# Remote sensing of sea ice

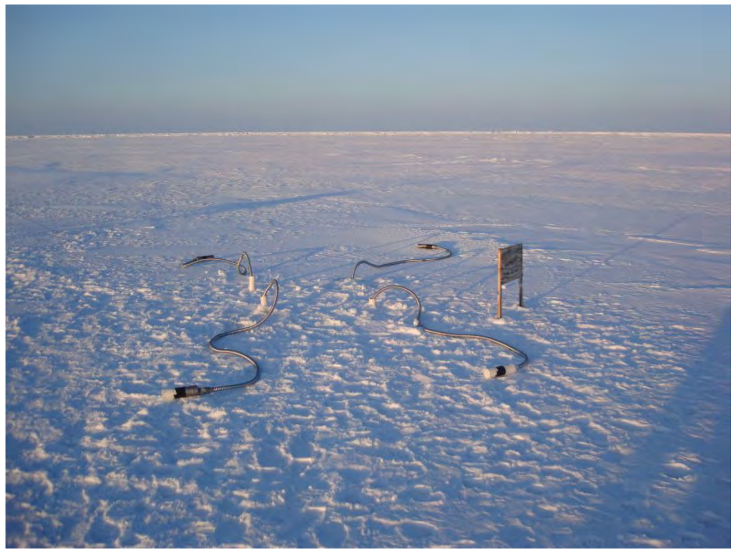
*sea ice thickness*
*ice concentration*

## *INVERSE PROBLEM*

Recover sea ice properties from electromagnetic (EM) data

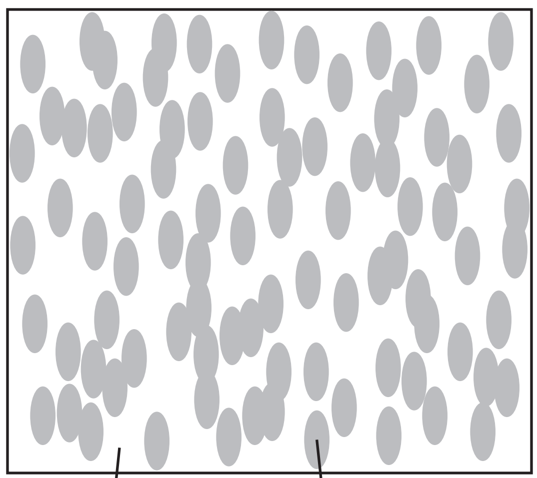
$\varepsilon^*$

effective complex permittivity
(dielectric constant, conductivity)



*brine volume fraction*
*brine inclusion connectivity*

# Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$\epsilon_1$ $\epsilon_2$

$\epsilon^*$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

$p_1$ , $p_2$ = volume fractions of the components

$$\epsilon^* = \epsilon^* \left( \frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry} \right)$$

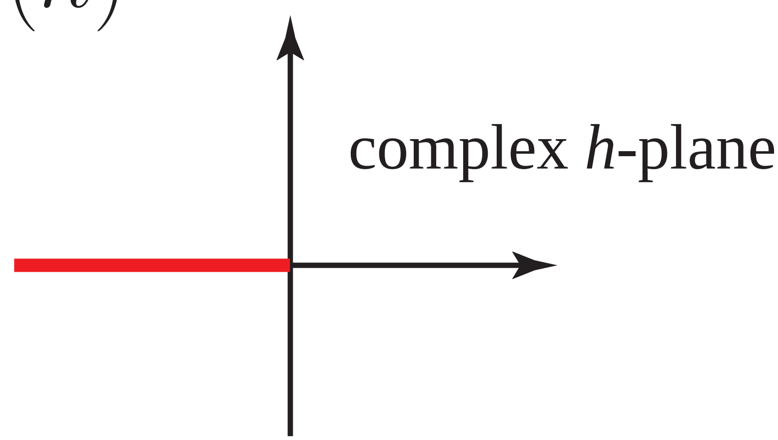
# **Analytic continuation method for bounding complex $\epsilon^*$**

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

$$m(h) = \frac{\epsilon^*}{\epsilon_2}\left(\frac{\epsilon_1}{\epsilon_2}\right) \qquad h = \frac{\epsilon_1}{\epsilon_2}$$

