Mathematics of Sea Ice

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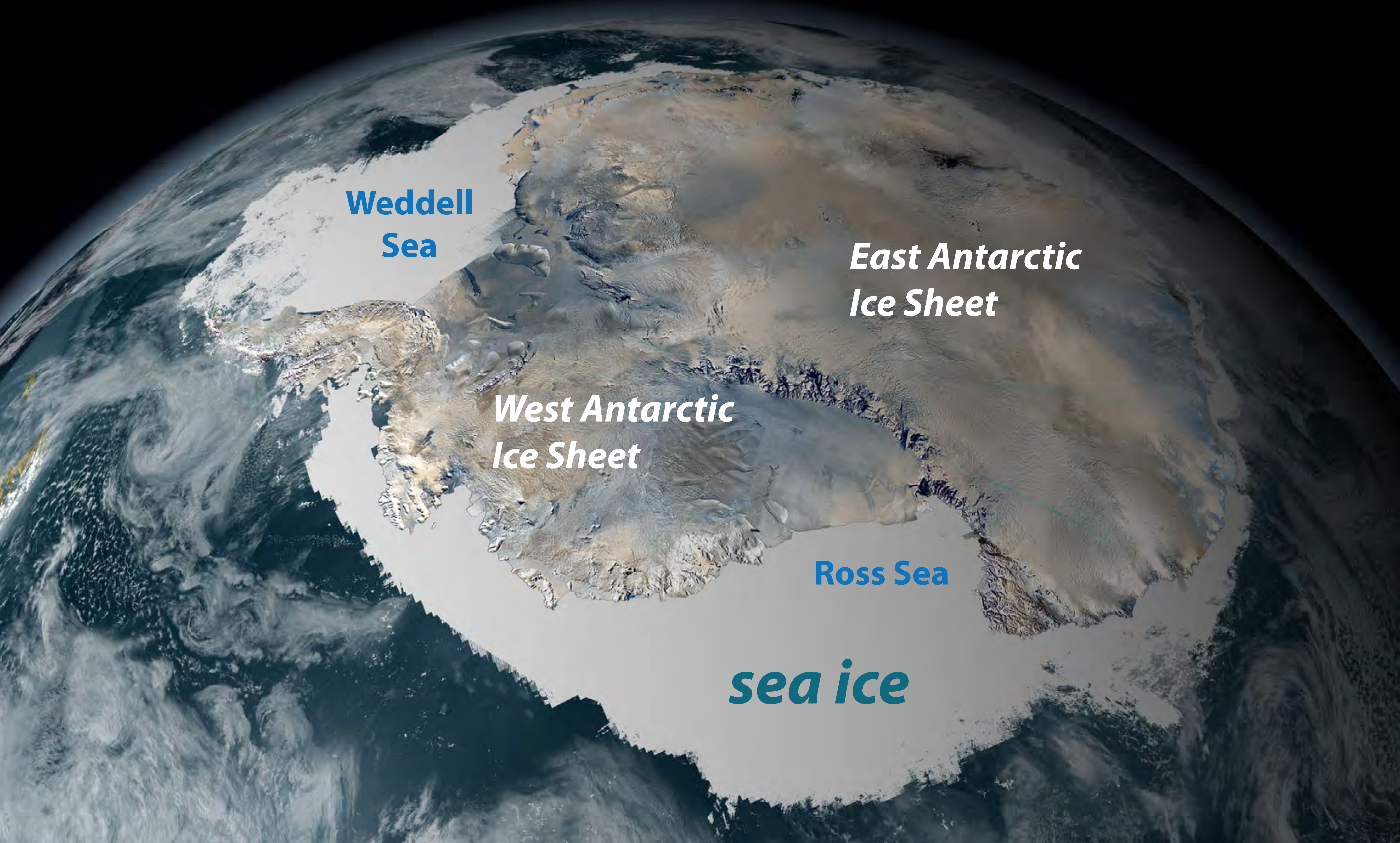
University of Utah

