

MODELING *the* MELT:

what math tells us about the shrinking polar ice caps

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***The Mathematics of Planet Earth
Joint Math Meetings January 11, 2015***

Frey

SEA ICE covers 7 - 10% of earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- indicator and agent of **climate change**



polar ice caps critical to global climate in reflecting incoming solar radiation



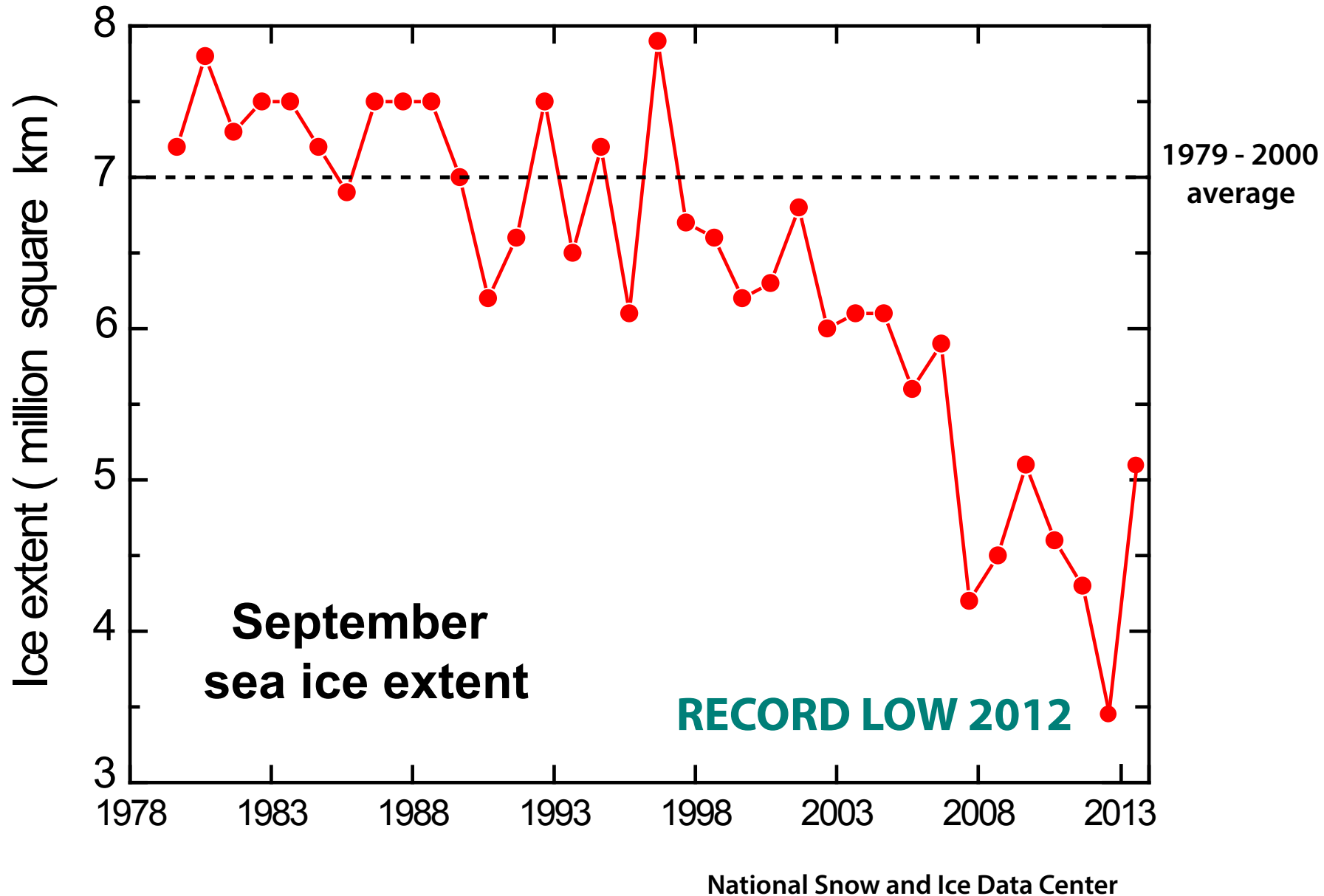
white snow and ice
reflect



dark water and land
absorb

$$\text{albedo } \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

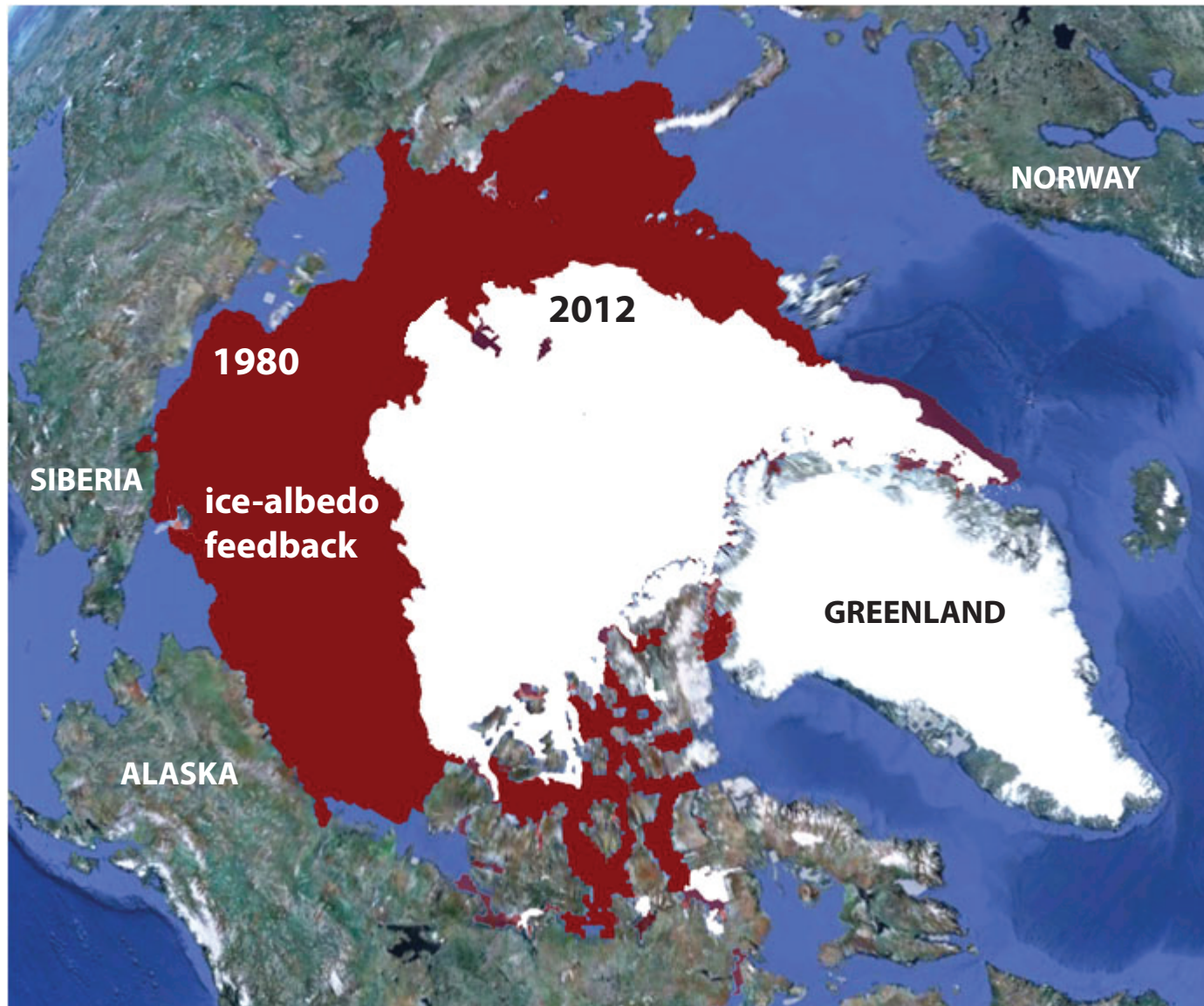
the summer Arctic sea ice pack is melting



Change in Arctic Sea Ice Extent

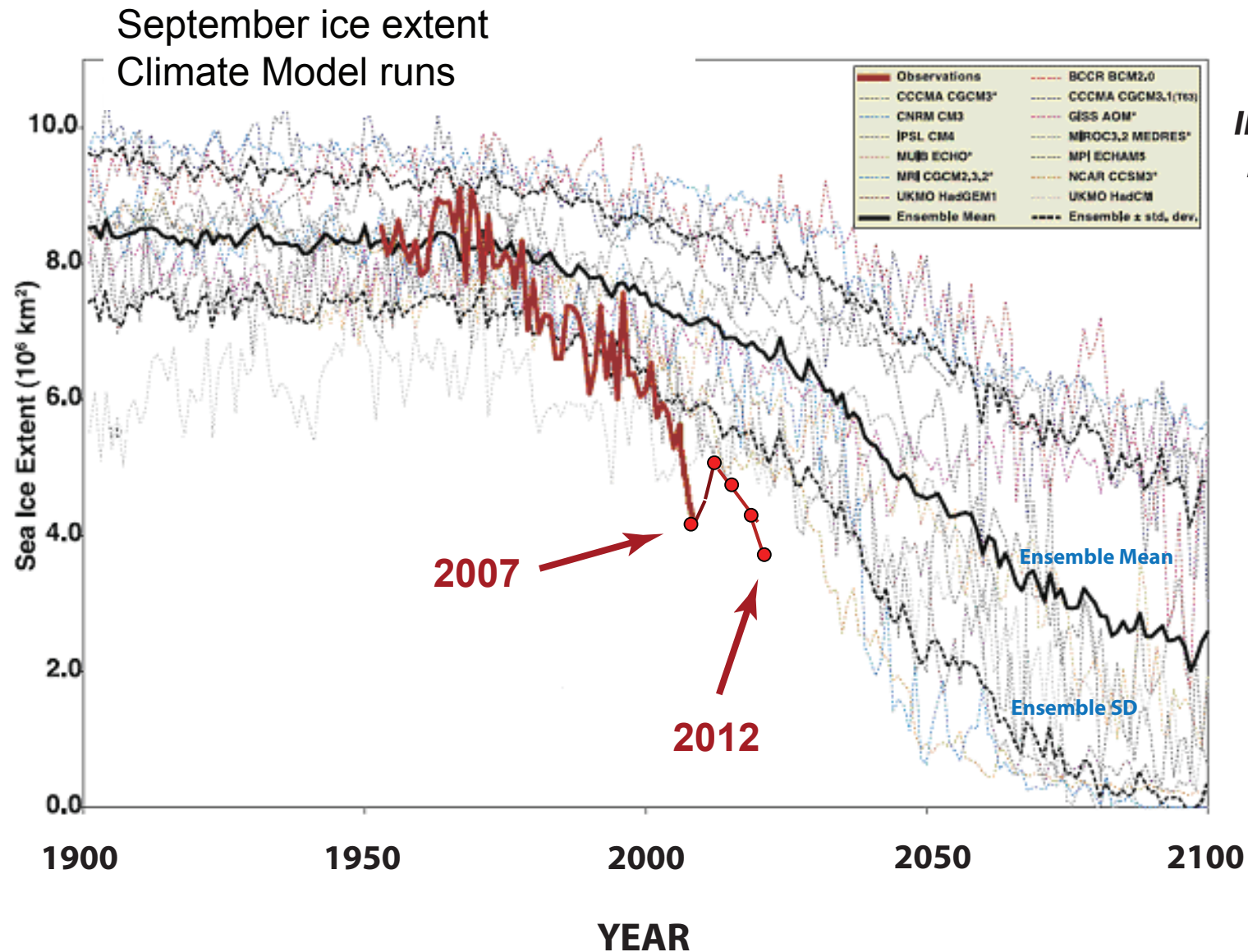
September 1980 -- **7.8** million square kilometers

September 2012 -- **3.4** million square kilometers



Arctic sea ice decline - faster than predicted by climate models

Stroeve et al., GRL, 2007



IPCC AR4
Models

challenge

represent sea ice more rigorously in climate models

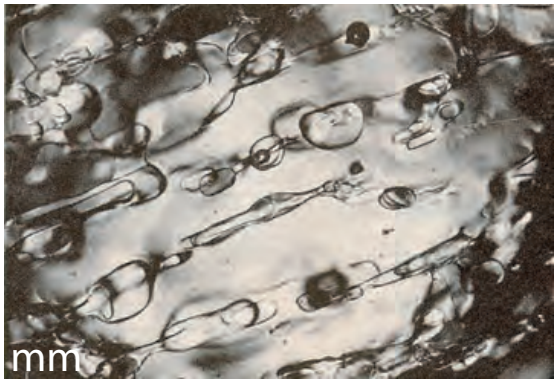
incorporate key processes

fundamental problem -- linkage of scales

sub-grid scale processes

sea ice displays *multiscale* structure over 10 orders of magnitude

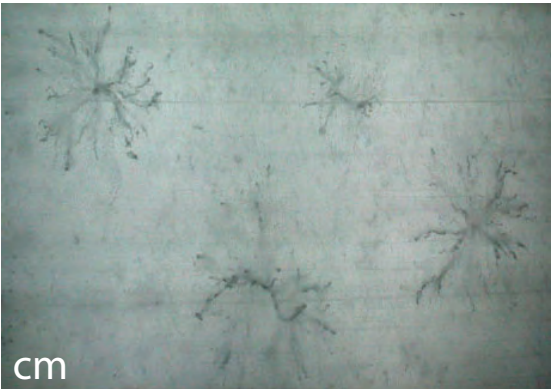
0.1 millimeter



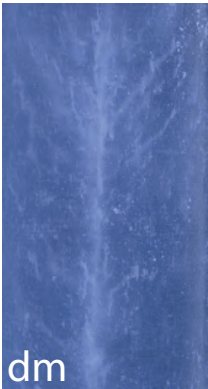
brine inclusions



polycrystals



horizontal

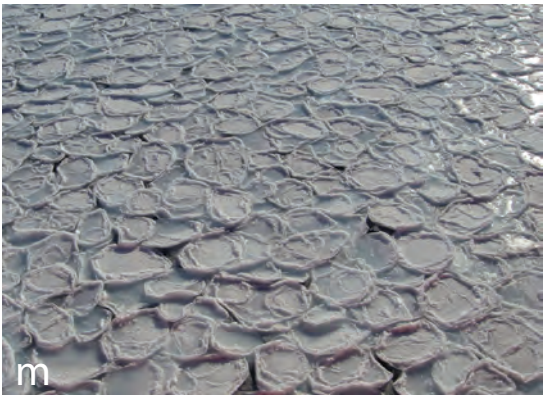
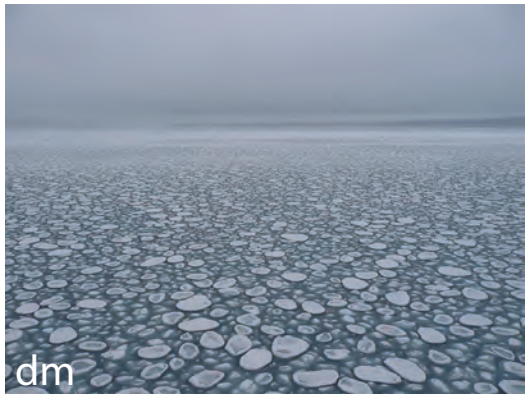


brine channels



vertical

1 meter

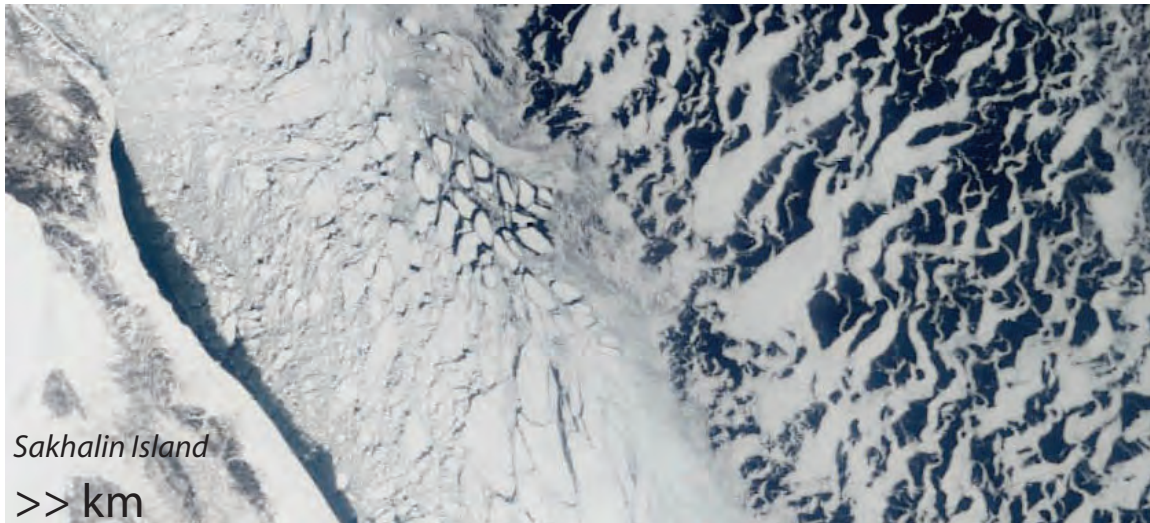


pancake ice

1 meter



100 kilometers



What is this talk about?

Using the mathematics of composite materials and statistical physics to study sea ice structures and processes ... to improve projections of climate change.