Exploit analytic properties of $m(h)$

- $m(h)$ analytic off negative real axis




- $m(h)$ : UHP $\longrightarrow$ UHP

# *Theory of Effective Electromagnetic Behavior of Composites*

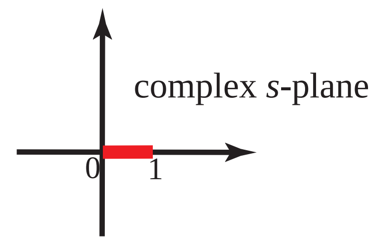
## *analytic continuation method*

**Forward Homogenization** Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

**composite geometry**
**(spectral measure μ)** $\longrightarrow$ $\varepsilon^*$

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z} \qquad s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$




**Inverse Homogenization** Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
(McPhedran, McKenzie, and Milton, 1982)

$\varepsilon^*$ $\longrightarrow$ **composite geometry**
**(spectral measure μ)**

recover brine volume fraction, connectivity, etc.

# Stieltjes integral representation

## separation of geometry from parameters

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

geometry

parameters

$\mu$

- spectral measure of self adjoint operator $\chi\Gamma\chi$
- mass = $p_1$
- higher moments depend on $n$-point correlations

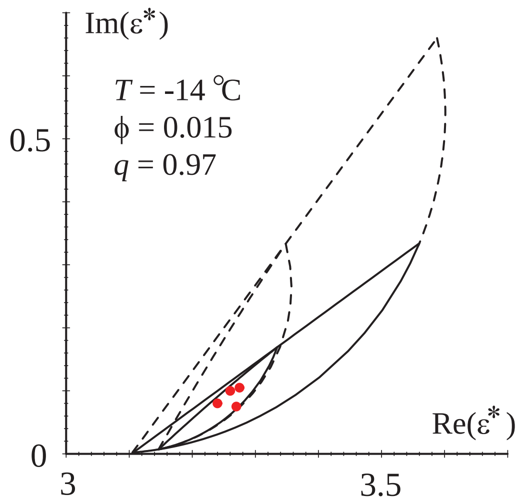
$$\Gamma = -\nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

*Golden and Papanicolaou, Comm. Math. Phys. 1983*

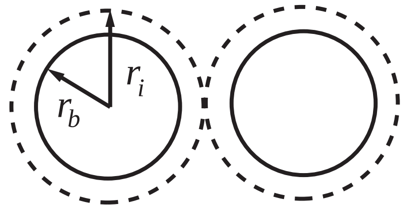
# *forward and inverse bounds on the complex permittivity of sea ice*