Grad Recruitment Weekend 24 March 2017

Frey

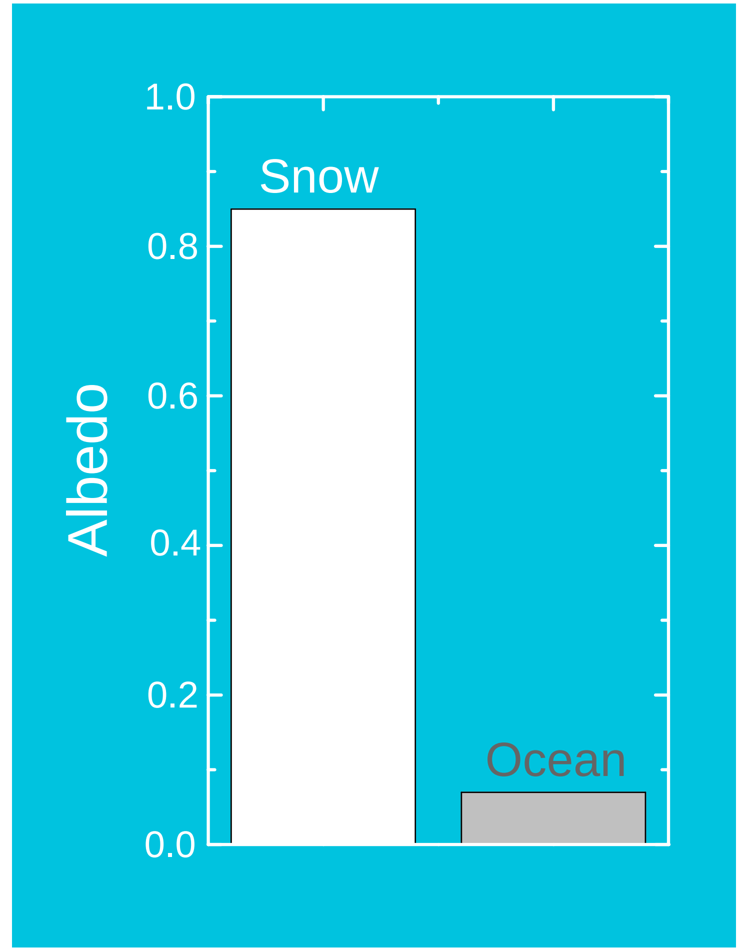
ANTARCTICA

southern cryosphere



polar ice caps critical to global climate in reflecting incoming solar radiation

white snow and ice reflect

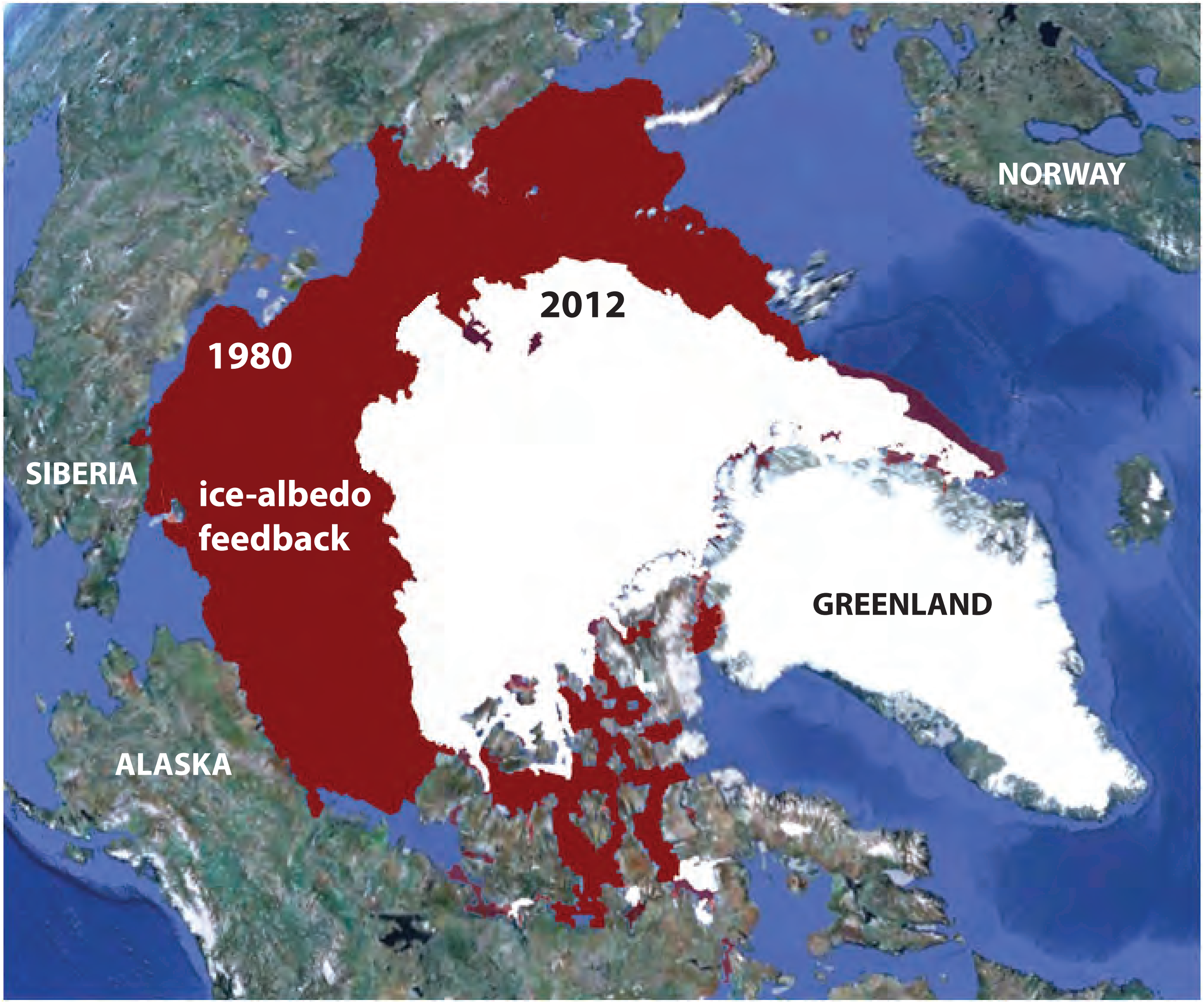


dark water and land absorb

$$\textbf{\textit{albedo}}\ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

Change in Arctic Sea Ice Extent

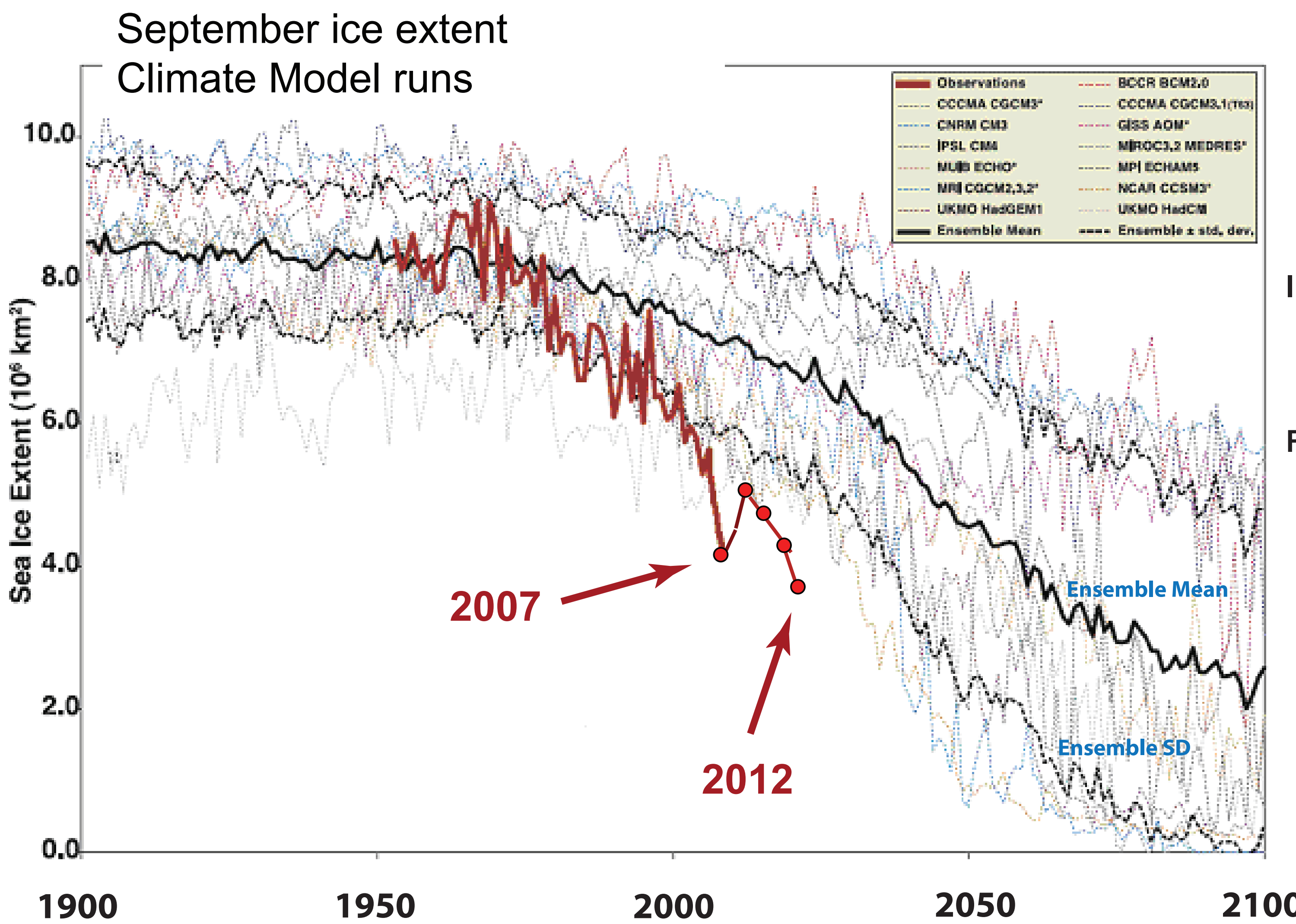
September 1980 -- 7.8 million square kilometers
September 2012 -- 3.4 million square kilometers



Perovich

Arctic sea ice decline: faster than predicted by climate models

Stroeve et al., GRL, 2007



IPCC AR4 Models

Intergovernmental Panel on Climate Change (IPCC)

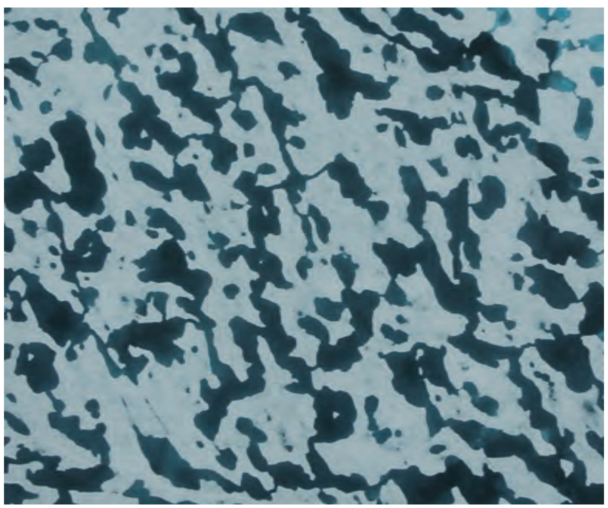
Fourth Assessment AR4, 2007

challenge

represent sea ice more rigorously in climate models

account for key processes

such as melt pond evolution



Impact of melt ponds on Arctic sea ice simulations from 1990 to 2007

Flocco, Schroeder, Feltham, Hunke, JGR Oceans 2012

For simulations with ponds September ice volume is nearly 40% lower.

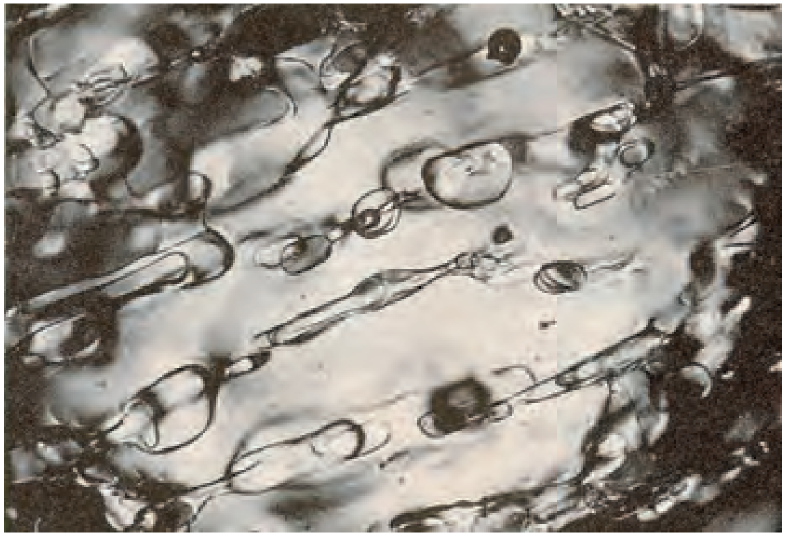
... and other sub-grid scale structures and processes

linkage of scales

sea ice is a multiscale composite

structured on many length scales

brine inclusions

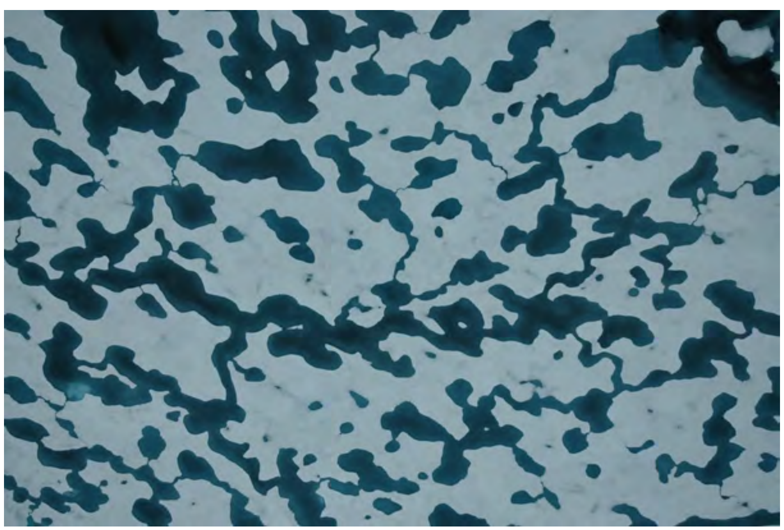


millimeters

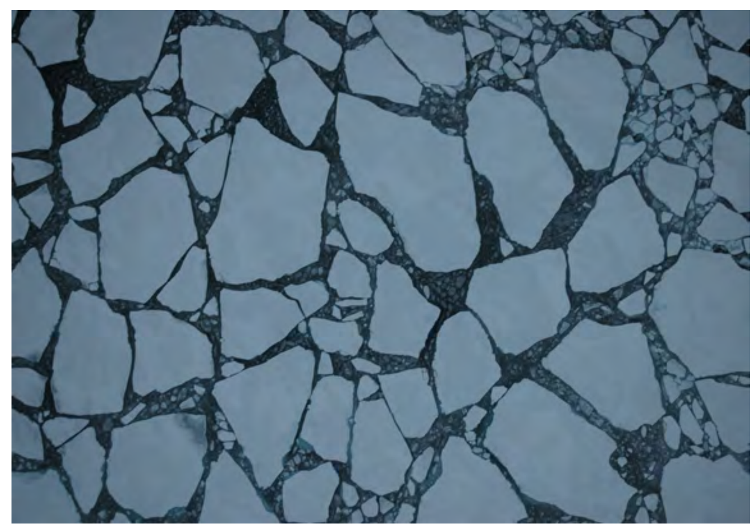
pancakes

centimeters

melt ponds



meters



ice floes

kilometers

What is this talk about?

Using mathematics and theoretical physics to study sea ice structures and processes ... to improve projections of climate change.

