SPECTRAL AND PROBABALISTIC ASPECTS OF MATRIX MODELS

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Abstract

The paper deals with the eigenvalue statistics of $n \times n$ random Hermitian matrices as $n \to \infty$. We consider a certain class of unitary invariant matrix probability distributions which have been actively studied in recent years in the quantum field theory (QFT). These ensembles are natural extensions of the archetype Gaussian ensemble well known and widely studied in the field called random matrix theory (RMT) and having applications in a number of areas of physics and mathematics. Our goal is to analyze the QFT motivated matrix ensembles from the point of view of the RMT. We consider the normalized counting functions of matrix eigenvalues (NCF), discuss the RMT content of various physical results (limiting form of the NCF, the eigenvalue spacing distribution, etc.), present rigorous versions and extensions some of them and other rigorous results, and discuss open mathematical problems, conjectures, and links with other areas.

1 Introduction

In recent years there has been considerable progress in describing important aspects of low–dimensional bosonic string theory and two-dimensional quantum gravity by matrix models (see e.g. recent review [1] and references therein). A number of deep links of these models with integrable systems, topological quantum field theory, spectral theory, algebraic geometry and other fields of theoretical physics and mathematics was found and studied.

The majority of these results came from the analysis of various integrals over the probability measure

$$p_n(M)dM = Z_n^{-1} \exp\{-nTrV(M)\}\, dM \tag{1.1}$$

defined on the space of $n \times n$ Hermitian matrices M. Here Z_n is the normalization factor, $V(\lambda)$, $\lambda \in \mathbb{R}$, is a real valued, bounded below and growing fastly enough at infinity function (an even polynomial in the quantum field theory (QFT) studies) and

$$dM = \prod_{k=1}^{n} dM_{kk} \prod_{k<j} dReM_{kj}d\ Im\ M_{kj} \tag{1.2}$$

is the "Lebesgue" measure for the Hermitian matrices. Thus

$$p_n(M) = Z_n^{-1} \exp\{-nTrV(M)\} \tag{1.3}$$

is the density of probability distribution (1.1) with respect to measure (1.2).

In this paper we are going to discuss several questions concerning the eigenvalues statistics of the random matrix ensembles defined by (1.1) and (1.2). These questions belong to the more traditional part of random matrix theory (RMT), whose modern history was initiated by Winger's works in the fifties (see their reprints and a discussion in [2]), was motivated by nuclear physics and has been actively developing since that time (see review works [3]–[5] for results and references). Subsequent rather strong motivations were provided by quantum chaology and condensed matter theory (see reviews [6] and [7] respectively). The range of problems and results of this part of the RMT is also rather broad but there is a quantity that is of considerable interest for a wide variety of studies. This is the normalized counting measure (NCM)

$$N_n(\Delta) = \sharp \left\{ \lambda_i^{(n)} \in \Delta \right\} \cdot n^{-1} \tag{1.4}$$

of eigenvalues

$$\lambda_1^{(n)} \le \lambda_2^{(n)} \le \ldots \le \lambda_n^{(n)} \tag{1.5}$$

of a random matrix M and $\Delta = (a, b)$ is an interval of the real axis. In particular, we will be interested in the expectation value

$$E\{N_n(\Delta)\} \tag{1.6}$$

of $N_n(\Delta)$, in its variance

$$D_n = E\left\{N_n^2(\Delta)\right\} - E^2\{N_n(\Delta)\}\ , \tag{1.7}$$

and in the probability

$$R_n(\Delta) = Pr\{N_n(\Delta) = 0\} \tag{1.8}$$

that a given interval Δ does not contain eigenvalues. In (1.6)–(1.8) the expectation and the probability are determined by distribution (1.1)–(1.2).

These problems have been studied rather completely for the archetype Gaussian case

$$V(\lambda) = \frac{\lambda^2}{2} \tag{1.9}$$

(see reviews [3]-[6]). In this paper we are going to discuss results that were recently obtained for other (mostly polynomial) forms of $V(\lambda)$ in (1.3). Following the QFT terminology we will call $V(\lambda)$ the potential. Thus we view the QFT motivated ensembles (1.1)–(1.2) as a natural generalization of the Gaussian ensemble (1.9). All these ensembles share with the Gaussian one the property to be invariant under the unitary transformations $M \to UMU^*$, $\forall U \in U(n)$, and we will call them the unitary ensembles (UE).

According to the simple theorem [2], which is a matrix analog of the Maxwell theorem for random vectors, an UE is the Gaussian unitary ensemble (GUE) if and only if all the functionally independent matrix elements $\{M_{ij},\ 1 \le j < k \le n\}$ are independent random variables. Thus from the probabilistic and spectral points of view unitary ensembles allow us to study manifestations of statistical dependency (correlations) of matrix elements on spectral properties of random matrices. It is easy to find that most general unitary invariant probability distribution density is given by an arbitrary nonnegative and integrable with respect to dM function $g_n(\lambda_1 \dots \lambda_n)$, while in (1.3) the role of g_n plays the product

$$g_n(\lambda_1 \dots \lambda_n) = \prod_{l=1}^{n} \exp\{-nV(\lambda_l)\}\,. \tag{1.10}$$

Nevertheless, as we shall see this seemingly rather special form of the ensemble probability density provides a wide variety of problems and results that have no analogues in the Gaussian case and widens considerably the frameworks of the RMT and its areas of applications.

Since we are interested in spectral properties of random matrices we have to express probability distribution (1.1)–(1.2) of the collection $\overline{M} = \{M_{jk}\,,\ 1 \le j \le k \le n\}$ of functionally independent entries via the collection

$$\overline{\wedge}_n = \left\{\lambda_1^{(n)} \le \ldots \le \lambda_n^{(u)}\right\} \tag{1.11}$$

of eigenvalues and

$$\Psi_n = \left\{\psi_l^{(n)} = \left(\psi_{1l}^{(n)}, \ldots, \psi_{nl}^{(n)}\right)\ ,\ \psi_{1l} \ge 0\right\}_{l=1}^{n} \tag{1.12}$$

of orthonormal eigenvectors of a Hermitian matrix M. These two collections provide two parametrizations of M. They are related by the spectral theorem

$$M_{kj} = \sum_{l=1}^{n} \lambda_l^{(n)} \psi_{kl}^{(n)} \psi_{jl}^{(n)}\ , \tag{1.13}$$

which can be viewed as the change of variable formula. The respective Jacobian $J(\overline{\wedge},\, \Psi)$ is known since the prewar time (see [8]):

$$J(\overline{\wedge},\, \Psi) = \text{const}\, \Delta^2(\overline{\wedge})\,, \tag{1.14}$$

where

$$\Delta(\overline{\wedge}) = \prod_{j<R} (\lambda_j - \lambda_k)\,. \tag{1.15}$$

Thus the distribution (1.1)–(1.2) expressed via $(\overline{\wedge},\, \Psi)$ has the form

$$p_n(\overline{\wedge},\, \Psi) = \left(\overline{Z}_n\right)^{-1} \exp\left\{-n \sum_{1}^{n} V(\lambda_l)\right\} \Delta^2(\overline{\wedge})\, d\overline{\wedge} d\Psi \tag{1.16}$$

where $d\overline{\wedge} = \prod_1^n d\lambda_l$ and $d\Psi$ is the Haar measure on $U(n)$. Formula (1.16) shows that eigenvalues and eigenvectors of random matrix ensembles (1.1)–(1.2) are statistically independent and that eigenvectors are always distributed "uniformly" over the part of the unitary group specified by (1.12). The latter property is a consequence of the unitary invariance of distribution (1.1)–(1.2).

We will be interested in probability properties of eigenvalues of random matrices, more precisely, in the normalized counting measure (1.4). Thus it suffices to know only the probability distribution of eigenvalues. The latter can be easily obtained from (1.16) because the integration over the "angle" variables (1.12) will not change $\overline{\wedge}$-dependence of (1.16), contributing only to the normalization constant. But before to write explicitly the eigenvalue distribution we will take into account that (1.4) and (1.16) can be easily extended from the part of $\mathbb{R}^n$ specified by the r.h.s. of (1.11) onto the whole $\mathbb{R}^n$ as symmetric functions. This allows us to consider in what follows the unordered collection

$$\wedge_n = \left\{ \lambda_l^{(n)} \right\}_{l=1}^{n} \tag{1.17}$$

of the random variables whose probability distribution has the form

$$p_n\left(\wedge\right) d\wedge \tag{1.18}$$

with

$$p_n(\wedge) = Q_n^{-1} \exp\left\{ -\sum_1^l V(\lambda_l) \right\} \Delta^2\left(\wedge\right) \ , \tag{1.19}$$

$$\wedge = \{\lambda_e\}_{l=1}^n \in \mathbb{R}^n \ , \quad d\wedge = \prod_1^n d\lambda_l \ , \tag{1.20}$$

$$Q_n = \int_{\mathbb{R}^n} \exp\left\{ -n \sum_1^n V\left(\lambda_l\right) \right\} d\wedge \ . \tag{1.21}$$

In this paper we will be concerned with unitary invariant ensembles of the Hermitian random matrices. The RMT deals also with two more invariant ensembles that consist from real symmetric and quaternion-real matrices [3]. The role of unitary invariance plays invariance with respect to orthogonal and symplectic transformations. One can easily write the orthogonal invariant and the symplectic invariant analogues of distribution (1.1)–(1.2), i.e. define the orthogonal and the symplectic ensembles (OE and SE). They will have the same form as (1.1)–(1.2) provided that the volume element of dM of respective matrix space is properly defined. Concerning these definitions we refer the reader to the book [3], where the Gaussian cases (1.9) of these ensembles (i.e. the GOE and the GSE) are considered in great detail, in particular, the respective Jacobians are given. The latter result allows us to write the eigenvalue probability density of the OE and the SE (cf. (1.19)–(1.21)):

$$p_{n\beta}(\Lambda) = Q_{n\beta}^{-1} \exp\left\{-\sum_{1}^{n} V(\lambda_l)\right\} |\Delta(\Lambda)|^{\beta} , \tag{1.22}$$

where β is equal to 1 and 4 for the OE and the SE respectively. In these notations the UE distribution density (1.18)–(1.21) is p_{n2}. In physical applications β is determined by symmetry of a system with respect to time reversal and rotations and by total spin.

The paper is organized as follows. In Section 2 we consider the density of states (DOS), more precisely, expectation (1.6) and variance (1.7) of the counting measure (1.4) for any n-independent interval Δ. We prove that (1.6) has an absolutely continuous limit $N(\Delta)$ as $n \to \infty$ and we study property of the limiting measure density $\rho(\lambda)$ known as the density of states:

$$\lim_{n\to\infty} E\left\{N_u(\Delta)\right\} \equiv N(\Delta) = \int_{\Delta} \rho(\lambda) d\lambda. \tag{1.23}$$

The limiting measure $N(\Delta)$ is called the integrated density of states (IDS).

We also prove that variance (1.7) vanishes as $n \to \infty$ and therefore the counting measure itself converges in probability to the limit (1.23) of its expectation. We discuss three approaches which allow us to see a number of interesting properties of the DOS and various links of the RMT with other branches of mathematics. In Section 3 we consider the asymptotic behavior of probability (1.8) that a given interval of length $O(1/n)$ is empty and we discuss the universality conjecture of the RMT which implies in particular that the leading term of this probability does not depend upon the form of potential in (1.3) but is determined by the parameter β only.