## forward bounds




4.75 GHz data

***matrix particle***




$$q = r_b \,/\, r_i$$

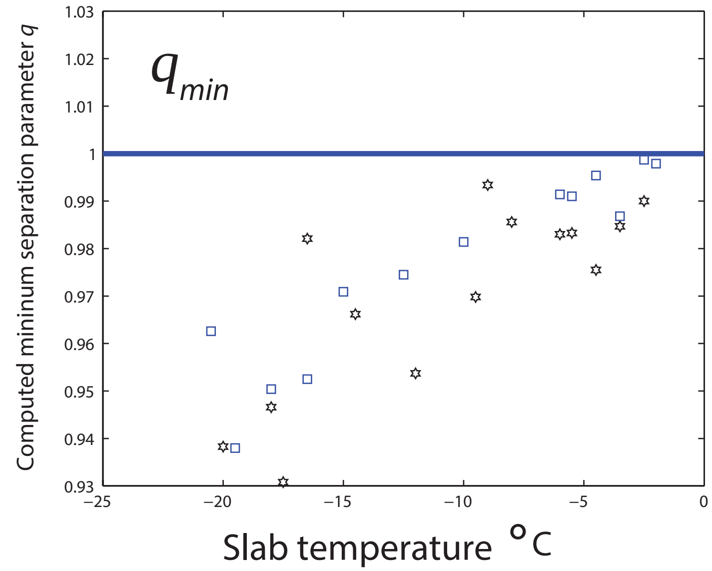
$$0 < q < 1$$

***Golden 1995, 1997***

**inverse bounds and recovery of brine porosity**

***Gully, Backstrom, Eicken, Golden***
***Physica B, 2007***

## inverse bounds




**inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε***

**rigorous inverse bound on spectral gap**

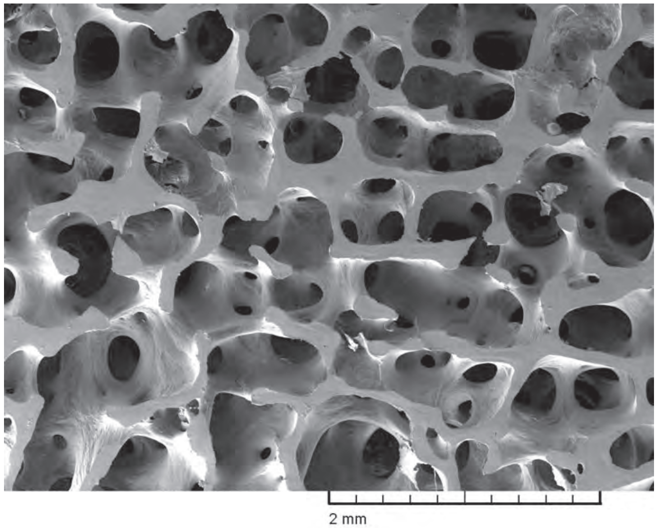
*construct algebraic curves which bound admissible region in (p,q)-space*

***Orum, Cherkaev, Golden***
***Proc. Roy. Soc. A, 2012***

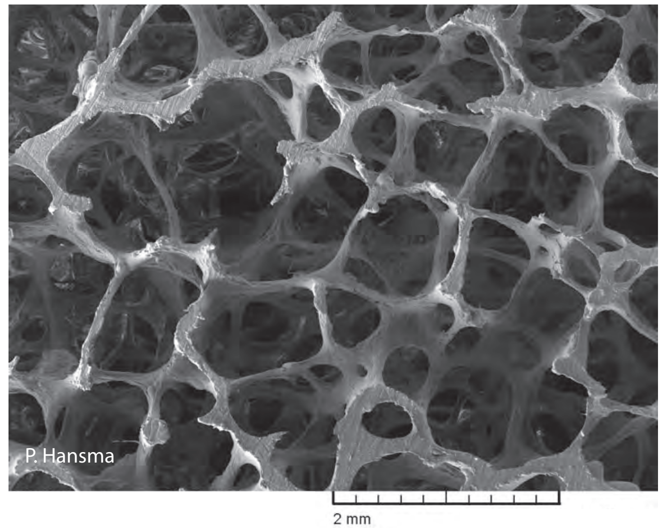
# spectral characterization of porous microstructures in bone