1. *Opposite poles of climate modeling*

partial differential equations, ODE's and dynamical systems

2. *Sea ice microphysics and composite structure*

homogenization, fluid flow, diffusion processes, percolation theory

3. *Electromagnetic monitoring of sea ice*

complex analysis, spectral measures, random matrix theory

4. *Fractal geometry of Arctic melt ponds*

continuum percolation theory, statistical physics

critical behavior ***cross - pollination***

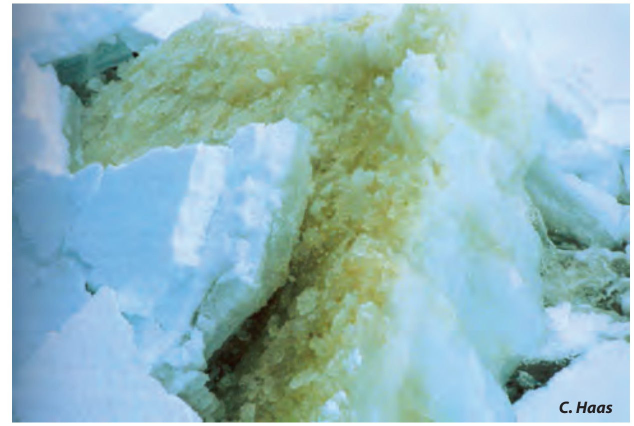
sea ice microphysics

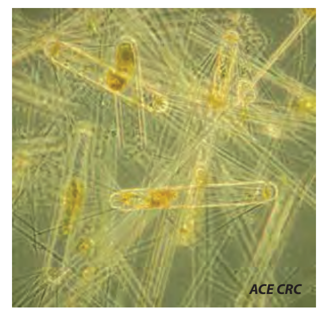
fluid transport

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities

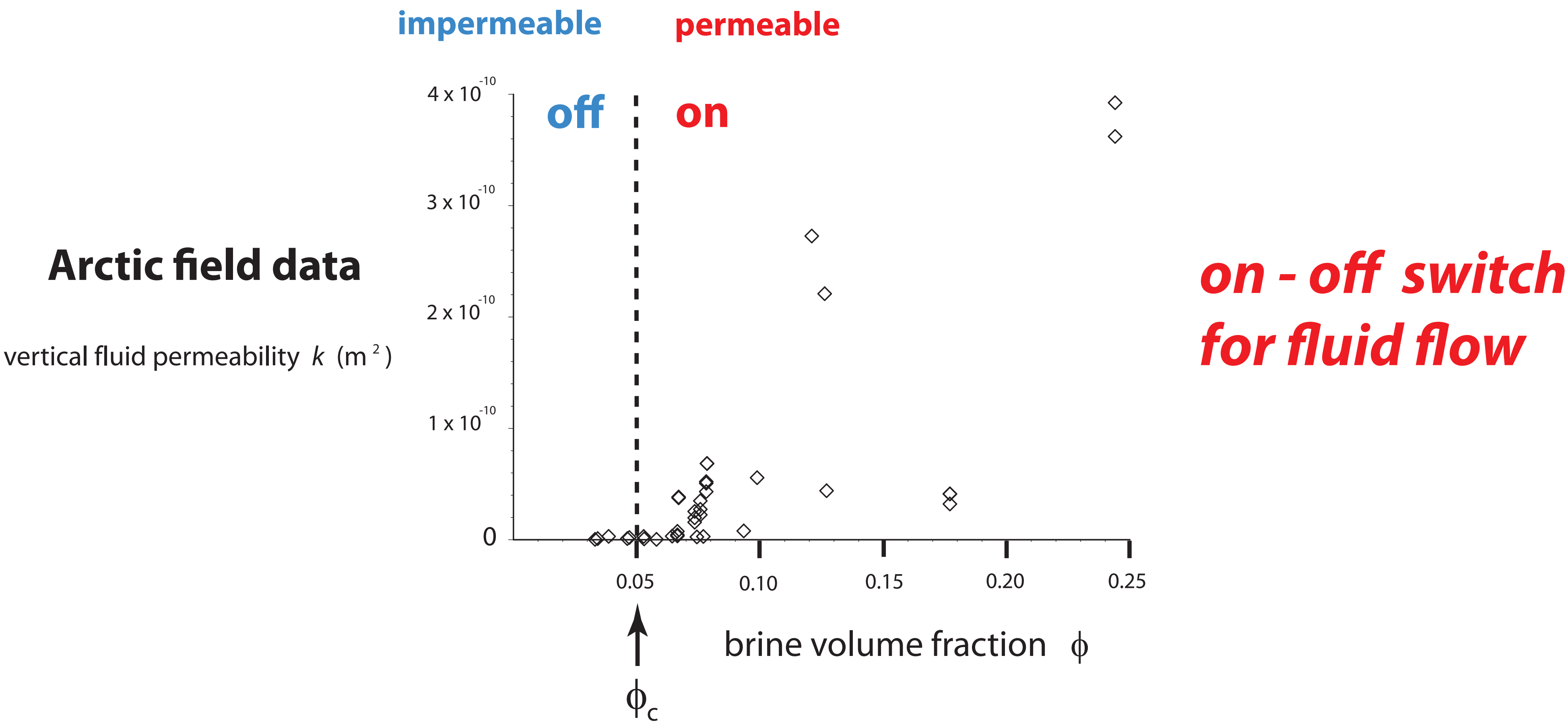




- *drainage of brine and melt water*
- *ocean-ice-air exchanges of heat, CO_2*
- *Antarctic surface flooding and snow-ice formation*
- *evolution of salinity profiles*

linkage of scales

Critical behavior of fluid transport in sea ice



critical brine volume fraction $\phi_c \approx 5\%$ $\longleftrightarrow$ $T_c \approx -5^\circ C$, $S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

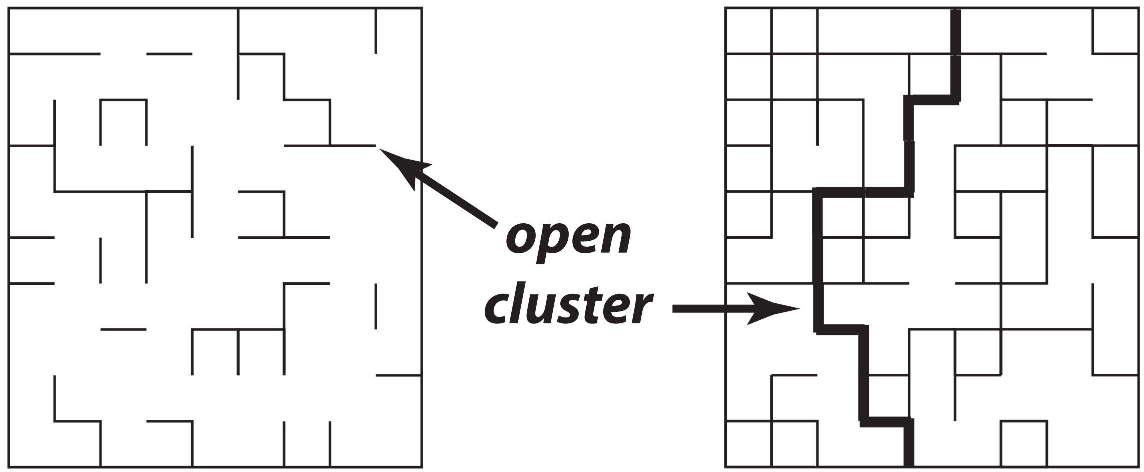
Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

percolation theory

probabilistic theory of connectedness

impermeable | permeable



$p = 1/3$ | $p = 2/3$

bond $\longrightarrow$ **open** *with probability* p
closed *with probability* $1\text{-}p$

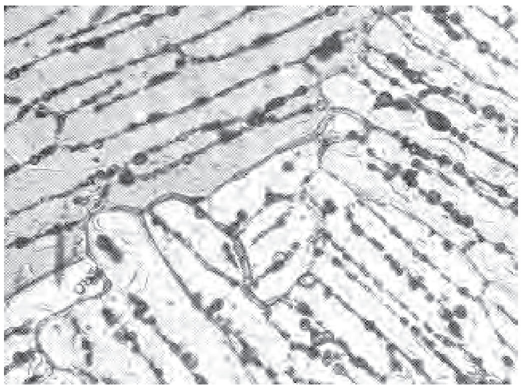
percolation threshold

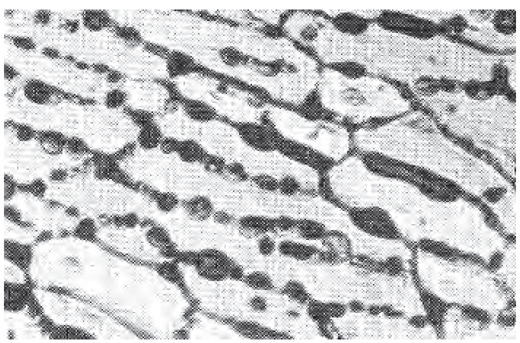
$$p_c = 1/2 \quad \text{for} \quad d = 2$$

smallest *p* for which there is an infinite open cluster

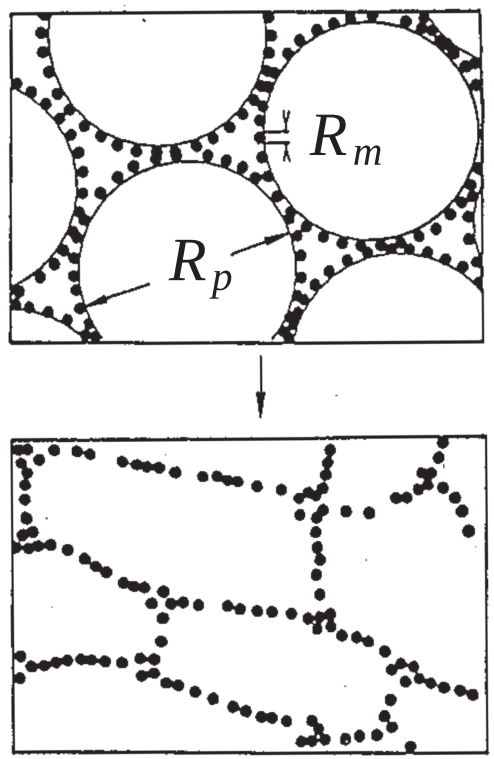
Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

