Name Solutions

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Class (circle one) 9:40 10:45

Mathematics 1210
Fall 2006
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FINAL EXAM

Monday, December 11, 2006

Problem	Points	Score
1.	25	
2.	15	
3.	10	
4.	5	
5.	15	
6.	15	
7.	10	
8.	10	
9.	10	
10.	10	
11.	10	
12.	15	
	TOTAL	

(25 points) 1. Calculate the following. Be sure to show all of your work.

(a) $\lim_{x\to 2} \frac{x^3-8}{x-2}$

We know $(x^3-8) = (x-2)(x^2+2x+4)$ so we get:

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)}$$
$$= \lim_{x\to 2}(x^2+2x+4) = 2^2+2(2)+4 = 12$$

(b) $\lim_{x\to 0} \frac{\int_0^x (\cos t\ -1)\, dt}{x^3}$

If we plug in 0 for the above limit we get $\frac{0}{0}$, which is indeterminate. We can use L'Hospital's rule and take the derivative of the top and the bottom, using the funadmental theorem of calculus to take the derivative of the top, to get:

$$\lim_{x\to 0} \frac{\int_0^x (\cos t - 1) dt}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x}$$
$$= \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

Where he we have just repeatedly applied L'Hospital's rule until we have an answer that was not indeterminate.

(c) $\lim_{x\to 0} \frac{\sin(x^2)}{\pi x}$

Again, if we plug in 0 for x we get $\frac{0}{0}$. So, again applying L'Hospital's rule we get:

$$\lim_{x\to 0} \frac{\sin(x^2)}{\pi x} = \lim_{x\to 0} \frac{2x\cos(x^2)}{\pi} = 0$$

(d) $\lim_{x\to+\infty} \dfrac{\sqrt{4x^2+1}}{x+3}$

If we take an x^2 term outside the square root, and divide the top and bottom by x we get:

$$\frac{\sqrt{4x^2+1}}{x+3} = \frac{x\sqrt{4+\frac{1}{x^2}}}{x+3} = \frac{\sqrt{4+\frac{1}{x^2}}}{1+\frac{3}{x}}$$

If we rewrite it like that, it's clear that if we take the limit as x goes to ∞ we get:

$$\frac{\sqrt{4}}{1} = 2$$

The terms with the x values in the denominators just go to 0.

(e) $\lim_{x\to 0} f(x), \qquad \text{where } f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

No matter how close we get to 0 we can find rational points and irrational points. So, this limit does not exist due to (very) wild oscillations. In fact, the limit as x goes to any number does not exist for this function!

(15 points) 2. A population of frogs in a swamp is found to grow at a rate proportional to the square root of the population size. The initial population P is 400 frogs, and 5 years later there are 900 of them. Write the differential equation for the frog population $P(t)$ with the two corresponding conditions, and find the particular solution which incorporates both conditions.

The differential equation and initial conditions for this problem are:

$$\frac{dP}{dt} = k\sqrt{P} \;\; P(0) = 400 \;\; P(5) = 900$$

We can rewrite the equation and then integrate to get:

$$\frac{dP}{\sqrt{P}} = kdt \Rightarrow 2\sqrt{P} = kt + C \Rightarrow P = (\frac{kt + C}{2})^2$$

If we apply our initial conditions we have $(\frac{C}{2})^2 = 400 \rightarrow C = 40$ and so $(\frac{5k + 40}{2})^2 = 900 \rightarrow k = 4$. So,

$$P(t) = (\tfrac{4t+40}{2})^2 = (2t + 20)^2$$

(10 points) 3. Use the differential to approximate $\sqrt{50}$, showing all your work.

We note that $\sqrt{49} = 7$, and so we use the function $\sqrt{x}$ and the derivative of the function at $x = 49$ to approximate the function at $x = 50$. Given the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$ we get:

$$\sqrt{50} \approx \sqrt{49} + (50 - 49)\frac{1}{2\sqrt{49}} = 7 + \frac{1}{7} = \frac{50}{7}$$

Most people who got the problem right did it this way. However, you could set this up using a reference point different than 7, or even a different function, and still do what the problem asks. If you did this, you were given full credit.