***Golden, Murphy, Cherkaev, J. Biomechanics 2011***

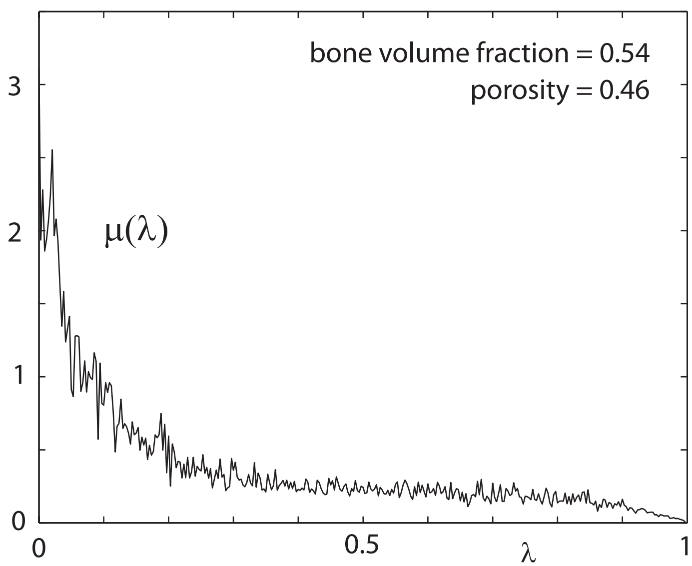
(a) young healthy trabecular bone



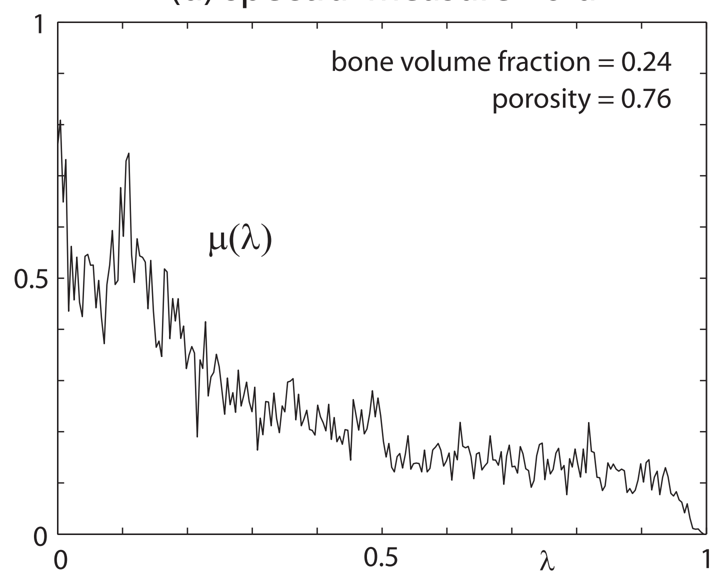

(b) old osteoporotic trabecular bone




(c) spectral measure - young




(d) spectral measure - old




\+

reconstruction of spectral measures from complex permittivity data

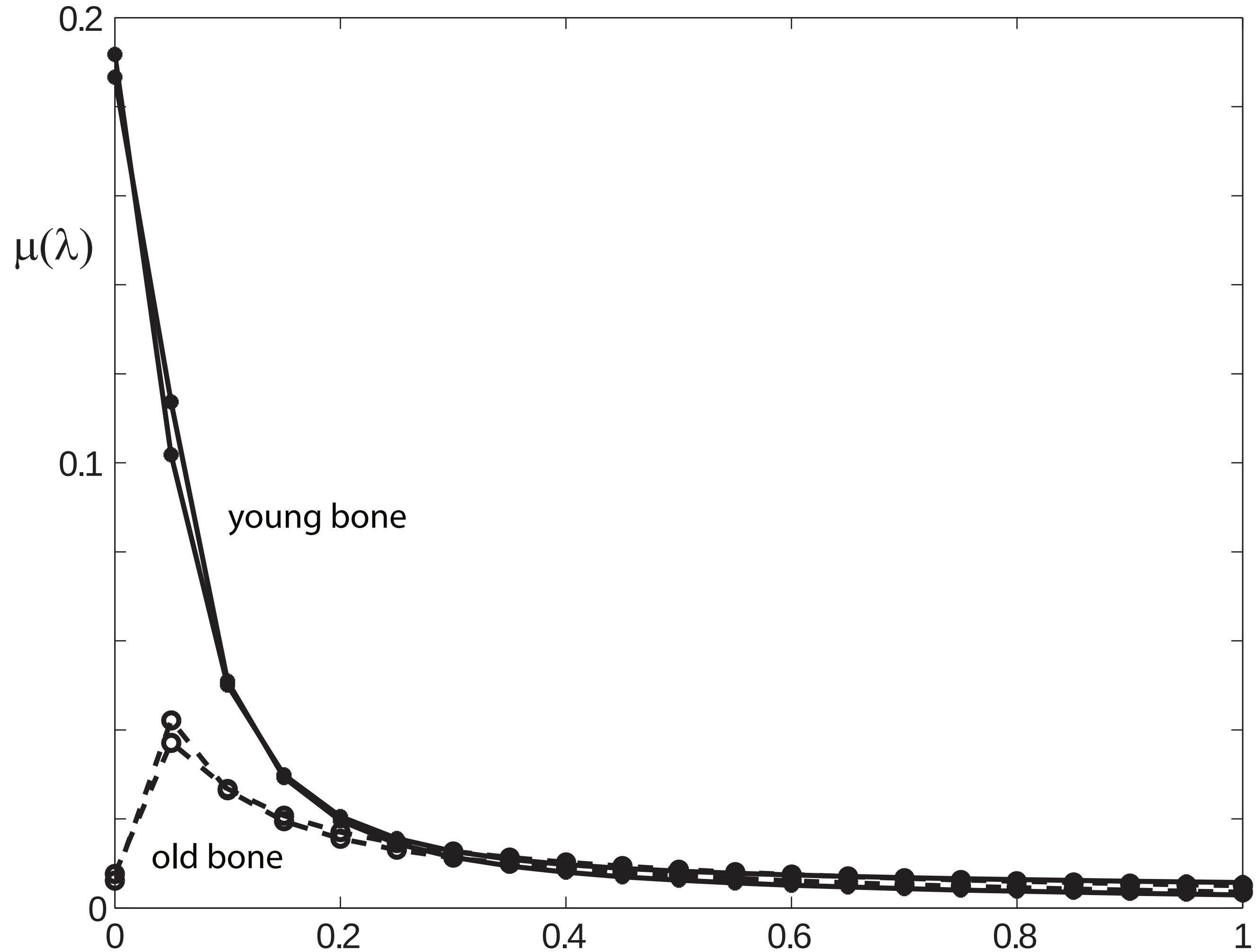
***using regularized inversion scheme***