$\phi_c \approx 5\,\%$ Golden, Ackley, Lytle, *Science*, 1998

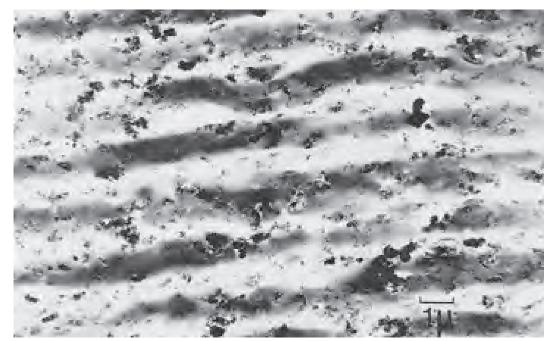




sea ice



compressed powder



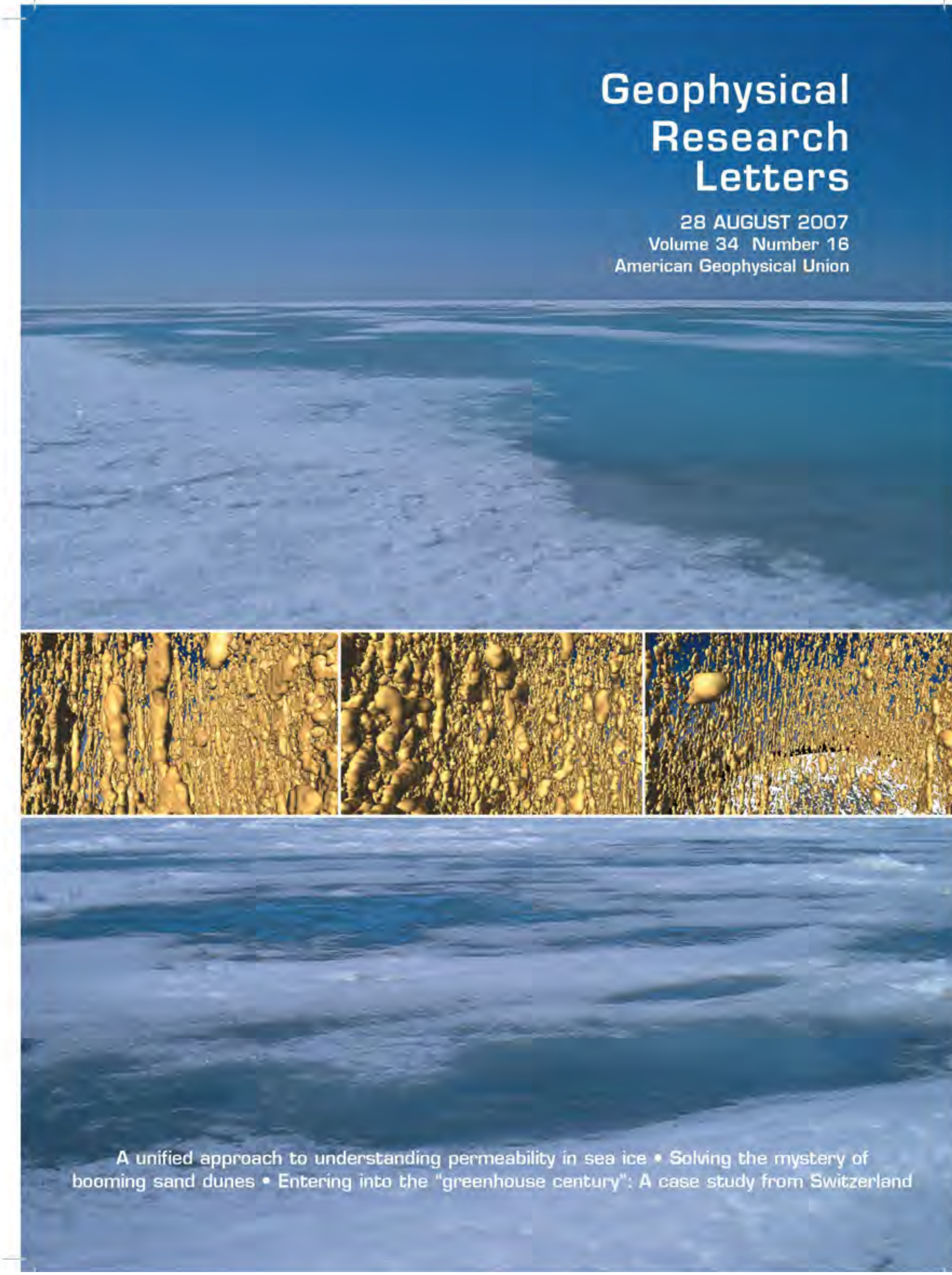
radar absorbing composite

sea ice is radar absorbing

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu

GRL 2007



rigorous bounds
percolation theory
hierarchical model
network model

field data

micro-scale

controls

macro-scale processes

X-ray tomography for brine inclusions

unprecedented look at thermal evolution of brine phase and its connectivity

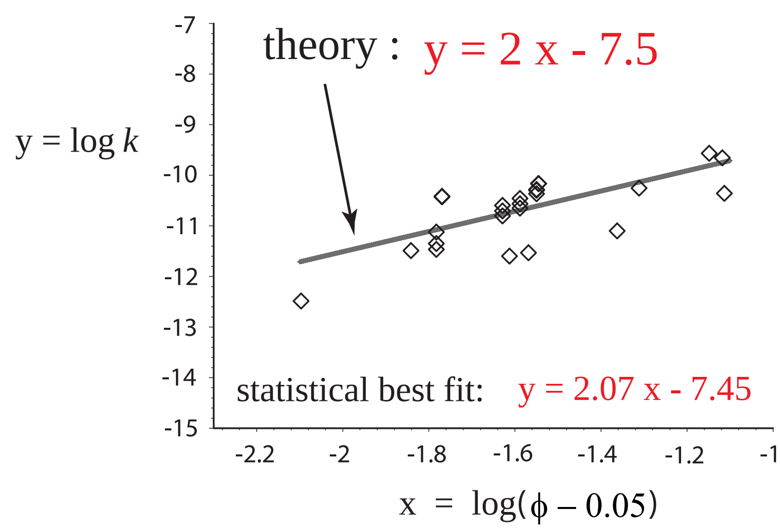
lattice and continuum percolation theories yield:

$$k\,(\phi) = k_0\,(\,\phi - 0.05\,)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8}\ \mathrm{m}^2$$

- exponent is UNIVERSAL lattice value $t \approx 2.0$
- sedimentary rocks like sandstones also exhibit universality
- critical path analysis -- developed for electronic hopping conduction -- yields scaling factor k_0



Remote sensing of sea ice

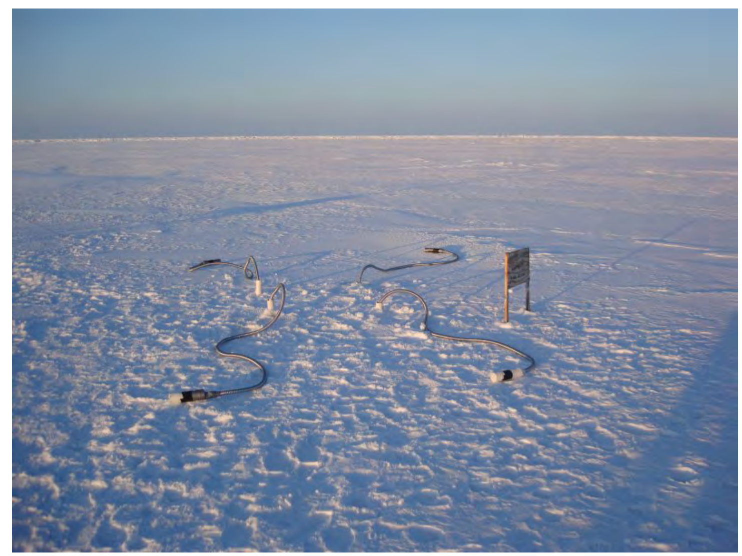
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

ε^*

effective complex permittivity (dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Theory of Effective Electromagnetic Behavior of Composites

analytic continuation method

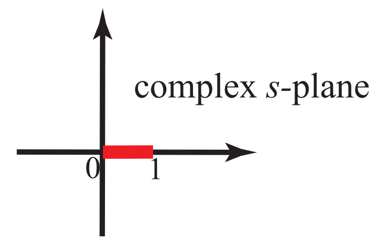
Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)
Theory of Composites, Milton (2002)

composite geometry
(spectral measure μ) $\longrightarrow$ ε^*

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* $\longrightarrow$ ***composite geometry***
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

Stieltjes integral representation

separation of geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

parameters

μ
- spectral measure of self adjoint operator $\chi\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = -\nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

Golden and Papanicolaou, Comm. Math. Phys. 1983

direct calculation of spectral measure

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).

2. The fundamental operator $\chi\Gamma\chi$ becomes a random matrix depending only on the composite geometry.

3. Compute the eigenvalues λ_i and eigenvectors of $\chi\Gamma\chi$ with inner product weights α_i

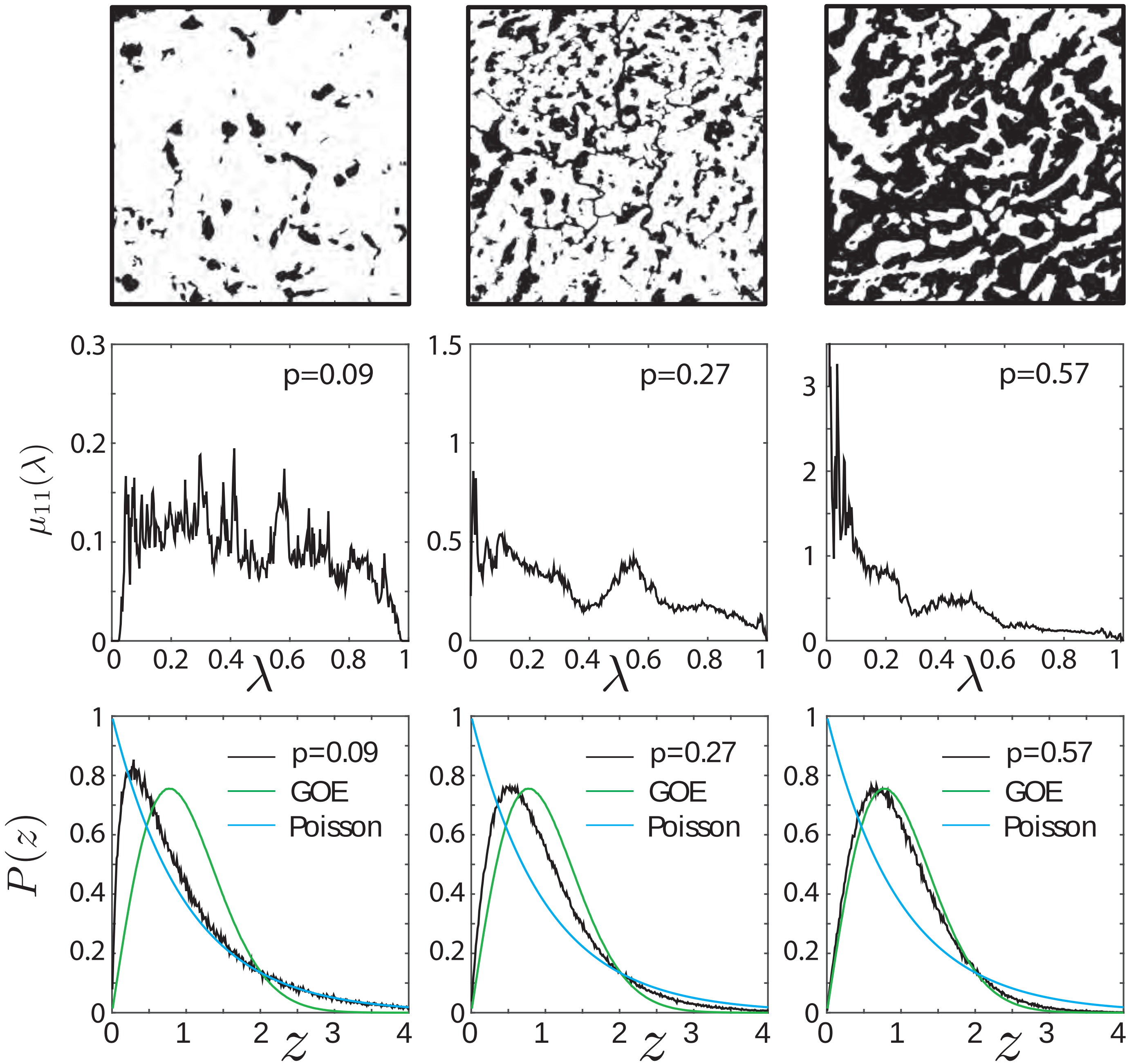
$$\mu(\lambda) = \sum_i \alpha_i \, \delta(\lambda - \lambda_i)$$

