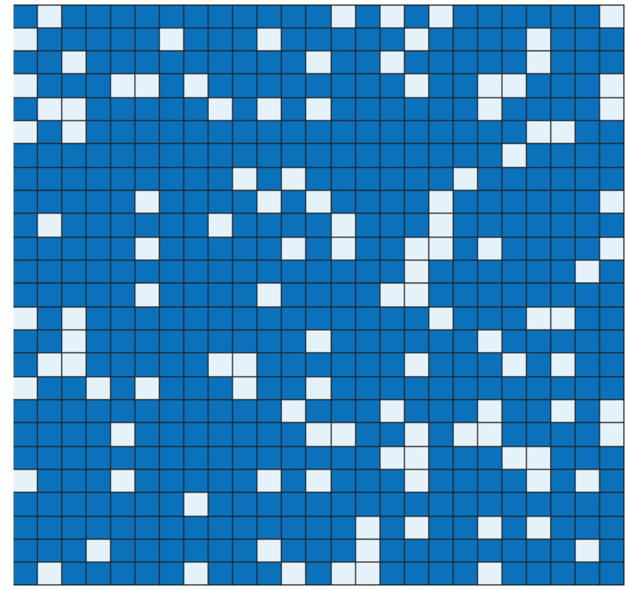
$\leftarrow h = 0.2$

$\leftarrow h = 0.1$

$\leftarrow h = 0$

$$h = \frac{c_i}{c_w}$$

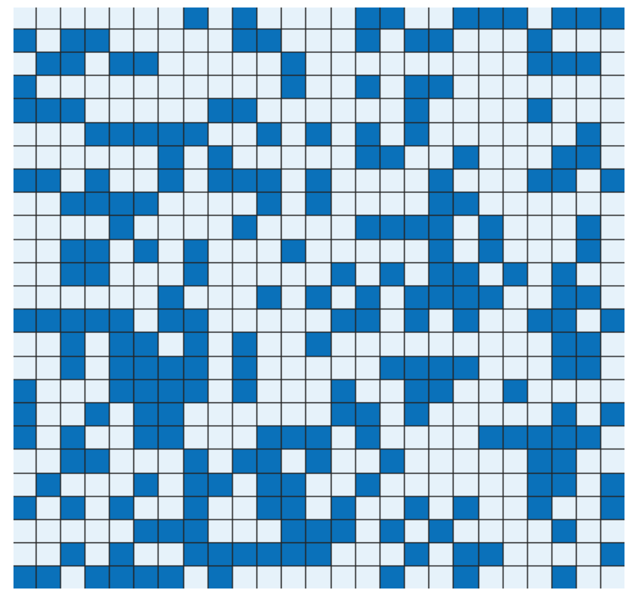
ratio of local "conductivities"



