Mathematics 1210-2,3 Prof. Ken Golden
PRACTICE FINAL EXAM ANSWER KEY Fall 2009

Problems From Varberg, Purcell, Rigdon (Chapter.Section.Number)

1. (5.1.6.) The boundary curves intersect when $x^2 - 2 = x + 4$ or $x^2 - x - 6 = (x - 3)(x + 2) = 0$ so at $x = -2$ and $x = 3$. We integrate the vertical slices whose height at x is the (positive) distance from the lower boundary to the upper boundary there, $(x + 4) - (x^2 - 2)$ and whose width is dx.

$$\int_{-2}^{3} -x^2 + x + 6dx = |_{-2}^{3} - \frac{x^3}{3} + \frac{x^2}{2} + 6x = \frac{27}{2} - (-\frac{22}{3}) = \frac{103}{6} = 17\frac{1}{6}.$$

2. (5.2.17.) The upper and lower boundary curves intersect when $y = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so when $x = \pm a$.

   If we slice vertically, the long direction of the slices is perpendicular to the axis of revolution and they sweep out circular discs. The volume of the disc at x is its thickness, dx, times its area πr^2. The radius is just $y = b(1 - \frac{x^2}{a^2})^{1/2}$, the height of the region being revolved about its lower boundary curve.

   So we integrate

$$V = \int_{-a}^{a} \pi b^2(1 - \frac{x^2}{a^2})dx = \pi b^2 \int_{-a}^{a} 1 - \frac{x^2}{a^2}dx = \pi b^2|_{-a}^{a} x - \frac{x^3}{3a^2} = \frac{4}{3}\pi ab^2.$$

   The prolate spheroid has a circular cross section with radius b in the $y - z-$plane, and extends a distance a from the origin in either direction in the $x-$direction. When $a = b = r$, it becomes a sphere of radius r and the formula gives its volume, $V = \frac{4}{3}\pi r^3$.

3. (5.3.2.) The upper and lower boundary curves intersect when $x^2 = 0$, so when $x = 0$ so the region extends from $x = 0$ to $x = 1$. If we slice vertically, the long direction of the slices is parallel to the axis of revolution and they sweep out cylindrical shells. The volume of the slice at x is its thickness, dx times its height, the distance from the lower to the upper boundary $x^2 - 0$, times its circumference, 2π times x, the 'radius' of that shell, given by the distance of the generating slice from the axis of revolution. So we integrate

$$V = \int_{0}^{1} 2\pi x(x^2)dx = |_0^1 2\pi\frac{x^4}{4} = \frac{\pi}{2}.$$

   If we slice horizontally, the long direction of each slice is perpendicular to the axis of revolution and they sweep out circular annuli or 'washers'. The volume of the washer at height y is its thickness, dy, times the difference of the area of the outer disc and the cut-out inner disc, $\pi r_o^2 - \pi r_i^2$. The height of the slices extends from the lowest point of the lower boundary, 0 everywhere, to the highest point of the upper boundary, $y = 1$ at $x = 1$. At height y, the inner radius is the x corresponding to $y = x^2$ which is both the upper *and* inner boundary, so $x = y^{1/2}$, and the outer radius is 1 at every y. So we integrate

$$V = \int_0^1 \pi(1^2 - (\sqrt{y})^2 dy = \pi \int_0^1 1 - y dy = \pi_0^1 y - \frac{y^2}{2} = \frac{\pi}{2}.$$

You can also see from the graph of $1 - y$ that the final integral is just the area of a triangle with base and height 1, so the area is $1/2$, recalling something we did at the very beginning of the class.

4. (5.4.10.) The arclength of a section of a parametric curve, $(x(t), y(t))$ is computed by integrating the infinitesimal hypotenuse lengths,

$$\sqrt{dx^2 + dy^2} = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} dt$$

between the t-values of interest.

$$x = \sqrt{5}\sin 2t - 2 \qquad y = \sqrt{5}\cos 2t - \sqrt{3}; \qquad 0 \le t \le \frac{\pi}{4},$$

In this case, using $\frac{dx}{dt} = 2\sqrt{5}\cos 2t$ and $\frac{dy}{dt} = -2\sqrt{5}\sin 2t$,

$$S = \int_0^{\frac{\pi}{4}} \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} dt = \int_0^{\frac{\pi}{4}} \sqrt{20(\cos^2 2t + \sin^2 2t} dt == 2\sqrt{5} \int_0^{\frac{\pi}{4}} dt = \frac{\sqrt{5}\pi}{2}.$$

5. (5.4.21a.) The arclength of a section of a graph, $y = f(x)$ is computed by integrating the infinitesimal hypotenuse lengths,

$$\sqrt{dx^2 + dy^2} = \sqrt{1 + \frac{dy}{dx}^2} dx = \sqrt{1 + f'(x)^2} dx$$

between the x-values of interest.