Dirac point measure (Dirac delta)

earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral computations for Arctic melt ponds



Ben Murphy
Elena Cherkaev
Ken Golden

***PRL*, 2017**

spectral measures

eigenvalue spacing distributions

uncorrelated → **level repulsion**

TRANSITION

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION TRANSITION → **transition to universal eigenvalue statistics (GOE)**
extended states, mobility edges

-- without wave interference or quantum effects --

surprising analog of

Anderson transition in wave physics - quantum, optics, ...

disorder ~ connectedness

metal / insulator transition at critical disorder

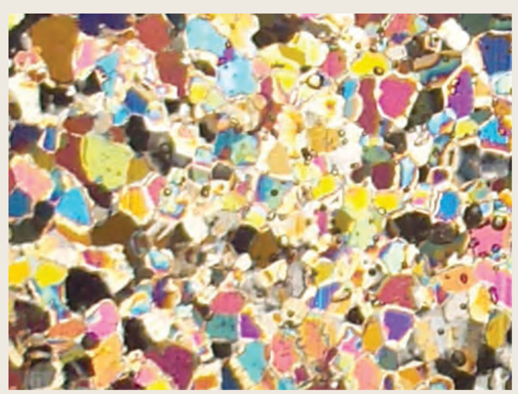
low disorder	**extended**	**GOE**
high disorder	**localized**	**Poisson**

Anderson, 1958
Shklovshii et al, 1993
Evangelou, 1992

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**

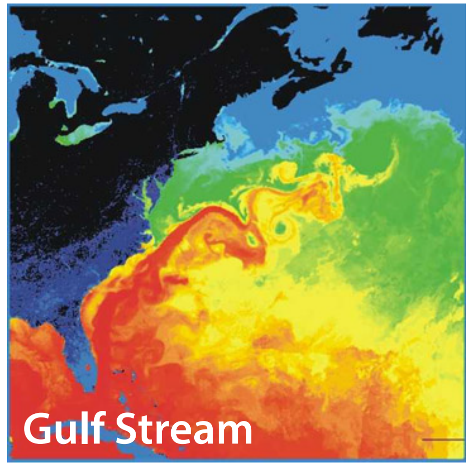


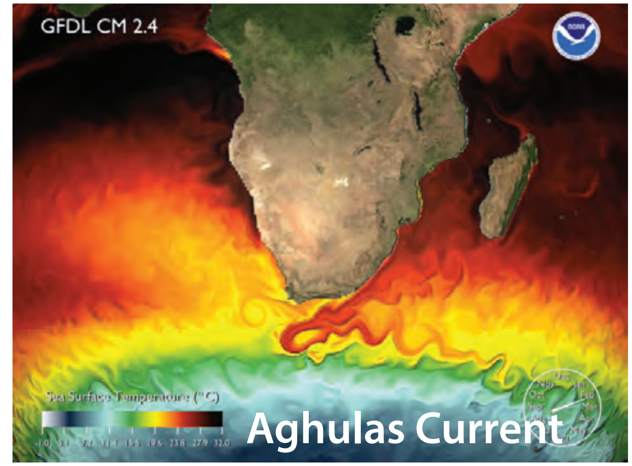
Proc. Roy. Soc. A 8 Feb 2015

advection enhanced diffusion

effective diffusivity

tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere
salt and heat transport in ocean
heat transport in sea ice with convection





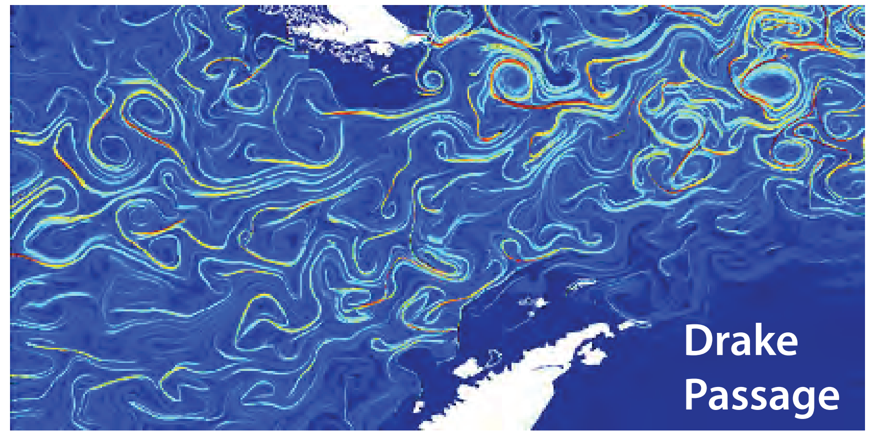
advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$\downarrow$ ***homogenize***

$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$



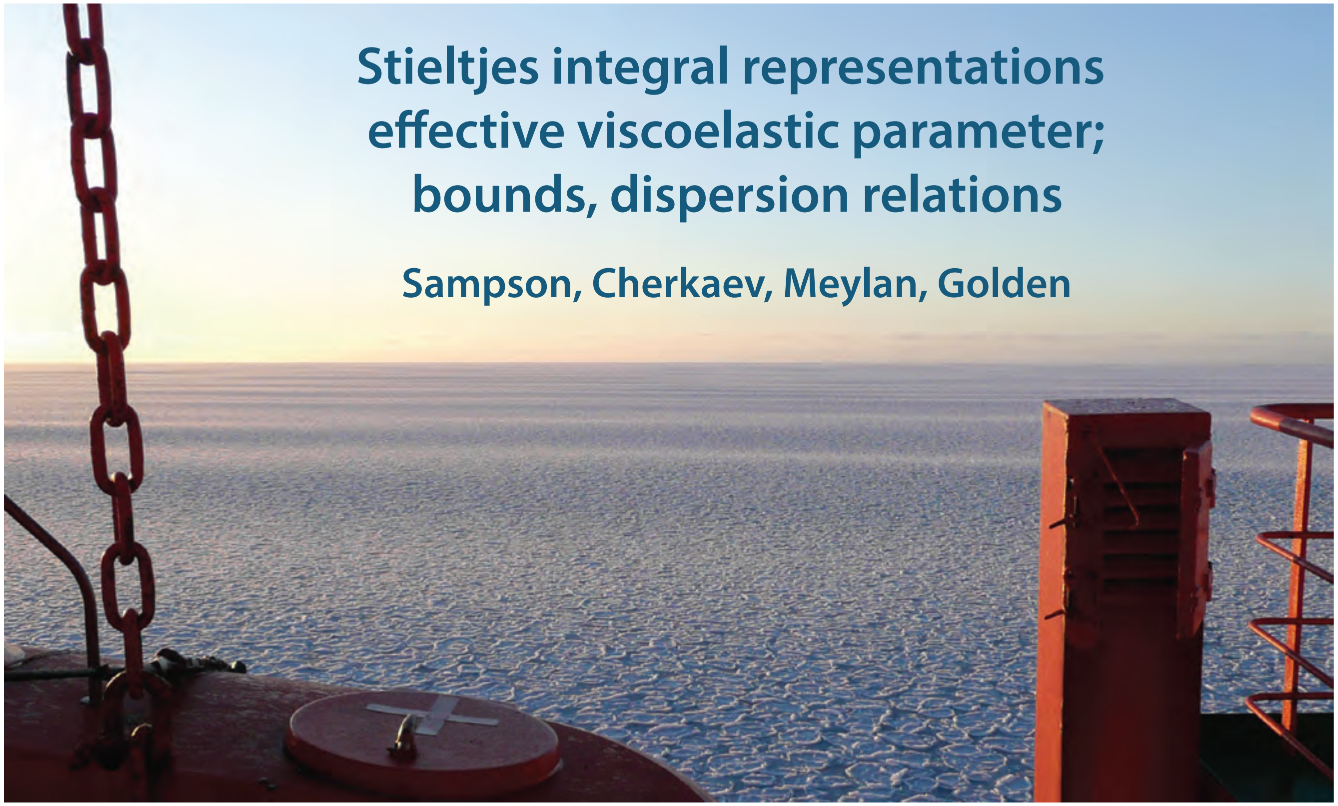
κ^* *effective diffusivity*

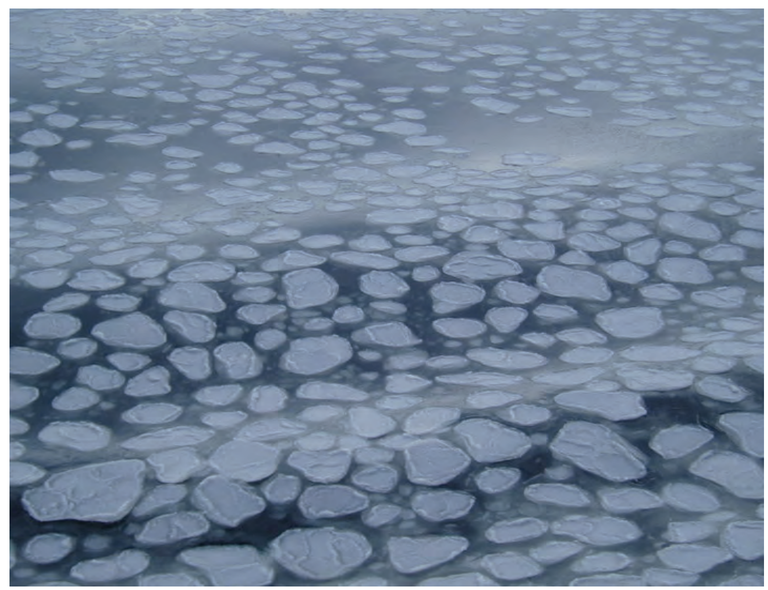
Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, 2017