(5 points) 4. Suppose a ball is propelled upward from the ground $x_0 = 0$ with an initial velocity of $v_0 = 64$ feet/second. Its position $x(t)$ after t seconds is then given by $x(t) = -16t^2 + 64t$. When does the ball reach its maximum height? What is the maximum height reached by the ball?

The ball will reach its maximum height at a time when the derivative of the position function (the velocity) is 0. It will be a maximum if the second derivative is negative at this point. If we differentiate the position function we get:

$$\frac{dx}{dt} = -32t + 64$$

This is 0 when $t = 2$. The second derivative is:

$$\frac{d^2x}{dt^2} = -32$$

Which is always negative. So, at time $t = 2s$ the ball reaches its maximum height. The height at that time will be:

$$x(2) = -16(2^2) + 64(2) = 64$$

So, the maximum height is 64 feet.

(15 points) 5. Find the dimensions of the rectangle with maximum area inscribed in a circle of unit radius. Be sure to show all your work, and verify your result with the second derivative test.

If we inscribe the rectangle in the circle as indicated in the picture above, then the area of the rectangle is:

$$A = height \times width = (2x)(2y) = 4xy$$

Now, x and y are related by the equation for the unit circle:

$$x^2 + y^2 = 1$$
$$\rightarrow y = \sqrt{1 - x^2}$$

And so, we get the area equation:

$$A = 4xy = 4x\sqrt{1 - x^2}$$

We want to find the value of x that maximizes this area. We know this value will occur at a critical point. Now, x can range between 0 and 1, but we can see that for both of these values the area is 0. So, the maximum must be a point at which the derivative is 0.

We take the derivative and solve for where it's 0:

$$A(x) = 4x\sqrt{1-x^2}$$
$$A'(x) = 4\sqrt{1-x^2} - \frac{4x^2}{\sqrt{1-x^2}}$$

Setting this equal to 0 and solving for x we find:

$$0 = 4\sqrt{1-x^2} - \frac{4x^2}{\sqrt{1-x^2}}$$

Multiplying both sides by $\sqrt{1-x^2}$ and dividing both sides by 4:

$0 = 1 - x^2 - x^2 \rightarrow 2x^2 = 1 \rightarrow x = \frac{1}{\sqrt{2}}$. So, we get $x = \frac{1}{\sqrt{2}}$ and given $y = \sqrt{1-x^2}$ we get $y = \frac{1}{\sqrt{2}}$ also.

We want to use the second derivative test to determine that this critical point is indeed a minimum. Taking the second derivative of $A(x)$ we get:

$$A''(x) = \frac{-4x}{\sqrt{1-x^2}} - \frac{4x^3}{(1-x^2)^{\frac{3}{2}}} - \frac{8x}{\sqrt{1-x^2}} = \frac{-12x}{\sqrt{1-x^2}} - \frac{4x^3}{(1-x^2)^{\frac{3}{2}}}$$

So, at $x = \frac{1}{\sqrt{2}}$ we get:

$$A''(\frac{1}{\sqrt{2}}) = -16 < 0.$$

So, the second derivative at $x = \frac{1}{\sqrt{2}}$ is negative, and therefore $A(x)$ is a local maximum at that point. Given this is the only critical point outside the endpoints, we conclude this is the global maximum.

The dimensions of the square will then be: $(2x, 2y) = (2\frac{1}{\sqrt{2}}, 2\frac{1}{\sqrt{2}}) = (\sqrt{2}, \sqrt{2})$.

(15 points) 6. Find the temperature profile $T(x)$ on the interval $[0, 2]$ satisfying the steady state heat equation

$$\frac{d^2T}{dx^2} = 0$$

and the boundary conditions $T(0) = -10$ and $T(2) = -2$.

To find $T(x)$ we integrate the above equation twice to get:

$$\frac{dt}{dx} = C$$
$$T(x) = Cx + D$$

Based on the initial conditions we get $T(0) = D = -10$ and $T(2) = -2 = 2C - 10$ and so $C = 4$. Therefore, the temperature function is:

$$T(x) = 4x - 10$$

(10 points) 7. For a given spring, the force required to keep it stretched x feet is given by $F = 9x$ pounds. How much work in ft·lb is done in stretching the spring 1 foot from its unstretched equilibrium state? Be sure to show all your work.