1. Fluid flow through sea ice - percolation

homogenization for composite materials

2. Electromagnetic monitoring of sea ice

homogenization for larger scale structures

3. Fractal geometry of Arctic melt ponds

sea ice microphysics

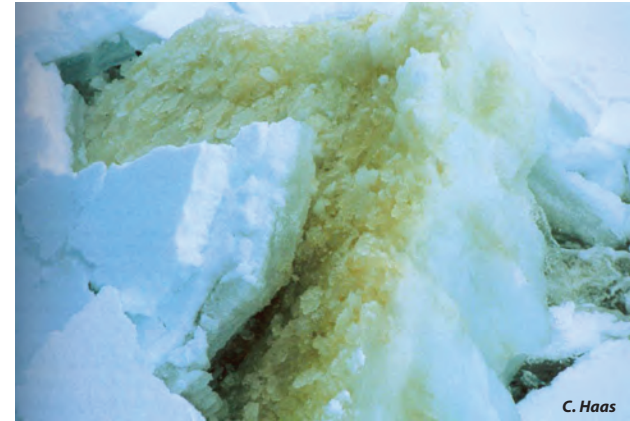
fluid transport

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo



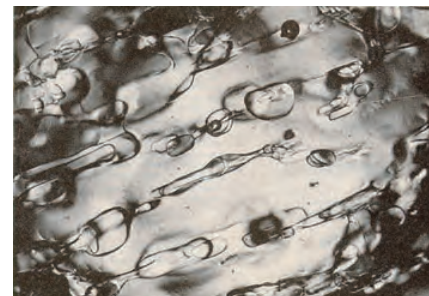
nutrient flux for algal communities



- *drainage of brine and melt water*
- *ocean-ice-air exchanges of heat, CO₂*
- *Antarctic surface flooding and snow-ice formation*
- *evolution of salinity profiles*



linkage of scales



Darcy's Law for slow viscous flow in a porous medium

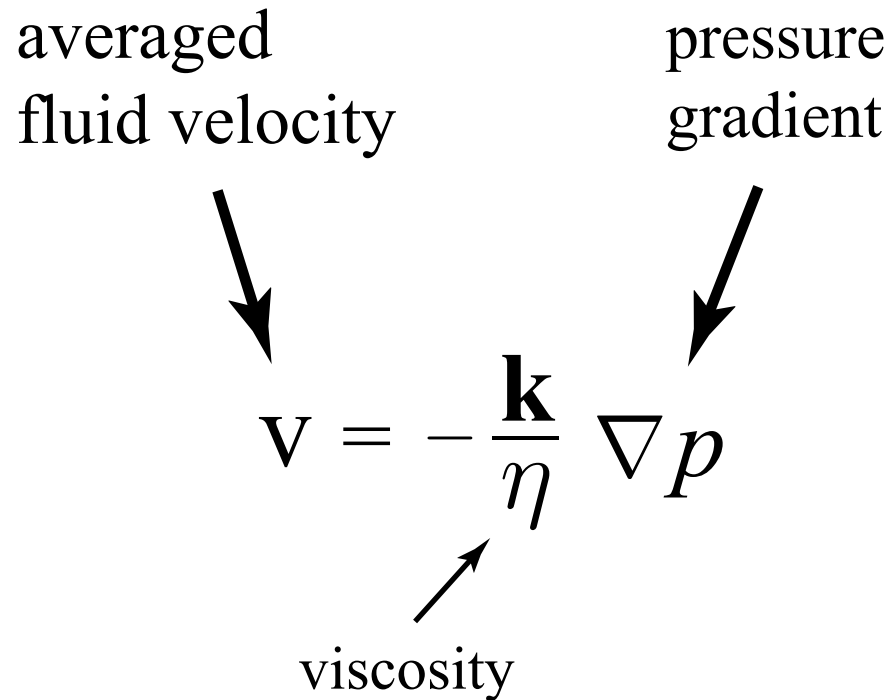


Diagram illustrating Darcy's Law for slow viscous flow in a porous medium. The equation is shown as $\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$. Arrows point from the labels to the corresponding terms in the equation: "averaged fluid velocity" points to \mathbf{v} , "pressure gradient" points to ∇p , and "viscosity" points to η .

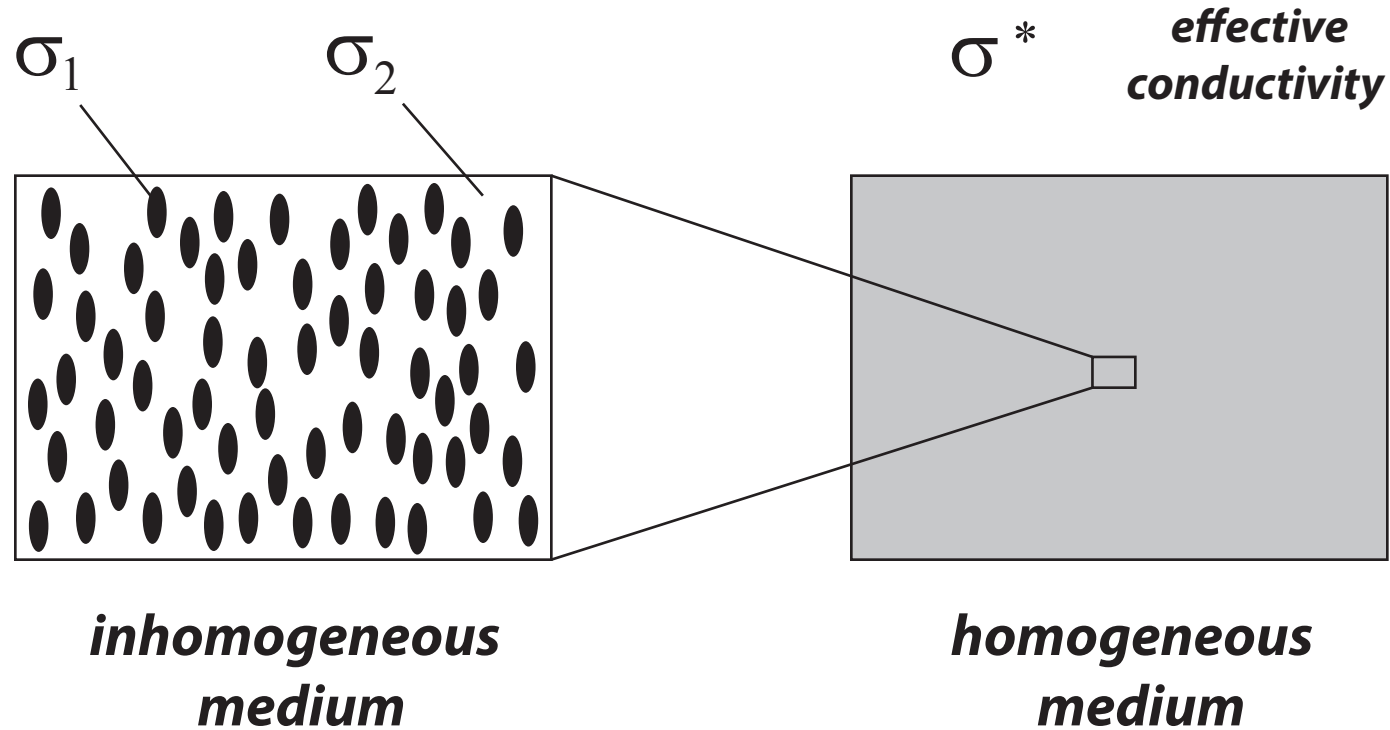
\mathbf{k} = fluid permeability tensor

example of *homogenization*

mathematics for analyzing effective behavior of heterogeneous systems

e.g. transport properties of composites - electrical conductivity, thermal conductivity, etc.

HOMOGENIZATION



**find the homogeneous medium which
behaves macroscopically the same as
the inhomogeneous medium**

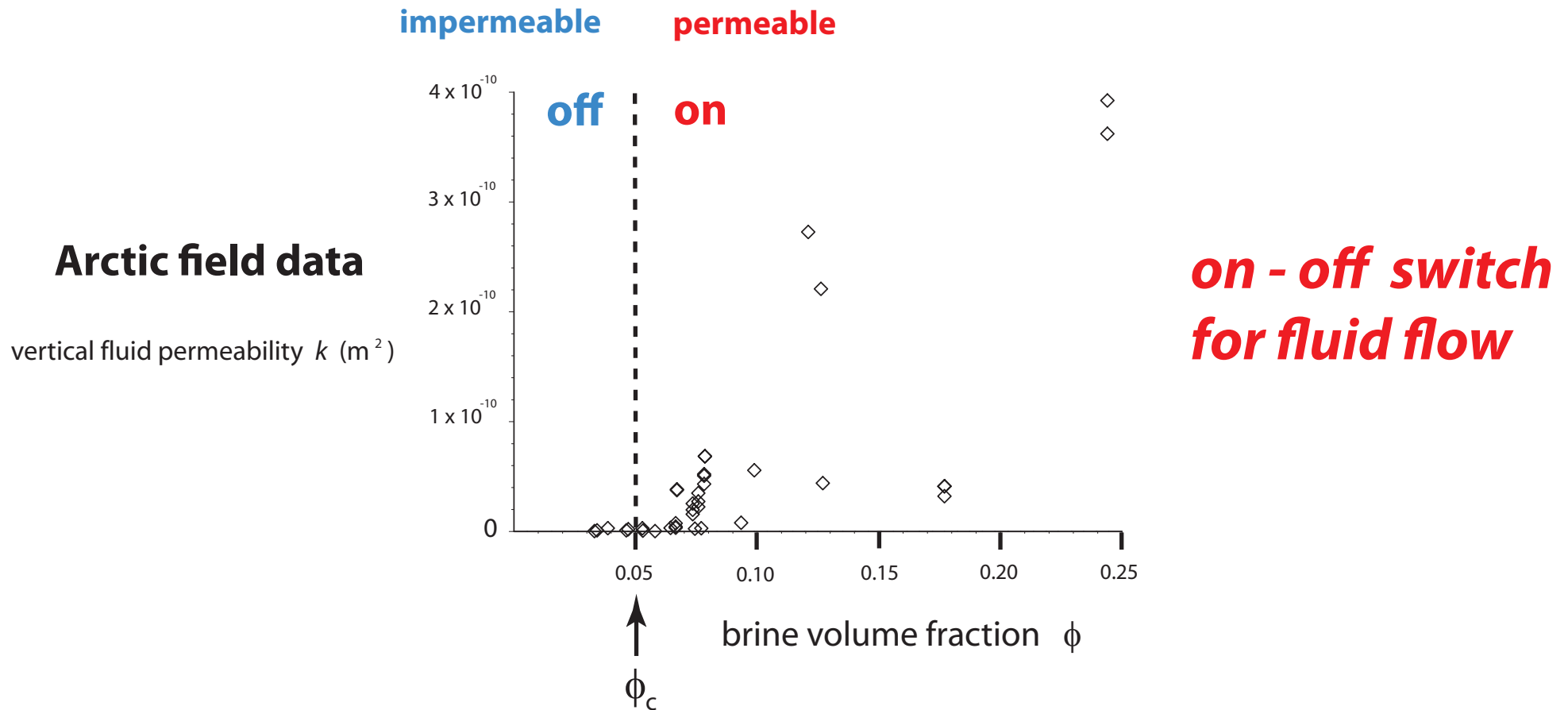
Maxwell 1873 : effective conductivity of a dilute suspension of spheres

Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*

*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

Critical behavior of fluid transport in sea ice



critical brine volume fraction $\phi_c \approx 5\%$ \longleftrightarrow $T_c \approx -5^\circ \text{C}$, $S \approx 5 \text{ ppt}$

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

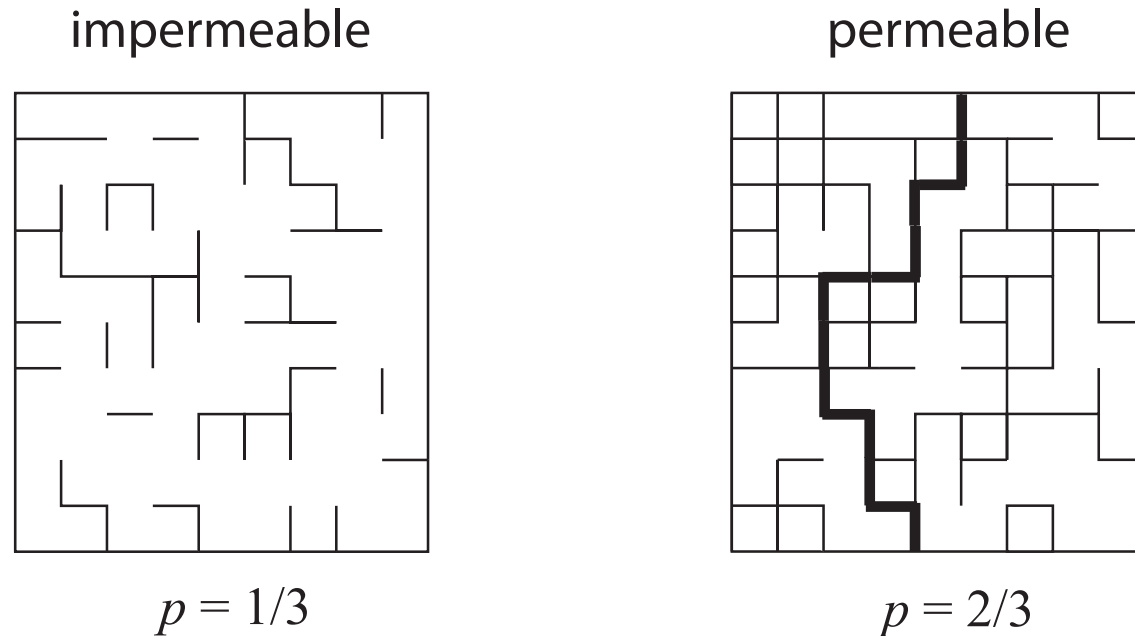
Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

Why is the rule of fives true?

percolation theory

mathematical theory of connectedness



bond \longrightarrow *open* with probability p
closed with probability $1-p$

percolation threshold

$$p_c = 1/2 \quad \text{for } d = 2$$

first appearance of infinite cluster

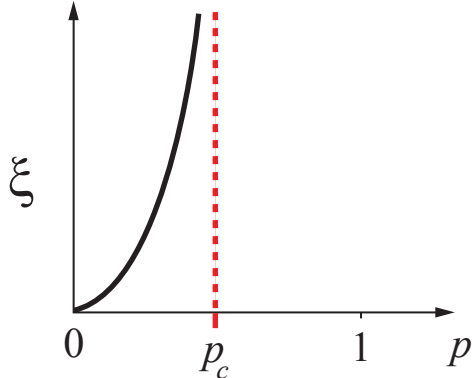
“tipping point” for connectivity

order parameters in percolation theory

geometry

correlation length

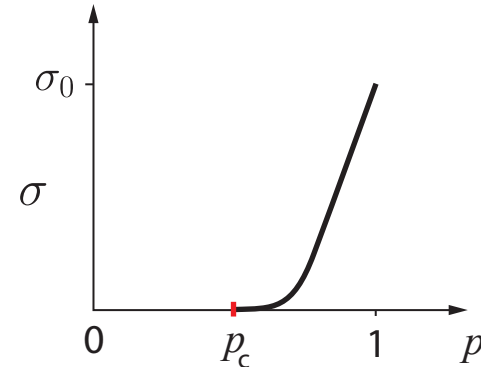
(characteristic scale of connectedness)



$$\xi(p) \sim |p - p_c|^{-\nu}$$

transport

effective conductivity or fluid permeability



$$\sigma(p) \sim \sigma_0 (p - p_c)^t$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

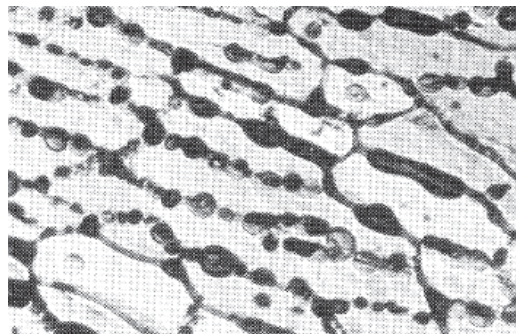
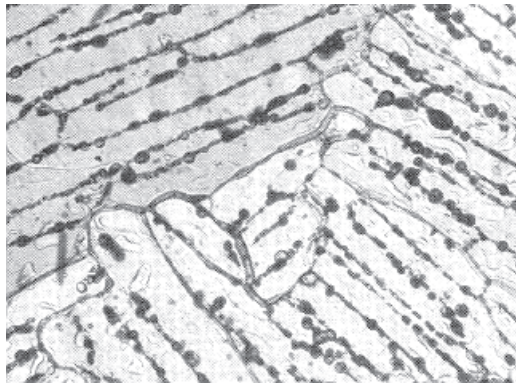
($1 \leq t \leq 2$, Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992)

non-universal behavior in continuum

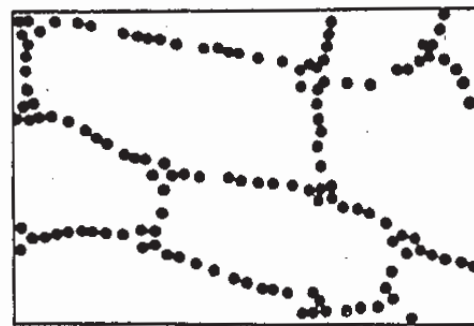
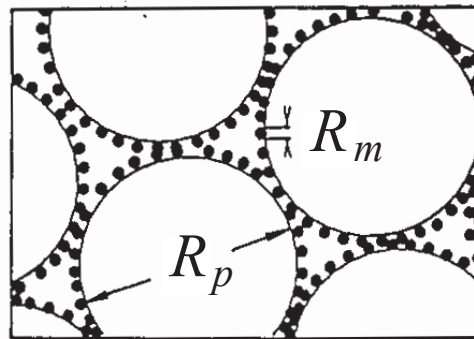
Continuum percolation model for **stealthy** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on **ice production** and **algal growth**

$$\phi_c \approx 5 \%$$

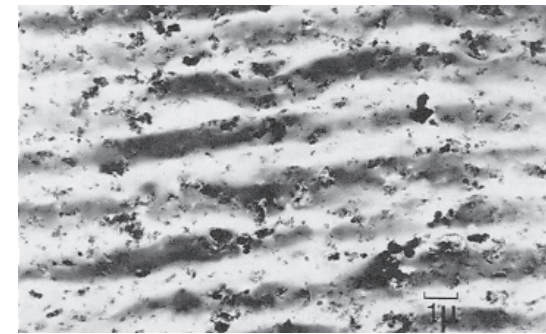
Golden, Ackley, Lytle, *Science*, 1998



sea ice



compressed
powder



radar absorbing
composite

sea ice is radar absorbing



***rigorous bounds
percolation theory
hierarchical model
network model***

field data

X-ray tomography for
brine inclusions

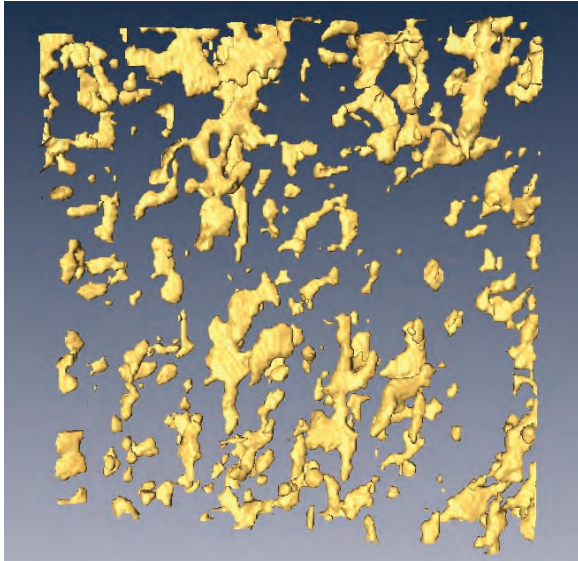
***unprecedented look
at thermal evolution
of brine phase and
its connectivity***

micro-scale
controls
macro-scale
processes

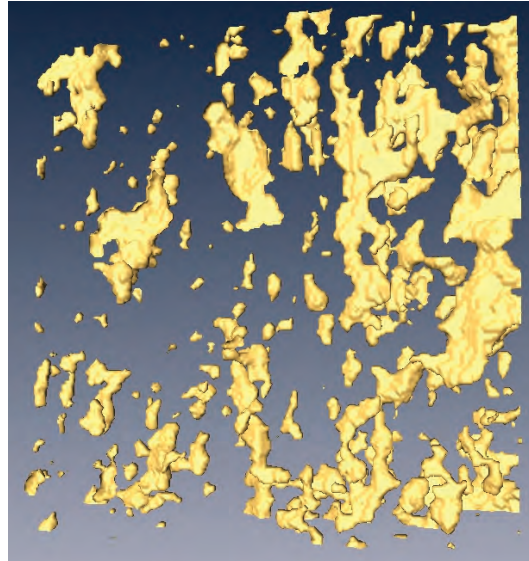
A unified approach to understanding permeability in sea ice • Solving the mystery of
booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

brine connectivity (over cm scale)