wave propagation in the marginal ice zone





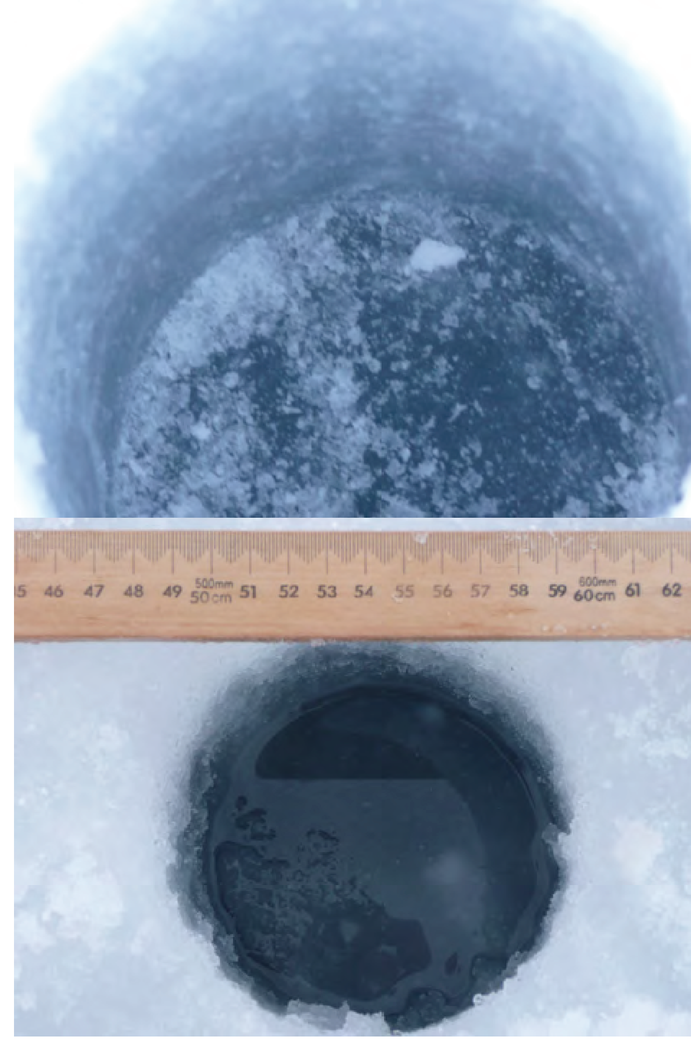
Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	*Antarctic*	*SIPEX*
2010	*Antarctic*	*McMurdo Sound*
2011	*Arctic*	*Barrow AK*
2012	*Arctic*	*Barrow AK*
2012	*Antarctic*	*SIPEX II*
2013	*Arctic*	*Barrow AK*
2014	*Arctic*	*Chukchi Sea*

photo by Jan Lieser



measuring fluid permeability of Antarctic sea ice

SIPEX 2007

melt pond formation and albedo evolution:

- ***major drivers in polar climate***
- ***key challenge for global climate models***

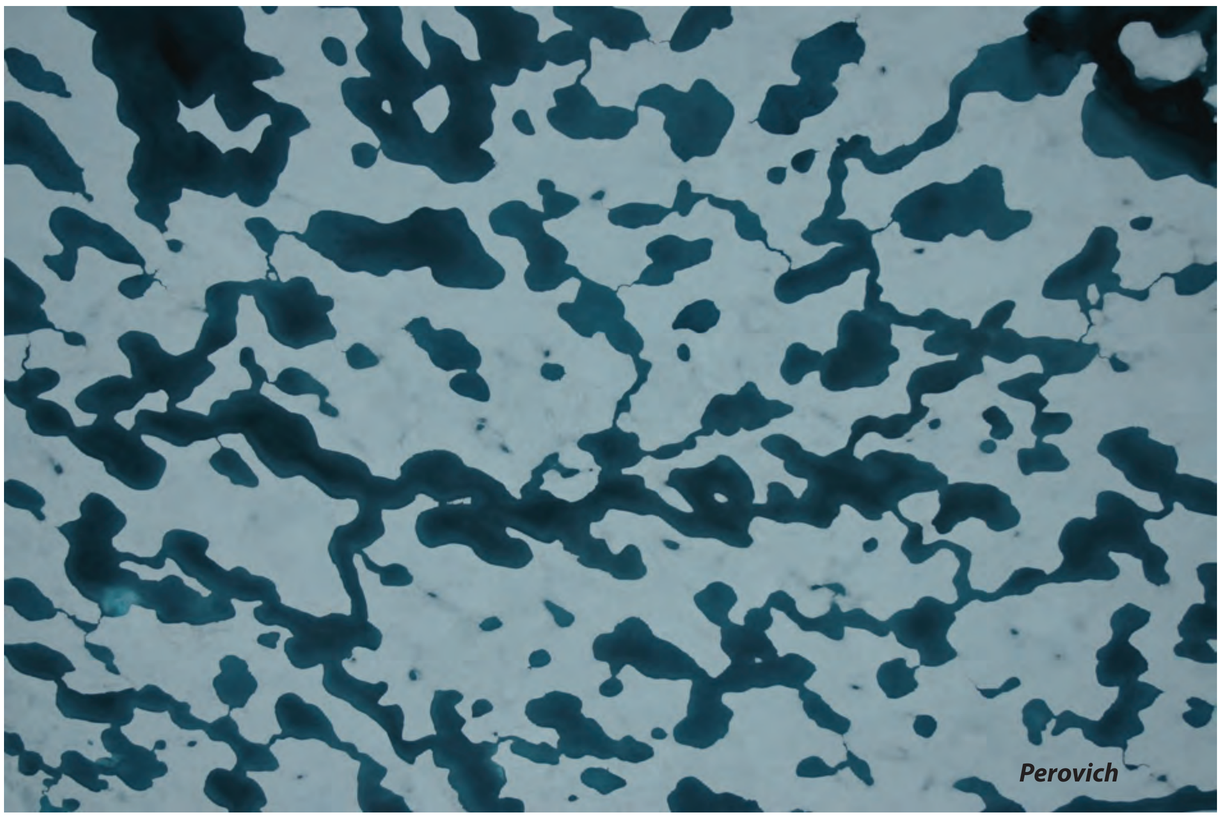
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

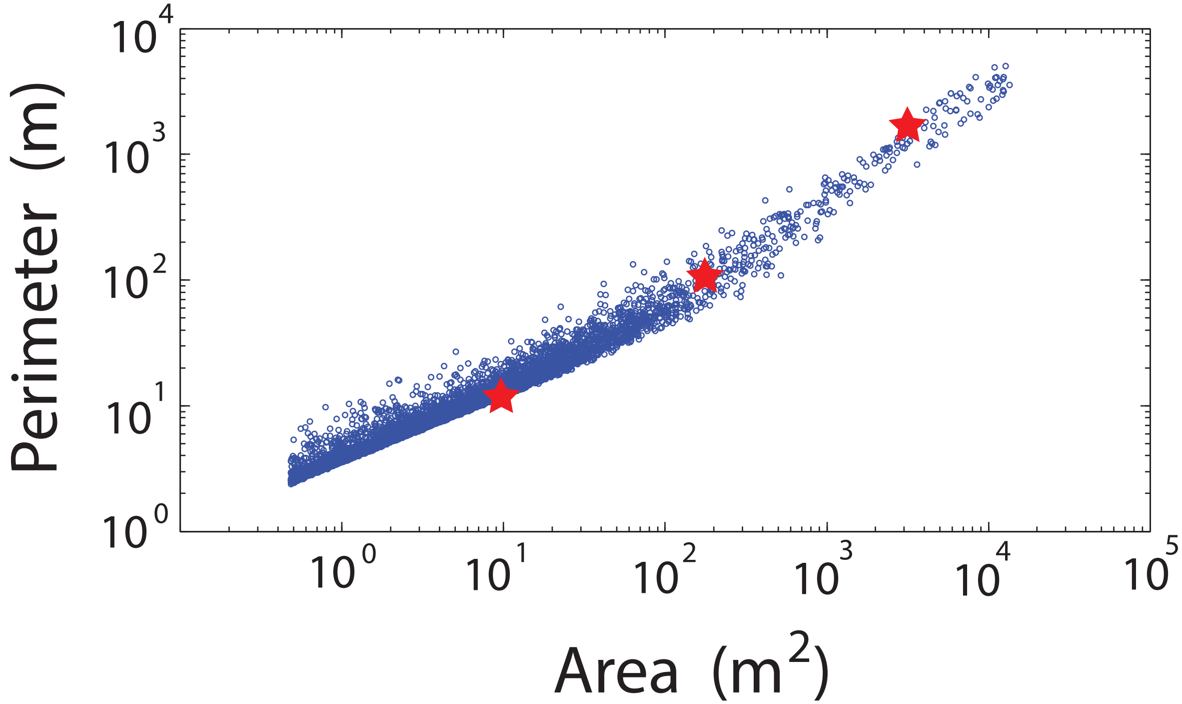
Flocco, Feltham, Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds *The Cryosphere, 2012*

