Mathematics 1210 PRACTICE EXAM III Fall 2004

ANSWER KEY

1. (a) $\frac{0}{0}$ so apply L'Hopital twice or simplify first to $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ and apply L'Hopital twice: $=\lim_{x\to 0}\frac{\sin x}{2x}$ $(\frac{0}{0})=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$.

   (b) $\frac{0}{0}$ so apply L'Hopital, $=\lim_{x\to 0}\frac{10/3x^{7/3}}{\cos x-1}$ $(\frac{0}{0})=\lim_{x\to 0}\frac{70/9x^{4/3}}{-\sin x}$ $(\frac{0}{0})=\lim_{x\to 0}\frac{280/27x^{1/3}}{-\cos x}=0$

   (c) $\frac{0}{0}$ so apply L'Hopital, $=\lim_{x\to\pi/2}\frac{-\sin x}{1}=-1$

   (d) $\frac{0}{0}$ so apply L'Hopital, and use the Fundamental theorem to take the derivative of the integral with respect to its upper endpoint, i.e., plug in x to the integrand, $=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$ $(\frac{0}{0})=\lim_{x\to 0}\frac{\sin x}{6x}$ $(\frac{0}{0})=\lim_{x\to 0}\frac{\cos x}{6}=\dfrac{1}{6}$

   (e) This is the definition of $\int_0^1 \sin x dx$ as limit of a Riemann sums, so $|_0^1 -\cos x$ or $1-\cos 1$.

2. (a) Separating variables, $y^{1/3}dy = x^{1/3}dx$ and integrating $y^{4/3} = (x^{4/3}+C)$ or $y=(x^{4/3}+C)^{3/4}$.

   (b) As in the previous practice exam solutions, $x(t)=A\sin\omega t+B\cos\omega t$.

   (c) Integrating once, $\frac{dx}{dt}=-gt+C$, and calling $\frac{dx}{dt}|_{t=0}=v_0$. Integrating again, $x(t)=-\dfrac{g}{2}t^2+v_0t+x_0$ where $x_0=x(0)$.

3. 22: $x(t)=-\dfrac{k}{2}t^2+v_0t$, set $v(t^*)=-kt^*+v_0=0$ solve for $t^*=\dfrac{v_0}{k}$, $x(t^*)=\dfrac{{v_0}^2}{2k}$

   35: (a) $\dfrac{dV}{dt}=kd^{1/2}$ and $V=\pi(\frac{10}{\sqrt{\pi}}{}^2 d=100d$ so $\dfrac{dV}{dt}=KV^{1/2}$ where $K=k/10$ (or just observe if cross-sectional area is constant, volume is proportional to depth.) Also, $V(0)=1600, V(40)=0$.

   (b) Separating variables, $V^{-1/2}dV=Kdt$ and integrating, $\frac{V^{1/2}}{1/2}=Kt+C$. From $V=1600$ when $t=0$, we get $C=80$ and from $V=0$ when $t=40$ we get $K=-2$. So $V=(\frac{1}{2}(-2t+80))^2=(40-t)^2$, and (c) $V(10)=900$.

   36: (a) $\dfrac{dP}{dt}=kP^{1/3}$. Also, $P(1980)=1000, P(1990)=1700$.

(b) Separating variables, $P^{-1/3}dP = kdt$ and integrating, $\frac{P^{2/3}}{2/3} = kt + C$, or $P(t) = (\frac{2}{3}(kt + C))^{3/2}$ From the two conditions, we get $1980k + C = \frac{3}{2}1000^{2/3}$ and $1990k + C = \frac{3}{2}1700^{2/3}$. Subtracting, we get

$$k = \frac{3}{20}(1700^{2/3} - 1000^{2/3} \approx 6.36.$$

Multiplying the first by 1990 and the second by 1980 and subtracting, we get

$$C = \frac{3}{20}(19901000^{2/3} - 19801700^{2/3}) \approx -12454.74$$

(c) Finally, setting $P = 4000$, and using $t = \frac{1}{k}(\frac{3}{2}P^{2/3} - C)$, the wolf population will reach 4000 at $t = \frac{1}{k}(\frac{3}{2}4000^{2/3} - C) \approx 2017.72$ in decimal years, or mid-August of the year 2017.

