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LETTER TO THE EDITOR

A solvable random matrix model for disordered conductors

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Abstract. We propose a random transfer matrix model for disordered systems which describes the distribution of conductance in the metallic as well as insulating regimes. The n-point correlation function of the eigenvalues can be obtained in terms of a known family of orthogonal polynomials. The model gives the well-known log–normal distribution of conductance in one dimension, and the universal conductance fluctuation in the metallic regime. The metal–insulator transition in this model is related to the opening of a gap in the density of eigenvalues near the origin, as a function of one parameter.

Since the discovery of universal conductance fluctuation in mesoscopic conductors [1], it has become clear that conductance is not a self-averaging quantity and therefore the transport properties of a disordered system should be described in terms of the probability distribution of conductance rather than its ensemble average. In particular, the disorder-induced Anderson transition from a metal to an insulator has been discussed so far only in terms of the ensemble averaged conductance [2], and needs to be re-examined in the presence of these fluctuations. One approach which provides a possible framework for such a description is based on models of random transfer matrices [3]. In these models a disordered conductor (of length L and cross sectional area L^{d-1} in d dimensions) attached to perfectly ordered leads with N propagating channels is characterized by a $2N\times 2N$ multiplicative transfer matrix $\mathbf{T}$ which gives the flux amplitudes to the right of the conductor in terms of the incoming and outgoing fluxes on the left. The conductance can be directly related to the N non-degenerate eigenvalues $x_n \geqslant 0$ of the matrix $\mathbf{X} = \frac{1}{4}(\mathbf{T}^\dagger\mathbf{T} + (\mathbf{T}^\dagger\mathbf{T})^{-1} - 2)$, and the distribution of the eigenvalues for given symmetries of the matrices $\mathbf{X}$ can be calculated using theories of random matrices [4]. Such models provide a simple explanation of universal conductance fluctuation in terms of the well-known spectral rigidity characteristic of random matrix ensembles [5]. The approach is especially appealing because it incorporates the effect of change in symmetries due to the presence of magnetic field or spin–orbit scattering in a simple way.

A critical quantitative test of the validity of the random transfer matrix approach in the weak disorder regime was recently carried out [6] by comparing exact analytic

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calculation of the two-point eigenvalue correlation function for a given finite-size system, within the random matrix model, with a completely independent numerical evaluation of the same correlation function using the tight-binding Anderson model. The excellent overall agreement shows the general validity of the random matrix model. This confirms the hypothesis that the only interacting term in the eigenvalue distribution is a logarithmic repulsion between eigenvalues which is universal and arises from symmetry considerations alone. The eigenvalues are bounded by a single-particle confining potential which might be thought of as a Lagrange multiplier function arising from some additional constraints, e.g. the given mean value of the conductance, and depends on system parameters. However, from numerical investigations [6], the functional form of the potential is found to be quite insensitive to changes in disorder, size or dimension. Although this single-particle potential is not known from any microscopic considerations, attempts have been made [3, 7] to consider simple potentials suggested by numerical studies in restricted regimes. For example it is now well-known that a linear confining potential (with a possible logarithmic repulsion from the origin) with disorder-dependent slope describes the metallic regime (small eigenvalues) very well [3] where eigenvalue correlations provided by the strong confining term results in universal conductance fluctuation. On the other hand, in the insulating regime, one can check for the potential by considering a system in 1D, where there is only one eigenvalue, and the distribution of conductance is entirely determined by the potential term. Since the conductance in 1D is known to have a log–normal distribution [8], it is clear that the confining potential for exponentially large eigenvalues which describes the insulating regime must behave as a square of a logarithm. We expect insulators in higher dimensions to be qualitatively similar. This behaviour for the potential at large eigenvalues is also well supported by extensive numerical studies [6]. Thus, a potential that might describe a system on both sides of the metal–insulator transition is unfortunately not a simple polynomial, and so far it has not been possible to address the problem of the transition itself in such random matrix models. It is therefore of great interest to investigate whether an analytically tractable model can be constructed with a single-particle potential term which has the desirable qualitative features stated above. One can then study the eigenvalue density and higher-order correlation functions to address the nature of metal–insulator transition in that model.