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

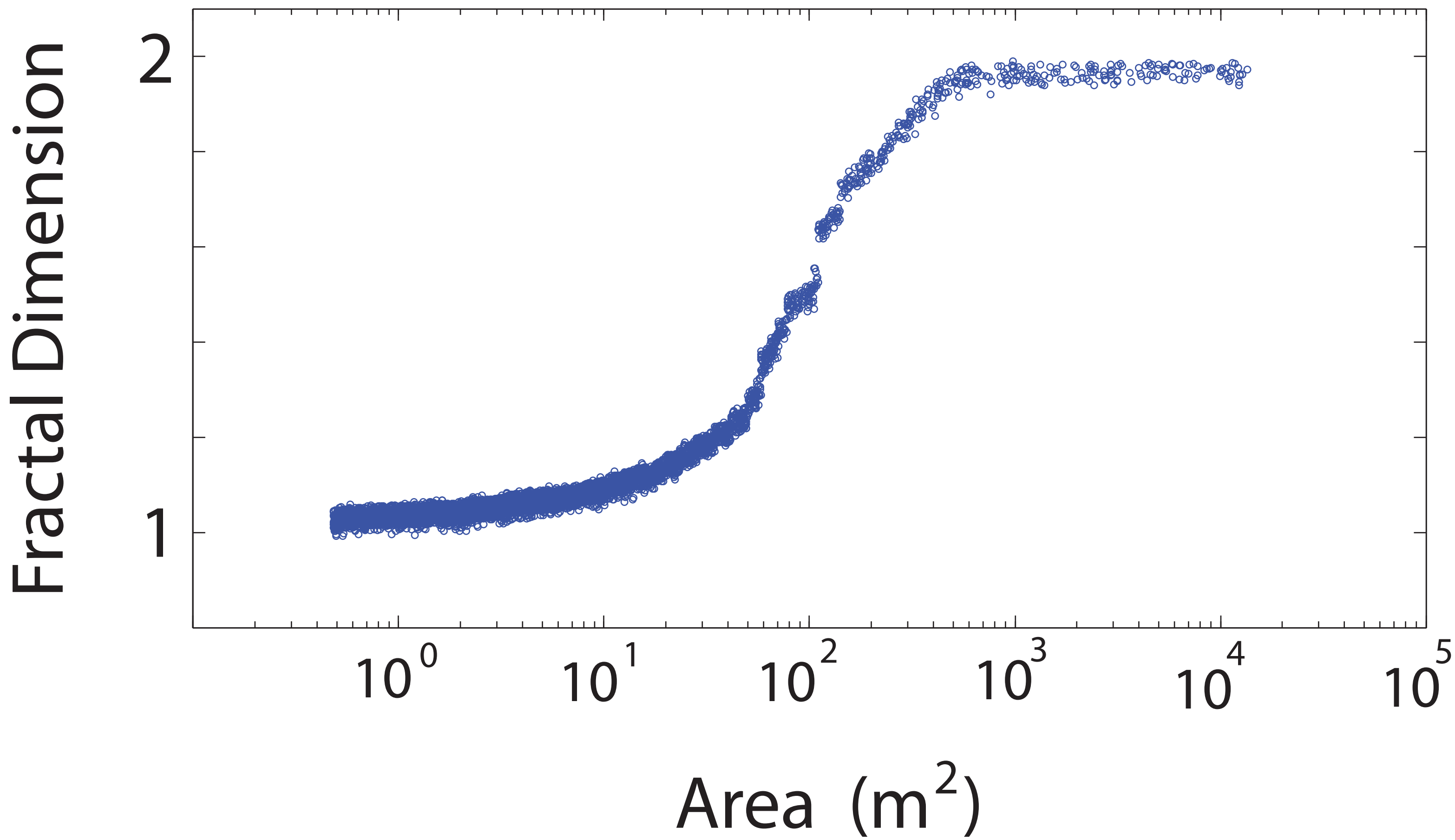


data from
5269 Arctic
melt ponds

simple pond ***transitional pond*** ***complex pond***

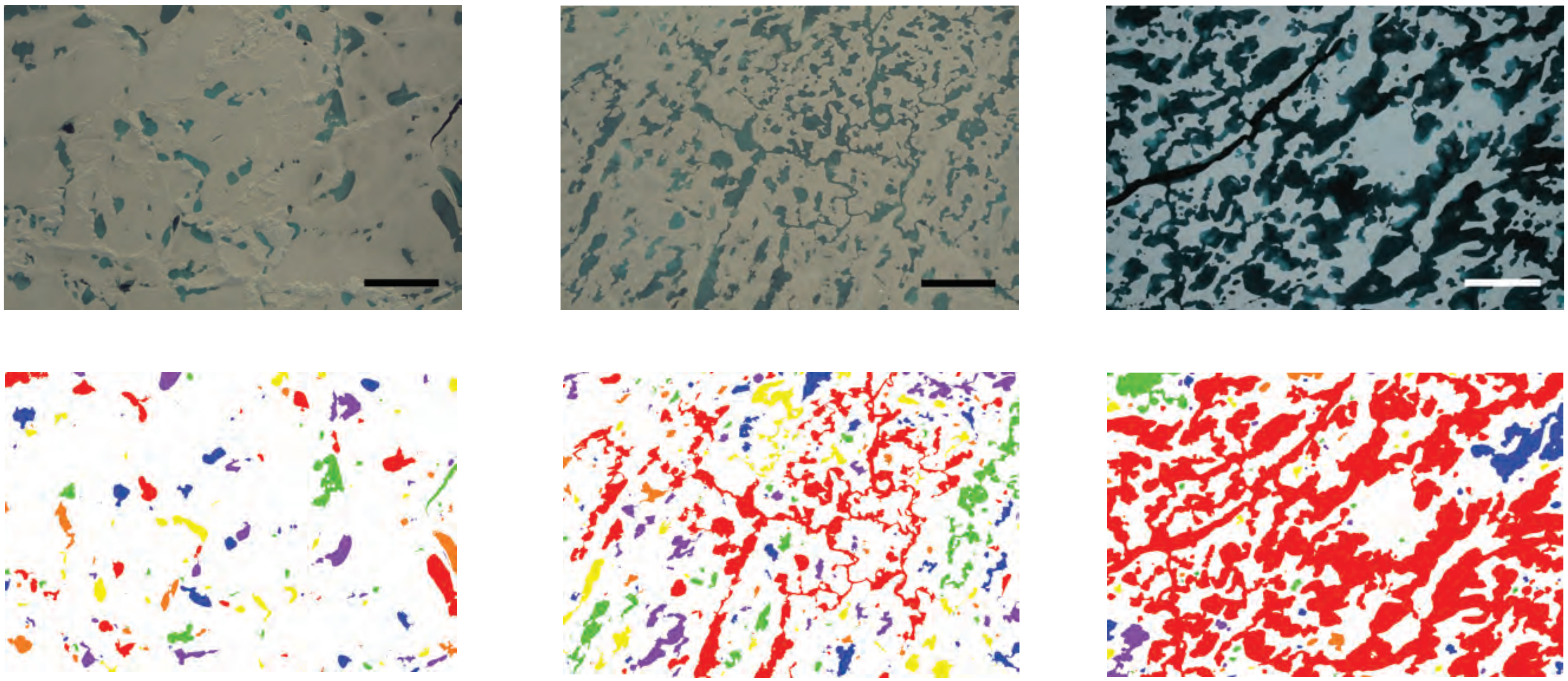
transition in the fractal dimension

complexity grows with length scale



compute "derivative" of area - perimeter data

small simple ponds coalesce to form large connected structures with complex boundaries



melt pond percolation

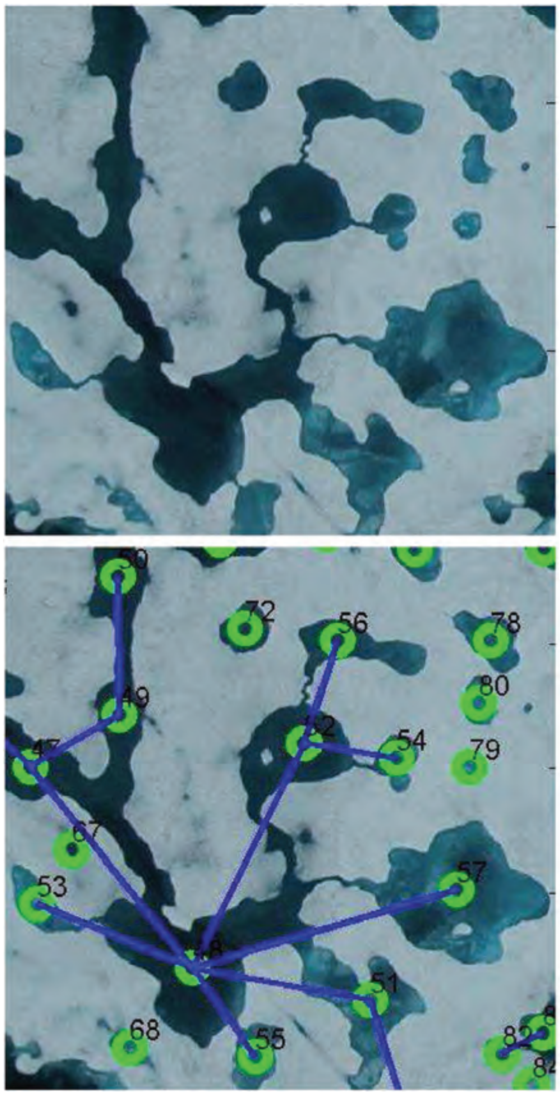
results on percolation threshold, cluster behavior

Anthony Cheng (Hillcrest HS), Bacim Alali, Ken Golden

Network modeling of Arctic melt ponds

Barjatia, Tasdizen, Song, Sampson, Golden

Cold Regions Science and Tecnology, 2016



develop algorithms to map images of melt ponds onto

random resistor networks

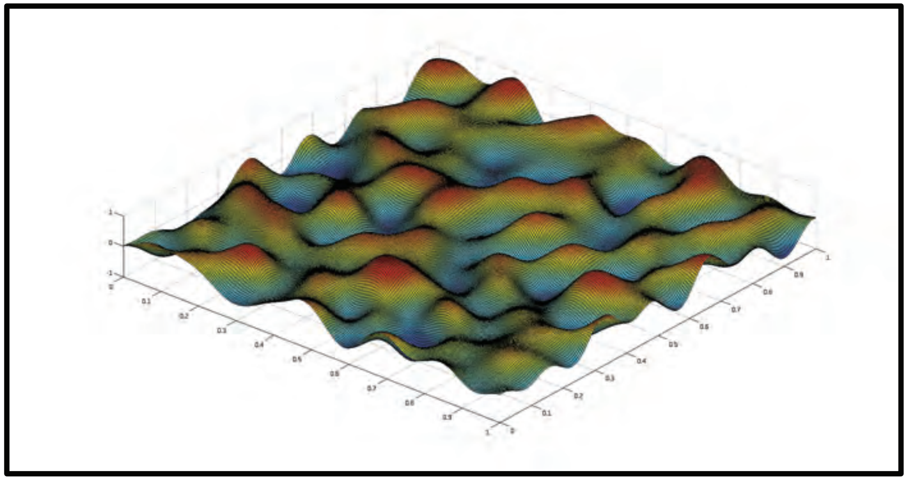
graphs of nodes and edges with edge conductances

edge conductance ~ neck width

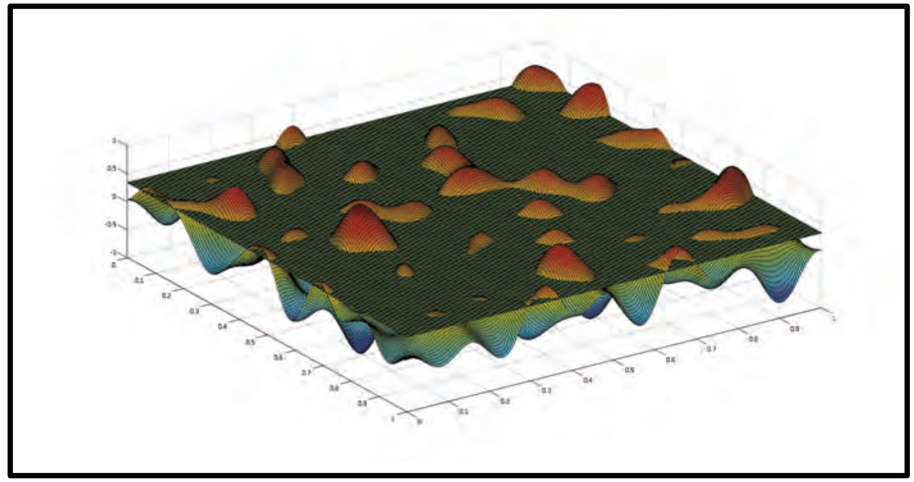
compute effective horizontal fluid conductivity

