Modeling the Melt

What math tells us about sea ice and polar ecosystems in a warming climate

Ken Golden, University of Utah

MathAcrossCampus
University of Washington
February 24, 2023

SEA ICE covers ~12% of Earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- hosts rich ecosystem
- indicator of **climate change**

polar ice caps critical to global climate in reflecting incoming solar radiation

Earth's refrigerator

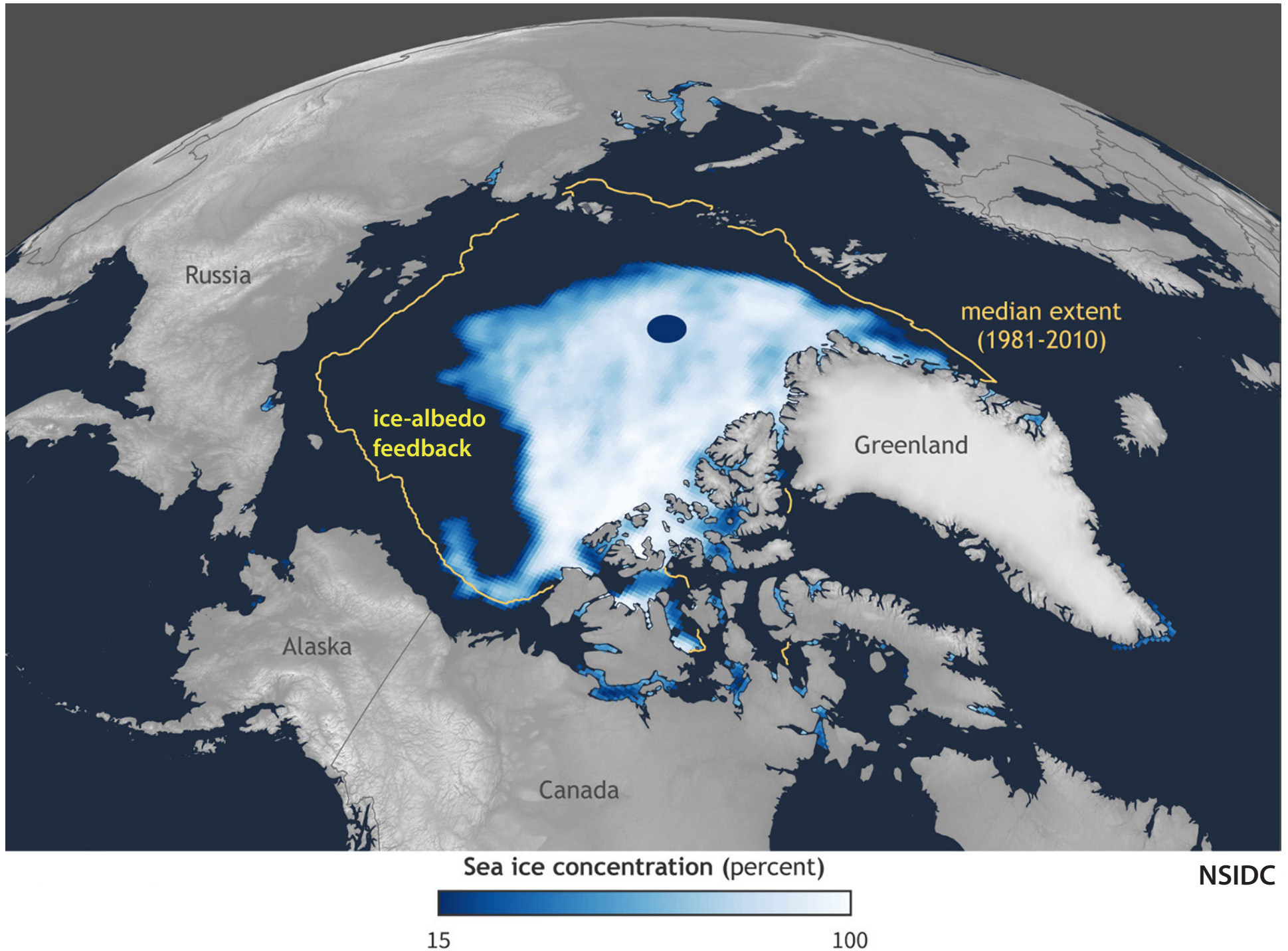NASA

NASA

white snow and ice reflect

dark water and land absorb

$$\textit{albedo}\ \ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

Arctic sea ice extent   September 15, 2020
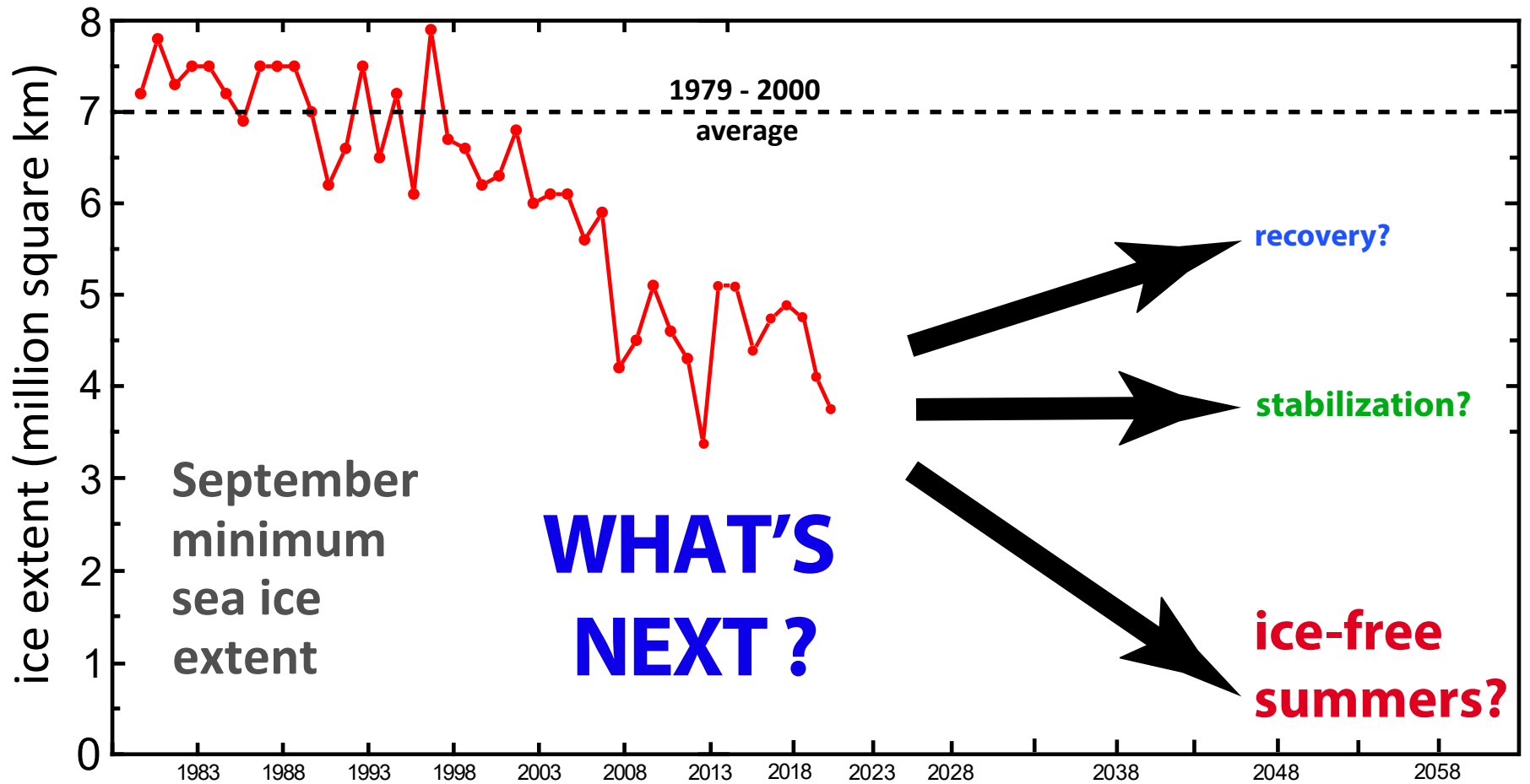

Russia

ice-albedo
feedback

Greenland

median extent
(1981-2010)

Alaska

Canada

Sea ice concentration (percent)

NSIDC

15   100

recent losses
in comparison to
the United States

Perovich

Predicting what may come next requires lots of math modeling.

September minimum sea ice extent

WHAT'S NEXT ?

recovery?

stabilization?

ice-free summers?

1979 - 2000 average

National Snow and Ice Data Center (NSIDC)

ANTARCTICA

southern cryosphere
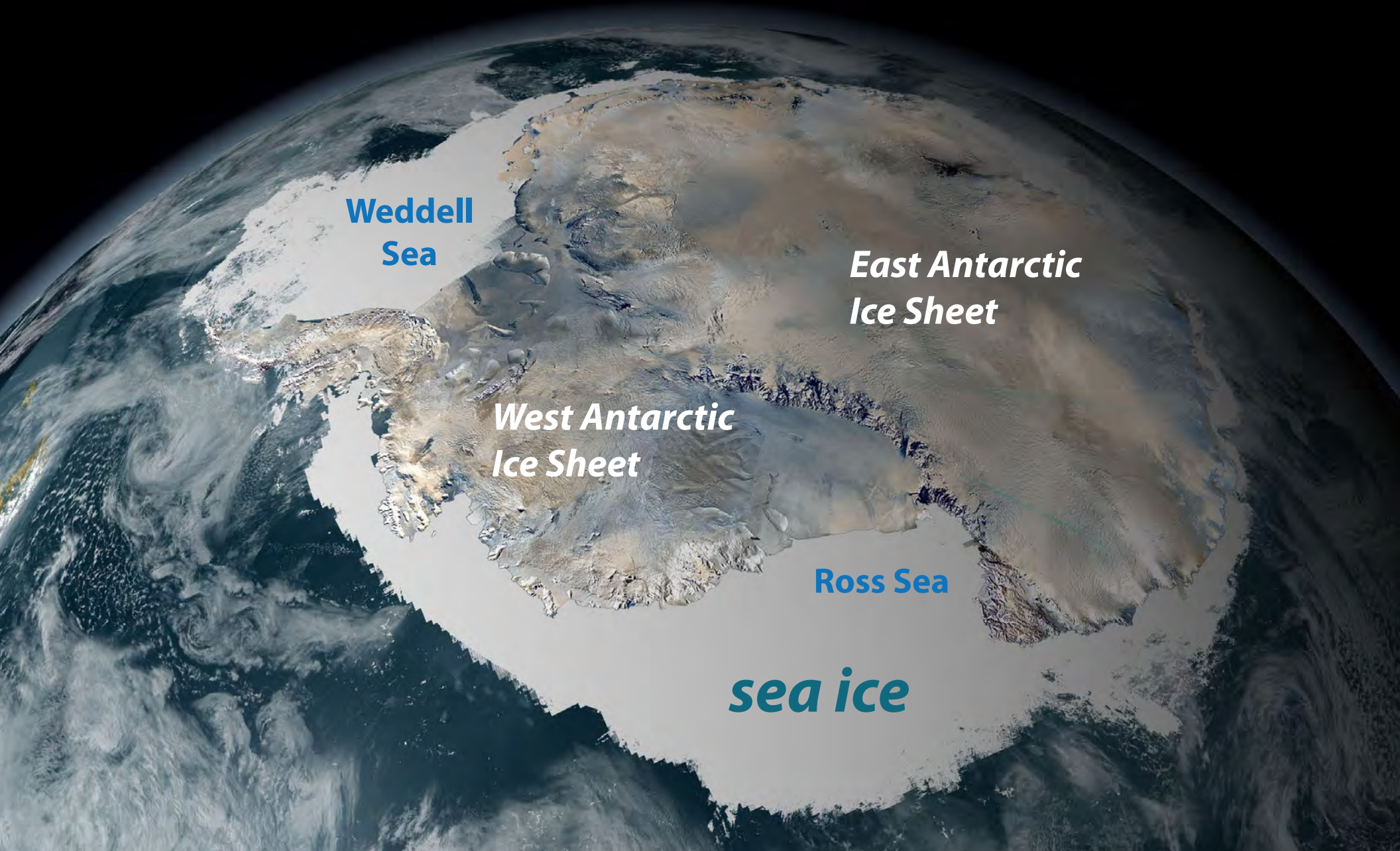
Weddell Sea

East Antarctic Ice Sheet

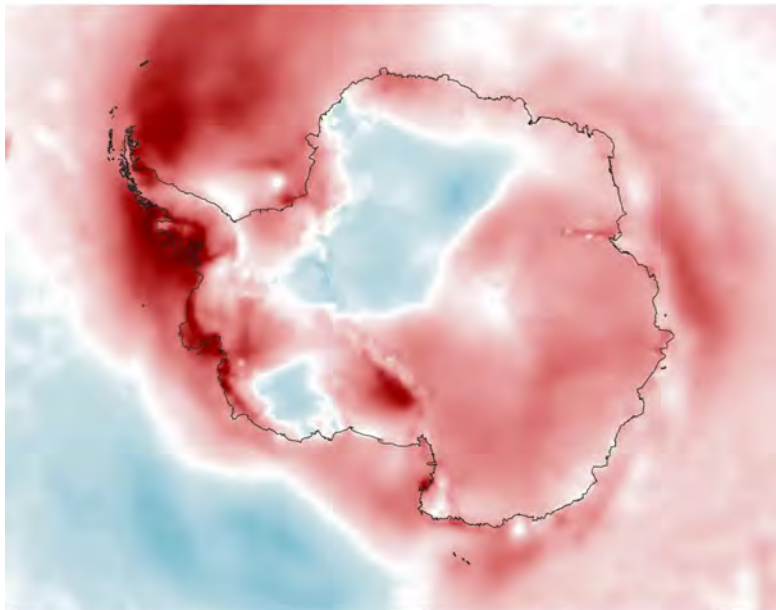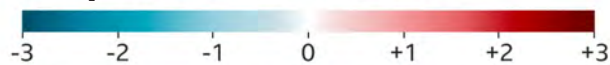*West Antarctic Ice Sheet*

Ross Sea

sea ice

New Record Low for Antarctic Sea Ice
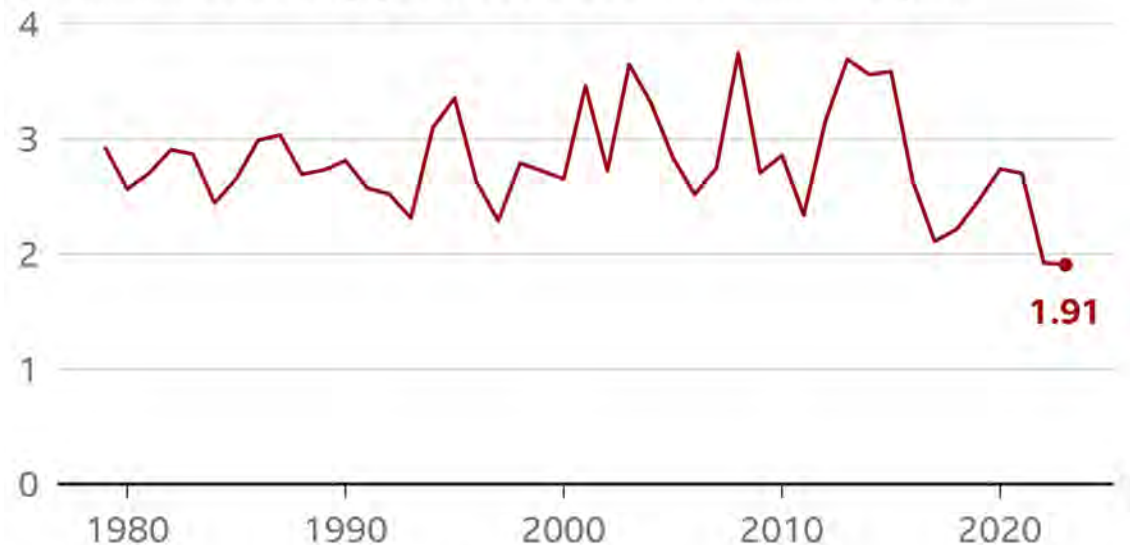
February 13, 2023

Much of Antarctica warmer than average

Mean 2022 surface air temp compared with 1991-2022 (°C)
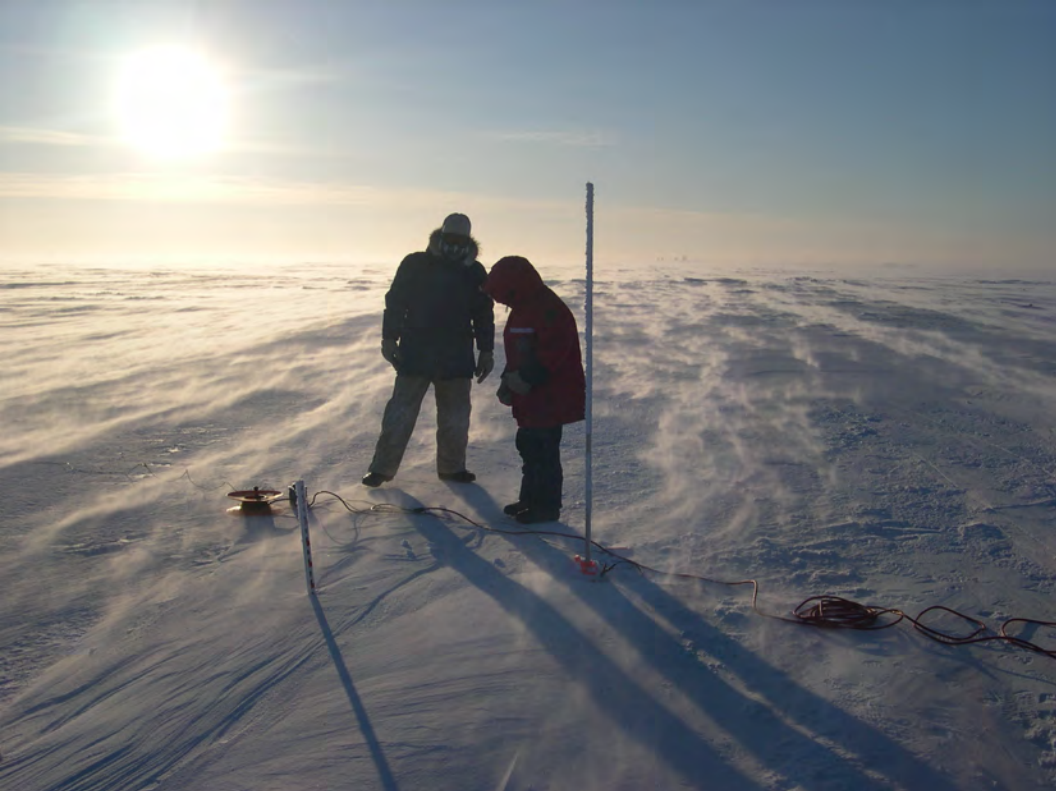
Source: ECMWF ERA5

Minimum extent 1979-2023 (million sq km)

1.91

Five-day rolling average of sea-ice extent
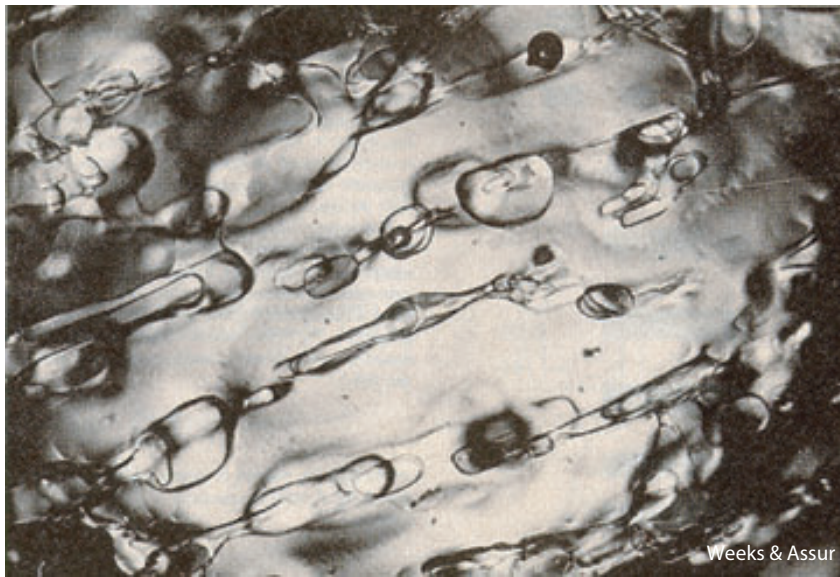
Source: National Snow and Ice Data Center (NSIDC)

sea ice may appear to be a barren, impermeable cap ...

Golden

brine inclusions in sea ice (mm)

100 microns

R. Obbard
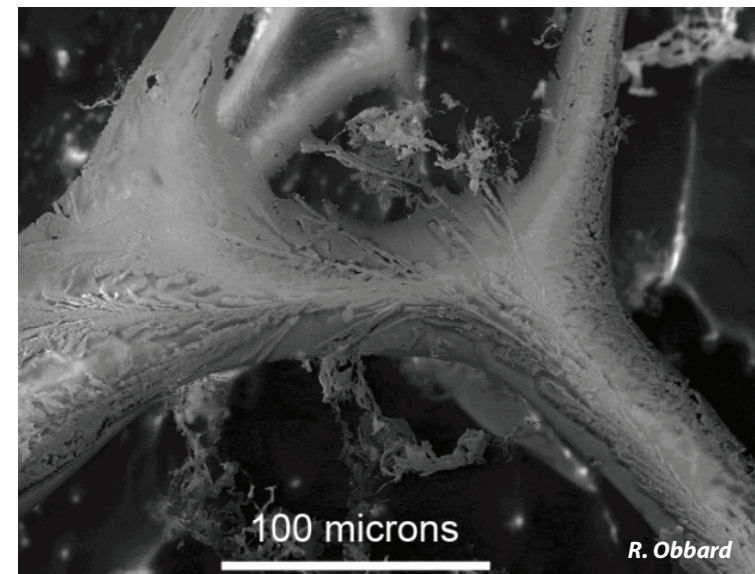
micro - brine channel (SEM)

brine channels (cm)

sea ice is a porous composite

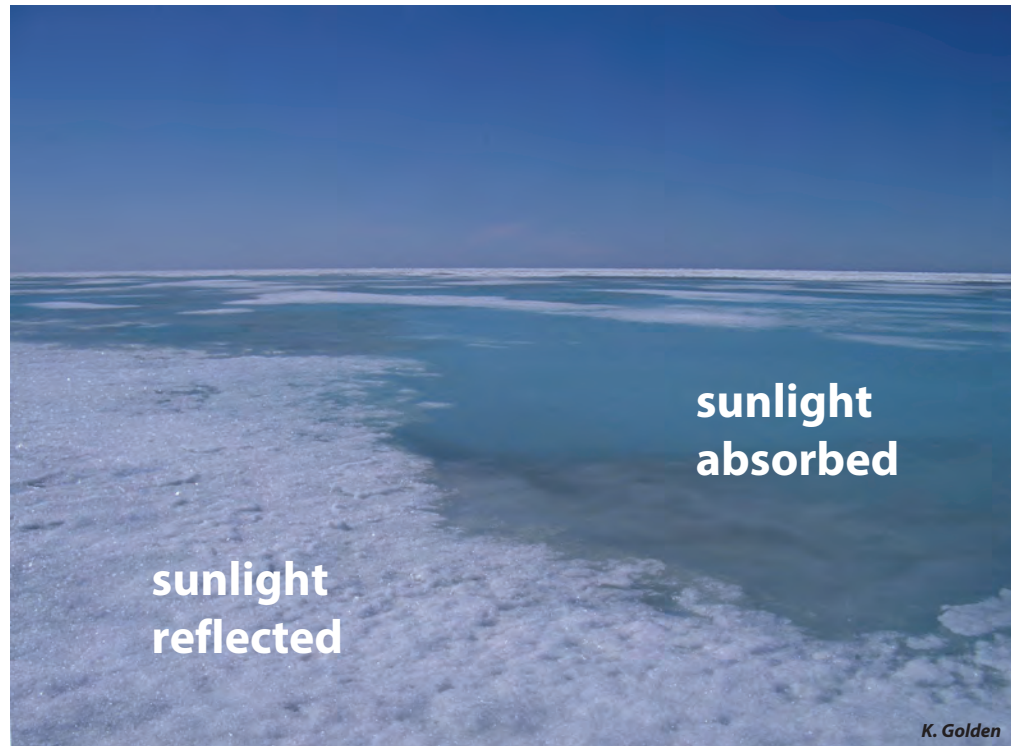pure ice with brine, air, and salt inclusions

D. Cole

horizontal section

Golden

vertical section

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

sunlight absorbed

sunlight reflected

K. Golden

nutrient flux for algal communities

C. Haas

ACE CRC

C. Krembs

22%

28%

26%

27%

September snow-ice estimates

0   25   50   75   100
percent snow ice

T. Maksym and T. Markus, 2008
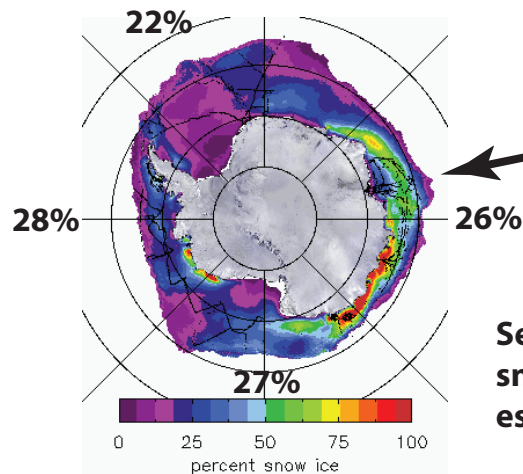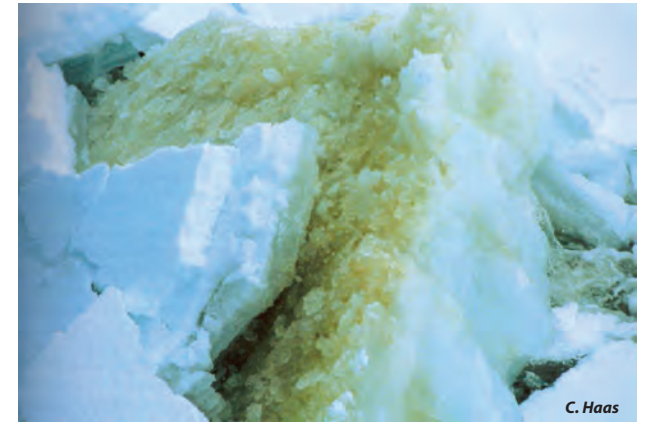
Antarctic surface flooding and snow-ice formation

