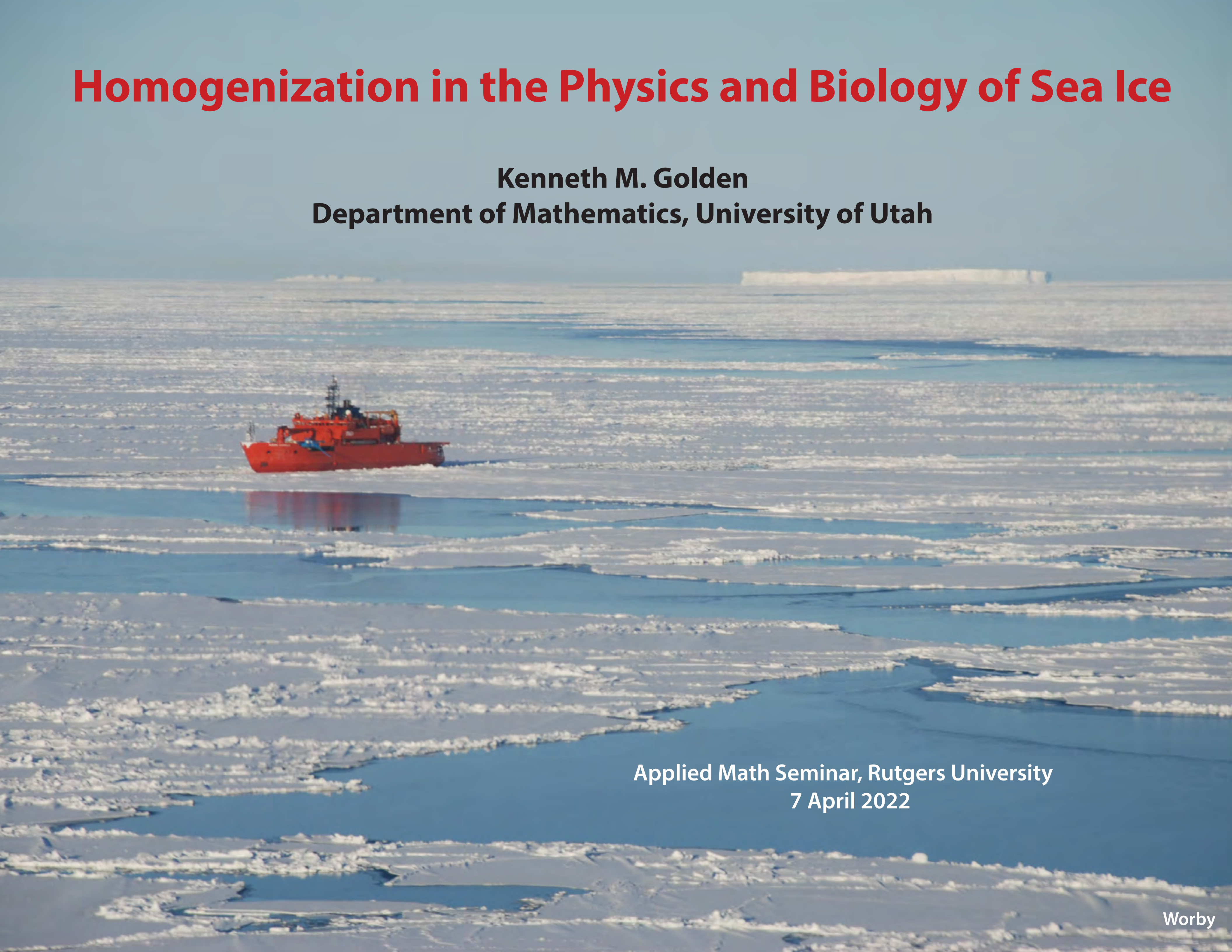
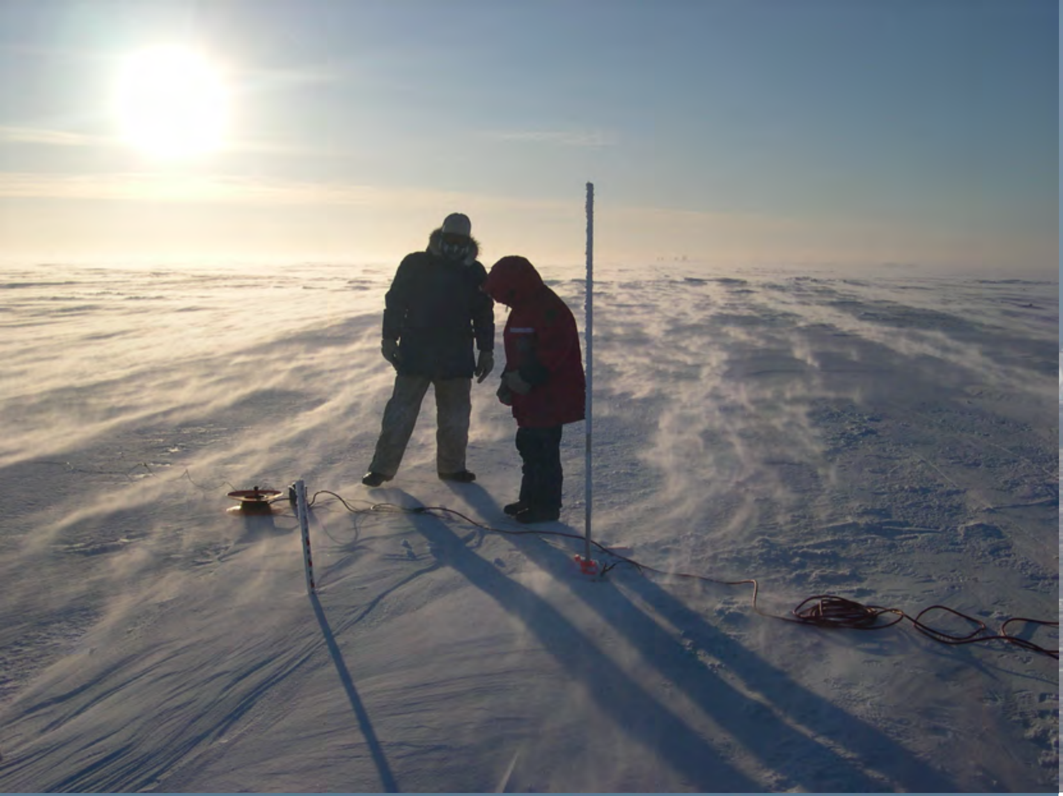
Homogenization in the Physics and Biology of Sea Ice

Kenneth M. Golden
Department of Mathematics, University of Utah



Applied Math Seminar, Rutgers University
7 April 2022

Worby

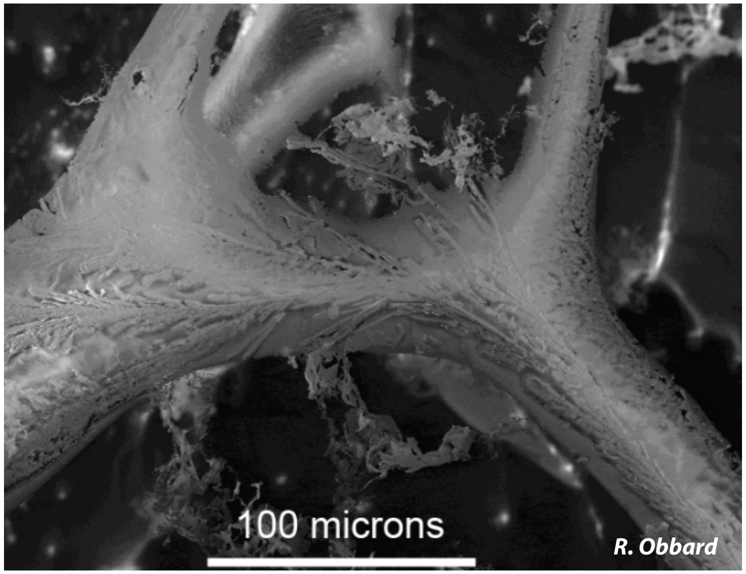


sea ice may appear to be a barren, impermeable cap ...

Golden


Weeks & Assur

brine inclusions in sea ice (mm)



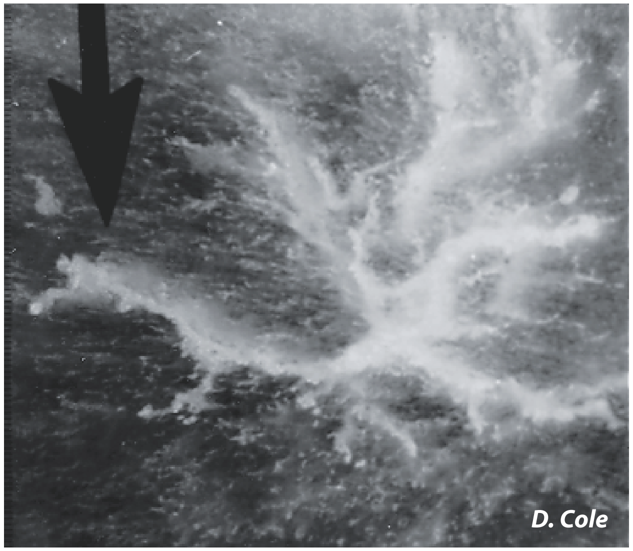
R. Obbard

micro - brine channel (SEM)

sea ice is a porous composite

pure ice with brine, air, and salt inclusions

brine channels (cm)


D. Cole

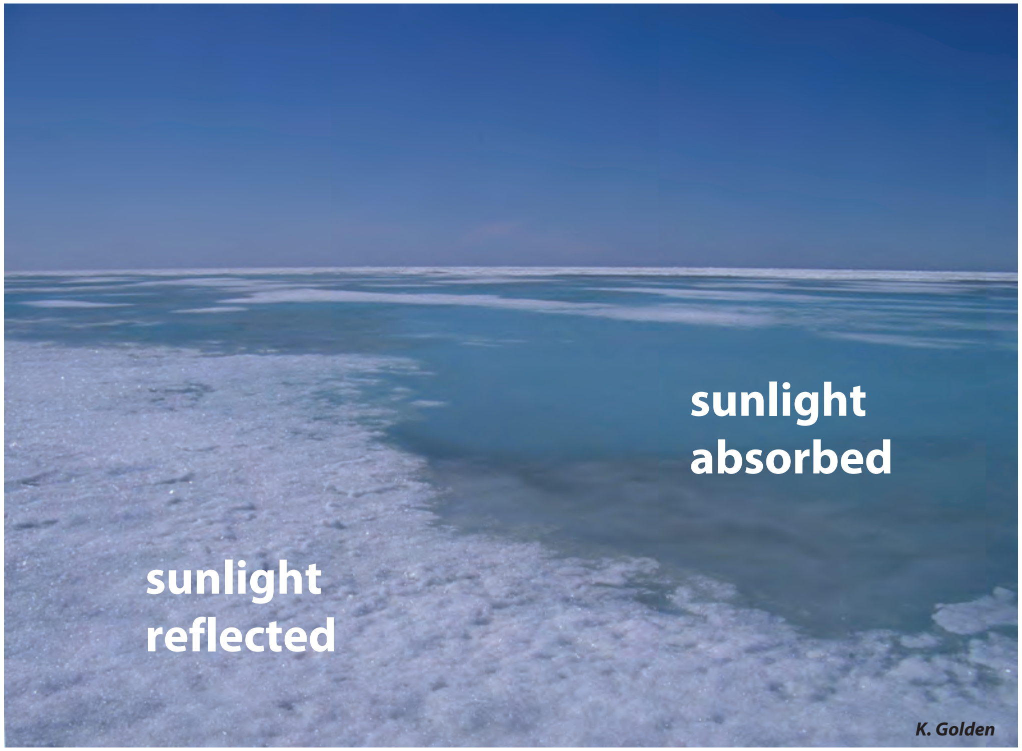
horizontal section


Golden

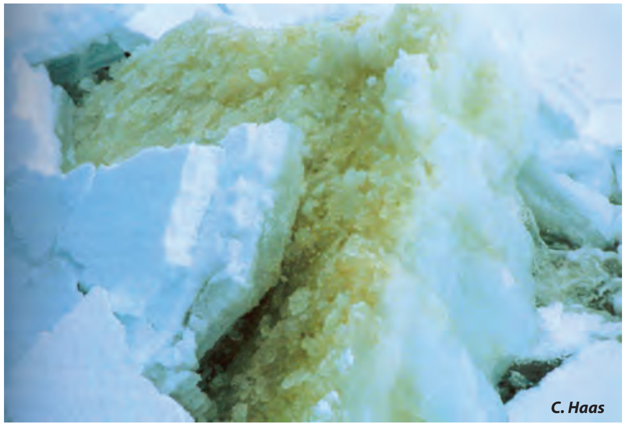
vertical section

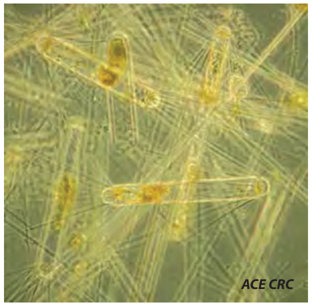
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

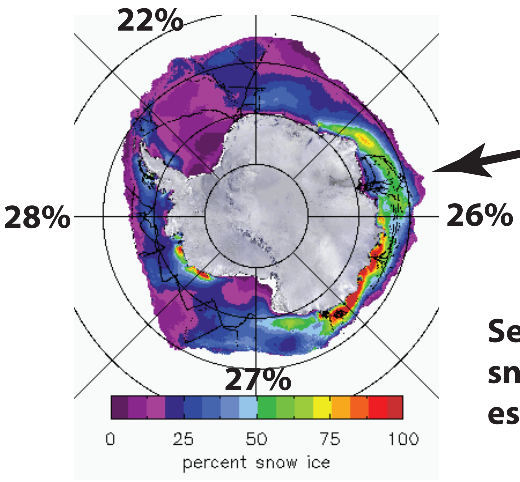
evolution of Arctic melt ponds and sea ice albedo



nutrient flux for algal communities







T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

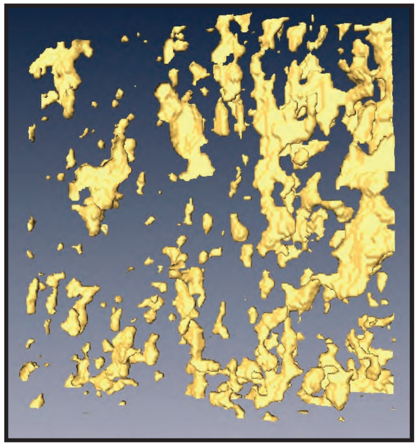
- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

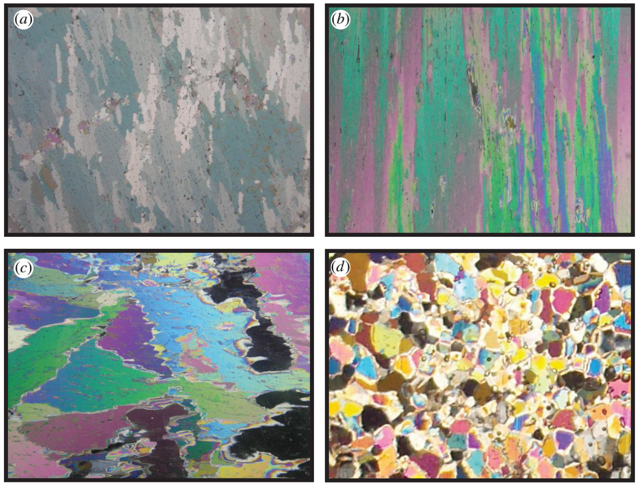
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

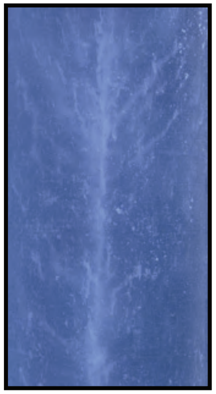
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

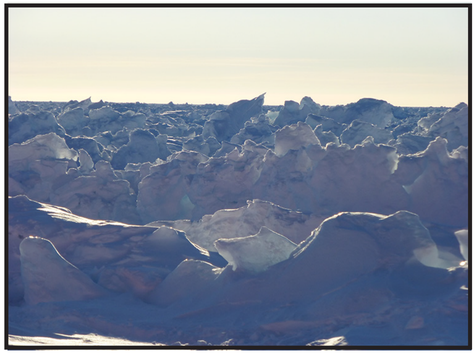
centimeters

mesoscale

Arctic melt ponds

K. Frey

Antarctic pressure ridges



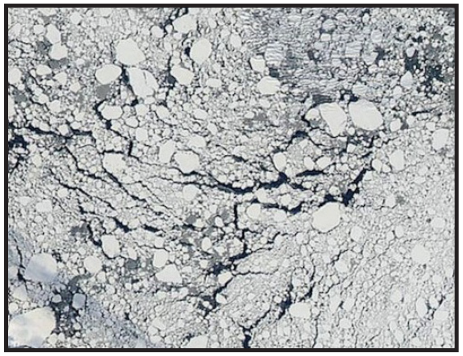
K. Golden

macroscale

sea ice floes

J. Weller

sea ice pack

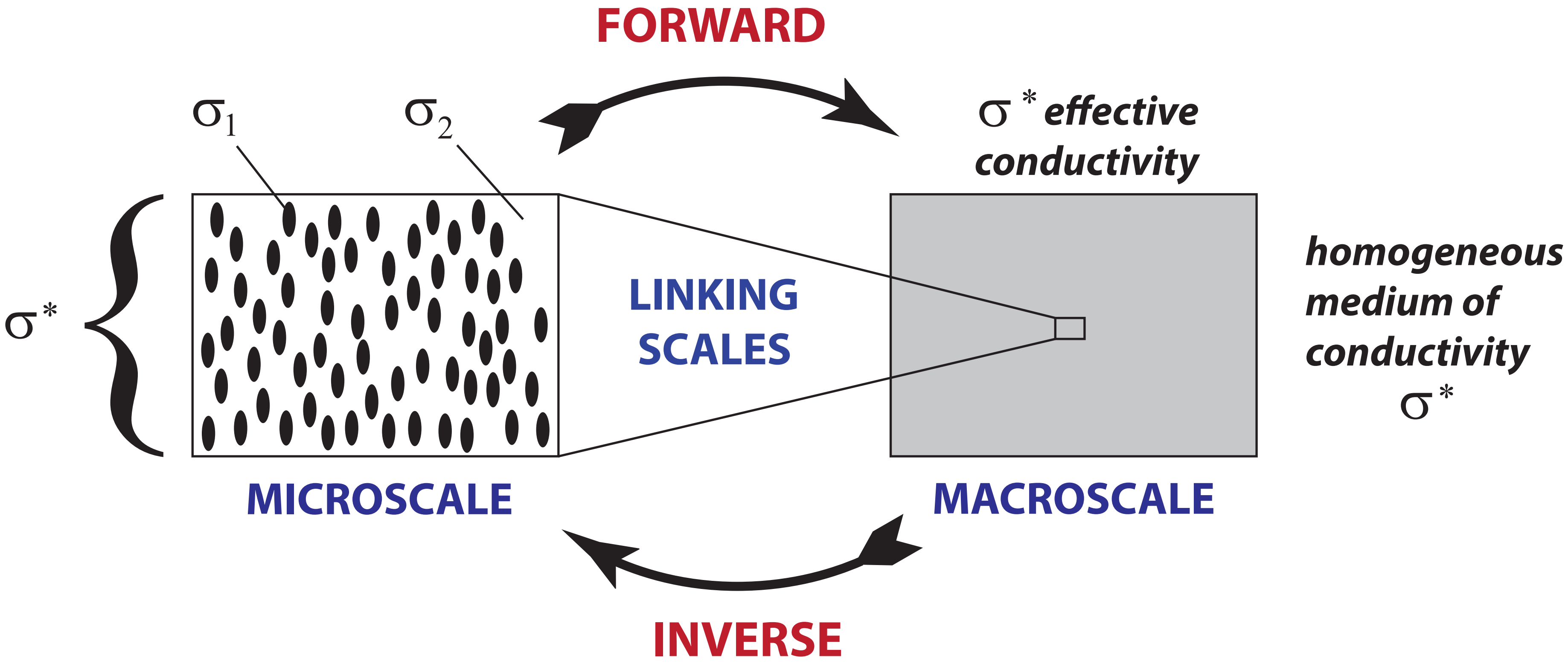


NASA

meters

kilometers

HOMOGENIZATION for Composite Materials



Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*
*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

What is this talk about?

Using methods of **homogenization and statistical physics** to model sea ice effective behavior and advance representation of sea ice in climate models, process studies, ...

MODELING SEA ICE

microscale

↓

mesoscale

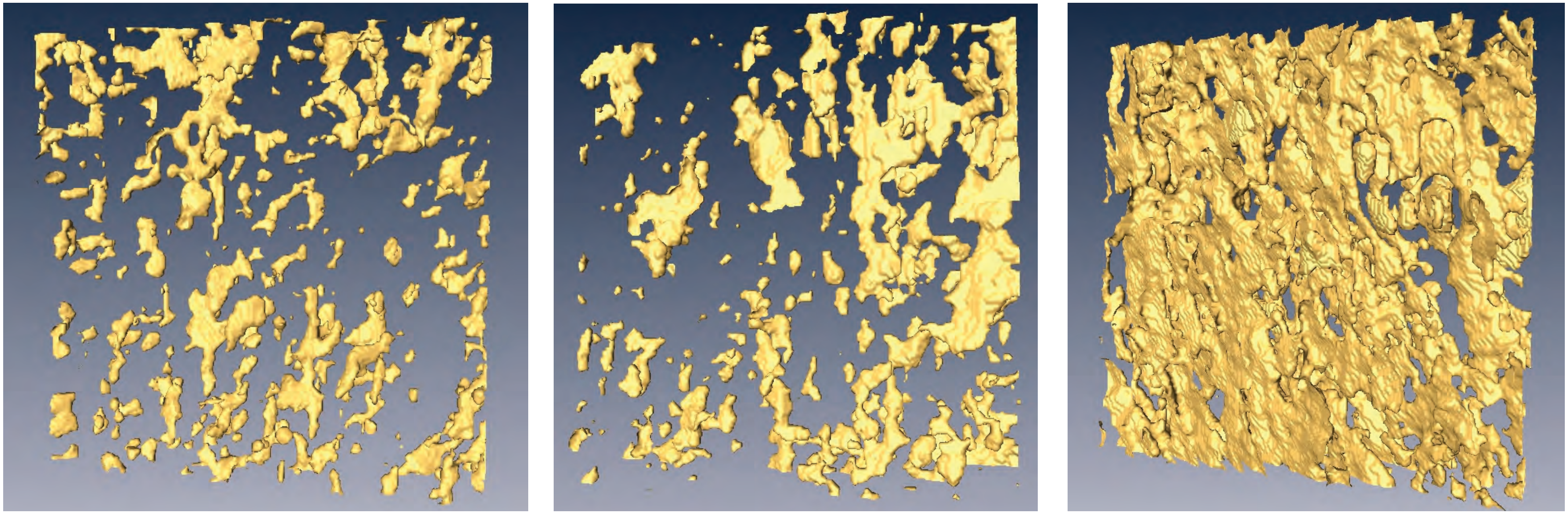
↓

macroscale

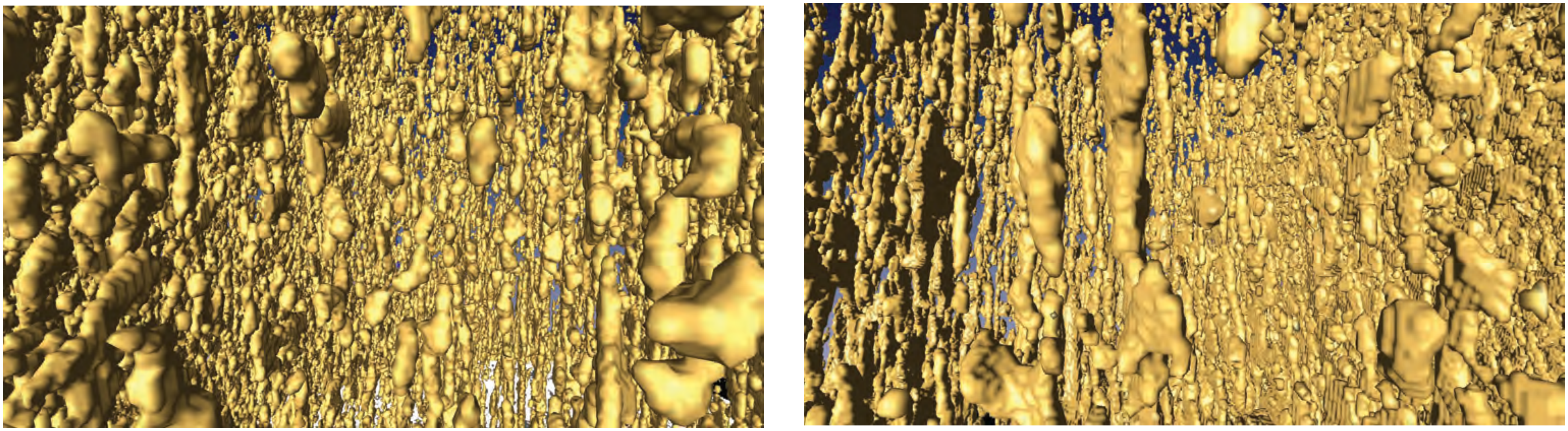
A tour of key sea ice processes on micro, meso, and macro scales.

