

Supplemental Material: Percolation Theory for Sea Ice

Consider the classical lattice percolation model described in section 3. Here we use this model to relate electrical transport to fluid flow through sea ice, and obtain a percolation theory prediction for the vertical component of the electrical conductivity of sea ice as a function of brine volume fraction. At the end of this section we demonstrate that the four probe Wenner method we developed to obtain the Antarctic conductivity data gives results very close to a parallel plate experiment, thus establishing this simple technique as a quick, viable field method for measuring sea ice conductivity.

Let $\sigma^*(p)$ be the effective conductivity of the square or cubic lattice percolation model in the vertical direction [Stauffer and Aharony, 1992]. For $p < p_c$, $\sigma^*(p) = 0$. For $p > p_c$ and near p_c , $\sigma^*(p)$ exhibits power law behavior,

$$\sigma^*(p) \sim \sigma_0(p - p_c)^t, \quad p \rightarrow p_c^+, \quad (3)$$

where t is the conductivity critical exponent. For lattices, t is believed to be universal, depending only on d . In $d = 2$, $t \approx 1.3$, and in $d = 3$, $t \approx 2.0$ [Stauffer and Aharony, 1992]. There is also a rigorous bound for an idealized model of the percolation cluster [Golden, 1990] that $1 \leq t \leq 2$ in $d = 2$ and $d = 3$. Since $\sigma^*(p) \rightarrow 0$ as $p \rightarrow p_c^+$, the effective resistivity $\rho^*(p) = 1/\sigma^*(p)$ diverges as $p \rightarrow p_c^+$, with a vertical asymptote at $p = p_c$, as shown in Figure 4 b. For two phase composites with finite component resistivities, like sea ice, the behavior only approximates the asymptote, and for $p < p_c$, ρ^* remains finite.

The fluid permeability $\kappa^*(p)$ corresponding to (1), where the open bonds are pipes of fluid conductivity $\kappa_0/\eta = r_0^2/8\eta$ and radius r_0 , behaves like $\kappa^*(p) \sim \kappa_0(p - p_c)^e$ as $p \rightarrow p_c^+$, with e the fluid permeability exponent and η the fluid viscosity. For lattices, it is believed [Stauffer

and Aharony, 1992] that $e = t$. In the continuum, the exponents e and t can take non-universal values, and need not be equal, such as for the three dimensional Swiss cheese model [Halperin et al., 1985; Stauffer and Aharony, 1992]. However, the lognormal distribution of brine inclusion cross-sectional areas in sea ice leads to *universal* behavior [Golden et al., 2007; Berkowitz and Balberg, 1992]. Thus for sea ice, $t = e \approx 2$.

It is interesting to note that sea ice is a continuum whose percolation threshold of 0.05 can be explained with a *continuum* percolation model, known as a “compressed powder.” Surprisingly, the lognormally distributed brine inclusions give rise to universal, *lattice* critical behavior for transport near the percolation threshold.

In order to use percolation theory to quantitatively describe the vertical conductivity $\sigma_v^*(\phi)$, and to provide a link between fluid and electrical transport in sea ice, we recall our result [Golden et al., 2007] for the vertical fluid permeability

$$k_v^*(\phi) \sim 3 (\phi - \phi_c)^2 \times 10^{-8} \text{ m}^2, \quad \phi \rightarrow \phi_c^+. \quad (4)$$

The scaling factor $k_0 = 3 \times 10^{-8}$ is estimated using critical path analysis [Stauffer and Aharony, 1992; Friedman and Seaton, 1998]. The effective behavior of media with a broad range of local conductances is dominated by a critical *bottleneck* conductance related to the minimal radius in a connected pathway of appropriate scale. To relate σ_v^* to k_v^* , we use the following relation from critical path analysis [Friedman and Seaton, 1998]. With r_c denoting the critical radius for our centimeter scale electrical experiments, then

$$k_v^* = \frac{r_c^2}{8} \frac{\sigma_v^*}{\sigma_b}, \quad (5)$$

where σ_b is the conductivity of brine, which depends [Stogryn and Desargant, 1985] on temperature T . By measuring the radii of vertical pathways in X-ray tomography images [Golden et al., 2007; Pringle et al., 2009], we estimate a range in mm of $0.1 \leq r_c \leq 0.2$.

It is useful to consider the vertical conductivity formation factor $F = \sigma_v^*/\sigma_b$, which removes the dependence of the effective parameter on the changing conductivity of the brine, and depends only on the pore volume fraction and geometry. In view of (3) and (5),

$$F(\phi) \sim F_0 (\phi - \phi_c)^2, \quad \phi \rightarrow \phi_c^+, \quad F_0 = \frac{8k_0}{r_c^2}. \quad (6)$$

The estimates of 0.1 mm to 0.2 mm for r_c yield a range for F_0 of $6 \leq F_0 \leq 24$.