***EM monitoring of osteoporosis***

***loss of bone connectivity***

***the math doesn't care if it's sea ice or bone!***

# reconstruction of spectral measures from simulated complex permittivity data




***regularized inversion scheme***

***Golden, Murphy, Cherkaev, J. Biomech. 2011***

## *direct calculation of spectral measure*

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).

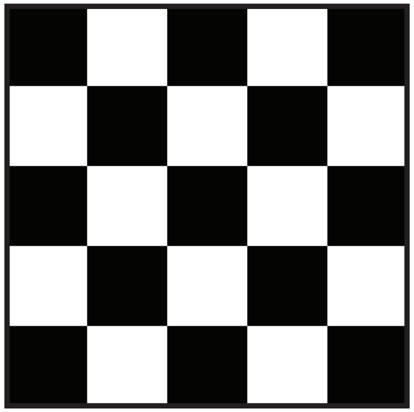
2. The fundamental operator $\chi\Gamma\chi$ becomes a random matrix depending only on the composite geometry.

3. Compute the eigenvalues $\lambda_i$ and eigenvectors of $\chi\Gamma\chi$ with inner product weights $\alpha_i$

$$\mu(\lambda) = \sum_i \alpha_i \, \delta(\lambda - \lambda_i)$$

Dirac point measure (Dirac delta)