microscale

brine volume fraction and ***connectivity*** increase with temperature



$T = -15\,^\circ\mathrm{C}, \ \phi = 0.033$ $T = -6\,^\circ\mathrm{C}, \ \phi = 0.075$ $T = -3\,^\circ\mathrm{C}, \ \phi = 0.143$



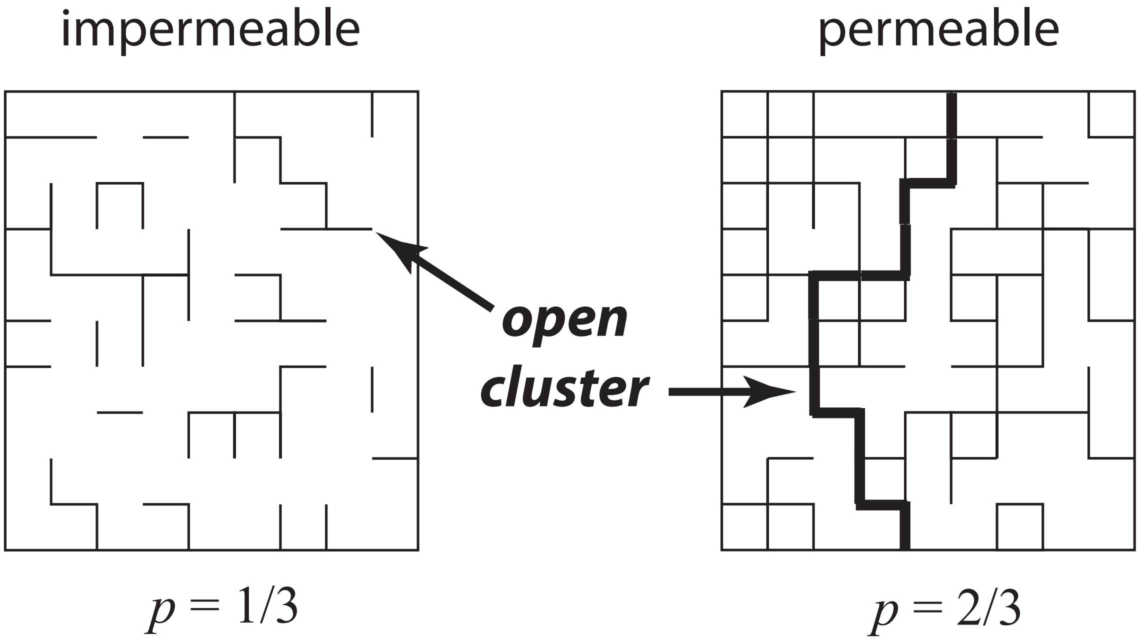
$T = -8^\circ\mathrm{C}, \ \phi = 0.057$ $T = -4^\circ\mathrm{C}, \ \phi = 0.113$

X-ray tomography for brine in sea ice

Golden *et al.*, *Geophysical Research Letters*, 2007

percolation theory

probabilistic theory of connectedness



bond → ***open*** *with probability* p
closed *with probability* $1-p$

percolation threshold

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

smallest *p* for which there is an infinite open cluster

Critical behavior of fluid transport in sea ice

Arctic field data

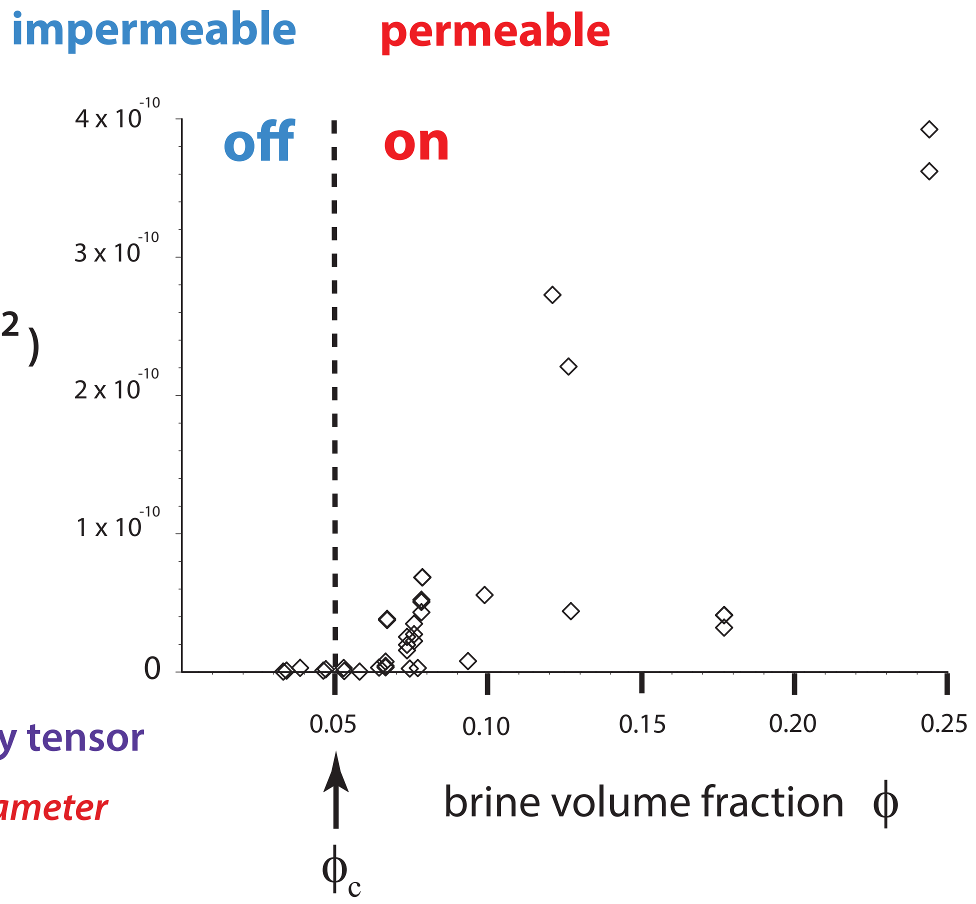
vertical fluid permeability k (m^2)

Darcy's Law

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\nabla p$$

$\mathbf{k}$ = fluid permeability tensor

homogenized parameter



"on - off" switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ\mathrm{C},\ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

sea ice ~ compressed powder in stealthy composites

sea ice algal communities

D. Thomas 2004

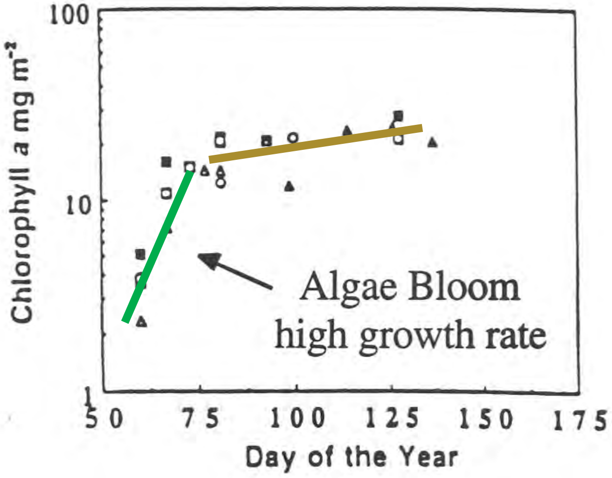
nutrient replenishment
controlled by ice permeability

biological activity turns on
or off according to
rule of fives

Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

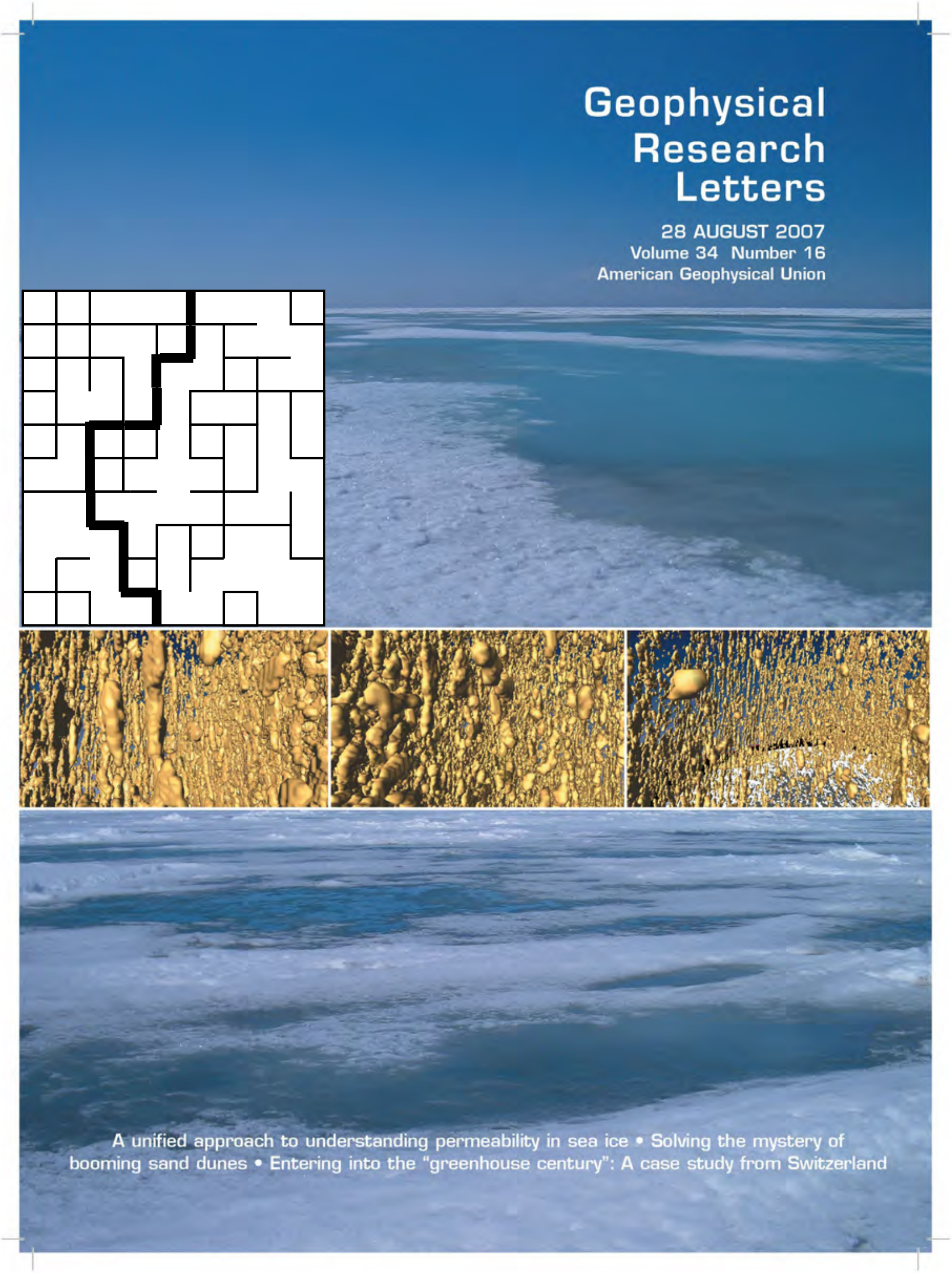
critical behavior of microbial activity



Convection-fueled algae bloom
Ice Station Weddell

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton*, Miner, Pringle, Zhu,   Geophysical Research Letters   2007



Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

microscale

governs

mesoscale
processes

melt pond evolution

percolation theory for fluid permeability

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical
exponent
t

$$k_0 = 3 \times 10^{-8}\ \mathrm{m}^2$$

from critical path analysis
in hopping conduction

hierarchical model

rock physics

network model

rigorous bounds

X-ray tomography for
brine inclusions

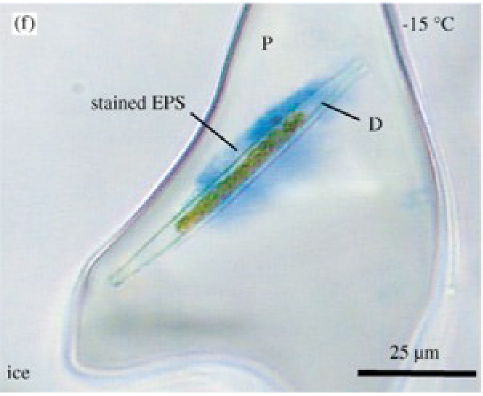
confirms rule of fives

Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

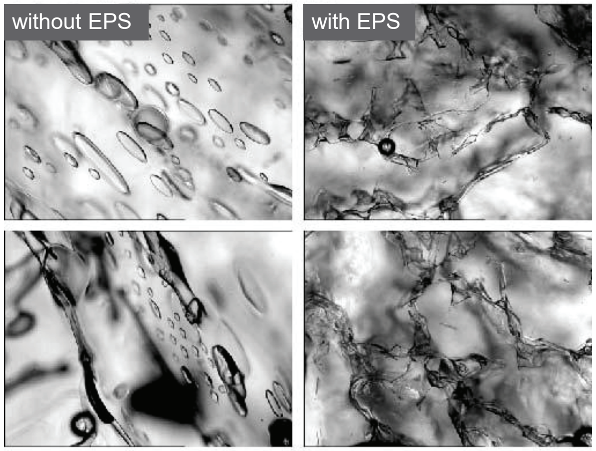
theories agree closely
with field data

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

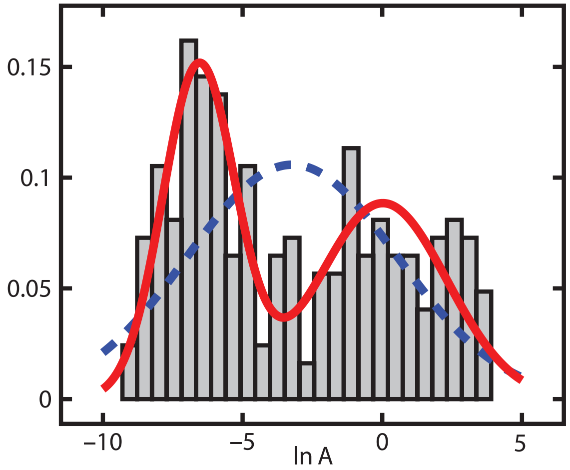
How does EPS affect fluid transport? **How does the biology affect the physics?**



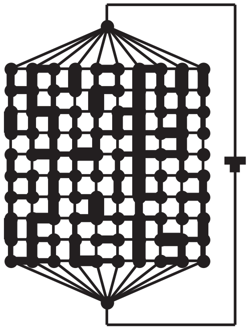
Krembs

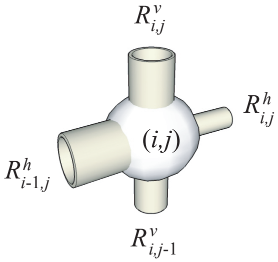


Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL





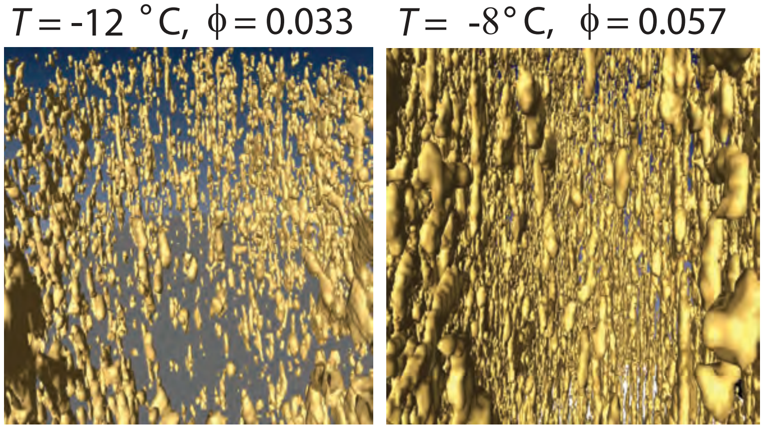
Zhu, Jabini, Golden,
Eicken, Morris
Ann. Glac. 2006

- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability *k;* results predict observed drop in *k*

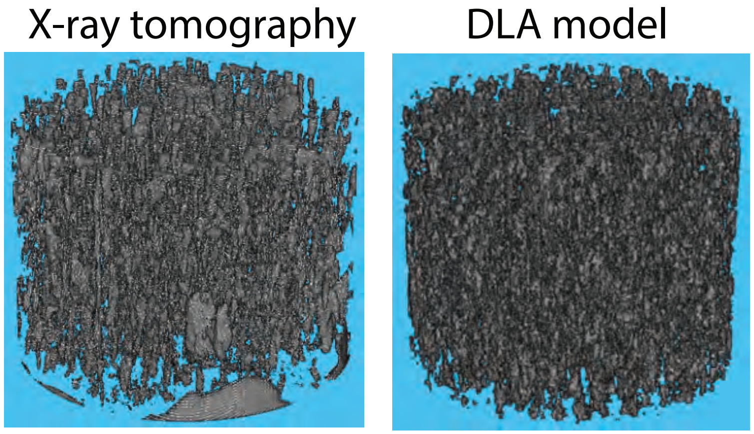
Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

Thermal evolution of the fractal geometry of the brine microstructure in sea ice

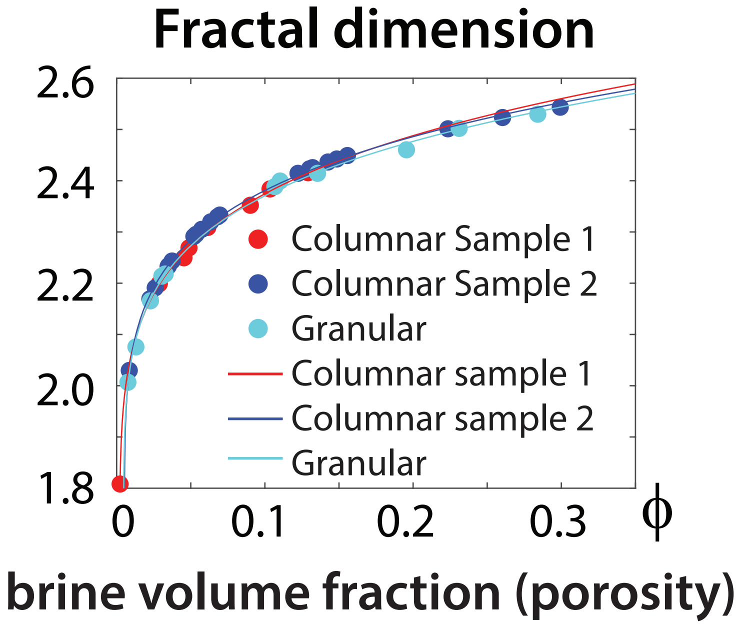
N. Ward, D. Hallman, J. Reimer, H. Eicken, M. Oggier and K. M. Golden, 2022



$T = -12\ ^\circ\mathrm{C},\ \phi = 0.033$ $\quad T = -8^\circ\mathrm{C},\ \phi = 0.057$



X-ray tomography DLA model



theory of porosity as a function of fractal dimension

invert

excellent correspondence with data

Katz and Thompson, *PRL*, 1985

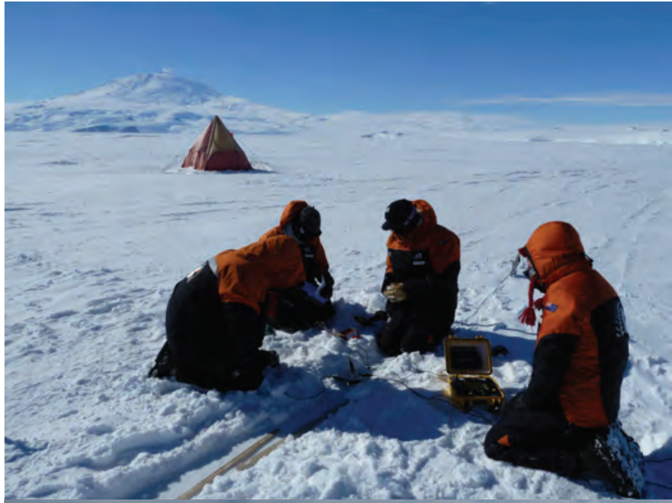
+ implications for brine phase as a habitat

Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	***Antarctic***	***SIPEX***
2010	***Antarctic***	***McMurdo Sound***
2011	***Arctic***	***Barrow AK***
2012	***Arctic***	***Barrow AK***
2012	***Antarctic***	***SIPEX II***
2013	***Arctic***	***Barrow AK***
2014	***Arctic***	***Chukchi Sea***



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Notices

of the American Mathematical Society

May 2009 **Volume 56, Number 5**

Climate Change and the Mathematics of Transport in Sea Ice

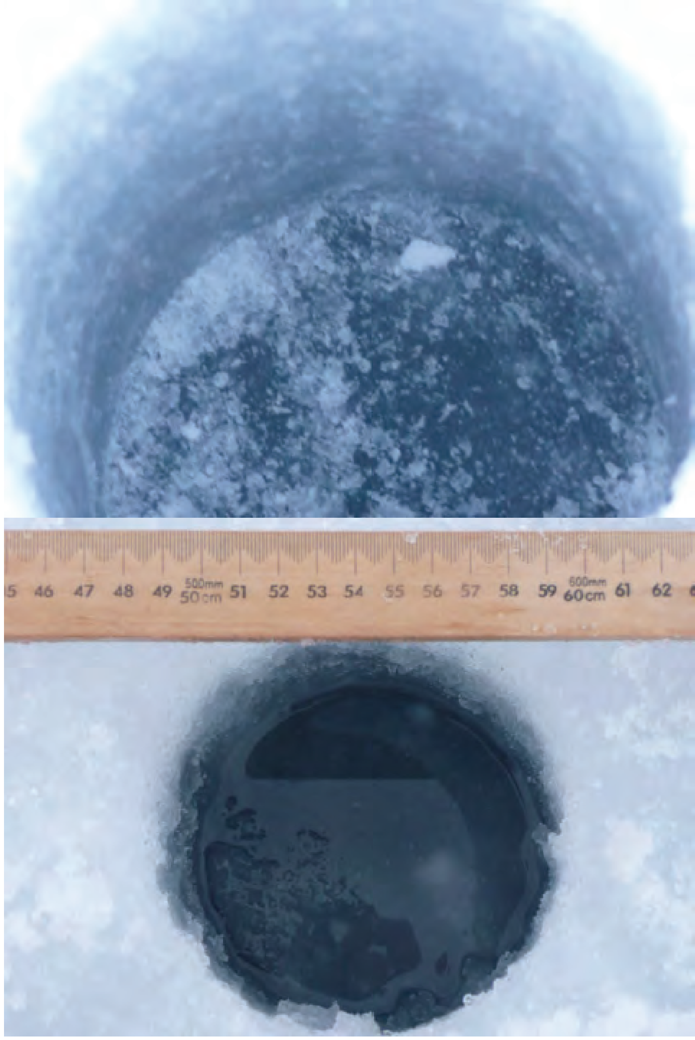
page 562

Mathematics and the Internet: A Source of Enormous Confusion and Great Potential

page 586

Real analysis in polar coordinates (see page 613)