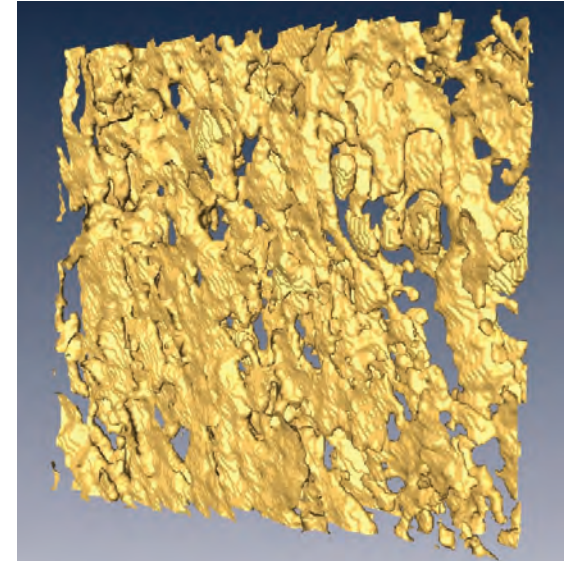
8 x 8 x 2 mm



-15 °C, $\phi = 0.033$



-6 °C, $\phi = 0.075$



-3 °C, $\phi = 0.143$

X-ray tomography confirms percolation threshold

3-D images
pores and throats



3-D graph
nodes and edges

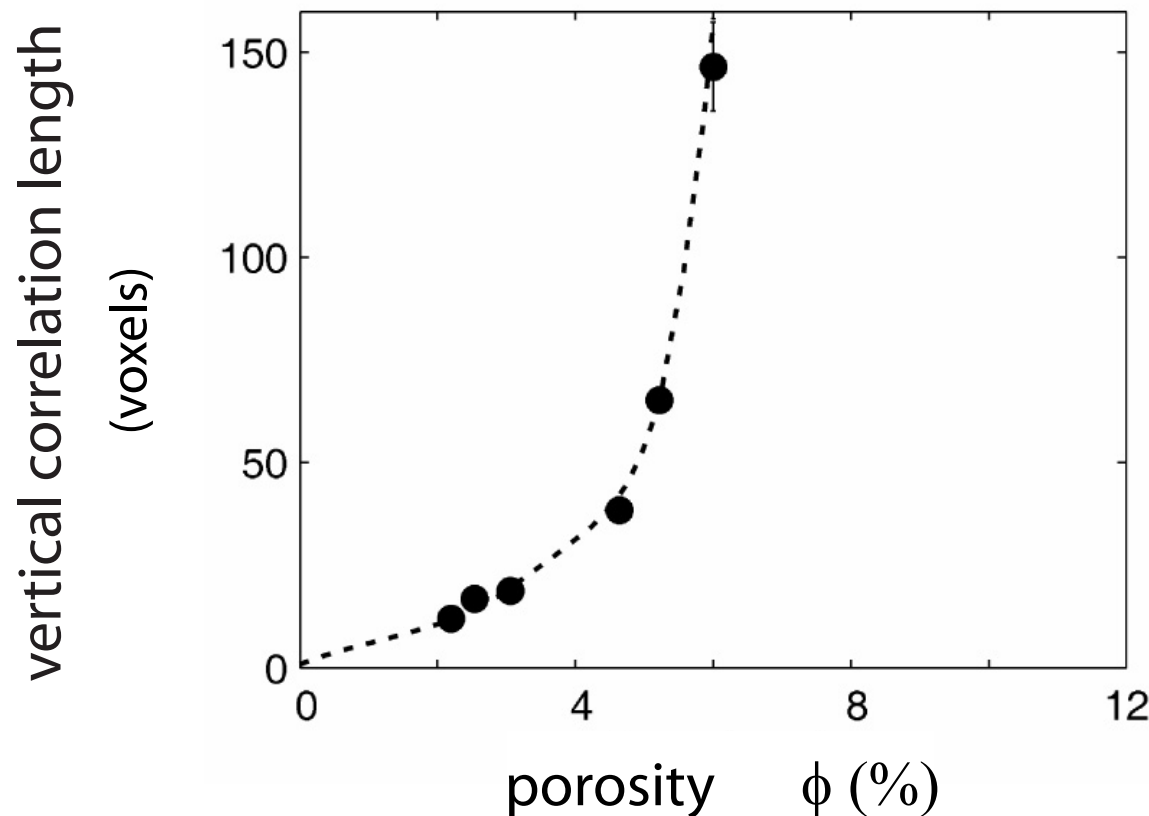
analyze graph connectivity as function of temperature and sample size

- ***use finite size scaling techniques to confirm rule of fives***
- ***order parameter data from a natural material***

The key connectivity functions of percolation theory have been computed **extensively** for many lattice models, but **NOT** for natural materials.

We have calculated them for sea ice single crystals and estimated anisotropic percolation thresholds.

Pringle, Miner, Eicken, Golden, JGR (Oceans) 2009



divergence of
correlation length
for single crystal data

lattice and continuum percolation theories yield:

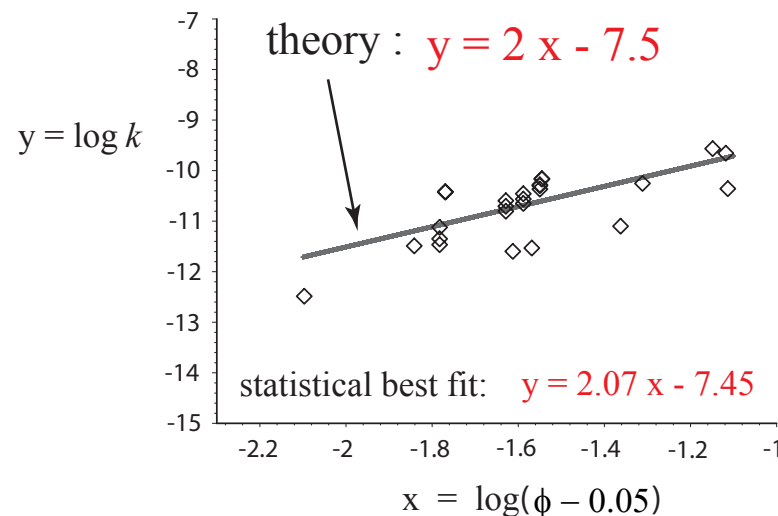
$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical
exponent

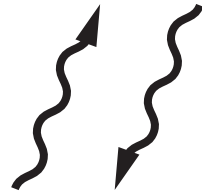
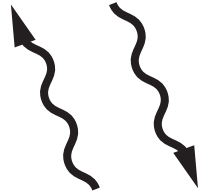
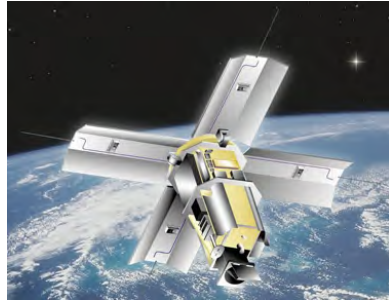
$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

t

- exponent is **UNIVERSAL** lattice value $t \approx 2.0$
- **sedimentary rocks** like sandstones also exhibit universality
- **critical path analysis** -- developed for electronic hopping conduction -- yields scaling factor k_0



Remote sensing of sea ice



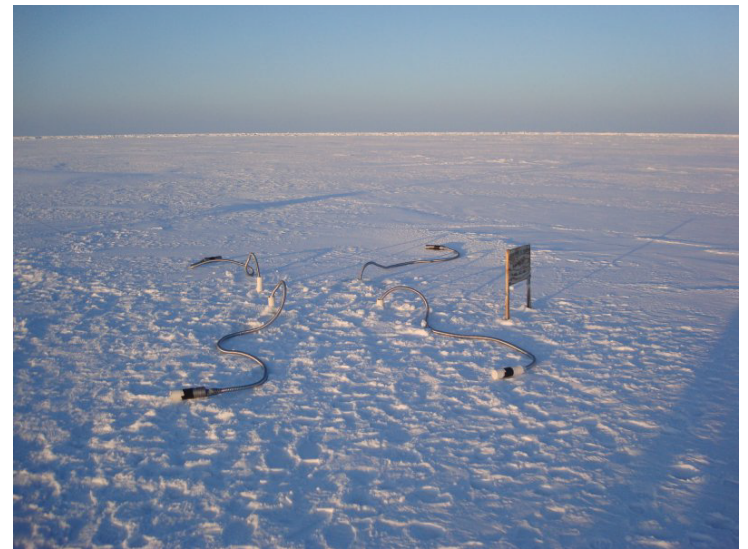
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice
properties from
electromagnetic
(EM) data

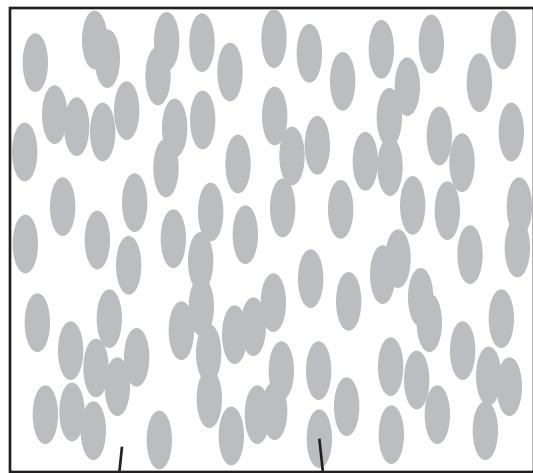
$$\epsilon^*$$

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



ϵ_1

ϵ_2



ϵ^*

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1, p_2 = volume fractions of
the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2}, \text{ composite geometry} \right)$$

Theory of Effective Electromagnetic Behavior of Composites

analytic continuation method

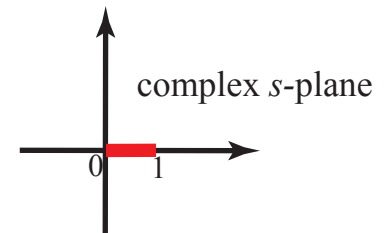
Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

composite geometry
(spectral measure μ) $\longrightarrow \epsilon^*$

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
(McPhedran, McKenzie, and Milton, 1982)

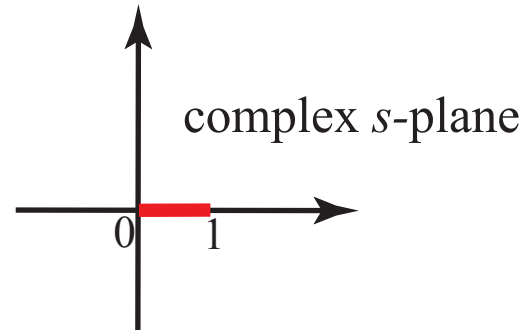
ϵ^* \longrightarrow **composite geometry**
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

Stieltjes integral representation

separation of geometry from parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

- μ /
- spectral measure of self adjoint operator $\Gamma\chi$
 - mass = p_1
 - higher moments depend on n -point correlations

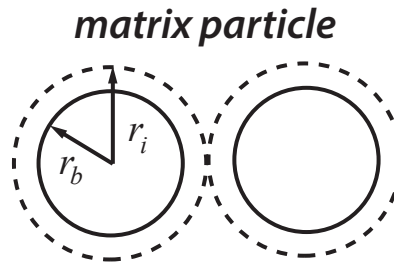
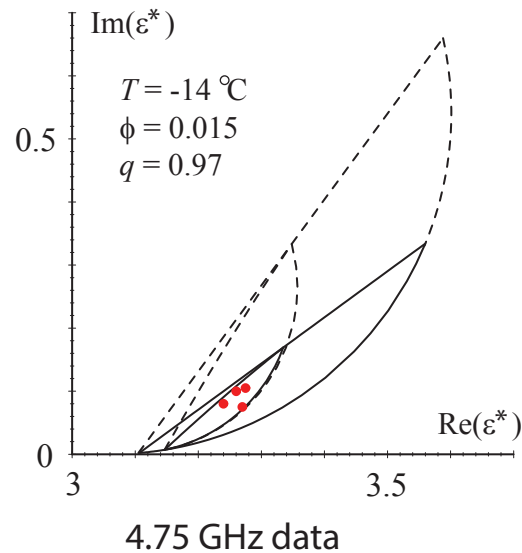
$$\Gamma = \nabla(-\Delta)^{-1}\nabla.$$

χ = characteristic function of the brine phase

$$E = (s + \Gamma\chi)^{-1}e_k$$

forward and inverse bounds for sea ice

forward bounds

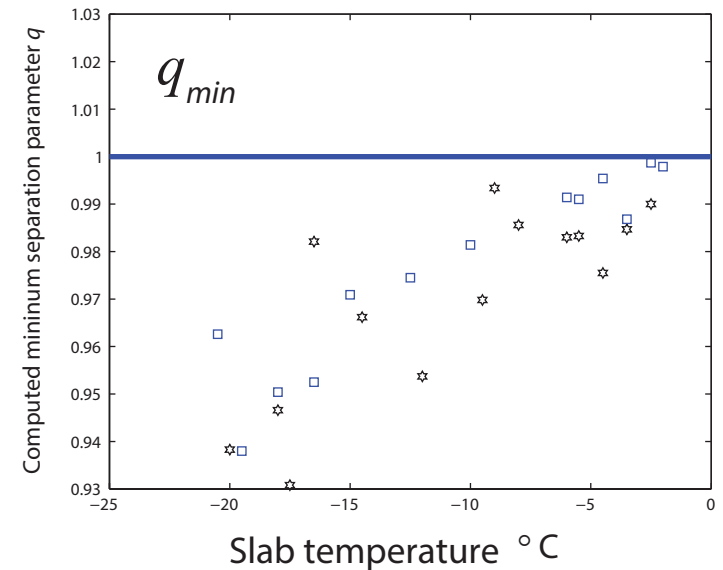


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



inverse bounds and recovery of brine porosity

**Gully, Backstrom, Eicken, Golden
Physica B, 2007**

polycrystalline bounds two-scale homogenization

Gully, Lin, Cherkaev, Golden, 2014

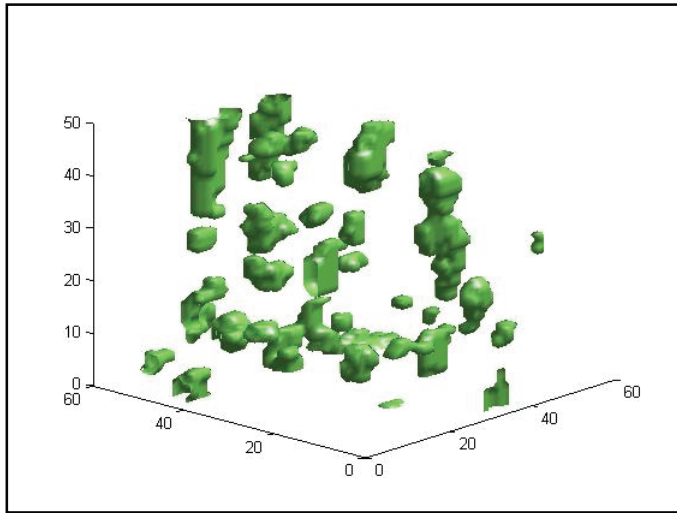
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ϵ^*

rigorous inverse bound
on spectral gap

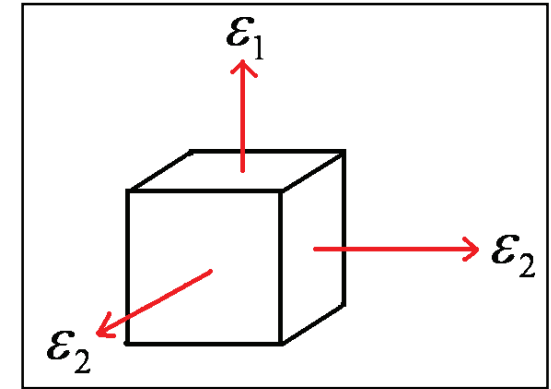
construct algebraic curves which bound
admissible region in (p, q) -space

**Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012**

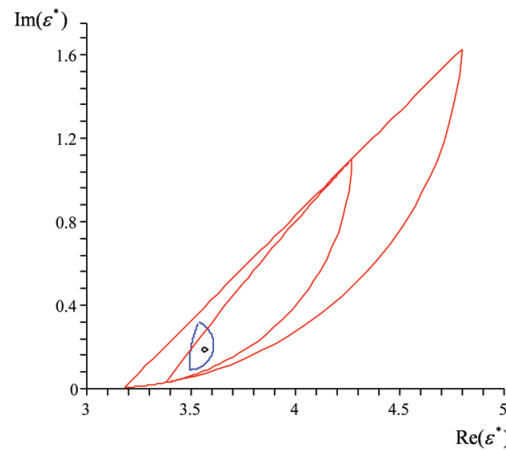
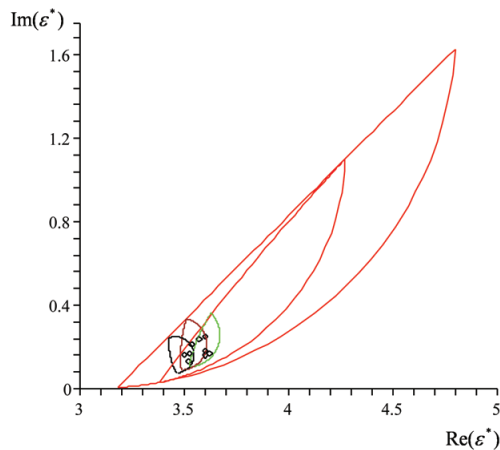
two scale homogenization for polycrystalline sea ice