Continuum composite

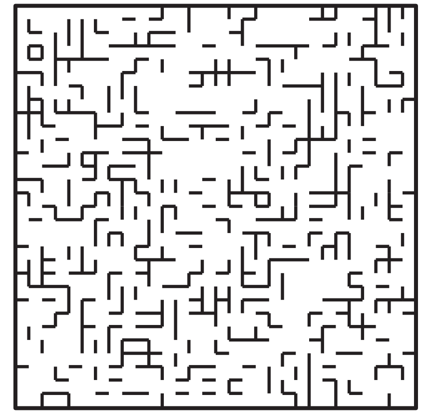


# *Spectral measures of*

$$\chi_1 \Gamma \chi_1$$

Murphy, Hohenegger, Cherkaev, Golden
Comm. Math. Sci. 2015

Discrete composite



***Integro-differential projection operator***

$$\Gamma = \vec{\nabla}(\Delta^{-1})\vec{\nabla}\cdot$$

***Point-wise indicator function***

$$\chi_1$$

***Resolvent representation of electric field***

$$\chi_1 \vec{E} = sE_0 (sI - \chi_1 \Gamma \chi_1)^{-1} \chi_1 \vec{e}_k$$

***Integral representation***

$$F(s) = \int_0^1 \frac{d\mu(\lambda)}{s - \lambda}$$

***Projection matrix***

$$\Gamma = \nabla(\nabla^{\mathrm{T}}\nabla)^{-1}\nabla^{\mathrm{T}}$$

***Diagonal projection matrix***

$$\chi_1$$

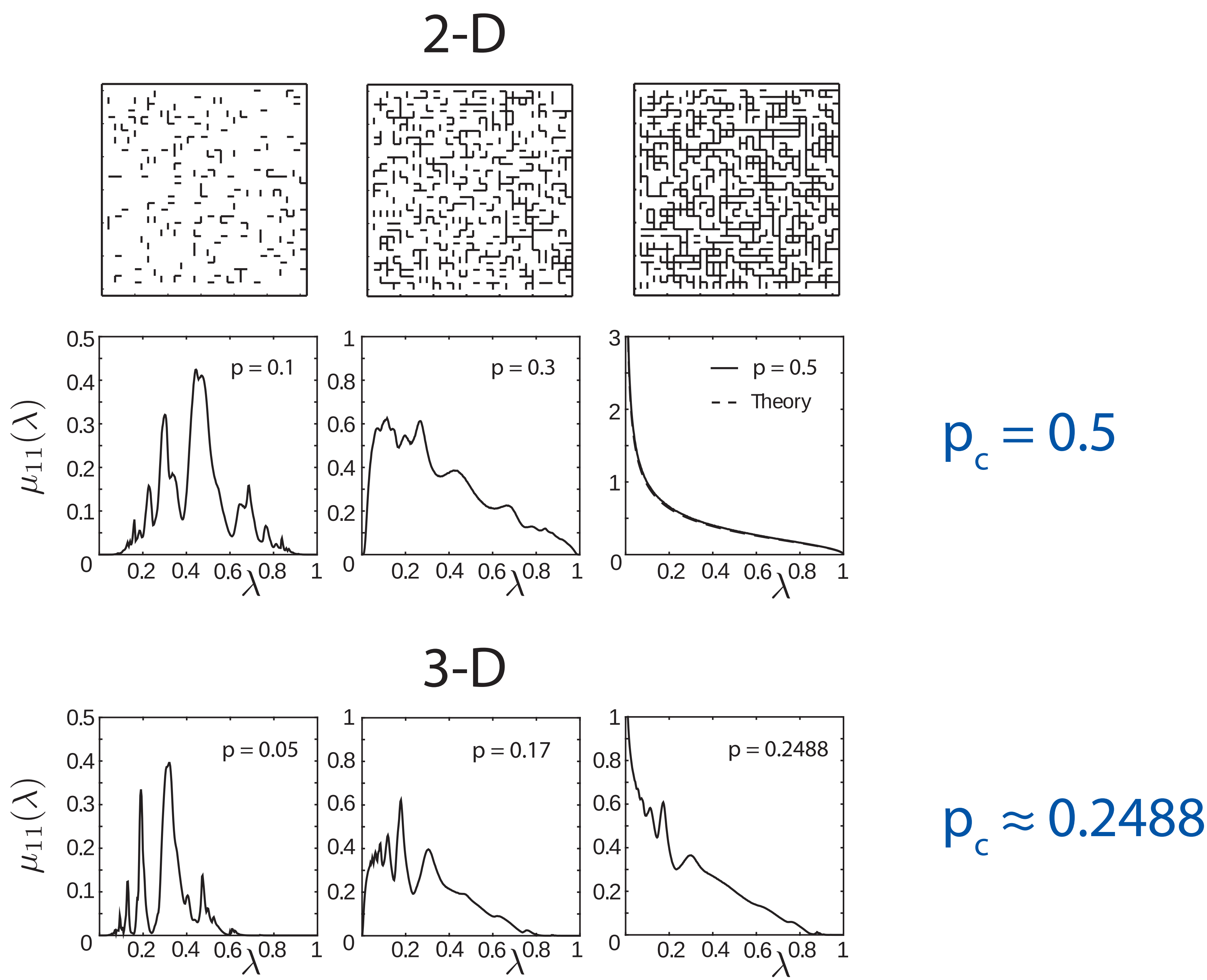
***Series representation of electric field***

$$\chi_1 \vec{E} = sE_0 \sum_j \frac{\vec{v}_j \cdot \chi_1 \vec{e}_k}{s - \lambda_j} \vec{v}_j$$