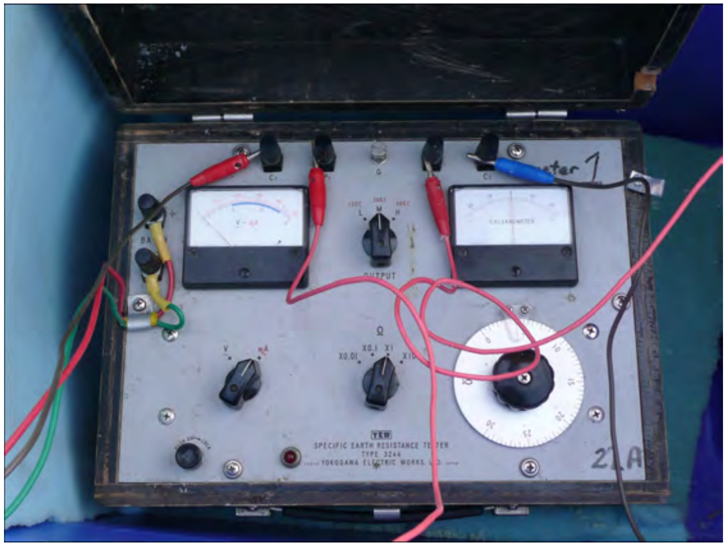
photo by Jan Lieser



measuring
fluid permeability
of Antarctic sea ice

SIPEX 2007

electrical measurements



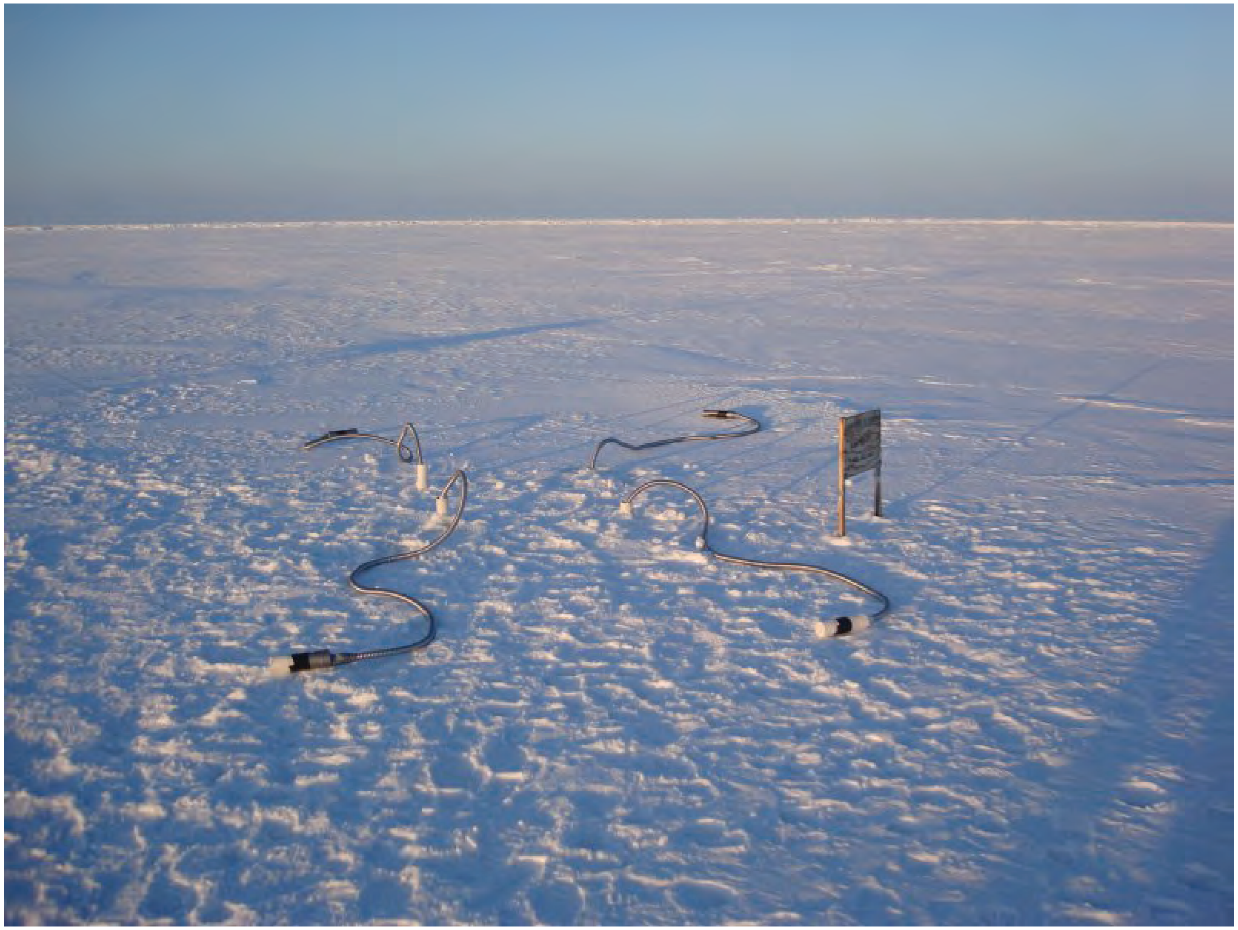
Wenner array

vertical conductivity

Zhu, Golden, Gully, Sampson **_Physica B_** 2010

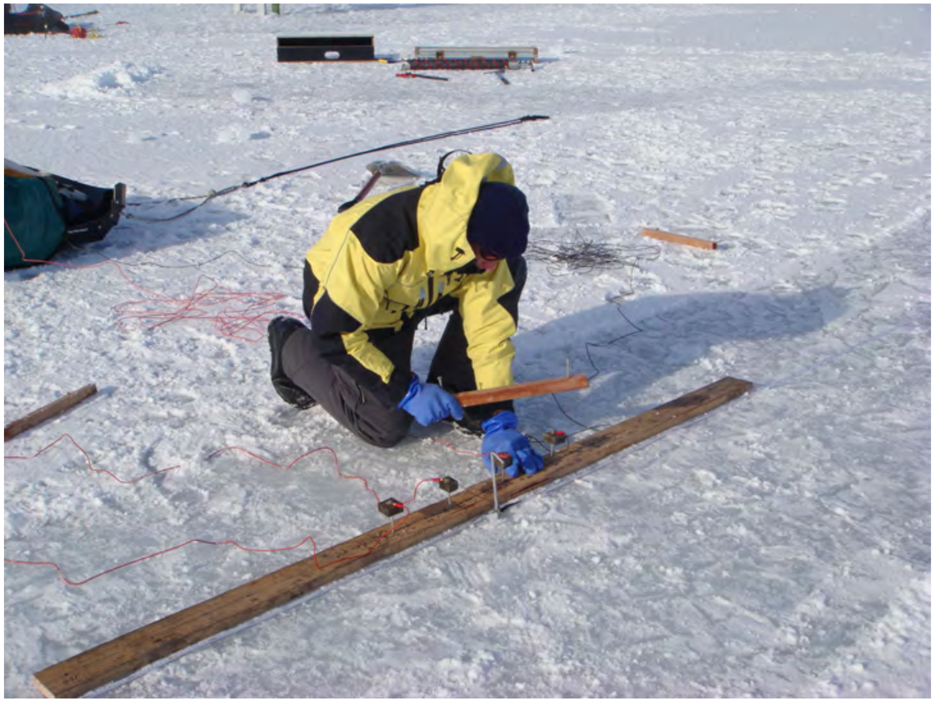
Sampson, Golden, Gully, Worby **_Deep Sea Research_** 2011

cross borehole tomography



Ingham, Jones, Buchanan
Victoria University, Wellington, NZ

Measuring sea ice thickness



Golden

Remote sensing of sea ice

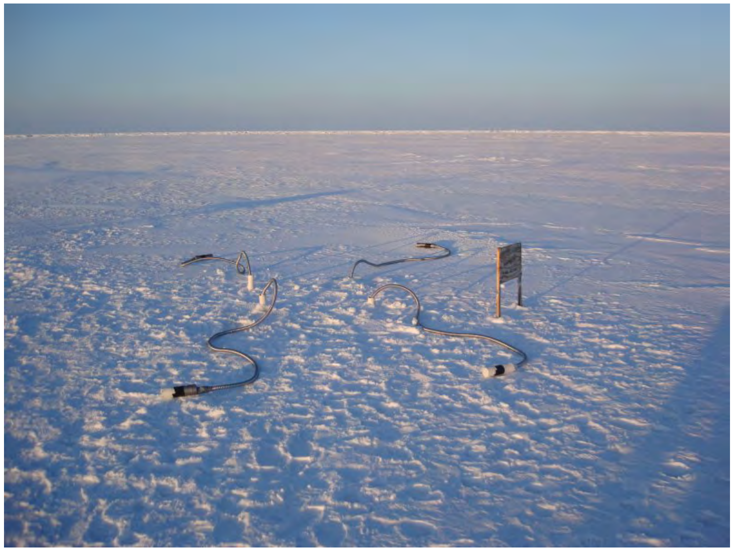
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

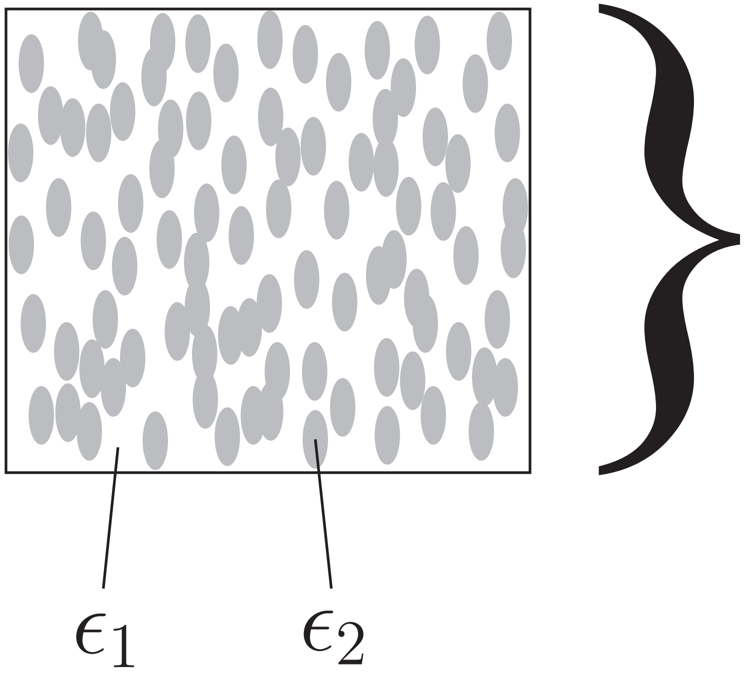
ε^*

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \;, \text{ composite geometry } \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z} \qquad s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

material parameters

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

χ = characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

$\Gamma\chi$: microscale $\longrightarrow$ macroscale

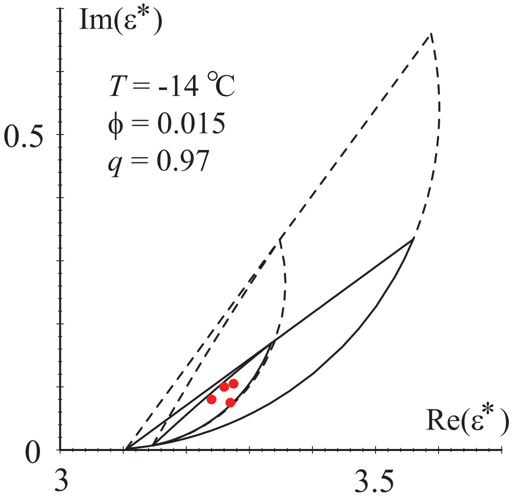
$\Gamma\chi$ links scales

Golden and Papanicolaou, Comm. Math. Phys. 1983

This representation distills the complexities of mixture geometry into the spectral properties of an operator like the Hamiltonian in physics.

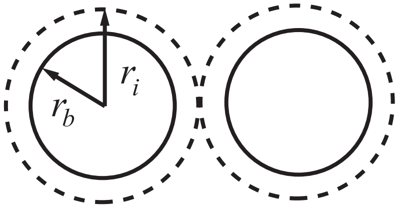
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle

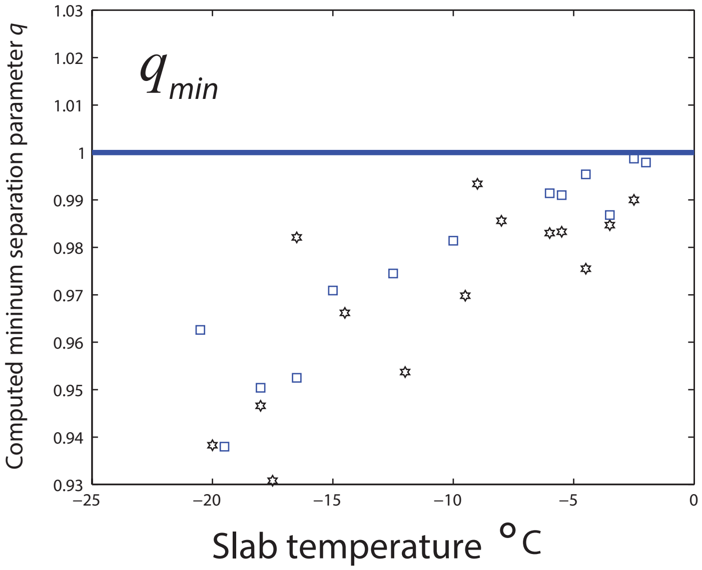


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



Inverse Homogenization

Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001), McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* → ***composite geometry***
(spectral measure μ)

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

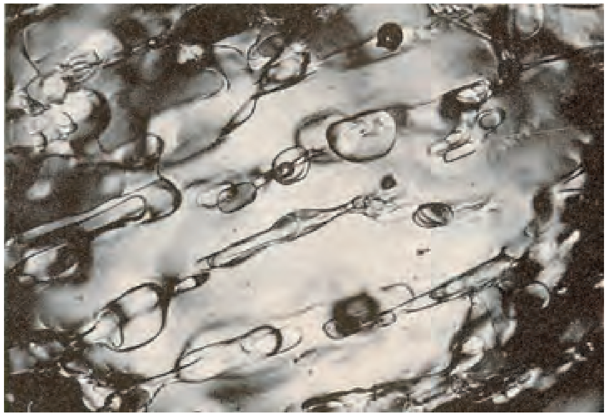
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

rigorous inverse bound on spectral gap

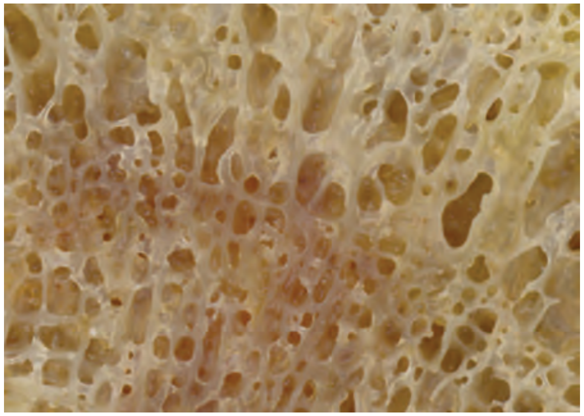
construct algebraic curves which bound admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

SEA ICE

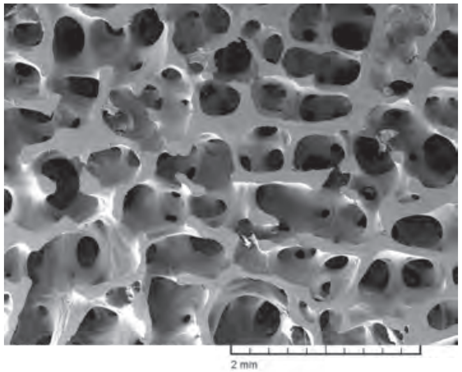


HUMAN BONE

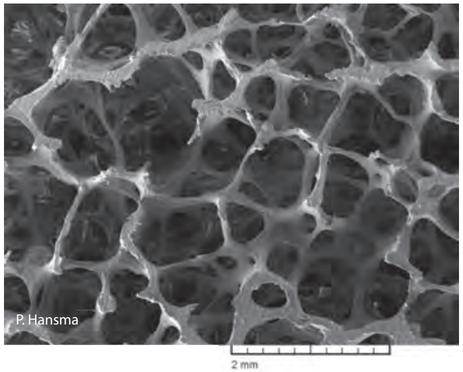


spectral characterization of porous microstructures in human bone

young healthy trabecular bone

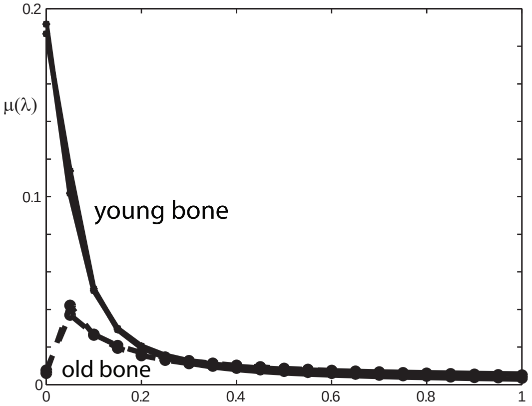


old osteoporotic trabecular bone



P. Hansma

reconstruct spectral measures from complex permittivity data



use regularized inversion scheme

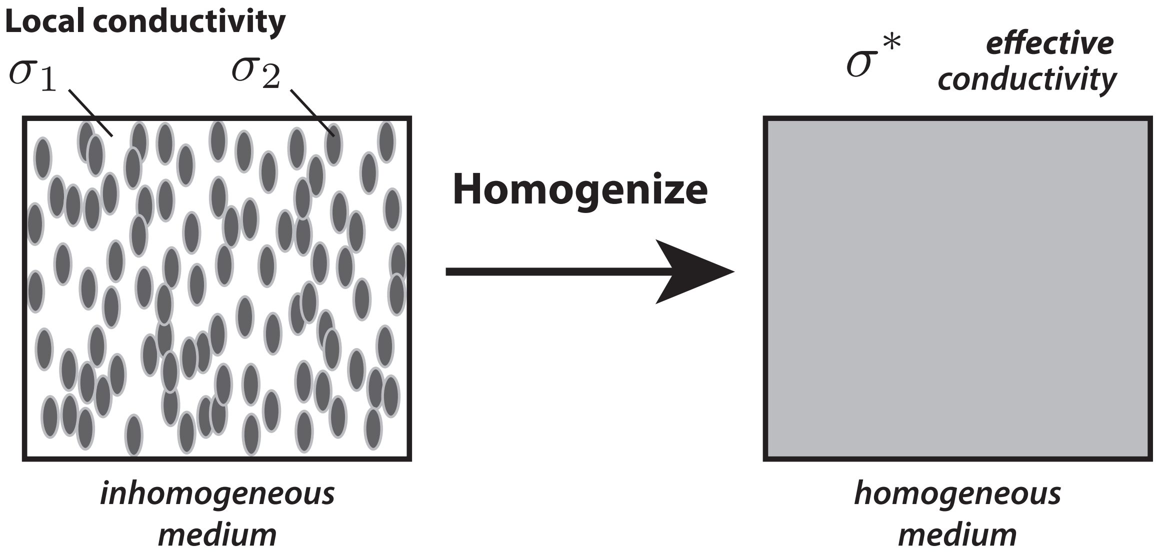
apply spectral measure analysis of brine connectivity and spectral inversion to electromagnetic monitoring of osteoporosis

Golden, Murphy, Cherkaev, J. Biomechanics 2011

the math doesn't care if it's sea ice or bone!

Homogenization for polycrystalline materials

Two-component composites



Local conductivity

σ_1 σ_2

inhomogeneous medium

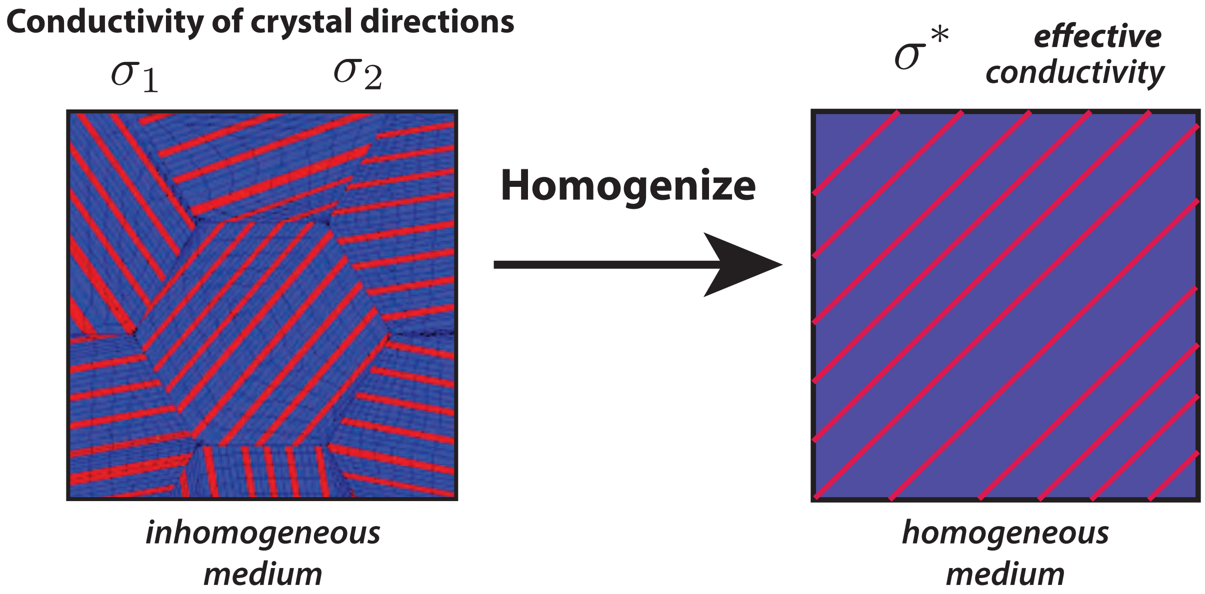
Homogenize

σ^* ***effective* conductivity**

homogeneous medium

Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Polycrystalline media



Conductivity of crystal directions

σ_1 σ_2

inhomogeneous medium

Homogenize

σ^* ***effective* conductivity**

homogeneous medium

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

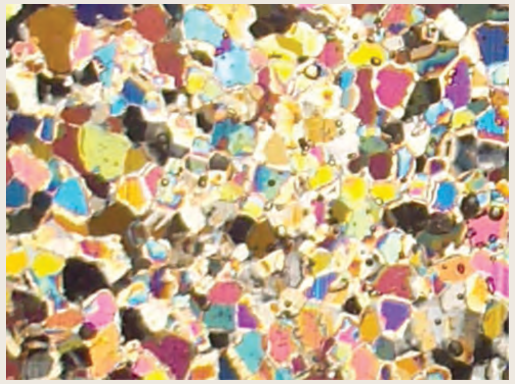
  Milton (1981, 2002), Barabash and Stroud (1999), ...