We notice in conclusion that the considerable part of our paper is a review of results obtained recently in the QFT for various particular classes of potentials. Some of these results can be proven rigorously and for wider classes of potentials. These results we formulate as theorems even in the cases when their proofs are simple. This is done in order to make some mathematical reference points in this branch of the RMT and to make the paper more readable and accessible for the mathematically-oriented reader.

2 Density of States

2.1 Orthogonal Polynomials Approach.

Denote by $P_k^{(n)}(\lambda)$, $k = 0, 1, \ldots$, the system of polynomials that are orthogonal with respect to the weight

$$w_n(\lambda) = e^{-nV(\lambda)} \tag{2.1}$$

so that

$$\int_{\mathbb{R}} P_j^{(n)}(\lambda) P_k^{(n)}(\lambda) w_n(\lambda) d\lambda = \delta_{jk}, \tag{2.2}$$

and let $\psi_k^{(n)}(\lambda)$, $k = 0, 1, \ldots$ be respective orthogonal system in $L_2(\mathbb{R})$:

$$\int_{\mathbb{R}} \psi_j^{(n)}(\lambda) \psi_k^{(n)}(\lambda) d\lambda = \delta_{jk}. \tag{2.3}$$

Then, according to [3], the joint probability density (1.19) can be written as

$$p_n(\wedge) = (n!)^{-1} \det \psi_n \tag{2.4}$$

$$= (n!)^{-1} \det k_n \tag{2.5}$$

where $n \times n$ matrices ψ_n and k_n are

$$\psi_n = \left\| \psi_{j-1}^{(n)}(\lambda_k) \right\|_{j,\,k=1}^{n}, \quad k_n = \left\| k_n\left(\lambda_j,\, \lambda_k\right) \right\|_{j,\,k=1}^{n} \tag{2.6}$$

and

$$k_n(\lambda,\, \mu) = \sum_{j=0}^{n-1} \psi_j^{(n)}(\lambda) \psi_j^{(n)}(\mu)\ . \tag{2.7}$$

This function is well known in classical analysis as the reproducing kernel. The function $[k_n(\lambda,\, \lambda)]^{-1}$ is known in the approximation theory as the Christoffel function. We have

$$k_n(\lambda,\, \mu) = k_n(\mu,\, \lambda)\,, \quad \int_{\mathbb{R}} k_n(\lambda,\, \nu) k(\nu,\, \mu) d\nu = k_n(\lambda,\, \mu), \tag{2.8}$$

$$\int_{\mathbb{R}} k_n^2(\lambda,\, \mu) d\lambda d\mu = n. \tag{2.9}$$

By using these formulas it can be shown that the joint probability distribution $p_n^{(l)}(\lambda_1, \ldots, \lambda_l)$ of $l \leq n$ eigenvalues is [3]

$$p_l^{(n)}(\lambda_1, \ldots \lambda_l) \equiv \int\limits_{\mathbb{R}^{n-l}} p_n\,(\lambda_1, \ldots \lambda_l,\, \lambda_{l+1}, \ldots \lambda_n)\, d\lambda_{l+1} \ldots d\lambda_n \tag{2.10}$$

$$= [n(n-1)\ldots(n-l+1)]^{-1} \det \left\| k_n(\lambda_j,\, \lambda_k) \right\|_{j,\,k=1}^{l} . \tag{2.11}$$

This formula allows us to rewrite the expectation (1.6) and the variance (1.7) of the NCM (1.4) in terms of the orthogonal polynomials:

$$E\,\{N_n(\Delta)\} = \int\limits_{\Delta} \rho_n\,(\lambda)\, d\lambda\ , \tag{2.12}$$

$$\rho_n(\lambda) \equiv p_1^{(n)}(\lambda) = K_n\,(\lambda,\, \lambda)\ , \tag{2.13}$$

$$K_n\,(\lambda,\, \mu) \equiv \frac{1}{n}\; k_n\,(\lambda,\, \mu) = \frac{1}{n}\sum_{j=0}^{n-1} \psi_j^{(n)}\,(\lambda)\, \psi_j^{(n)}\,(\mu)\ , \tag{2.14}$$

$$D_n\,\{N_n\,(\Delta)\} = \frac{2}{n}\int\limits_{\Delta} \rho_n\,(\lambda)\, d\lambda - \frac{2}{n^2}\int\limits_{\Delta^2} k_n^2\,(\lambda,\, \mu)\, d\lambda d\mu\ . \tag{2.15}$$

Now we are able to formulate our first simple but important statement.

Theorem 2.1 *Let an ensemble of $n \times n$ Hermitian matrices is defined by distribution (1.2) and (1.3) in which $V(\lambda)$ is a real valued bounded below function such that*

$$V(\lambda) \geq (2+\varepsilon)\ln|\lambda|\ , \quad |\lambda| \geq L\ , \quad \varepsilon > 0\ , \quad L < \infty\ . \tag{2.16}$$

Then

$$D\,\{N_n(\Delta)\} = O\left(\frac{1}{n}\right) \tag{2.17}$$

The proof of (2.17) follows immediately from (2.15).

The relation similar to (2.17) is also known in other branches of spectral theory of random operators and in mathematical physics of disordered systems [9]. This relation implies that the fluctuations of a corresponding random spectral or physical characteristic vanish as $n \to \infty$ (the large parameter n may have different origin and meaning). As for the existence of the limit of the expectation of a corresponding characteristic it usually requires additional conditions and arguments. We begin the discussion of this problem for the NCM (1.4) of ensembles (1.1)–(1.2) from the Gaussian case (1.1). In this case polynomials (2.1) are

$$P_l^{(n)}(\lambda) = \left(\frac{2}{n}\right)^{1/4} H_l\left(\lambda\sqrt{\frac{n}{2}}\right) \ , \tag{2.18}$$

where $H_l(\xi)\,,\ l = 0, 1, \ldots$ are the Hermite polynomials that are orthogonal with the weight $e^{-\xi^2}$. By using (2.13) and (2.14) and the Plancherel-Rotah asymptotic formula for the Hermite polynomials [10]

$$e^{-\frac{\xi^2}{2}} H_l(\xi)\bigg|_{\xi=\sqrt{2l+1}\cdot\cos\theta} = \left(\frac{2}{\pi^2 l \sin^2\theta}\right)^{1/4} \sin\left(l\Gamma(\theta) + \gamma(\theta)\right) + O\left(\frac{1}{l}\right) \ , \tag{2.19}$$

$$l \to \infty,$$

$$\Gamma(\theta) = \frac{\sin 2\theta}{2} - \theta \ , \quad \gamma(\theta) = \Gamma(\theta) + \frac{3\pi}{4} \ , \tag{2.20}$$

we prove for the GUE (1.9) the result (1.23) in which

$$\rho(\lambda) = \frac{1}{2\pi} \,{}_{+}\!\sqrt{4-\lambda^2} \tag{2.21}$$

and

$${}_{+}\!\sqrt{a} = \sqrt{\max(a,\, 0)} \ .$$

This is the well known semicircle law of Wigner [3].

Unfortunately, rigorous asymptotic formulas for the orthogonal polynomials specified by (2.1) and (2.2), where $V(\lambda)$ is a general enough function, are not know (see however physical papers [11, 12]). Thus we consider here the case

$$V(\lambda) = \frac{|\lambda|^\alpha}{\alpha}, \tag{2.22}$$

where $\alpha > 1$ is a real number. Respective asymptotic formulas were found in [13] and [14]. They yield

$$\rho_\alpha(\lambda) = a^{-1} v_\alpha(\lambda/a), \tag{2.23}$$

where

$$a^\alpha = 2\alpha \int_0^1 \frac{t^{\alpha-1}dt}{\sqrt{1-t^2}} \tag{2.24}$$

and

$$v_\alpha(t) = \frac{\alpha}{\pi} \begin{cases} \int_{|t|}^1 \frac{\tau^{\alpha-1}d\tau}{\sqrt{\tau^2-t^2}} , & |t| \le 1, \\ 0 , & |t| \ge 1. \end{cases} \tag{2.25}$$

The function $v_\alpha(t)$ is known in the theory of orthogonal polynomials as the Ulman-Nevai density. It describes the so-called contracted zero distribution of polynomials orthogonal on $\mathbb{R}$ with the weight $\exp\{-Q(x)\}$ where $Q(x)$ is an even nonnegative and smooth enough function that behaves as $|x|^\alpha$ for $|x| \to \infty$ [13].

If $\alpha = 2p$ where p is a positive integer, then

$$\rho_\alpha(\lambda) = t_{\alpha-2}(\lambda) \ {}_+\sqrt{a^2-\lambda^2} \ , \tag{2.26}$$

where $t_{\alpha-2}(\lambda)$ is a polynomial of degree $\alpha - 2$ in λ and a which is positive for $|\lambda| \le a$. (2.26) is an analogue of the semicircle law for more general than (1.9) monomial potential

$$V(\lambda) = \frac{\lambda^{2p}}{2p} \tag{2.27}$$

We demonstrate now one more formula for the DOS. This formula uses another asymptotic characteristic of orthogonal polynomials. Recall that an arbitrary system of orthogonal polynomials $P_l(\lambda)$, $l = 0, 1, \ldots$ satisfies the second order finite-difference equations of the form [10]:

$$r_{l+1}P_{l+1}(\lambda) + r_l P_{l-1}(\lambda) = \lambda P_l(\lambda), \quad l \ge 0, \tag{2.28}$$

where

$$r_{-1} = 0 \ , \quad r_l = \int \lambda P_l(\lambda) P_{l-1} w(\lambda) d\lambda \ge 0. \tag{2.29}$$

In our case the weight (2.1) depends on n. Thus coefficients (2.29) are also dependent on n: $r_l = r_l^{(n)}$. The following theorem is a rigorous version of one of the results of the seminal physical paper [15]. Our proof is based on an idea different from that of [15].