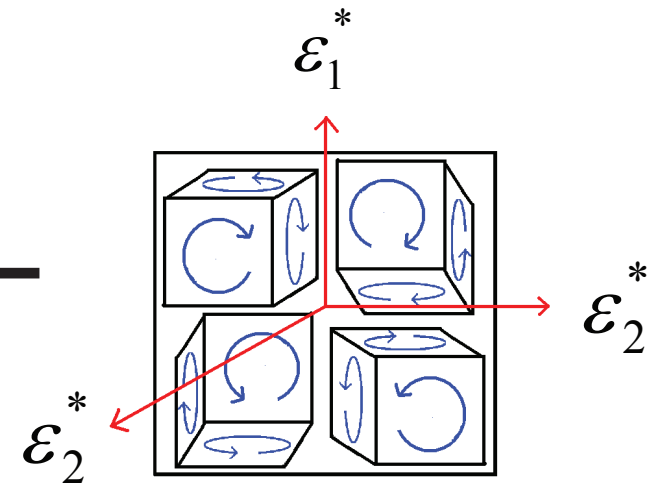
numerical homogenization
for single crystal



analytic continuation
for polycrystals



bounds



direct calculation of spectral measure

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).
2. The fundamental operator $\Gamma\chi$ becomes a random matrix depending only on the composite geometry.
3. Compute the eigenvalues λ_i and eigenvectors of $\Gamma\chi$ with $(\text{length})^2 = \alpha_i$

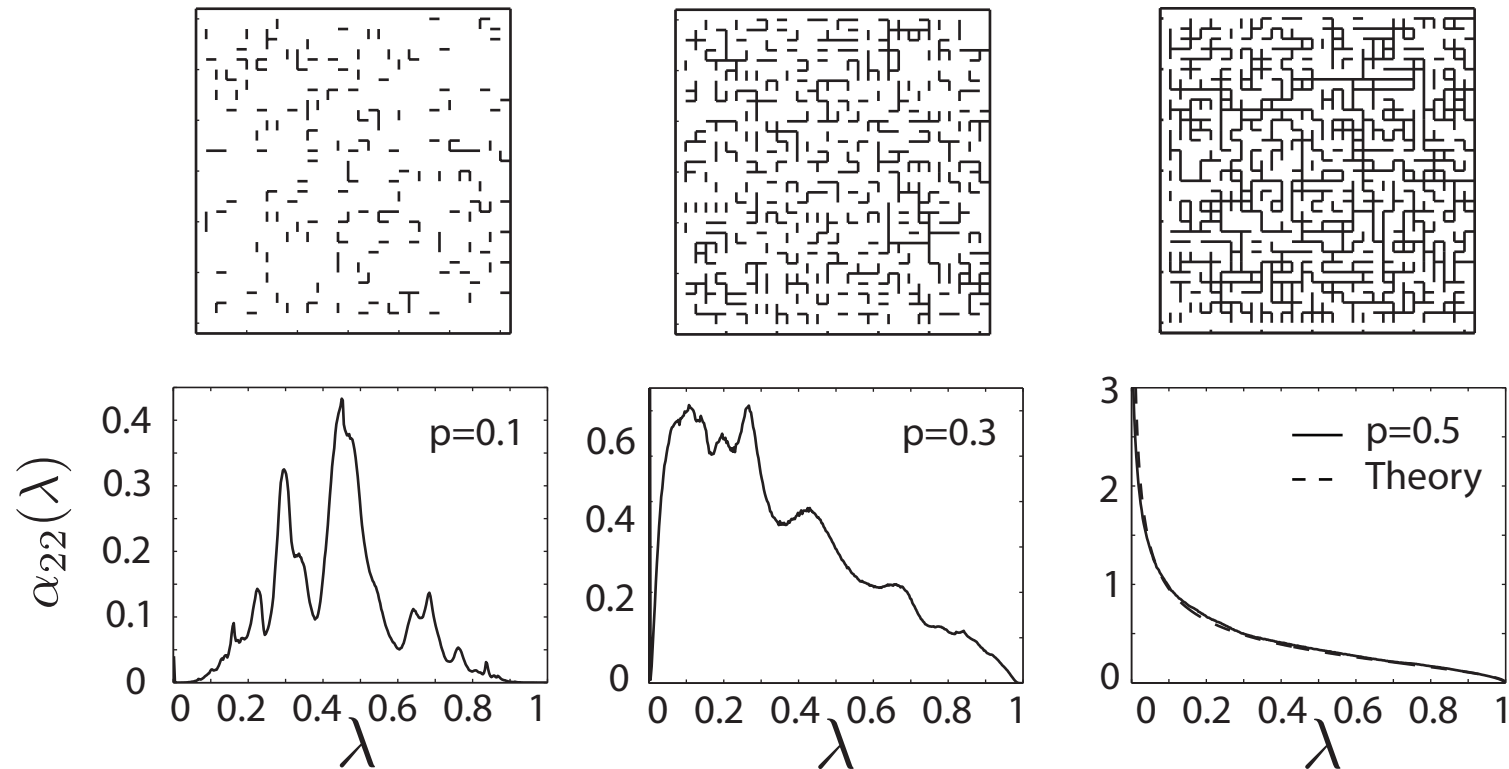
$$\mu(\lambda) = \sum_i \alpha_i \delta(\lambda - \lambda_i)$$



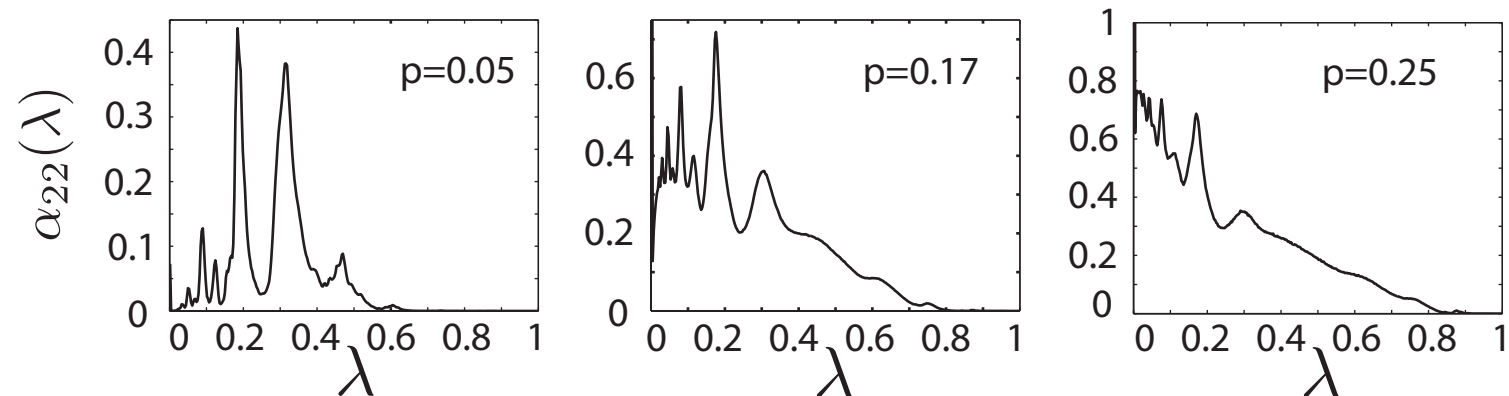
Dirac point measure (Dirac delta)

The Spectral Measures for Random Resistor Networks

2-D



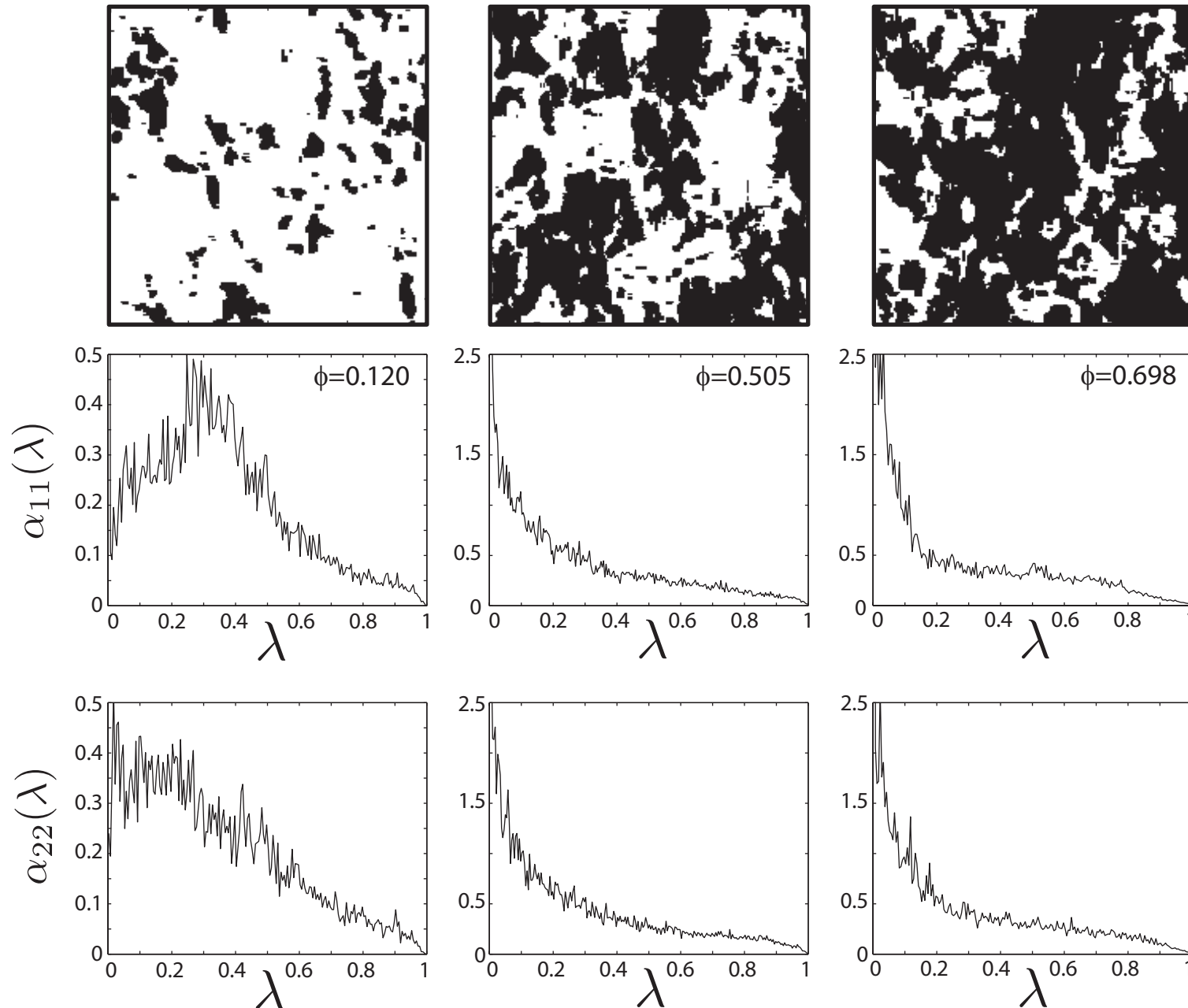
3-D



spectral gaps collapse at the percolation transitions

Murphy and Golden, J. Math. Phys. (2012)

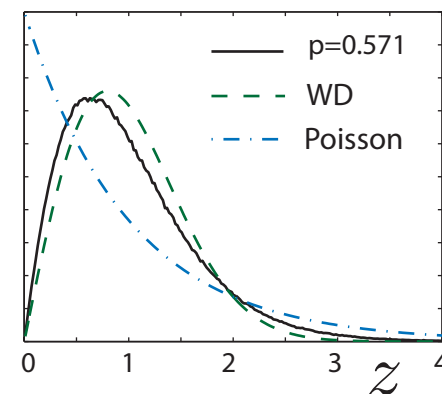
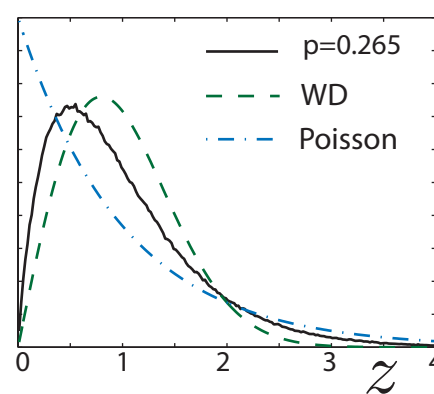
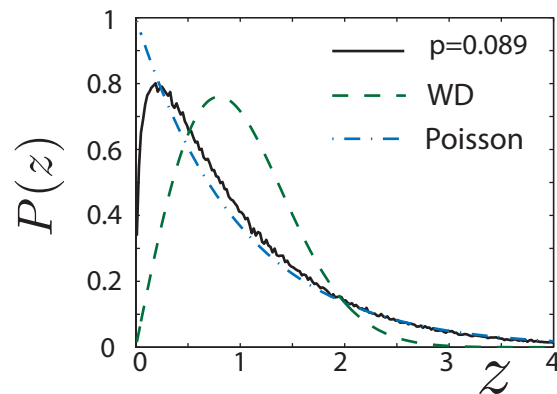
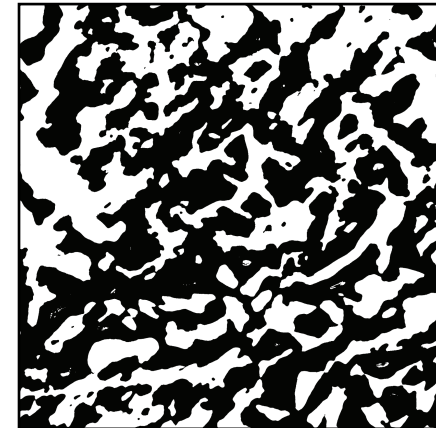
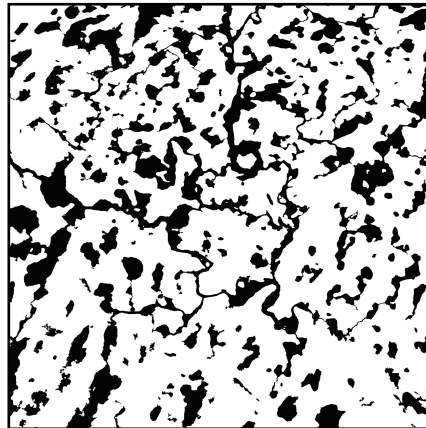
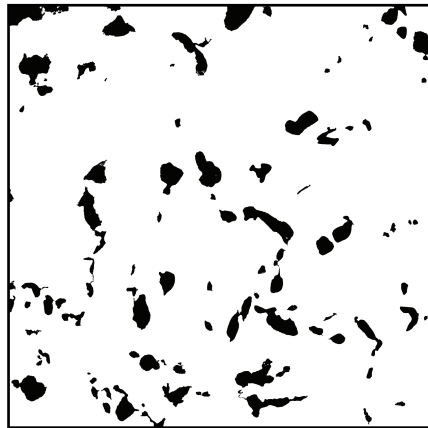
Spectral Measures for Sea Ice Structures: Brine Inclusions



random matrix characterization of connectedness transition -- discretization of $\chi\Gamma\chi$

Unfolded Eigenvalue Spacing Distribution

ARCTIC MELT PONDS



*eigenvalue statistics for transport tend toward the **UNIVERSAL Wigner-Dyson distribution** as the “conducting” phase becomes connected over large scales*

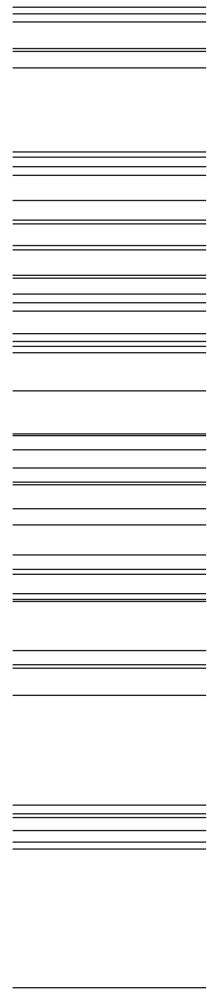
uncorrelated \longrightarrow “level repulsion”