4. (a) integrable – $f(x)$ is bounded and continuous
   (b) not integrable – $f(x) \sim \dfrac{1}{x^{3/2}}$ as $x \to 0,\ 3/2 > 1$
   (c) not integrable – different approximations to the integral have different limits depending on the choice of sample points

5. Using $x_i = 1 + \dfrac{i}{n}$, $\Delta x = \dfrac{1}{n}$,

$$\int_1^2 (3x^2 - 2)dx = \lim_{n\to\infty} \sum_{i=1}^{n} (3{x_i}^2 - 2)\Delta x ,$$

$$= \lim_{n\to\infty} 3 \sum_{i=1}^{n} (1 + 2\frac{i}{n} + \frac{i}{n}^2)\Delta x - 2 \sum_{i=1}^{n} \Delta x ,$$

$$= 3(\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} + \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n}^2 \frac{1}{n}) - 2 \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} ,$$

Using

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} ,$$

we get

$$\sum_{i=1}^{n} \frac{i}{n}^2 \frac{1}{n} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n} ,$$

whose limit as n goes to infinity is 1/3.

Using

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2} ,$$

we get

$$\sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} = \frac{1}{2} + \frac{1}{2n} ,$$

whose limit as n goes to infinity is 1/2.

Most easily, $\sum_{i=1}^{n} \frac{1}{n} = 1$ so the final answer is $3(1 + 2(1/2) + 1/3) - 2(1) = 5$.

6. (a) $|_0^4 \frac{x^{3/2}}{3/2} = (2/3)2^3 - 0 = \quad 16/3 \qquad .$

   (b) $|_0^{\pi/2} - \cos x = 0 - (-1) = \quad 1$

   (c) Rewrite the integrand as $x^{-2} - 3x$ and integrate to get

$$|_1^3 \frac{x^{-1}}{-1} - 3\frac{x^2}{2} = (-1/3 - 27/2) - (-1 - 3/2) = \quad -11\frac{1}{3}.$$

   (d) $\displaystyle\int_0^\pi \sin^2 x dx = \frac{1}{2}\int_0^\pi (\sin^2 x + \cos^2 x)dx = \pi/2$

   Or subtract $\cos 2x = \cos^2 x - \sin^2 x$ from $1 = \cos^2 x + \sin^2 x$ to get $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and integrate to get

$$|0^\pi \frac{1}{2}x - \frac{1}{4}\sin 2x = \frac{\pi}{2}$$

   which may be about as involved as showing

$$\int_0^\pi \sin^2 x dx = \frac{1}{2}\int_0^\pi (\sin^2 x + \cos^2 x)dx.$$

   (e)

   Use $u = x^2 + \pi$ so $du = 2xdx$ to get

$$\int_{u=\pi}^{u=\pi^2+\pi} \frac{\cos u du}{2} = \frac{1}{2}|_{u=\pi}^{u=\pi^2+\pi} \sin u = \frac{1}{2}(\sin(\pi^2 + \pi))$$

   (f) Either explicitly, $|{-3}^3 \frac{x^4}{4} = \frac{3^4}{4} - \frac{(-3)^4}{4} = 0$ or just the fact that the integral of any odd function over an interval which is symmetric about the origin is zero. The integral of an even function over an interval which is symmetric about the origin is always twice the integral of the same function from zero to the right endpoint of the interval.

7. Use the Fundamental theorem and the chain rule. You may want to substitute $u = x^2$ then compute

$$(\frac{d}{du}\int_0^u \tan\theta d\theta)(\frac{d}{dx}x^2) = \tan u)(2x).$$

   The final answer should be in terms of x: $2x\tan x^2$.