Theorem 2.2 *Let* $r_n(x)\,,\quad 0\le x\le 1\,,$ *be the piece-wise linear function with vertices* $r_l^{(n)}$ *at* $x=\dfrac{l}{n}\,,\quad l=0,1,\ldots n\,.$ *Assume that*

$$\begin{array}{ll}(i) & \sup\limits_{0\le l\le n} r_l^{(n)}\le C<\infty\ ; \\ (ii) & \textit{there exists a piece-wise continuous function} \\ & r(x)\ \textit{such that uniformly in}\ x\in[0,\,1]\end{array} \tag{2.30}$$

$$\lim_{n\to\infty} r_n(x)=r(x)\ . \tag{2.31}$$

Then the DOS defined in (1.23) has the form

$$\rho(\lambda)=\int\limits_{X(\lambda)}\frac{dx}{\sqrt{4r^2(x)-\lambda^2}} \tag{2.32}$$

where $X(\lambda)=\{x\in[0,\,1]\ :\quad 2r(x)\ge\lambda\}\,.$

Proof (scheme). We will use the Stieltjes transform of all measures involved and some simple properties of the resolvent of the selfadjoint operator $J^{(n)}$ defined in $l_2(\mathbb{Z}_+)$ by (2.28). Recall that the Stieltjes transform $f_\nu(z)$ of a nonnegative measure $\nu(d\lambda)\,,\ \nu(\mathbb{R})=1$ is defined as

$$f_\nu(z)=\int\limits_{\mathbb{R}}\frac{\nu(d\lambda)}{\lambda-z}\ ,\quad Im\ z\ne 0\ . \tag{2.33}$$

f_ν is an analytic function for $Im\ z\ne 0$,

$$Im\ f(z)\cdot Im\ x>0\ ,\quad Im\ z\ne 0\ , \tag{2.34}$$

$$\sup_{\eta\ge 1}\eta|f\,(i\eta)\,|=1\ , \tag{2.35}$$

and any function possessing these properties can be represented in the form (2.33). $f_\nu(z)$ determines uniquely the measure $\nu(d\lambda)$. If $\Delta=(a,\,b)$ and a and b are continuity points of $\nu(d\lambda)$, then

$$\nu(\Delta)=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int\limits_a^b Im\ f(\lambda+i0)d\lambda. \tag{2.36}$$

Besides if $\{\nu_m\}_{m=1}^{\infty}$ is a sequence of measures that converges weakly to a measure ν, then

$$\lim_{m\to\infty} f_{\nu_m}(z) = f_\nu(z) \tag{2.37}$$

uniformly on compact sets of $\{z: \ Im\ z \neq 0\}$. The converse statement is also true. If $\mathcal{E}^{(n)}(d\lambda) = \left\{\mathcal{E}_{kl}^{(n)}(d\lambda)\right\}_{k,\,l=0}^{\infty}$ is the resolution of the identity of the selfadjoint operator $J^{(n)}$ defined by (2.28), i.e. by the semi-infinite Jacobi matrix

$$J_{kl}^{(n)} = r_{k+1}^{(n)}\delta_{k+1,l} + r_{k-1}^{(n)}\delta_{k-1,l}\ , \tag{2.38}$$

then [16]

$$\mathcal{E}_{kl}^{(n)}(d\lambda) = \psi_k^{(n)}(d\lambda)\psi_l^{(n)}(\lambda)d\lambda \equiv e_{kl}^{(n)}(\lambda)d\lambda\ . \tag{2.39}$$

Thus we can rewrite (2.13) as

$$\rho_n(\lambda) = \frac{1}{n}\sum_{l=0}^{n-1} e_{ll}^{(n)}(\lambda)\ . \tag{2.40}$$

This relation and the spectral theory for selfadjoint operators imply that the Stieltjes transform $f_{\rho_n}(z)$ is

$$f_{\rho_n}(z) = \frac{1}{n}\sum_{l=0}^{n-1} G_{ll}^{(n)}(\lambda)\ , \tag{2.41}$$

where $G^{(n)}(z) = \left(J^{(n)} - z\right)^{-1}$ is the resolvent of $J^{(n)}$. By using (2.38) and the resolvent identity we can prove that $f_{\rho_n}(z)$ differs by $O\left(\frac{1}{n|Im\ z|^2}\right)$ from the Stieltjes transform $f_{\nu_n}(z)$ of the eigenvalue counting measure $\nu_n(d\lambda)$ of the $n\times n$ Jacobi matrix defined by $r_0^{(n)}, r_1^{(n)}, \dots, r_{n-1}^{(n)}$. Then we set $n = pq$, divide the "interval" $[0,\, n]$ on p subintervals of the length q each, and show by similar arguments that $f_{\nu_n}(z)$ differs by $O\left(\frac{1}{n|Im\ z|^2}\right)$ from the arithmetic mean of p Stieltjes transforms of the eigenvalue counting measures of the $q\times q$ "block" Jacobi matrices defined by $r_{sq+t}^{(n)},\quad t = 1,\dots,q-1,\quad s = 0,1,\dots,p-1$. Then we choose p to be so large as to guarantee the inequality $|r(x) - r(x')| \le \varepsilon,\quad |x - x'| \le p - 1$ for a given $\varepsilon > 0$ (see condition (ii) of theorem). This allows us to replace $r_l^{(n)}$ within the s-th "block" by the constant $r\left(\frac{s}{p}\right)$. If

q is also large enough then the contribution of each block is close enough to the Stieltjes transform $f\left(r\left(\frac{s}{p}\right), z\right)$ of the semi-infinite Jacobi matrix $J\left(r\left(\frac{s}{p}\right)\right)$, where $J(r)$ is defined by (2.38) in which $r_l^{(n)} \equiv r$. This means that uniformly in z, $|Im\ z| \geq \eta_0 > 0 \quad \lim_{n\to\infty} f_{\rho n}(z) = \int_0^1 f(r(x), z)dx$. It is easy to show that $f(r, z) = (z^2 - 4r^2)^{-1/2}$ where the branch of the square root is defined by (2.34) and (2.35). The latter expression, (2.36) and (2.37) imply (2.32).

Remark. Formula (2.32) illustrate the "slow varying" character of the coefficients $r_l^{(m)}$ of the second order finite-difference operator (2.38). Concerning similar formulas for a wide variety of finite-difference and differential operators see [17, 18].

Theorem 2.2 reduces the problem of the computing the DOS of the UE (1.1)-1.2) to the proof of asymptotic relations given in its conditions (i) and (ii) just as the more general formula (2.13) reduces this problem to proof of asymptotic formulas for the orthogonal polynomials (2.2). The next theorem treats a simple case where conditions (i) and (ii) of Theorem 2.2 can be justified and the function $r(x)$ defined in (2.31) can be found. This theorem again is a rigorous version of some statement from [15].

Theorem 2.3 *Assume that the potential $V(\lambda)$ in (1.1) is an even polynomial with nonnegative coefficients. Then conditions (i) and (ii) of Theorem 2.2 are valid. In particular, there exists the even polynomial $W(t)$ of the same degree such that $r(x)$ is a unique positive solution of the algebraic equation*

$$x = W(r(x)). \tag{2.42}$$

Proof. We consider the simplest nontrivial case when

$$V(t) = g_1 \frac{t^2}{2} + \frac{t^4}{4}, \quad g_1 > 0 \tag{2.43}$$

and we use the formula [15]

$$r_l V'_{l-1,\,l}\left(J^{(n)}\right) = \frac{l}{n}\,. \tag{2.44}$$

This formula is known now as the string (pre-string) equation [1] and can be easily obtained by using integration by parts, the orthogonality property of polynomials $P_l^{(n)}(\lambda)$ and relations (2.29) and (2.38). By using (2.38) and (2.44) we find the following recurrence relation

$$\frac{l}{n} = R_l \left[g_1 + (R_{l+1} + R_{l-1} + R_l) \right] \ , \quad r_l^2 = R_l \ . \tag{2.45}$$

Since $g_1 \geq 0$, this relation immediately implies (2.30). Now, to prove (2.31) we have to show that if

$$\delta_l^{(n)} = R_{l+1}^{(n)} - R_l^{(n)} \ ,$$

then

$$\sup_{o \leq l \leq n} n|\delta_l^{(n)}| \leq C_1 \leq \infty. \tag{2.46}$$

Subtracting (2.45) from the same relation written for $l+1$ we find that $\delta_l^{(n)}$ satisfies the linear equation

$$R_{l+1}^{(n)}\, \delta_{l+1}^{(n)} + R_l^{(n)}\, \delta_{l-1}^{(n)} + \left(g_1 + 2R_l^{(n)} + 2R_{l+1}^{(n)} \right) \delta_l^{(n)} = \frac{1}{n} \ .$$

Iterating this equation we prove (2.31).

Thus the sequence $\{r_n(x)\}_{n=1}^{\infty}$ defined in Theorem 2.2 is compact in $C[0,\, 1]$ and in view of (2.45) any convergent sequence is a positive solution of (2.45) in which

$$W = r^2 \left(g_1 + 3r^2 \right). \tag{2.47}$$

Since $g_1 \geq 0$, this equation has a unique positive solution. Theorem 2.3 is proved.

Remark. According to [19], in the general polynomial case

$$W(r) = \frac{1}{2\pi i} \cdot \oint V' \left(x + \frac{r^2}{z} \right) dz \tag{2.48}$$

It is easy to see that for the monomial specified by (2.27) the latter formula and general formula (2.32) imply (2.23)-(2.25) for $\alpha = 2p$.

2.2 Integral Identity Approach

We outline here another method to compute the DOS (1.23). This method was proposed in paper [15] and is applicable to polynomial potentials whose coefficients can be negative. This method is based on the identity

$$E\left\{\dot{\phi}_B(M) - n\phi(M)TrV'(M)B\right\} = 0 \tag{2.49}$$

where $\phi(M)$ is a differentiable scalar function of a matrix M,

$$\dot{\phi}_B(M) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\left[\phi(M+\varepsilon B) - \phi(M)\right] \ , \tag{2.50}$$

and B is an arbitrary Hermitian matrix. Identity (2.49) can be obtained by computing the derivative with respect to ε at $\varepsilon = 0$ of the ε-independent integral.

$$Z_\phi = \int e^{-nTrV(M+\varepsilon B)}\phi(M+\varepsilon B)dM \ . \tag{2.51}$$

Applying (2.49) to

$$\phi(M) = G_{jk}(z) \equiv \left[(M-z)^{-1}\right]_{jk} \tag{2.52}$$

and choosing properly B we find the relation

$$E\left\{g_n^2(z)\right\} + V'(z)E\left\{g_n(z)\right\} + Q_n(z) = 0 \tag{2.53}$$

where

$$g_n(z) = n^{-1}Tr(M-z)^{-1}$$

and

$$Q_n(z) = E\left\{\frac{1}{n}\,Tr\left[V'(M) - V'(z)\right](M-z)^{-1}\right\} \ . \tag{2.54}$$

$Q_n(z)$ is a polynomial of degree $2p-2$ where $2p$ is the degree of $V(z)$. It can be written as

$$Q_n(z) = \int \frac{V'(\lambda) - V'(z)}{\lambda - z}\ E\left\{N_n(d\lambda)\right\} \ . \tag{2.55}$$

Both formulas show that the coefficients of $Q_n(z)$ are linear combinations of quantities (moments)

$$m_l^{(n)} = \int \lambda^l E\left\{N_n(d\lambda)\right\} \tag{2.56}$$

for $l \leq p-2$.