In the present work we consider the simplest one-parameter potential (with another hidden parameter that defines the scale and relates the eigenvalues to the conductance) which has the required limiting forms stated above, and is exactly solvable. The distribution of the eigenvalues for this model can be calculated exactly in the entire parameter range in terms of a known family of orthogonal polynomials. In one limit of the parameter the potential becomes linear and therefore [3] describes the metallic regime and the associated universal conductance fluctuation. We calculate explicitly the eigenvalue density in this regime, which is large near the origin (small eigenvalues contribute more to the conductance) and falls off as an inverse power. As a check, we evaluate the mean conductance taking into account the scale factor mentioned above and identify the scale factor with physical parameters by demanding that Ohm's law be obeyed in this regime. This agrees with the previous saddle-point calculation for the most probable conductance [7]. In the opposite limit, we find that a gap appears in the density of the eigenvalues whose width is a function of the parameter of the potential, giving rise to an insulator. Clearly the metal–insulator transition in this model is associated with the opening of the gap in the density of eigenvalues as a function of the parameter. We also show that the model recovers

the well-known log–normal distribution [8] for conductance in one dimension.

We define the random transfer matrix model with N eigenvalues $0 \leqslant x_n \leqslant \infty$, $n = 0, 1, \ldots N-1$, whose distribution according to the maximum entropy hypothesis [3] can be written as

$$P\{x_0, , , x_{N-1}\} = \prod_{m<n,0}^{N-1} |x_m - x_n|^\beta \prod_{k=0}^{N-1} \exp(-V(x_k)) \tag{1}$$

where β is the symmetry parameter and is equal to 1, 2 or 4 for the case with no magnetic field, in the presence of magnetic field, and in the presence of spin–orbit scattering, respectively. For technical simplicity, we will restrict ourselves to the case of $\beta = 2$ only. (We will assume that the magnetic field is sufficiently small so that we are not in the quantum Hall regime.) The two-probe conductance g is related to the eigenvalues by the relation [9]

$$g = \sum_{n=0}^{N-1} 1/(1 + x_n). \tag{2}$$

While the β-dependent product term (which becomes a logarithmically repulsive interaction term when written as an exponential) arises from symmetry considerations alone, the single-particle potential $V(x)$ depends on various physical parameters, and is model-dependent.

As stated earlier, on the basis of physical as well as numerical considerations, we wish to consider a potential which is linear in x for small x, with a possible logarithmic repulsion near the origin, and behaves as $(\ln x)^2$ for large x. We consider the potential given by

$$w(x; q) \equiv e^{-V(x;q)} = x^\alpha / (-(1-q)tx; q)_\infty \tag{3}$$

where $0 < q < 1$, $\alpha > -1$, and we have used the conventional notation

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n). \tag{4}$$

It can be shown using the q-binomial theorem [10] that $V(x; q)$ has a power series expansion in x for small x, and $\alpha = 0$. On the other hand for sufficiently large x and small q, one an show using the Jacobi triple-product identity [10] that $V(x; q) \sim (\ln x)^2$. Thus the one-parameter potential V satisfies the criteria stated above. Although we will show later that the log–normal distribution in one dimension as given by Mel'nikov [8] is recovered for $\alpha = 1$, we will calculate the eigenvalue density and the mean conductance for $\alpha = 0$ only. This is because firstly, numerical investigations have not, so far, indicated a non-zero α for two and three dimensions, so one possibility is that $\alpha = 1/N$ (note that $N \propto L^{d-1}$). Secondly, all our qualitative results including in particular the appearance of the gap in the density are determined by the parameter q, and the $\alpha = 0$ case will illustrate that. We also observe that the $q \to 1^-$, $w \to e^{-tx}$, and such a linear potential is known to describe the metallic regime very well [3, 7], with t as a disorder-dependent parameter. However, since t is only a scale factor, we can ignore it in the potential for simplicity, and include its effect at the end by appropriately scaling x in the definition of the conductance g in (2). We shall therefore evaluate the eigenvalue density for $\alpha = 0$ and $t = 1$, although we will describe the formulation for general α.

Given w, we wish to calculate the corresponding eigenvalue density, which will give us the mean conductance. We define a family of orthogonal polynomials $P_n(x;q)$ with w as the measure such that

$$\int_0^\infty \mathrm{d}x\, w(x;q) P_m P_n = \delta_{mn} h_n .$$

It is then possible to express the eigenvalue density as well as higher point correlation functions in terms of these polynomials [4]. Our model is solvable because for the choice of w given in (3) the polynomials are known explicitly for the entire parameter range. They are the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ introduced by Hahn and more recently investigated by Moak [12]. In the limit $q \to 1^-$ and for $\alpha \neq 0$, $L_n^{(0)}(x;q)$ reduces to the ordinary Laguerre polynomials. In this limit, the confining potential dominates at large x over the logarithmic interaction term, thereby pushing the eigenvalues towards the origin. Using an asymptotic expansion of the Laguerre polynomials for large N [11], we obtain the eigenvalue density