20% Ice

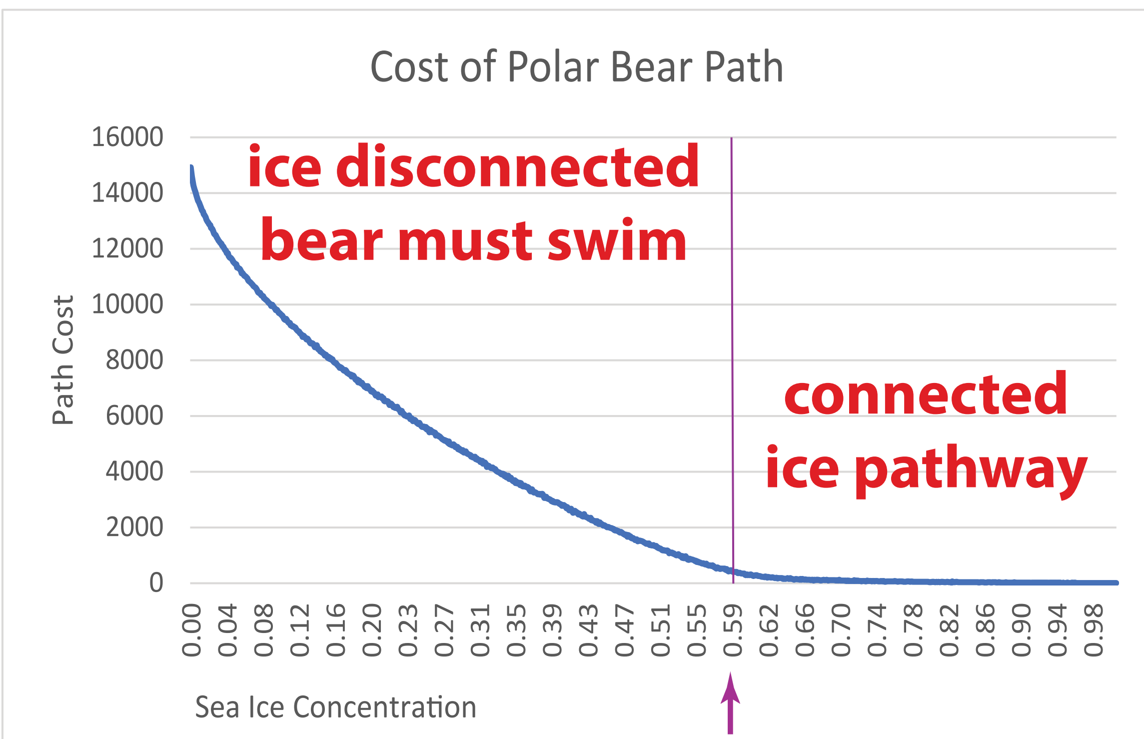
site percolation threshold

$p_c = 0.59$ for $d = 2$

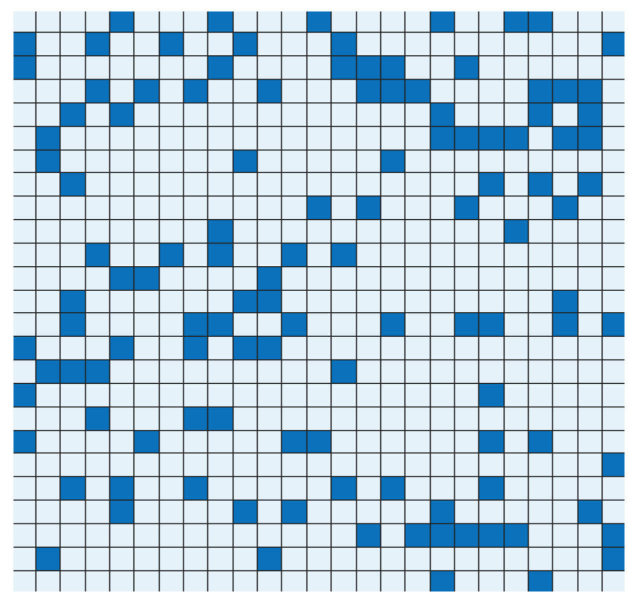


60% Ice

$C(p)$

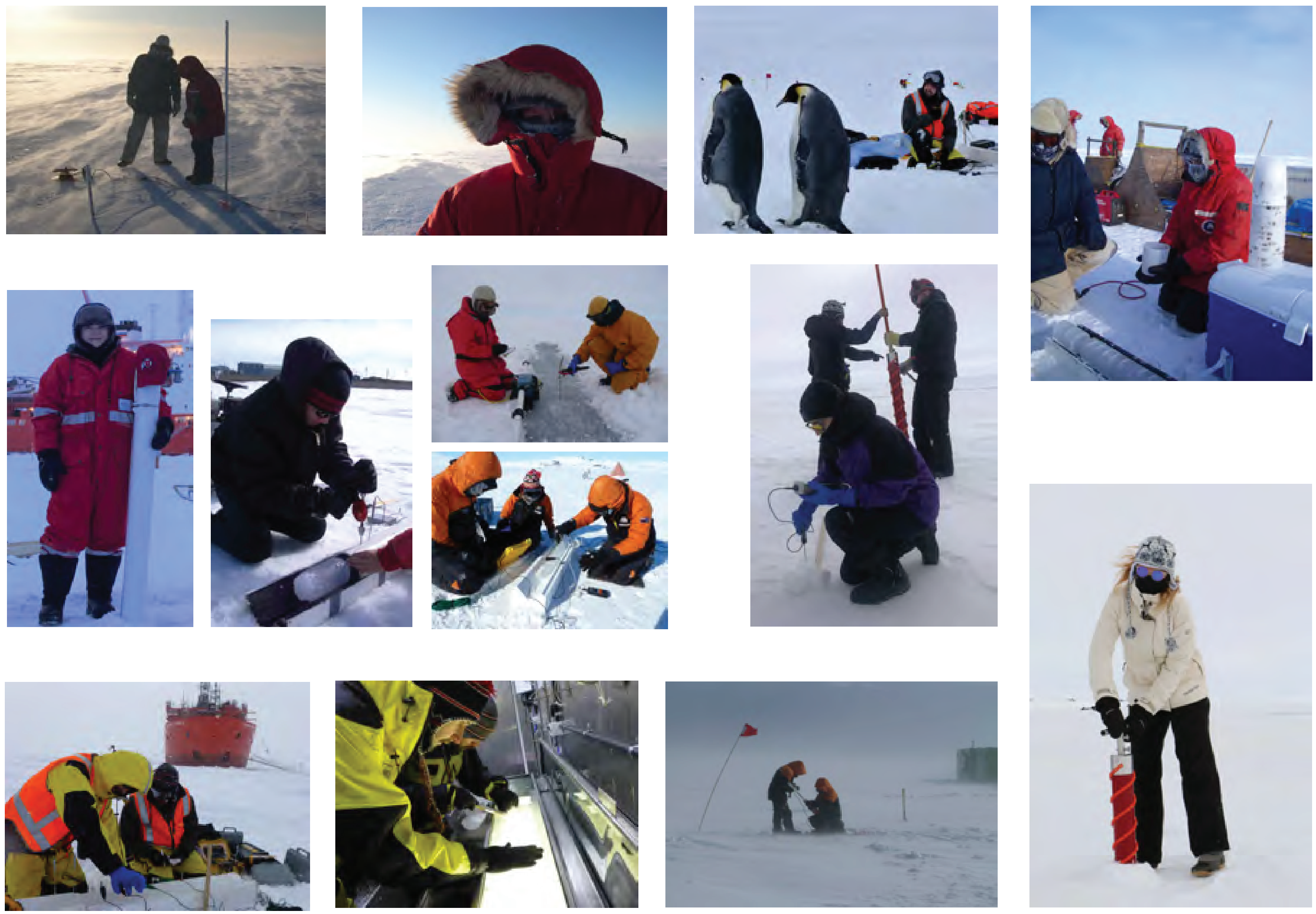


$\leftarrow h = 0$



80% Ice

Thank you to so many postdocs, graduate students, undergraduates, high school students and colleagues who contributed to this work!



U. of Utah students in the Arctic and Antarctic (2003-2022): closing the gap between theory and observation - making math models come alive and experiencing climate change firsthand.

ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices

of the American Mathematical Society

November 2020 Volume 67, Number 10

AMERICAN MATHEMATICAL SOCIETY

Advancing research. Creating connections.

The cover is based on "Modeling Sea Ice," page 1535.

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999