By using identity (2.49) in which $\phi(M)$ is a properly chosen polynomial of M_{kj} we can prove that $m_l^{(n)}$ in (2.56) is bounded uniformly in n for $l \geq 2p$. Besides the general formula (2.11) imply that (cf(2.17))

$$E\left\{|g_n(z) - E\left\{g_n\right\}|^2\right\} \leq \frac{4}{n|Im\ z|^2}. \tag{2.57}$$

Therefore, standard compactness arguments imply that there exists a subsequence $f_{n_j}(z)$ of the sequence

$$f_n(z) \equiv E\left\{g_n(z)\right\} = \int_{\mathbb{R}} \frac{E\left\{N_n(d\lambda)\right\}}{\lambda - z} , \tag{2.58}$$

converging to a limit $f(z)$ uniformly on compact sets of

$$C_{\pm} = \{z : \pm Im\ z > 0\} , \tag{2.59}$$

and corresponding subsequence $E\left\{N_{n_j}(z)\right\}$ of measures converging weekly to a limiting measure $N(d\lambda)$, such that

$$f(z) = \int_{\mathbb{R}} \frac{N(d\lambda)}{\lambda - z} \quad Im\ z \neq 0 , \tag{2.60}$$

$$f^2(z) + V'(z)f(z) + Q(z) = 0 , \tag{2.61}$$

where

$$Q(z) = \int_{\mathbb{R}} \frac{V'(z) - V'(\lambda|}{z - \lambda}\ N(d\lambda) . \tag{2.62}$$

The quadratic equation (2.61) with $V(z) = \frac{z^2}{2}$ corresponding to the Gaussian case, i.e. to the semicircle law (2.21) for $N(d\lambda)$, was obtained for the first time in [20] as an equation determining the Stieltjes transform of the IDS of random matrices with independent or weakly dependent but not necessary Gaussian distributed matrix elements.

According to (2.62) in the GUE case $Q \equiv 1$ and equation (2.61) can be easily solved in the class (2.34). In general case $Q(z)$ is a nontrivial polynomial whose coefficients are functionals of the IDS $N(d\lambda)$ (like the Stieltjes transform $f(z)$) and all of them have to be found selfconsistently from equations (2.60)-(2.62).

The next theorem demonstrates some properties of a solution of these equations belonging to the class (2.34)-(2.35).

Theorem 2.4 *Let $V(\lambda)$ in (1.1)-(1.2) be an even polynomial of degree $2p$ with the real coefficients*

$$V(\lambda)=\sum_{l=1}^{p} g_l \frac{\lambda^{2l}}{2l}\ , \quad g_p=1 \tag{2.63}$$

and $f(z)$ be a solution of (2.60)-(2.62) satisfying (2.34)-(2.35). Then respective IDS $N(d\lambda)$ has the properties:
(i) $N(d\lambda)$ is absolutely continuous and its derivative $\rho(x)$ (DOS) admits the bound

$$\rho(\lambda)\le \pi^{-1}|Q(\lambda)|^{1/2}\ ; \tag{2.64}$$

(ii) $\rho(\lambda)$ is a Hölder continuous function;
(iii) supp $\rho(\lambda)$ is a bounded set of $\mathbb{R}$;
(iv) $\rho(\lambda)$ satisfies the following singular integral equation:

$$2 \fint_{\sigma} \frac{p(\lambda)d\lambda}{\lambda-\mu}=V'(\mu)\ , \quad \mu\in\sigma\ , \quad \sigma = supp\ \ \rho(\lambda) \tag{2.65}$$

Proof. Let us rewrite (2.61) as

$$f(z)=-\frac{Q(z)}{f(z)+V'(z)} \tag{2.66}$$

and denote $R(z)=Re\ f(z),\ I(z)=Im\ f(z)$. According to (2.36)

$$\rho(\lambda)=\pi^{-1}I(\lambda+i0)\ge 0\ , \quad \lambda\in \text{supp}\ \ \rho \tag{2.67}$$

if $I(\lambda)\equiv I(\lambda+i0)$ exists and is bounded. On the other hand, according to (2.66)

$$0\le I(z)\le |f(z)|\le \frac{|Q(z)|}{I(z)-|Im\ V'(z)|}.$$

Since $Im\ V'(\lambda+i0)=0$, this inequality and (2.67) imply (2.64). Furthermore, (2.66) yields for $z=\lambda+i0$

$$R=\frac{Q(R+V')}{|f+V'|^2}\ , \qquad I=\frac{IQ}{|f+V'|^2}\ . \tag{2.68}$$

The second relation implies that if $\lambda \in \text{supp } \rho$, then

$$1 = \frac{Q(\lambda)}{|f + V'|^2}. \tag{2.69}$$

It is easy to see that the converse statement is also true. Thus combining (2.68) and (2.69) we obtain for $\lambda \in \text{supp } \rho$

$$R(\lambda) = -\frac{V'(\lambda)}{2}\ , \tag{2.70}$$

$$I^2(\lambda) = Q(\lambda) - R^2(\lambda) = Q(\lambda) - \frac{V'^2(\lambda)}{2}\ . \tag{2.71}$$

Since $V'(\lambda)$ and $Q(\lambda)$ are polynomials of degrees $2p-1$ and $2p-2$ respectively, it follows from (2.71) that supp ρ cannot be unbounded and that $\rho(\lambda)$ is a Hölder-continuous function. This allows us to use the classical Sokhotskii-Plemelj formula [21] for the real part of the Cauchy integral and to present (2.70) in the form (2.65). Theorem 2.4 is proved.

Remark. Relation (2.65) can be viewed as a singular integral equation for the DOS $\rho(\lambda)$. This equation was considered for the first time by Wigner [2][3] for the Gaussian case (2.9). Its general form (2.65) was introduced in the important physical paper [22]. In both papers the authors used the "steepest descent" arguments whose rigorous form was recently given in [23] (see also Theorem 2.5) below.

In what follows we call supp ρ the spectrum. One of the important implications of the above formulas is that they illustrate a property of the spectrum which we have not seen before considering only potentials with nonnegative coefficients. Namely, if some of the coefficients are negative, then according to (2.62) $Q(\lambda)$ may be negative on some intervals and according to (2.69)-(2.71) these intervals cannot belong to the spectrum. In other words, the spectrum that corresponds to (2.63) with negative coefficients may consist of several intervals.

We demonstrate this property by using equation (2.65). Assume that in (2.63) $g_l \geq 0, \quad l = 1, \ldots p$. Then Theorem 2.3 and Theorem 2.4 imply that the spectrum is an interval $(-a, a), \quad a = 2r(1)$. According to [21], the singular integral equation (2.65) with $\sigma = (-a, a)$ has the unique bounded solution

$$\rho(\lambda) = \frac{1}{2\pi^2}\ {}_+\sqrt{a^2 - \lambda^2} \int_{-a}^{a} \frac{V'(\mu)}{\sqrt{a^2 - \mu^2}} \frac{d\mu}{\mu - \lambda} \tag{2.72}$$

provided that

$$\int_{-a}^{a} \frac{V'(\mu)d\mu}{\sqrt{a^2-\mu^2}} = 0 \ . \tag{2.73}$$

In our case $\rho(\lambda)$ has also to satisfy the normalization condition

$$\int_{\sigma} \rho(\lambda)d\lambda = 1. \tag{2.74}$$

Since $V(\lambda)$ defined in (2.63) is an even polynomial, condition (2.73) is trivial and condition (2.74) determines the endpoints $\pm a$ of the spectrum $(-a,\, a)$. It is easy to show that (2.72) and (2.73) are equivalent to (2.26), (2.32), and (2.42). Consider now the case (2.43). Here the polynomial $t_2(\lambda)$ in the representation (2.26) is

$$t_2(\lambda) = \frac{1}{2\pi}\left(\lambda^2 + g_1 + \frac{a^2}{2}\right) \ , \quad 3a^4 + 4g_1a^2 - 16 = 0 \ . \tag{2.75}$$

We see that $t_2(\lambda)$ is strictly positive for $|\lambda| \le a$ not only when $g_1 > 0$ but also for $g_1 \ge g_1^{(c)} = -2$. If $g_1 < g_1^{(c)}$ then our one-interval ansatz (2.72) is not correct and we have to try the two-interval ansatz, i.e. to assume that $\sigma = (-b,\, -a) \cup (a,\, b)\,, \quad 0 < a < b < \infty$. According to [21] in this case the solution of the singular integral equation exists provided that

$$\int_{\sigma} \frac{\mu^l V'(\mu)d\mu}{\sqrt{X(\mu)}} = 0 \tag{2.76}$$

where $l = 0,\, 1$ and $X(\lambda) = (b^2-\lambda^2)(\lambda^2-a^2)$. The solution has the form

$$\rho(\lambda) = \frac{1}{2\pi^2} + \sqrt{X(\lambda)} \int_{\sigma} \frac{V'(\mu)}{\sqrt{X(\mu)}}\, \frac{d\mu}{\mu-\lambda} \tag{2.77}$$

The solvability condition (2.76) for $l = 0$ is trivial because $V'(\mu)$ is an odd function and σ is a symmetric set. The second condition (2.76) and the normalization condition determines endpoints $\pm a$ and $\pm b$.

Thus we have demonstrated a possibility for the DOS to have the support consisting of several intervals provided that some coefficients of the potential are negative and their magnitude is large enough.

The two-interval formula (2.77) can also be obtained in the framework of the approach based on a proper extension of (2.32) and on (2.44). In this case, however, the latter approach is not unambiguous and requires additional rigorous analysis [1], [24]. We can also tune the coefficient of the potential to reach an "opposite" effect. Namely, it can be shown that if p in (2.63) is an odd number then there exists a potential for which

$$\rho_p(\lambda) = c_p \left(a^2 - \lambda^2\right)^{p-1/2} . \tag{2.78}$$

Respective potentials [1]

$$V_p(\lambda) = \sum_{l=1}^{p} (-l)^{l-1} \frac{(2p)!(l-1)!}{(2p-l)!(2l)!}\, t^{2l} \tag{2.79}$$

are known as critical ones. They play the important role in constructing the so-called "double-scaling limit" of the matrix models of QFT [1].

The results presented above were obtained from the integral equation (2.65). One of the advantages of this approach is that for polynomial potentials it provides a "closed form" of the DOS (cf. 2.26) and (2.77)

$$\rho(\lambda) = t_{2p-2}(\lambda) \,{}_{+}\sqrt{X_q(\lambda)}\ , \quad \lambda \in \sigma \tag{2.80}$$

where $X_q(\lambda) = \prod_{j=1}^{2q} |\lambda - a_j|$, and $a_j,\ j = 1, \ldots, 2q$ are the endpoints of the spectrum σ which consists of $q \leq p$ intervals. However, since the positivity of the solution is not incorporated in a natural way in the theory of singular integral equations, the procedure of the construction of (2.80) is partially heuristic and does not result in general in the unique solution. Indeed it is easy to check that if the potential is a polynomial of degree 6, then there exists the one-parameter family of solutions (2.80) satisfying the normalizing condition (2.74) and respective conditions having the form (2.76) with $l = 0, 1, 2$. Thus, in the general case we have to take into account the imaginary part of equation (2.61) as well (remember, that (2.65) is in fact the real part of this equation). These questions require an additional study. We refer the reader to the reviews [1, 24] for theoretical physics results and discussions.

2.3 Statistical Mechanics Approach

This approach was introduced in the RMT by Wigner and Dyson (see [2, 3]). It is based on the observation that the joint eigenvalue density (1.19) can be written as the canonical Gibbs distribution

$$p_n(\lambda_1, \ldots, \lambda_n) = Q_n^{-1} \exp\{-nH_n(\lambda_1, \ldots, \lambda_n)\} \ , \tag{2.81}$$

corresponding to a one-dimensional system of n "particles" with the Hamiltonian

$$H_n(\lambda_1, \ldots, \lambda_n) = \sum_{l=1}^{n} V(\lambda_l) - \frac{1}{n} \sum_{k \neq l} \log(\lambda_k - \lambda_l) \tag{2.82}$$

at the temperature of n^{-1}. The first term of the r.h.s. plays the role of the energy of particles due to the external field $V(\lambda)$ and the second one plays the role of the interaction (two-dimensional Coulomb) energy.