- evolution of salinity profiles

- ocean-ice-air exchanges of heat, CO_2

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

Weeks & Assur 1969

H. Eicken
Golden *et al.* GRL 2007

polycrystals

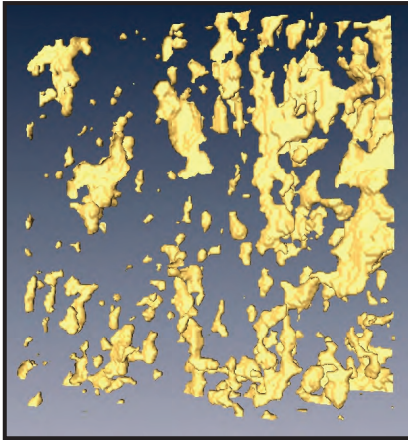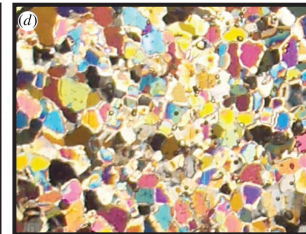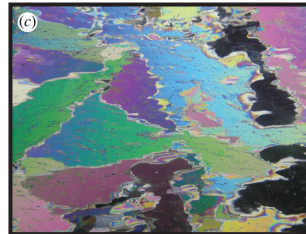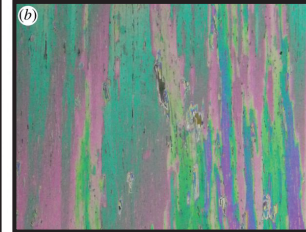Gully *et al. Proc. Roy. Soc. A* 2015

brine channels
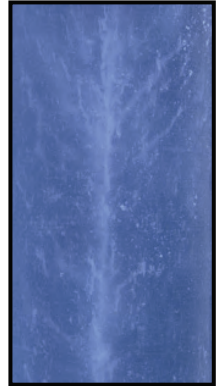
D. Cole   K. Golden

millimeters   **centimeters**

mesoscale   *macroscale*

Arctic melt ponds

K. Frey

Antarctic pressure ridges
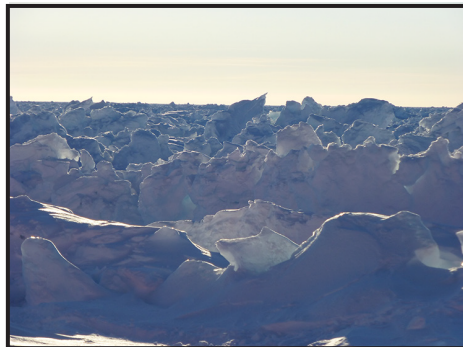
K. Golden

sea ice floes

J. Weller

sea ice pack
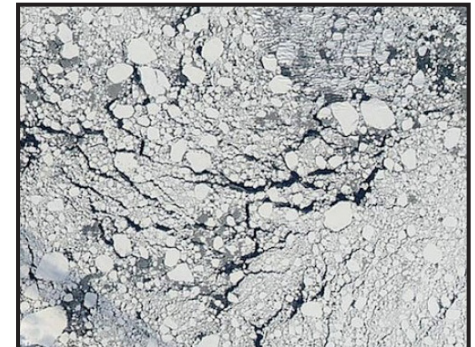
NASA

meters   **kilometers**

HOMOGENIZATION for Composite Materials

FORWARD

σ_1 σ_2

σ^* *effective conductivity*

LINKING SCALES

σ^*

homogeneous medium of conductivity σ^*

MICROSCALE

MACROSCALE

INVERSE
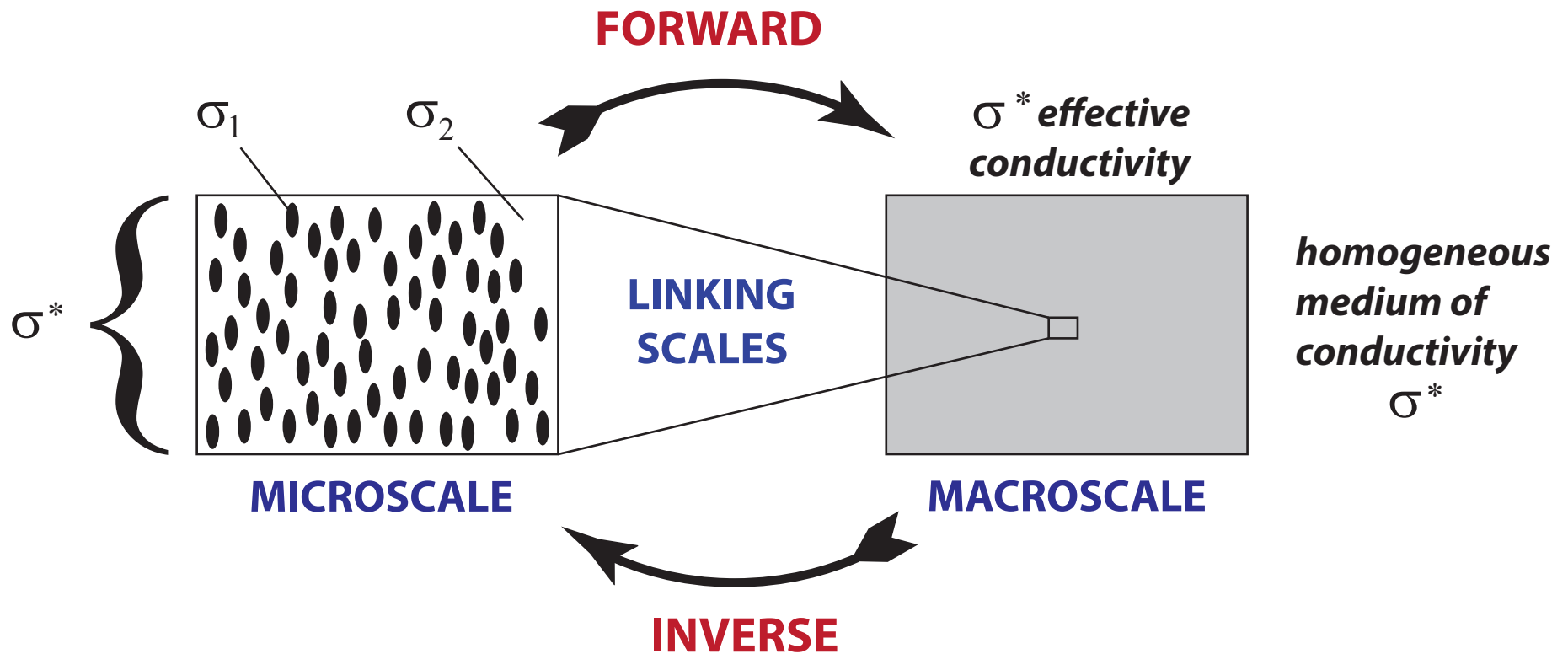
Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

Wiener 1912 : arithmetic and harmonic mean bounds on effective conductivity
Hashin and Shtrikman 1962 : variational bounds on effective conductivity

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

What is this talk about?

A tour of recent results on multiscale modeling of physical and ecological processes in the sea ice system, with a focus on novel mathematics.

microscale

mesoscale

macroscale

Microbial Ecology and the Physics of Sea Ice

How do the physical properties of sea ice affect microbial communities?

How does the presence of microbial life in sea ice affect its physical properties?

Golden
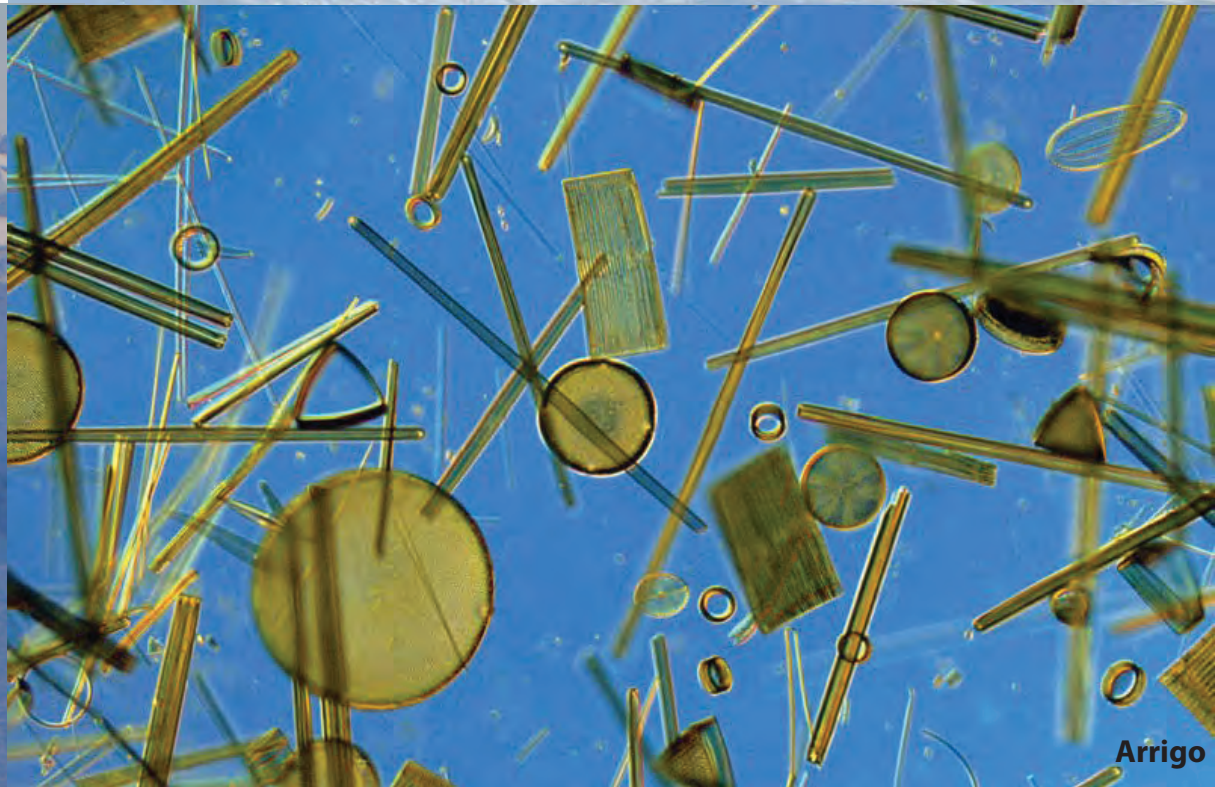
Arrigo

microscale

brine volume fraction and *connectivity* increase with temperature

$T = -15\,°\mathrm{C}, \quad \phi = 0.033$
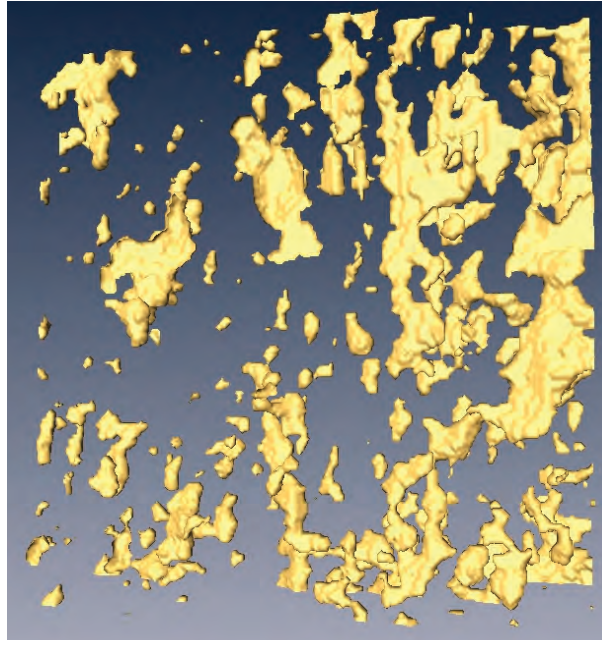
$T = -6\,°\mathrm{C}, \quad \phi = 0.075$
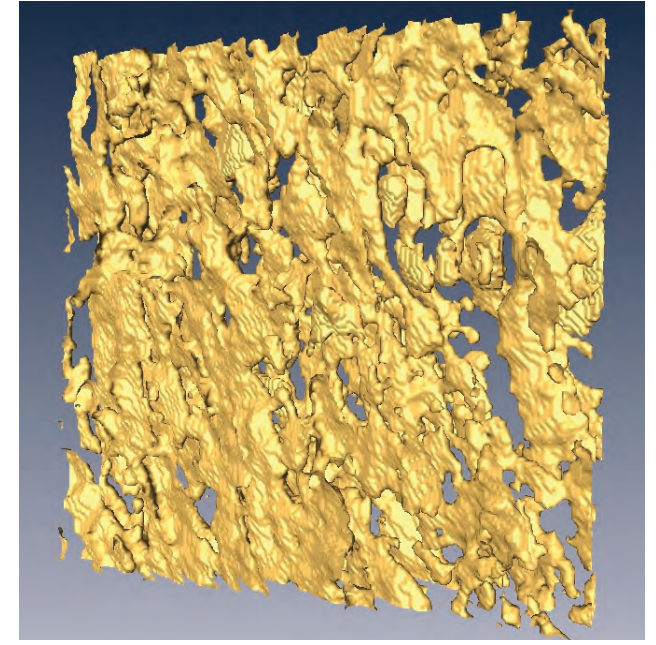
$T = -3\,°\mathrm{C}, \quad \phi = 0.143$

$T = -8\,°\mathrm{C}, \quad \phi = 0.057$
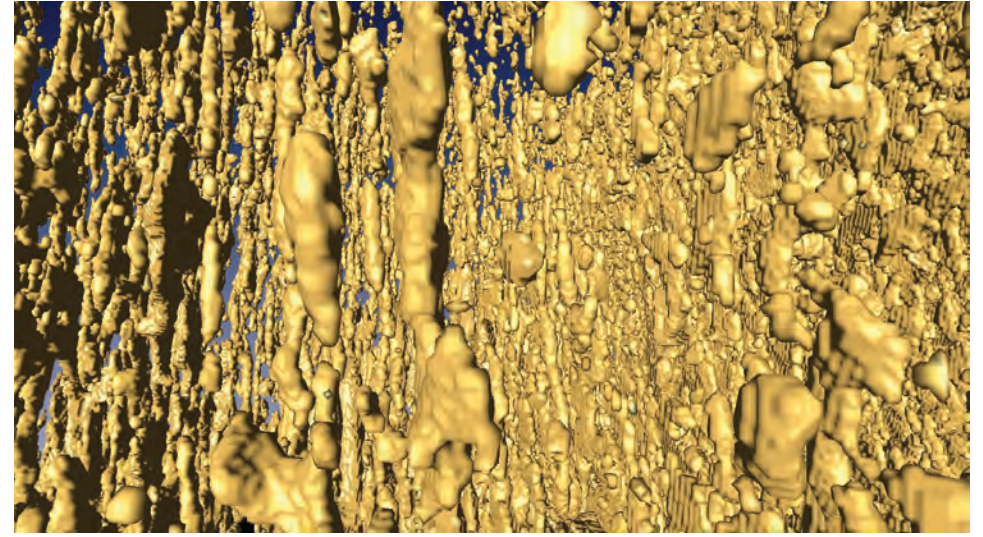
$T = -4\,°\mathrm{C}, \quad \phi = 0.113$

X-ray tomography for brine in sea ice

Golden *et al.*, *Geophysical Research Letters*, 2007

percolation theory

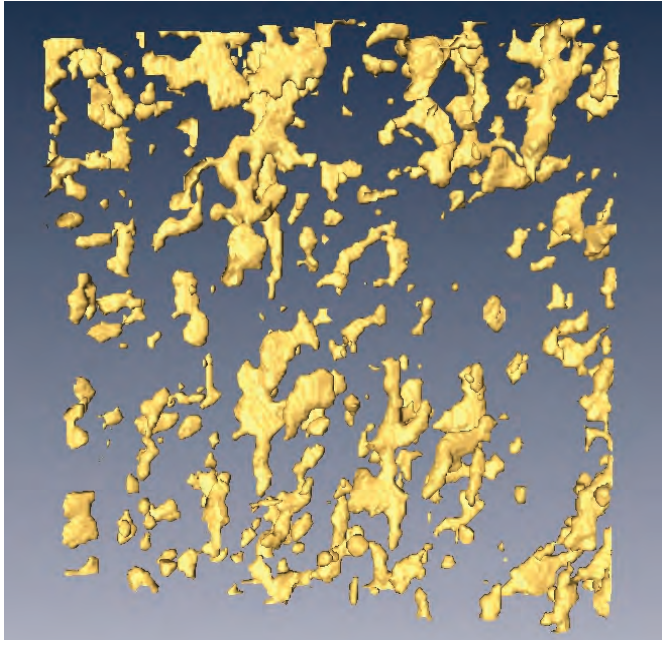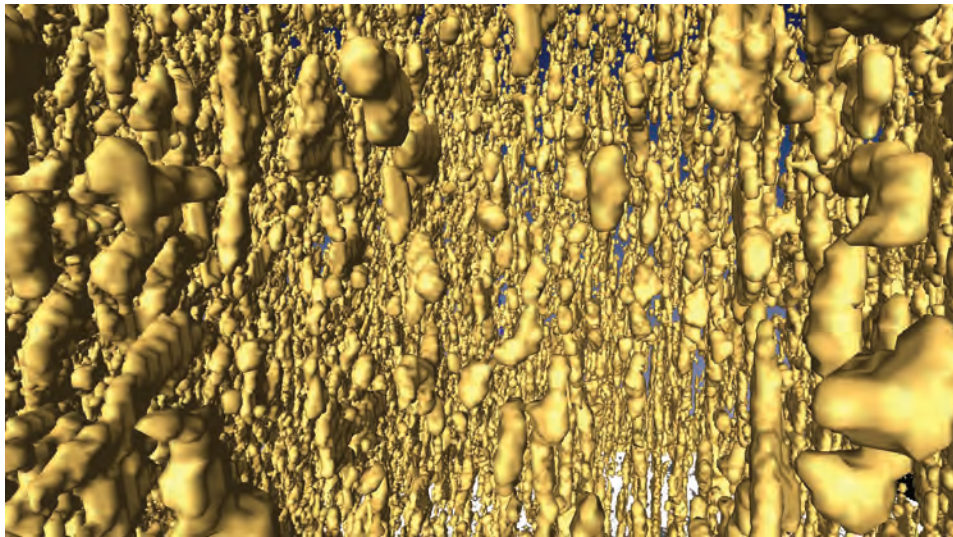## *probabilistic theory of connectedness*

impermeable

permeable

correlation length

open cluster

development of long range order

characteristic scale of connectedness

$p = 1/3$

$p = 2/3$

ξ

$0 \qquad p_c \qquad 1 \qquad p$

percolation threshold
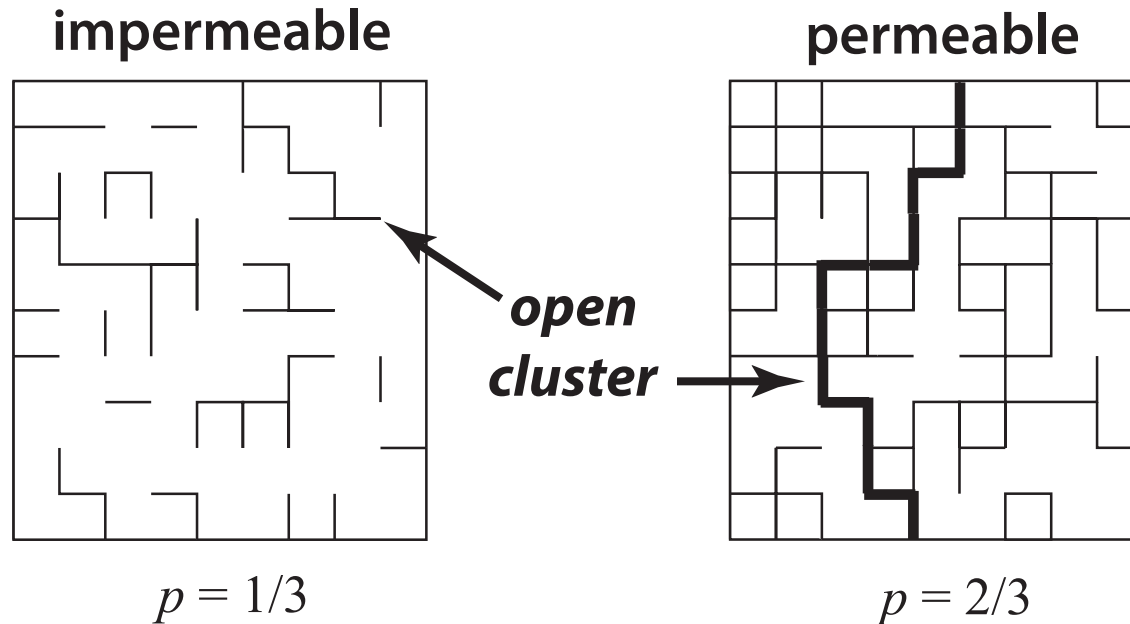
$bond \longrightarrow$ **open** *with probability* p
closed *with probability 1-p*
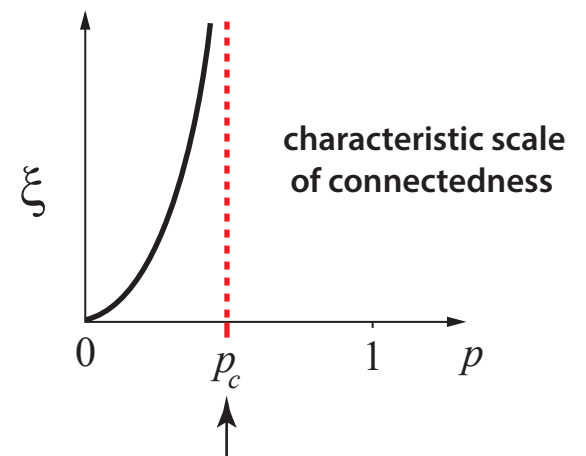
$\xi\,(p) \sim |p - p_c|^{-\nu} \qquad p \to p_c$

ν **universal: depends only on** *d*

percolation threshold

p_c **depends on type of lattice and** *d*

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

smallest *p* for which there is an infinite open cluster

transport in percolation theory

MICRO → **MACRO**

lattice homogenization

local conductivity (electrical or fluid)   **effective** conductivity or fluid permeability

insulator   **conductor**
off : **on**

bond → $\begin{cases} \sigma_0 & \text{probability } p \\ 0 & \text{probability } 1-p \end{cases}$
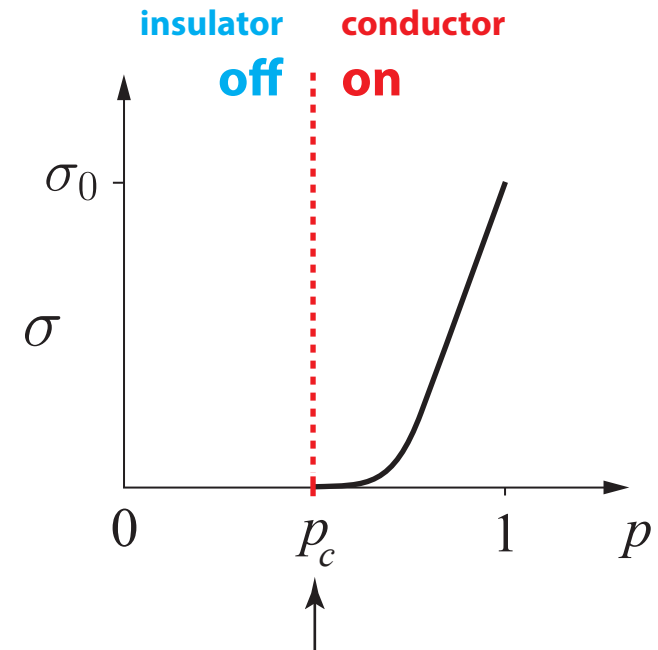
percolation threshold

consider local conductivities
1 and $h > 0$
smooths, softens transition

$$\sigma(p) \sim \sigma_0 (p - p_c)^t \quad p \to p_c^+$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

$1 \leq t \leq 2$ (for idealized model), Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992

non-universal behavior in continuum

Critical behavior of fluid transport in sea ice

impermeable   **permeable**

"on - off" switch
for bulk fluid flow

off   **on**

Arctic field data

vertical fluid permeability k (m^2)

Darcy's Law

$$\mathbf{v} = - \frac{\mathbf{k}}{\eta} \nabla p$$

k = fluid permeability tensor

homogenized parameter

brine volume fraction ϕ

ϕ_c

impermeable

permeable

$p = 1/3$ $p = 2/3$

lattice percolation
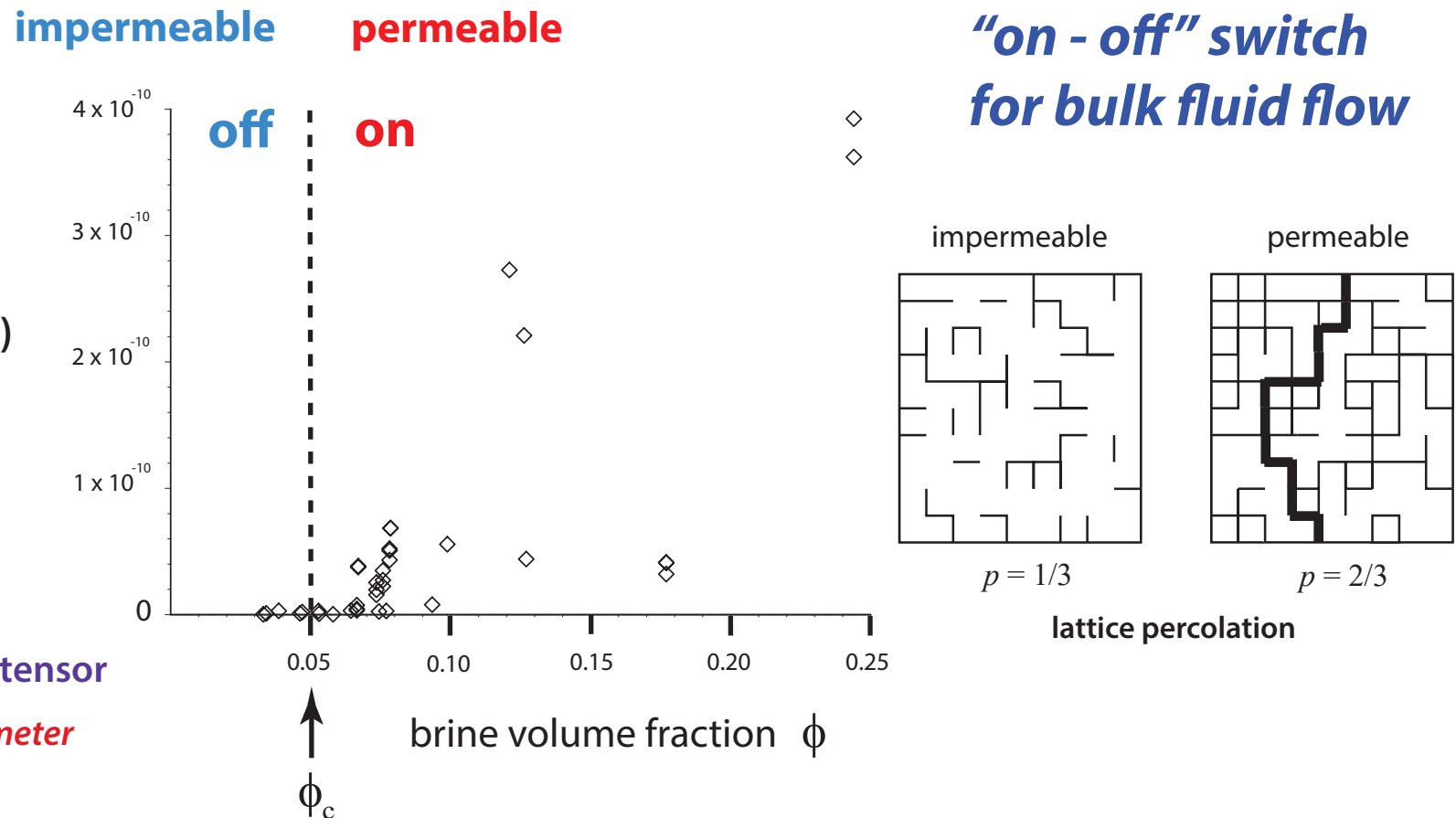
PERCOLATION THRESHOLD $\phi_c \approx 5\%$ $\longleftrightarrow$ $T_c \approx -5°C,\ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

