Math 1210 Fall 2004 K. M. Golden

EXAM III Solutions

(20 points) 1. Calculate the following. Be sure to show all of your work.

(a) $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

This is Indeterminate $\frac{0}{0}$ so use L'Hopital:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = ds \lim_{x\to 0} \frac{\cos x - 1}{3x^2}$$

Intdeterminate $\frac{0}{0}$ so use L'Hopital.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = ds \lim_{x\to 0} \frac{-\sin x}{6x}$$

Intdeterminate $\frac{0}{0}$ so use L'Hopital.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = ds \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}.$$

(b) $\lim_{x\to 1} \frac{x^3 - 1}{x - 1}$

Factor $\frac{x^3-1}{x-1} = (x^2 + x + 1)$ for $x \neq 1$ so

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1} x^2 + x + 1 = 3.$$

or, this is Indeterminate $\frac{0}{0}$ so use L'Hopital:

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1} \frac{3x^2}{1} = 3$$

or this limit is equivalent to the definition of the derivative of $f(x) = x^3$ at $x = 1$ (so technically you shouldn't be using L'Hopital's rule which uses its derivative to find its derivative!

(c) $\lim_{x\to 0} \frac{\int_0^x \sin(t^2)\, dt}{x^3}$

This is Indeterminate $\frac{0}{0}$ so use L'Hopital and the Fundamental Theorem:

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x \sin\left(t^2\right)\, dt}{x^3} = \lim_{x\to 0} \frac{\sin\left(x^2\right)}{3x^2}$$

This is Indeterminate $\frac{0}{0}$ so use L'Hopital:

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x \sin\left(t^2\right)\, dt}{x^3} = \lim_{x\to 0} \frac{2x\cos\left(x^2\right)}{6x}$$

This is Indeterminate $\frac{0}{0}$ so use L'Hopital:

$$\lim_{x\to 0} \frac{\displaystyle\int_0^x \sin\left(t^2\right)\, dt}{x^3} = \lim_{x\to 0} \frac{2\cos\left(x^2\right) - 2x\sin\left(x^2\right)}{6} = \frac{2}{6} = \frac{1}{3}$$

or use $\sin x \sim x$ so $\sin\left(t^2\right) \sim t^2$ and $\int_0^x \sin\left(t^2\right) dt \sim \frac{t^3}{3}$ gives the same result.

(20 points) 2. A population of frogs in a swamp is found to grow at a rate proportional to the square root of the population size. The initial population P is 400 frogs, and 5 years later there are 900 of them.

(a) Write the differential equation for the frog population $P(t)$ with the two corresponding conditions.

$$P'(t) = kP^{1/2} \qquad P(0) = 400 \qquad P(5) = 900.$$

(b) Solve this differential equation, that is, find the particular solution which incorporates both conditions.

$$\int \frac{dP}{P^{1/2}} = \int k dt$$

so

$$2P^{1/2} = kt + C$$

or

$$P(t) = \frac{(kt + C)^2}{4}.$$

Plugging in $P = 400$ and $t = 0$ gives $C = 2(400)^{1/2} = 40$ and plugging in $P = 900$ and $t = 5$ gives $k = \frac{2(900)^{1/2} - 40}{5} = 4$.

(c) How long does it take for the frog population to quadruple (reach 1600) from its intitial value of 400?

Setting $P = 1600$ and solving for t,

$$t = \frac{2(1600)^{1/2} - 40}{4} = 10$$

so it takes 10 years.

(10 points) 3. Determine whether or not $f(x) = \dfrac{1}{\sqrt{\sin x}}$ is Riemann integrable on the interval (0,1], and **fully** explain your result (you may want to do an explicit calculation to explain your result).

Using $\sin x \sim x$ as $x \to 0$ we get $\frac{1}{\sqrt{\sin x}} \sim x^{-1/2}$ or, more precisely, using $\sin x > \frac{x}{2}$ for $x \in (0, 1]$ we get $\frac{1}{\sqrt{\sin x}} < 2x^{-1/2}$ for $x \in (0, 1]$.

Because of this

$$\int_a^1 \frac{1}{\sqrt{\sin x}}\, dx < 2 \int_a^1 x^{-1/2}\, dx$$

for any $a \in (0, 1]$.

Then because

$$\lim_{a \to 0^+} \int_a^1 x^{-1/2}\, dx = \lim_{a \to 0^+} |_a^1 2x^{1/2} = 2$$

is finite, so is

$$\lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{\sin x}}\, dx$$

so by comparison, $\int_0^1 \dfrac{1}{\sqrt{\sin x}}$ is a convergent improper integral, i.e. $f(x) = \dfrac{1}{\sqrt{\sin x}}$ is Riemann integrable on the interval $(0, 1]$.