Transitions in Eigenvalue Correlations

$$P(z) = \exp(-z)$$

Eigenvalue Spacing Distribution

**Poisson
Spectra**

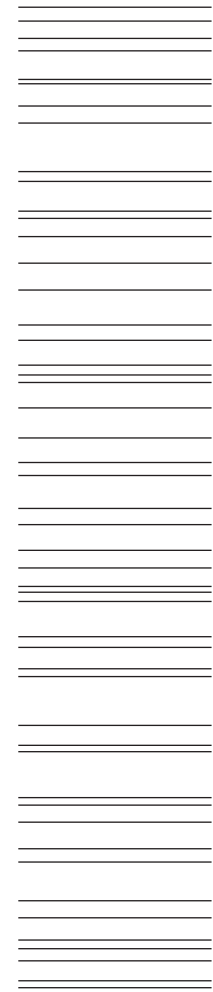


Uncorrelated

$$P(z) \approx \frac{\pi z}{2} \exp(-\pi z^2/4)$$

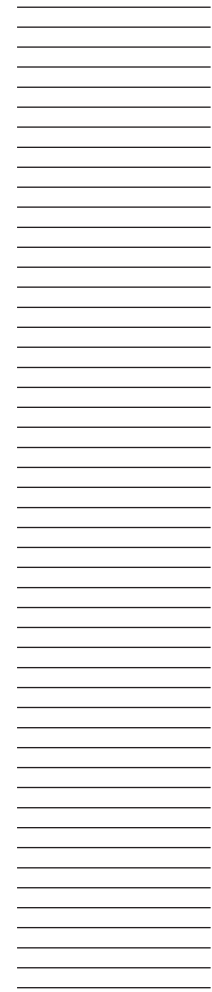
Eigenvalue Spacing Distribution

**WD
Spectra**



**Highly
Correlated**

**Picket
Fence**



**Completely
Correlated**

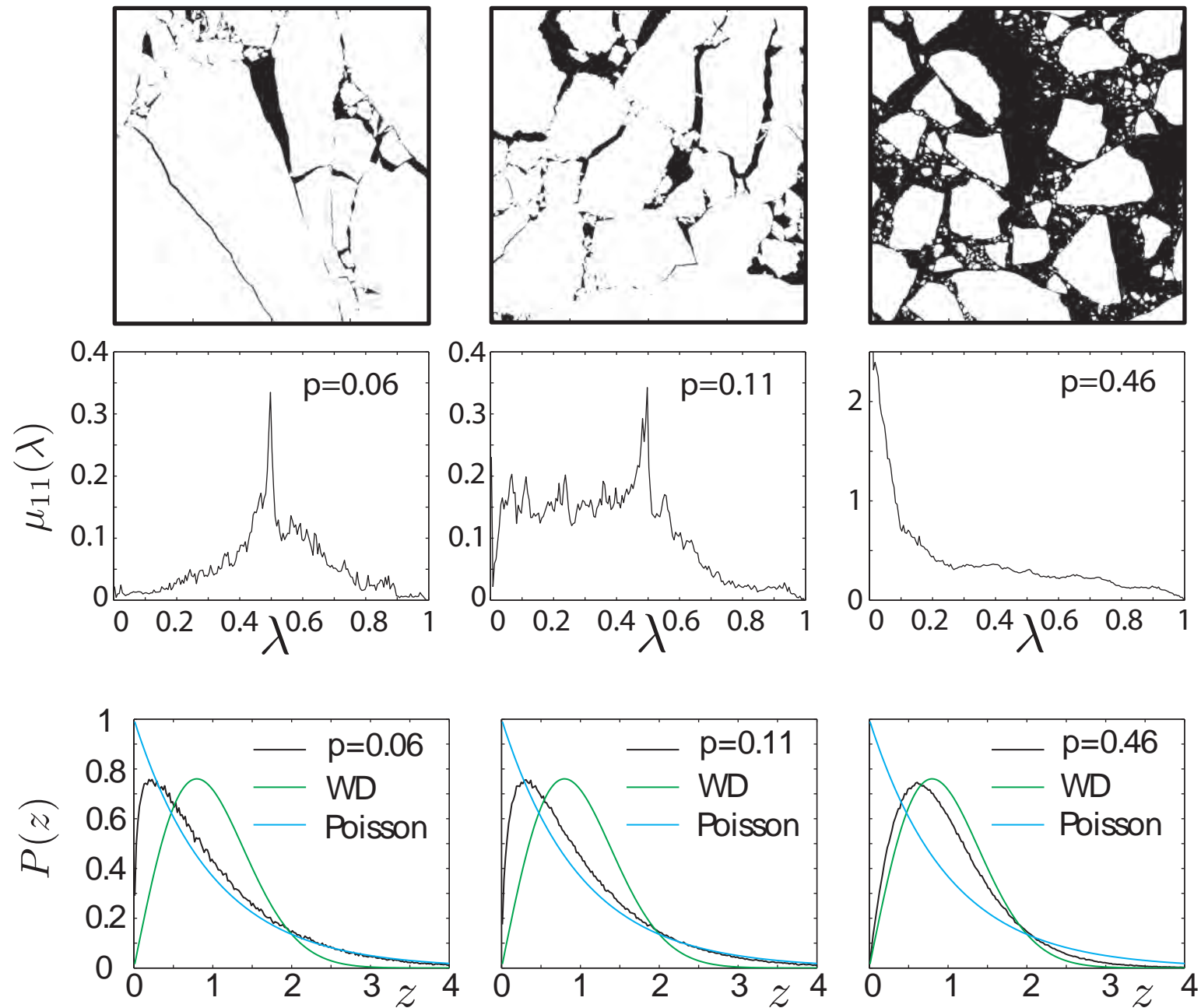
**Connectedness
Phase Transition**



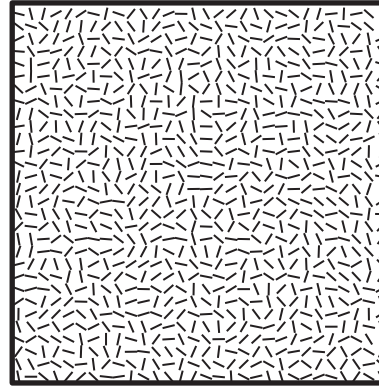
**Eigenvalues
More Correlated**



Spectral computations for Arctic sea ice pack

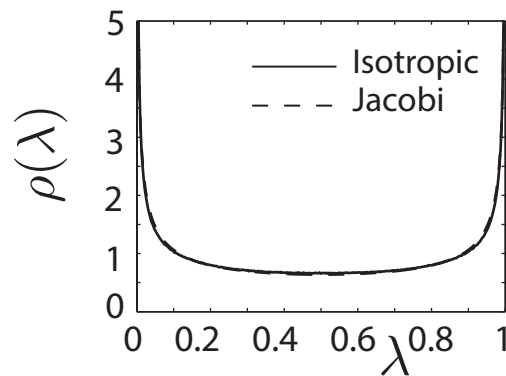


spectral data for isotropic 2-D polycrystalline materials

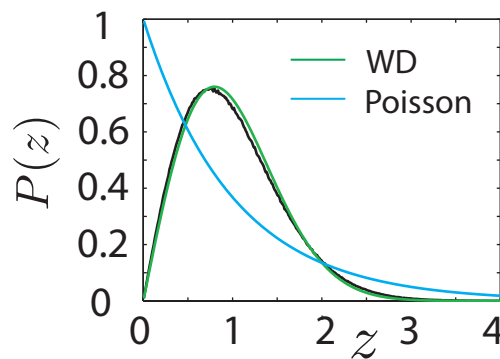
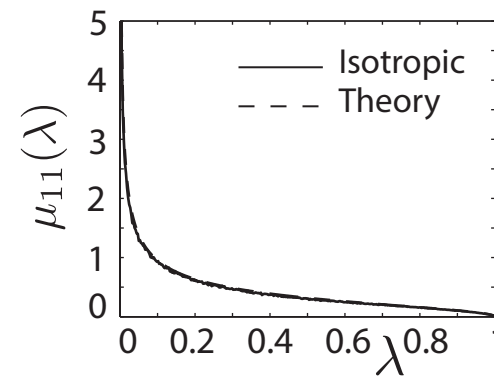


Murphy, Cherkaev, Golden

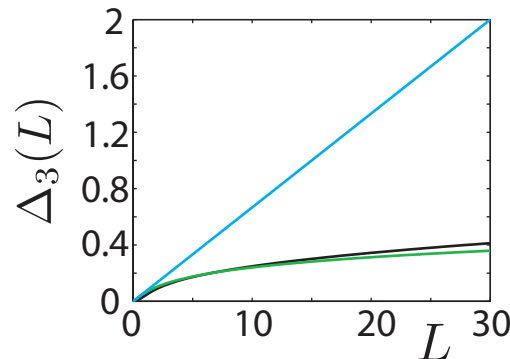
spectral density



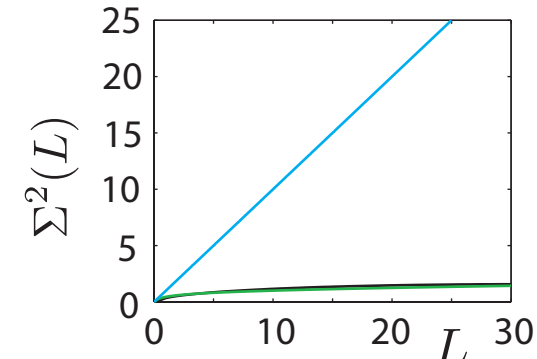
spectral measure



eigenvalue spacings



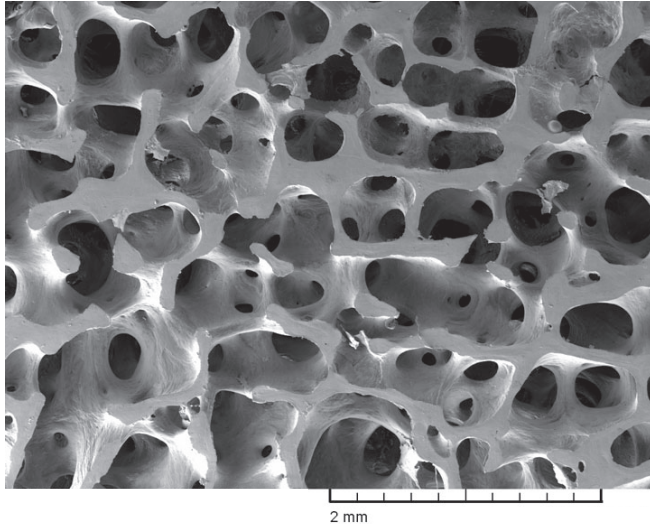
long range correlations



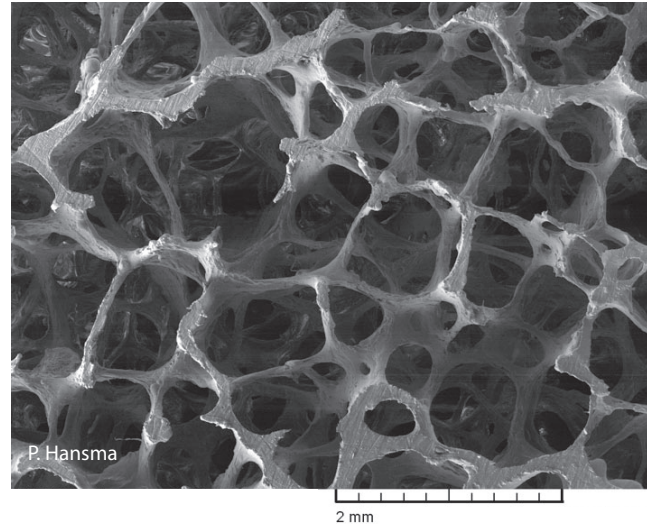
spectral characterization of porous microstructures in bone

Golden, Murphy, Cherkaev, J. Biomechanics 2011

(a) young healthy trabecular bone



(b) old osteoporotic trabecular bone



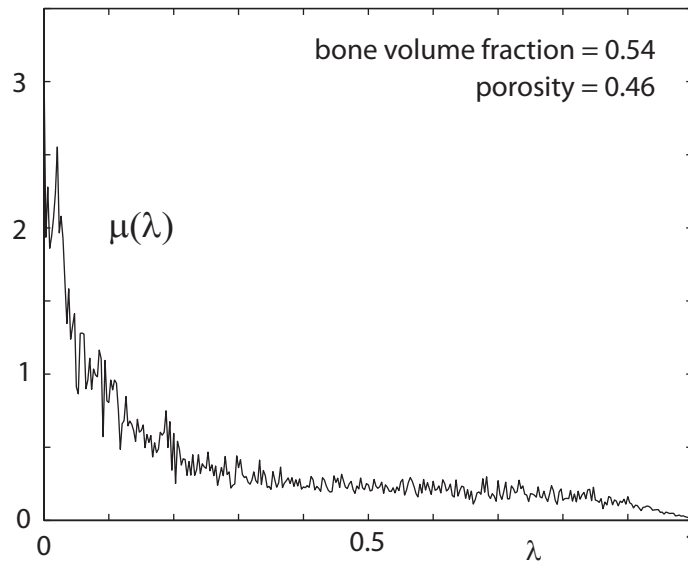
+

reconstruction of spectral
measures from complex
permittivity data

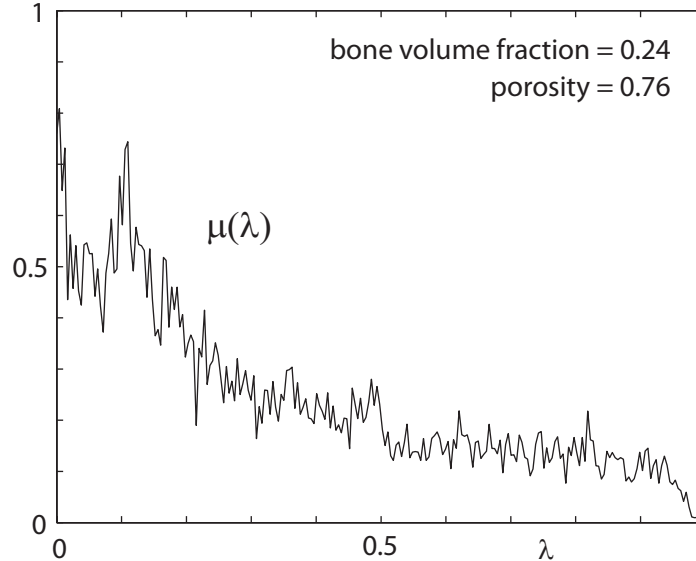
*using regularized
inversion scheme*



(c) spectral measure - young



(d) spectral measure - old

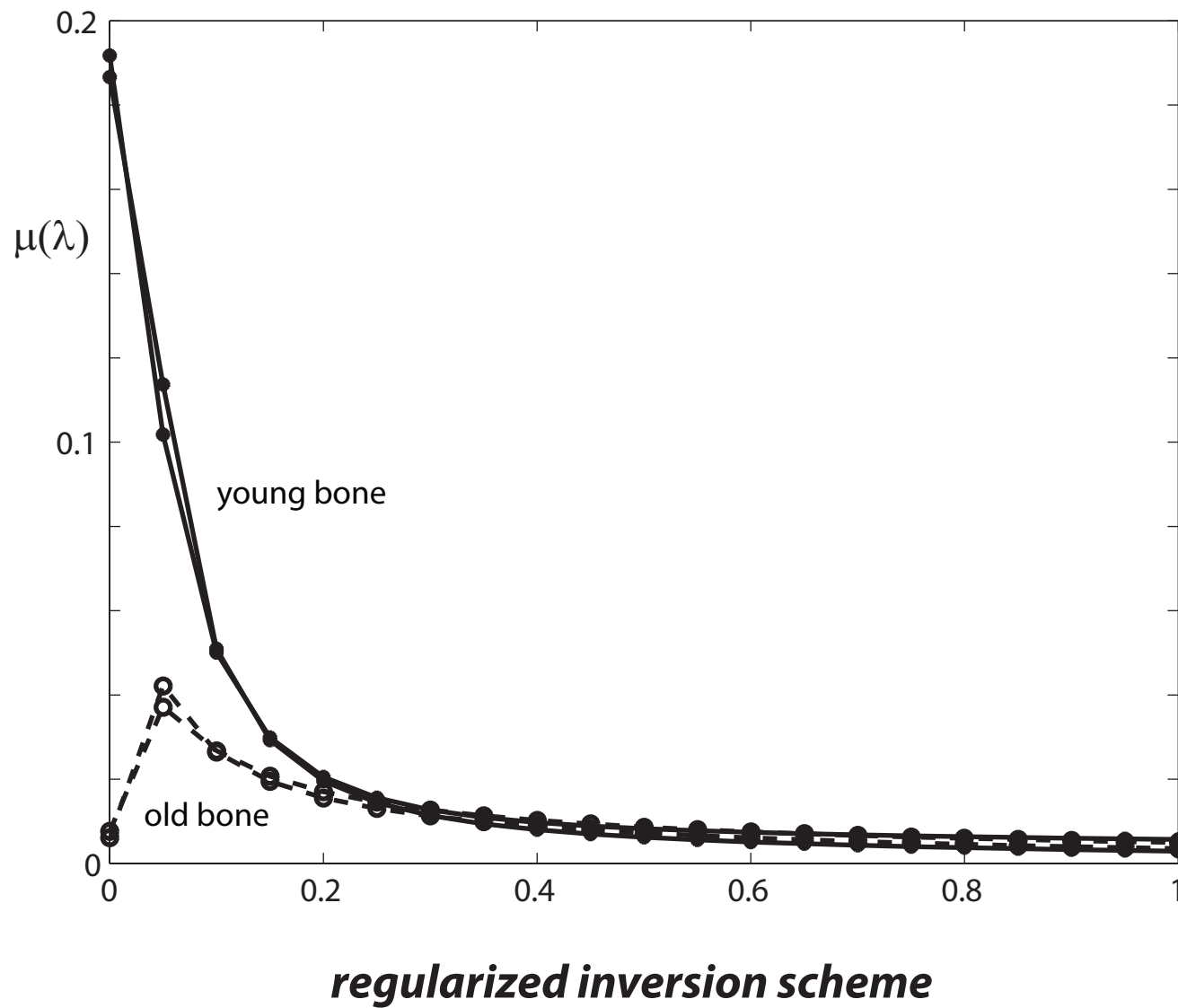


***EM monitoring
of osteoporosis***

***loss of bone
connectivity***

the math doesn't care if it's sea ice or bone!