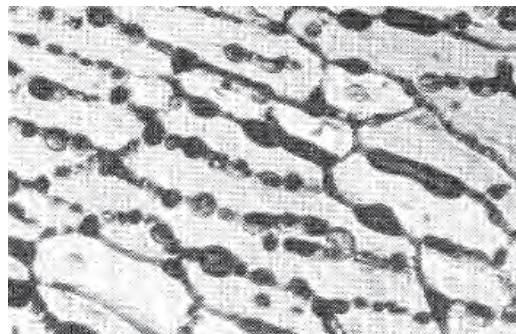*Continuum* percolation model for *stealthy* materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

$\phi_c \approx 5\%$ Golden, Ackley, Lytle, *Science,* 1998

R_m

R_p

B-2 Stealth Bomber
F-117 Nighthawk
Stealthy Helicopters
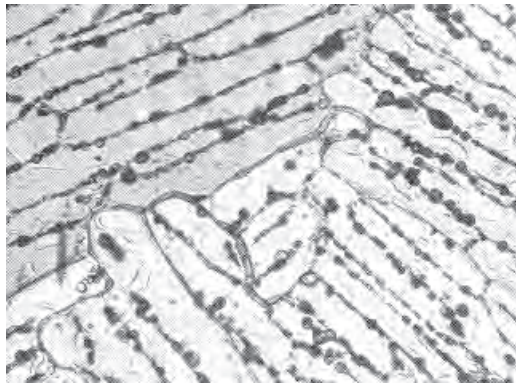
sea ice   **compressed powder**   **radar absorbing composite**
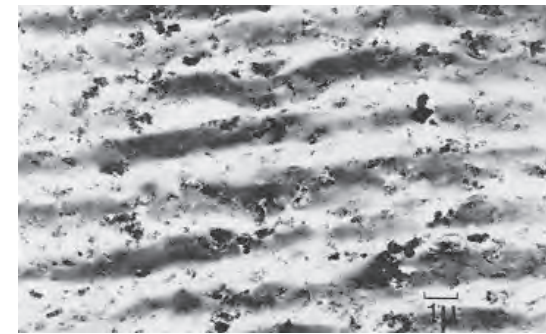
sea ice is radar absorbing

sea ice algal communities

D. Thomas 2004

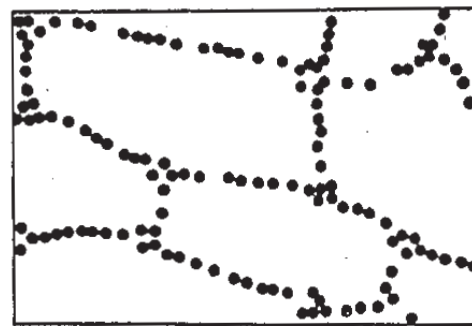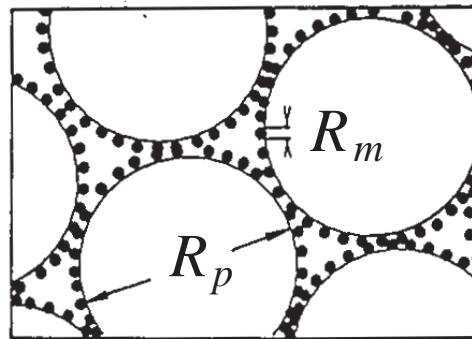nutrient replenishment controlled by ice permeability

biological activity turns on or off according to **rule of fives**

Golden, Ackley, Lytle   *Science 1998*

Fritsen, Lytle, Ackley, Sullivan   *Science 1994*

critical behavior of microbial activity

Convection-fueled algae bloom
Ice Station Weddell

sea ice ecosystem

sea ice algae
support life in the polar oceans

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu,   Geophysical Research Letters   2007

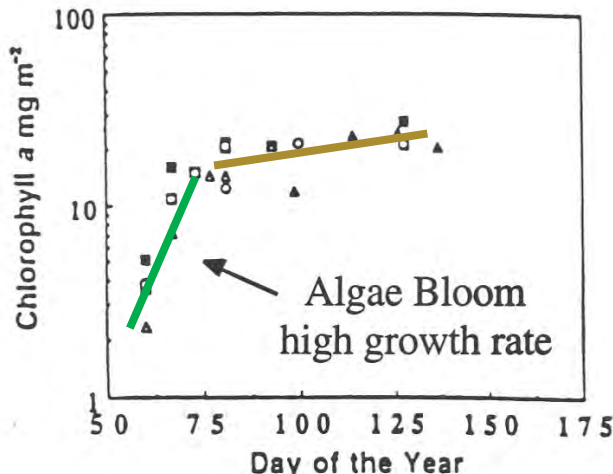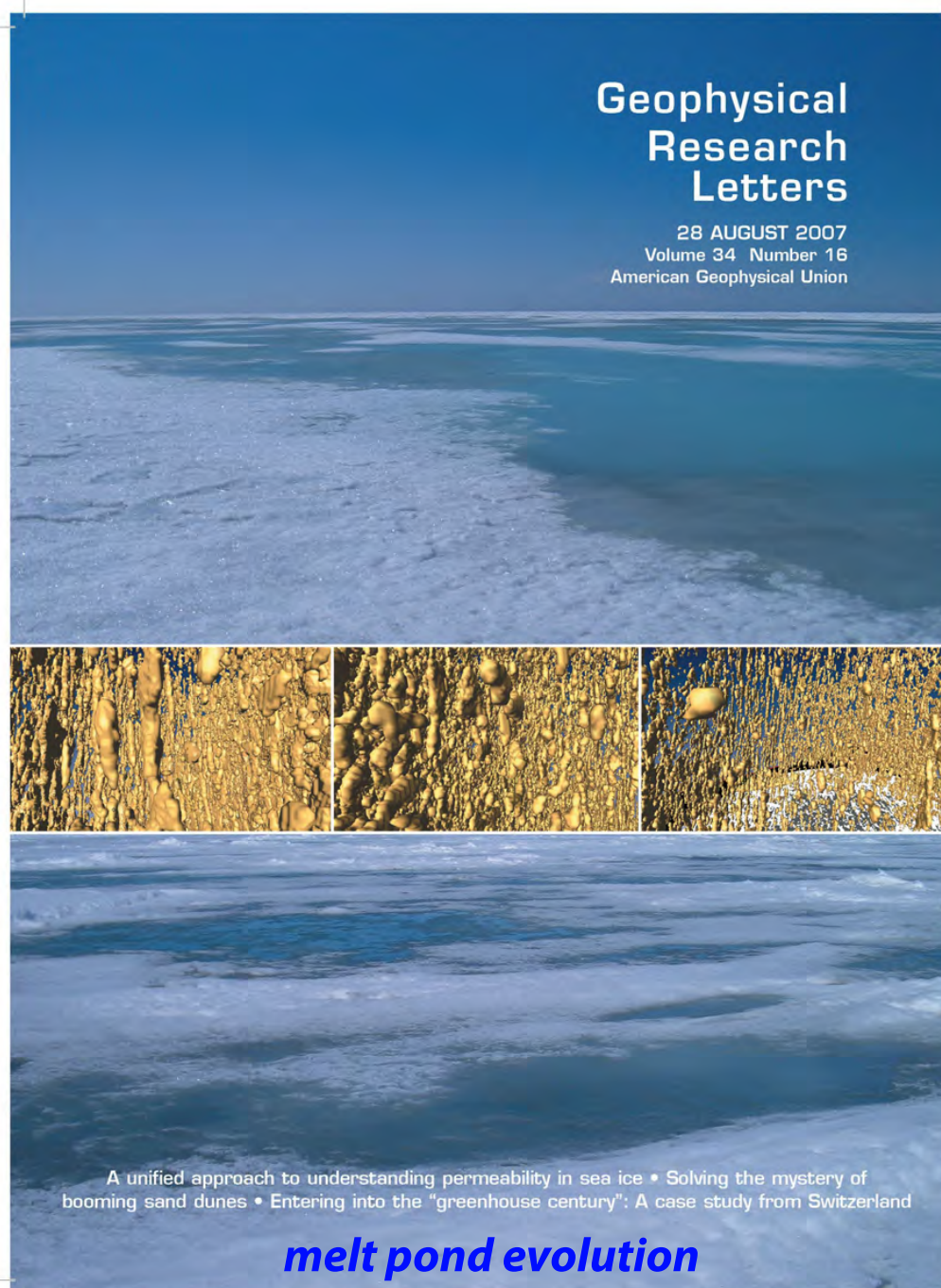

percolation theory
for fluid permeability

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent

t

$$k_0 = 3 \times 10^{-8} \; m^2$$

from critical path analysis
in hopping conduction

hierarchical model
rock physics
network model
rigorous bounds

X-ray tomography for
brine inclusions

confirms rule of fives

brine percolation threshold
of $\phi = 5\%$ for bulk fluid flow

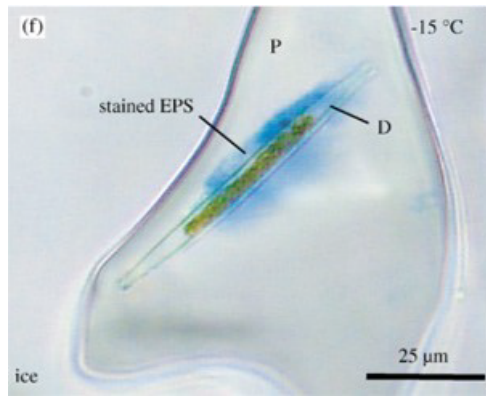*Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009*

theories agree closely
with field data

microscale

governs

mesoscale
processes

melt pond evolution

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

How does EPS affect fluid transport? How does the biology affect the physics?

Krembs
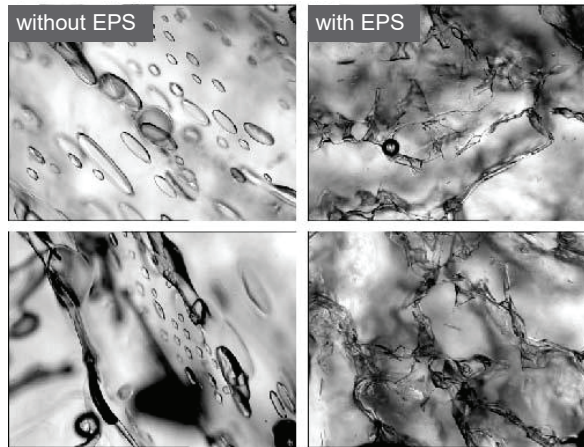
without EPS / with EPS
Krembs, Eicken, Deming, PNAS 2011
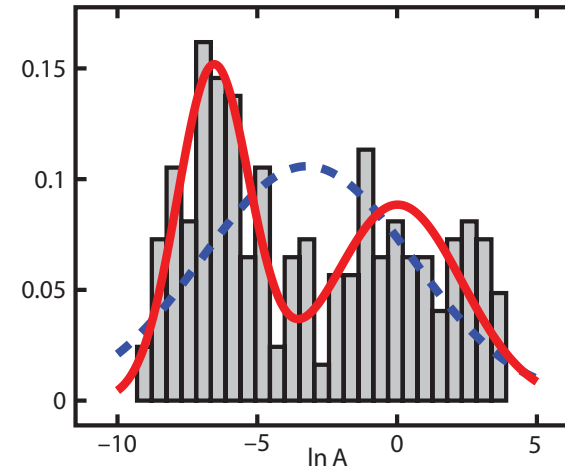
RANDOM PIPE MODEL
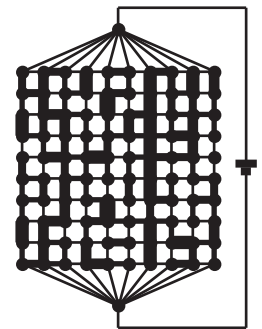
- **2D random pipe model with bimodal distribution of pipe radii**

- **Rigorous bound on permeability *k*; results predict observed drop in *k***
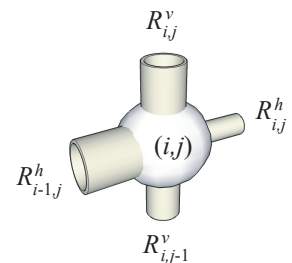
Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
Multiscale Modeling and Simulation, 2018

$R_{i,j}^v$
$R_{i,j}^h$
$R_{i-1,j}^h$
$R_{i,j-1}^v$
(i,j)

Zhu, Jabini, Golden,
Eicken, Morris
Ann. Glac. 2006
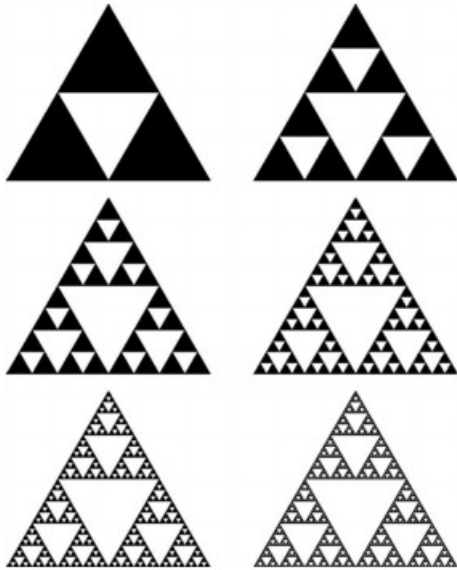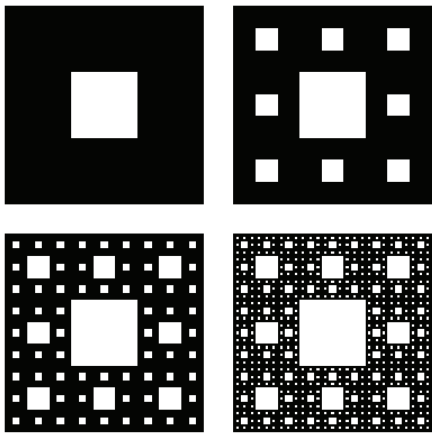
fractals

**self-similar structure
non-integer dimension**

$$D = \frac{\log 3}{\log 2} = 1.585...$$
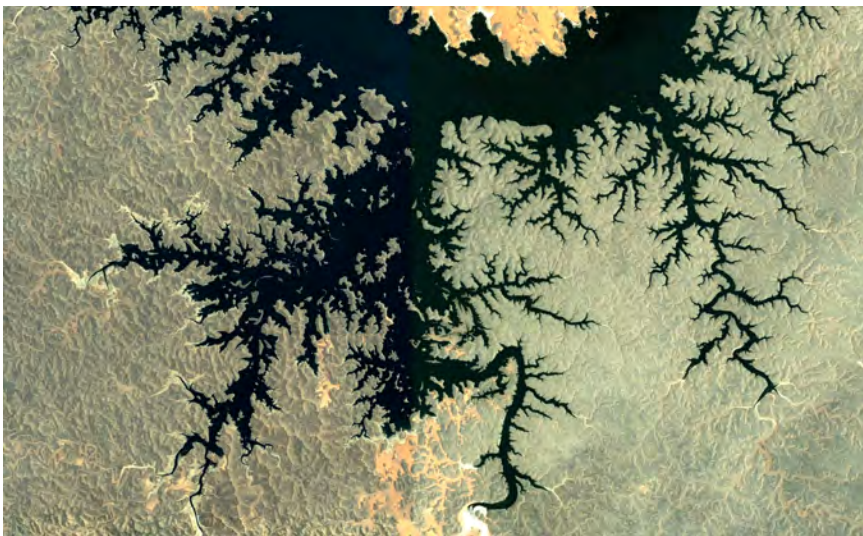
Thermal Evolution of Brine Fractal Geometry in Sea Ice

Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O'Sadnick, Elena Cherkaev and Kenneth Golden, 2023

Prokofiev
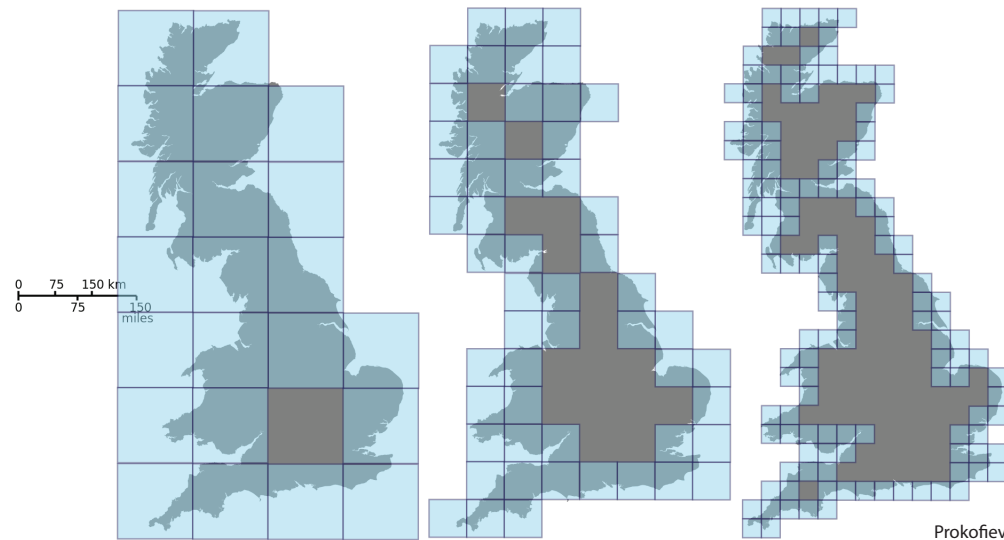
fractal dimension of the British coastline by box counting
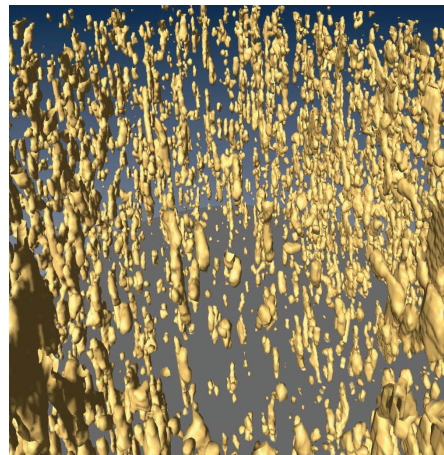
$T = -12°\,C, \;\; \phi = 0.033$ $T = -8°\,C, \;\; \phi = 0.057$
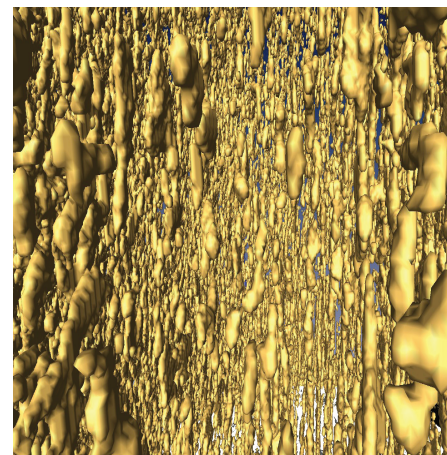
brine channels and inclusions "look" like fractals (from 30 yrs ago)

X-ray computed tomography of brine in sea ice

columnar and granular

Golden, Eicken, et al. *GRL*, 2007

The first comprehensive, quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.
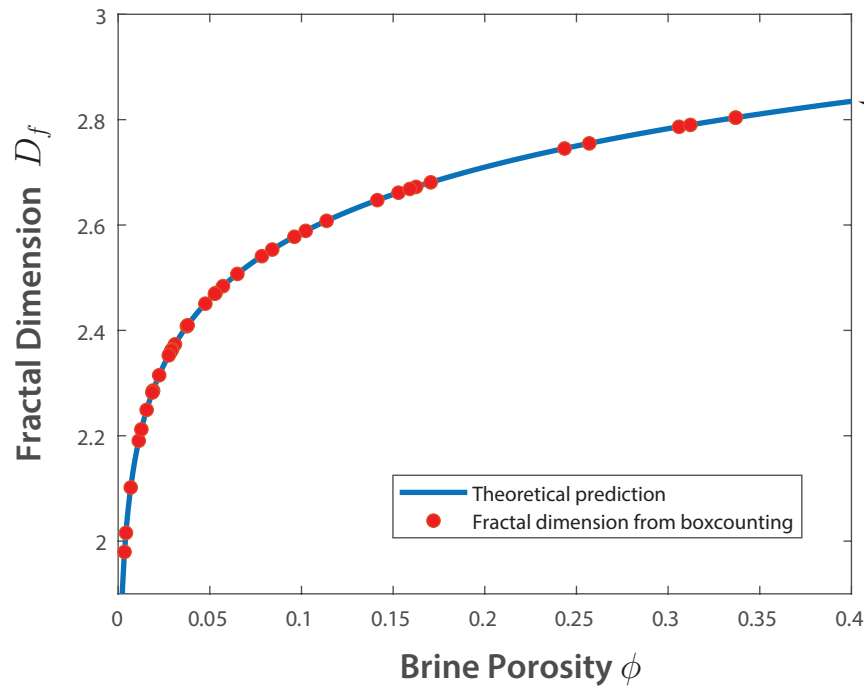
$$D_f = 3 - \frac{\ln \phi}{\ln(\lambda_{min}/\lambda_{max})}$$

The blue curve is exact for the Sierpinski gasket (an exactly self-similar geometry); discovered for sandstones - statistically self-similar porous media.

Katz and Thompson, 1985
Yu and Li, 2001
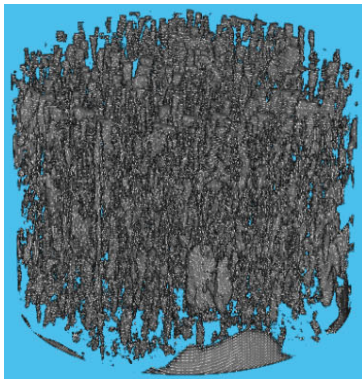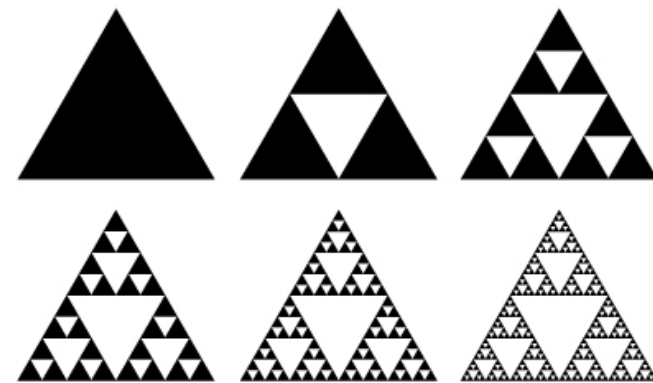
X-ray tomography
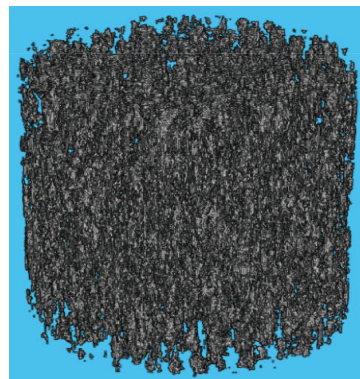
DLA model
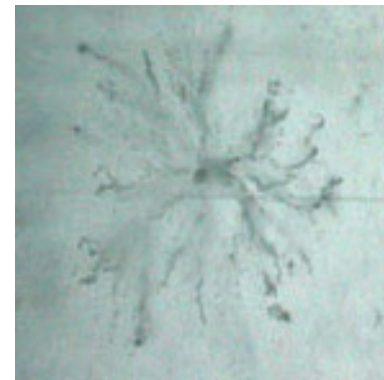
brine channel in sea ice
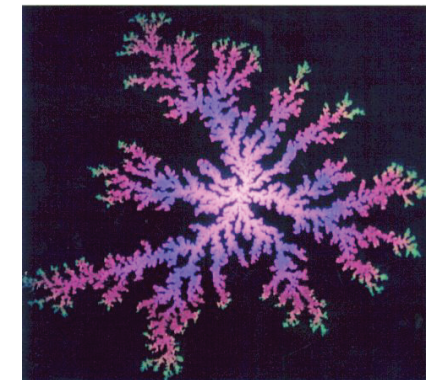
diffusion limited aggregation

Implications of brine fractal geometry on sea ice ecology and biogeochemistry

10 cm   1 cm   2 mm
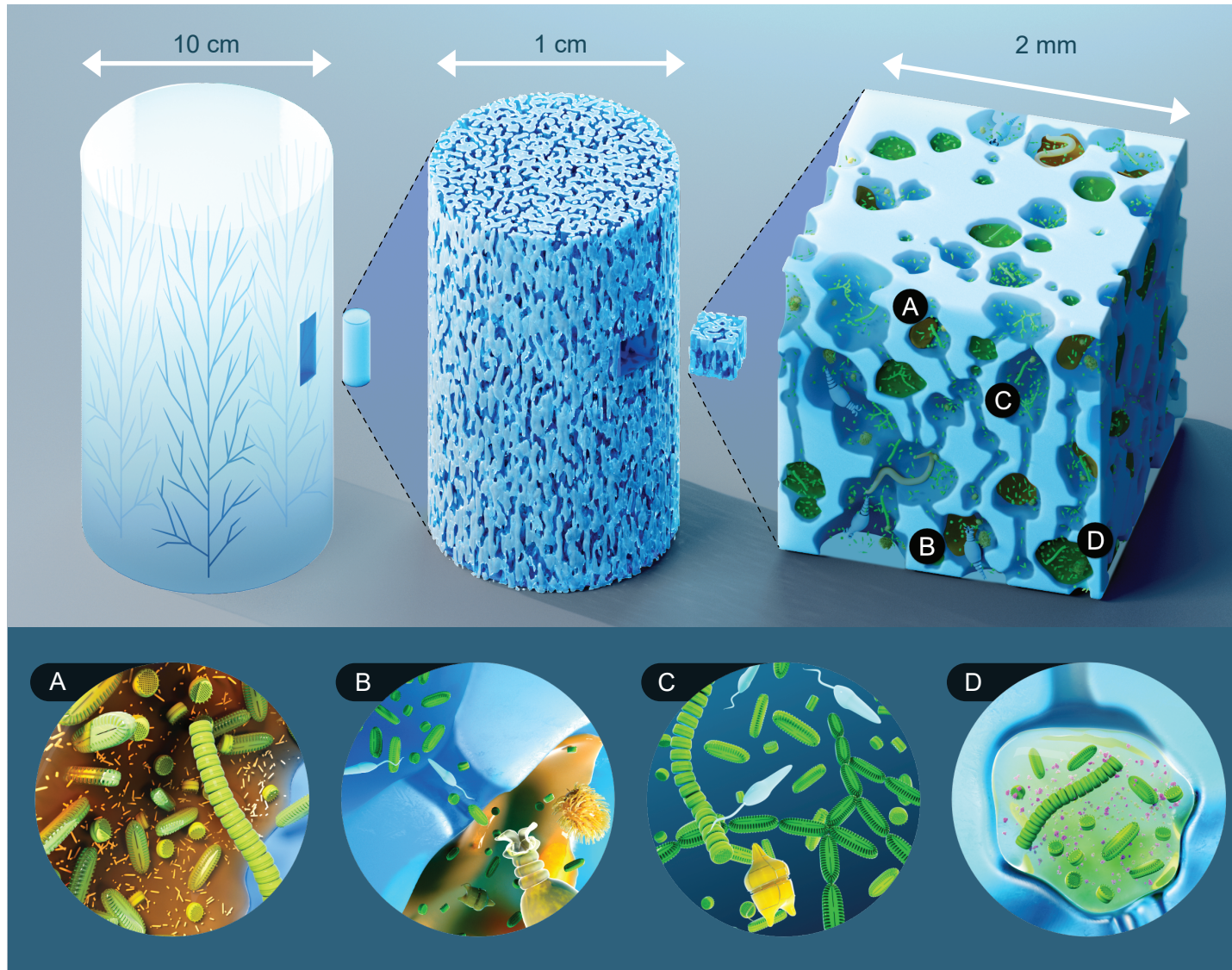
Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

(A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.