(10 points) 4. Suppose the velocity $v(t)$ of a mass at the end of a spring is given by $v(t) = \cos t$. If the position $x(t)$ of the mass and its velocity are related by $v(t) = \frac{dx}{dt}$, find $x(t)$, given that its initial position is $x(0) = 1$.

$$x(t) - x(0) = \int_0^t \frac{dx}{dt} dt = \int_0^t v(t) dt = \int_0^t \cos t dt$$

so integrating and solving for $x(t)$ using $x(0) = 1$,

$$x(t) = 1 + \sin t.$$

(20 points) 5. Using the fact that $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$, calculate $\int_0^2 x^2\ dx$ directly from the **definition** of the integral, i.e., using Riemann sums. *Check* your result using the **fundamental theorem of calculus**.

Divide the interval $[0, 2]$ into n equal intervals of length $\Delta x = \frac{2}{n}$ with endpoints $x_i = i\Delta x$, $i = 1, \ldots, n$, and define the integral as the limit of the Riemann sums corresponding to those intervals, $\sum_{i=1}^{n} f(x_i)\Delta x$:

$$\int_0^2 x^2\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{2i}{n})^2 \frac{2}{n}.$$

Pulling out three factors of 2 out of the sum and the limit, and using the given explicit form of the sum,

$$\int_0^2 x^2\ dx == 8 \lim_{n\to\infty} \sum_{i=1}^{n} i^2 \frac{1}{n^3} = 8 \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} \frac{1}{n^3}$$

and simplifying,

$$\int_0^2 x^2\ dx = 8 \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} = \frac{8}{3}.$$

Checking our result,

$$\int_0^2 x^2\ dx = F(2) - F(0)$$

where $F'(x) = x^2$, i.e., F is an antiderivative of $f(x) = x^2$. We know that if $F(x) = \frac{x^3}{3} + C$ for any C, then $F'(x) = x^2$, and $F(2) - F(0) = \frac{8}{3}$.

(20 points) 6. Calculate the following integrals:

In the definite integral problems, to find $\int_a^b f(x)dx$, find an antiderivative $F(x)$ such that $F'(x) = f(x)$ and then $\int_a^b f(x)dx = |_a^b F = F(b) - F(a)$. All antiderivatives of the same function differ by a constant, which cancels out in the subtraction, so typically we use a 'simplest' F for definite integrals, e.g., for $f(x) = x^n$, $F(x) = \frac{x^{n+1}}{n+1}$, that is, there is no point in writing $+C$ or $+7$ when it will just disappear after we subtract.

(a) $\displaystyle\int_1^3 (4x^3 - 2x^2 + x - 5)\ dx$

$$\int_1^3 (4x^3 - 2x^2 + x - 5)\ dx = |_1^3\, x^4 - 2\frac{x^3}{3} + \frac{x^2}{2} - 5x$$

or

$$\int_1^3 (4x^3 - 2x^2 + x - 5)\ dx = (81 - 18 + \frac{9}{2} - 15) - (1 - \frac{2}{3} + \frac{1}{2} - 5) = 56\frac{1}{3} = 169/3.$$

(b) $\displaystyle\int \tan^2 x\ \sec^2 x\ dx$

This is an indefinite integral, so just use substitution to find the general antiderivative, and in this case you must include the $+C$. Let $u = \tan x$ and $du = \sec^2 x dx$. Then

$$\int \tan^2 x\ \sec^2 x\ dx = \int u^2\ du\ = \frac{u^3}{3} + C = \frac{\tan x^3}{3} + C.$$

You can always check an antiderivative for both indefinite and definite integrals by taking its derivative.

(c) $\displaystyle\int_0^1 \frac{u(u+1)^2}{\sqrt{u}}\ du$ Multiplying the integrand out to a more convenient form,

$$\int_0^1 u^{1/2} + 2u^{3/2} + u^{5/2}\ du = |_0^1\ \frac{u^{3/2}}{3/2} + 2\frac{u^{5/2}}{5/2} + \frac{u^{7/2}}{7/2}.$$

Plugging in and subtracting (the $u = 0$ terms all vanish)

$$\int_0^1 \frac{u(u+1)^2}{\sqrt{u}}\ du == \frac{184}{105}.$$

(d) $\int_0^\pi \sin^2 x \, dx$

Either use the sin-cos reverse symmetry on $[0, \pi]$ so that

$$\int_0^\pi \sin^2 x \, dx = \frac{1}{2} \int_0^\pi (\sin^2 x + \cos^2 x) \, dx = \frac{1}{2} \int_0^\pi 1 \, dx = \frac{\pi}{2}$$

or use $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$ so

$$\int_0^\pi \sin^2 x \, dx = \int_0^\pi (\frac{1}{2} - \frac{1}{2} \cos 2x) \, dx.$$

After substituting $u = 2x$ in the second term and antidifferentiating,

$$\int_0^\pi \sin^2 x \, dx = |_0^\pi \frac{1}{2} x - \frac{1}{4} \sin 2x$$

and since $\sin 0 = \sin \pi = 0$,

$$\int_0^\pi \sin^2 x \, dx = \frac{1}{2}(\pi - 0) = \frac{\pi}{2}.$$