It is important that the Hamiltonian (2.82) contains explicitly the "number of particles" n. This allows us to regard (2.81) as an analogue of molecular field models of statistical mechanics. This analogy was used in a number of physical papers.

The rigorous treatment of the molecular field models of a rather general form was given by several authors (see e.g. [25]). In particular, papers [26, 27] contain the approach whose extension allows us to carry out a rigorous analysis of Hamiltonian (2.82). The result of this analysis is as follows.

Theorem 2.5 *Let an ensemble of random matrices be specified by (1.1)-(1.2) where the function $V(\lambda)$ satisfies the condition of Theorem 2.1 and in addition*

$$|V(\lambda_1) - V(\lambda_2)| \leq const\, |\lambda_1 - \lambda_2|^\gamma \ , \quad |\lambda_1|, |\lambda_2| \leq L \tag{2.83}$$

for some $\gamma > 0$ and $L \leq \infty$. Then the normalized counting measure (1.4) of this ensemble converges in probability to the nonrandom absolutely continuous IDS $N(\lambda)$ whose density is uniquely determined by the conditions

$$\rho(\lambda) \geq 0 \ , \tag{2.84}$$

$$\int \rho(\lambda) d\lambda = 1 \ , \tag{2.85}$$

$$-\int \ln|\lambda_1-\lambda'|\rho(\lambda_1)\rho(\lambda_2)d\lambda_1 d\lambda_2 < \infty \ , \tag{2.86}$$

the function

$$u(\lambda)=\int \ln|\lambda-\lambda'|\rho(\lambda')d\lambda' - V(\lambda) \tag{2.87}$$

is bounded from above, and

$$supp \ \ \rho(\lambda)\subset \left\{\lambda : u(\lambda)=\max_{\lambda'} u(\lambda')\right\} \ . \tag{2.88}$$

Remarks. *(i).* The analogues of Theorem 2.5 for the orthogonal invariant and symplectic ensembles (1.10) are also valid [23] if we introduce the factor β in front of the integral of the r.h.s. of (2.87). The analogues of formula (2.82) in these cases have the factor β in front of the double sums, i.e. β plays the role of a coupling constant of the respective n-particle system.

(ii). Equation (2.88) is just the zero temperature case of the selfconsistent equation for the particle density and well known in the molecular field theory. Indeed, we have mentioned before that the large parameter n plays different roles in formulas (2.81) and (2.82). In the former one n plays the role of the inverse temperature while in the latter one the factor n^{-1} allows us to treat it as a molecular field type Hamiltonian. Thus, if the factor n in (2.81) were replaced by the inverse temperature $(kT)^{-1}$, then the arguments which we used to prove Theorem 2.5 would lead to the standard molecular field equation for the particle density

$$\rho(\lambda)=\frac{\exp\left\{-(kT)^{-1}u(\lambda)\right\}}{\int \exp\left\{-(kT)^{-1}u(\mu)\right\}d\mu}$$

Now, if in this equation we perform the limiting transition $T\to 0$ we obtain (2.88).

Thus, from the statistical mechanics point of view, Theorem 2.5 asserts that the zero temperature case of the molecular field equation for our model can be obtained not only after subsequent limiting transitions $n\to\infty$ and then $T\to 0$, but also as a result of simultaneous limiting transitions $n\to\infty\,,\ T\to 0$, provided that the product nT is fixed.

(iii). By the method of Theorem 2.5 one can also show that the ground state energy of the statistical mechanics model, i.e. $E=\lim_{n\to\infty} n^{-2}\ln Q_n$

has the following form:

$$E = \frac{\beta}{2}\int \ln|\lambda-\lambda'|\rho(\lambda)\rho(\lambda')d\lambda d\lambda' + \int \rho(\lambda)V(\lambda)d\lambda \ , \qquad (2.89)$$

where $\rho(\lambda)$ is given by Theorem 2.5. Moreover, in total agreement with statistical mechanics, E can be obtained as the minimum value of the "electrostatic" energy

$$E = \min_{\nu}\left\{-\frac{\beta}{2}\int \ln|\lambda-\lambda'|\nu(d\lambda)\nu(\lambda') + \int V(\lambda)\nu(d\lambda)\right\} \qquad (2.90)$$

of two dimensional (line) charges whose distribution on the real line is described by the measure $\nu(\cdot)$, $\nu(R) = 1$. Then Theorem 2.5 implies that under its conditions a minimizing measure has the density $\rho(\lambda)$ satisfying (2.84)–(2.88). This density is the unique solution of the extremum equation of the variational problem

$$\beta \int_{\text{supp }\rho} \ln|\lambda-\lambda'|\rho(\lambda')d(\lambda') = V(\lambda) + \text{const} \ , \quad \lambda \in \text{supp } \rho \ . \qquad (2.91)$$

If we differentiate (2.91) with respect to λ, we obtain the singular integral equation (2.65). However, now it is valid for any function $V(\lambda)$ satisfying conditions of Theorem 2.5, while in Section 2.2 we proved this equation for polynomial V's. This equation has a simple electrostatic interpretation: it is just the equilibrium condition for the continuously distributed charges of strength $\beta^{1/2}$ subjected to the external electrostatic potential. *(iv)*. Repeating almost literally the arguments which where used to prove Theorem 2.5 we can also prove analogous result for a more general ensemble of random matrices with an unitary invariant density (cf.(1.10))

$$p_n(M) = Z_n^{-1}\exp\left[-nV_n(\lambda_1,\ldots,\lambda_n)\right] \ , \qquad (2.92)$$

where the function V_n is

$$V_n(\lambda_1.\ldots,\lambda_n) = \sum_{i=1}^{n} V(\lambda_i) + \sum_{k=2}^{n}\frac{1}{k!n^{k-2}}\sum_{i_1\neq i_2\neq\ldots\neq i_k} V^{(k)}\left(\lambda_{i_1},\ldots,\lambda_{i_k}\right) \qquad (2.93)$$

with bounded, symmetric, and Hölder continuous functions $V^{(k)}$, $k = 2, 3, \ldots$ satisfying the following condition. The functional

$$U(c) = \sum_{k=2}^{n}\frac{1}{k!}\int V^{(k)}(\lambda_1,\ldots,\lambda_{2k})\prod_{i=1}^{k} c(\lambda_i)d\lambda_i \qquad (2.94)$$

is convex in the space of smooth functions with compact supports.

Theorem 2.6 *Let the ensemble of random matrices be specified by (2.92) and (2.93) in which a real-valued function $V(\lambda)$ satisfies the condition (2.83). Then the normalized eigenvalue counting measure corresponding to this ensemble converges in probability to the nonrandom absolutely continuous IDS whose density is uniquely determined by conditions (2.84), (2.86), and (2.88) where now $u(\lambda)$ is*

$$u(\lambda) = \int \ln|\lambda - \lambda'|\rho(\lambda')d\lambda' - V(\lambda) \\ - \sum_{k=2}^{\infty} \frac{1}{k!} V_k(\lambda, \lambda_1, \ldots, \lambda_k) \prod_{i=1}^{k} \rho(\lambda_i)d\lambda_i$$

and as before has to be bounded from above.

We mention here two examples where (2.94) is convex. The first one corresponds to $V^{(k)} = 0,\ k \geq 3$, and $V^{(2)}(\lambda_1, \lambda_2)$ defining a positive operator in the space $L_2(-l, l)$, where l is large enough. In particular, if $F(\lambda) \in L_1(R)$ has a nonnegative Fourier transform, then we take $V^{(2)}(\lambda_1, \lambda_2) = F(\lambda_1 - \lambda_2)$. In the second example we take the sequence $\left\{V^{(k)}\right\}_{k=2}^{\infty}$ to be a sequence of moments of some random process $\xi(\lambda),\ \lambda \in R :\ V^{(k)}(\lambda_1, \ldots, \lambda_k) = M\left\{\xi(\lambda_1), \ldots, \xi(\lambda_k)\right\},\ k = 2, 3, \ldots$ where the symbol $M\{\ldots\}$ denotes the mathematical expectation with respect to this process. We assume that the generating functional

$$M\left\{\exp\left\{\int \xi(\lambda)c(\lambda)d\lambda\right\}\right\}$$

exists for any smooth function $c(\lambda)$ with compact support.

The proof, discussion, and applications of Theorem 2.5 and Theorem 2.6 are given in [23]. In particular, Theorem 2.5 allows us to establish the unicity of several-interval ansatzes for the solutio ns of (2.65) with polynomial V's which we discussed in Section 2.2 and extend the validity of (2.23) to the case $\alpha \in [0.1]$, which can not be treated by approaches discussed in the previous sections. In the latter case $\rho_\alpha(\lambda)$ is unbounded at zero.

2.4 Edge Behavior of the DOS

We start again from the GUE specified by (1.19) and (1.9). By using (2.61) and the Plancherel-Rotah asymptotic formula for the Hermite polynomials $H_l(\xi)\,,\ \xi = \sqrt{2l+1}\cdot \mathrm{ch}\theta\,,\ 0<\varepsilon<\theta<\infty$, which differs from (2.20) by the replacement sin with sh, we can derive the bound

$$\rho_n(\lambda) \le C_1 n^{-1} \exp\left\{-C_2(\varepsilon)\cdot n\,(|\lambda|-2)^2\right\}\ . \tag{2.95}$$

This bound implies that

$$\lim_{n\to\infty} P\left\{\|M\| \ge 2+\varepsilon\right\} = 0\ , \quad \varepsilon = 0 \tag{2.96}$$

i.e. that maximum and minimum eigenvalues of the GUE matrices converge in probability to the edges ± 2 of the DOS (2.21) support. Indeed, the probability in the l.h.s. of (2.96) is equal to $Pr\left\{nN_n\left(I_\varepsilon\right)\ge 1\right\}$, where $I_\varepsilon = \mathbb{R}\setminus(-2-\varepsilon,\,2+\varepsilon)$. The latter probability in view of the Chebyshev inequality does not exceed $n\int\limits_{I_\varepsilon}\rho_n(\lambda)d(\lambda)$, or in view of (2.95)

$$Pr\left\{\|Mn\| \ge 2+\varepsilon\right\} \le C_3(\varepsilon) e^{-C_2(\varepsilon)\varepsilon^2 n}\ . \tag{2.97}$$

This bound proves (2.96). Moreover, if we consider the infinite family $M_{jk},\ j,k = 1,\ldots,\infty$, of the independent Gaussian variables defined on the same probability space and such that

$$E\left\{M_{jk}\right\} = 0\ , \quad E\left\{M_{jk}M_{lm}\right\} = \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}$$

then by using the Borel-Cantelli lemma and (2.97) we obtain that the number of eigenvalues of random matrix $M_n = \{M_jk\}_{j,\,k=1}^{n}$ lying outside of the interval $|\lambda|>2$ is bounded with probability 1 as $n\to\infty$.

Analogous results can be obtained for the ensembles specified by (2.22) by using asymptotic formulas proven in [13, 14].