(B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).

(D) Ice algae secrete extracellular polymeric substances (EPS) which alter incusion geometry and may further increase the fractal dimension.

Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	*Antarctic*	*SIPEX*
2010	*Antarctic*	*McMurdo Sound*
2011	*Arctic*	*Barrow AK*
2012	*Arctic*	*Barrow AK*
2012	*Antarctic*	*SIPEX II*
2013	*Arctic*	*Barrow AK*
2014	*Arctic*	*Chukchi Sea*

Notices

of the American Mathematical Society

Climate Change and
the Mathematics of
Transport in Sea Ice

page 562

Mathematics and the
Internet: A Source of
Enormous Confusion
and Great Potential

page 586

photo by Jan Lieser

Real analysis in polar coordinates (see page 613)
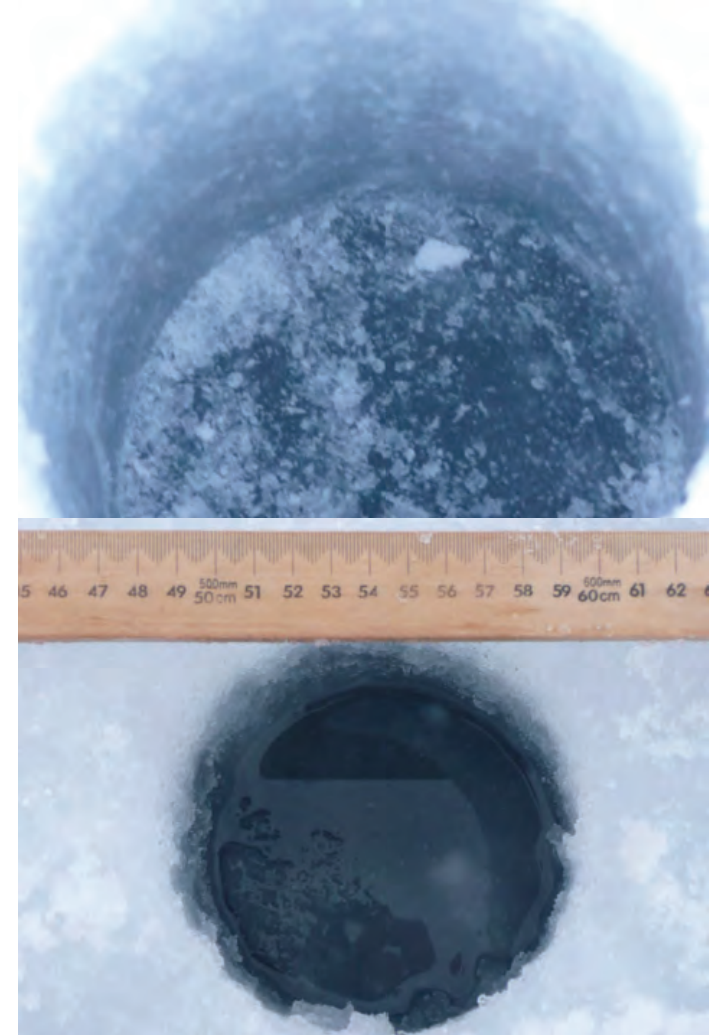
*measuring
fluid permeability
of Antarctic sea ice*

SIPEX 2007

Golden

Remote sensing of sea ice

sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

$$\varepsilon^*$$

effective complex permittivity (dielectric constant, conductivity)
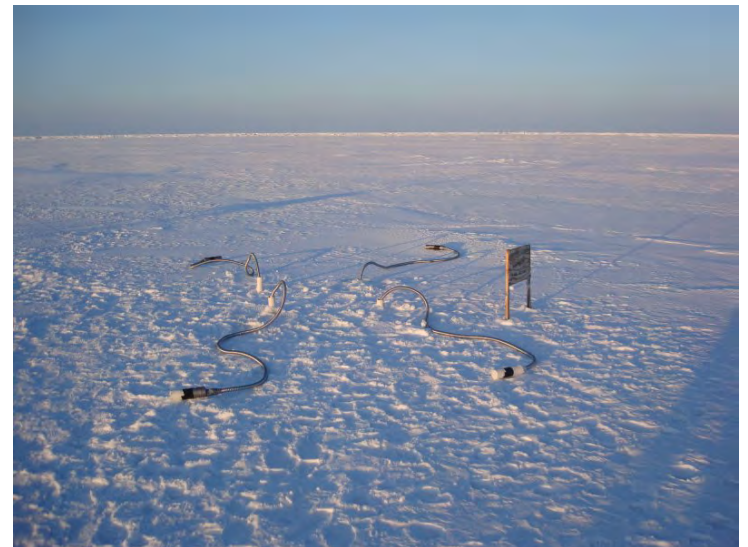
brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit
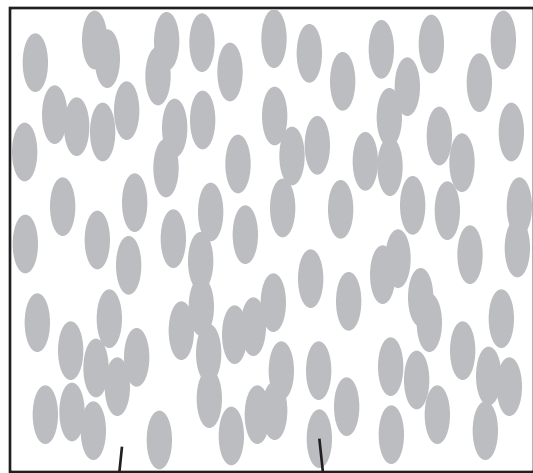
ϵ_1 ϵ_2

$\left. \right\} \epsilon^*$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

$p_1 , p_2 =$ volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} , \text{composite geometry} \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1/\epsilon_2}$$

material parameters

μ
- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi = $ characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1}e_k$$

$\Gamma\chi$: **microscale** $\longrightarrow$ **macroscale**

$\Gamma\chi$ *links scales*

Golden and Papanicolaou, Comm. Math. Phys. 1983

complexities of mixture geometry

Particles   Short fibers or whiskers   Continues fibers   Sheet laminates   Bartleby.com

distilled   distilled

p=0.05

2D lattice
spectral
measure μ

gap

0   1
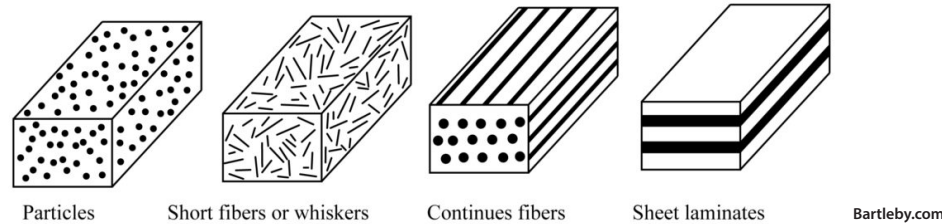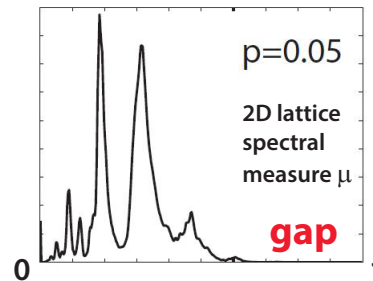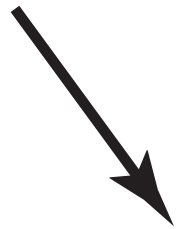
spectrum of an operator like the Hamiltonian in quantum physics

EXTEND Stieltjes / Spectral Framework

polycrystals
advection diffusion
waves through ice pack

forward and inverse bounds on the complex permittivity of sea ice

forward bounds

$\text{Im}(\varepsilon^*)$

$T = -14\ ^\circ\text{C}$
$\phi = 0.015$
$q = 0.97$

0.5

$\text{Re}(\varepsilon^*)$

0

3   3.5

4.75 GHz data

matrix particle

r_i

r_b

$q = r_b / r_i$

$0 < q < 1$

Golden 1995, 1997

inverse bounds

Computed minimum separation parameter q

q_{min}

Slab temperature $^\circ\text{C}$

Inverse Homogenization
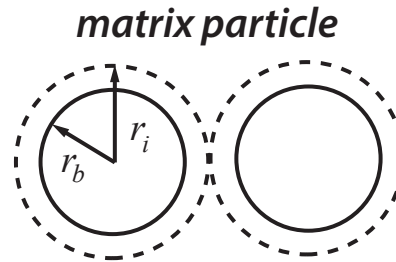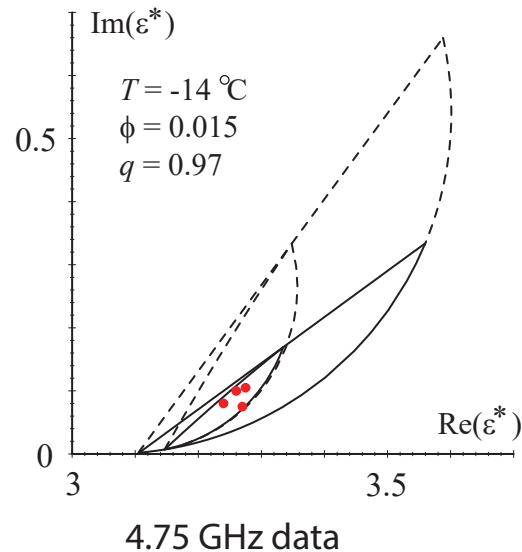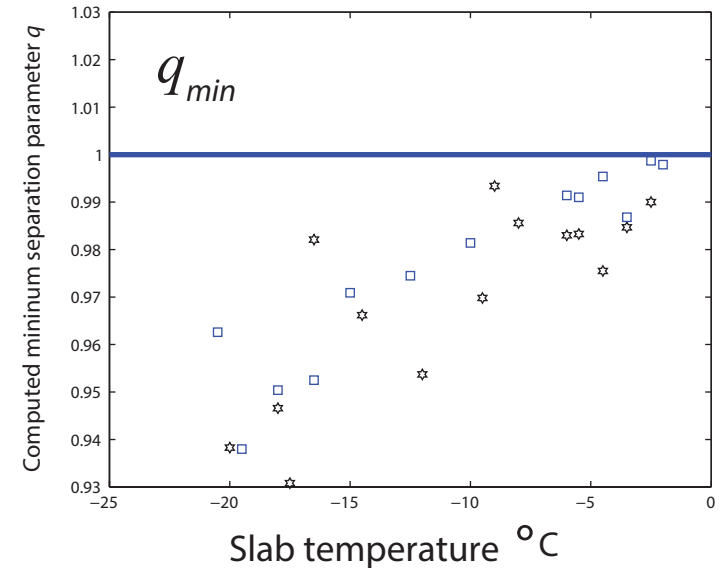
Cherkaev and Golden (1998), Day and Thorpe (1999),
Cherkaev (2001), McPhedran, McKenzie, Milton (1982),
Theory of Composites, Milton (2002)

$\varepsilon^* \longrightarrow$ **composite geometry**
(spectral measure μ)

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
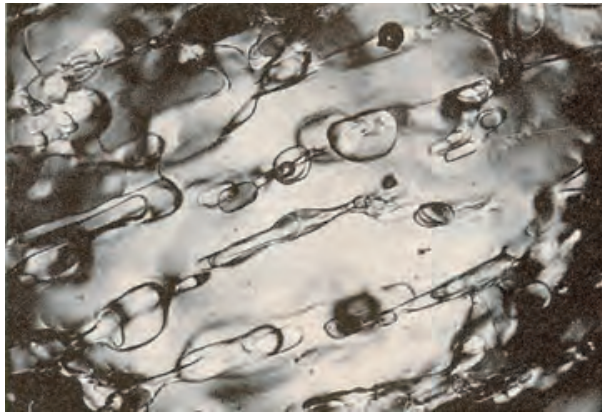***Physica B, 2007***

inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

**rigorous inverse bound
on spectral gap**

*construct algebraic curves which bound
admissible region in (p,q)-space*

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

SEA ICE

HUMAN BONE
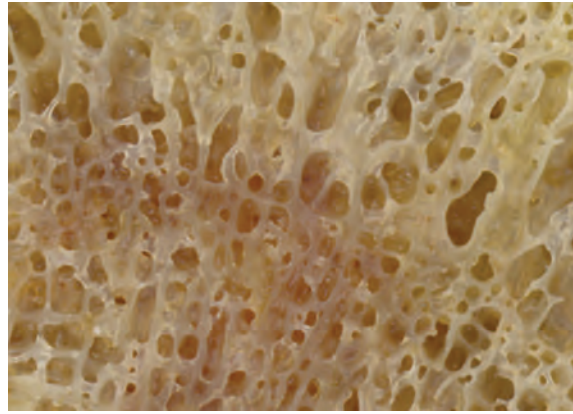
spectral characterization of porous microstructures in human bone

young healthy trabecular bone
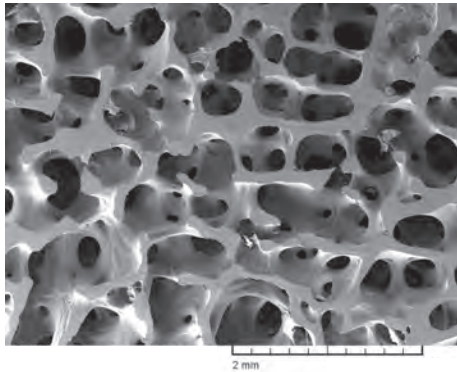
old osteoporotic trabecular bone
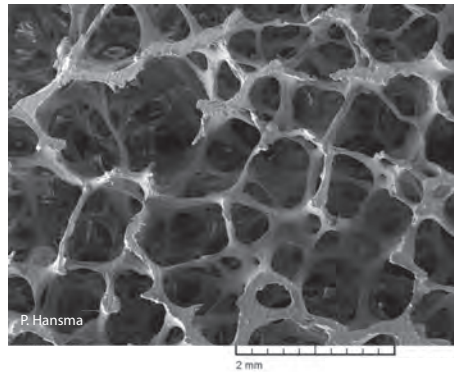
reconstruct spectral measures from complex permittivity data
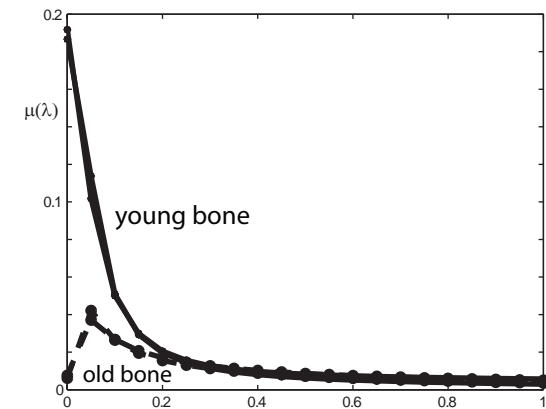
use regularized inversion scheme

apply spectral measure analysis of brine connectivity and spectral inversion to electromagnetic monitoring of osteoporosis

Golden, Murphy, Cherkaev, J. Biomechanics 2011

the math doesn't care if it's sea ice or bone!

Homogenization for polycrystalline materials

Two-component composites

Local conductivity

σ_1 σ_2

Homogenize

σ^* *effective conductivity*

inhomogeneous medium

homogeneous medium
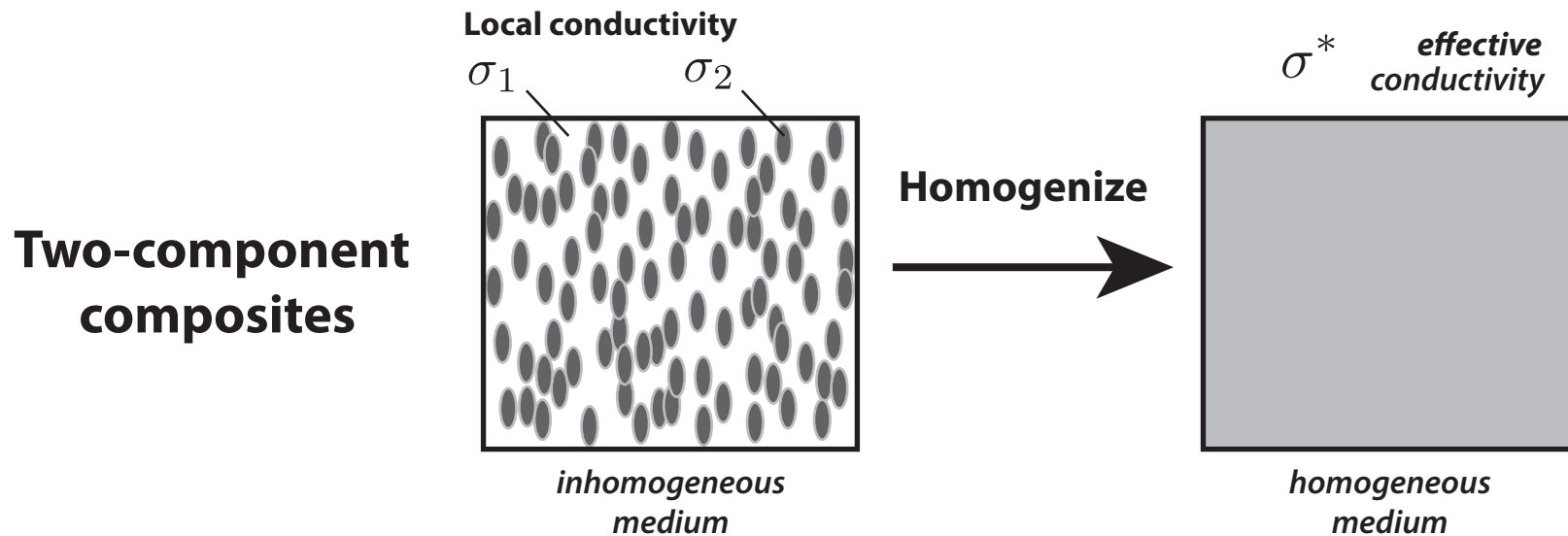
Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Polycrystalline media

Conductivity of crystal directions

σ_1 σ_2

Homogenize

σ^* *effective conductivity*

inhomogeneous medium

homogeneous medium
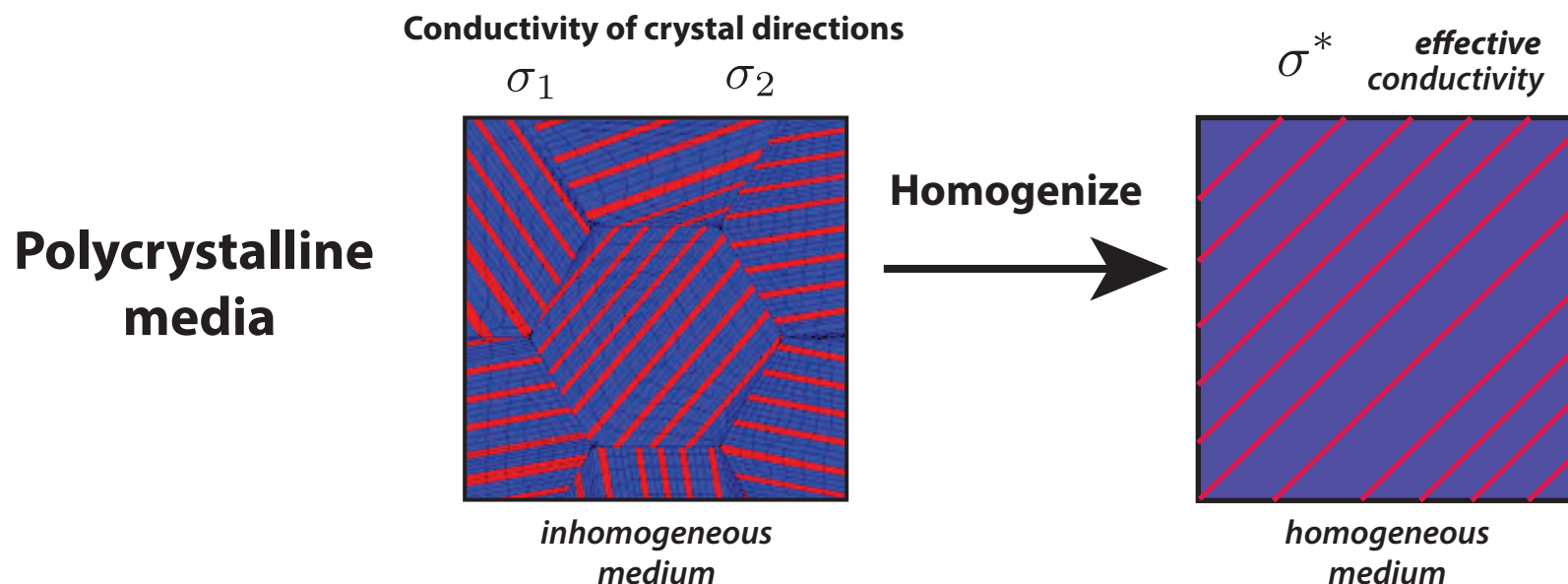
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin, Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...

- **Forward and inverse bounds**

  orientation statistics

- **Applied to sea ice using two-scale homogenization**

- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**
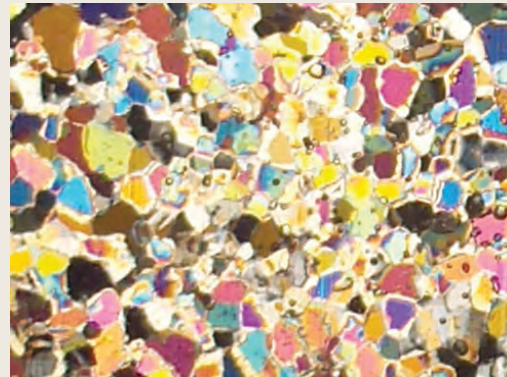
PROCEEDINGS A

350 YEARS OF SCIENTIFIC PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

THE ROYAL SOCIETY PUBLISHING

21/01/15 1:47 PM

higher threshold for fluid flow in granular sea ice
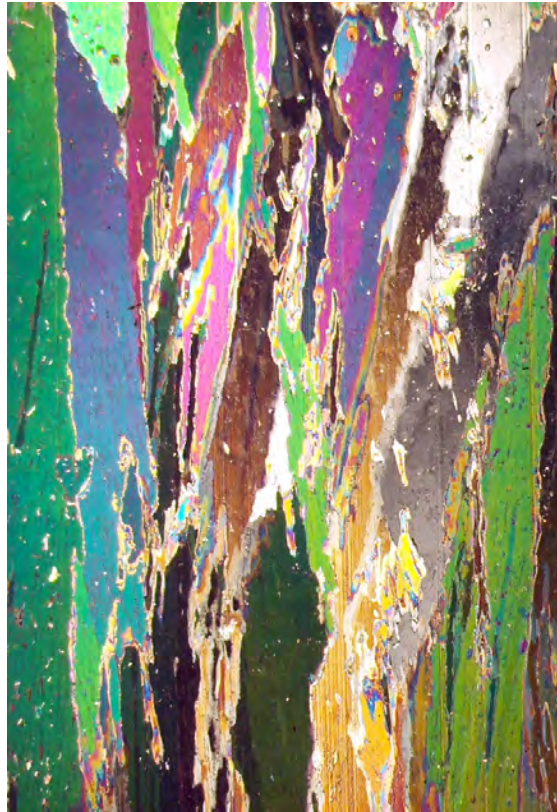
microscale details impact "mesoscale" processes

nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar   granular

5%   10%

Golden, Sampson, Gully, Lubbers, Tison 2023
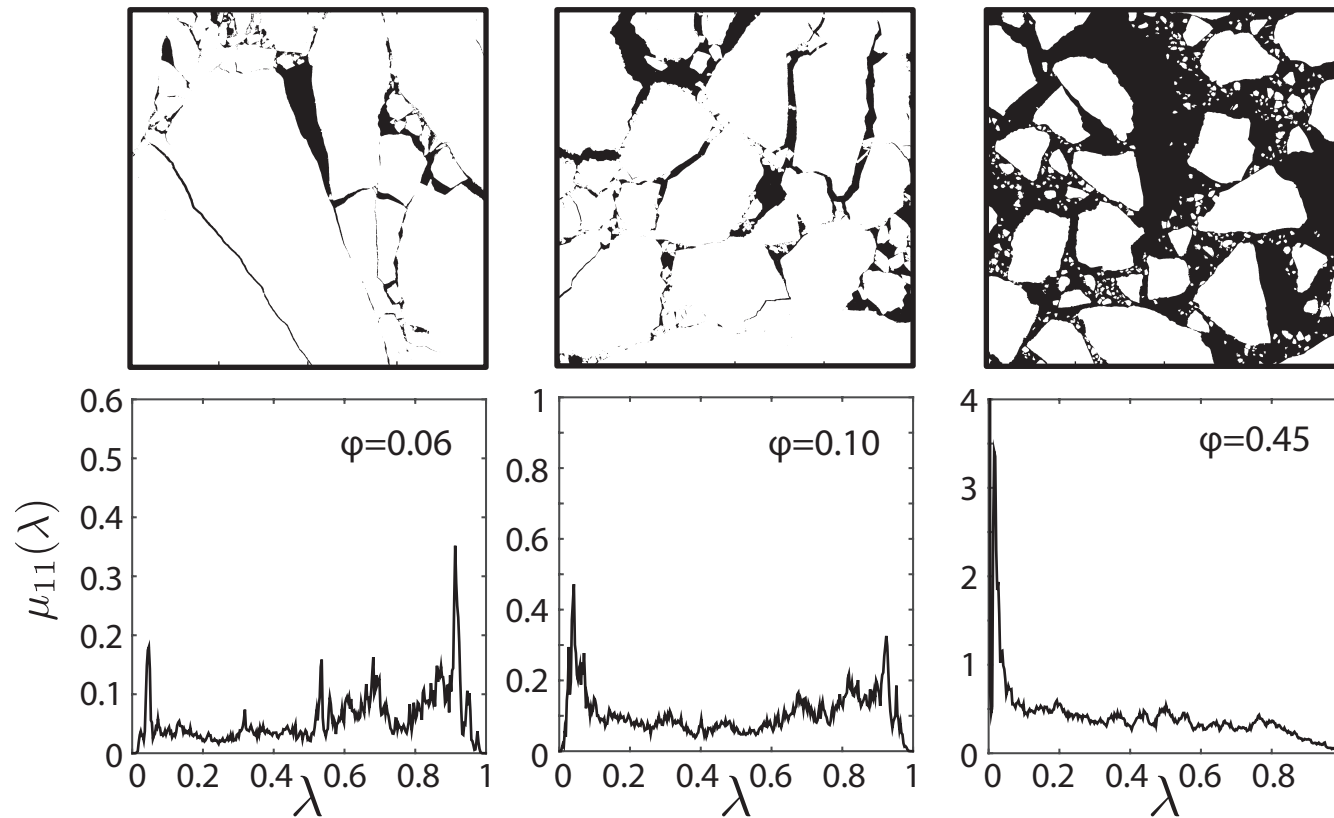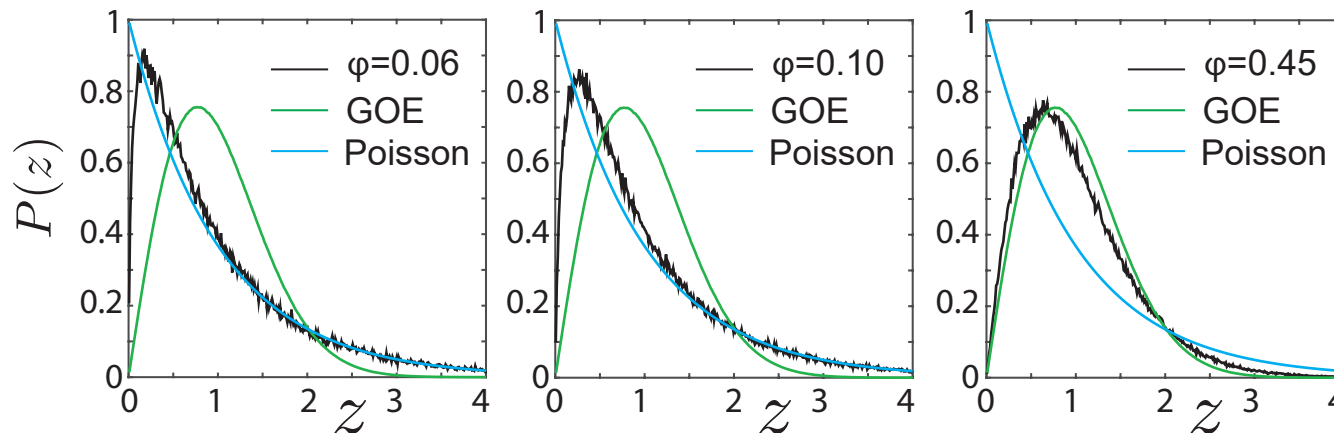
electromagnetically distinguishing ice types
Kitsel Lusted, Elena Cherkaev, Ken Golden