- **Forward and inverse bounds**

  orientation statistics

- **Applied to sea ice using two-scale homogenization**

- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS OF SCIENTIFIC PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

THE ROYAL SOCIETY PUBLISHING

21/01/15 1:47 PM

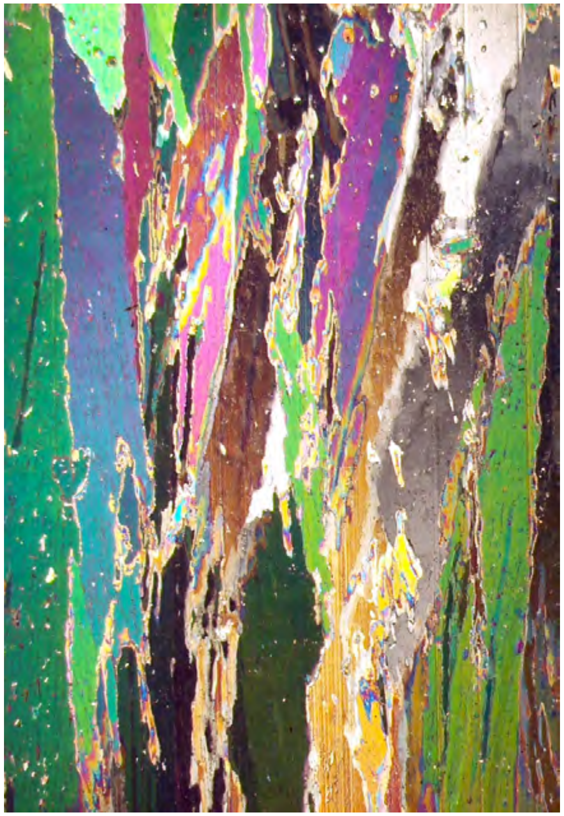
higher threshold for fluid flow in granular sea ice

microscale details impact "mesoscale" processes

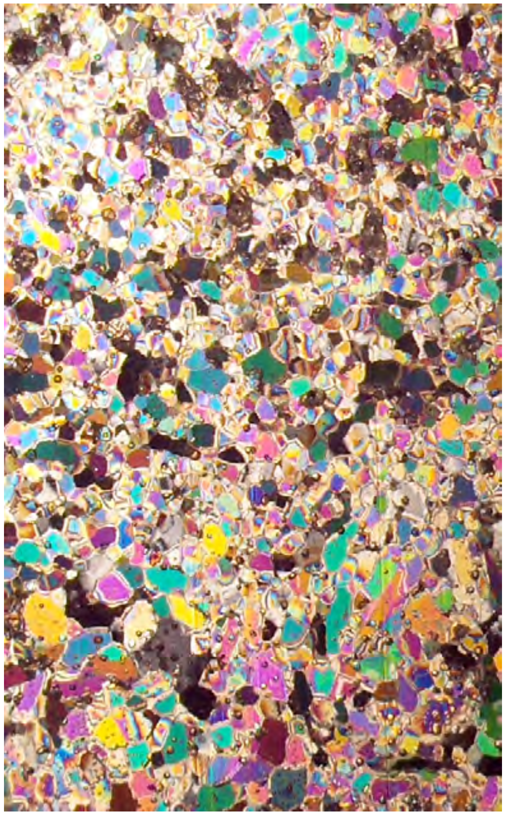
nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar

5%



granular



10%

Golden, Sampson, Gully, Lubbers, Tison 2022

electromagnetically distinguishing ice types
Kitsel Lusted, Elena Cherkaev, Ken Golden

wave propagation in the marginal ice zone (MIZ)

Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Cherkaev, Golden 2022

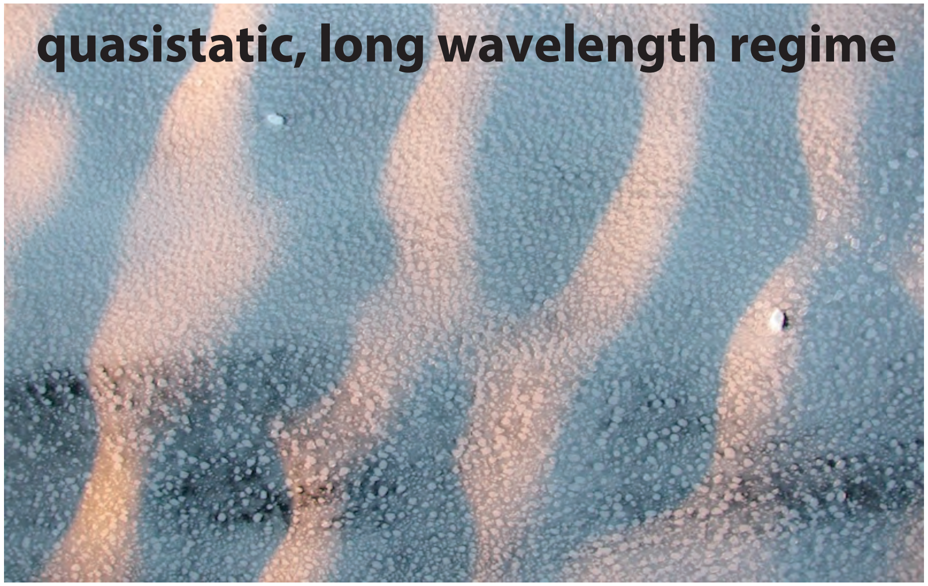
first theory of key parameter in wave-ice interactions only fitted to wave data before

Keller, 1998
Mosig, Montiel, Squire, 2015
Wang, Shen, 2012

Analytic Continuation Method

Bergman (78) - Milton (79)
integral representation for ε^*
Golden and Papanicolaou (83)

Milton, *Theory of Composites* (02)



quasistatic, long wavelength regime

homogenized parameter depends on sea ice concentration and ice floe geometry

like EM waves

direct calculation of spectral measures

Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

once we have the spectral measure μ it can be used in Stieltjes integrals for other transport coefficients:

electrical and thermal conductivity, complex permittivity, magnetic permeability, diffusion, fluid flow properties

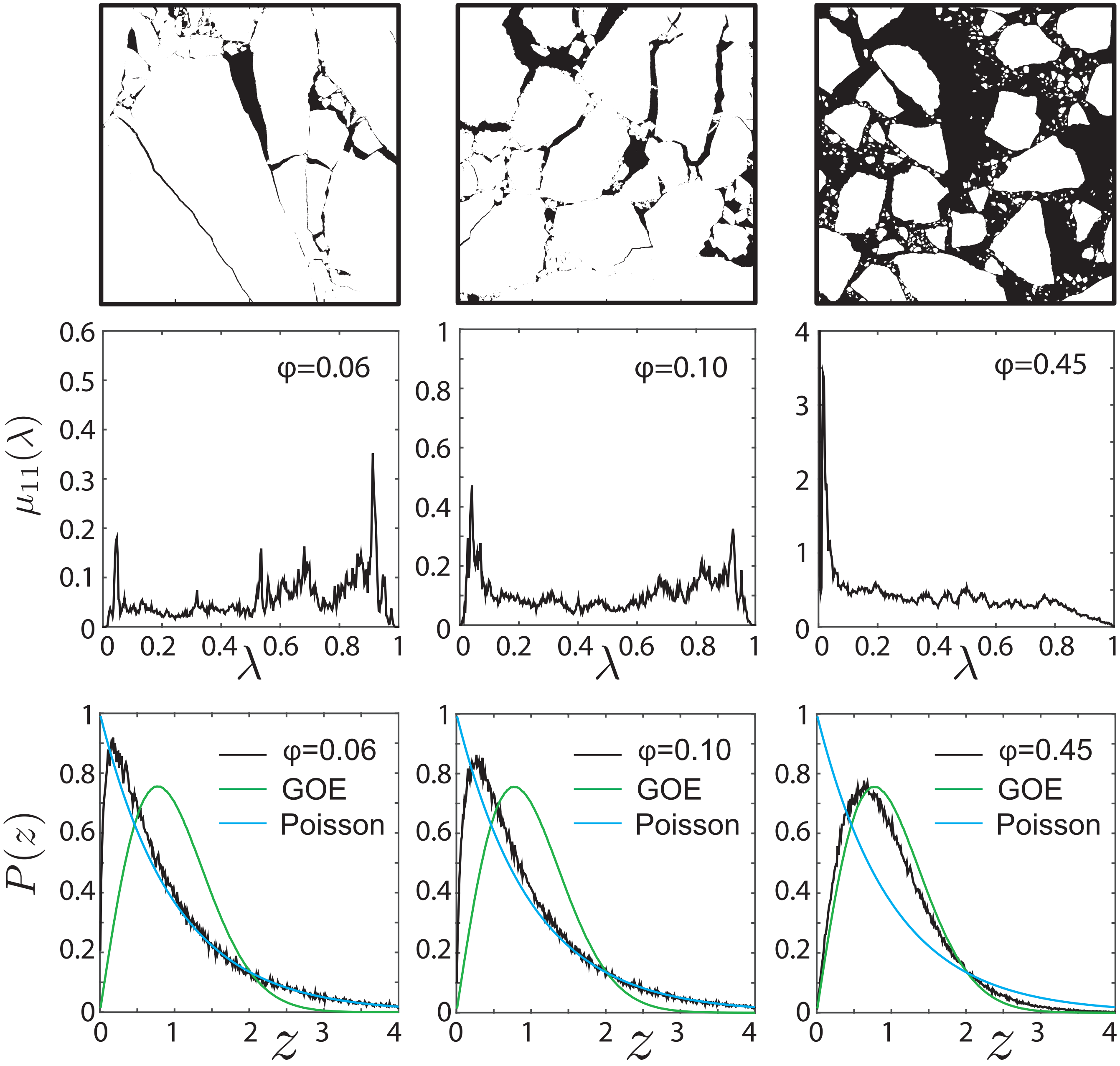
earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral computations for sea ice floe configurations

spectral measures

eigenvalue spacing distributions



UNIVERSAL Wigner-Dyson distribution

uncorrelated → **level repulsion**

Eigenvalue Statistics of Random Matrix Theory

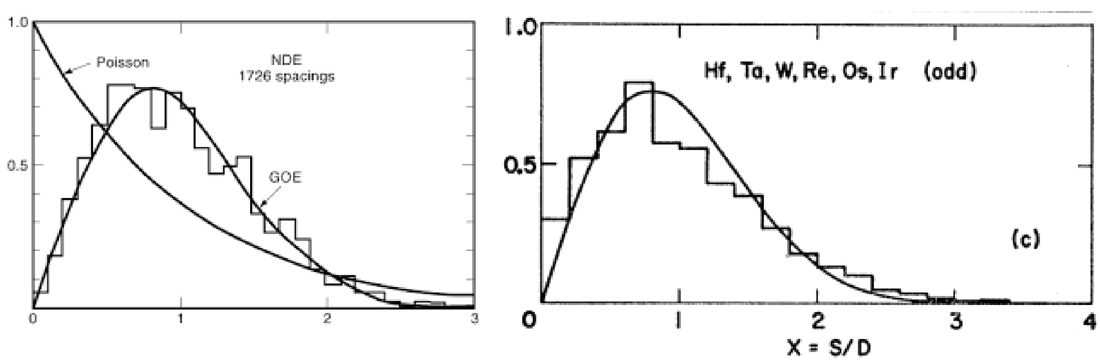
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^T)/2$ ***Gaussian orthogonal ensemble (GOE)***

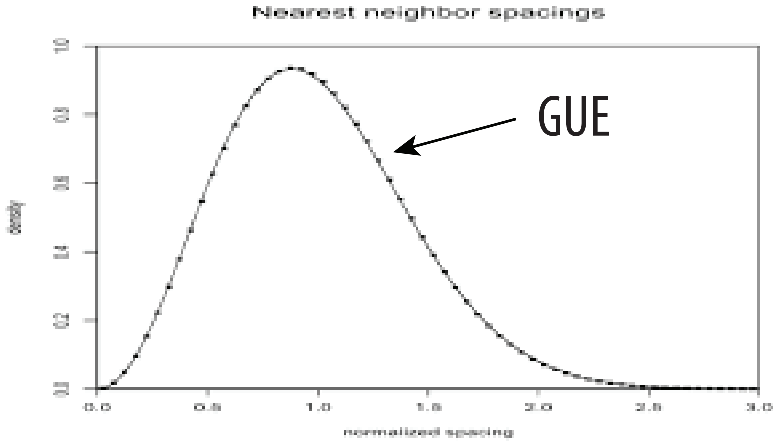
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^\dagger)/2$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.

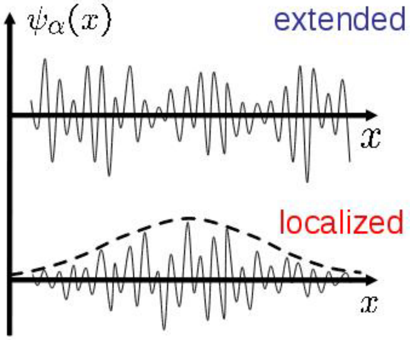
Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



Universal eigenvalue statistics arise in a broad range of "unrelated" problems!



electronic transport in semiconductors

metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

from analysis of spectral measures for brine, melt ponds, ice floes

we find percolation-driven

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION TRANSITION → **universal eigenvalue statistics (GOE)**
extended states, mobility edges

-- but with NO wave interference or scattering effects ! --

local conductivity in 1D inhomogeneous material

$$\sigma(x) = 3 + \cos x + \cos kx$$

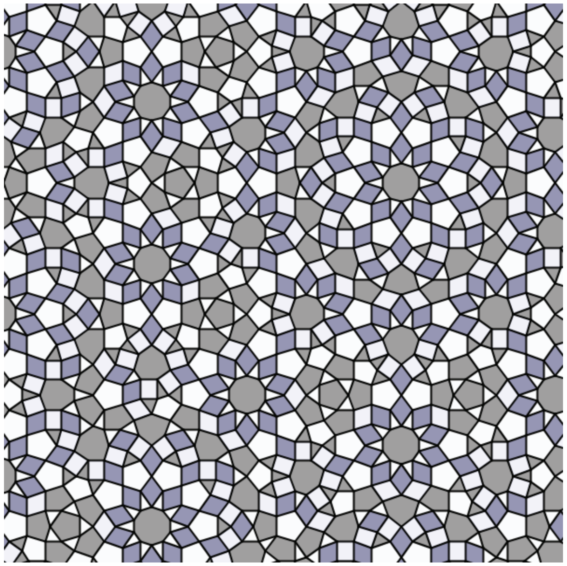
effective conductivity

$$\sigma^*(k) = \begin{cases} \text{constant} & k \ \text{ irrational} \qquad \text{quasiperiodic} \\ f(k) & k \ \text{ rational} \qquad \text{periodic} \end{cases}$$

Golden, Goldstein, Lebowitz, Phys. Rev. Lett. 1985

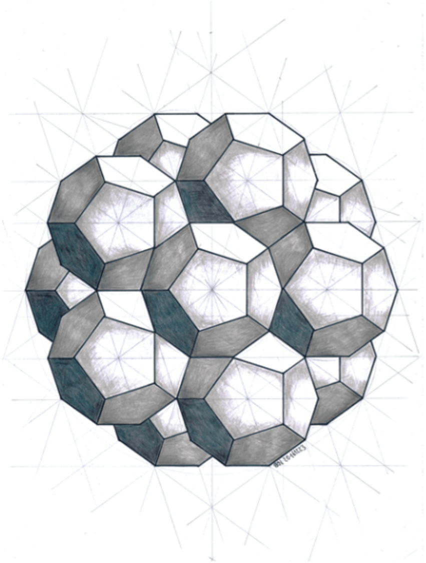
Order to Disorder in Quasiperiodic Composites

D. Morison (Physics), N. B. Murphy, E. Cherkaev, K. M. Golden, *Communications Physics* 2022



quasiperiodic checkerboard

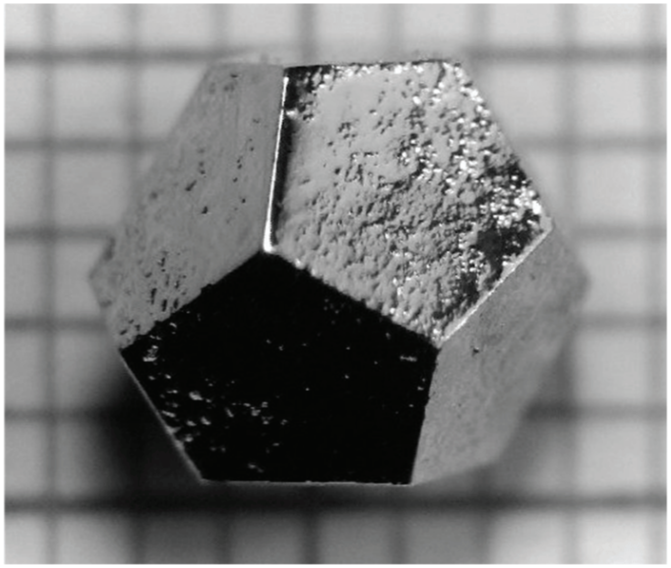
Stampfli, 2013



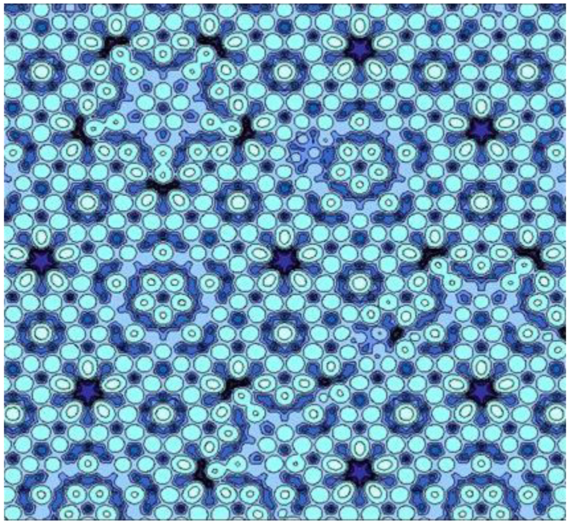
dense packing of dodecahedra

3D Penrose tiling

Tripkovic, 2019



Holmium–magnesium–zinc quasicrystal



energy surface Al-Pd-Mn quasicrystal

Unal et al., 2007

quasiperiodic crystal

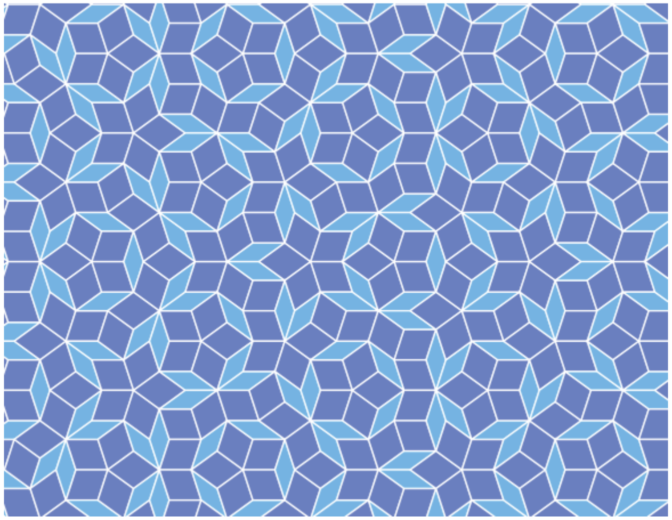
quasicrystal

ordered but aperiodic

lacks translational symmetry

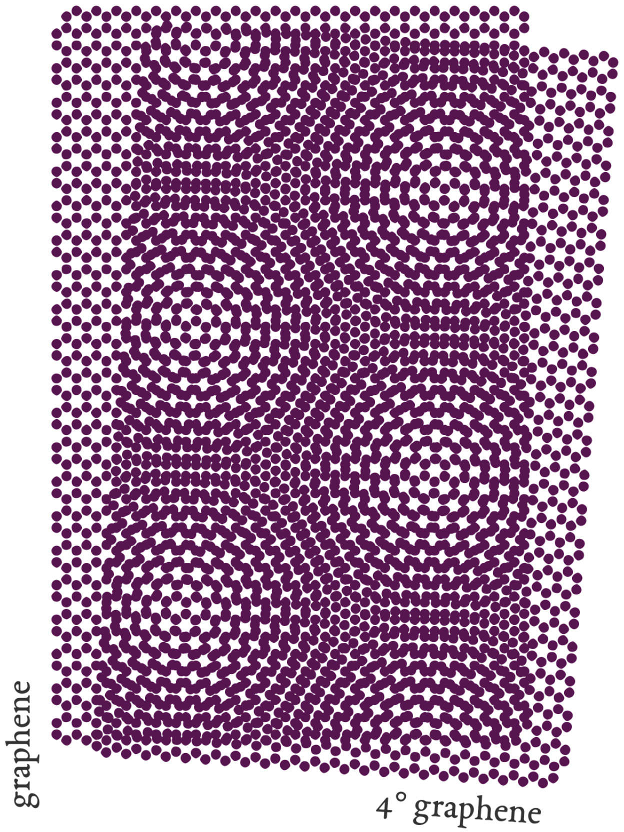
Schechtman et al., 1984

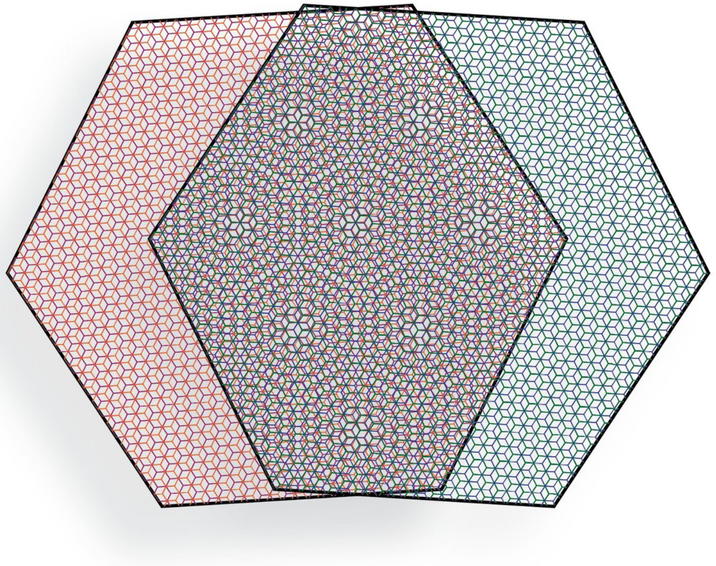
Levine & Steinhardt, 1984

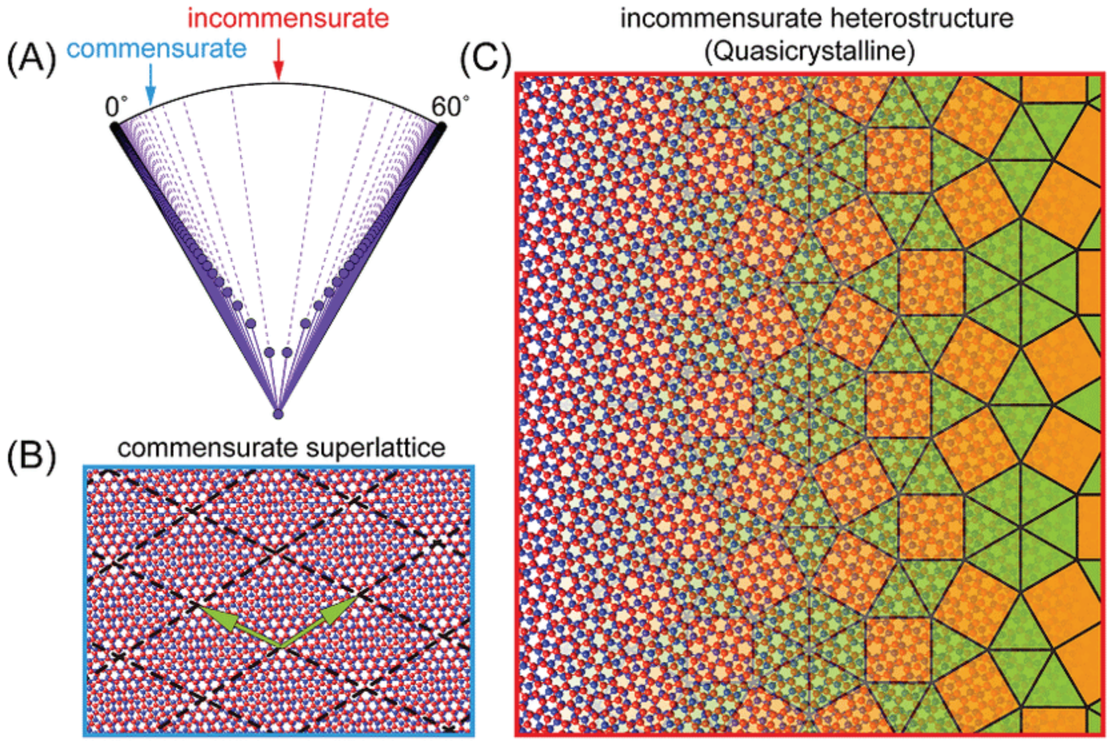


aperiodic tiling of the plane - R. Penrose 1970s

twisted bilayer graphene

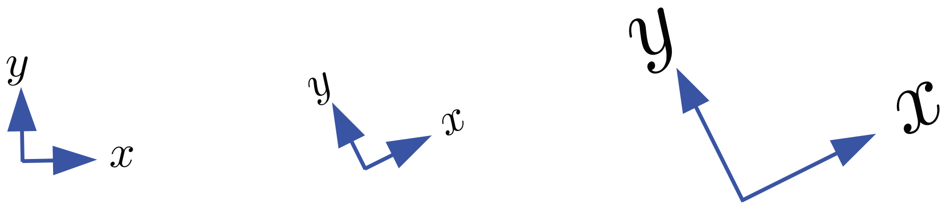


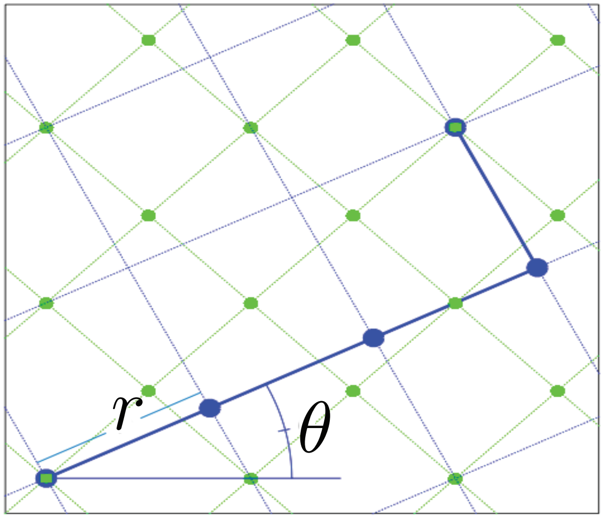




Yao et al., 2018

Moiré patterns generate two component composites





$$\begin{pmatrix} x' \\ y' \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\psi\left(x', y'\right) = \cos 2\pi x' \cos 2\pi y'$$