For the more general form of $V(\lambda)$ we only know that the probability in the l.h.s. of (2.97) is exponentially small in n if ε is large enough [23]. Having established that $\rho_n(\lambda)$ converges to a nonzero limit (2.21) for $|\lambda|<2$ and converges exponentially fast to zero for $|\lambda|>2$ it is natural to study the "crossover" of these two asymptotic behaviors, i.e. to find a variable

$$t = c_\nu(\lambda+2)n^\nu \tag{2.98}$$

and a function $A(t)$ such that

$$\lim_{n\to\infty} \rho_n \left(2 + (c_\nu n^\nu)^{-1} t\right) = A(t) \tag{2.99}$$

and $A(t) = (2\pi)^{-1}\sqrt{2|t|}\,(1 + o(1))$, $t \to -\infty$ and $A(t)$ vanishes exponentially fast as $t \to \infty$. In other words the asymptotic behavior of the crossover function $A(t)$ as $t \to \pm\infty$ has to "match" the asymptotic form of $\rho_n(\lambda)$ in small (on the "initial" scale λ) left and right neighborhoods of the DOS support endpoint $\lambda = 2$ (and the same for $\lambda = -2$ by symmetry).

This asymptotic study for the GUE can be carried out by using the "turning point" asymptotic formula for the Hermite polynomials [10]

$$e^{-\frac{\xi^2}{2}} H_l(\xi) = \pi^{1/4}\, 2^{\frac{l+2}{4}} (h!)^{1/2}\, l^{-1/2} \left[Ai\left(-t3^{-1/3}\right) + O\left(l^{-2/3}\right)\right] \tag{2.100}$$

where

$$\xi = (2l+1)^{1/2} + 2^{-1/2}\, 3^{-1/3}\, l^{-1/6}\, t \tag{2.101}$$

and $Ai(t)$ is the Airy function. It solves the Schrodinger equation

$$-\,u''(y) + yu(y) = 0\ , \tag{2.102}$$

oscillates as $y \to -\infty$ and decays exponentially as $y \to +\infty$. Relations (2.100), (2.101), and (2.13) yield for (2.98) and (2.99) [28]:

$$\nu = 2/3\ , \quad A(x) = -X\left[Ai(X)\right]^2 + \left[Ai'(X)\right]^2\ , \quad X = -t^{-1/3} \tag{2.103}$$

and a certain value for c_ν. The latter formula and asymptotics for the Airy function [10] imply that

$$A(X) = \begin{cases} \dfrac{\sqrt{|X|}}{\pi} - \dfrac{\cos\left(4|X|^{3/2}/3\right)}{4\pi|X|} + O\left(|X|^{-5/2}\right)\ , & X \to -\infty\ ; \\ \dfrac{17}{96\pi}\, X^{-1/2} \exp\left(-4X^{3/2}/3\right)(1 + o(1))\ , & X \to \infty\ . \end{cases} \tag{2.104}$$

According to the physical paper [29] the similar behavior of $\rho_n(\lambda)$ is the case for an arbitrary polynomial potential provided that $\rho_n(\lambda)$ has the square root behavior near a given spectrum edge. The recent progress in the so-called "double scaling limit" of two-dimensional quantum gravity in the frameworks of the matrix models (see review [1]) suggests the form of analogues of (2.102) and (2.103) in the case of critical potentials (2.79) and the DOS (2.78). Set $a = 2$ in (2.78). Then the exponent that determines the crossover neighborhood of the endpoint $\lambda = 2$ is

$$\nu = \frac{2}{2p+1} \tag{2.105}$$

and the crossover functions $u_p(y)$ that plays the role of the Airy function (i.e. $Ai(y) = u_1(y)$) is the solution of the Schrodinger equation (cf. (2.102))

$$-u''(y) + q_p(y)u(y) = 0 \tag{2.106}$$

which is bounded as $y \to +\infty$. The "potential" $q_p(t)$ in (2.106) is the solution of the equation

$$H_p\left[q_p\right] = y \tag{2.107}$$

where $H_p\left[q\right]$ is a certain polynomial of degree p with respect to q and its derivatives of the order at most p. In particular

$$H_1 = x \ , \quad H_2 = +\frac{1}{3}\, q'' + q^2 \ , \quad H_3 = \frac{1}{10}\, q'' + \frac{1}{5}\, q'^2 + qq'' + q^3.$$

The polynomials $H_p[q]$ appear in a number of areas. In the spectral theory t hey are coefficients in the semi-classical asymptotic expansion of the Schrodinger operator resolvent

$$\left(-\frac{d^2}{dy^2} + q(y) - z\right)^{-1}(x, x) = \sum_{p=0}^{\infty} \frac{H_p[q]}{(-2z)^{p+1/2}} \ , \quad |z| \to \infty \ ,$$

in the theory of integrable systems they are densities of the infinite system of conservation laws for the Korteweg-de-Vries equation and are the Hamiltonians of the infinite hierarchy of all higher Korteveg-de-Vries equations written in the Hamiltonian form

$$\frac{\partial v}{\partial \tau} = (-1)^p \frac{\partial}{\partial y}\, \frac{\delta Hp[v]}{\delta v(y)} \ .$$

If we introduce the sequence $R_p[q]$ as

$$H_p[q] = (-1)^p \frac{2^{p+1} p!}{(2p-1)!!}\, R_p[q],$$

then $R_p[q]$ are the Gelfand-Dikii polynomials which can be found from the recurrent relation

$$R'_{l+1} = \frac{1}{4}\, R'''_l - qR'_l - \frac{1}{2}\, q'R_l\ , \quad R_o = \frac{1}{2}\ .$$

A heuristic explanation of the origin of above formulas, (2.106) in particular, is as follows. According to (2.32) the edge behavior of the DOS is due to the behavior of $r(x)$ near $x = 1$ or $r_l^{(n)}$ for $l = O(n)$. Thus, in addition to (2.99) we have to set in (2.28) and (2.44)

$$y = n^{\nu}\left(1 - \frac{l}{n}\right)\ , \quad R_l^{(n)} = 1 + n^{-\nu} q_p(y)\ , \quad [r_l^{(n)}]^2 = R_l^{(n)}.$$

(recall that $r(1) = 1$), and $P_n^{(l)}(\lambda) = u_p(y, t)$. Then in the limit $n \to \infty$, (2.28) reduces to Schrodinger equation (2.106), and (2.44) reduces to the equation (2.107) if $\nu = \frac{2p}{2p+1}$. The formal derivation of the (2.106) and (2.44) directly from (2.28) and (2.50) is rather tedious. Elegant and efficient schemes of formal derivation of the above relations are proposed in [19, 30].

This study of the behavior of the DOS in the neighborhood of endpoints of its support reveals important and interesting links of the RMT with the theory of integrable systems. In the next section we discuss more links of similar nature which appear in studies of other characteristics of the random matrix eigenvalue statistics.

3 Spacing Distribution

3.1 Generalities and the GUE

We discuss now some results concerning the asymptotic behavior of probability (1.8) as $n \to \infty$. This quantity is important because it determines the probability distribution of distances between the nearest neighbor eigenvalues of random matrices. Indeed, if $\lambda_1, \ldots, \lambda_n$ is a set of random variables whose joint distribution has a symmetric density $p_n(\lambda_1, \ldots, \lambda_n)$ and

$$\rho_n(\lambda) \equiv p_n^{(1)}(\lambda) = \int_{\mathbb{R}^{n-1}} p_n(\lambda, \lambda_2, \ldots \lambda_n) d\lambda_2, \ldots, d\lambda_n,$$

then for $\Delta = (a,\, b)\,, \quad \overline{\Delta} = \mathbb{R} \setminus \Delta$

$$\rho_n^{-1}(a) \int\limits_{\overline{\Delta}^{n-1}} p_n(a, \lambda_2, \ldots, \lambda_n) d\lambda_2, \ldots, d\lambda_n \tag{3.1}$$

is the conditional probability of the event $\{\lambda_i \overline{\in} \Delta, \quad i = 1, \ldots, n\}$, provided that at the left endpoint a of Δ there is an eigenvalue. Since the probability (1.8) is

$$R_n(\Delta) = \int\limits_{\overline{\Delta}^n} p_n(\lambda_1, \ldots, \lambda_n) d\lambda, \tag{3.2}$$

then the probability density $p_a^{(n)}(b-a)$ of the conditional probability distribution (3.1) is

$$p_a^{(n)}(b-a) = -\frac{1}{n\rho_n(a)} \frac{\partial^2}{\partial a \partial b} R_n((a,\, b)). \tag{3.3}$$

Formulas (3.1)–(3.3) are valid for an arbitrary set of symmetrically distributed random values $\lambda_1, \ldots, \lambda_n$. The simplest case corresponds to independent identically distributed (i.i.d.) λ_i's with the common density $\rho(\lambda)$. Then obviously

$$\rho_n(\lambda) = \rho(\lambda)\ , \quad R_n(\Delta) = \left(1 - \int\limits_{\Delta} p(\lambda) d\lambda\right)^n . \tag{3.4}$$

Thus, if Δ is a n-independent interval, $\lim\limits_{n\to\infty} R_n(\Delta) = 0$. This simple fact is of general nature because if the relations analogous to (1.23) and (2.17) are valid (this is obvious in our case of i.i.d. λ's), then with probability close to 1, Δ contains $nN(\Delta) \sim n\Delta\rho(a)$ eigenvalues as $n \to \infty$. Thus to obtain nontrivial asymptotic behaviors of $R_n(\Delta)$ as $n \to \infty$ we have to consider intervals whose length is of the order of n^{-1}, i.e. of the "typical" distance $[n\rho(\lambda)]^{-1}$ between eigenvalues in the neighborhood of a given $\lambda \in \operatorname{supp}\ \rho$. That makes natural introducing the following variables

$$a = \lambda\ , \quad b = \lambda + \frac{s}{n\rho_n(\lambda)} \tag{3.5}$$

where s is the "scaling" variable, measuring the length of our interval (window) relative to the typical eigenvalue spacing in the $O\left(\frac{1}{n}\right)$-neighborhood of a given spectral point λ. Combining (3.4) and (3.5) we find that

$$r_\lambda(s) = e^{-s} , \quad p_\lambda(s) = e^{-s} \tag{3.6}$$

where

$$r_\lambda(s) = \lim_{n\to\infty} R_n \left(\left(\lambda,\ \lambda + \frac{s}{n\rho_n(\lambda)} \right) \right) \tag{3.7}$$

is the scaling limit of probability (1.8) and

$$p_\lambda(s) = \lim_{n\to\infty} \frac{\partial^2}{\partial\lambda\partial s}\ R_n \left(\left(\lambda,\ \lambda + \frac{s}{n\rho_n(\lambda)} \right) \right) \tag{3.8}$$

is the same limit for the spacing probability density, both expressed via the scaled spacing (3.5).