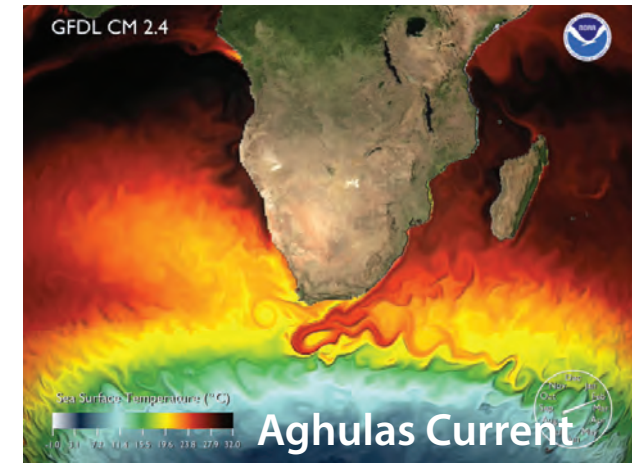
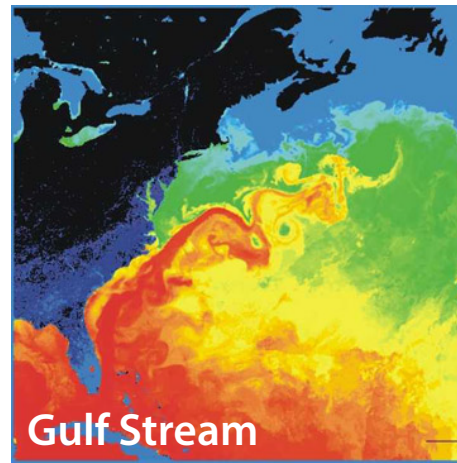
reconstruction of spectral measures from simulated complex permittivity data



advection enhanced diffusion

effective diffusivity

tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere
salt and heat transport in ocean



advection diffusion equation with a velocity field \vec{u}

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$



homogenize

$$\frac{\partial \bar{T}}{\partial t} = \kappa^* \Delta \bar{T}$$

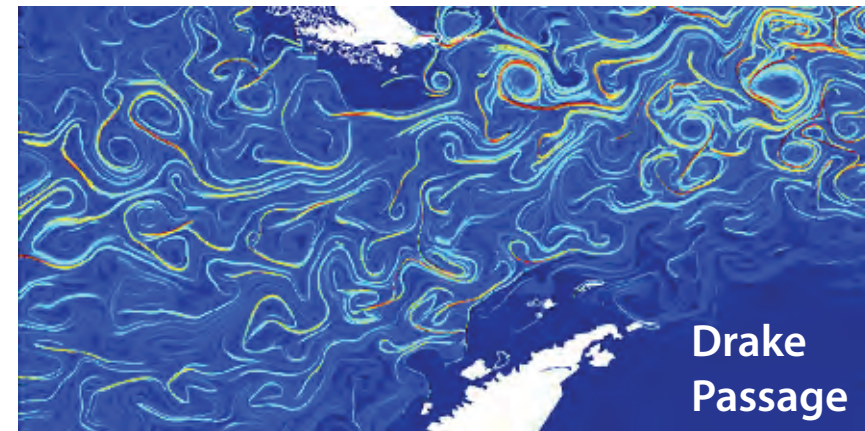
κ^* **effective diffusivity**

analytic function
of Péclet number

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Zhu, Golden 2014



Stieltjes integral for κ^* with spectral measure

composites

Golden and Papanicolaou, CMP 1983

$$\frac{\epsilon^*}{\epsilon_2} = 1 - \int_0^1 \frac{d\mu(\lambda)}{s - \lambda}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

- computations of spectral measures and effective diffusivity for model flows

$$i\Gamma H\Gamma \quad \vec{u} = \kappa_0 \xi \vec{\nabla} \cdot \mathbf{H}$$

\mathbf{H} antisymmetric vector potential

Murphy, Zhu, Golden 2014

- rigorous bounds and computations on convection enhanced thermal conductivity of sea ice

Wang, Liu, Zhu, Golden 2014

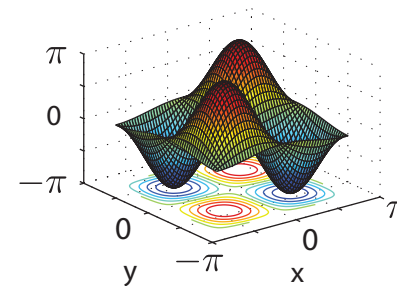
advection diffusion

Avellaneda and Majda, PRL 89, CMP 91

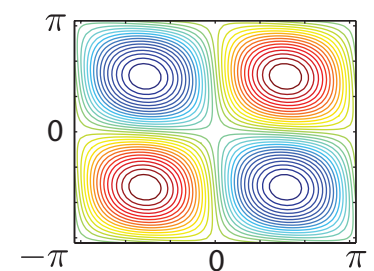
$$\frac{\kappa^*}{\kappa_0} = 1 - \int_0^\infty \frac{d\rho(z)}{t - z}$$

$$t = -1/\xi^2, \quad \xi = \text{Péclet number}$$

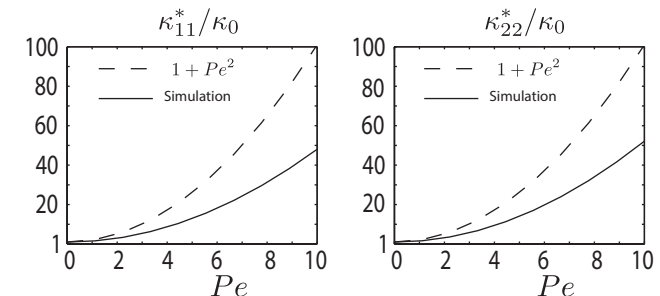
stream function



streamlines



effective diffusivities



Arctic and Antarctic field experiments

*develop electromagnetic methods
of monitoring fluid transport and
microstructural transitions*

extensive measurements of fluid and
electrical transport properties of sea ice:

2007 Antarctic SIPEX

2010 Antarctic McMurdo Sound

2011 Arctic Barrow AK

2012 Arctic Barrow AK

2012 Antarctic SIPEX II

2013 Arctic Barrow AK

2014 Arctic Chukchi Sea



Notices

of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and
the Mathematics of
Transport in Sea Ice

page 562

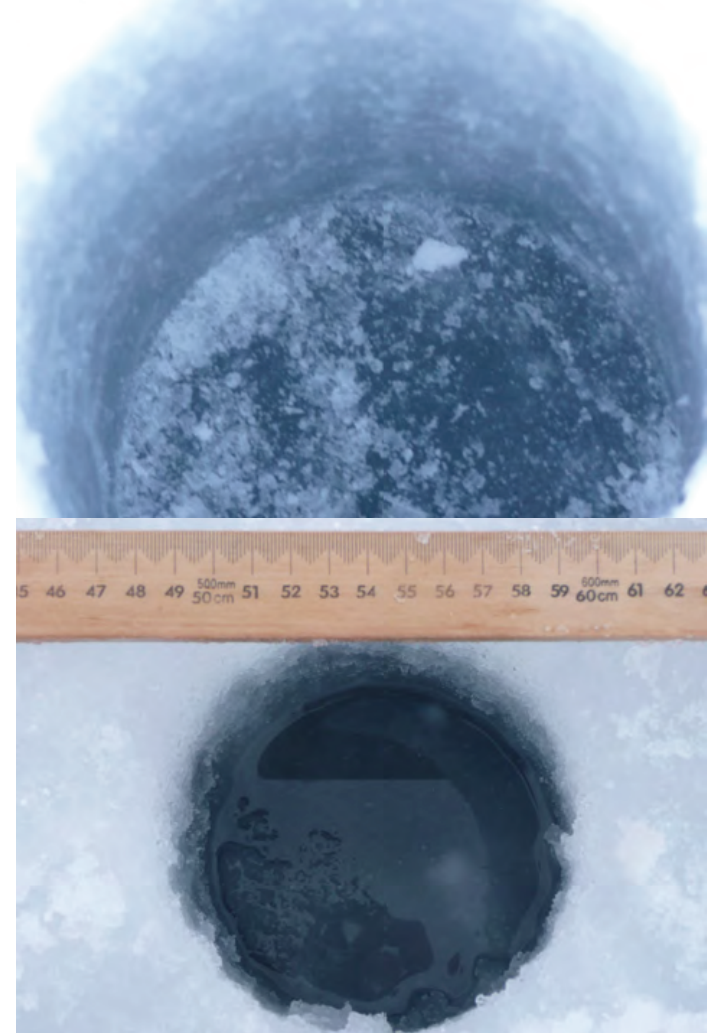
Mathematics and the
Internet: A Source of
Enormous Confusion
and Great Potential

page 586



photo by Jan Lieser

Real analysis in polar coordinates (see page 613)



***measuring
fluid permeability
of Antarctic sea ice***

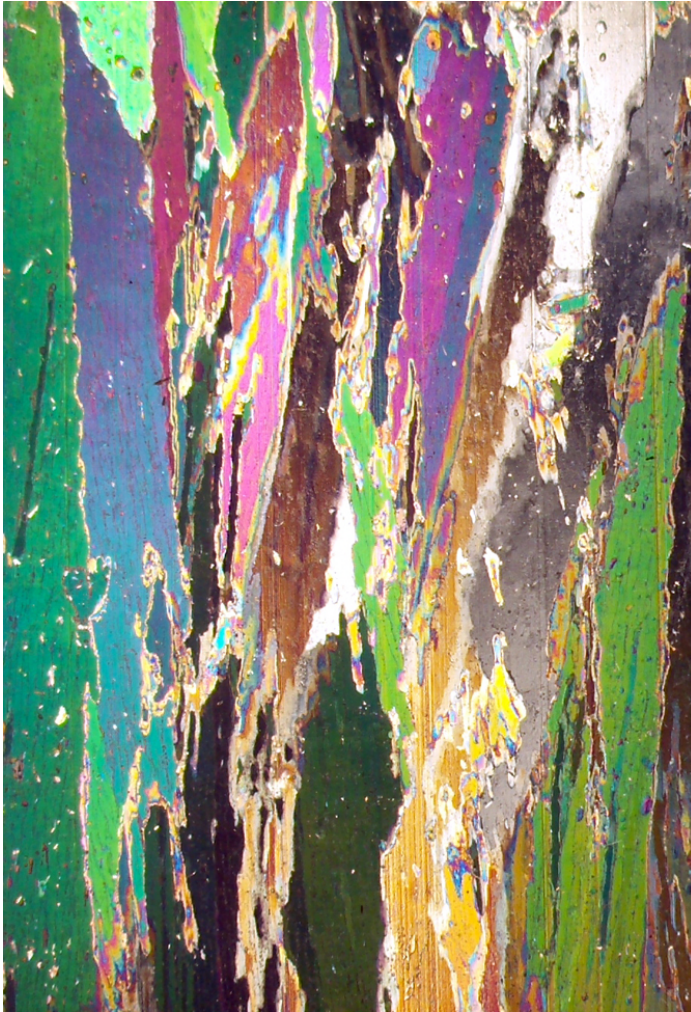
SIPEX 2007

higher threshold for fluid flow in Antarctic granular sea ice

columnar

granular

5%

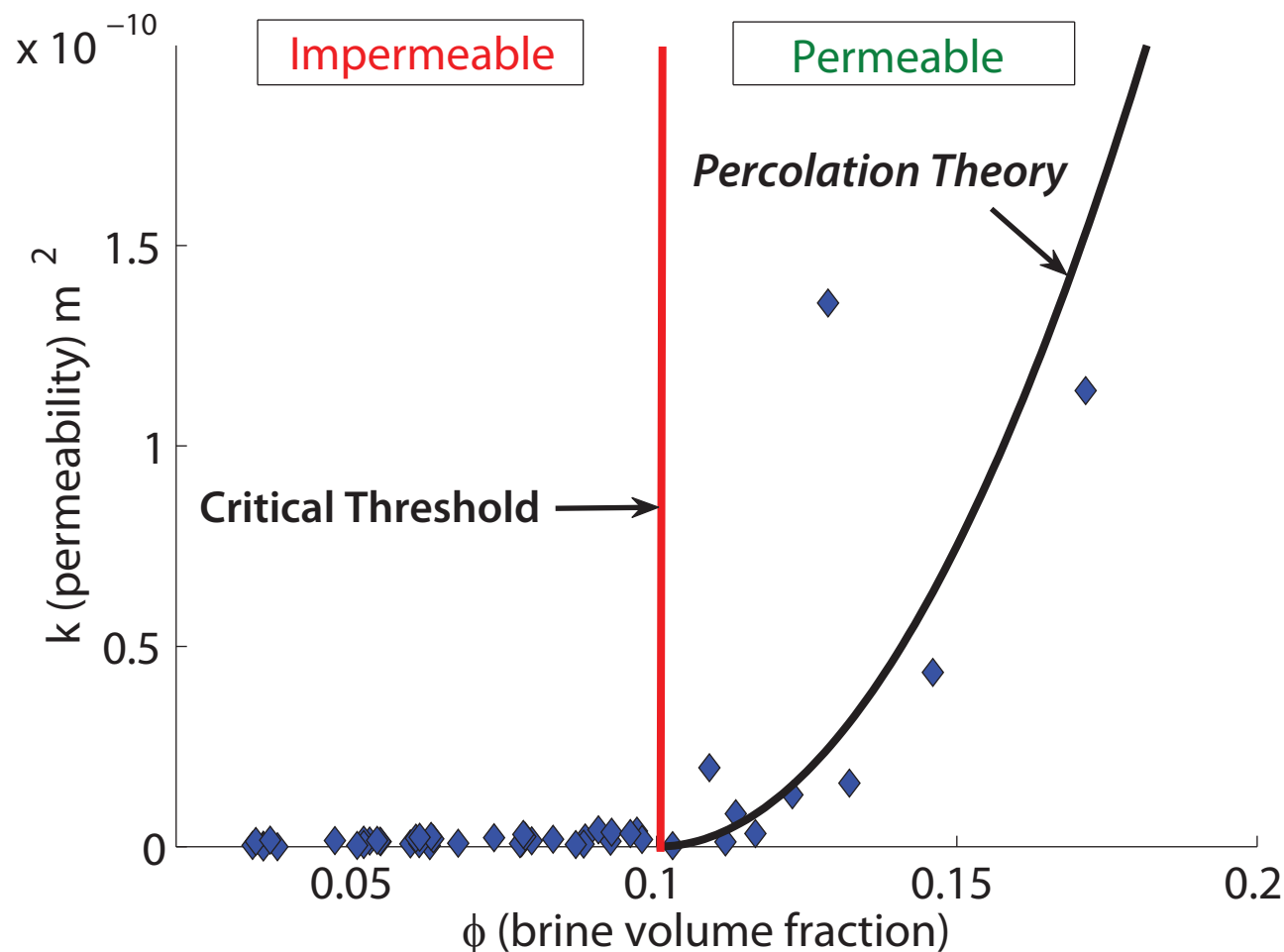


10%



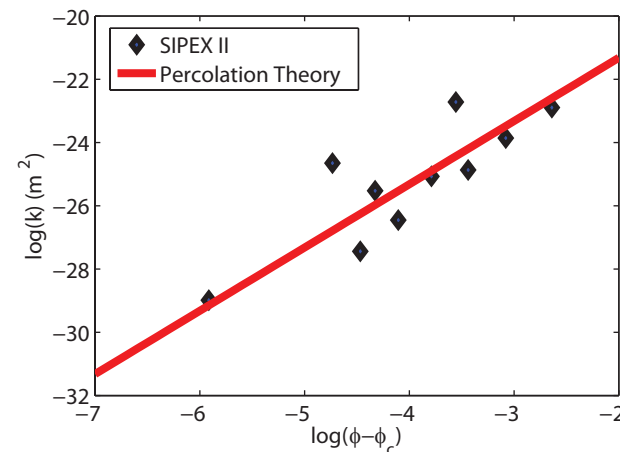
Golden, Gully, Lubbers, Sampson, Tison 2014

SIPEX II vertical permeability data



*same universal
critical exponent
as lattice models*

data above threshold

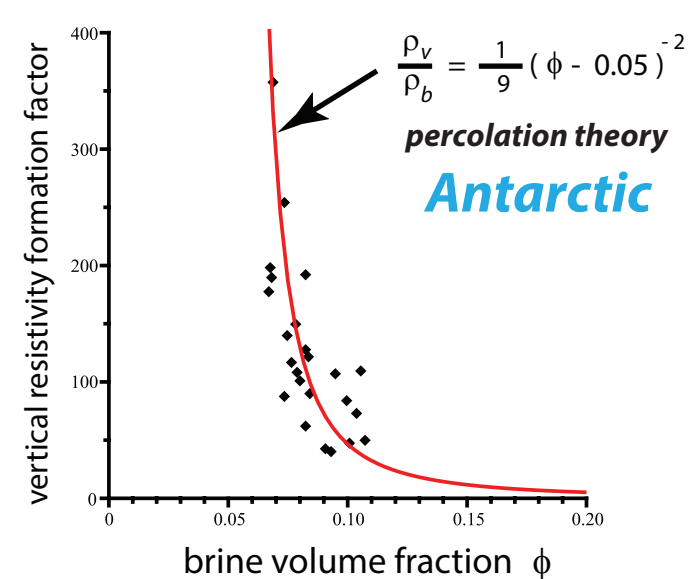
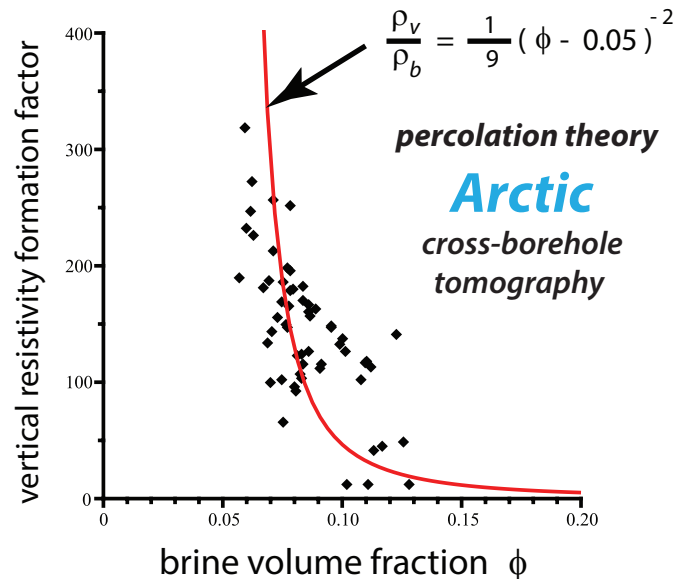
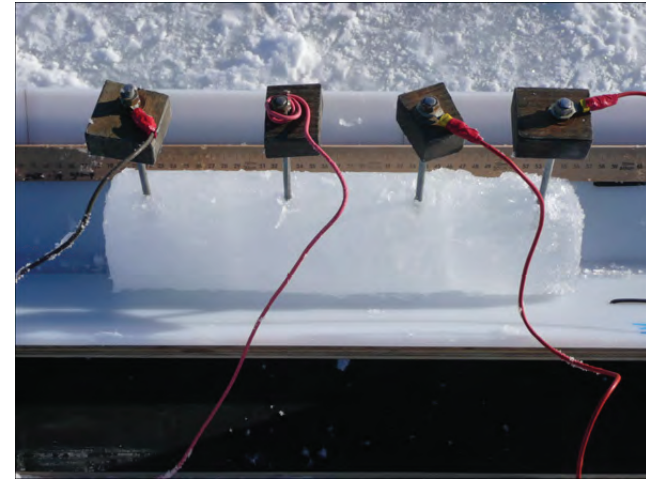
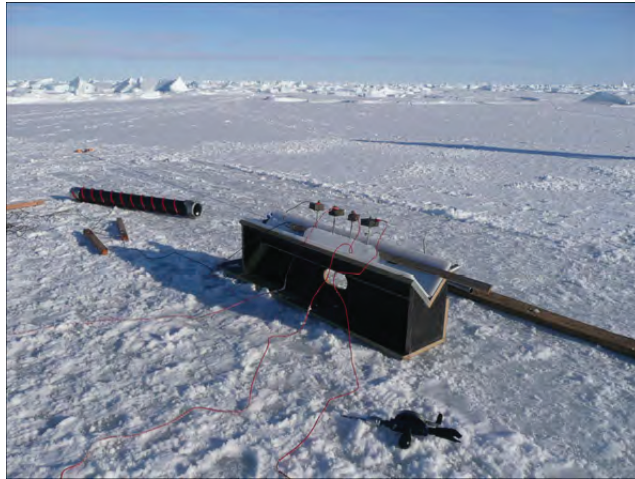