Spectral computations for sea ice floe configurations

spectral measures

eigenvalue spacing distributions

$\varphi=0.06$

$\varphi=0.10$

$\varphi=0.45$

$\mu_{11}(\lambda)$

$P(z)$

GOE

Poisson

UNIVERSAL Wigner-Dyson distribution

uncorrelated → **level repulsion**

Eigenvalue Statistics of Random Matrix Theory

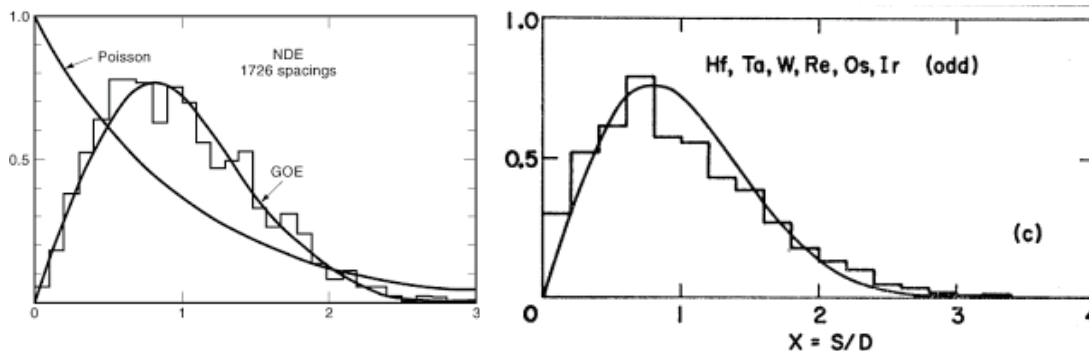*Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.*

$[\mathbf{N}]_{ij} \sim N(0,1),$ $\qquad$ $\mathbf{A} = (\mathbf{N} + \mathbf{N}^{T})/2$ $\qquad$ *Gaussian orthogonal ensemble (GOE)*

$[\mathbf{N}]_{ij} \sim N(0,1) + iN(0,1),$ $\quad$ $\mathbf{A} = (\mathbf{N} + \mathbf{N}^{\dagger})/2$ $\qquad$ *Gaussian unitary ensemble (GUE)*

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.
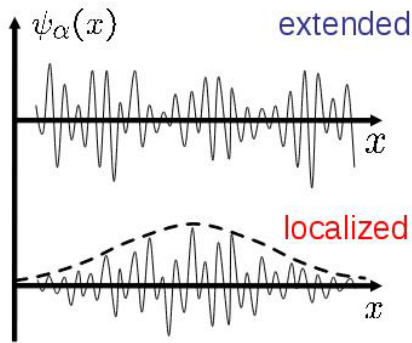
Spacing distributions of the first billion zeros of the Riemann zeta function

Spacing distributions of energy levels for heavy atomic nuclei
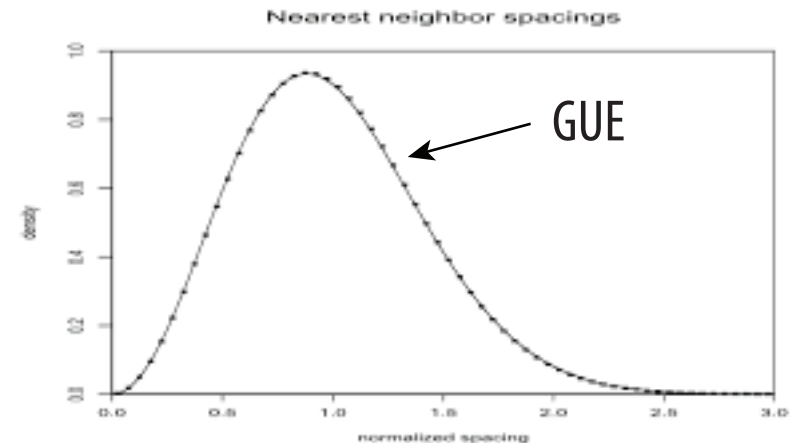
Universal eigenvalue statistics arise in a broad range of "unrelated" problems!

electronic transport in semiconductors

metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

from analysis of spectral measures for brine, melt ponds, ice floes

we find percolation-driven

Anderson transition for classical transport in composites
Murphy, Cherkaev, Golden   Phys. Rev. Lett.   2017
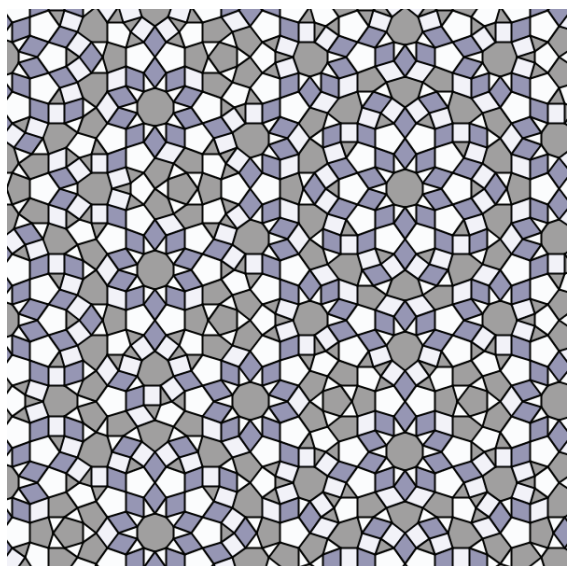
PERCOLATION
TRANSITION   →   universal eigenvalue statistics (GOE)
extended states, mobility edges

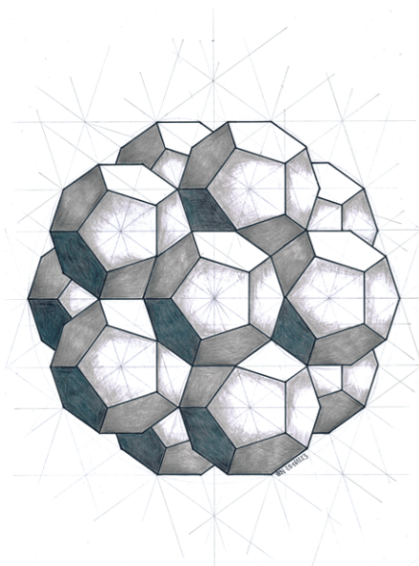-- but with NO wave interference or scattering effects ! --

Order to Disorder in Quasiperiodic Composites

D. Morison (Physics), N. B. Murphy, E. Cherkaev, K. M. Golden, *Communications Physics* 2022
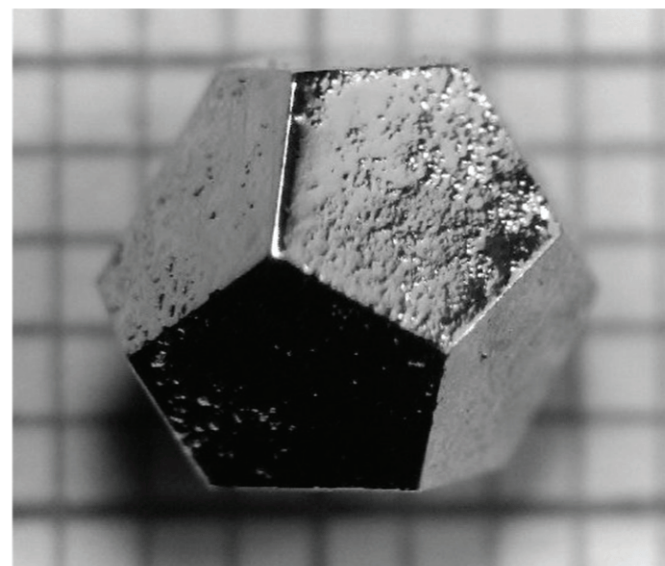

quasiperiodic checkerboard
Stampfli, 2013
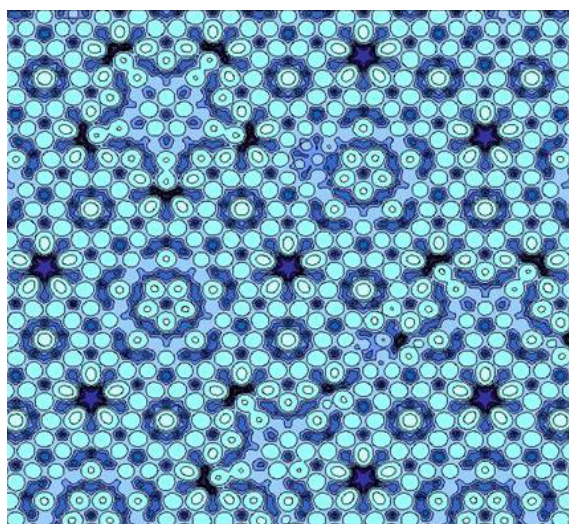

dense packing of dodecahedra
3D Penrose tiling   Tripkovic, 2019

Holmium−magnesium−zinc quasicrystal

energy surface Al-Pd-Mn quasicrystal
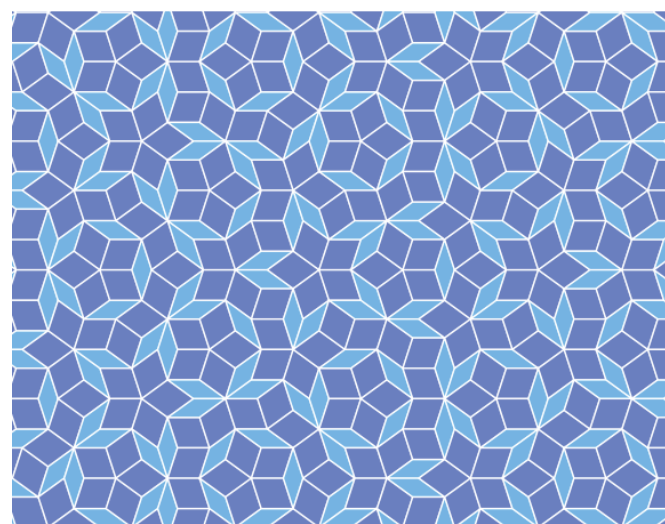Unal et al., 2007

quasiperiodic crystal
quasicrystal

ordered but aperiodic

lacks translational symmetry

Schechtman et al., 1984
Levine & Steinhardt, 1984

aperiodic tiling of the plane - R. Penrose 1970s

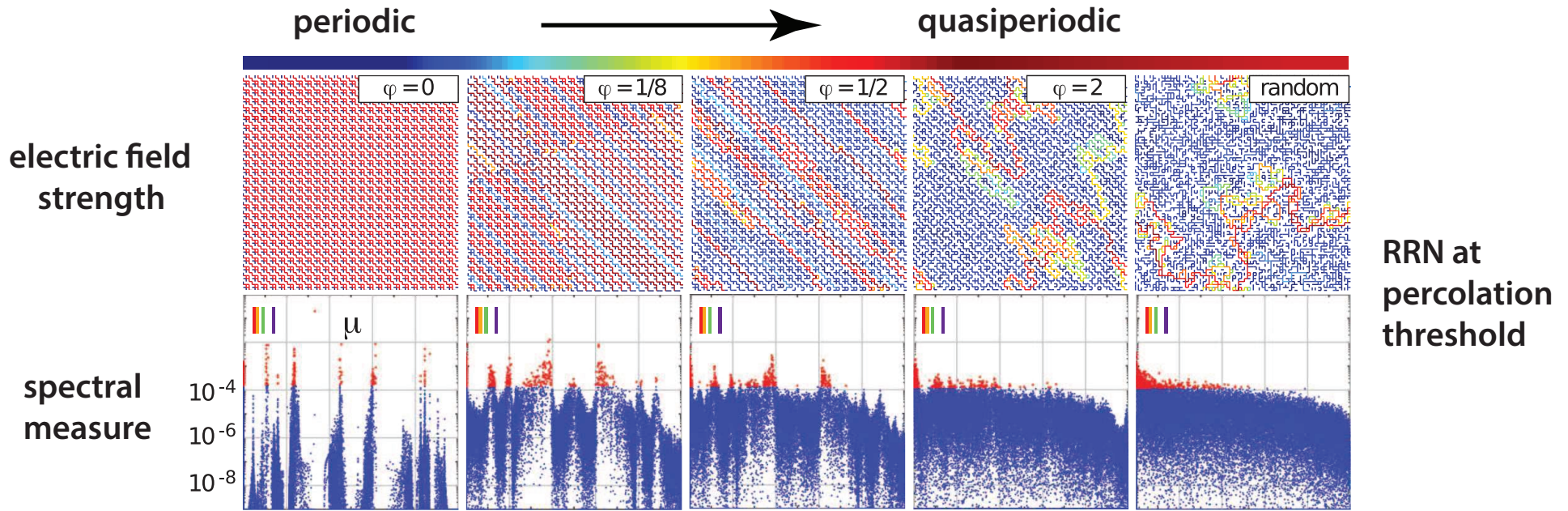# Order to disorder in quasiperiodic composites

Morison, Murphy, Cherkaev, Golden, Comm. Phys. 2022

sea ice inspired - high tech spin off

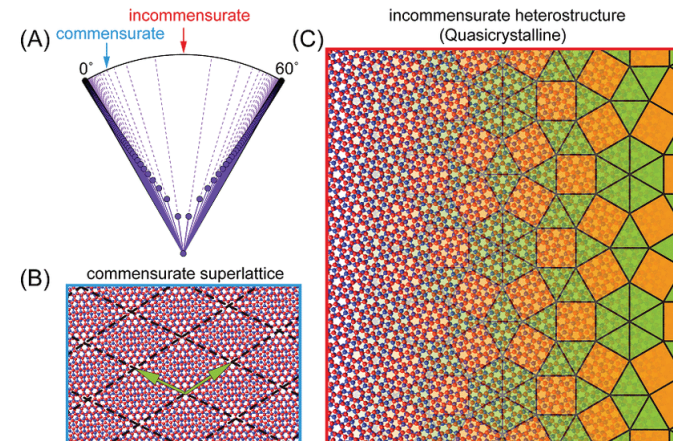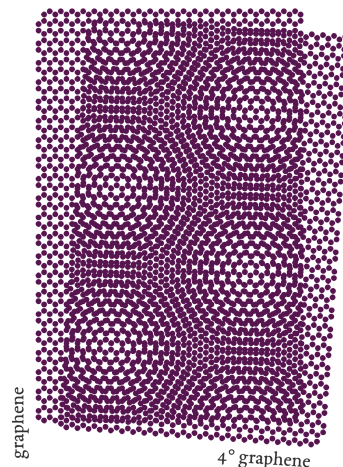**tunable quasiperiodic composites with exotic properties**

(optical, electrical, thermal, ...), Anderson localization; our Moiré patterned geometries are similar to twisted bilayer graphene

increasing twist angle between two lattices

periodic ⟶ **quasiperiodic**

electric field strength

$\varphi = 0$ $\varphi = 1/8$ $\varphi = 1/2$ $\varphi = 2$ random

RRN at percolation threshold

spectral measure

μ

10^{-4}
10^{-6}
10^{-8}

twisted bilayer graphene

superconducting magic twist angle

graphene

4° graphene

(A) commensurate   incommensurate   (C) incommensurate heterostructure (Quasicrystalline)

0°   60°

(B) commensurate superlattice

Yao et al., 2018

constellation of periodic systems in a sea of randomness

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Order to disorder in quasiperiodic composites

David Morison, N. Benjamin Murphy ... Kenneth M. Golden
Article | 14 June 2022
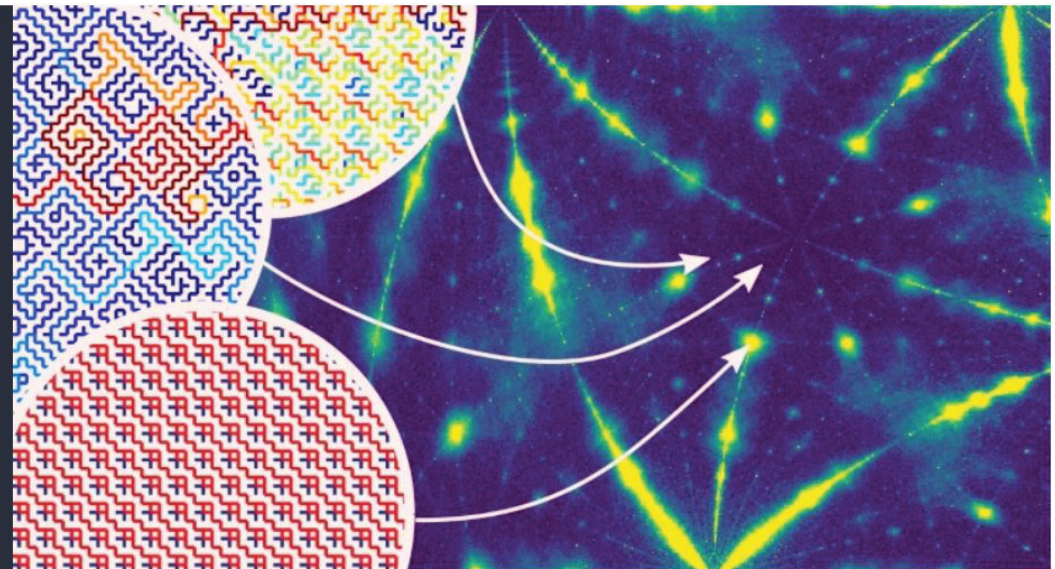
Featured

Article
10 Jan 2023

Versatile tuning of Kerr soliton microcombs in crystalline microresonators

High-repetition rate microresonator-based frequency combs offer powerful and compact optical frequency comb sources that are of great importance to various applications. Here, the authors extend the tunability of the Kerr soliton frequency combs by exploiting thermal effects and frequency stabilization techniques.

Shun Fujii, Koshiro Wada ... Takasumi Tanabe
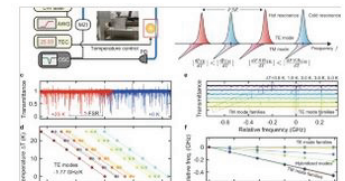
Article
12 Jan 2023

Compliant mechanical response of the ultrafast folding protein EnHD under force

Exhibiting low-energy (un)folding barriers and fast kinetics, ultrafast folding proteins are enticing models to study protein dynamics. The authors use single molecule force spectroscopy AFM to capture the compliant behaviour hallmarking the dynamics of ultrafast folding proteins under force.

Antonio Reifs, Irene Ruiz Ortiz ... Raul Perez-Jimenez
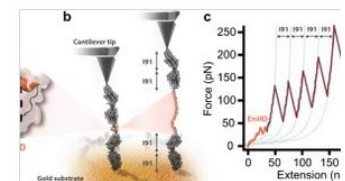
mesoscale

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere

advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

homogenize

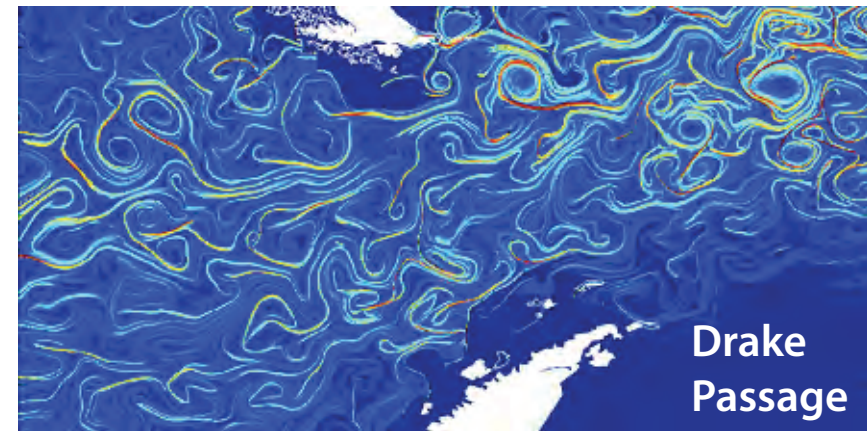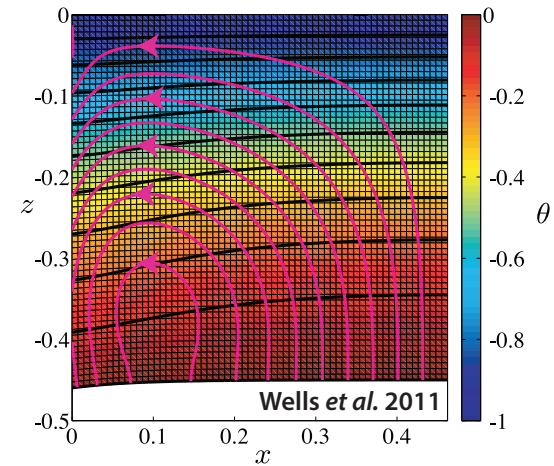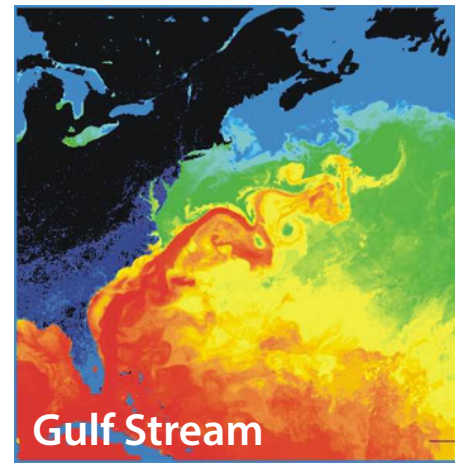$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* **effective diffusivity**

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
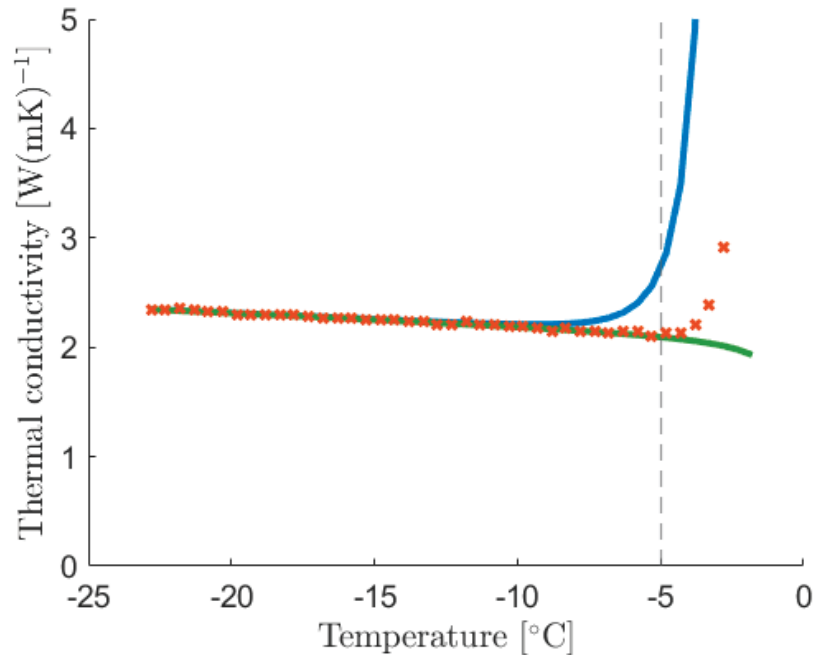Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

Gulf Stream

Wells *et al.* 2011

Drake Passage

sea ice vortices

Masters, 1989

tracers flowing through inverted sea ice blocks

Bounds on Convection Enhanced Thermal Transport

simulations

data [Trodahl et al., 2001]
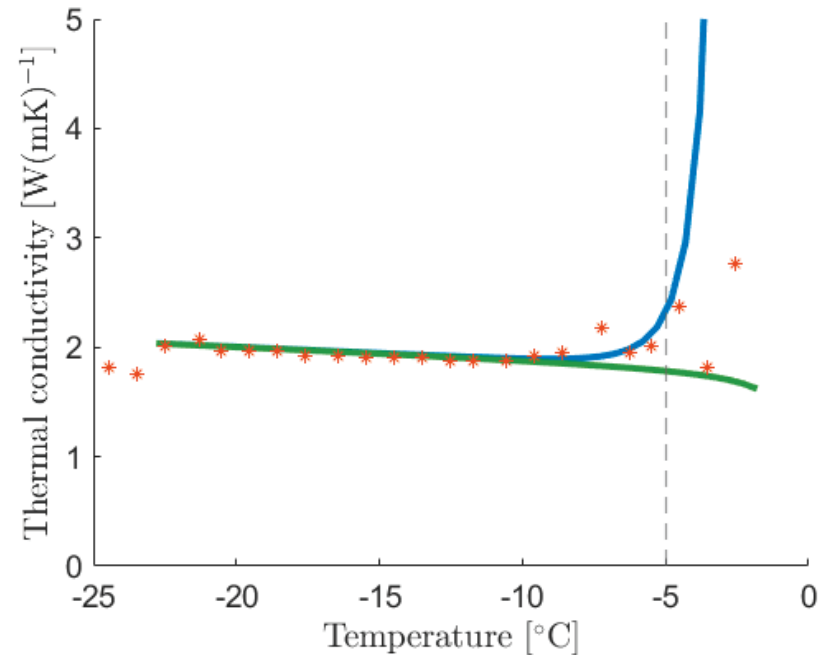

Monte-Carlo simulations of SDE with temperature dependent Péclet number P

strength of advection B = κP/2π

Euler-Maruyama and subsampling methods for SDE
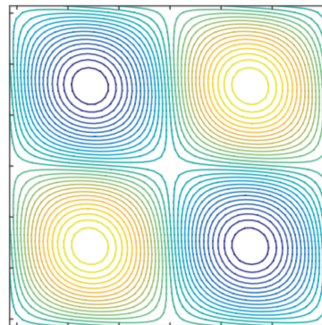
cat's eye flow model for brine convective flow