$$\frac{\sigma_N(x;\Gamma)}{N} = \frac{1}{\sqrt{x}}\left[\sqrt{N_+}J_0(\sqrt{x_-})J_1(\sqrt{x_+}) - \sqrt{N_-}J_0(\sqrt{x_+})J_1(\sqrt{x_-}) + O(N^{-3/2})\right] \tag{5}$$

where $N_\pm = N \pm 1/2$, $x_\pm = 4N_\pm x$, and $J_\nu(x)$ are the Bessel functions. For $x = 0$, $\sigma_N(x)/N = 1$. For $x \gg 1/N$, (5) falls off as $1/x$. The mean conductance $\langle g \rangle$ can now be determined, with the appropriate scale factor t included. For $t \to \infty$, all eigenvalues are compressed at the origin giving $\langle g \rangle = N$, which is the ballistic limit. In general, t will depend on physical parameters such as disorder, size and dimension. In the large N limit and for $Nt \gg 1$, $\langle g \rangle \sim N(1 - \mathrm{e}^{-\sqrt{t/N}})$. For $t/N \ll 1$, the leading term is $\sqrt{Nt}$, which suggests we identify $t = N(l/L)^2$, which reproduces Ohm's law, $g_0 = Nl/L$. This identification agrees with an earlier saddle-point calculation for the most probable conductance [7]. (The leading correction in this limit is given by $\delta g/g_0 = l/(2L)$, and it is tempting to identify it with the leading quantum correction in 3D, due to weak localization [2], where $Nt \gg 1$ is well satisfied. In 2D, in order to see the logarithmic size dependence, one needs to be in a limit where $\ln(L/l) \ll Nl/L$, and where $N \propto L$. This is outside our large-N limit. We mention that a logarithmic size dependence is obtained in the limit $Nt \ll 1$, using (5). However, since we only restricted ourselves to the case $\beta = 2$, valid in a finite magnetic field, we should also identify a magnetic length in the problem. The proper way to identify these weak localization effects is to calculate g for the case $\beta = 1$ as well.) It is also known [3] that for Laguerre polynomials, the variance of g is independent of N or t, giving rise to the universal conductance fluctuation. Thus the case $q = 1$ describes the metallic regime quite well.

On the other hand, for $q \ll 1$, the large-x potential is only very weakly confining. We therefore expect the logarithmic repulsion to push the eigenvalues away from the origin, thereby depressing the density there. In fact the q-Laguerre polynomials have an asymptotic expansion [13] qualitatively different from that of $q = 1$:

$$L_N^{(\alpha)}(x;q) = \left(1 + \frac{q^{\alpha+N+1}}{1-q}\right) L_\infty^{(\alpha)}(x;q) - \frac{q^{N+1}}{1-q} L_\infty^{(\alpha)}\left(\frac{x}{q};q\right) + O(q^{2N}) \tag{6}$$

where

$$L_\infty^{(\alpha)}(x;q) = [x(1-q)]^{-\alpha/2} J_\alpha^{(2)}\left(2\sqrt{x(1-q)};q\right) \tag{7}$$

is an entire function in x. Here $J_\alpha^{(2)}(x;q)$ is the q-Bessel function [10]. It is, in principle, possible to evaluate the eigenvalue density in the same way as for the case $q = 1$ using these asymptotic expressions, but we will follow a simpler way which exploits the fact that for $q < 1$, the measure w is not unique [12]. In this case there exist many different but equivalent measures (with the same moments, giving the same polynomials), and we will choose the one that allows us to calculate the density most easily. It is well known from the classical moment problem that the extreme measure in such cases is discrete and is supported at the zeros of an entire function [14]. Moak has constructed a family of equivalent discrete measures for the present problem, the explicit form for a representative one is

$$w_{\mathrm{dis}}(x;q) = \sum_{n=0}^{\infty} w_n \delta(x-\tau_n)$$
$$\frac{1}{w_n} = \frac{(1-q)q^\alpha (q^{\alpha+1};q)_\infty}{(q;q)_\infty} \left\{ \frac{\mathrm{d}}{\mathrm{d}x} \left[L_\infty^{(\alpha)}(x;q) L_\infty^{(\alpha)}\left(\frac{x}{q};q\right)\right]\right\}_{x=\tau_n} \tag{8}$$

where τ_n are the zeros of $L_\infty^{(\alpha)}(\kappa;q)$. We evaluate the density for $\alpha = 0$ using the measure, which gives

$$\sigma_\infty(x;q) = \lim_{N\to\infty} \sigma_N(x;q) = \frac{1-q}{\ln(1/q)} \sum_{n=0}^{\infty} \delta(x-\tau_n). \tag{9}$$