*higher threshold in granular ice predicted with
percolation theory by Golden, et al. (Science, 1998)*

not confirmed experimentally until SIPEX I (2007) and SIPEX II (2012)

critical behavior of electrical transport in sea ice

electrical signature of the on-off switch for fluid flow



cross-borehole tomography - electrical classification of sea ice layers

melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

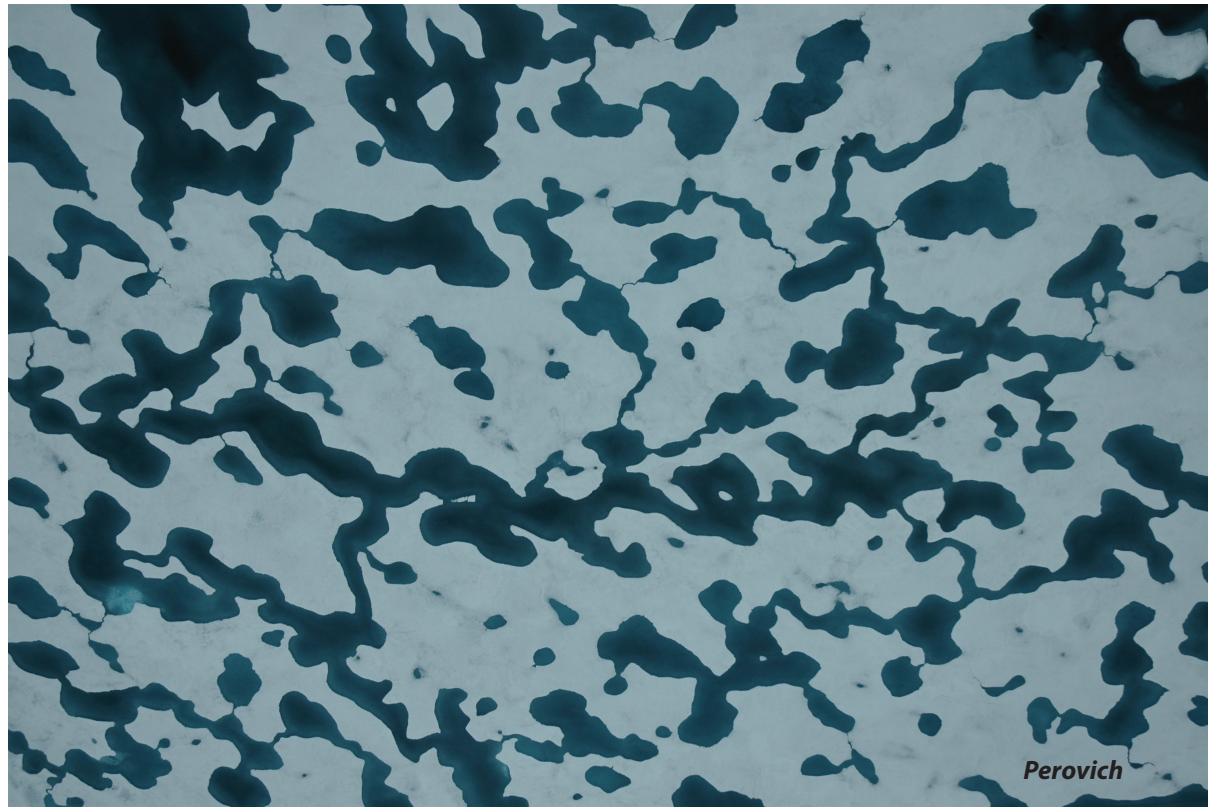
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham,
Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson,
Perovich 2009

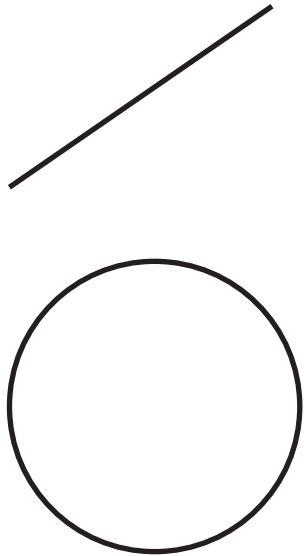
Flocco, Feltham,
Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

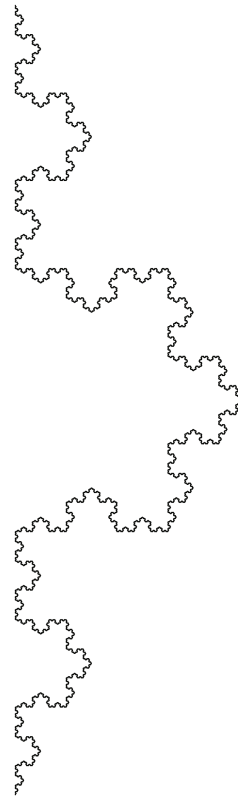
fractal curves in the plane

they wiggle so much that their dimension is >1



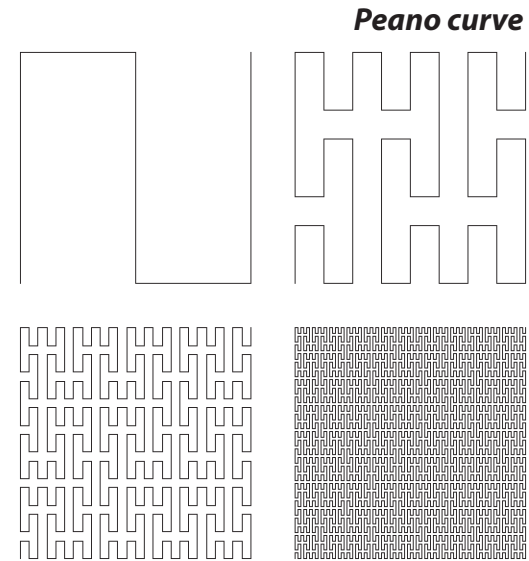
simple curves

$D = 1$



Koch snowflake

$D = 1.26$



Peano curve

Brownian motion

space filling curves

$D = 2$

clouds exhibit fractal behavior from 1 to 1000 km

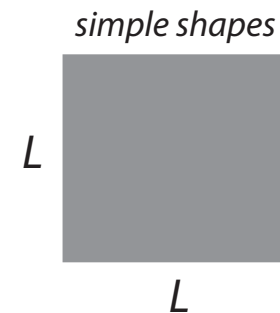
use **perimeter-area** data to find that
cloud and rain boundaries are fractals

$$D \approx 1.35$$

S. Lovejoy, Science, 1982



$$P \sim \sqrt{A}$$



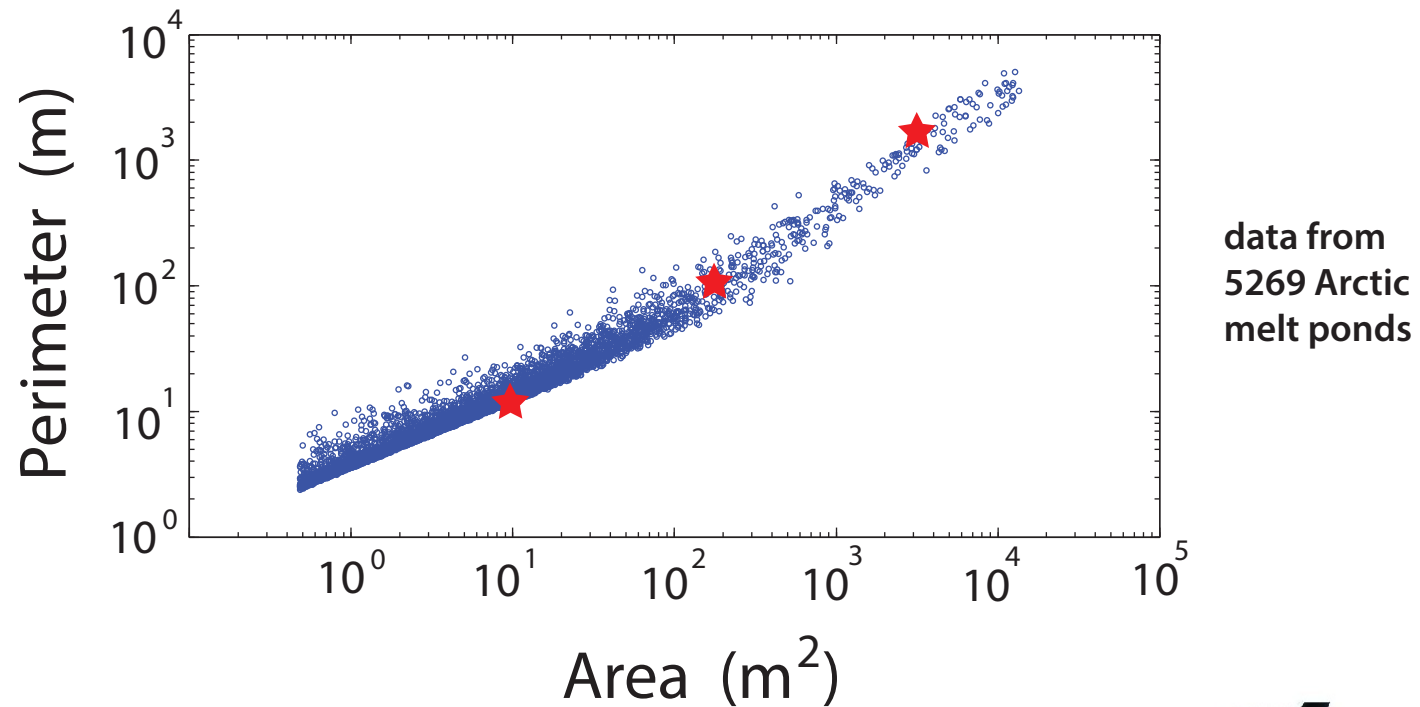
$$A = L^2$$
$$P = 4L = 4\sqrt{A}$$

$$P \sim \sqrt{A}^D$$

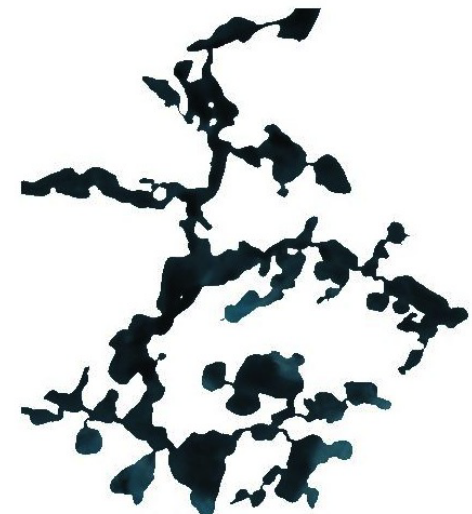


for fractals with
dimension D

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden



~ 30 m



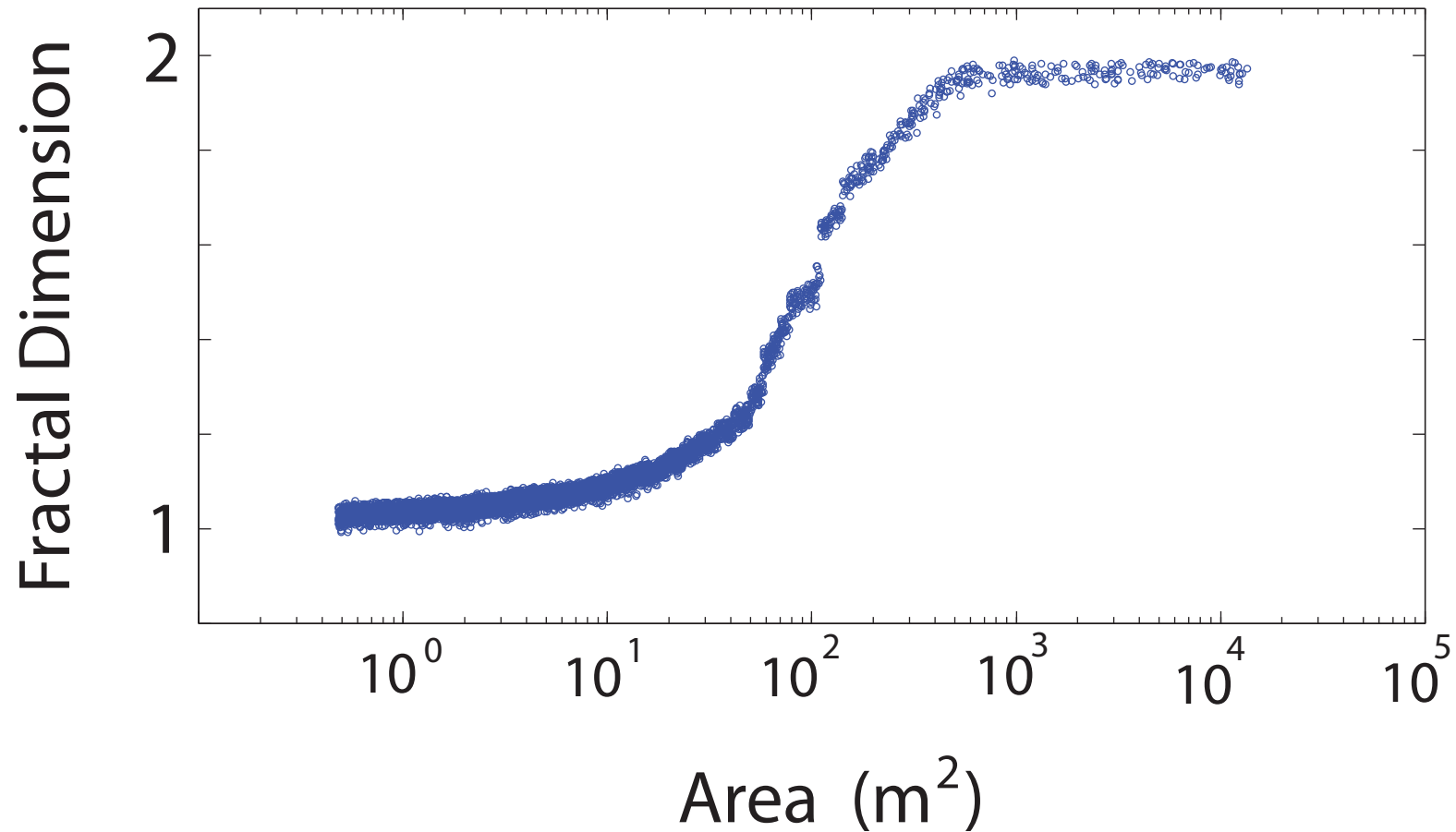
simple pond

transitional pond

complex pond

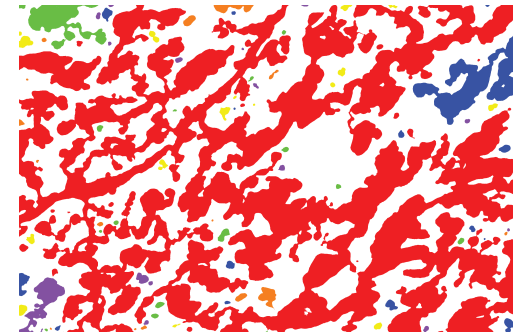
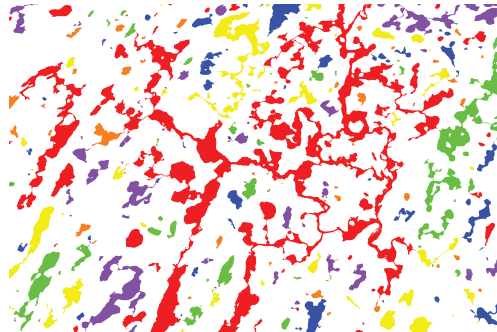
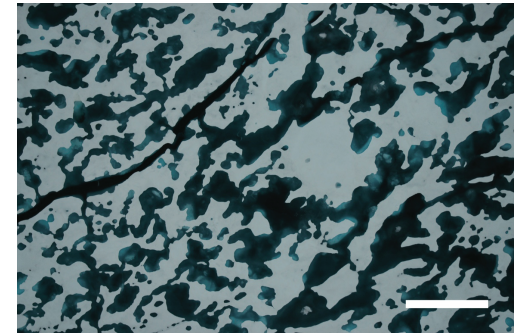
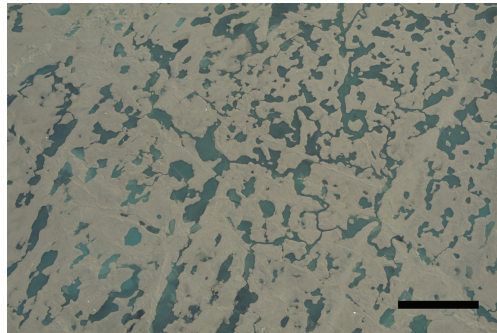
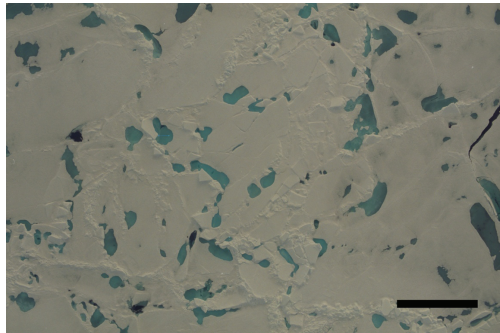
transition in the fractal dimension

complexity grows with length scale



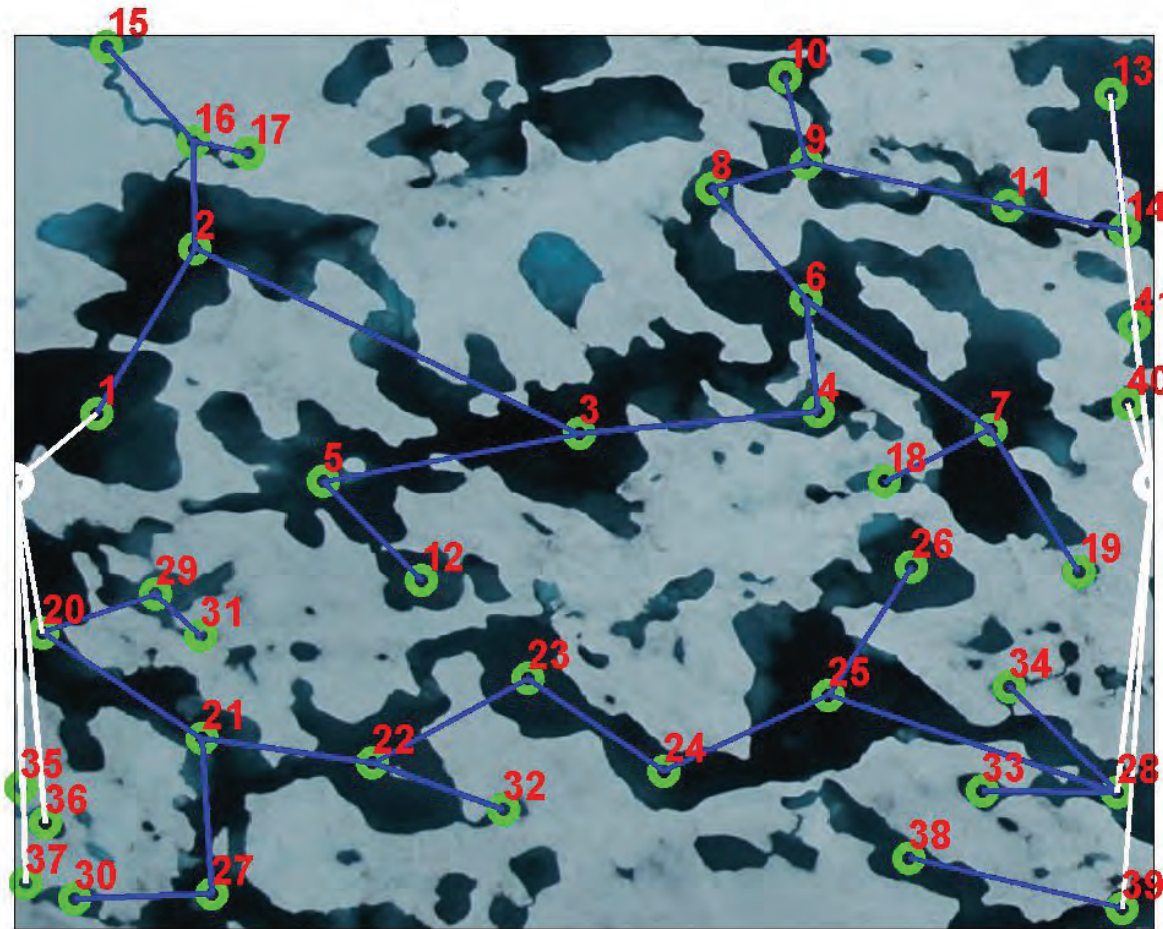
compute “derivative” of area - perimeter data

