Name ______________________________

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Class (circle one) 9:40 10:45

Math 1210
Fall 2006
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EXAM III
Friday, November 17, 2006

Problem	Points	Score
1.	15	
2.	10	
3.	20	
4.	15	
5.	15	
6.	15	
7.	10	
	TOTAL	

(15 points) 1. Calculate the following. Be sure to show all of your work.

(a) $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x}$$
$$= \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

L'Hospital's rule was applied three times.

(b) $\lim\limits_{x\to 1} \dfrac{x^3 - 1}{x - 1}$

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1} \frac{3x^2}{1} = 3$$

L'Hospital's rule was applied once.

(10 points) 2. Solve the differential equation

$$\frac{dx}{dt} = -32t + 48$$

for the function $x(t)$ with the initial condition $x(0) = 0$.

First, let's separate the variables of this differential equation, and integrate both sides: $x(t) = \displaystyle\int dx = \int -32t + 48dt = -16t^2 + 48t + C$ and $x(0) = C = 0$, so $x(t) = -16t^2 + 48t$.

(20 points) 3. A population of frogs in a swamp is found to grow at a rate proportional to the square root of the population size. The initial population P is 400 frogs, and 5 years later there are 900 of them.

(a) Write the differential equation for the frog population $P(t)$ with the two corresponding conditions.

$\frac{dP}{dt} = kP^{1/2}$, where k is the proportionality constant, $P(0) = 400$ and $P(5) = 900$

(b) Solve this differential equation, that is, find the particular solution which incorporates both conditions.

Using the differential equation above, $\int P^{-1/2} dP = \int k dt$, so $2P^{1/2} = kt + C$, where C is the integration constant. Using the initial condition $P(0) = 400$, we see $C = 2(400)^{1/2} = 40$. Then, the second condition $P(5) = 900$ tells us $2(900)^{1/2} = 5k + 40$, so we solve for k to find $k = 4$. Therefore, $2P^{1/2} = 4t + 40$, so the final solution is $P = (2t + 20)^2$.

(c) How long does it take for the frog population to quadruple (reach 1600) from its intitial value of 400?

When $P = 1600$, $(1600)^{1/2} = 2t + 20$, so $t = 10$ years.

(15 points) 4. Consider

$$f(x) = x + \frac{1}{x}\ .$$

Sketch the graph of $f(x)$, indicating any symmetry and asymptotes. Find where the function is increasing and decreasing, and where it is concave up and concave down. Indicate all local and global extrema, and any inflection points. Be sure to show all your work.

As can be seen from the graph, it's an odd function, meaning it's symmetric with respect to a 180 degree rotation about the origin. As $x \to 0$ the graph is asymptotic to the vertical line $x = 0$, approaching ∞ from the right and $-\infty$ from the left. As $x \to \pm\infty$ the graph is asymptotic to the line $y = x$.

The function, first derivative, and second derivative are:

$$f(x) = x + \frac{1}{x}$$
$$f'(x) = 1 - \frac{1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$

The derivative is positive on the intervals: $(-\infty, -1), (1, \infty)$. So, the function is increasing on these intervals.

The derivative is negative on the intervals: $(-1, 0), (0, 1)$. So, the function is decreasing on these intervals.

The derivative is 0 when $x = \pm 1$. When $x = -1$ the second derivative is negative, so there is a local maximum at that point. When $x = 1$ the second derivative is positive, so there is a local minimum at that point.

As $x \to -\infty$ $f(x) \to -\infty$, while as $x \to \infty$ $f(x) \to \infty$. So, there are no global maxima or minima.

The second derivative is negative when $x < 0$ and positive when $x > 0$, so the function is concave down for $x < 0$ and concave up for $x > 0$.

Note that there are no points at which $f''(x) = 0$ and while it is the case that $f(x)$ is concave down for $x < 0$ and concave up for $x > 0$ the point $x = 0$ is not an inflection point, as $f(0)$ is not defined. So, there are no inflection points.

(15 points) 5. Find the dimensions of the rectangle with maximum area inscribed in a circle of unit radius. Be sure to show all your work, and verify your result with the second derivative test.

If we inscribe the rectangle in the circle as indicated in the picture above, then the area of the rectangle is:

$$A = height \times width = (2x)(2y) = 4xy$$

Now, x and y are related by the equation for the unit circle:

$$x^2 + y^2 = 1$$
$$\rightarrow y = \sqrt{1 - x^2}$$

And so, we get the area equation:

$$A = 4xy = 4x\sqrt{1 - x^2}$$

We want to find the value of x that maximizes this area. We know this value will occur at a critical point. Now, x can range between 0 and 1, but we can see that for both of these values the area is 0. So, the maximum must be a point at which the derivative is 0.

We take the derivative and solve for where it's 0:

$$A(x) = 4x\sqrt{1-x^2}$$

$$A'(x) = 4\sqrt{1-x^2} - \frac{4x^2}{\sqrt{1-x^2}}$$

Setting this equal to 0 and solving for x we find:

$$0 = 4\sqrt{1-x^2} - \frac{4x^2}{\sqrt{1-x^2}}$$

Multiplying both sides by $\sqrt{1-x^2}$ and dividing both sides by 4:

$0 = 1 - x^2 - x^2 \rightarrow 2x^2 = 1 \rightarrow x = \dfrac{1}{\sqrt{2}}$. So, we get $x = \dfrac{1}{\sqrt{2}}$ and given $y = \sqrt{1-x^2}$ we get $y = \dfrac{1}{\sqrt{2}}$ also.

We want to use the second derivative test to determine that this critical point is indeed a minimum. Taking the second derivative of $A(x)$ we get:

$$A''(x) = \frac{-4x}{\sqrt{1-x^2}} - \frac{4x^3}{(1-x^2)^{\frac{3}{2}}} - \frac{8x}{\sqrt{1-x^2}} = \frac{-12x}{\sqrt{1-x^2}} - \frac{4x^3}{(1-x^2)^{\frac{3}{2}}}$$

So, at $x = \frac{1}{\sqrt{2}}$ we get:

$$A''(\frac{1}{\sqrt{2}}) = -16 < 0.$$

So, the second derivative at $x = \dfrac{1}{\sqrt{2}}$ is negative, and therefore $A(x)$ is a local maximum at that point. Given this is the only critical point outside the endpoints, we conclude this is the global maximum.

The dimensions of the square will then be: $(2x, 2y) = (2\frac{1}{\sqrt{2}}, 2\frac{1}{\sqrt{2}}) = (\sqrt{2}, \sqrt{2})$.

(15 points) 6. Calculate $\int_0^1 x\ dx$ from the **definition** of the integral – using a Riemann sum. Be sure to show all your work. Hint: use the fact that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} .$$

Check your result using the **fundamental theorem of calculus**.

$\int_0^1 xdx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}(\frac{i}{n}) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}$. Using the fundamental theorem of calculus, $\int_0^1 xdx = \frac{1}{2}x^2|_0^1 = \frac{1}{2}$.

(10 points) 7. Calculate the following integrals:

(a) $\int_1^2 (3x^2 + x + 1)\ dx = x^3 + \frac{1}{2}x^2 + x|_1^2 = 12 - \frac{5}{2} = \frac{19}{2}$

(b) $\int \sin^2 x \ \ \cos x\ \ dx = \int \sin^2 x \cos x dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C$. We used the u-substitution $u = \sin x$, $du = \cos x dx$