In this case, since f itself is given as an integral between 1 and x, we use the Fundamental Theorem of Calculus to compute $f'(x) = \sqrt{x^3 - 1}$ so that $\sqrt{1 + f'(x)^2} = x^{3/2}$ and

$$S = \int_1^2 \sqrt{1 + f'(x)^2} dx = \int_1^2 x^{3/2} dx = |_1^2 \frac{2}{5} x^{5/2} = \frac{2}{5}(2^{5/2} - 1).$$

6. (5.5.6.) Since $F = ks^{4/3}$ and $F = 2$ pounds when $s = 8$ inches, $k = \frac{1}{8} = 0.125$ when units of F are in pounds and s are in inches. Integrating the force of the spring times the infinitesimal displacement, $F(s)ds$, while stretching it from $s = 0$ (equilibrium) to 27 inches from equilibrium,

$$W = \int_0^{27} \frac{1}{8} s^{4/3} ds = |_0^{27} \frac{3}{56} s^{7/3} = \frac{3^8}{56} \sim 117.16$$

where the answer is in inch-pounds, and can be converted to the more common foot-pounds by dividing by 12.

7. (5.5.22.) Coulomb's Law may be written $F = K\frac{q_1 q_2}{r^2}$ where here $q_1 = q_2 = q$ and 10 dynes $= K\frac{q^2}{4}$. The work done bringing the charges from a distance $r_1 = 5$ centimeters apart to a distance $r_2 = 1$ centimeter apart is

$$W = -\int_{r_1}^{r_2} F(r)dr = -\int_5^1 Kq^2 r^{-2} dr,$$

where the minus sign is because the force you must apply to move the charge in closer is equal and opposite at each point to the repulsive force which is in the positive direction. Doing the integration, we get

$$W = -Kq^2|_5^1(-r^{-1}) = Kq^2(1 - \frac{1}{5}).$$

Finally, using $Kq^2 = 40$ in dyne-cm units, $W = 32$ dynes.

8. (5.6.25.) Using the method of cylindrical shells

Recall that slices parallel to the axis of revolution give cylindrical shells, slices perpendicular to the axis of revolution give discs or washers. Also notice that since the slices are rectangles, every slice is really both parallel AND perpendicular, so that the difference between a cylindrical shell and a washer is really just a matter of aspect ratio.

Each dx slice parallel to the y-axis at a particular value of x extends from the lower boundary line $y = 0$ to the upper boundary at $y = x^3$, and the x values extend from $x = 0$ where the upper and lower boundary curves intersect (at the axis of revolution) to $x = 1$.

The cylindrical shell generated by revolving such a slice about the y-axis is its circumference times its thickness times its height. (This is just length times width times height if the cylindrical shell is unrolled like a 'Ding-dong' into a rectangular slab.) The circumference is 2π times the radius from the axis to the slice, in this case just x, the height is the distance from the lower boundary to the upper boundary at x, $x^3 - 0$, and the thickness is dx. So the volume is given by

$$\int_{x=0}^{x=1} 2\pi x(x^3)dx = 2\pi|_0^1 \frac{x^5}{5} = \frac{2\pi}{5}.$$

The centroid of the region is obtained by averaging the value of x and y in the region, weighting each by the fraction of the region having that x or y value.

$$\bar{x} = \int_a^b x \frac{f(x)}{\int_a^b f(x)dx} dx$$

so in this case

$$\bar{x} = \frac{1}{\int_0^1 x^3 dx} \int_0^1 x^4 dx = \frac{1/5}{1/4} = \frac{4}{5}.$$

For $\bar{y}$ we can either slice along constant y-values and use

$$\bar{y} = \frac{1}{\int_0^1 1 - y^{1/3} dy} \int_0^1 y(1 - y^{1/3}) dy = \frac{2}{7}$$

and notice that the normalization factor is the same area of the region, 1/4, computed with dy slices instead of dx as before, so we really didn't need to do it twice.

The other way to compute the y-component of the centroid is to average the average y value in each dx slice, $\frac{1}{2}y$, weighted by the the fraction of the region having that average y value.

The area of the region is $\frac{1}{4}$. The distance swept out by the centroid is 2π times $\frac{4}{5}$, its distance from the axis of revolution. Multiplying these according to Pappus' volume theorem (there is also an area theorem) gives $\frac{1}{4}(2\pi\frac{4}{5} = \frac{2\pi}{5}$, confirming Pappus' Theorem in this case.

The alternate derivation of the y-coordinate of the centroid is

$$\bar{y} = \int_a^b \frac{1}{2} y \frac{f(x)}{\int_a^b f(x) dx} dx$$

$$\bar{y} = 4 \int_0^1 \frac{1}{2} x^6 dx = \frac{2}{7}.$$

9. To find $T(x)$ we integrate the differential equation twice to get:

$$\frac{dT}{dx} = C$$
$$T(x) = Cx + D$$

Based on the initial conditions we get $T(0) = D = 4$ and $T(1) = 2 = 2C + 4$ and so $C = -1$. Therefore, the temperature function is:

$$T(x) = -2x + 4$$