Continuum percolation model for melt pond evolution

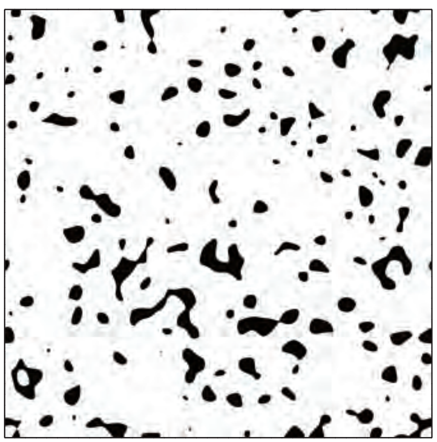
Brady Bowen, Court Strong, Ken Golden ***J. Fractal Geom. 2017***



random Fourier series representation of surface topography

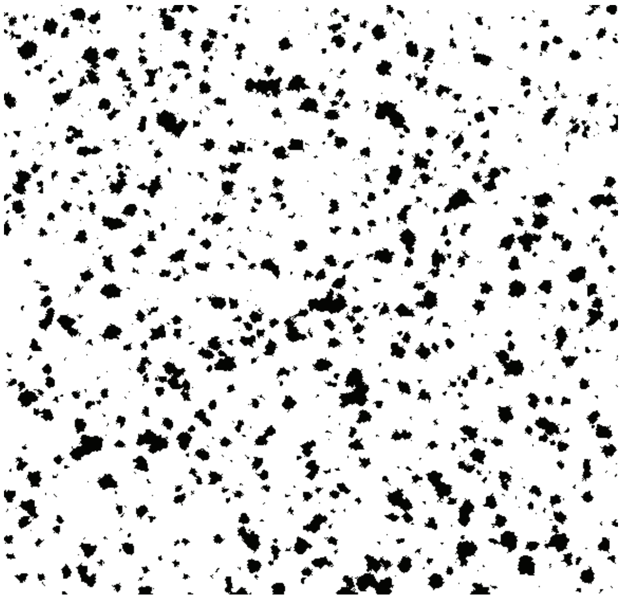


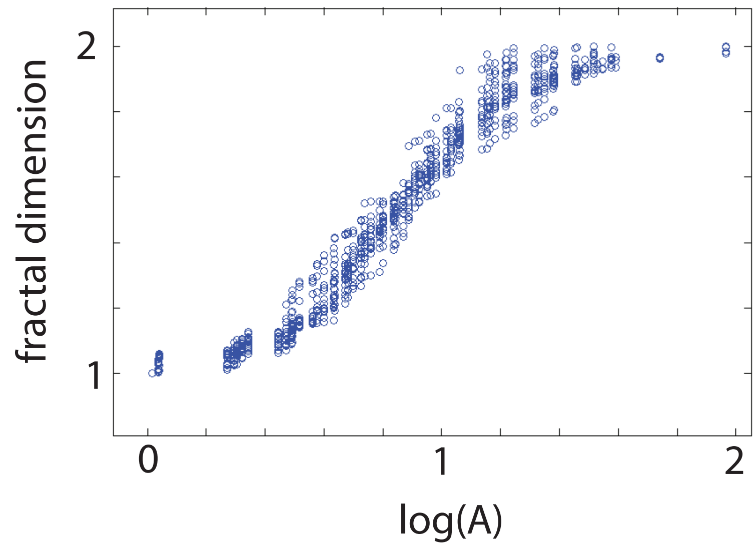
intersections of a plane with the surface define melt ponds

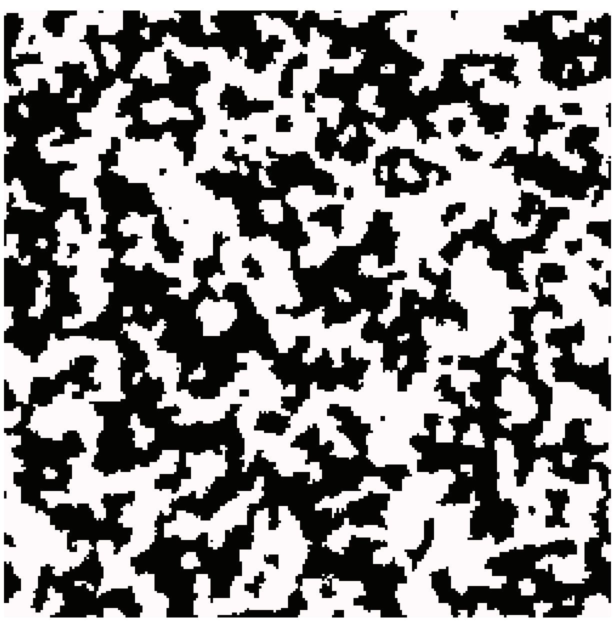


electronic transport in disordered media *diffusion in turbulent plasmas* *(Isichenko, Rev. Mod. Phys., 1992)*

simple stochastic growth model of melt pond evolution







voter model

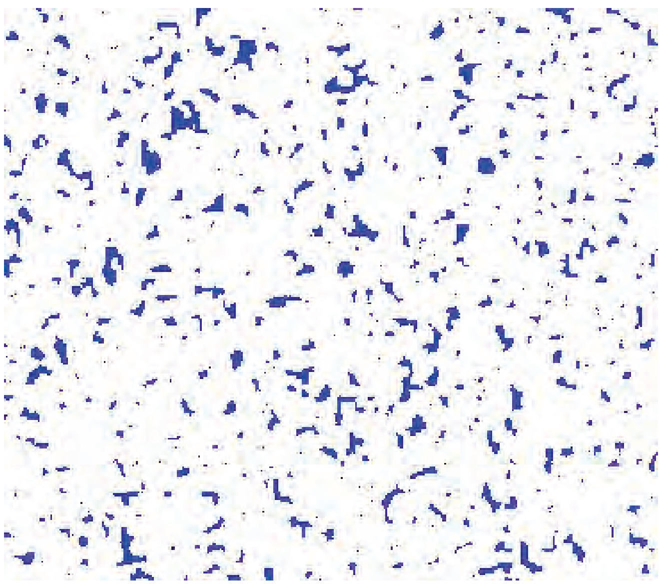
a square is more likely to melt if its neighbors have melted

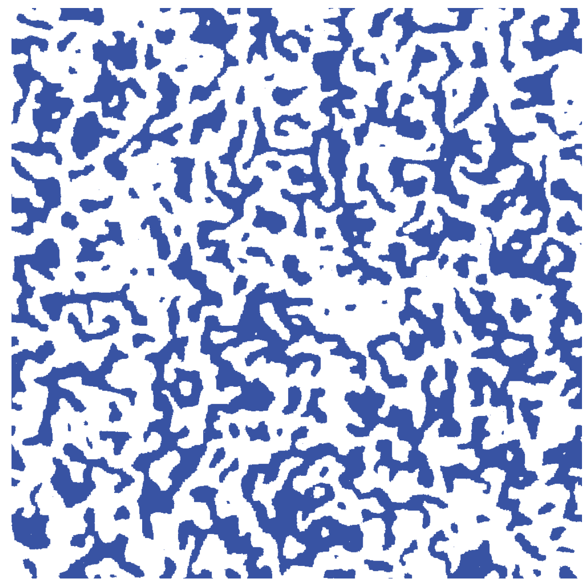
Rebecca Nickerson (West HS, Salt Lake City) and Ken Golden

Ising model for ferromagnets $\longrightarrow$ Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - H \sum_{i}^{N} s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

magnetization $M = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$ **pond coverage** $\frac{(M+1)}{2}$





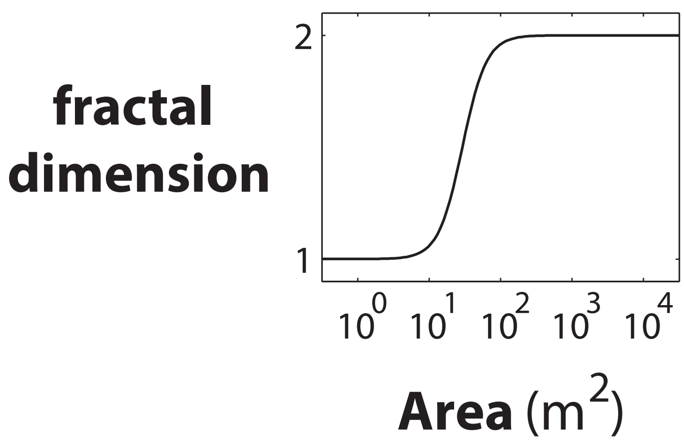
"melt ponds" are clusters of magnetic spins that align with the applied field

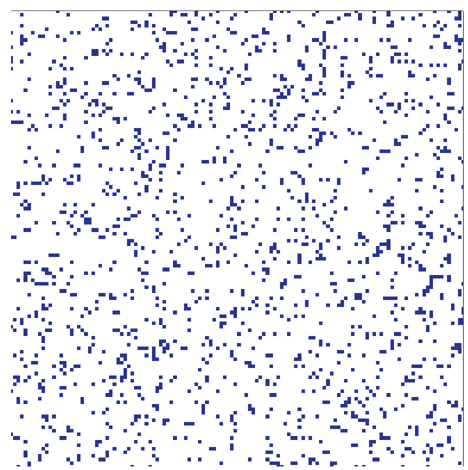
Ma, Sudakov, Strong, Golden 2017

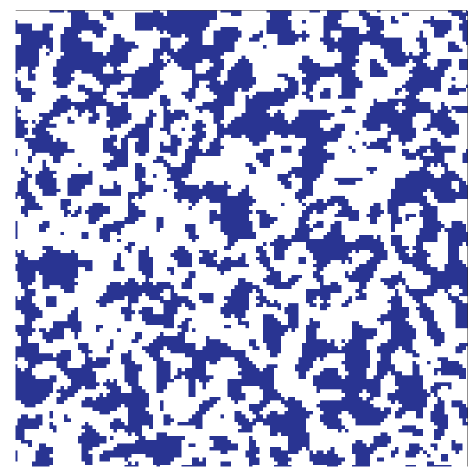
Melt Pond Ising Model

- Minimize an Ising Hamiltonian
  random magnetic field represents ice topography
  interaction term represents horizontal heat transfer

- Ice-albedo feedback incorporated by taking coupling constant in interaction term to depend on the pond coverage







- predicted fractal transition **40 - 90 m^2** vs. **86 m^2** observed

- predicted pond size distribution exponent **-1.51**
  vs. **-1.50** observed

Conclusions

1. Summer Arctic sea ice is melting rapidly, and melt ponds and other processes must be accounted for in order to predict melting rates.

2. Fluid flow through sea ice mediates melt pond evolution and many processes important to climate change and polar ecosystems.

3. Statistical physics and homogenization help *link scales*, provide rigorous methods for finding effective behavior, and advance how sea ice is represented in climate models.

4. Our research will help to improve projections of climate change and the fate of the Earth sea ice packs.

THANK YOU

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

Office of Naval Research

Arctic and Global Prediction Program

Applied and Computational Analysis Program

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999