***Series representation***

$$F(s) = \sum_j \frac{(\vec{v}_j \cdot \chi_1 \vec{e}_k)^2}{s - \lambda_j}$$

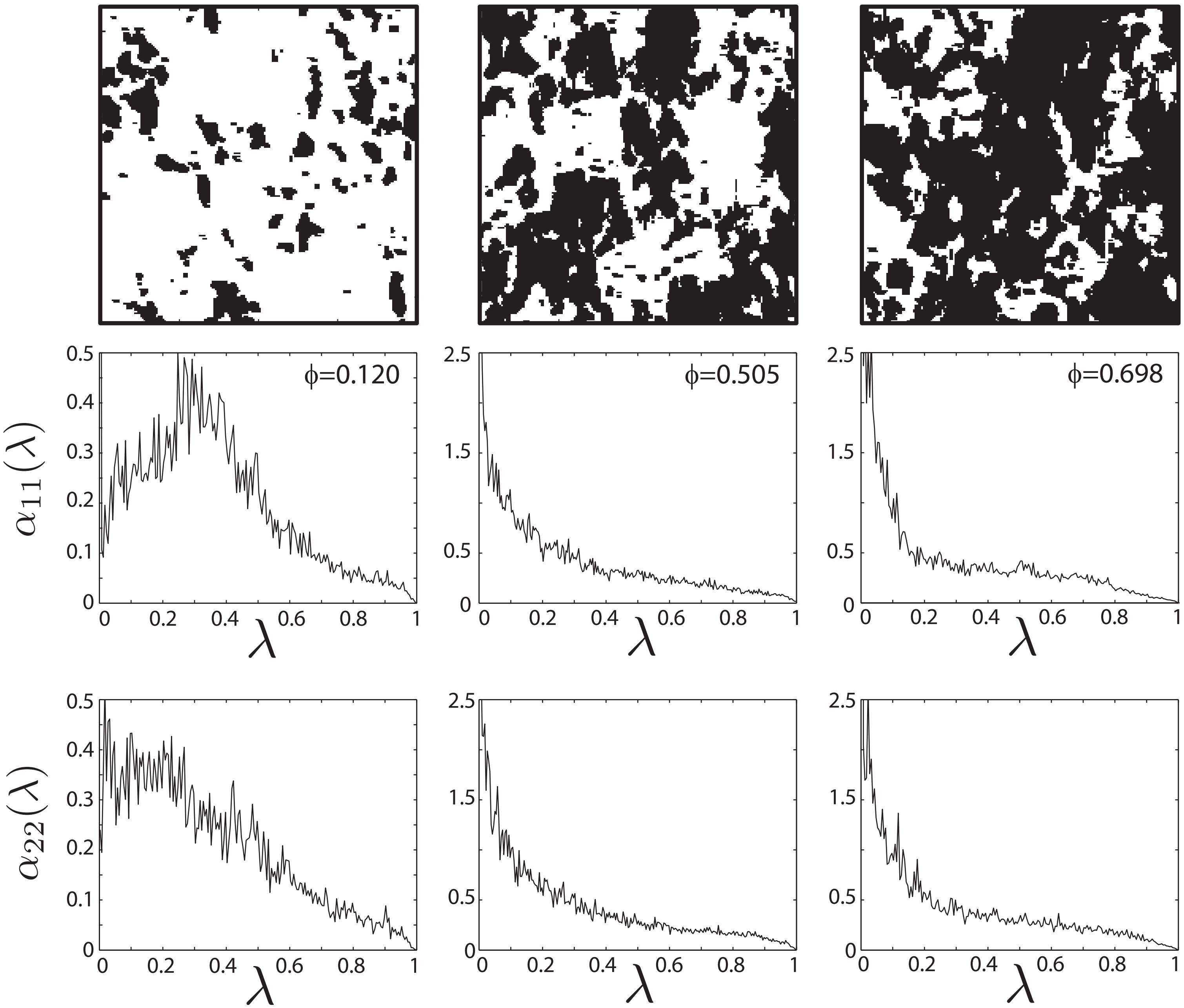
# Spectral Measures for Random Resistor Networks