Now we demonstrate that our adapted Wenner array method is a viable field technique for measuring the vertical conductivity of sea ice. Plate electrodes in contact with the ends of a cylinder generate parallel field lines which make measuring the conductivity of the cylinder material relatively straightforward, as illustrated in Figure 5 a. To assess the accuracy of our four probe method, the commercial package Comsol 3.5a was used to create a finite element model of cylindrical sea ice cores 0.09 m in diameter and 0.5 m in length. Four metal probes of 0.004 m in diameter and 0.09 m in length were inserted approximately 0.07 m into the core, similar to Figure 1 b. When the current is injected through the outer probes instead of parallel plates, as in Figure 5 b, the nearby field lines show significant curvature. However, in the boxed measurement region in Figure 5 b where the inner probes are located, the field lines are relatively straight, thus minimizing the error between the actual conductivity of the material and what is measured by the array. Numerical simulations show that if the outer probes are 5 cm or more from the inner measurement region, this error is less than 8.5%, and is less than 1.5% if the distance is 10 cm or more, as for much of our data.

375 When extracting a sea ice core to measure its properties, loss of brine is a principal concern.
376 However, for our experiments we did not see any evidence of significant brine loss during the
377 relatively short measurement periods with air temperatures ranging from about -6° C to -18°
378 C (with most below -9° C). Moreover, the probes are inserted deep into the core, minimizing
379 contact with potential brine surface films. Our numerical simulations and these observations
380 establish the Wenner array as a viable field method for *direct* resistivity measurements.

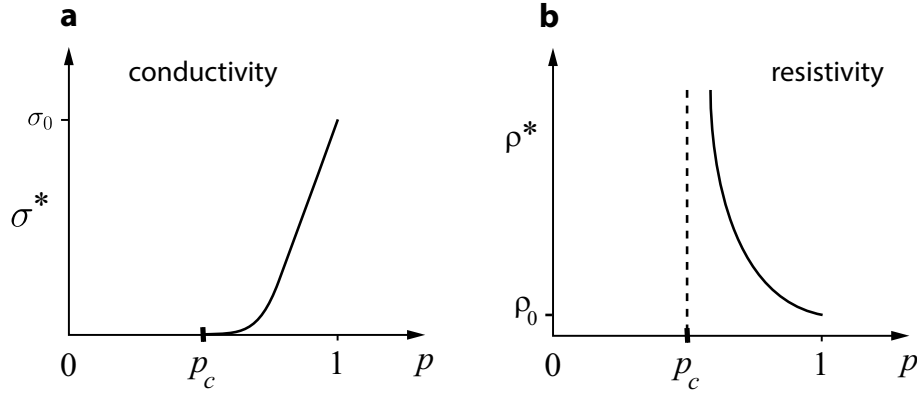


Figure 4. For the infinite, two dimensional square lattice there is no bulk transport below its percolation threshold p_c with the effective conductivity $\sigma^* = 0$ for $p \leq p_c$, while $\sigma^*(p)$ increases with power law behavior just above p_c , as shown in (a). In (b) the corresponding effective resistivity $\rho^* = 1/\sigma^*$ diverges as $p \rightarrow p_c^+$ with a vertical asymptote at $p = p_c$.

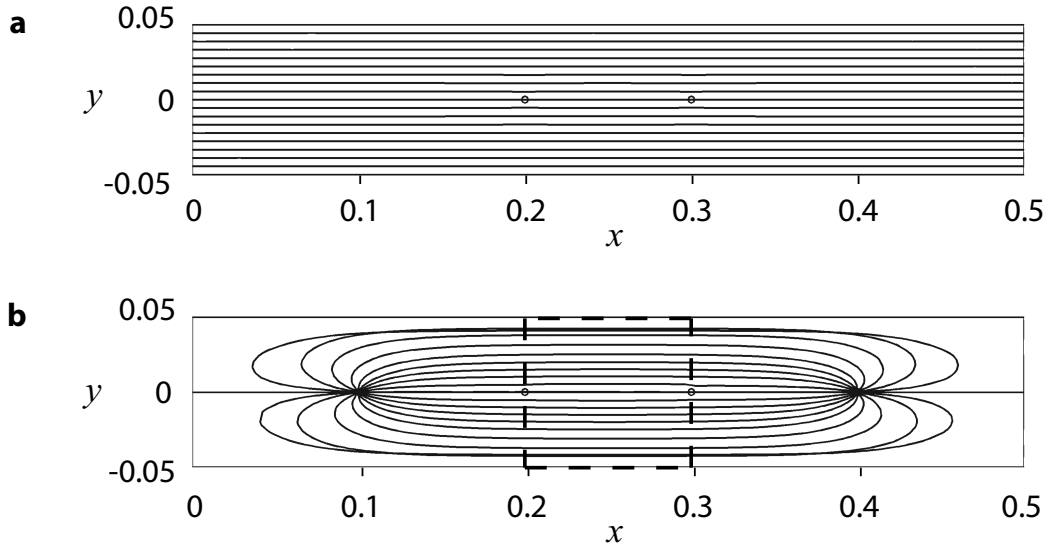


Figure 5. Comparison of field lines for a parallel plate configuration in (a) with those for a four probe Wenner array in (b), where the region of interest is inside the dotted box.