Rigorous Padé approximant bounds in terms of P using Stieltjes integral + analytic continuation method for the measure

Darcy velocity v = 0.5 [m/s]

Kraitzman, Hardenbrook, Dinh, Murphy, Cherkaev, Zhu, & Golden, 2023

wave propagation in the marginal ice zone (MIZ)

Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Cherkaev, Golden 2023

first theory of key parameter in wave-ice interactions only fitted to wave data before

Keller, 1998
Mosig, Montiel, Squire, 2015
Wang, Shen, 2012

Analytic Continuation Method
Bergman (78) - Milton (79)
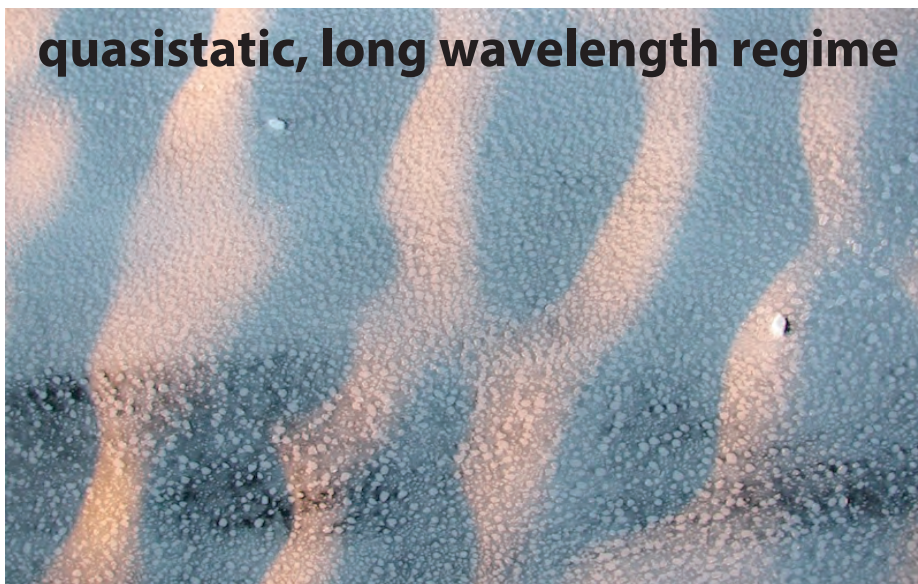**integral representation for ε^***
Golden and Papanicolaou (83)

Milton, *Theory of Composites* (02)

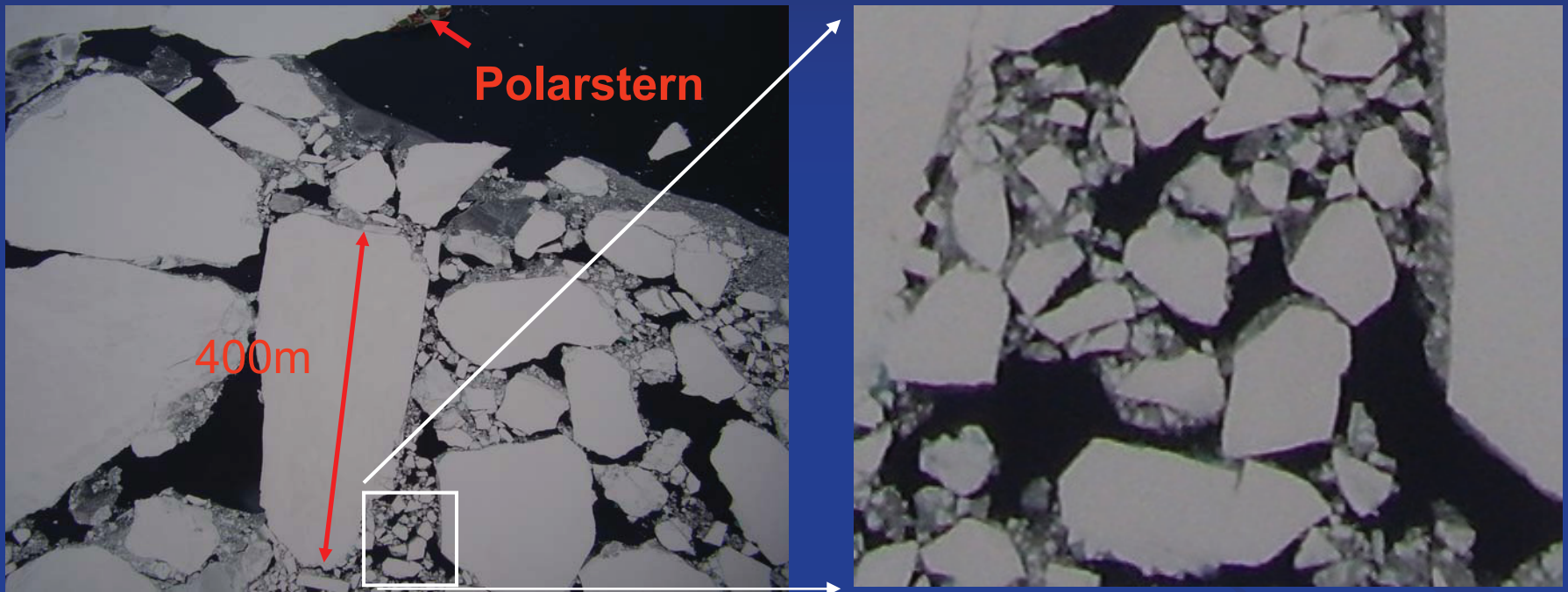**quasistatic, long wavelength regime**

homogenized parameter depends on sea ice concentration and ice floe geometry

like EM waves

fractal sea ice cover

Self-similarity of sea ice pack

Polarstern

400m

Weddell Sea, Antarctica
2006

spectral measures!

Takenobu Toyota

melt pond formation and albedo evolution:

- **major drivers in polar climate**
- **key challenge for global climate models**

numerical models of melt pond evolution, including topography, drainage (permeability), etc.
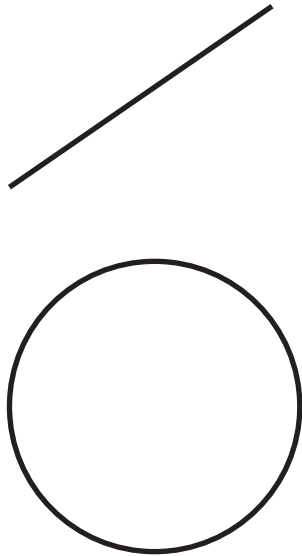
Lüthje, Feltham, Taylor, Worster 2006

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham 2007

Flocco, Feltham, Hunke 2012

Perovich
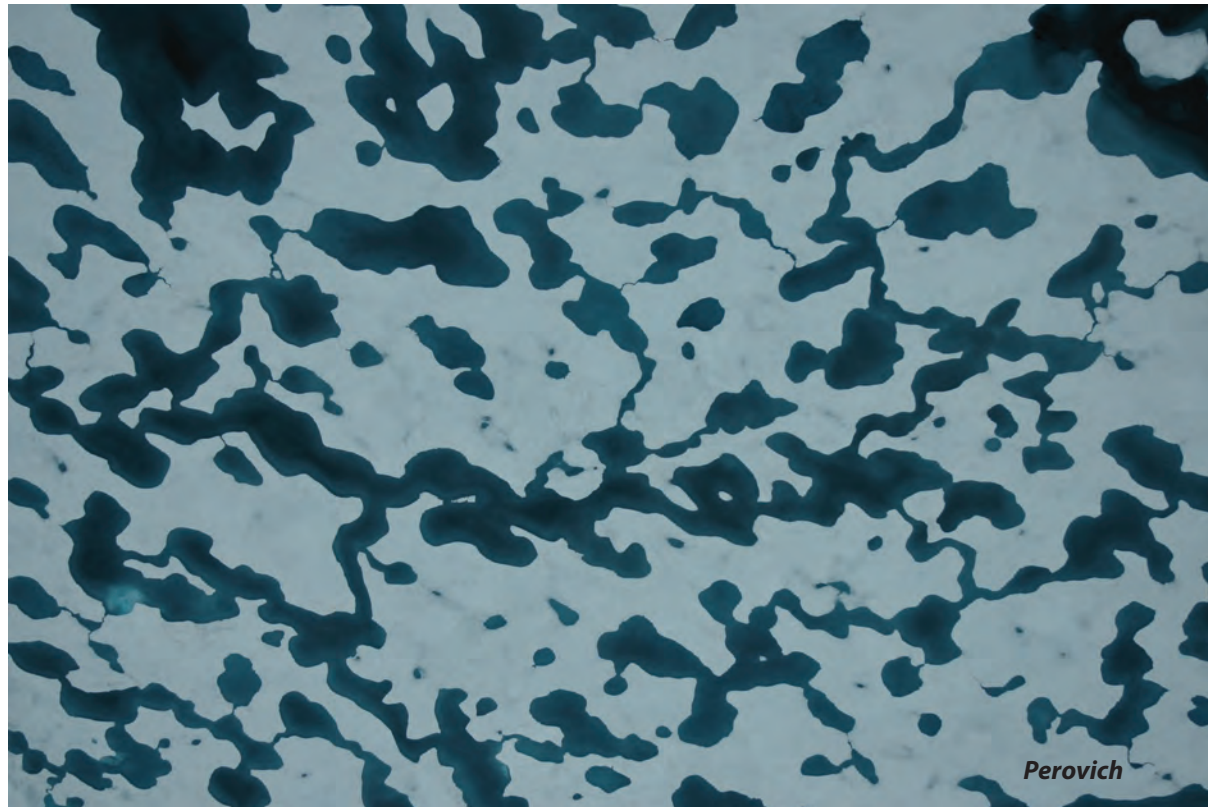
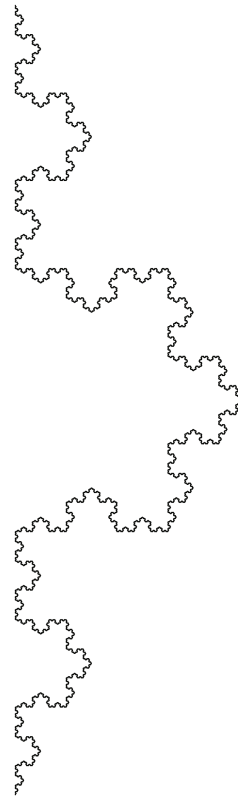**Are there universal features of the evolution similar to phase transitions in statistical physics?**

fractal curves in the plane
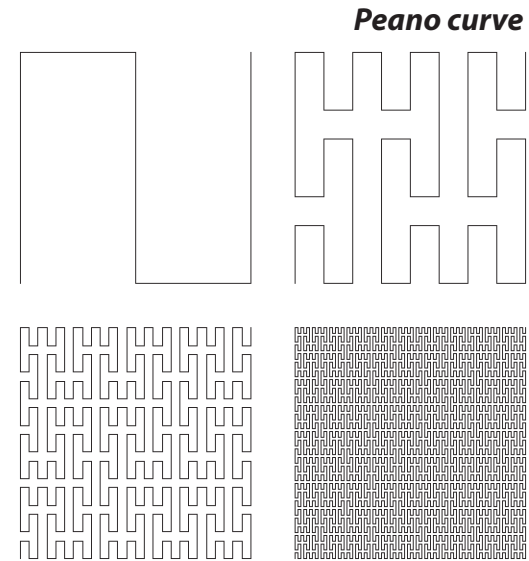
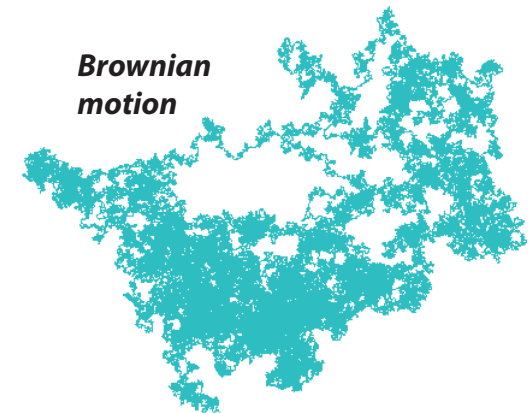## they wiggle so much that their dimension is >1

Peano curve

Brownian motion

simple curves

Koch snowflake
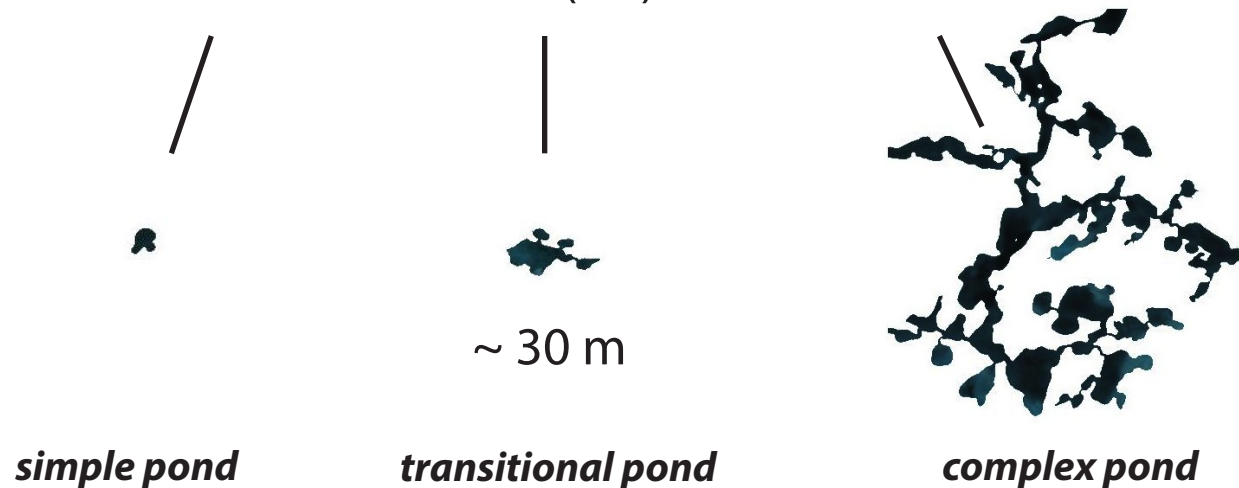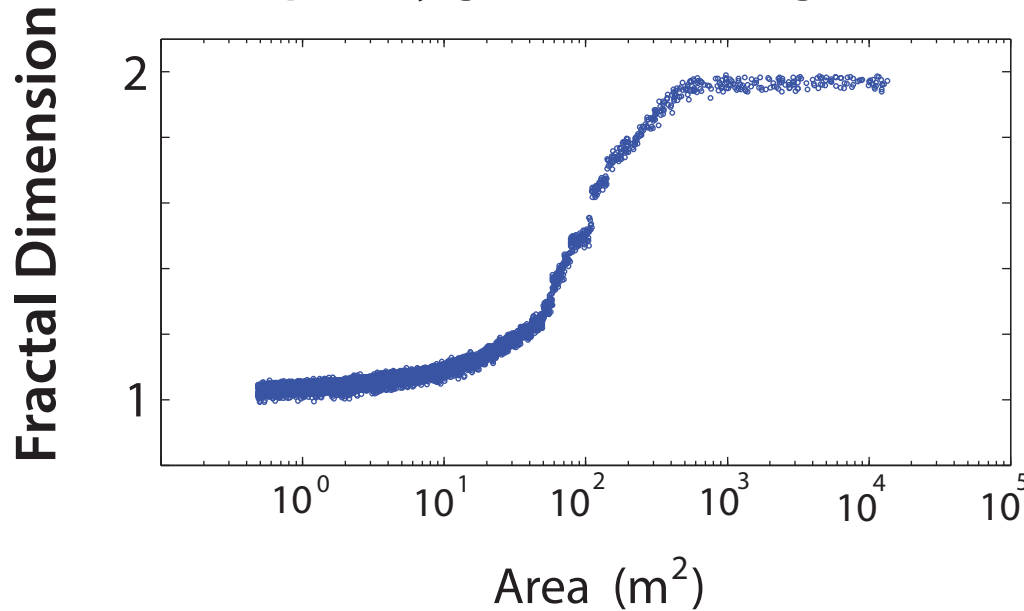
space filling curves

D = 1

D = 1.26

D = 2

Transition in the fractal geometry of Arctic melt ponds

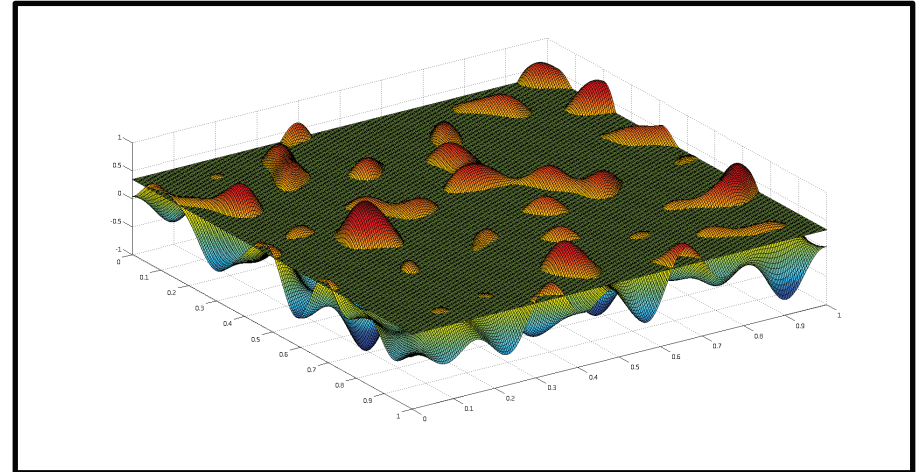Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

complexity grows with length scale

~ 30 m

simple pond   transitional pond   complex pond

Continuum percolation model for melt pond evolution
level sets of random surfaces

Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018
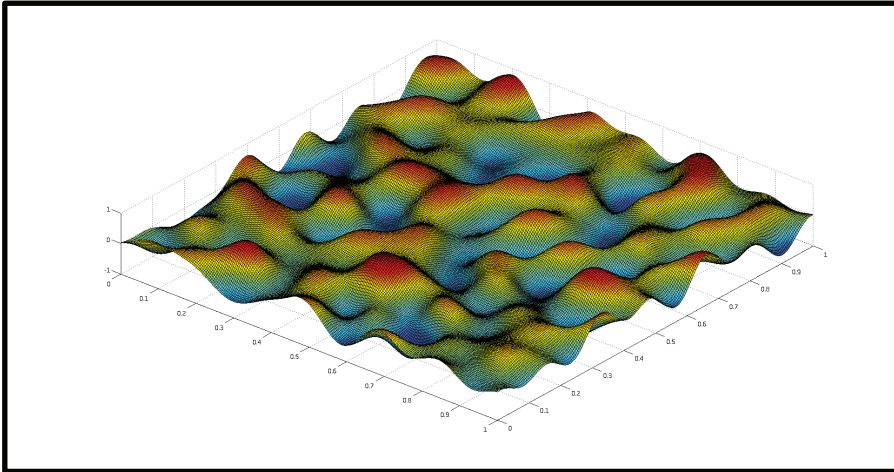
random Fourier series representation of surface topography

intersections of a plane with the surface define melt ponds
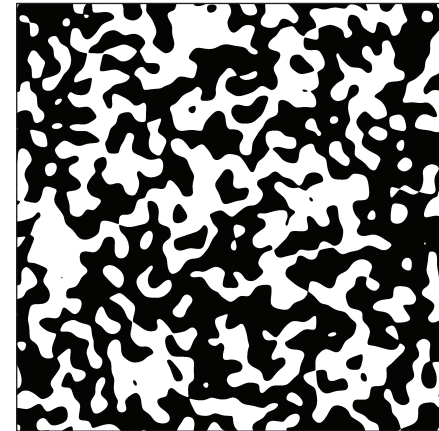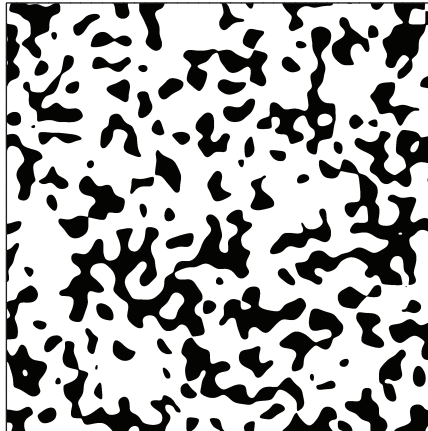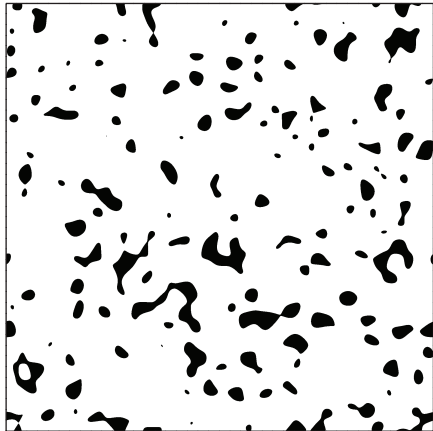
electronic transport in disordered media   *diffusion in turbulent plasmas*   *Isichenko, Rev. Mod. Phys., 1992*

fractal dimension curves depend on statistical parameters defining random surface
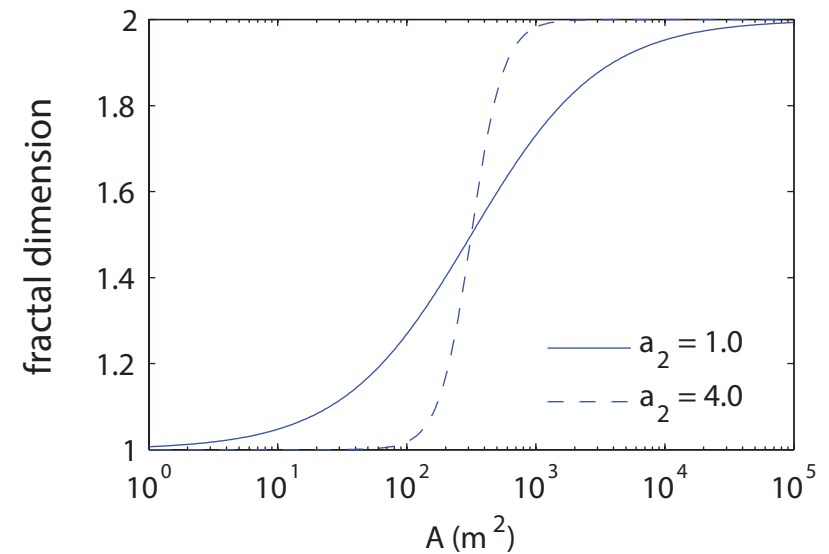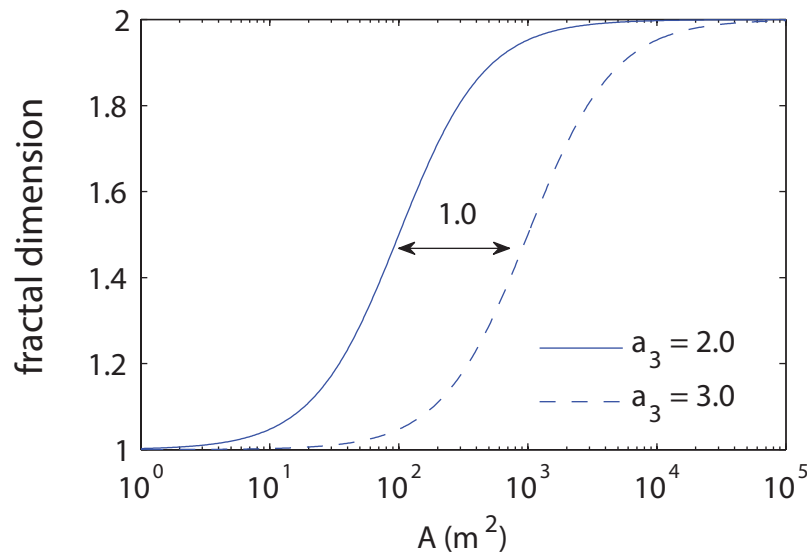
Topology of the sea ice surface and the fractal geometry of Arctic melt ponds

Ryleigh Moore, Jacob Jones, Dane Gollero,
Court Strong, Ken Golden

**Several models replicate the transition in
fractal dimension, but none explain how it arises.**

We use Morse theory applied to the random surface model
to show that saddle points play the critical role in the fractal transition.
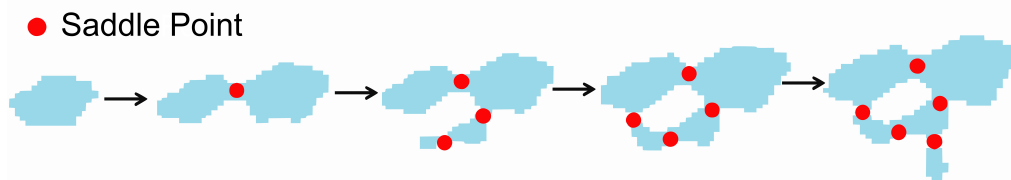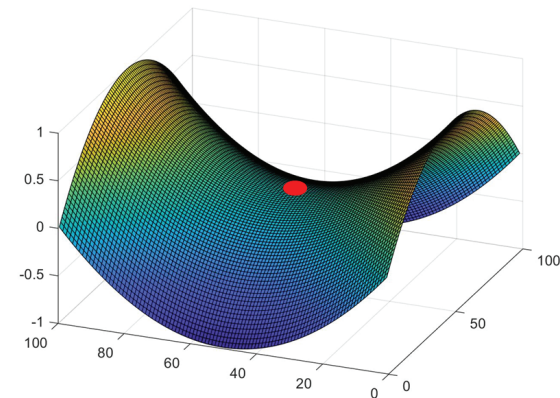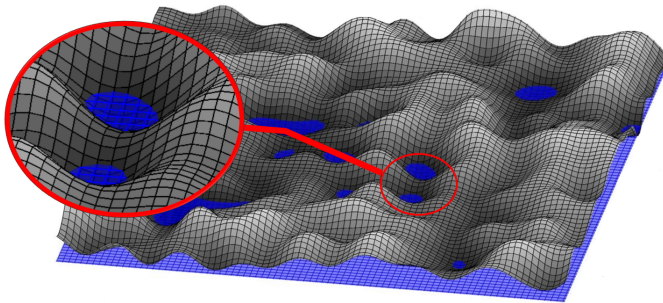
● Saddle Point

ponds coalesce
(change topology) and
complexify at saddle points

Morse theory
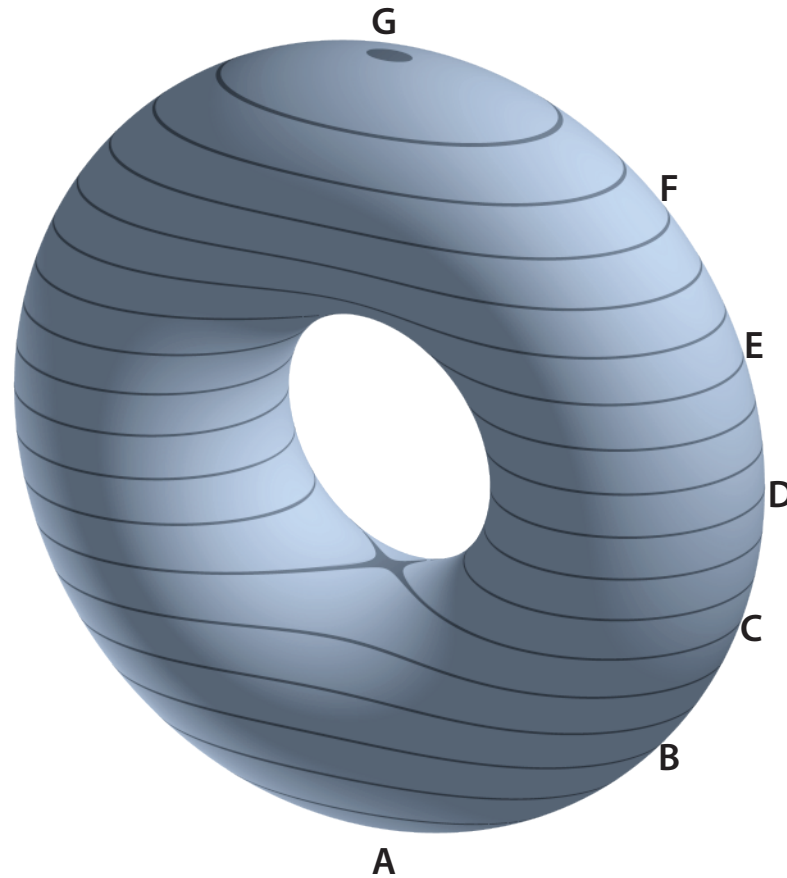
Morse theory tells us that changes in the topology of a surface occur at critical points of smooth functions on the surface: maxima, minima, and saddles.