$$\chi = \begin{cases} 1, & \psi \geq 0 \\ 0, & \psi < 0 \end{cases}$$

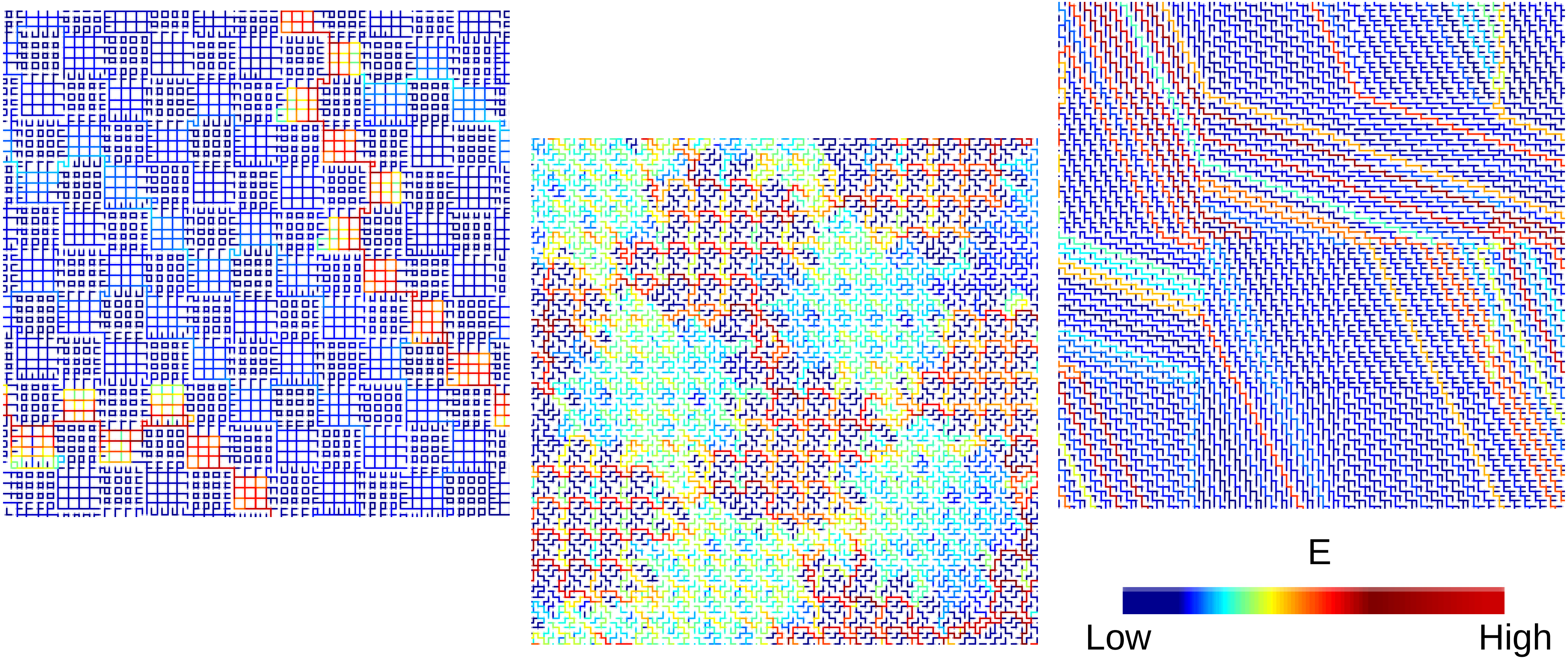
rotation and dilation

Small Difference in Moiré Parameters

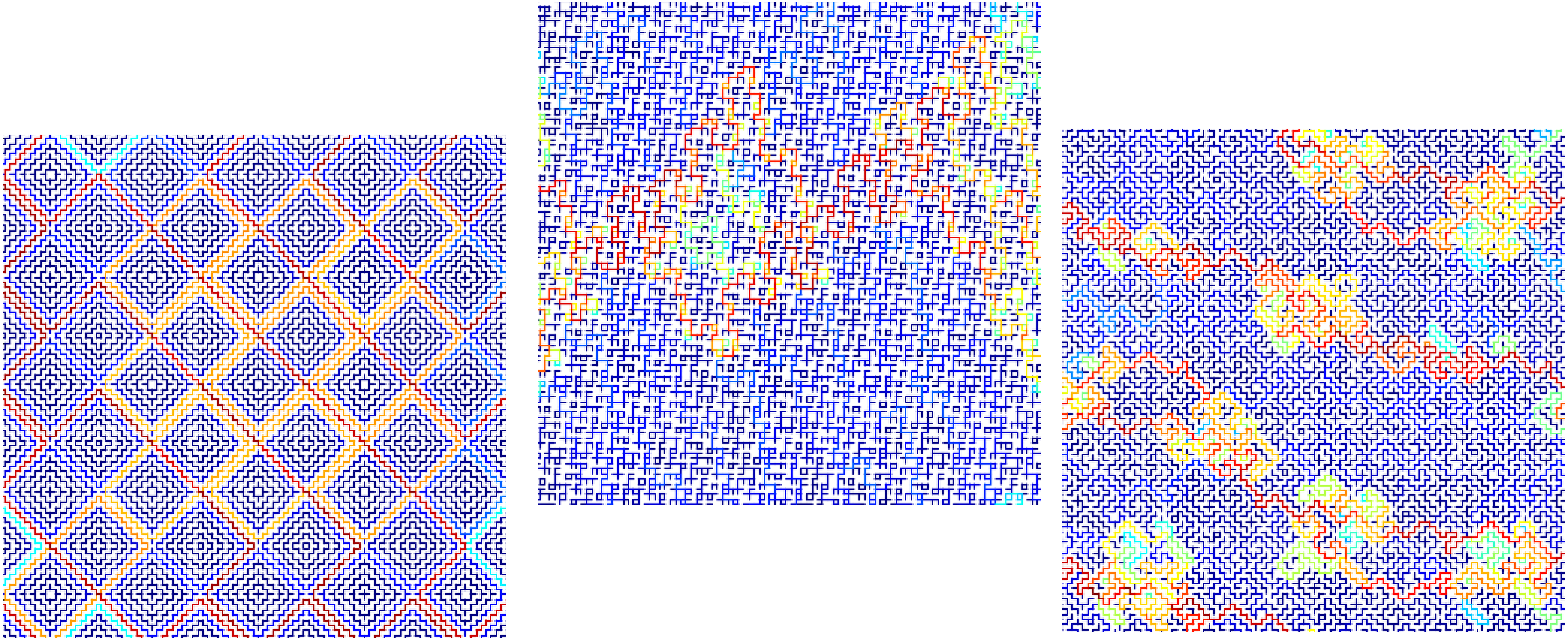
↓

Big Difference in Material Properties

Wide Variety of Microgeometries



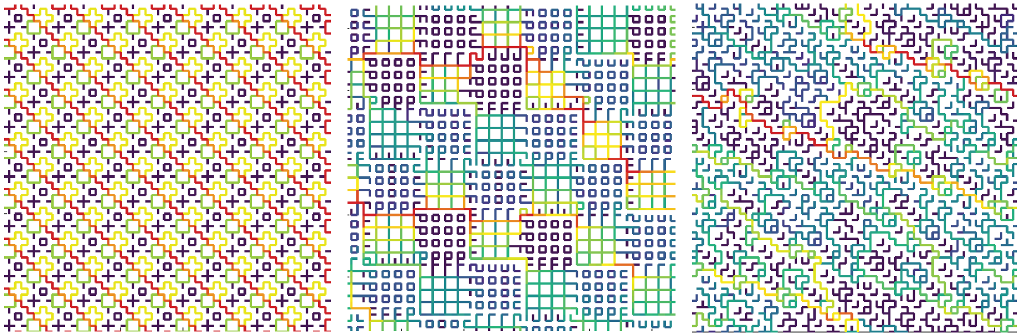
Wide Variety of Microgeometries



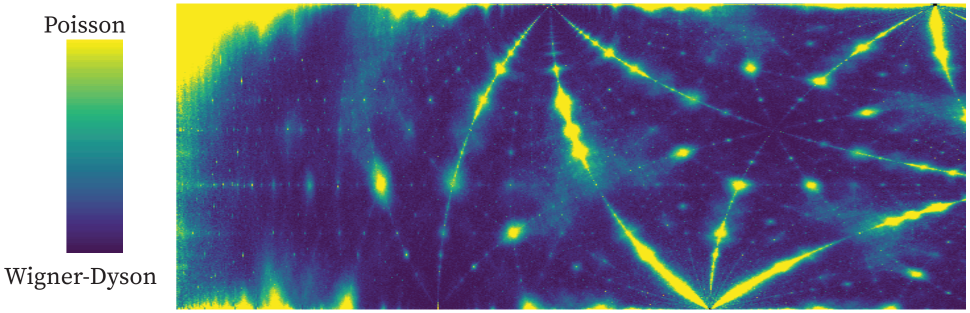
Order to disorder in quasiperiodic composites

Morison, Murphy, Cherkaev, Golden, *Commun. Phys.* 2022

Parameterized Moiré Pattern Creates Tunable Microgeometry

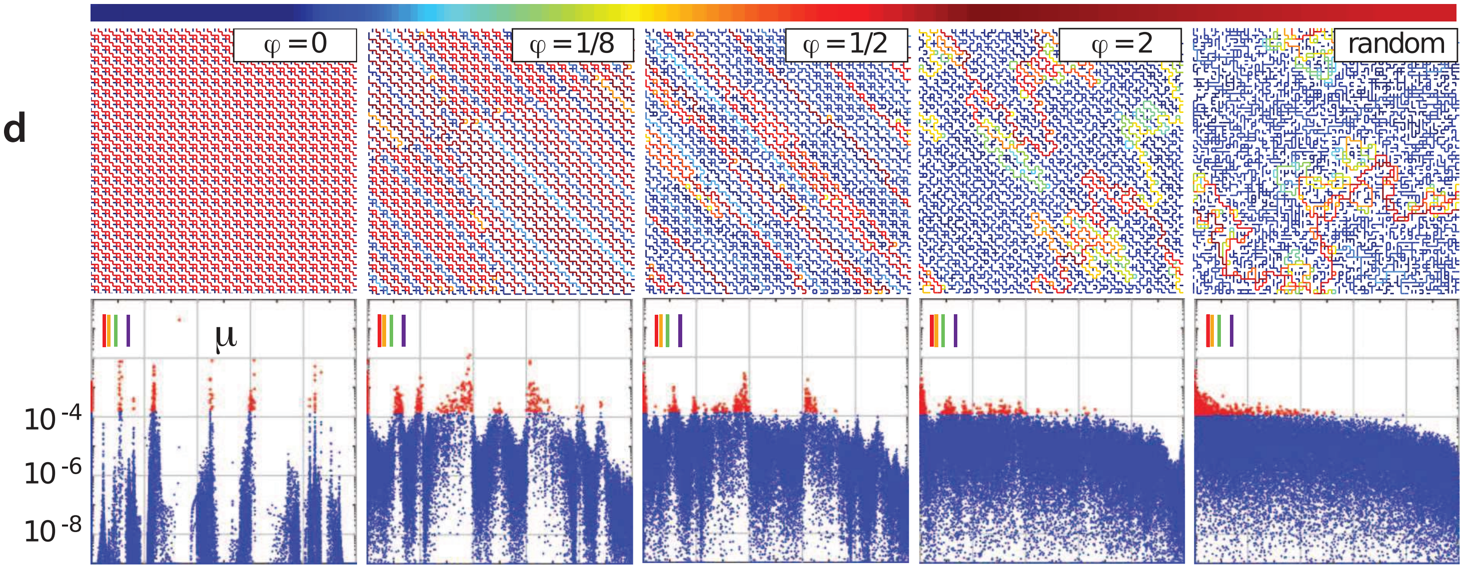


constellation of periodic systems in a sea of randomness



parameter space

periodic → **quasiperiodic**



electric field strength

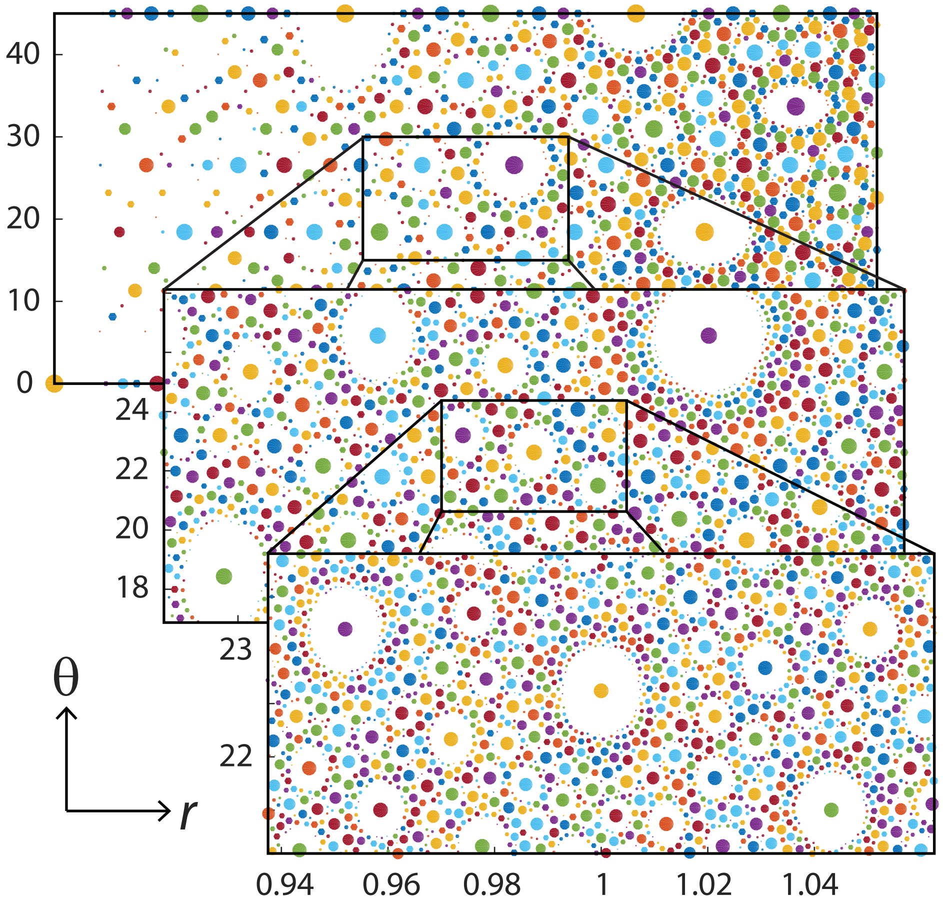
spectral measure

RRN at percolation threshold

we bring the framework of solid state physics of electronic transport and band gaps in semiconductors to classical transport in periodic and quasiperiodic composites

photonic crystals and quasicrystals

Fractal arrangement of periodic systems



Sequential insets zooming into smaller regions of parameter space.

size of the dots ~ length of period

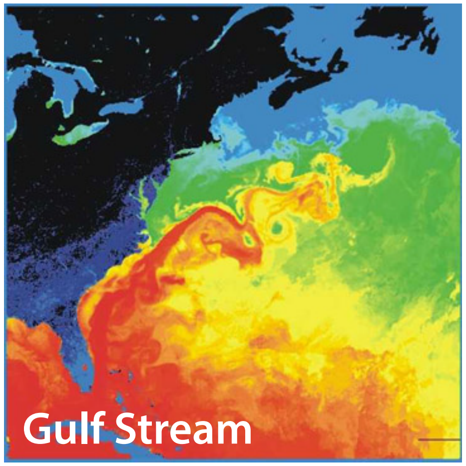
(large dot ~ small period; small dot ~ large period; white space ~ "infinite" period)

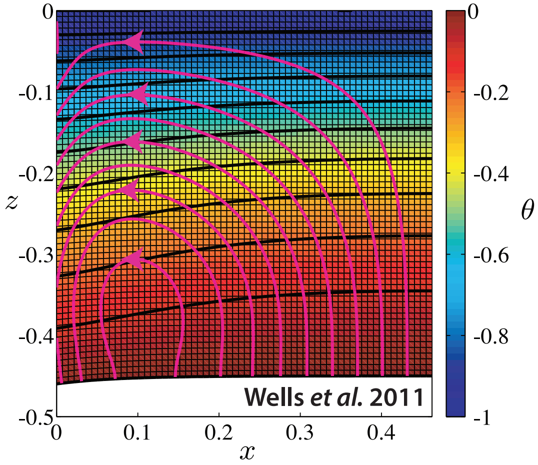
mesoscale

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

homogenize

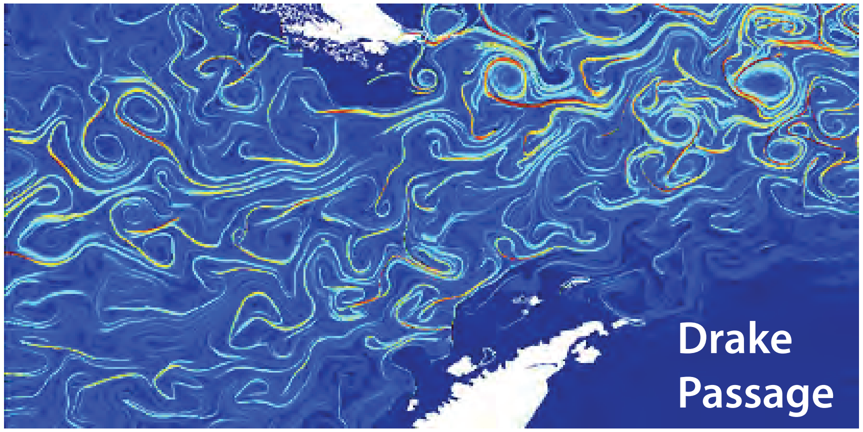
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

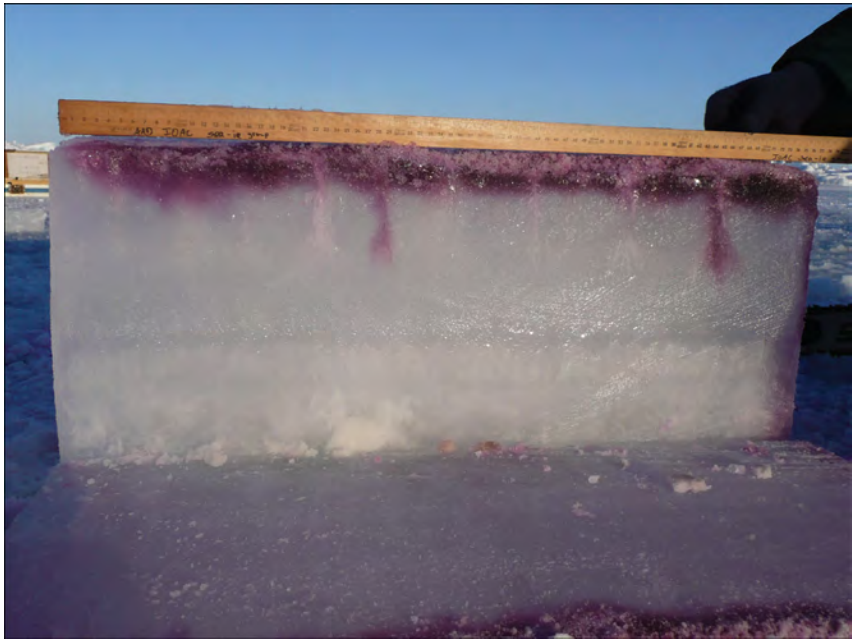
Stieltjes integral for κ^* ***with spectral measure***

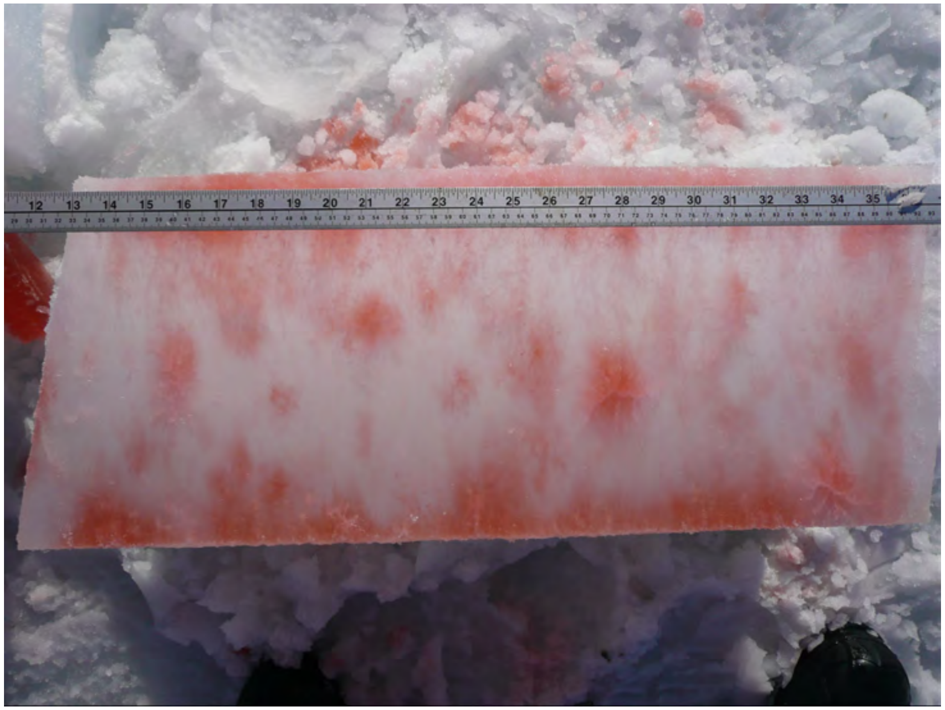
Avellaneda and Majda, PRL 89, CMP 91

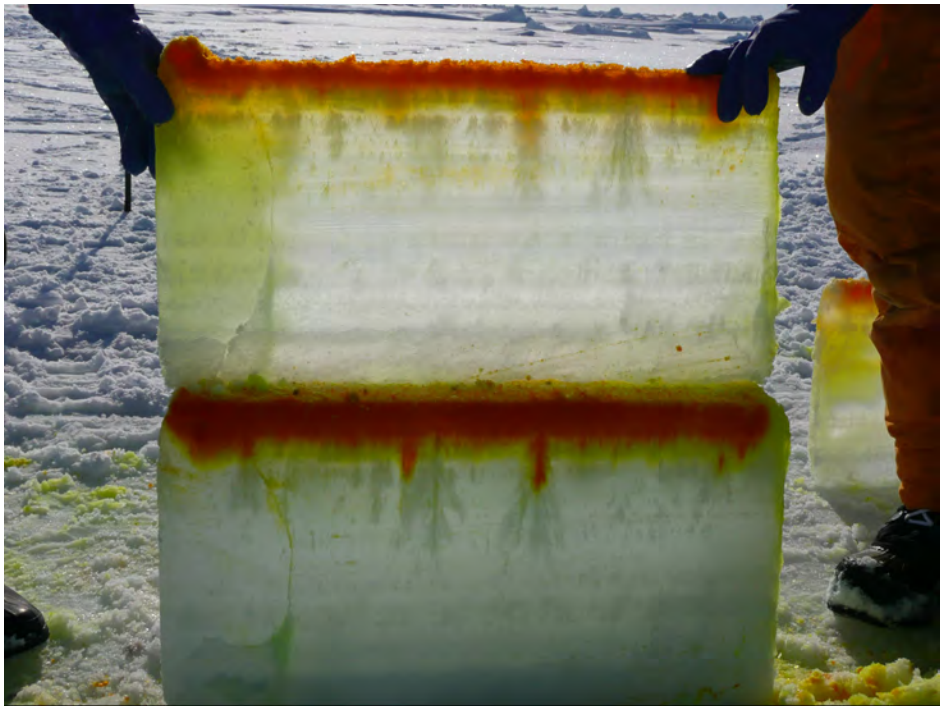
Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020



tracers flowing through inverted sea ice blocks







melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

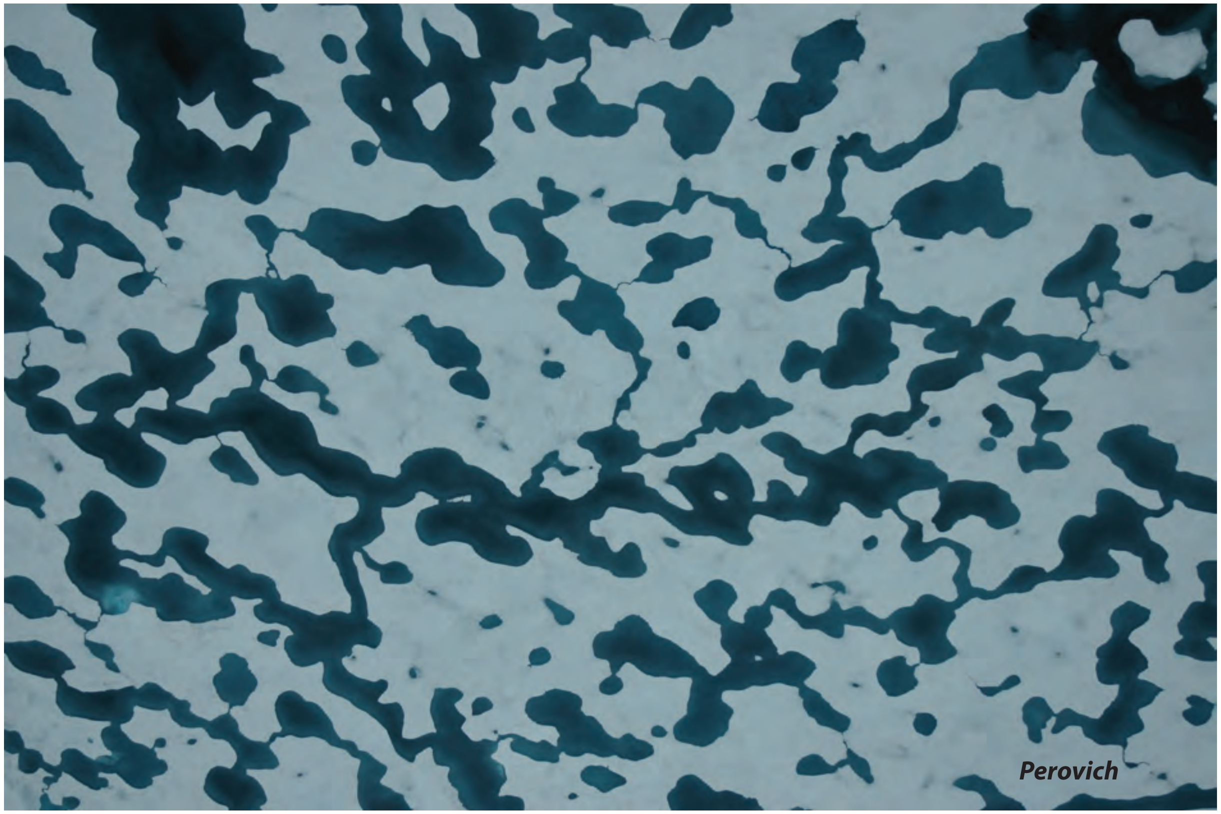
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012