We obtained the Poisson distribution for spacings. Notice that the r.h.s. of (3.6) does not depend on λ and $\rho(\lambda)$ provided that we use the scaled spacing (3.5). This simple result can be interpreted as the limiting form of the eigenvalue spacing distribution for diagonal random matrices. According to [31] the same Poisson distribution is valid in the much less trivial case of the discrete and continuous Schrodinger operator with a random potential. The former operator is just a three-diagonal matrix with i.i.d. entries on the principal diagonal and unities of the two nearest diagonals. Moreover, in all these cases we have more general than (3.6) limiting relations:

$$r(\delta; k) = \prod_{j=1}^{l} \frac{s_j^{k_j}}{k_j!}\ e^{-s_j} \tag{3.9}$$

where for any integer $l \geq 1$,

$$\delta = \bigcup_{j=1}^{l} \delta_j , \quad \delta_j = (\alpha_j, \beta_j) , \quad j = 1, \ldots, l \tag{3.10}$$

is an ordered set of disjoint intervals such that

$$a_j = \lambda + \frac{\alpha_j}{n\rho_n(\lambda)} , \quad b_j = \lambda + \frac{\beta_j}{n\rho_n(\lambda)} , \tag{3.11}$$

$k = \{k_j\}_{j=1}^{l}$ where k_j are integer numbers, $s_j = \beta_j - \alpha_j$, and

$$r(\delta; k) = \lim_{n\to\infty} P \left\{ nN_n(\Delta_1) = k_1, \ldots, nN_n(\Delta_l) = k_l \right\} . \tag{3.12}$$

In other words, in all these cases the random point process (the random point measure)

$$\nu_\lambda^{(n)}(t) = nN_n\left(\left(\lambda, \lambda + \frac{t}{n\rho_n(\lambda)}\right)\right) , \quad t \geq 0 \tag{3.13}$$

converges weakly (i.e. in the sense of convergence of all finite-dimensional distributions) to the λ-independent Poisson process. This allows us to introduce the Poisson universality class of the spacing distributions which includes diagonal random matrices with arbitrary i.i.d. random entries having continuous probability density and three-diagonal (Jacobi) random matrices with analogous entries on the principal diagonal.

Let us consider now the unitary invariant ensembles (1.1)-(1.2). Denote by $\chi_\Delta(\lambda)$ the indicator of an interval $\Delta = (a, b)$. Then, according to (1.8) and (2.11)

$$R_n(\Delta) = E\left\{\prod_{j=1}^n (1 - \chi_\Delta(\lambda_j))\right\} = 1 + \sum_{l=1}^n \frac{(-1)^l}{l!} \ .$$

$$\int_\Delta \det \|k_n(\lambda_j,\, \lambda_k)\|_{j,\,k=1}^l \quad d\lambda_1, \ldots, d\lambda_l \ .$$

The r.h.s. of this relation is the Fredholm determinant

$$R_n(\Delta) = \det(1 - k_n) \tag{3.14}$$

of the integral operator defined on the interval Δ by degenerate (rank n) kernel $k_n(\lambda,\, \mu)$ specified by (2.7):

$$((1 - k_n)f)(\lambda) = f(\lambda) - \int_\Delta k_n(\lambda,\, \mu) f(\mu) d\mu \ , \quad \mu \in \Delta. \tag{3.15}$$

Taking into account (2.7) and (3.5) we have to rewrite (3.15) in the form

$$\left((1 - Q_\lambda^{(n)})f\right)(\xi) = f(\xi) - \int_0^s Q_\lambda^{(n)}(\xi,\, \eta) d\eta \ , \quad 0 \leq \xi \leq s$$

where

$$Q_\lambda^{(n)}(\xi,\, \eta) = \frac{1}{n\rho_n(\lambda)} \sum_{l=0}^{n-1} \psi_l^{(n)}\left(\lambda + \frac{\xi}{n\rho_n(\lambda)}\right) \psi_l^{(n)}\left(\lambda + \frac{\eta}{n\rho_n(\lambda)}\right) \tag{3.16}$$

Thus, as in the case of the DOS (see (2.14)), we reduced the problem of the asymptotic study of $R_n(\Delta)$ to a certain asymptotical problem for orthogonal polynomials (2.1)-(2.3). The simplest case is again the GUE specified by (1.9). In this case we can use the Plancherel-Rotah asymptotic formula (2.19) for the Hermite polynomials and obtain that for $\lambda \in (-2, 2)$ [3]

$$\lim_{n\to\infty} Q_\lambda^{(n)}(\xi, \eta) = \frac{\sin \pi(\xi - \eta)}{\pi(\xi - \eta} \equiv S(\xi - \eta) \ . \tag{3.17}$$

By using this formula it is easy to prove that the determinant in the r.h.s. of (3.14) converges to the Fredholm determinant of the integral operator Q_s defined by kernel (3.17) on the interval $(0, s)$. Thus for the GUE the quantity (3.7) is

$$r(s) = \det(1 - Q_s). \tag{3.18}$$

Numerous properties of this determinant are presented in [3]. We mention here the small-s asymptotics of $r(s)$

$$r(s) = 1 - s + \frac{\pi^2 s^4}{36} + O(s^5) \ , \quad s \downarrow 0 \tag{3.19}$$

which implies the small-s asymptotics of (3.8) for the GUE:

$$p(s) = \frac{\pi^2}{3} \ s^2 + O(s^3) \ , \quad s \downarrow 0 \ . \tag{3.20}$$

This asymptotic behavior of the spacing probability density is a manifestation of the property known as the repulsion of eigenvalues of random matrices, i.e. strong short-range correlation between eigenvalues forbidding them to be arbitrarily close. The asymptotic behavior $O(s^2)$ of the spacing probability density for any finite $n \geq 2$ is evident from (1.15) and (1.19). It is important however that this behavior remains true after the scaling limit (3.8).

In recent years numerous links of the Fredholm determinant (3.17), (3.18) with exactly solvable models of statistical mechanisms, completely integrable systems having the Painlevé property, the Painlevé transcendents or, more generally, with the deformation equations of a compatible set of nonautonomous Hamiltonian systems have been found and studied (see [3],[32]-[34] and references therein).

We conclude this section presenting additional probabilistic aspects of results given above.

1). Consider marginal distributions (2.11) and set

$$\lambda_j = \lambda + \frac{\xi_i}{n\rho_n(\lambda)}. \tag{3.21}$$

Then, according to (2.11), (2.14), (3.16), and (3.17)

$$\lim_{n\to\infty} (n\rho_n(\lambda))^{-l}\, p_l^{(n)}(\lambda, \ldots, \lambda_l) = \det \|S(\xi_i - \xi_j)\|_{i,j=1}^{l} \, , \tag{3.22}$$

This formula is one more manifestation of strong correlations between eigenvalues in the scaling limit (3.21). In particular, the simplest correlation function

$$\kappa_2^{(n)}(\lambda_1, \lambda_2) = p_2^{(n)}(\lambda_1, \lambda_2)\left[p_1^{(n)}(\lambda_1) p_1^{(n)}(\lambda_2)\right]^{-1} - 1 \tag{3.23}$$

has the following form in this limit

$$\begin{aligned} \kappa_2(\xi_1, \xi_2) &= \lim_{n\to\infty} \kappa_2^{(n)}\left(\lambda + \frac{\xi_1}{n\rho_n(\lambda)}\, , \quad \lambda + \frac{\xi_2}{n\rho_n(\lambda)}\right) \\ &= -\left(\frac{\sin \pi(\xi_1 - \xi_2)}{\pi(\xi_1 - \xi_2)}\right)^2 . \end{aligned} \tag{3.24}$$

Thus, the eigenvalues of the GUE are strongly correlated not only on short distances (see repulsion property (3.20)), but also on long distances, because the correlation function (3.24) decays as an inverse square of the distance modulo oscillating numerator. By applying to (3.24) the smoothing procedure $\delta^{-1} \int_s^{s+\delta} \ldots d(\xi_1 - \xi_2)$ where $s \ll \delta \ll \pi^{-1}$ we find that the smoothed correlation function for large s is

$$\rho^2(\lambda)\kappa_2^{\text{smooth}}(s) = -\frac{1}{2\pi^2 s^2}\, , \quad s \to \infty \, . \tag{3.25}$$

2). Regard the set of random eigenvalues $\lambda_1, \ldots, \ldots \lambda_n$ as a finite-dimensional point process, i.e, random counting measure

$$\nu_n(\Delta) \equiv n N_n(\Delta) = \sharp \{\lambda_i \in \Delta\} \, . \tag{3.26}$$

We can define this process either by the system of its marginal distributions (2.10) or by the generating functional

$$\Phi_n[\varphi] = E\left\{\exp\left[\int \varphi(\lambda)\nu_n(d\lambda)\right]\right\} \ , \tag{3.27}$$

defined on a suitable space of test functions $\varphi(\lambda)$, $\lambda \in \mathbb{R}$. We use the simplest case of bounded piece-wise functions with a compact support. Then the arguments similar to those used in the derivation of (3.14) yield

$$\Phi_n[\varphi] = \det(1 - k_n[\varphi]) \tag{3.28}$$

where $k_n[\varphi]$ is the integral operator defined on the support of φ by the kernel

$$k_n(\lambda,\, \mu)\, \left(1 - e^{\varphi(\mu)}\right) \ . \tag{3.29}$$

The scaling limit of all marginal densities is given by (3.22). To find the same limit for the generating functional we have to use again the scaling variables (3.21) allowing us to study neighborhoods of a given λ (windows), that contains finitely many eigenvalues as $n \to \infty$.

Besides we have to replace a test function $\varphi(\lambda)$ by $\varphi_n(\xi) = \varphi\left(\dfrac{\xi}{n\rho_n(\lambda)}\right)$. Then in the limit $n \to \infty$ we obtain (cf.(3.18))

$$\lim_{n\to\infty} \Phi_n\left\{\varphi_n\right\} = \det\ (1 - Q_\varphi) \tag{3.30}$$

where Q_φ is the integral operator defined on the support σ_φ of $\varphi(\xi)$ by the formula

$$\left(Q_\varphi f\right)(\xi) = \int\limits_{\sigma_\varphi} S(\xi - \eta)\left(1 - e^{\varphi(\eta)}\right) f(\eta) d\eta, \quad \xi \in \sigma_\varphi. \tag{3.31}$$

These formulae contain in fact the same information as (3.22), saying that in the case of the GUE the point process (3.13) converges weakly as $n \to \infty$ to the random process defined by (3.30) and (3.31) or by (3.22).

Comparing (3.18) and (3.30) we see that if $\varphi(\xi)$ is the indicator $\chi_\delta(\xi)$ of the interval $\delta = [0, s]$ then

$$Q_\varphi = (1 - e)Q_s. \tag{3.32}$$

This relation makes natural introducing the parameter τ in front of the integral operator Q_s (3.18). Then we can rewrite (3.32) as

$$\det\ (1 - Q_{\chi_\delta}) = \det\ (1 - \tau Q_s)\,|_{\tau=1}. \tag{3.33}$$

Consider now the probability

$$R_n\left(\{\Delta_j\}_{j=1}^l\right) = Pr\left\{nN_n(\Delta_j) = 0,\quad j = 1, \ldots, l\right\} \tag{3.34}$$

that an ordered set of disjoint intervals (3.11) does not contain eigenvalues. Then in the limit $n = \infty$ (3.34) is

$$r(\delta) = \det\ (1 - Q_\delta) \tag{3.35}$$

where Q_δ is the integral operator defined by the kernel $S(\xi - \eta)$ on the union (3.10).