Topological Data Analysis

Euler characteristic = # maxima + # minima - # saddles

topological invariant

persistent homology

filtration - sequence of nested topological spaces, indexed by water level
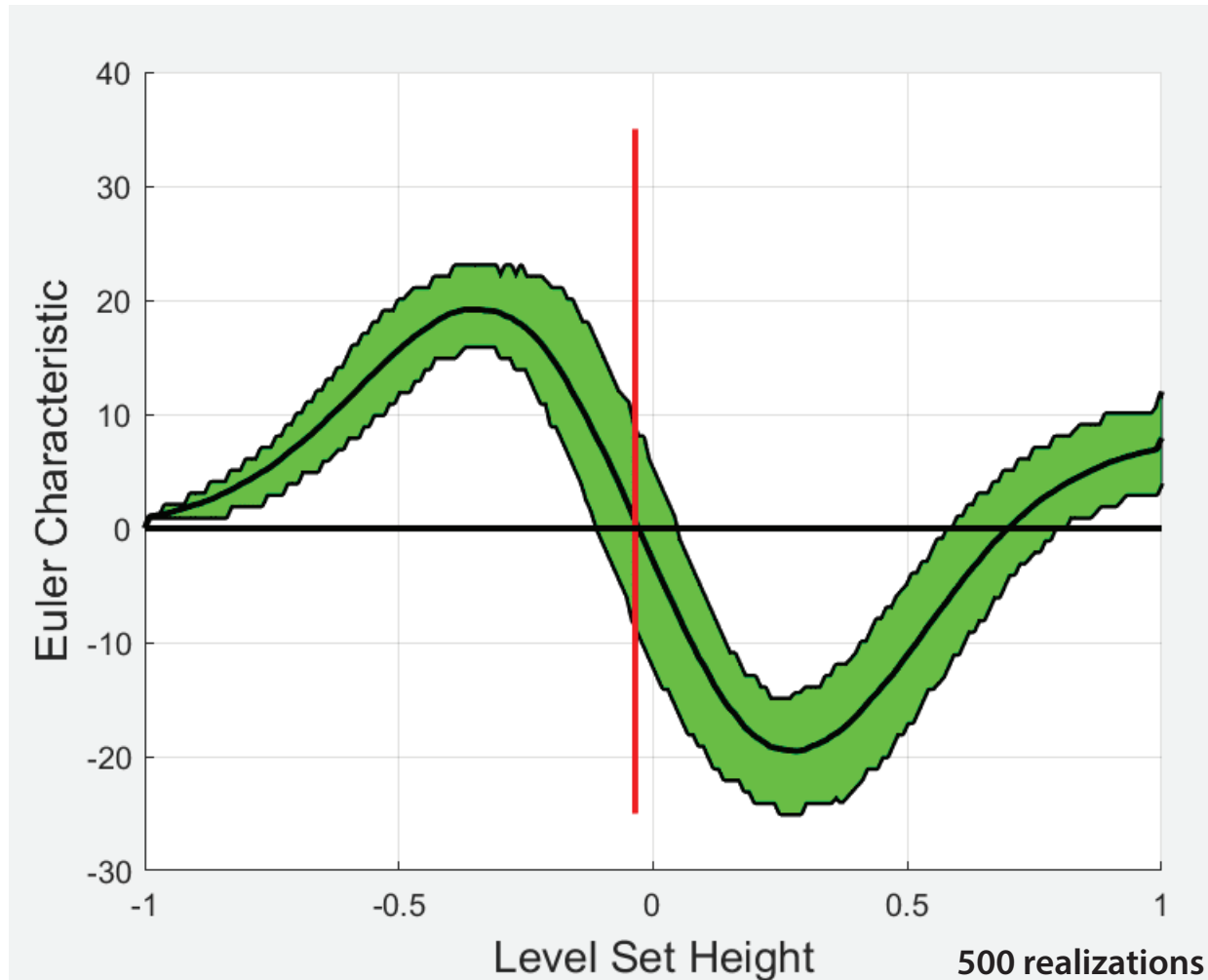
Expected
Euler Characteristic Curve (ECC)

tracks the evolution of the EC of the flooded surface as water rises

zero of ECC ~ percolation

percolation on a torus creates a giant cycle

Bobrowski & Skraba, 2020

Carlsson, 2009

Vogel, 2002   GRF

porous media
cosmology
brain activity

melt pond donuts

From magnets to melt ponds

100 year old model for magnetic materials used to explain melt pond geometry
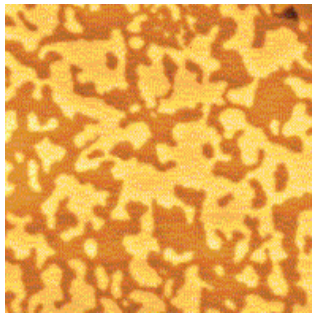
magnetic domains in cobalt
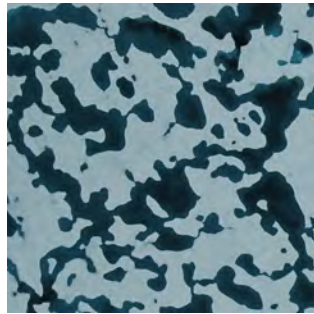
Arctic melt ponds
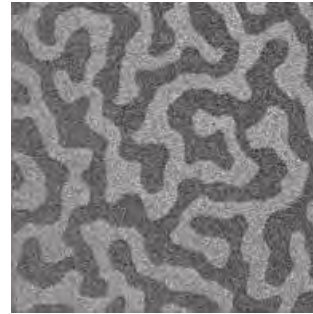
magnetic domains in cobalt-iron-boron
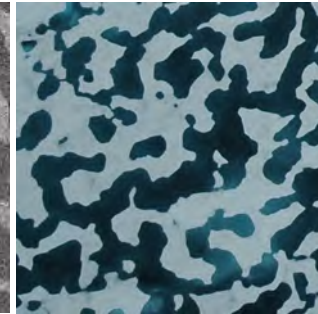
Arctic melt ponds

2D Ising model

spin down

spin up
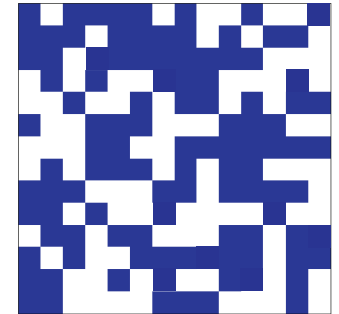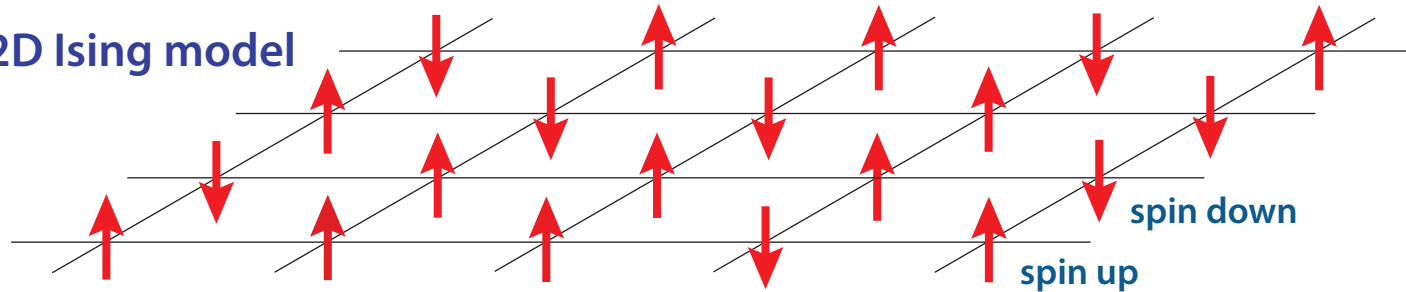
model
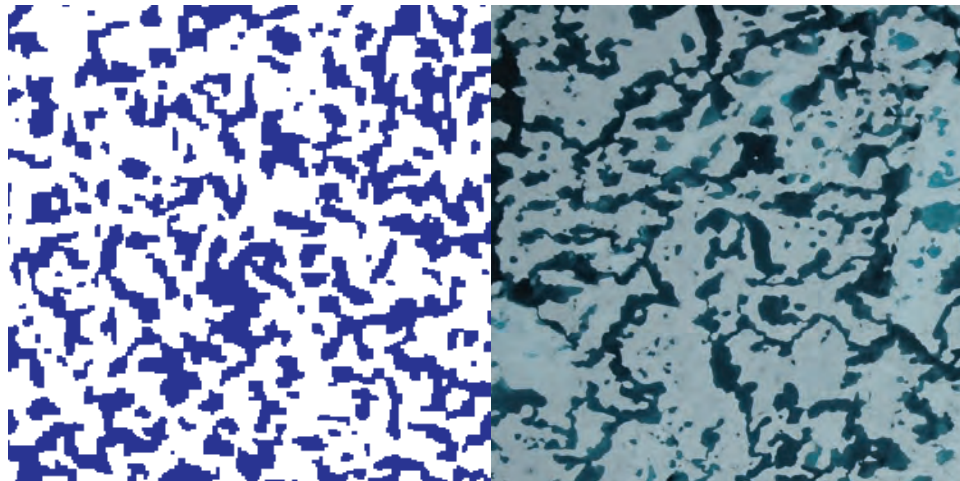
real ponds (Perovich)

Ma, Sudakov, Strong, Golden, *New J. Phys.* 2019

Scientific American, EOS, PhysicsWorld, ...

melt pond evolution depends also on large-scale "pores" in ice cover

drainage vortex

photo courtesy of C. Polashenski and D. Perovich

Melt pond connectivity enables vast expanses of melt water to drain down seal holes, thaw holes, and leads in the ice.

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS

Perovich
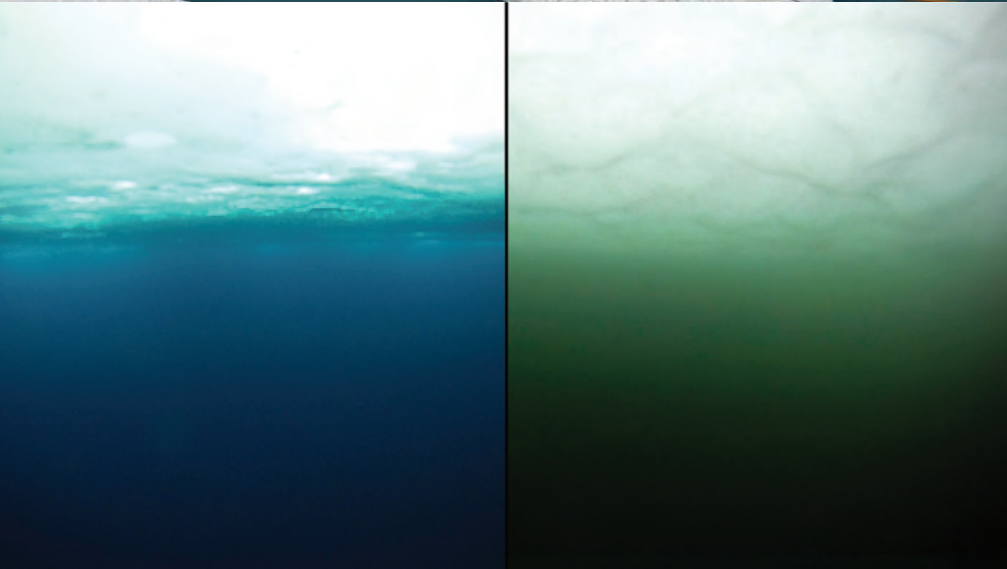
no bloom   bloom

massive under-ice algal bloom

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

Horvat, Rees Jones, Iams, Schroeder, Flocco, Feltham, *Science Advances* 2017

The effect of melt pond geometry on the distribution of solar energy under first year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden *Geophys. Res. Lett.* 2019

(2015 AMS MRC)

SEA ICE ALGAE

Photo: Meiners

Photo: Haas

Porous microstructure
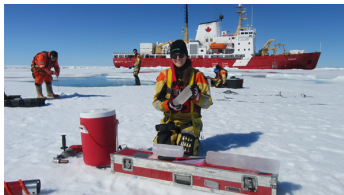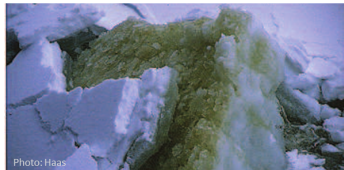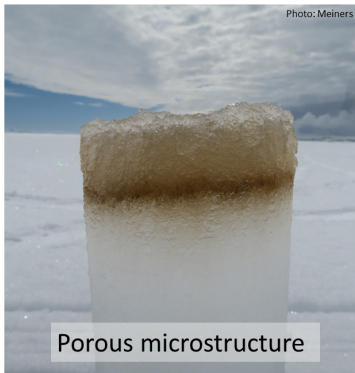
Can we improve agreement between algae models and data?

80% of polar bear diet can be traced to ice algae*.

[*]Brown TA, et al. (2018). *PloS one*, 13(1), e0191631

HETEROGENEITY

(Reimer)

(Thurber (CC))

(a) Snow depth
(b) Sea-ice thickness
(c) Ice freeboard
(d) Chl-a$_{ROV}$

Meiners, K.M., et al. (2017). *Geophysical Research Letters*, 44(14), 7382-7390

ECOLOGY LETTERS cnrs WILEY

Uncertainty quantification for ecological models with random parameters

Jody R. Reimer[1,2] | **Frederick R. Adler**[1,2] | **Kenneth M. Golden**[1] | **Akil Narayan**[1,3]

[1]Department of Mathematics, University of Utah, Salt Lake City, Utah, USA

[2]School of Biological Sciences, University of Utah, Salt Lake City, Utah, USA

[3]Scientific Computing and Imaging Institute, University of Utah, Salt Lake City, Utah, USA

Correspondences
Jody R. Reimer, Department of Mathematics and School of Biological Sciences, University of Utah, Salt Lake City, Utah, USA.
Email: reimer@math.utah.edu

Abstract

There is often considerable uncertainty in parameters in ecological models. This uncertainty can be incorporated into models by treating parameters as random variables with distributions, rather than fixed quantities. Recent advances in uncertainty quantification methods, such as polynomial chaos approaches, allow for the analy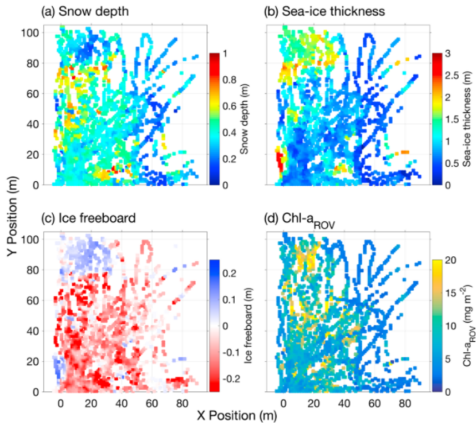sis of models with random parameters. We introduce these methods with a motivating case study of sea ice algal blooms in heterogeneous environments. We compare Monte Carlo methods with polynomial chaos techniques to help understand the dynamics of an algal bloom model with random parameters.

Introduce polynomial chaos approach to widely used ecological ODE models, but with random parameters.

ECOLOGICAL INSIGHTS
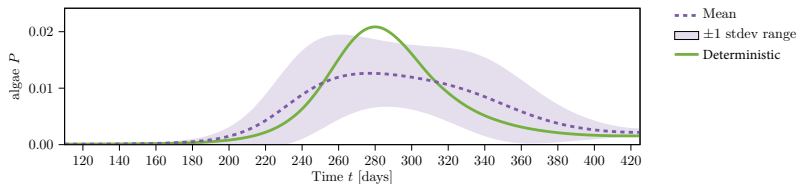
- lower peak bloom intensity
- longer bloom duration
- able to compare variance to data

macroscale

Marginal Ice Zone

MIZ

- biologically active region
- intense ocean-sea ice-atmosphere interactions
- region of significant wave-ice interactions

transitional region between
dense interior pack ($c > 80\%$)
sparse outer fringes ($c < 15\%$)
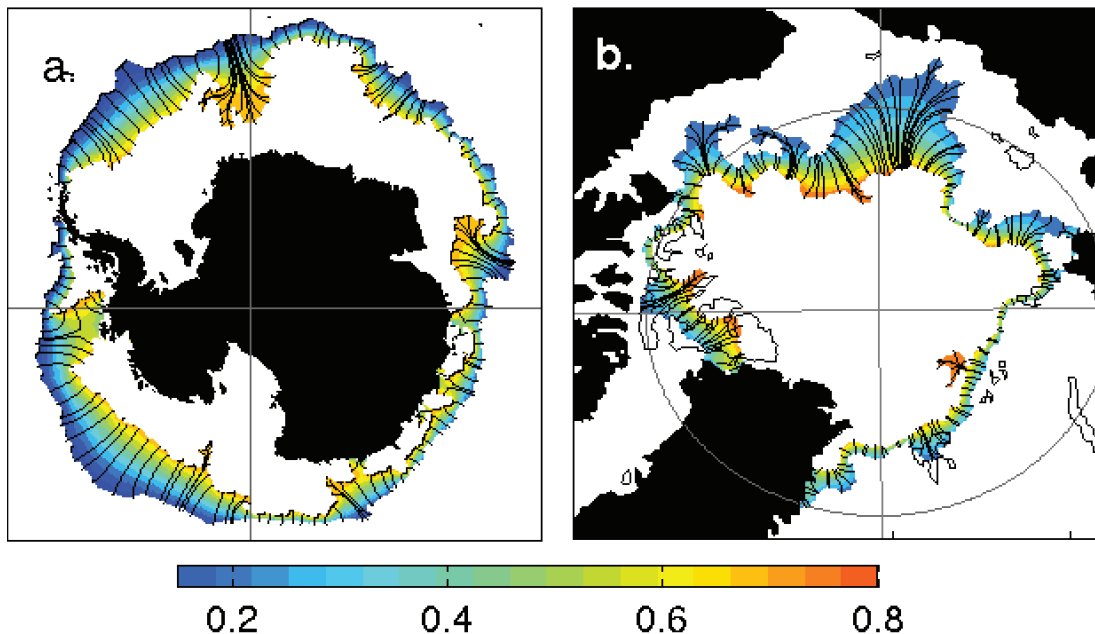
MIZ WIDTH
fundamental length scale of
ecological and climate dynamics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

How to objectively
measure the "width"
of this complex,
non-convex region?

Objective method for measuring MIZ width motivated by medical imaging and diagnostics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

39% widening
1979 - 2012

streamlines of a solution
to Laplace's equation

"average" lengths of streamlines

MIZ   pack ice
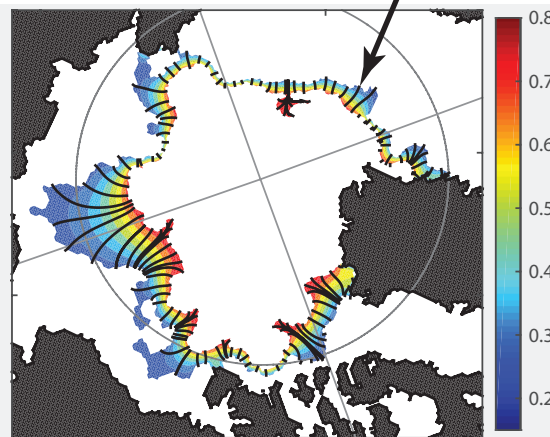
Arctic Marginal Ice Zone
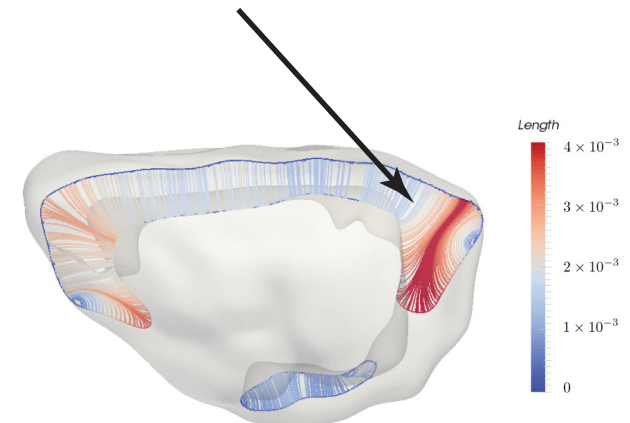
crossection of the
cerebral cortex of a rodent brain

analysis of different MIZ WIDTH definitions

Strong, Foster, Cherkaev, Eisenman, Golden
J. Atmos. Oceanic Tech. 2017

**Model larger scale effective behavior
with partial differential equations that
homogenize complex local structure and dynamics.**

Arctic MIZ

sea ice concentration ψ
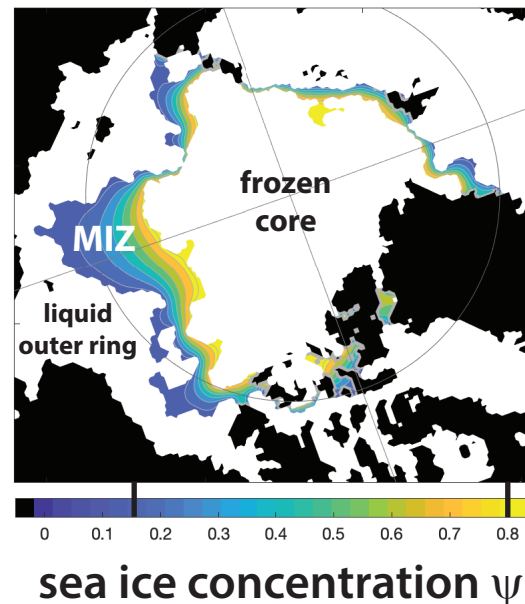
Predict MIZ width and location with basin-scale phase change model.

dynamic transitional region - mushy layer - separating two "pure" phases

seasonal and long term trends

**C. Strong, E. Cherkaev, and K. M. Golden,
Annual cycle of Arctic marginal ice zone location
and width explained by dynamic phase transition model, 2023**

Filling the polar data gap with partial differential equations

hole in satellite coverage of sea ice concentration field

previously assumed ice covered

Gap radius: 611 km

06 January 1985
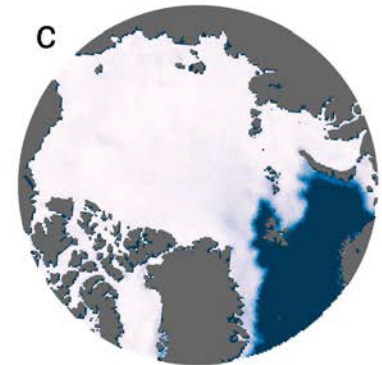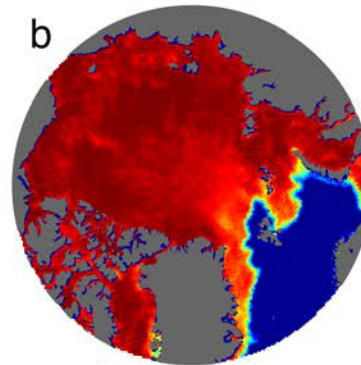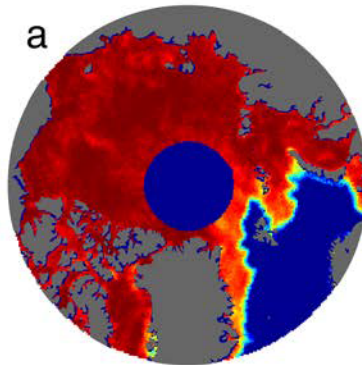
Gap radius: 311 km

30 August 2007
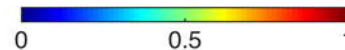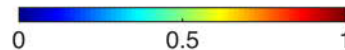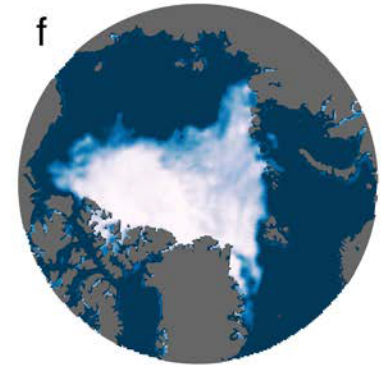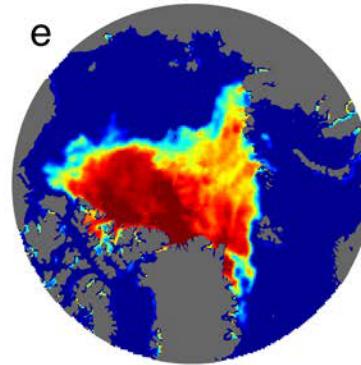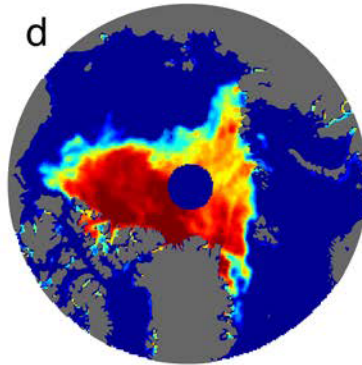
$$\Delta\psi=0$$

fill = harmonic function with learned stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

NOAA/NSIDC Sea Ice Concentration CDR product update will use our PDE method.

polar bear foraging in a fractal icescape

Nicole Forrester
Jody Reimer
Ken Golden

It costs the polar bear 5 times the energy to swim through water than to walk on sea ice.