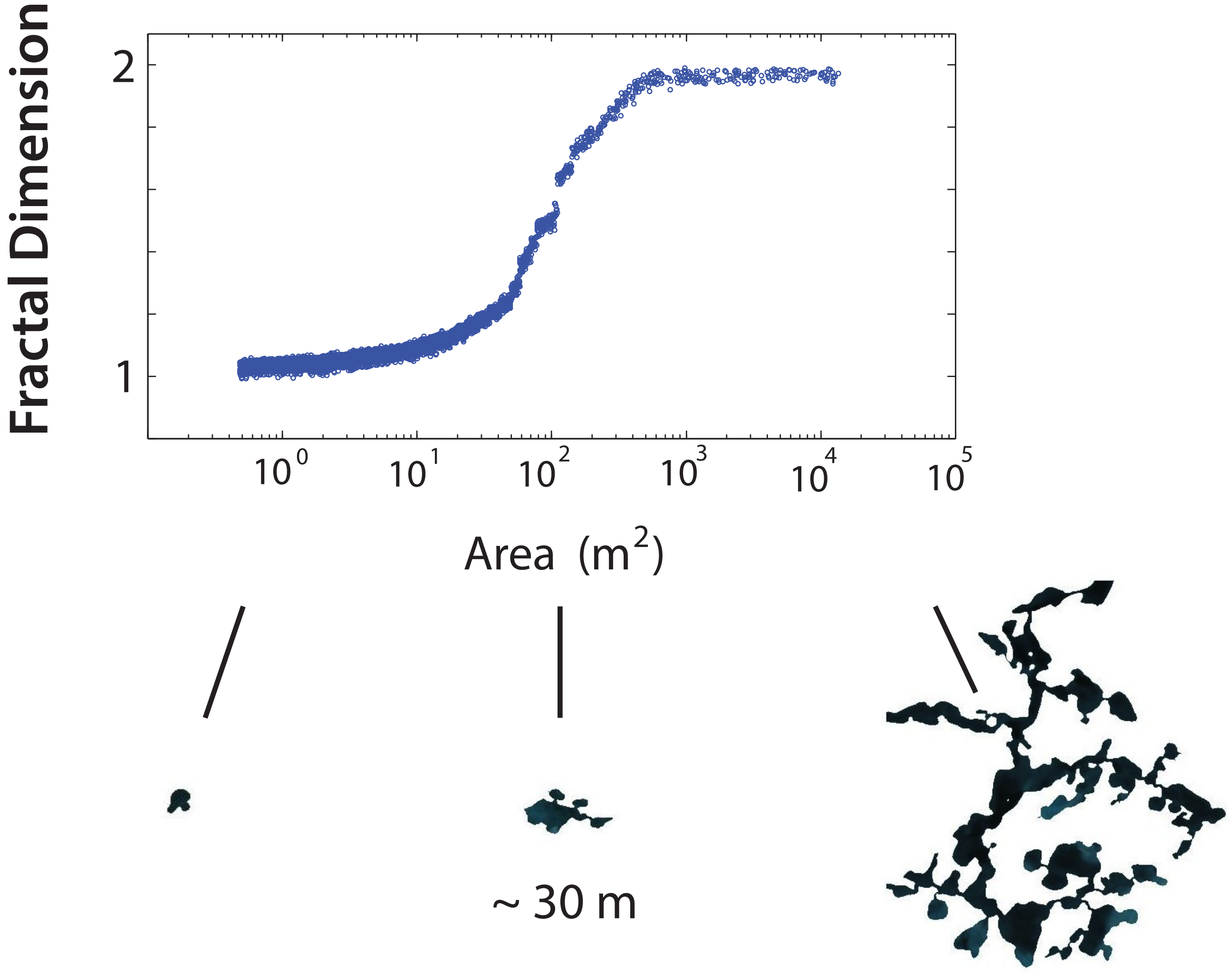
Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

complexity grows with length scale

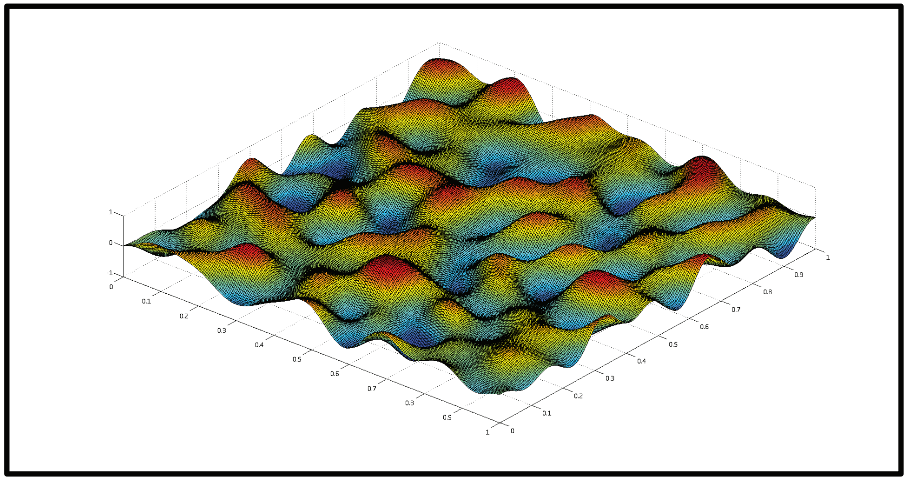


simple pond ***transitional pond*** ***complex pond***

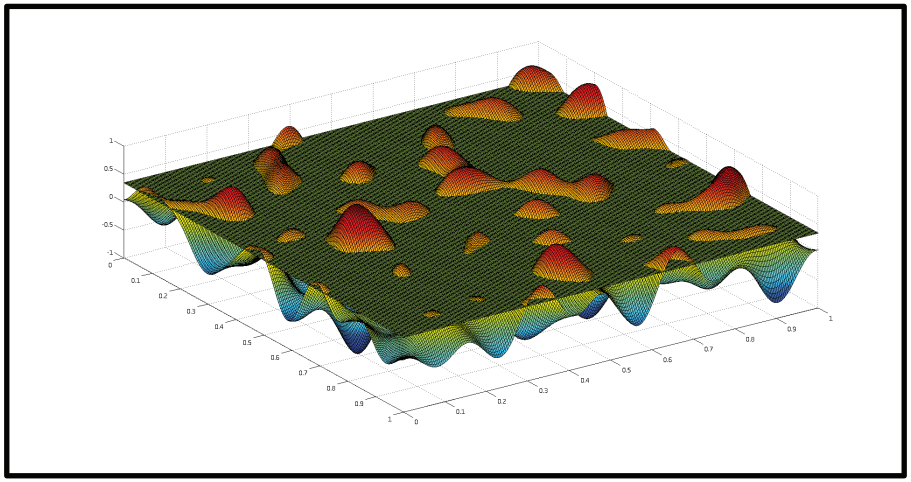
Continuum percolation model for melt pond evolution

level sets of random surfaces

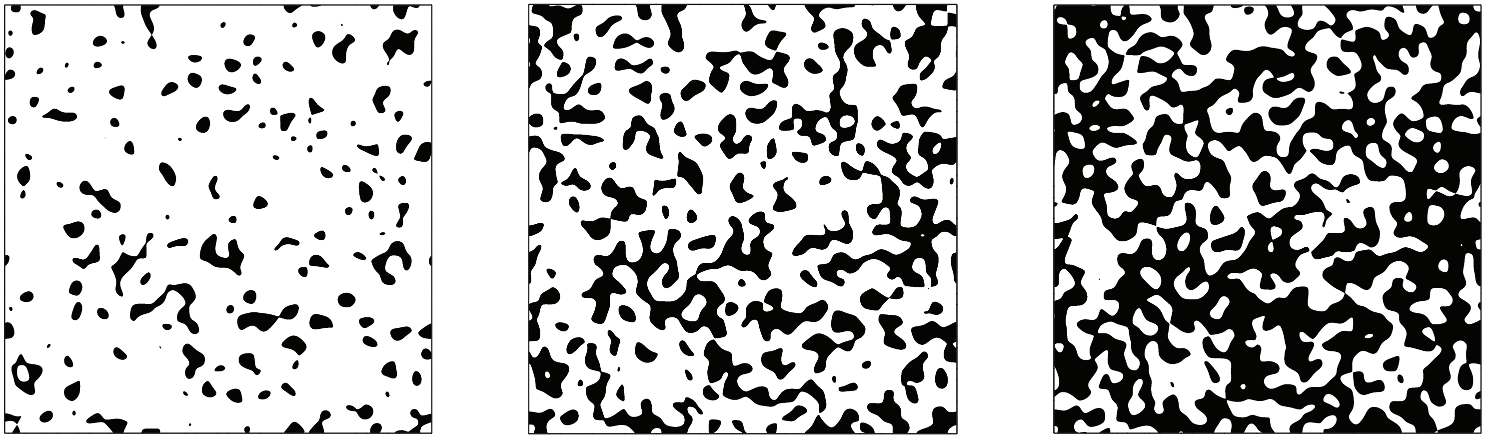
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography

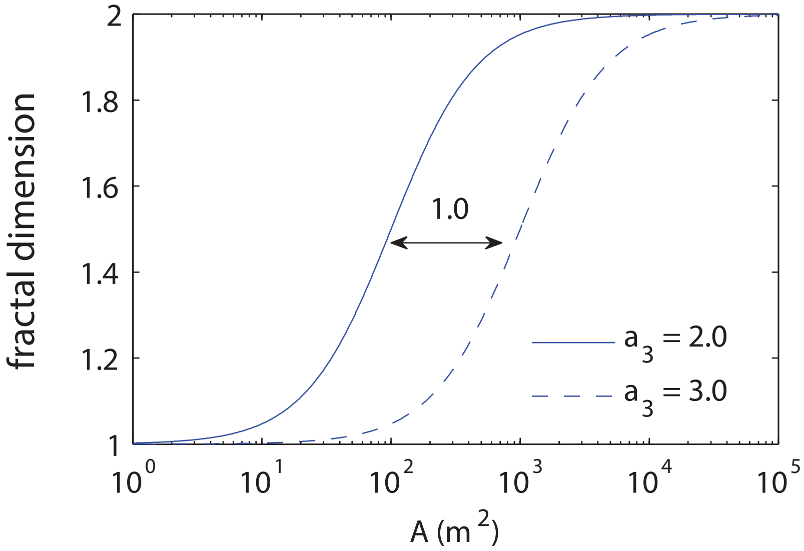


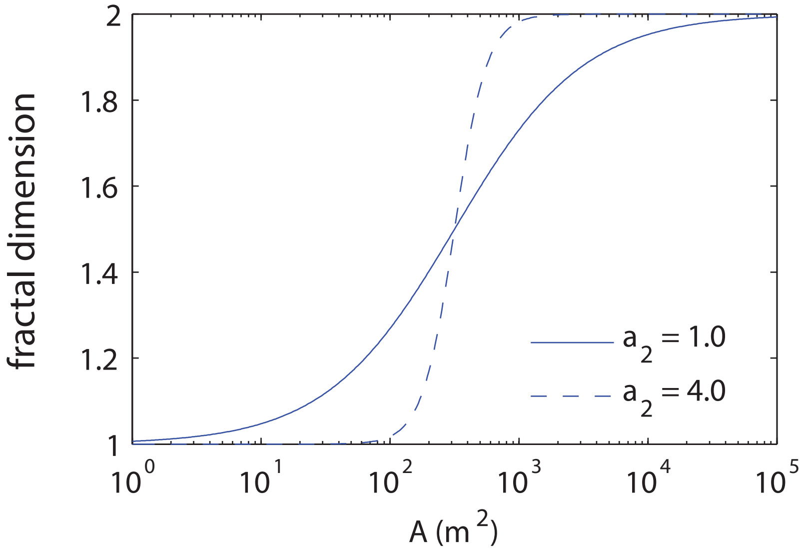
intersections of a plane with the surface define melt ponds



electronic transport in disordered media *diffusion in turbulent plasmas* *Isichenko, Rev. Mod. Phys., 1992*

fractal dimension curves depend on statistical parameters defining random surface





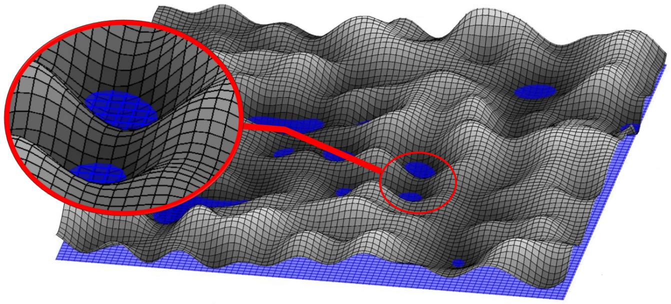
Topology of the sea ice surface and the fractal geometry of Arctic melt ponds

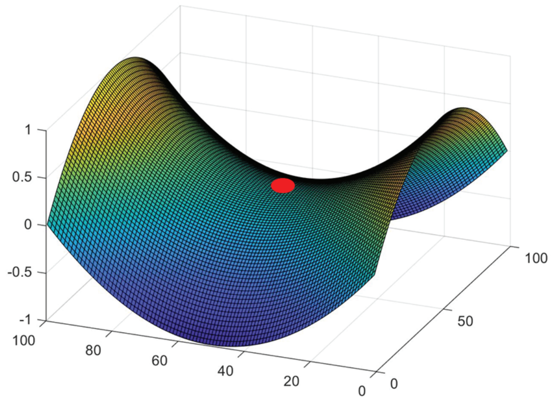
Physical Review Research (invited, under revision)

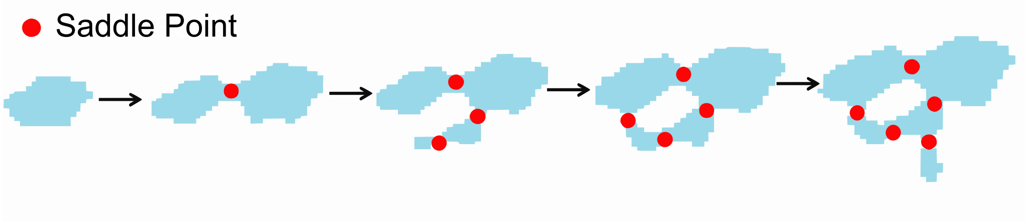
Ryleigh Moore, Jacob Jones, Dane Gollero,
Court Strong, Ken Golden

Several models replicate the transition in fractal dimension, but none explain how it arises.

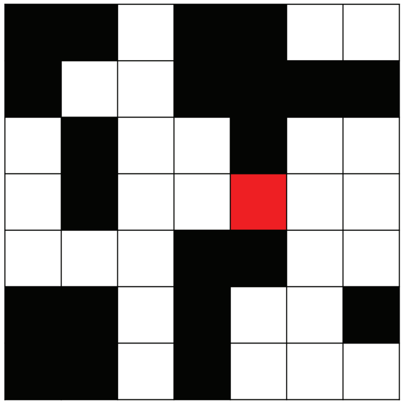
We use Morse theory applied to the random surface model to show that saddle points play the critical role in the fractal transition.



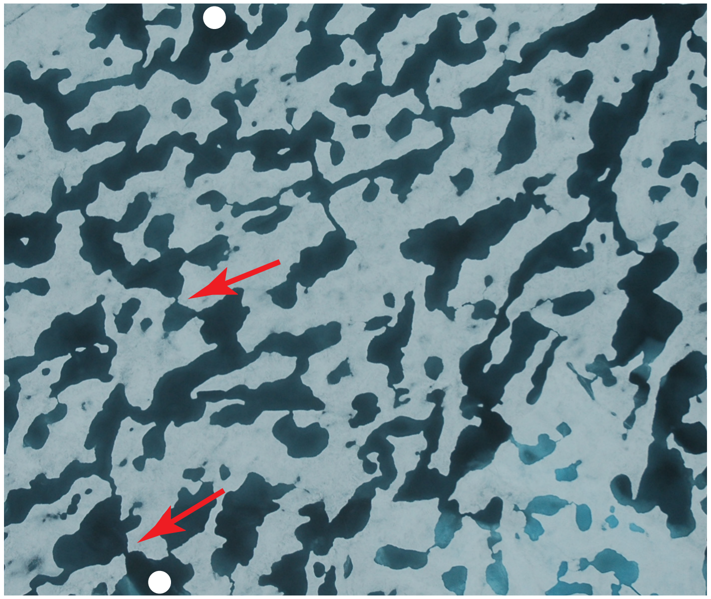




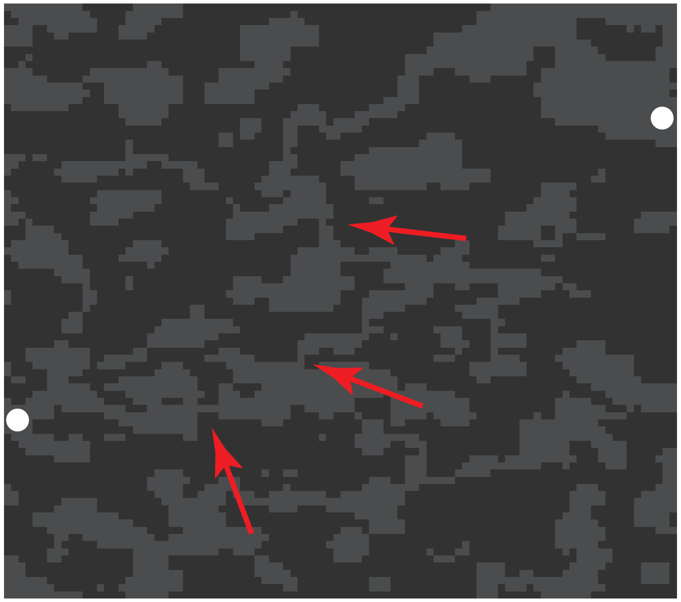
ponds coalesce
(change topology) and
complexify at saddle points



- Ponds connect through saddle points (Morse Theory).
- Red bonds in lattice percolation theory ~ saddle points.



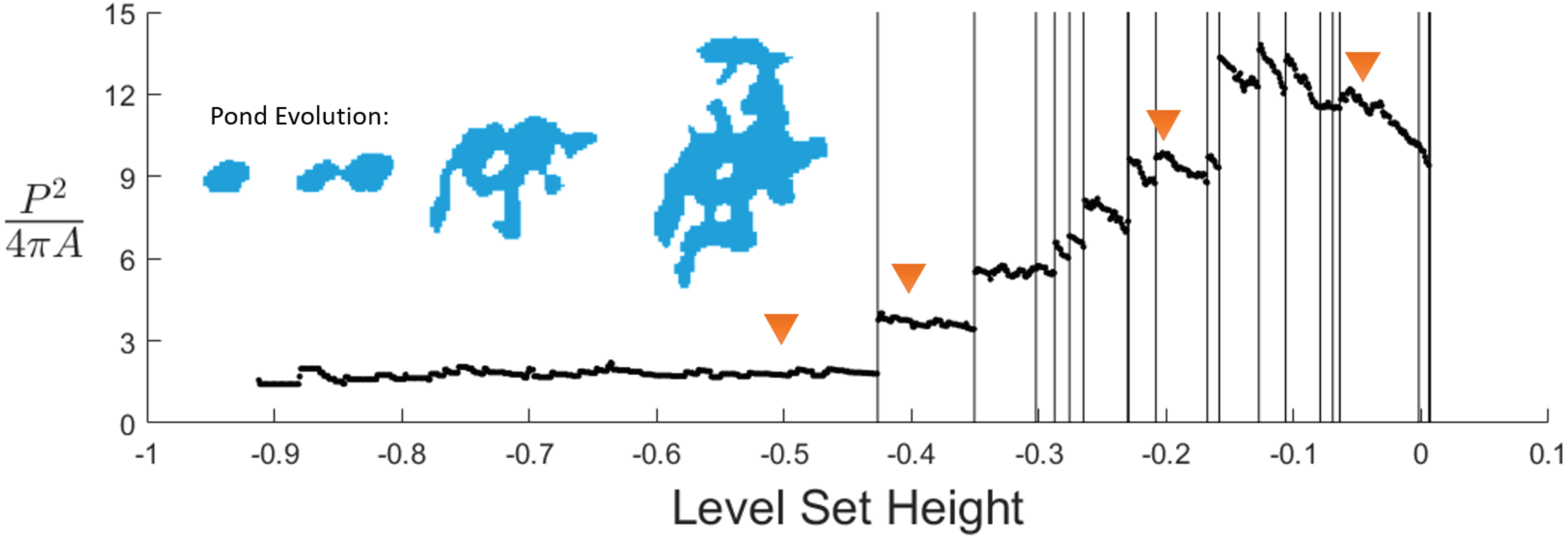
saddles



“red squares”

Main results

Isoperimetric quotient - as a proxy for fractal dimension - increases in discrete jumps when ponds coalesce at saddle points.