The word done is the integral of the force over the distance. So, if we stretch it from $x = 0$ to $x = 1$, where distance is measured in feet, we get:

$$W = \int_0^1 9x dx = \frac{9}{2}x^2|_0^1 = \frac{9}{2}1^2 - \frac{9}{2}0^2 = \frac{9}{2} \text{ foot-pounds}$$

(10 points) 8. Consider the region under the graph of $y(x) = \sin x$, over the $x-$axis between $x = 0$ and $x = \pi$. Revolve this region around the $x-$axis, and find the volume of the resulting solid.

Note - Please forgive my crude drawings. There's a reason I wasn't an art major. (Dylan)

The volume of this region can be calculated using the disk method:

$$V = \int_0^\pi \pi(\sin x)^2 dx = \int_0^\pi \pi \frac{1 - \cos 2x}{2} dx = \pi(\frac{x}{2} - \frac{\sin 2x}{4})|_0^\pi$$
$$= (\frac{\pi^2}{2} - \pi\frac{0}{4}) - (\frac{0}{2} - \frac{0}{4}) = \frac{\pi^2}{2}$$

(10 points) 9. Find the area of the finite region bounded by the graphs of $y = -x+2$ and $y = x^2$. Sketch the region.

The area of this region will just be the integral of the upper graph subtract the integral of the lower graph, where the limits of integration are determined by where the two graphs touch. First, we'll solve for the limits of integration using the quadratic equation :

$$-x + 2 = x^2 \rightarrow x^2 + x - 2$$

$$\rightarrow x = \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm 3}{2} = (-2, 1)$$

So, the area of the region will be:

$$\int_{-2}^{1} [(-x + 2) - x^2]dx = -\frac{x^2}{2} + 2x - \frac{x^3}{3}|_{-2}^{1}$$
$$= (-\frac{1}{2} + 2 - \frac{1}{3}) - (-\frac{4}{2} + 2(-2) - \frac{-8}{3}) = \frac{9}{2}$$

(10 points) 10. Find the volume of the solid generated by revolving about the $x-$axis the region bounded by the $x-$axis and the upper half of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Again for this problem we'll want to use the disk method. In the upper half plane we can express the curve for the ellipse as a function of x:

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

With limits of integration that go from $-a$ to a. If we apply the disk method to find the volume we get when we rotate the region defined by that curve around the x-axis we get:

$$V = \int_{-a}^{a} \pi b^2 (1 - \frac{x^2}{a^2}) dx = \pi b^2 (x - \frac{x^3}{3a^2})|_{-a}^{a} =$$
$$\pi b^2 [(a - \frac{a}{3}) - (-a + \frac{a}{3})] = \frac{4}{3}\pi a b^2$$

(10 points) 11. Consider a circle of radius 2 centered at the origin, parameterized by $(x(t), y(t)) = (2\cos t, 2\sin t)$. Find the circumference of this circle by calculating the arc length of the curve parameterized by $(x(t), y(t))$ with $t \in [0, 2\pi]$.

First noting that $x'(t) = -2\sin t$ and $y'(t) = 2\cos t$ we use the arc length formula for a parameterized curve and take the integral:

$$s = \int_0^{2\pi} \sqrt{4\sin t^2 + 4\cos t^2} dt = \int_0^{2\pi} 2dt = 4\pi$$

Where here we used the relation $\sin x^2 + \cos x^2 = 1$. We can check to see that this is indeed the circumference of a circle of radius 2 - $C = 2\pi r = 2\pi(2) = 4\pi$.

(15 points) 12. Use the Theorem of Pappus to find the volume of the doughnut, or *torus*, obtained when the region bounded by the circle of radius 1, centered at the point $(3, 0)$, is revolved around the $y-$axis. The equation of the circle is $(x - 3)^2 + y^2 = 1$. (Hint: where is the centroid of the circular region?)

We exploit the symmetry of the circle to note that the centroid must lie at the center of the circle, so at the point $(3, 0)$. This point is a distance of 3 away from the y-axis, so if we revolve the circle about the y-axis the centroid will travel the circumference of a circle of radius 3, or 6π. Multiplying this by the area of the unit circle $\pi(1)^2 = \pi$ we get $V = 6\pi^2$.