We can also introduce more general integral operator with the kernel

$$\sum_{j=1}^{l} \tau_k \chi_{\delta_k}(\xi) S(\xi - \eta) \chi_{\delta_k}(\eta) \tag{3.36}$$

which is a natural analogue of the operator τQ_s for the case of several intervals. Then the family of probabilities for all δ's and k's (3.12), that also defines the limit point process for (3.13) is [32]

$$r(\delta; k) = \frac{(-1)^k}{k_1! \ldots k_l!} \frac{\partial^k r(\delta; \tau)}{\partial \tau_1^{k_1} \ldots \partial \tau_l^{k_l}} \Bigg|_{\tau_1 = \ldots = \tau_l = 1}$$

where $k = k_1 + \ldots + k_l$ and $r(\delta; \tau)$ is the Fredholm determinant of the kernel (3.36).

The functional (3.30) resembles in parts the generating functional of the Poisson point process and the generating functional of the square of the Gaussian stationary process, whose correlation function is $S(\xi)$. Indeed, if we formally replace the kernel $S(\xi - \eta)$ in (3.30) by $c\delta(x - y)$, $\ c > 0$, then we obtain the expression

$$\exp\left[c \int\limits_{\sigma_\varphi} \left(1 - e^{\varphi(\xi)}\right) d\xi\right] \tag{3.37}$$

coinciding with the generating functional of the Poisson point process with density c. If we replace in (3.31) the factor $\left(1-e^{\varphi(\eta)}\right)$ by $\varphi(\xi)$ itself, we obtain the Fredholm determinant of the integral operator defined by the kernel $S(\xi-\eta)\varphi(\eta)$ on the support of $\varphi(\xi)$. This Fredholm determinant is the generating functional of the random process $g^2(\xi)$, where $g(\xi)$ is the Gaussian stationary process with zero mean and the correlation function $S(\xi)$.

We can also consider another asymptotic regime, making "windows" in $O(1/n)$ – neighbourhoods of different spectral points, i.e. considering joint probability distribution of the counting functions $\nu_{\lambda_1}^{(n)}(t_1),\dots,\nu_{\lambda_k}^{(n)}(t_k)$ for distinct and n – independent $\lambda_1,\dots,\lambda_k$. Take for simplicity $k=2$. Then we have to consider generating functional (3.27) on functions

$$\varphi(\mu)=\varphi_1\left(n\rho_n(\lambda_1)(\mu-\lambda_1)\right)+\varphi_2\left(n\rho_n(\lambda_2)(\mu-\lambda_2)\right). \tag{3.38}$$

Inserting this $\varphi(\mu)$ in (3.27) and using asymptotic formula (2.18) we obtain that

$$\lim_{n\to\infty}\Phi_n[\varphi]=\Phi[\varphi_1]\Phi[\varphi_2], \tag{3.39}$$

where $\Phi[\varphi]$ is defined by (3.30). We see that in the scaling limit the "local" statistics of eigenvalues lying in $O(1/n)$ – neighbourhoods of district spectral point are independent.

3.2 Other ensembles and the universality conjecture

Relations (3.7) and (3.8) that determine the eigenvalue spacing distribution are valid for all unitary invariant ensembles of form (1.1) – (1.2). They allow us to find the limiting form (3.7), (3.8) of this distribution provided that a respective asymptotic formula for the orthogonal polynomials (2.2) is known.

Consider first the "toy" model corresponding to the Chebyshev polynomials

$$T_l(\lambda)=\sqrt{\frac{2}{\pi}}\cos l\Theta, \qquad \lambda=\cos\Theta, \tag{3.40}$$

and associated system of functions orthogonal on the interval $[-1,1]$

$$\psi_l(\lambda)=w^{1/2}(\lambda)T_l(\lambda) \tag{3.41}$$

which corresponds to the weight

$$w(x) = (1 - \lambda^2)^{-1/2}, \qquad |\lambda| < 1.$$

By using (3.7) and (3.8) we can easily compute in this case the DOS (cf. (2.21))

$$\rho(\lambda) = \pi^{-1}(1 - \lambda^2)^{-1/2}, \qquad |\lambda| \leq 1 \tag{3.42}$$

and the limiting kernel for (2.16) provided that the "reference" point λ is not an endpoint of the spectrum $|\lambda| \leq 1$. We find that the limiting kernel for the Chebyshev polynomials coincides with respective kernel (3.17) for the Hermite polynomials, i.e. for the GUE. We conclude that for the "Chebyshev" ensemble the DOS (3.42) differs from the semicircle DOS (2.21) of the GUE, while the probability distribution of spacings is the same. This result is, in fact, just another form of the well known Dyson's result, who introduced and analyzed in great details the class of random matrix ensembles known as the circular ensembles (see book [3] for results and references). The simplest case of these ensembles is the circular unitary ensemble (CUE) consisting of unitary matrices whose distribution is unitary invariant. Therefore respective probability distribution coincides with the Haar measure on $U(n)$ and respective polynomials are orthogonal on the unit circle with the unit weight. It is well known that the latter polynomials are closely related to the Chebyshev ones [10].

The Chebyshev polynomials are simplest polynomials orthogonal on the finite interval. Consider now more general case of polynomials which are orthogonal on the finite interval (say $[-1, 1]$) with respect to some n – independent weight $w(\lambda)$, $|\lambda| \leq 1$. Here we can use the classical asymptotic formula by Bernstein- Szego- Akhiezer [10] which is valid under rather weak conditions on the weight $w(\lambda)$ (the Lipshitz- Dini continuity, finitely many zeroes and integrable singularities). The leading term of this asymptotic formula coincide with the r.h.s. of (3.40) in which $l\Theta$ is replaces by $l\Theta + \gamma(\Theta)$, where the phase shift $\gamma(\Theta)$ (the "scattering" phase) is uniquely determined by the weight function $w(\lambda)$. This results in the same formulae (3.42) and (3.17) for the DOS and the spacing distribution. Thus we can introduce one more (in addition to the Poisson one) universality class of random matrix ensembles having the same probability spacing distribution (3.8), (3.18). This class includes Hermitian matrices distributed according to the Gaussian law (1.1), (1.9), ensembles defined by potential (2.22) for $\alpha \geq 1$ (in the latter case for $\alpha \neq 2, 4, 6$ (3.18) is proven only for $\lambda = 0$ [35]), the ensemble of positive defined Hermitian matrices related to Laguerre polynomials [36, 28] which is important in the solid state theory [7], and all ensembles related to orthogonal polynomials whose weight has a compact support. This class includes also the Dyson circular ensemble of unitary matrices and all similar ensembles related to polynomials that are orthogonal on the unit circle with a n – independent weight satisfying rather weak smoothness conditions (the latter fact can be easily checked by using respective asymptotic formulae from [10]). This class is known as the Wigner-Dyson universality class.

We conclude this section by the discussion of the universality conjecture of the random matrix theory [3].

Given an ensemble of random $n \times n$ matrices whose eigenvalue set is $\Lambda_n = (\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ let us consider some probabilistic characteristic A_n of the set Λ_n and assume that A_n depends on k spectral parameters $\lambda_1, \ldots, \lambda_k$:

$$A_n = A_n(\lambda_1, \ldots, \lambda_k). \tag{3.43}$$

We say that (3.43) is in the global regime as $n \to \infty$ (is a global quantity) if k is an n – independent number and parameters $\lambda_1, \ldots, \lambda_k$ vary in an n – independent domain, and we say that (3.43) is in the local regime (is a local quantity) if $\lambda_j, \quad j = 1, 2, \ldots, k$ are scaling variables specified by (3.21), where $\rho_n(\lambda)$ is the mean density defined by (1.23) (we assume for simplicity of the formulation that $\rho_n(\lambda)$ exists).

The simplest example of the global quantity is $\rho_n(\lambda)$ itself. More generally, the joint probability distribution of $N_n(\Delta_1), \ldots, N_n(\Delta_q)$, where $N_n(\Delta)$ is the NCM (1.4) and

$$\Delta_j = (a_j, b_j), \quad j = 1, \ldots, q \tag{3.44}$$

are n – independent intervals of the spectral axis is in the global regime. The role of spectral parameters $\lambda_1, \ldots, \lambda_k$ play the endpoints $a_j, b_j, \quad j = 1, \ldots, q$ of intervals (3.44).

The same joint probability distribution in the case when (a_j, b_j) are specified by (3.11) is in the local regime. In particular, the probability (1.8) that defines the spacing distribution density (3.3) is a typical and important example of a local quantity. The global regime for $R_n(\Delta)$ is trivial (at least in the leading order) because if Δ does not depend on n, then, as a rule, $N_n(\Delta)$ tends in probability to a nonzero limit as $n \to \infty$ and thus

$\lim_{n\to\infty} R_n(\Delta) = 0$. On the other hand, the local regime for $R_n(\Delta)$ is rather nontrivial and respective limiting form is given by (3.6) and (3.18) for the Poisson and Wigner–Dyson classes respectively.

One more couple of examples is given by the joint probability density (2.11). If $\lambda_1, \dots, \lambda_k$ are n – independent variables then we are in the global regime and according to (2.11)

$$\lim_{n\to\infty} p_k^{(n)}(\lambda_1, \dots, \lambda_k) = p_1(\lambda_1) \dots p_1(\lambda_k) \tag{3.45}$$

where

$$p_1(\lambda) \equiv \rho(\lambda) = \lim_{n\to\infty} p_n^{(1)}(\lambda), \tag{3.46}$$

provided that the latter limit exists (see e.g. [23] for the proof of (3.45) and (3.46) in the case of the UE's (2.92) – (2.94)). This is the global regime answer.

The local regime answer, corresponding to λ_j's of the form (3.21) is given by (3.22) in the case of the GUE, and, in fact, for all ensembles which we mentioned above as belonging to the Wigner – Dyson universality class.

As we have seen above for all ensembles studied so far the normalizing counting measure (1.4) tends to a nonrandom limit as $n \to \infty$ in probability or even with probability 1 (see [18] and [37] for results and discussions of other numerous cases). In all these cases global quantities describe probabilistic properties of $O(n)$ eigenvalues while local quantities describe properties of finitely many eigenvalues.

The universality conjecture of the random matrix theory says that the limiting form of a local quantity corresponding to a given ensemble of $n \times n$ matrices for $n \to \infty$ does not depend on the particular form of the ensemble probability distribution (the potential $V(\lambda)$ in (1.1), probability distributions of matrix elements for ensembles of random Hermitian matrices with independent for $j < k$ matrix elements, etc.).

On the other hand the limiting form of global quantities depends as a rule on the form of the ensemble distribution.

We have seen above a number of results supporting the validity of the universal ity conjecture and other results demonstrating diversity of limiting forms of global quantities, the DOS in particular. This allows us to finish the paper by citing F.Dyson [38]: "... there are two styles of science..., unifying and diversifying. They are complementary, giving us two views of the universe which are both valid...". Dyson's contribution to the random matrix theory is well-known. These words, however, were written by Dyson in much more general context of discussion of "... the diversity of the natural world and... the diversity of human reaction to it". The author hopes very much that the reader will not blame him too strongly for using these words in the rather special and concrete context of the random matrix theory, whose origins and developments were motivated by a number of branches of mathematics and physics and which demonstrates numerous ideas and results common for all these branches.

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