Horizontal fluid permeability "controlled" by saddles ~ electronic transport in 2D random potential.

drainage processes, seal holes

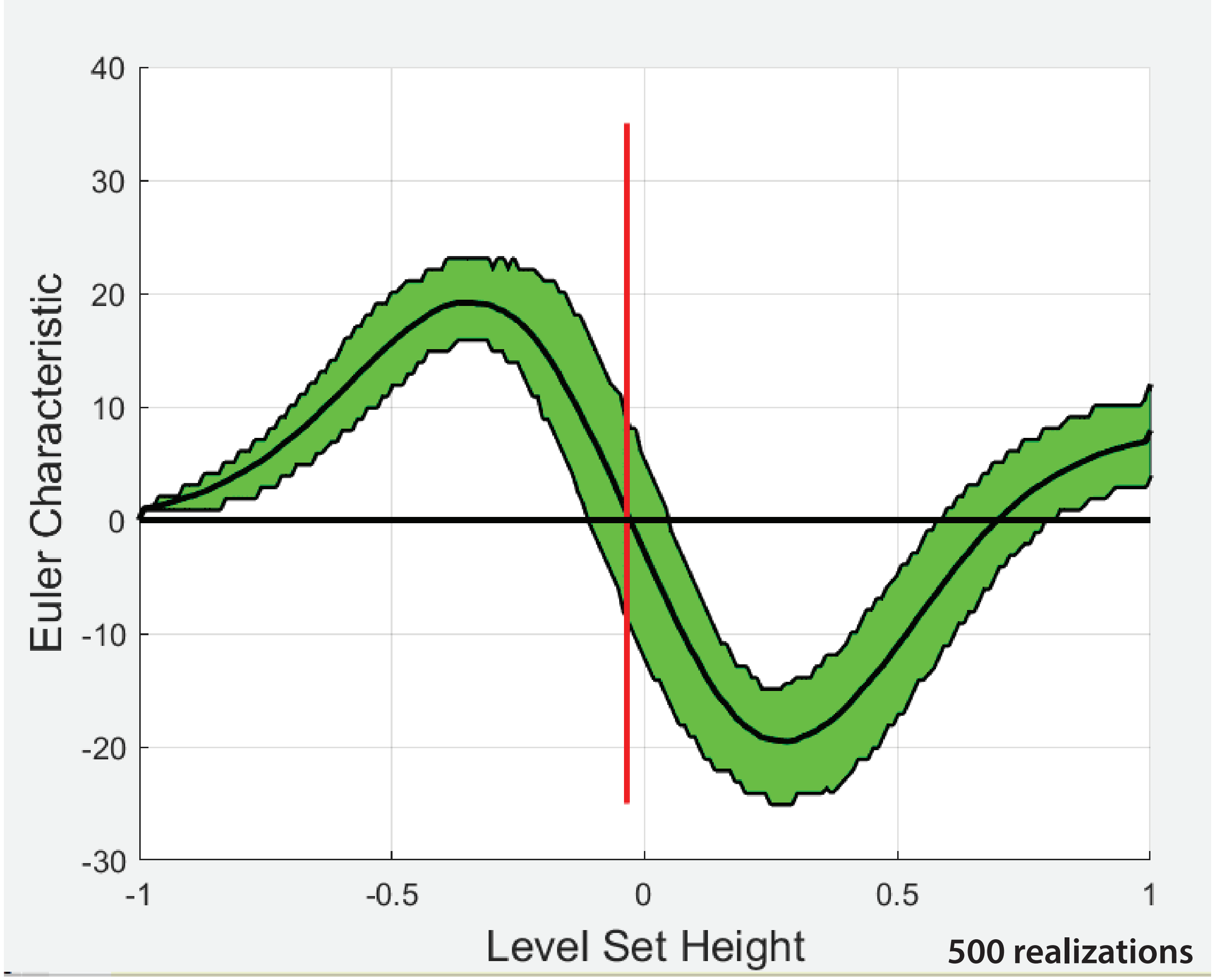
Topological Data Analysis

Euler characteristic = # maxima + # minima - # saddles

topological invariant

persistent homology

filtration - sequence of nested topological spaces, indexed by water level



Expected
Euler Characteristic Curve (ECC)

tracks the evolution of the EC of the flooded surface as water rises

zero of ECC ~ percolation

percolation on a torus
creates a giant cycle

Bobrowski &
Skraba, 2020

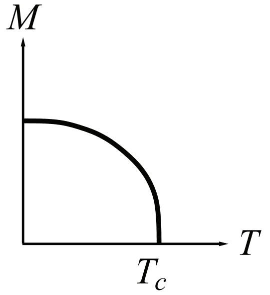
Carlsson, 2009

Vogel, 2002 GRF

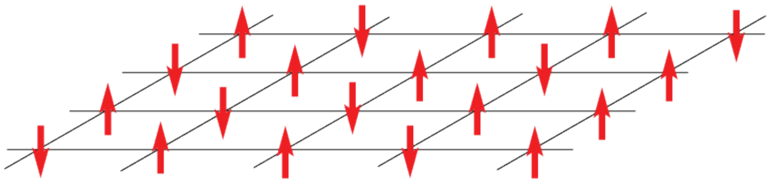
porous media
cosmology
brain activity

melt pond donuts

Ising Model for a Ferromagnet



Curie point
critical temperature



applied
magnetic
field

$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

blue
white

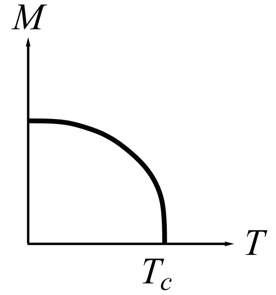
$$\mathcal{H} = -H\sum_i s_i - J\sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

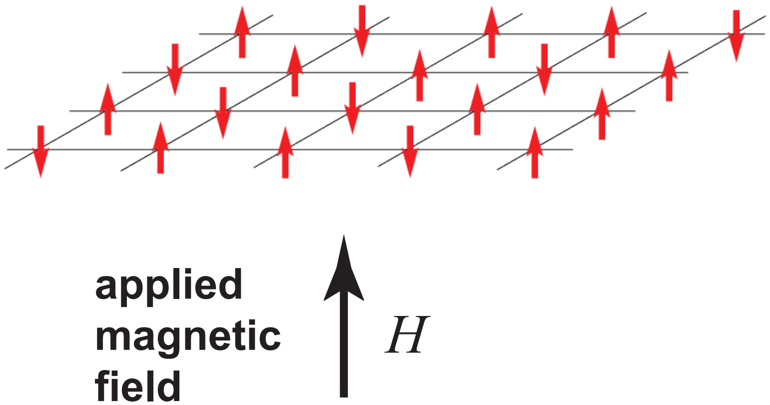
$$M(T,H) = \lim_{N\to\infty}\frac{1}{N}\left\langle \sum_j s_j \right\rangle$$

effective magnetization

Ising Model for a Ferromagnet



Curie point critical temperature



$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

blue

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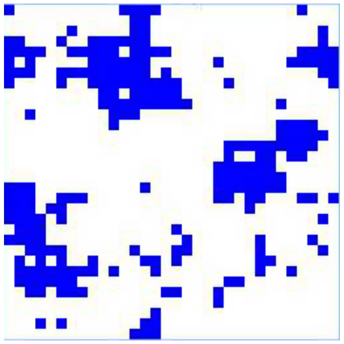
$$\mathcal{H} = -H \sum_i s_i - J \sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

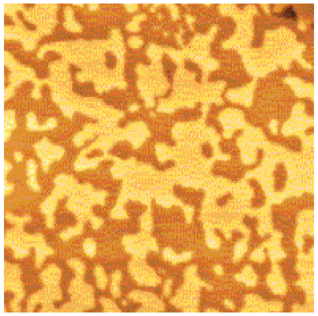


islands of like spins

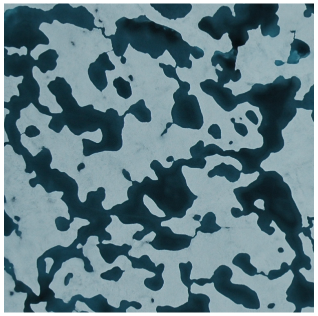
$$M(T,H) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$

effective magnetization

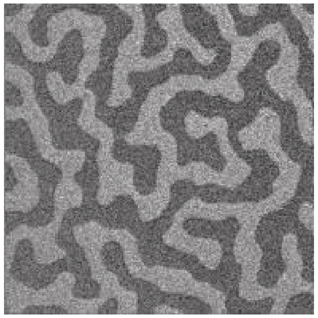
energy is lowered when nearby spins align with each other, forming **magnetic domains**



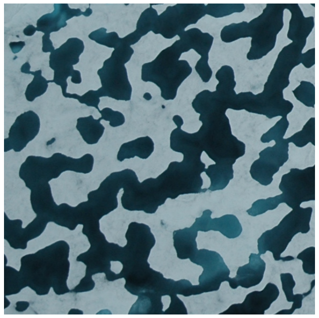
magnetic domains in cobalt



melt ponds (Perovich)



magnetic domains in cobalt-iron-boron



melt ponds (Perovich)

Ising model for ferromagnets → Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

$$\mathcal{H} = -\sum_{i}^{N} H_i \, s_i - J \sum_{<i,j>}^{N} s_i s_j \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

random magnetic field
represents snow topography

magnetization M

pond area fraction $F = \frac{(M+1)}{2}$

~ albedo

only nearest neighbor
patches interact

Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system “flows” toward metastable equilibria.

Order from Disorder

Ising model for ferromagnets → Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

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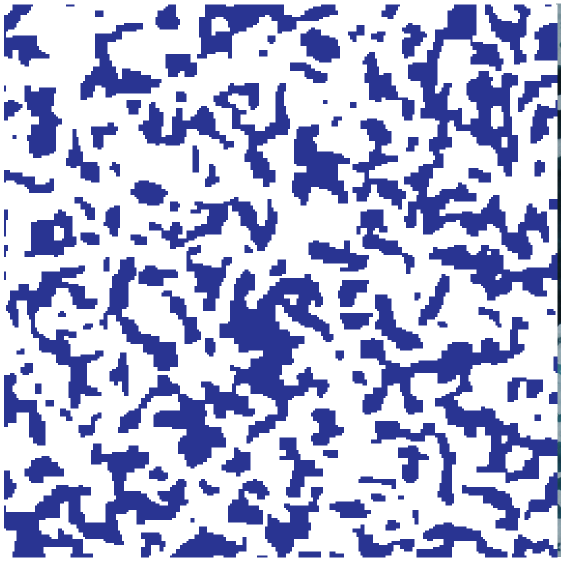
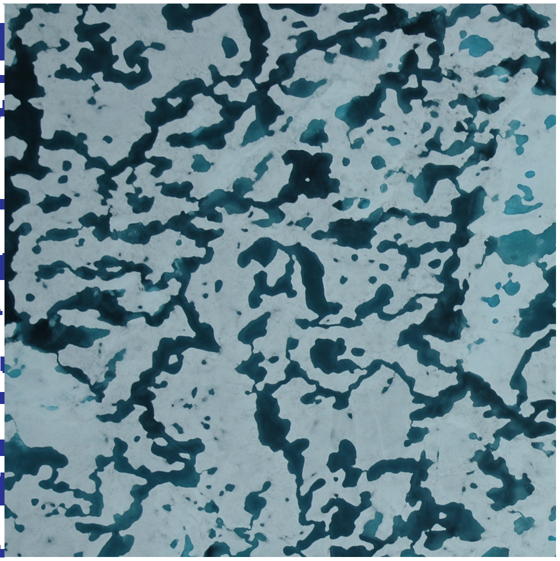
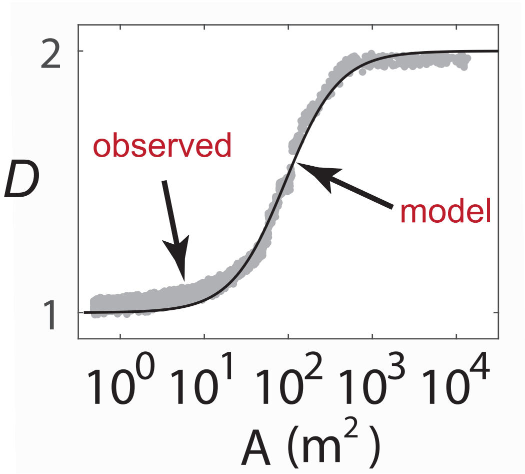
Order from Disorder



Ising model



melt pond photo (Perovich)



pond size distribution exponent

observed -1.5

(Perovich, *et al.* 2002)

model -1.58

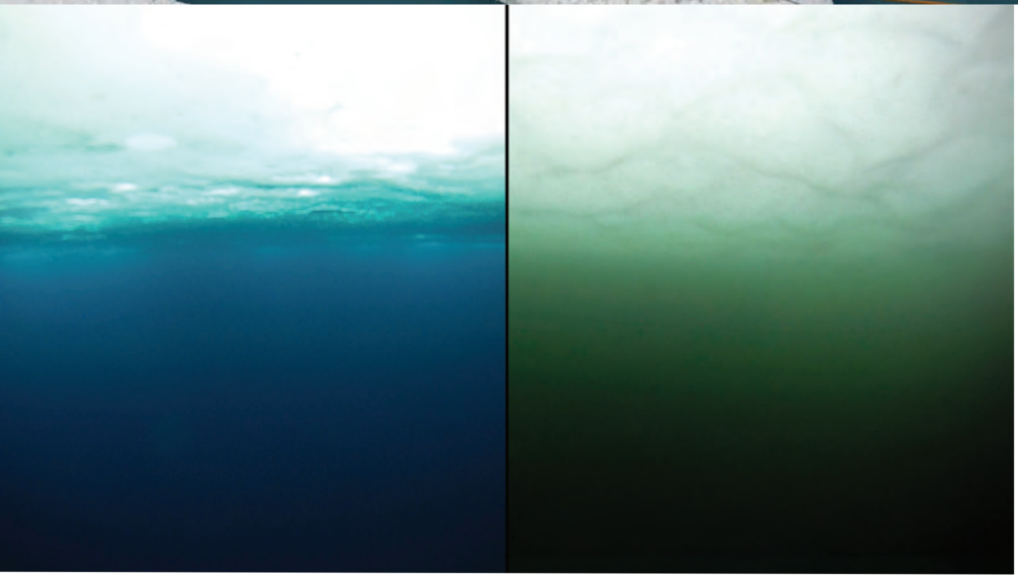
Scientific American
EOS, PhysicsWorld, ...

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS

Perovich



no bloom

bloom

massive under-ice algal bloom

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

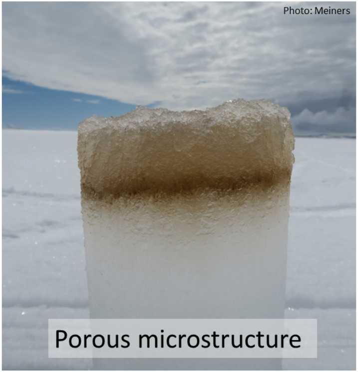
Horvat, Rees Jones, Iams, Schroeder,
Flocco, Feltham, ***Science Advances*** 2017

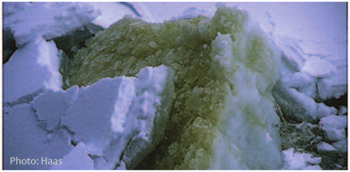
The effect of melt pond geometry on the distribution of solar energy under first year sea ice

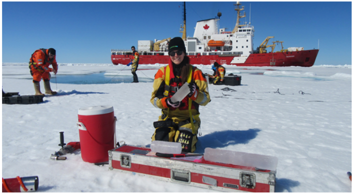
Horvat, Flocco, Rees Jones, Roach, Golden
Geophys. Res. Lett. 2019

(2015 AMS MRC)

SEA ICE ALGAE



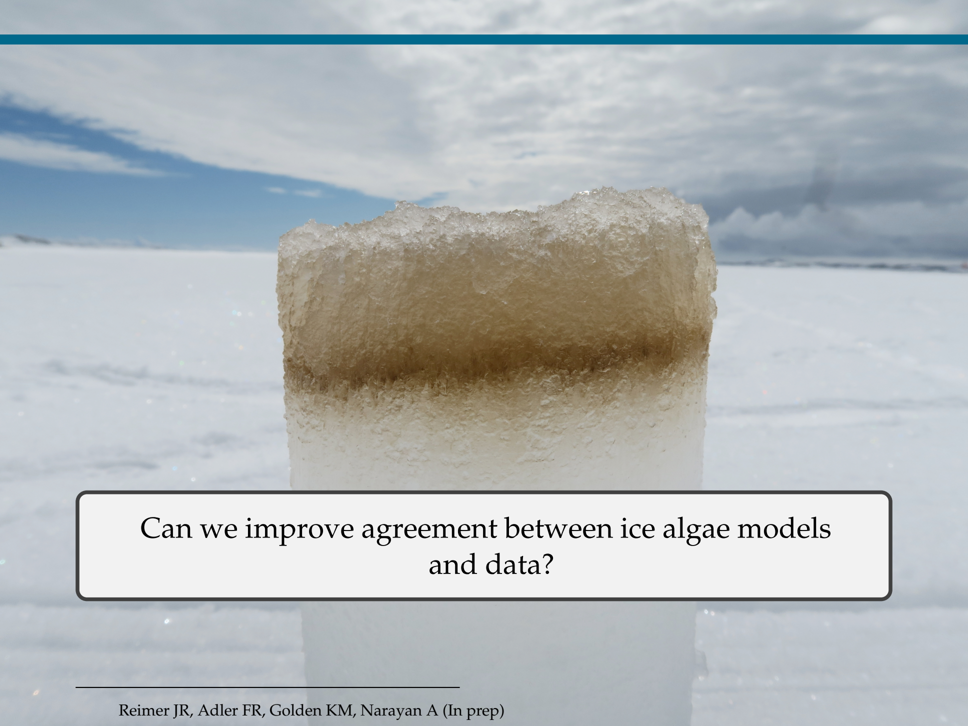




80% of polar bear diet can be traced to ice algae*.

---

[*] Brown TA, et al. (2018). *PloS one*, 13(1), e0191631



Can we improve agreement between ice algae models and data?

Reimer JR, Adler FR, Golden KM, Narayan A (In prep)

ALGAL BLOOM MODEL*

$$\text{nutrients:} \qquad \frac{dN}{dt} = \underbrace{\alpha}_{\text{input}} - \underbrace{\beta NP}_{\text{uptake}} - \underbrace{\eta N}_{\text{loss}}$$

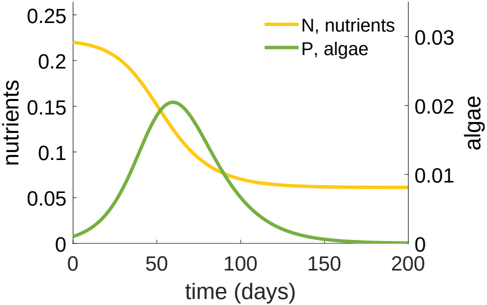
$$\text{algae:} \qquad \frac{dP}{dt} = \underbrace{\gamma\beta NP}_{\text{growth}} - \underbrace{\delta P}_{\text{death}},$$

$$N(0) = n_0, \qquad P(0) = p_0$$

---

[*] Huppert, A., et al. (2002). *American Naturalist*, 159(2), 156-171

ALGAL BLOOM MODEL

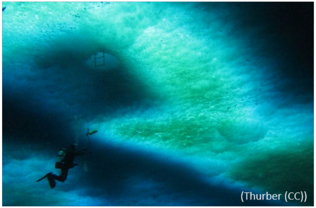


- poor agreement with data
- poor agreement between models

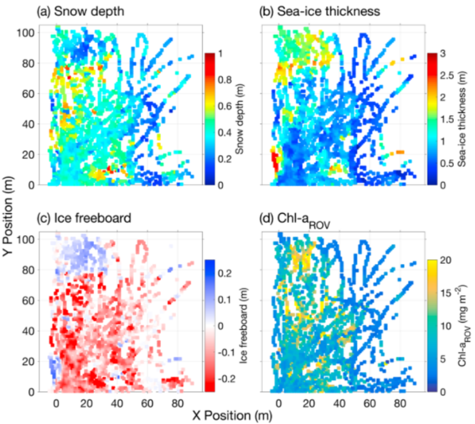
Steinacher, M., et al. (2010). *Biogeosciences*, 7(3), 979-1005

HETEROGENEITY

(Reimer)



(Thurber (CC))



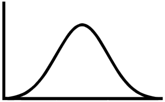
Meiners, K.M., et al. (2017). *Geophysical Research Letters*, 44(14), 7382-7390

HETEROGENEITY IN INITIAL CONDITIONS

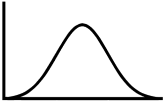
At each location within a larger region, we could consider

$$\frac{dN}{dt} = \alpha - BNP - \eta N$$
$$\frac{dP}{dt} = \gamma BNP - \delta P$$

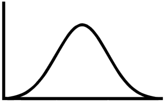
$$N(0) = N_0, \qquad P(0) = P_0$$



growth rate, B



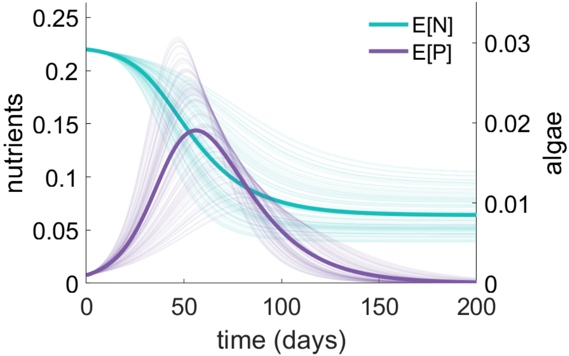
Initial nutrients, N_0



Initial algae, P_0

HOW DO WE ANALYZE THIS MODEL?

Monte Carlo simulations?



Too slow! Full algae model takes **8 hours** (cloud computing).

Uncertainty quantification and ecological dynamics in a model of a sea ice algae bloom, in prep. 2022

Jody Reimer, Fred Adler, Ken Golden, and Akil Narayan

POLYNOMIAL CHAOS EXPANSIONS

$$N(t; B, P_0, N_0) \approx N_V(t; B, P_0, N_0) \coloneqq \sum_{j=1}^{n} \widetilde{N}_j(t)\phi_j(B, P_0, N_0),$$

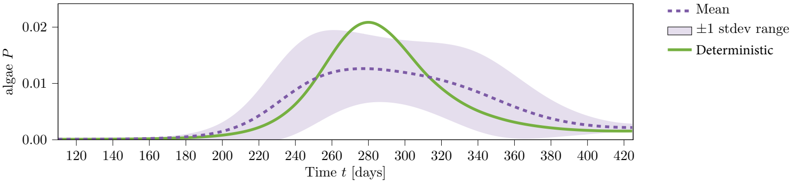
$$P(t; B, P_0, N_0) \approx P_V(t; B, P_0, N_0) \coloneqq \sum_{j=1}^{n} \widetilde{P}_j(t)\phi_j(B, P_0, N_0),$$

where

- $V \coloneqq \text{span}\{\phi_j\}_{j=1}^{n}$
- ϕ_j are orthogonal polynomials that form a basis for V
- $(\widetilde{N}_j, \widetilde{P}_j)$ need to be computed

Xiu, D. (2010). Numerical methods for stochastic computations. Princeton university press.