What pathway to a seal minimizes energy spent?

Polar Bear Percolation

Optimal Movement of a Polar Bear in a Heterogenous Icescape

Cost of Polar Bear Path (top chart)

$C(p)$

Path Cost axis: 0, 2000, 4000, 6000, 8000, 10000, 12000, 14000, 16000
Sea Ice Concentration axis: 0.00 ... 0.98

← $h = 0.2$
← $h = 0.1$
← $h = 0$

Ice: 0 Water: 100 — Ice: 10 Water: 100 — Ice: 20 Water: 100
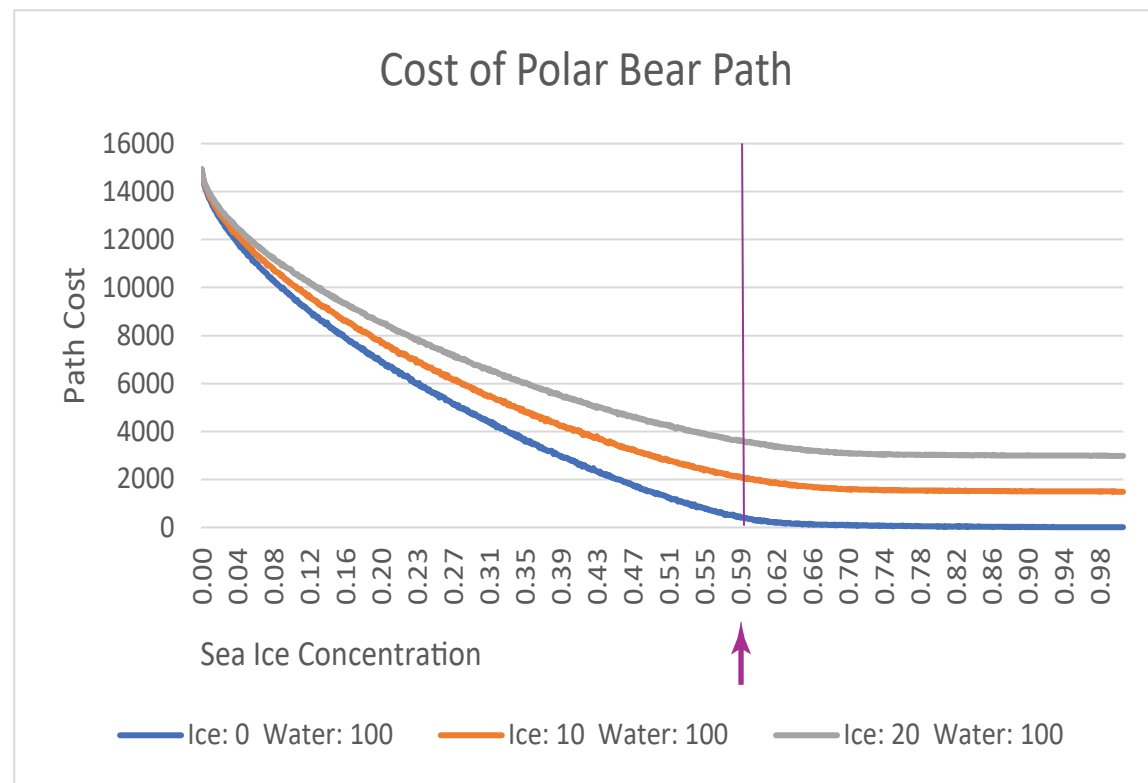
$$h = \frac{c_i}{c_w}$$

ratio of local "conductivities"

site percolation threshold

$p_c = 0.59$ for $d = 2$

Cost of Polar Bear Path (bottom chart)

$C(p)$

ice disconnected
bear must swim

connected
ice pathway

← $h = 0$

Path Cost axis: 0, 2000, 4000, 6000, 8000, 10000, 12000, 14000, 16000
Sea Ice Concentration axis: 0.00 ... 0.98
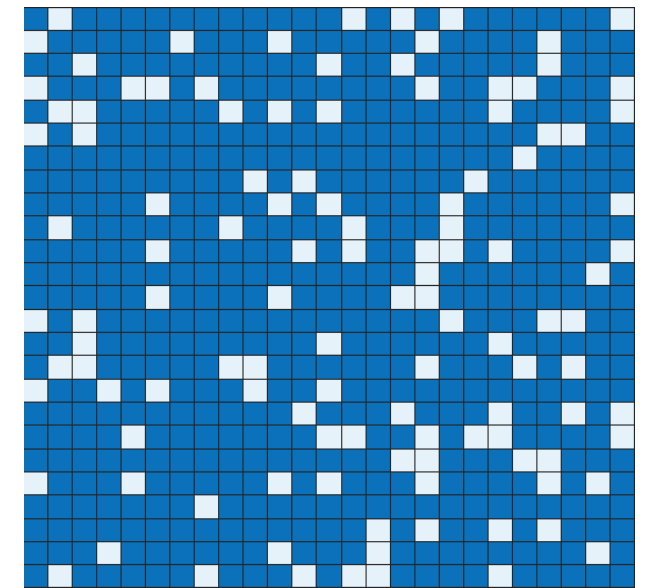
20% Ice
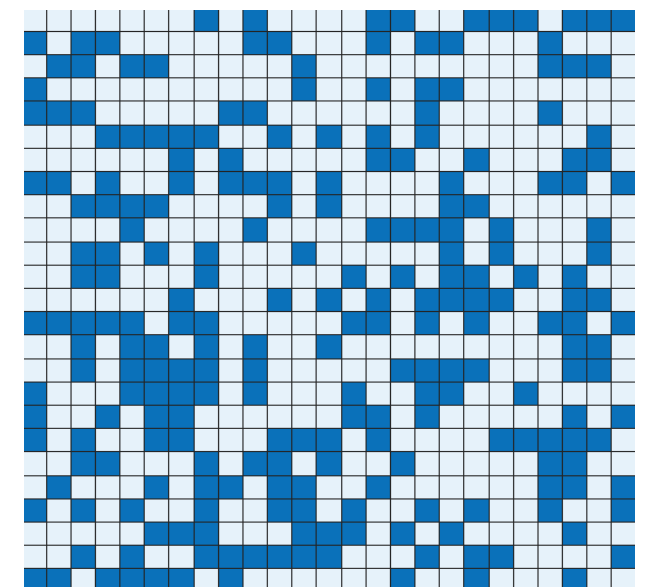
60% Ice
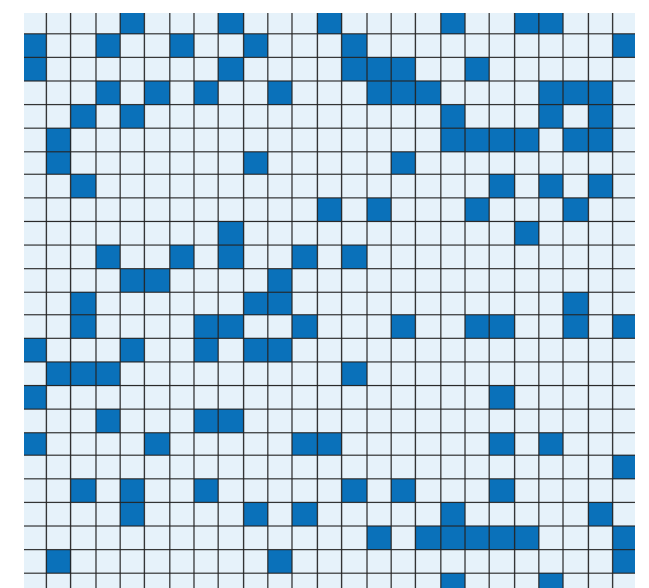
80% Ice
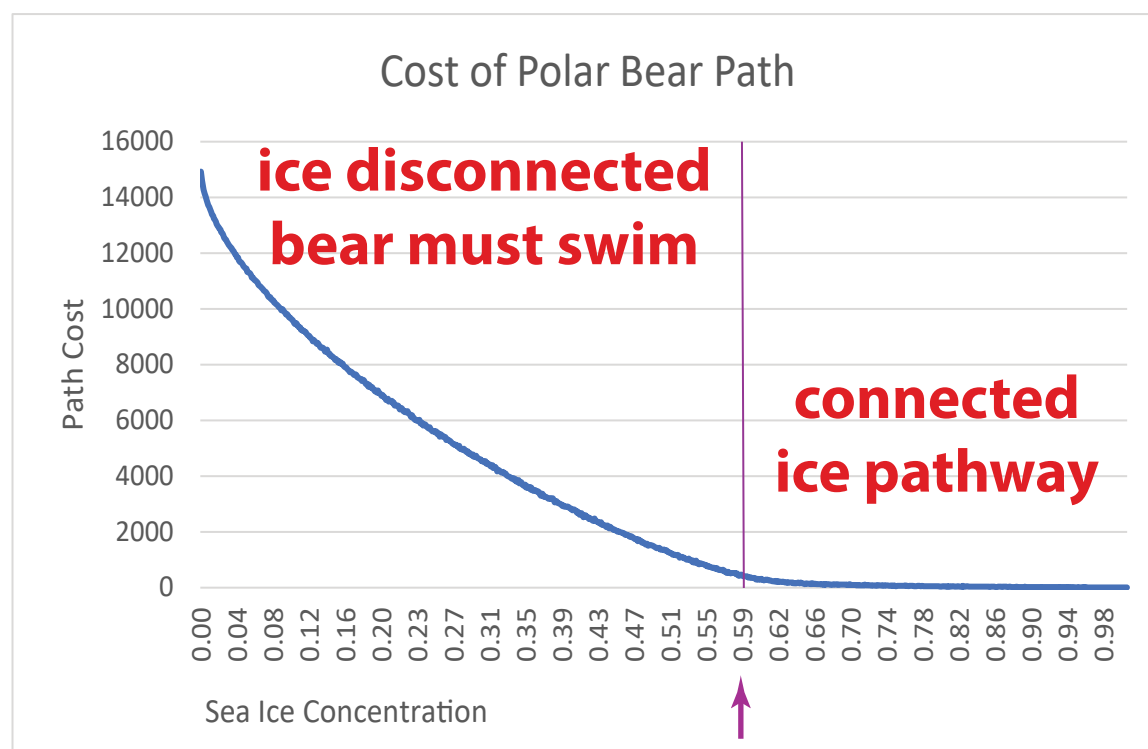
Conclusions

Our research is helping to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

Mathematics for sea ice advances the theory of composites, inverse problems, and other areas of science and engineering.

Sea ice is really cool! Modeling it leads to unexpected areas of math and physics.

Notices

of the American Mathematical Society

AMS

AMERICAN
MATHEMATICAL
SOCIETY

Advancing research. Creating connections.

The cover is based on "Modeling Sea Ice,"
page 1535.

University of Utah Sea Ice Modeling Group (2017-2023)

Senior Personnel: Ken Golden, Distinguished Professor of Mathematics
Elena Cherkaev, Professor of Mathematics
Court Strong, Associate Professor of Atmospheric Sciences
Ben Murphy, Adjunct Assistant Professor of Mathematics

Postdoctoral Researchers: Noa Kraitzman, Jody Reimer, Bohyun Kim

Graduate Students: Kyle Steffen (now at UT Austin)
Christian Sampson (now at NCAR)
Huy Dinh (MURI sea ice Postdoc at NYU/Courant)
Rebecca Hardenbrook (-> Dartmouth Postdoc)
David Morison (Physics Department)
Ryleigh Moore
Delaney Mosier, Daniel Hallman, Julie Sherman

Undergraduate Students: Kenzie McLean, Jacqueline Cinella Rich,
Dane Gollero, Samir Suthar, Anna Hyde,
Kitsel Lusted, Ruby Bowers, Kimball Johnston,
Jerry Zhang, Nash Ward, David Gluckman,
Kayla Stewart, Nicole Forrester, Megan Long

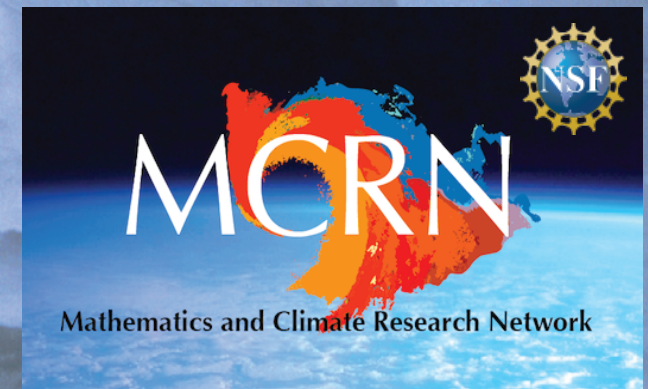**High School Students:** J. Chapman, T. Quah, D. Webb, A. Lee, A. Dorsky

THANK YOU

Office of Naval Research

Applied and Computational Analysis Program

Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999

Fire endangers Hobart's ice ship

BY DAVID CARRIGG

AN engine-room fire has left the Hobart-based Antarctic research ship Aurora Australis without power in dangerous sea ice off the Antarctic coast.

None of the 79 people on board was injured in the blaze, which broke out early yesterday morning while the ship was in deep water 185km off the coast.

The extent of the damage is not known.

Australian Antarctic Division director Rex Moncur said the fire was extinguished by flooding the engine room with an inert gas.

The gas had to be cleared before crew wearing breathing apparatus could enter and assess the situation.

He said it could be some time before the extent of damage was known.

The 25 crew and 54 expeditioners, mostly from Hobart, would wear thermal clothing and stay below decks to keep warm.

"There is always a risk of becoming ice-bound in these waters at this time of the year but at this stage we don't expect to launch a rescue mission from Hobart," Mr Moncur said.

The ship was in regular radio contact with the Antarctic Division's Hobart office.

He expected the expeditioners and crew to abandon the pioneering winter voyage and return the ship to Hobart for repairs in about a week.

The Antarctic Division, which hires the ship from P&O Australia, would not be hiring another vessel for the expedition.

"It's a pretty specialist vessel so you couldn't get the sort of research capability that this ship has got readily available," Mr Moncur said.

"We hope the next voyage can still proceed on schedule, which is early September."

The Aurora Australis is owned by P&O Australia and charted by the Antarctic Division for about $11 million a year.

P&O Australia managing director Richard Hein said yesterday the company was assessing the situation and a number of rescue options were being considered.

It was too early to say whether P&O would be liable for the cost of the aborted mission.

The vessel left Hobart last Wednesday for a seven-week voyage mainly to study a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

The ship was nearing the polynya when the fire broke out.
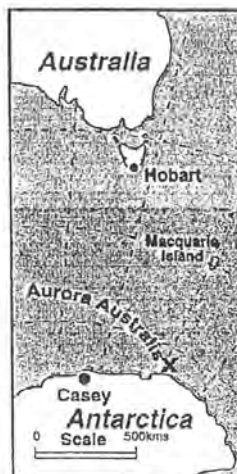
Oceanographers believe a closer study of the phenomenon will lead to a better understanding of climate change.

CSIRO Marine Research oceanographer Steve Rintoul said the dense bottom water, created only in a few places in Antarctica and to a lesser extent in the North Atlantic, was critical to the chemistry and biology of the world's oceans.

A file photo of the Aurora Australis in Antarctica.

Fire strands Antarctic ship in sea ice

AN engine room fire has disabled the icebreaker Aurora Australis in sea ice, deep in Antarctic waters.

There were no injuries and the ship was not in danger after Tuesday night's fire.

Australian Antarctic Division director Mr Rex Moncur said. But Mr Moncur said he expected it would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent and return to Hobart for repairs.

The cause of the fire was not known but the engines have been turned off, with the ship 100 nautical miles from the Antarctic coast.

Antarctic voyage stopped by fire

HOBART: An engine room fire has disabled the Australian icebreaker Aurora Australis in sea ice, deep in Antarctic waters.

Australian Antarctic Division director Rex Moncur said there were no injuries and the ship was not in danger after Tuesday night's fire.

But Mr Moncur said he expected Aurora Australis would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent to return to Hobart for repairs.

The fire had been extinguished and the engines were turned off, leaving the ship in sea ice about 100 nautical miles from the Antarctic coast, he said. The weather was good.

Crew had to wear breathing apparatus to enter the engine room and it was likely to be 24 hours before the damage could be fully assessed.

The Aurora, with 54 expeditioners and 25 crew, left Hobart last Wednesday for a seven-week voyage which was to have focused on a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

Mr Moncur said, the cause of the fire was not yet known.

Sydney Morning Herald
23 July, 1998

ICEBREAKER BURNS

A pioneering $2 million Australian scientific voyage to the mid-winter Antarctic pack ice is expected to be scrapped following an engine-room fire on the Aurora Australis yesterday. The 54 people on board were forced on deck in the...

2:45 am July 22, 1998

``Please don't be alarmed but we have an uncontrolled fire in the engine room ….''

about 10 minutes later …

``Please don't be alarmed but we're lowering the lifeboats ….''

AURORA AUSTRALIS
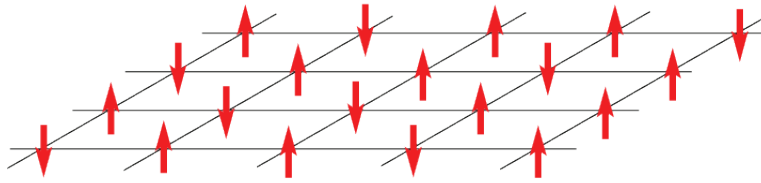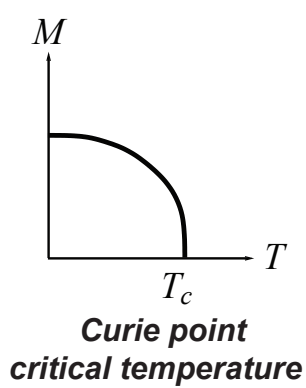
Main results

Isoperimetric quotient - as a proxy for fractal dimension - increases in discrete jumps when ponds coalesce at saddle points.
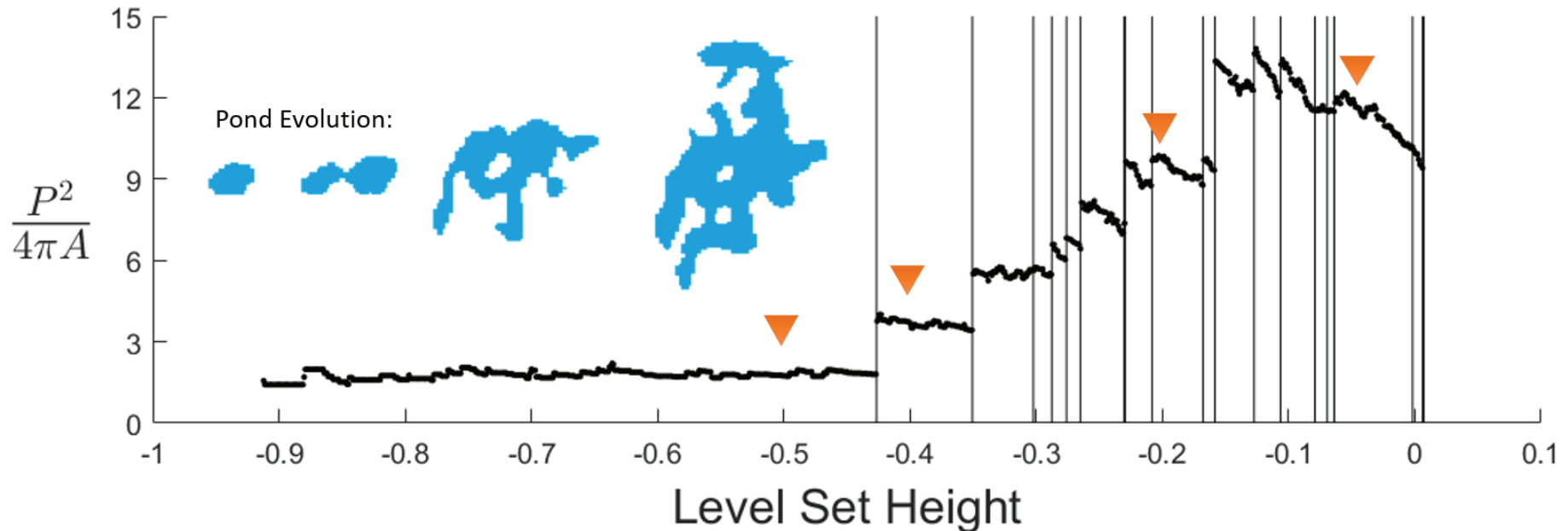
Horizontal fluid permeability "controlled" by saddles ~ electronic transport in 2D random potential.

drainage processes, seal holes

Ising Model for a Ferromagnet

M

T_c

T

Curie point critical temperature

$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

blue

white

applied magnetic field H

$$\mathcal{H} = -H \sum_i s_i - J \sum_{<i,j>} s_i s_j$$
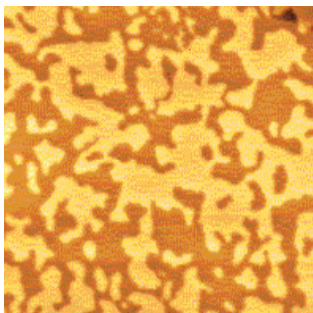
nearest neighbor Ising Hamiltonian

islands of like spins

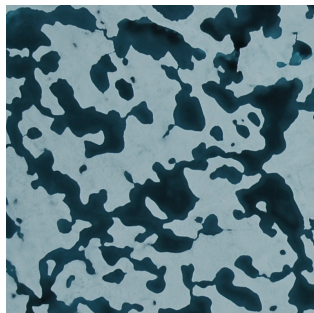$$M(T, H) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$
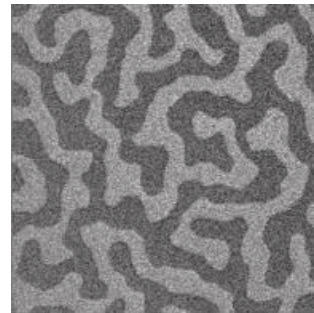
effective magnetization
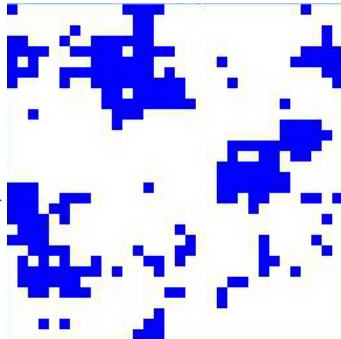
energy is lowered when nearby spins align with each other, forming **magnetic domains**

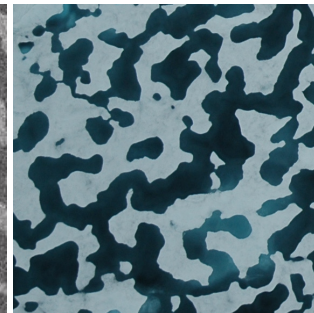**magnetic domains in cobalt**

melt ponds (Perovich)

magnetic domains in cobalt-iron-boron

melt ponds (Perovich)

Ising model for ferromagnets → Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

$$\mathcal{H} = - \sum_{i}^{N} H_i\, s_i - J \sum_{<i,j>}^{N} s_i s_j \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad\ \ \text{(spin down)} \end{cases}$$

random magnetic field represents snow topography

magnetization M **pond area fraction** $F = \dfrac{(M+1)}{2}$ only nearest neighbor patches interact
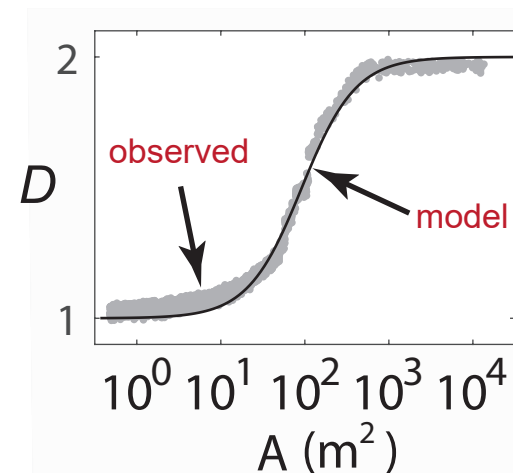
~ albedo

Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system "flows" toward metastable equilibria.

Order from Disorder
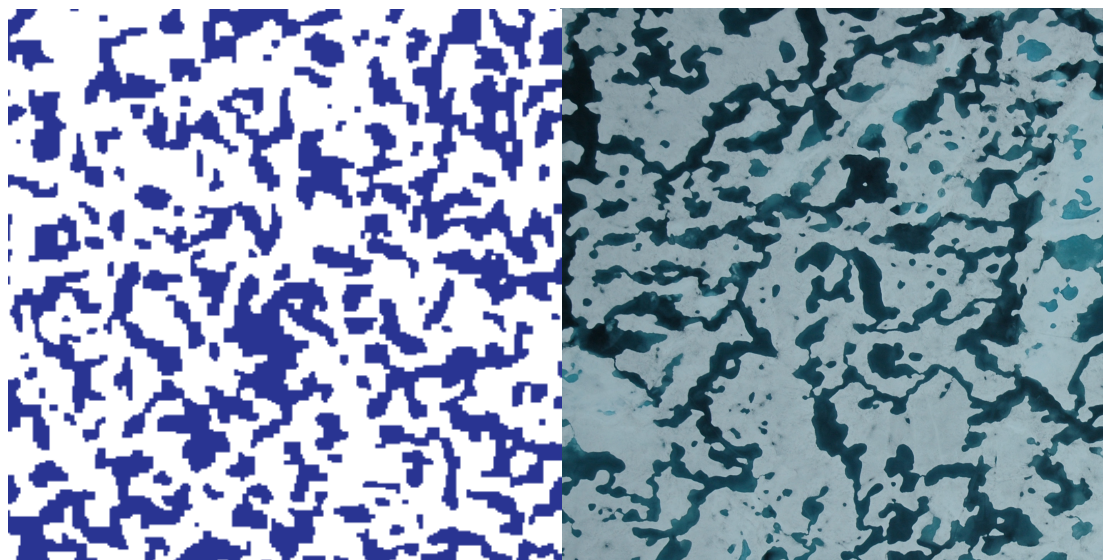
Ising model

melt pond photo (Perovich)

pond size distribution exponent

observed   -1.5

(Perovich, *et al.* 2002)

model   -1.58

Scientific American EOS, PhysicsWorld, ...

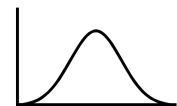**ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data**

HETEROGENEITY IN INITIAL CONDITIONS

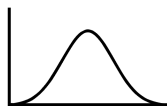At each location within a larger region, we could consider

Nutrients $\quad \dfrac{dN}{dt} = \alpha - BNP - \eta N$

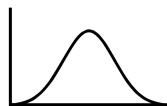Algae $\quad \dfrac{dP}{dt} = \gamma BNP - \delta P$

$$N(0) = N_0, \qquad P(0) = P_0$$

growth rate, **B**   Initial nutrients, N_0 Initial algae, P_0