***small simple ponds coalesce to form
large connected structures with complex boundaries***



melt pond percolation

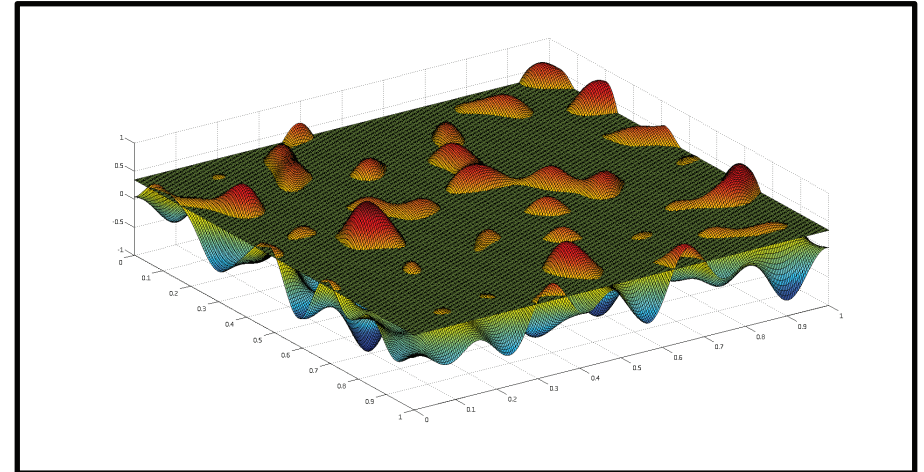
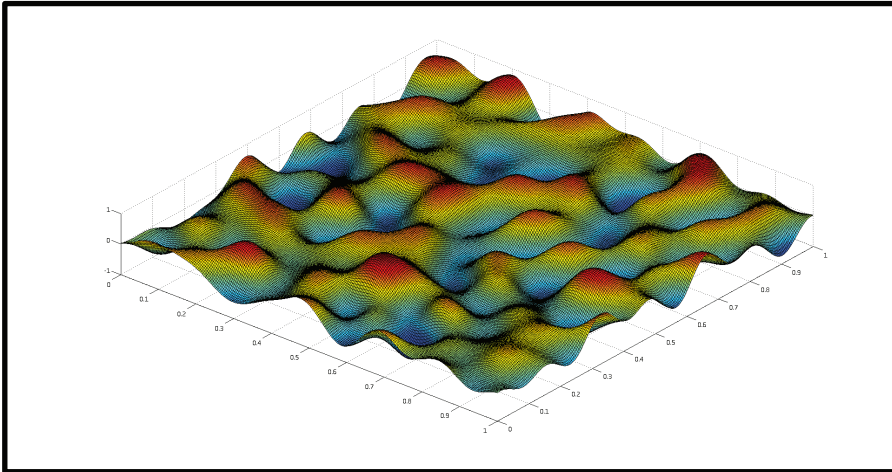
map melt pond configurations onto resistor networks
compute horizontal fluid permeability



4 August 2005, Healy–Oden Trans Arctic Expedition (HOTRAX)

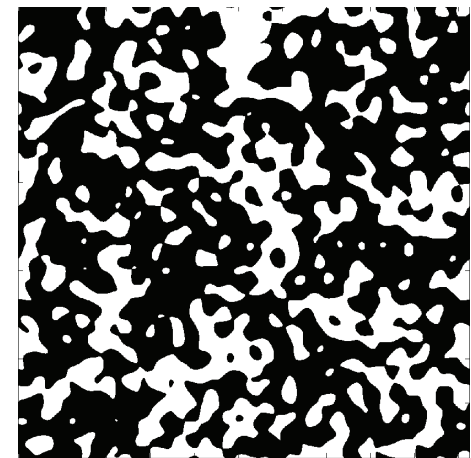
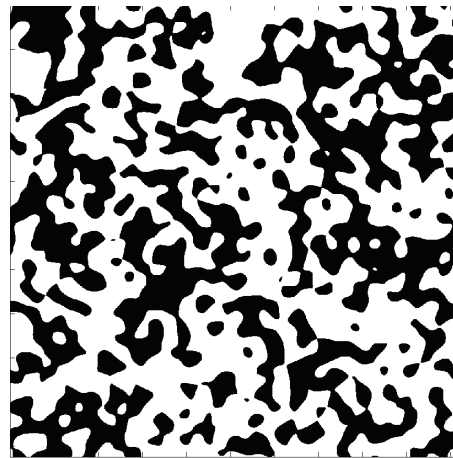
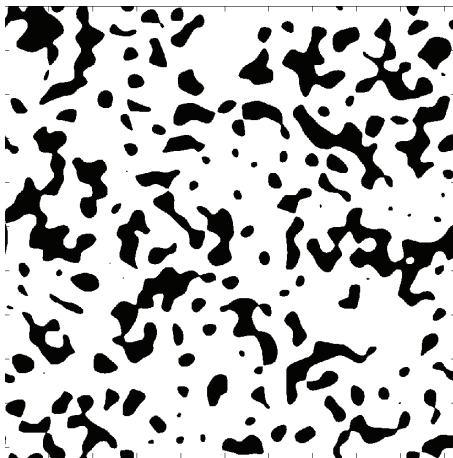
Continuum percolation model for melt pond evolution

Brady Bowen, Court Strong, Ken Golden, 2014



random Fourier series representation of surface topography

intersections of a plane with the surface define melt ponds

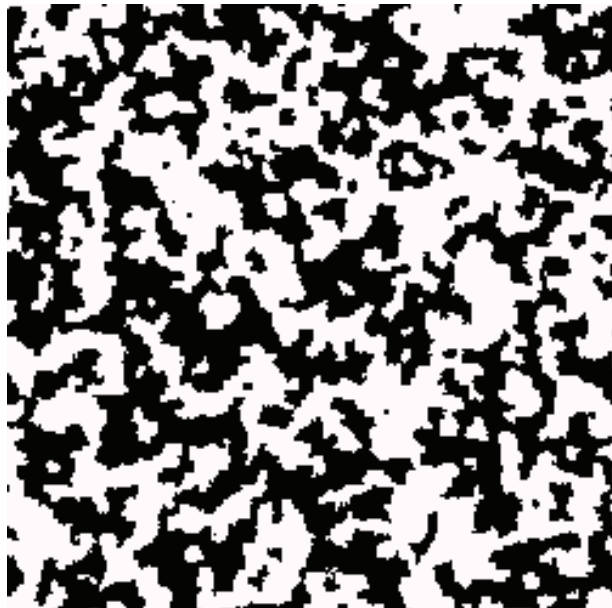
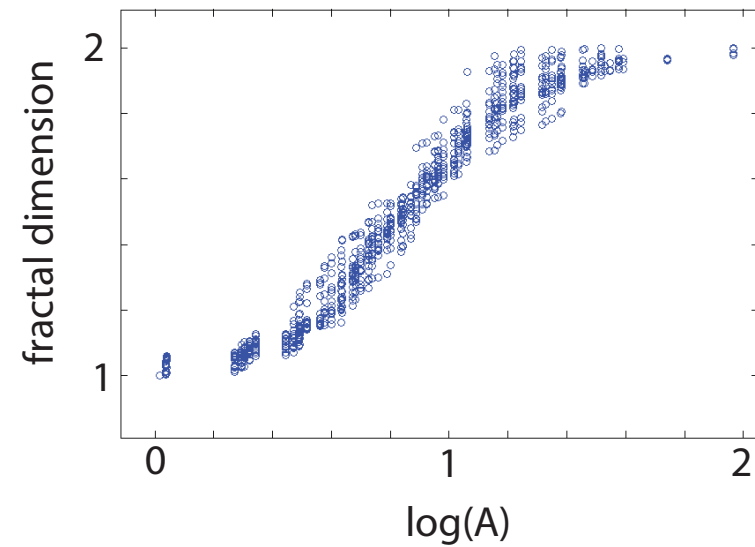
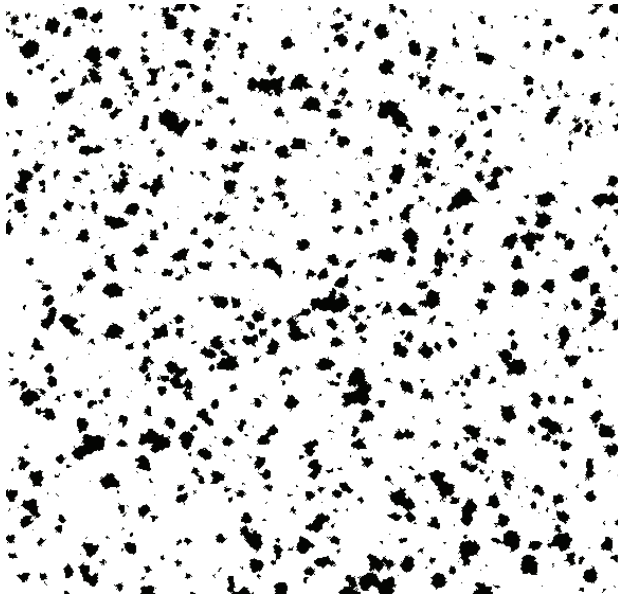


electronic transport in disordered media

diffusion in turbulent plasmas

(Isichenko, Rev. Mod. Phys., 1992)

simple stochastic growth model of melt pond evolution



voter
model

*a square is more likely to melt
if its neighbors have melted*

Ising model for ferromagnets



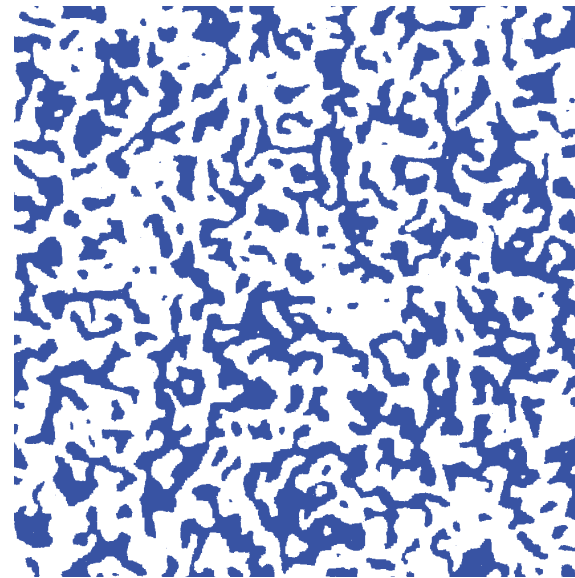
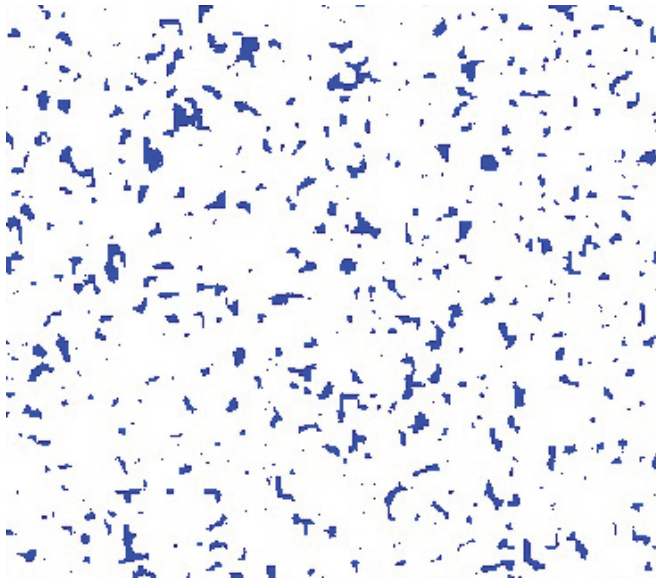
Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{\langle i,j \rangle}^N s_i s_j - H \sum_i^N s_i$$

$$s_i = \begin{cases} \uparrow & +1 & \text{water} & (\text{spin up}) \\ \downarrow & -1 & \text{ice} & (\text{spin down}) \end{cases}$$

magnetization $M = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$

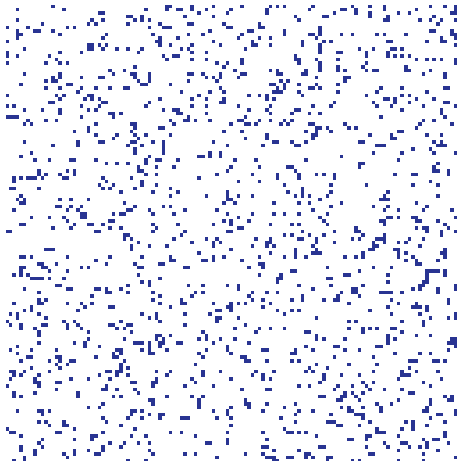
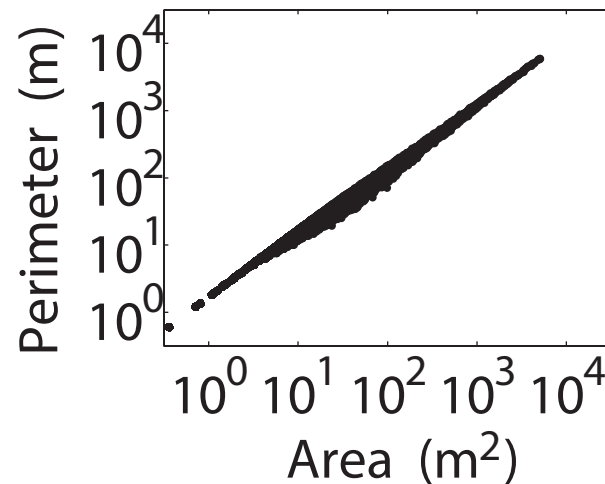
pond coverage $\frac{(M+1)}{2}$



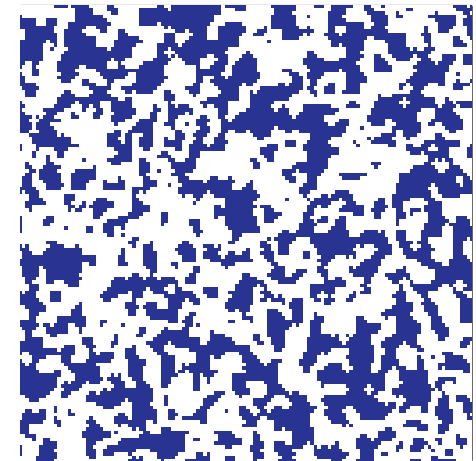
“melt ponds” are clusters of magnetic spins that align with the applied field

Melt Pond Ising Model

- Minimize an Ising Hamiltonian
random magnetic field represents the initial ice topography
interaction term represents horizontal heat transfer
- Ice-albedo feedback incorporated by taking coupling constant in interaction term to depend on the pond coverage



*predicted length scale
of fractal transition
agrees well with data*



Arctic melt ponds and bifurcations in the climate system

I. Sudakov, S. A. Vakulenko, and K. M. Golden

Communications in Nonlinear Science and Numerical Simulation, in press, 2014.

*investigate effect of fractal transition in melt pond geometry
on conceptual climate models, bifurcations, multiple equilibria, etc.*

Conclusions

- 1. Summer Arctic sea ice is melting rapidly.**
- 2. Fluid flow through sea ice mediates many processes of importance to understanding climate change and the response of polar ecosystems.**
- 3. Mathematics of composite materials and statistical physics help us understand sea ice, and suggest rigorous frameworks for representing sea ice in climate models .**
- 4. Field experiments are essential to developing relevant mathematics.**
- 5. This research will help to improve projections of climate change and the fate of Earth's sea ice packs and their ecosystems.**

THANK YOU

National Science Foundation

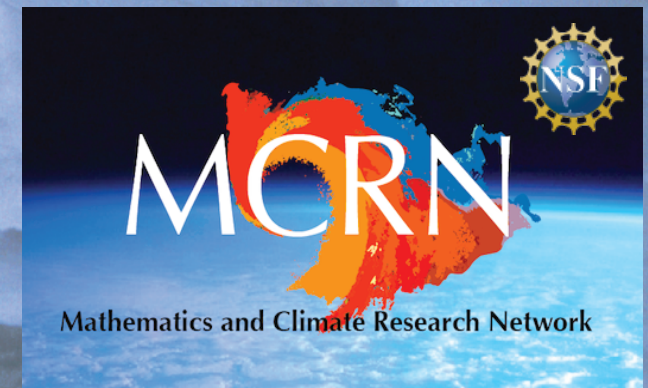
Division of Mathematical Sciences

Division of Polar Programs

Office of Naval Research

Arctic and Global Prediction Program

Applied and Computational Analysis Program



Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999