ECOLOGICAL INSIGHTS

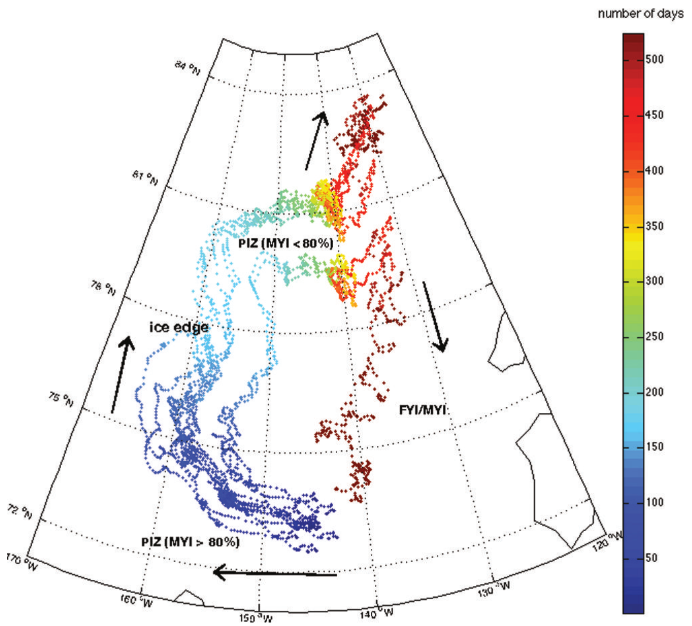


- lower peak bloom intensity
- longer bloom duration
- able to compare variance to data

macroscale

Anomalous diffusion in sea ice dynamics

Ice floe diffusion in winds and currents



observations from GPS data:

Jennifer Lukovich, Jennifer Hutchings,
David Barber, *Ann. Glac.* 2015

- On short time scales floes observed (buoy data) to exhibit Brownian-like behavior, but they are also being advected by winds and currents.
- Effective behavior is purely diffusive, sub-diffusive or super-diffusive depending on ice pack and advective conditions - Hurst exponent.

modeling:

Huy Dinh, Ben Murphy, Elena Cherkaev,
Court Strong, Ken Golden 2022

floe scale model to analyze transport regimes in terms of ice pack crowding, advective conditions

Delaney Mosier, Jennifer Hutchings, Jennifer Lukovich,
Marta D'Elia, George Karniadakis, Ken Golden 2022

learning fractional PDE governing diffusion from data

Floe Scale Model of Anomalous Diffusion in Sea Ice Dynamics

Huy Dinh, Ben Murphy, Elena Cherkaev, Court Strong, Ken Golden 2022

$$\left\langle \left|\mathbf{x}(t)-\mathbf{x}(0)-\langle\mathbf{x}(t)-\mathbf{x}(0)\rangle\right|^2\right\rangle \sim t^{\alpha}$$

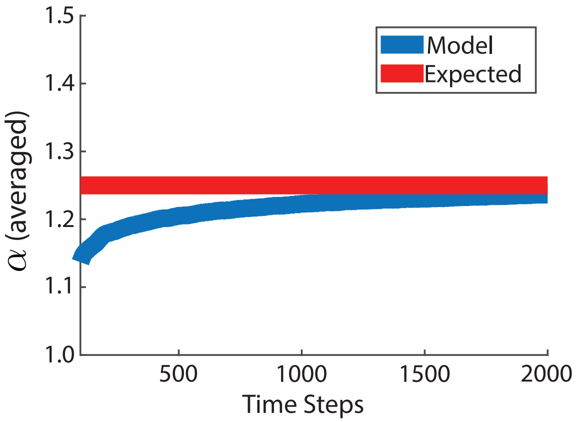
$\alpha =$ Hurst exponent

diffusive $\alpha = 1$

sub-diffusive $\alpha < 1$

super-diffusive $\alpha > 1$

Sparse Packing, Shear Dominated Drift



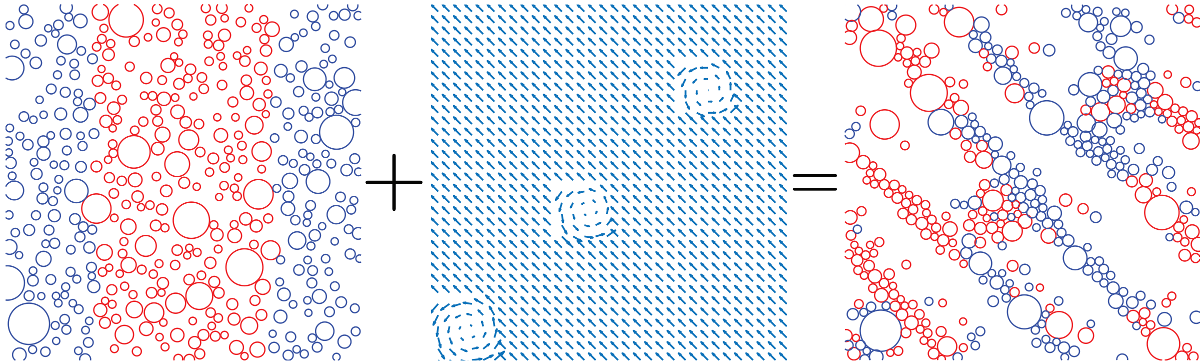
Model Approximations

Power Law Size Distribution: $N(D) \sim D^{-k}$

D. A. Rothrock and A. S. Thorndike Journal of Geophysical Research 1984

Floe-Floe Interactions: Linear Elastic Collisions

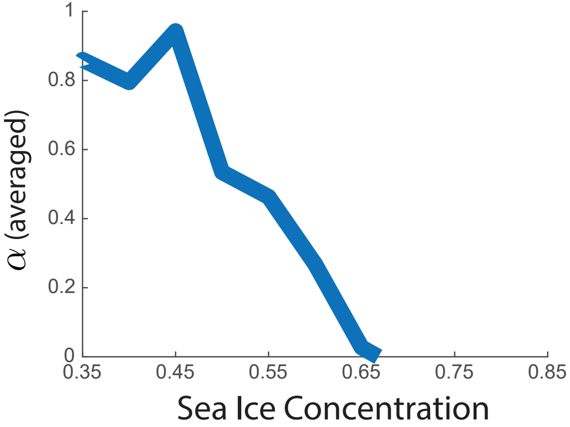
Advective Forcing: Passive, Linear Drag Law



$\psi = 0.3$

Vorticity Dominated Drift

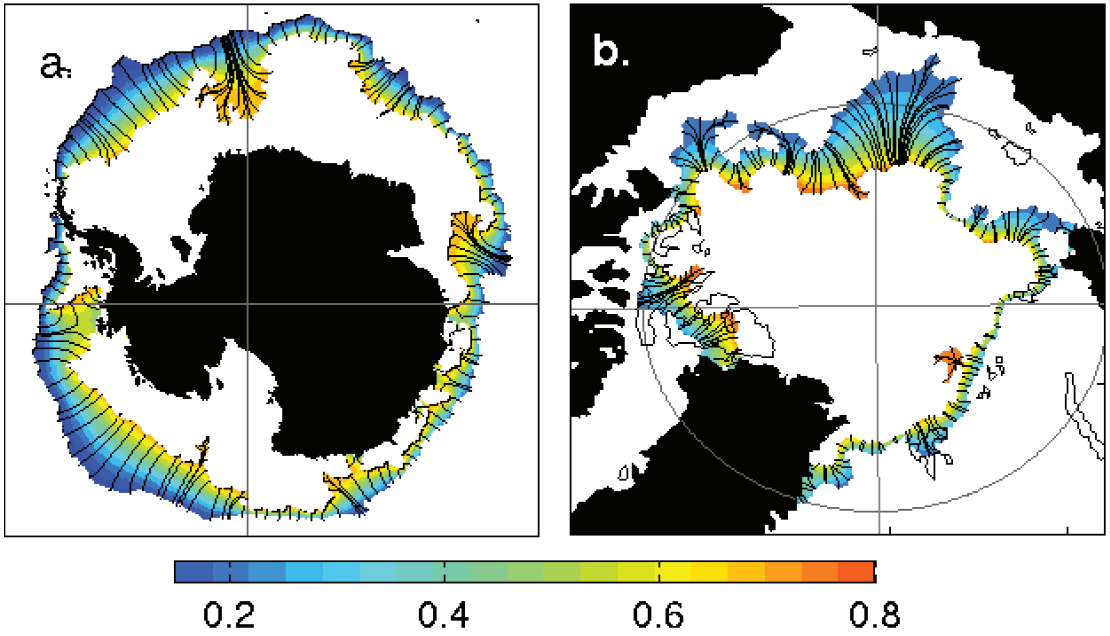
Crowding, Diffusive Drift



Marginal Ice Zone

MIZ

- biologically active region
- intense ocean-sea ice-atmosphere interactions
- region of significant wave-ice interactions



transitional region between dense interior pack ($c > 80\%$) sparse outer fringes ($c < 15\%$)

MIZ WIDTH

fundamental length scale of ecological and climate dynamics

Strong, *Climate Dynamics* 2012

Strong and Rigor, *GRL* 2013

How to objectively measure the "width" of this complex, non-convex region?

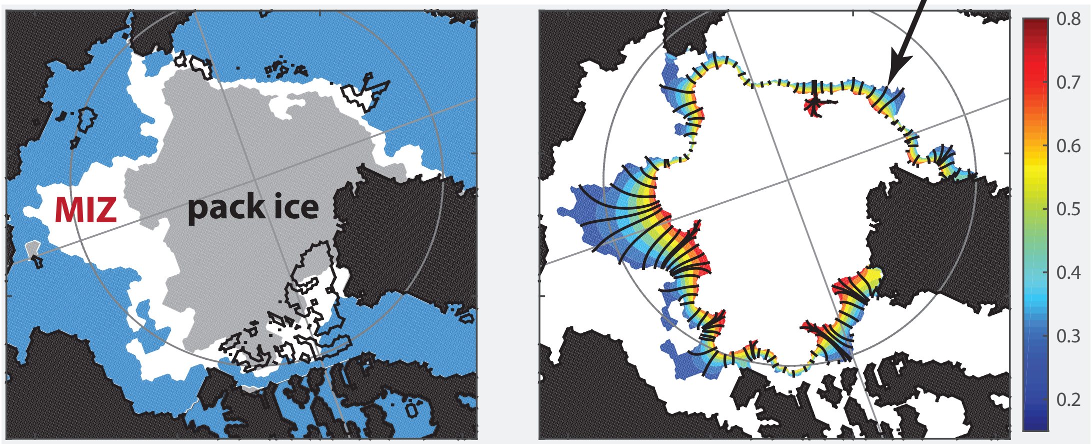
Objective method for measuring MIZ width motivated by medical imaging and diagnostics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

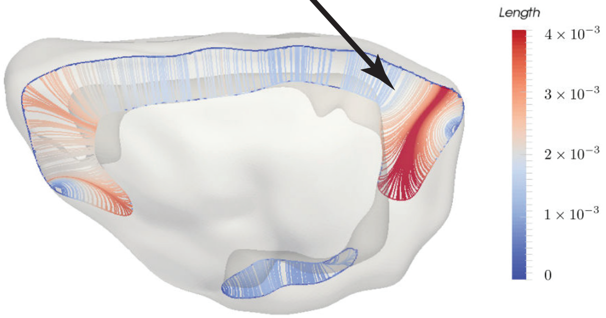
39% widening
1979 - 2012

"average" lengths of streamlines

streamlines of a solution to Laplace's equation



Arctic Marginal Ice Zone



crossection of the cerebral cortex of a rodent brain

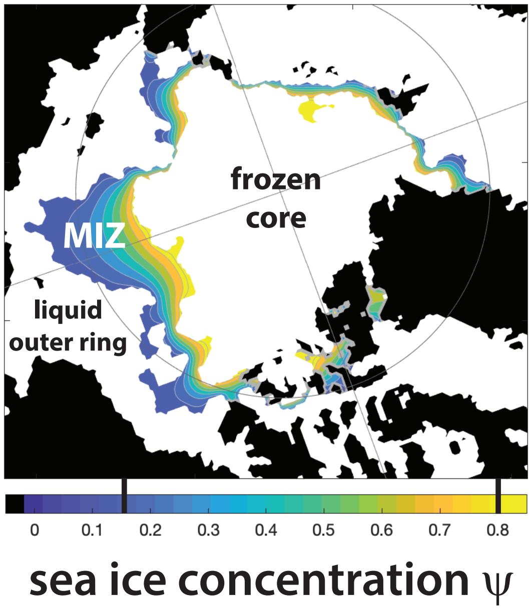
analysis of different MIZ WIDTH definitions

Strong, Foster, Cherkaev, Eisenman, Golden
J. Atmos. Oceanic Tech. 2017

Strong and Golden
Society for Industrial and Applied Mathematics News, April 2017

Model larger scale effective behavior with partial differential equations that *homogenize* complex local structure and dynamics.

Arctic MIZ



sea ice concentration ψ

Predict MIZ width and location with basin-scale phase change model.

dynamic transitional region - mushy layer - separating two "pure" phases

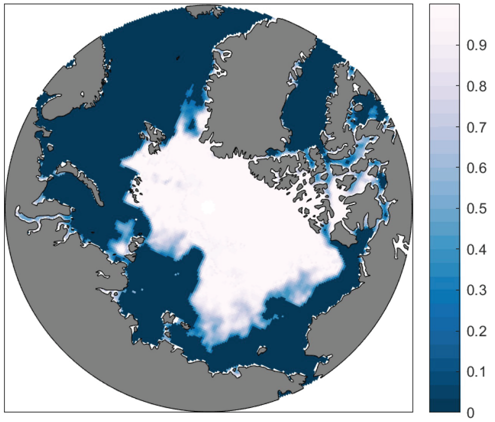
seasonal and long term trends

C. Strong, E. Cherkaev, and K. M. Golden,
Annual cycle of Arctic marginal ice zone location
and width explained by phase change front model, 2022

Learning the velocity field in an advection diffusion model for sea ice concentration

Eric Brown, Delaney Mosier, Bao Wang, Ken Golden, 2022

Goal: Develop PDE model to describe evolution of sea ice concentration field.



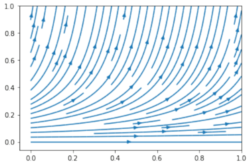
National Snow and Ice Data Center

discretized satellite concentration data

Figure 1. Arctic sea ice concentration in early August 2012.

advection diffusion model for sea ice concentration:

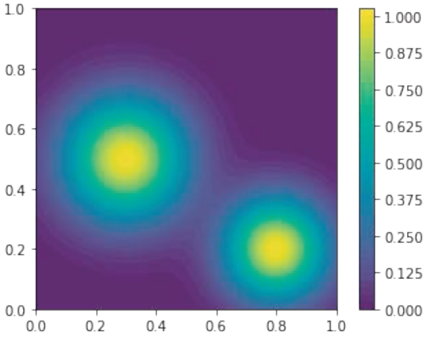
$$\frac{\partial\psi}{\partial t} = -\mathbf{v}\cdot\nabla\psi + k\Delta\psi$$



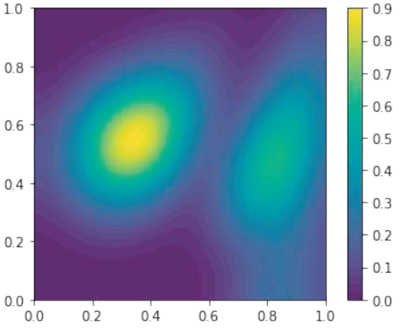
learned velocity

2.5% absolute error in preliminary study

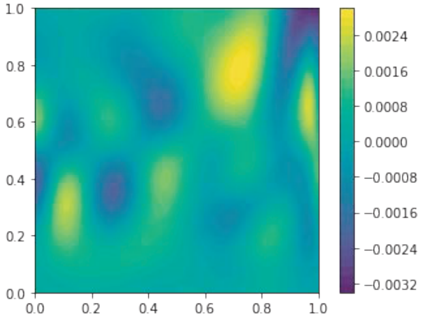
Use two-layer neural network to infer advective fields based on satellite imagery



initital test concentation



predicted concentation



error

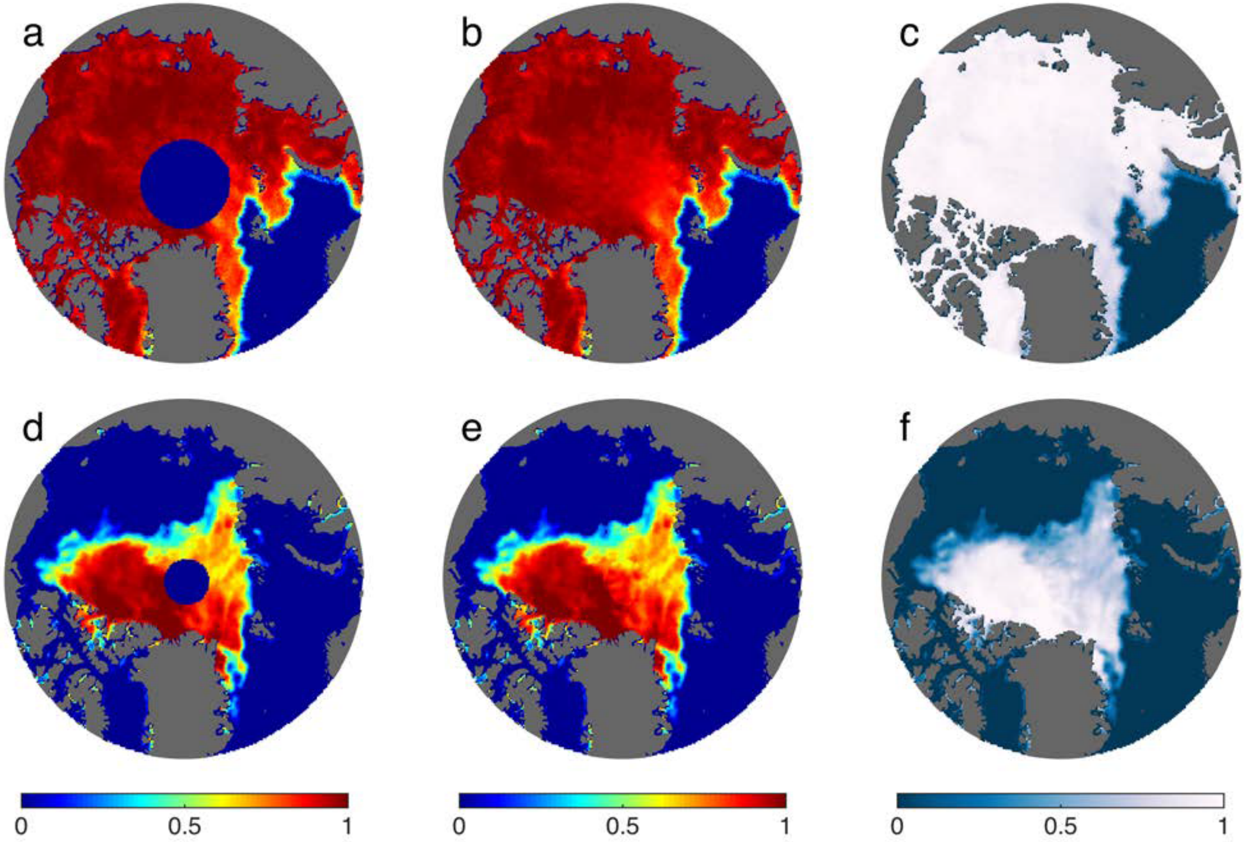
Filling the polar data gap with partial differential equations

hole in satellite coverage of sea ice concentration field

previously assumed ice covered

Gap radius: 611 km
06 January 1985

Gap radius: 311 km
30 August 2007



$$\Delta\psi=0$$

fill = harmonic function with learned stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

NOAA/NSIDC Sea Ice Concentration CDR product update will use our PDE method.

Conclusions

1. Sea ice is a fascinating multiscale composite with structure similar to many other natural and man-made materials.

2. Mathematical methods developed for sea ice advance theories of composites and inverse problems in science and engineering.

3. Homogenization and statistical physics help ***link scales in sea ice and composites***; provide rigorous methods for finding effective behavior; advance sea ice representations in climate models.

4. Inverse problems of many types arise naturally in studying sea ice and the impact of climate change in Earth's polar regions.

5. Field experiments are essential to developing relevant mathematics.

6. Our research is helping to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

University of Utah Sea Ice Modeling Group (2017-2021)

Senior Personnel: Ken Golden, Distinguished Professor of Mathematics
Elena Cherkaev, Professor of Mathematics
Court Strong, Associate Professor of Atmospheric Sciences
Ben Murphy, Adjunct Assistant Professor of Mathematics

Postdoctoral Researchers: Noa Kraitzman (now at ANU), Jody Reimer

Graduate Students: Kyle Steffen (now at UT Austin with Clint Dawson)
Christian Sampson (now at UNC Chapel Hill with Chris Jones)
Huy Dinh (now a sea ice MURI Postdoc at NYU/Courant)
Rebecca Hardenbrook
David Morison (Physics Department)
Ryleigh Moore
Delaney Mosier
Daniel Hallman

Undergraduate Students: Kenzie McLean, Jacqueline Cinella Rich,
Dane Gollero, Samir Suthar, Anna Hyde,
Kitsel Lusted, Ruby Bowers, Kimball Johnston,
Jerry Zhang, Nash Ward, David Gluckman

High School Students: Jeremiah Chapman, Titus Quah, Dylan Webb

Sea Ice Ecology Group Postdoc Jody Reimer, Grad Student Julie Sherman,
Undergraduates Kayla Stewart, Nicole Forrester

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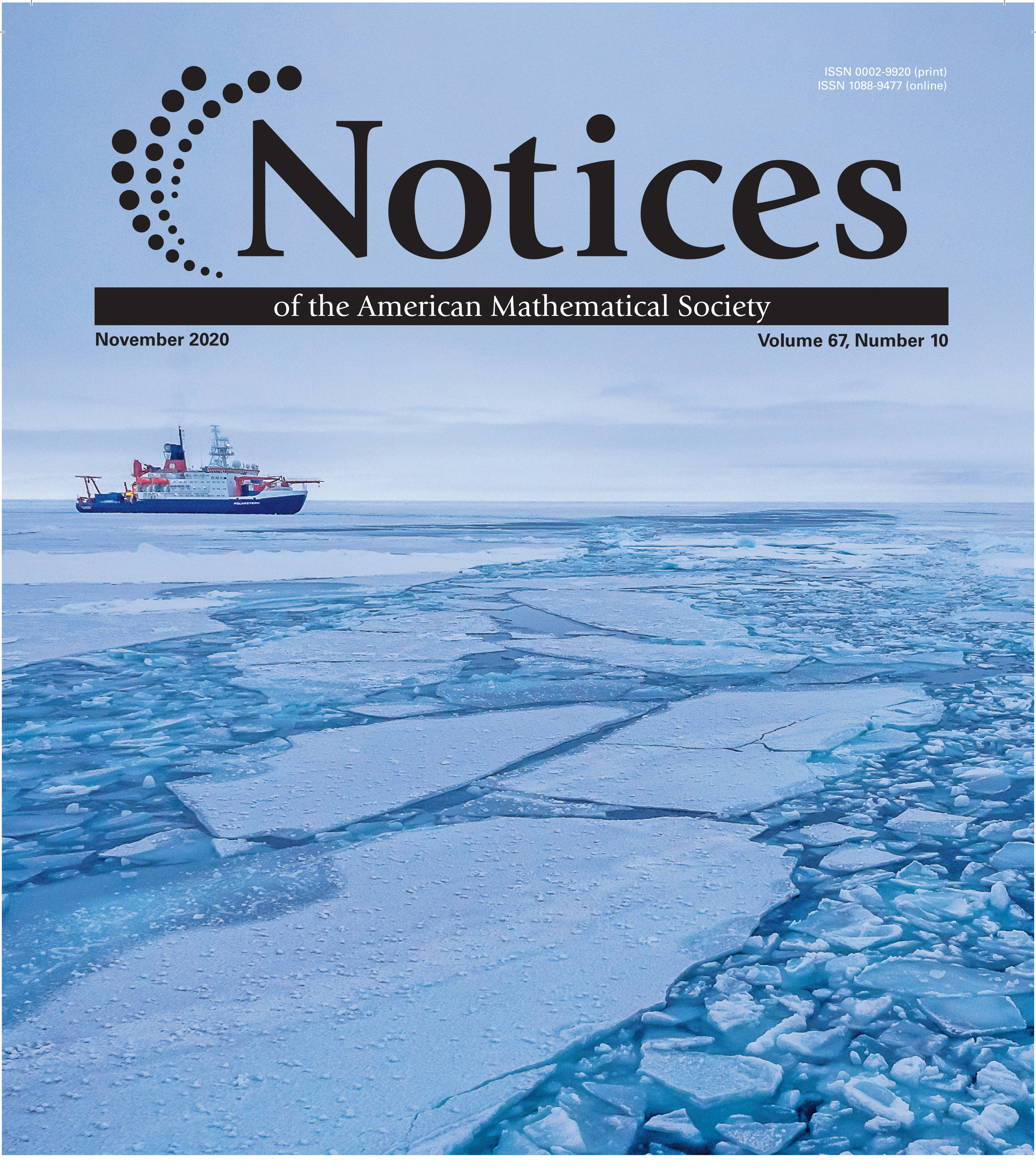

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Advancing research. Creating connections.

The cover is based on "Modeling Sea Ice," page 1535.

THANK YOU

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Applied and Computational Analysis Program

Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

UNITED STATES COAST GUARD
1790

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Australian Antarctic Division

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ANTARCTIC CLIMATE & ECOSYSTEMS
COOPERATIVE RESEARCH CENTRE

Antarctica New Zealand

UAF
UNIVERSITY OF ALASKA FAIRBANKS

THE UNIVERSITY OF UTAH

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999