One can now anticipate the qualitatively different behaviour of the density for $q = 1$ and $q < 1$. It is clear that $\sigma_\infty(x;q)$ can have only exponentially small support up to the first zero of $L_\infty^{(0)}(x;q)$. For sufficiently small q and large enough x such that $x(1-q) \gg \sqrt{q} \gg q^{3/2}$, we use the asymptotic expansion of the q-Bessel function [15] in (7) to obtain

$$L_\infty^{(\alpha)}(x;q) \sim \sqrt{4\pi/\beta}\cos(\pi y/\beta)\exp\left[(y^2-\pi^2/4)/\beta\right] \tag{10}$$

where we have defined β (not to be confused with the symmetry parameter in (1)) and y as $q = \mathrm{e}^{-\beta}$, $y = \frac{1}{2}\ln[x(1-q)q^\alpha]$. Thus the zeros of $L_\infty^{(\alpha)}(x;q)$ are found to be the zeros of $\cos(\pi y/\beta)$, and are given by $\tau_n \approx ((1-q)q^\alpha q^{2n+1})^{-1}$, which agrees with the known fact that the zeros are very well separated [12]. Using a parametric representation of the delta function and the Euler approximation for the sum, the eigenvalue density for $\alpha = 0$ can now be evaluated:

$$\sigma_\infty(x;q) = \begin{cases} 0 & x(1-q) \leqslant 1/q \\ (1-q)/(2\beta^x) & x(1-q) > 1/q. \end{cases} \tag{11}$$

The appearance of the gap in the density is a clear signature of an insulator. Indeed from the definition of $\langle g\rangle$, an elementary quadrature gives $\langle g\rangle(q) \approx (t/2\beta^2)\mathrm{e}^{-\beta}$, $\beta \gg 1$ so that the mean conductance is exponentially small. We can now identify the parameter β as $\beta = L/\xi$, i.e. $q = \mathrm{e}^{-L/\xi}$. Thus as the localization length becomes smaller than the system size, q changes from 1 to a value smaller than 1 and the eigenvalue density at the origin gets depressed. At sufficiently small q, a gap opens up, indicating a metal to insulator transition.

In the insulating regime essentially only the first eigenvalue contributes, and the channel number N which goes as L^{d-1} in d dimensions does not appear in the exponent. We therefore expect the fluctuations in this regime to be qualitatively similar to

that in a 1D insulator, where $N = 1$. (We remark at this point that going from one to higher dimensions is highly non-trivial, as seen from the presence of the long-range logarithmic interaction between eigenvalues representing different channels in higher dimensions.) The distribution of resistance (the inverse of conductance) $P(\rho; q)$ for the 1D problem in the large resistance limit, $\rho \sim x/t$ can be simply written as

$$P(\rho; q) = \int_0^\infty \mathrm{d}x\, w_{\mathrm{dis}}(x; q)\delta(x/t - \rho).$$

For $\alpha = 1$, we get

$$P(\rho; q) \sim \exp[-\{\ln \rho - \ln(q^{-1/2})\}^2/4 \ln(q^{-1/2})] \tag{12}$$

which is the log–normal distribution of Mel'nikov [8] with our previous identification $q = \mathrm{e}^{-L/\xi}$. Note that for $\alpha = 0$, the $\ln^2(\rho)$ dependence of the exponent is correctly reproduced, but the mean is shifted. Let us point out that although the discrete measure w_{dis} is quite distinct from the continuum measure w, their asymptotics in fact are essentially the same, and both give identical results for the resistance distribution. The moments of the resistance distribution may also be determined easily from either measure w_{dis} or w, and for $\alpha = 0$, $\langle \rho^n \rangle \sim \exp[n(n+1)L/\xi]$ $(q \ll 1, n \gg 1)$. Thus the distribution function itself cannot be uniquely reconstructed from the given moments because of the non-uniqueness of the measure.

In summary, we propose a solvable random transfer matrix model with a particular choice of the single-particle confining potential which describes the known features of the conductance distribution both in the metallic and in the insulating regimes. The metal–insulator transition in this approach is shown to be related to the appearance of a gap in the density of eigenvalues, which is very different from the conventional Hamiltonian formulation where the transition shows up only at the level of a two-point correlation function. Various higher-order correlation functions in the present model can be investigated in detail (e.g. if the transition is associated with a change in the level spacing distribution from the known Wigner-type [3] in the metallic regime to an uncorrelated Poisson-type distribution in the insulating side) in terms of the same parameters and the same known family of orthogonal polynomials, which we leave for future studies.

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