spectral gaps collapse at the percolation transitions

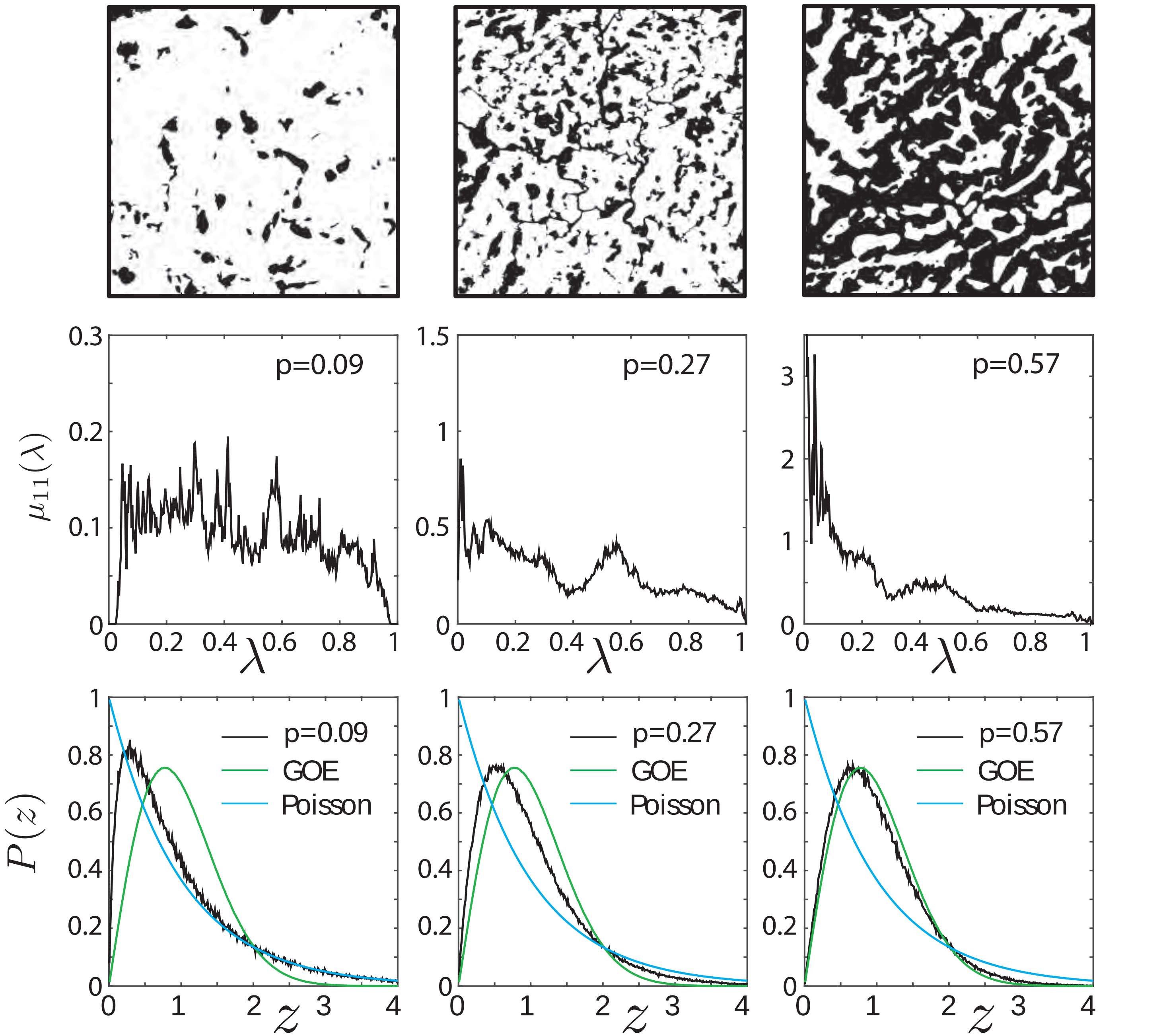
Murphy and Golden, J. Math. Phys. (2012)

# Spectral Measures for Sea Ice Structures: Brine Inclusions




N. B. Murphy, C. Hohenegger, C. S. Sampson, D. K. Perovich, H. Eicken, E. Cherkaev, B. Alali, and K. M. Golden

# Spectral computations for Arctic melt ponds




**Ben Murphy**
**Elena Cherkaev**
**Ken Golden**
**2016**

**spectral measures**

**eigenvalue spacing distributions**

**uncorrelated** → **level repulsion**

**TRANSITION**

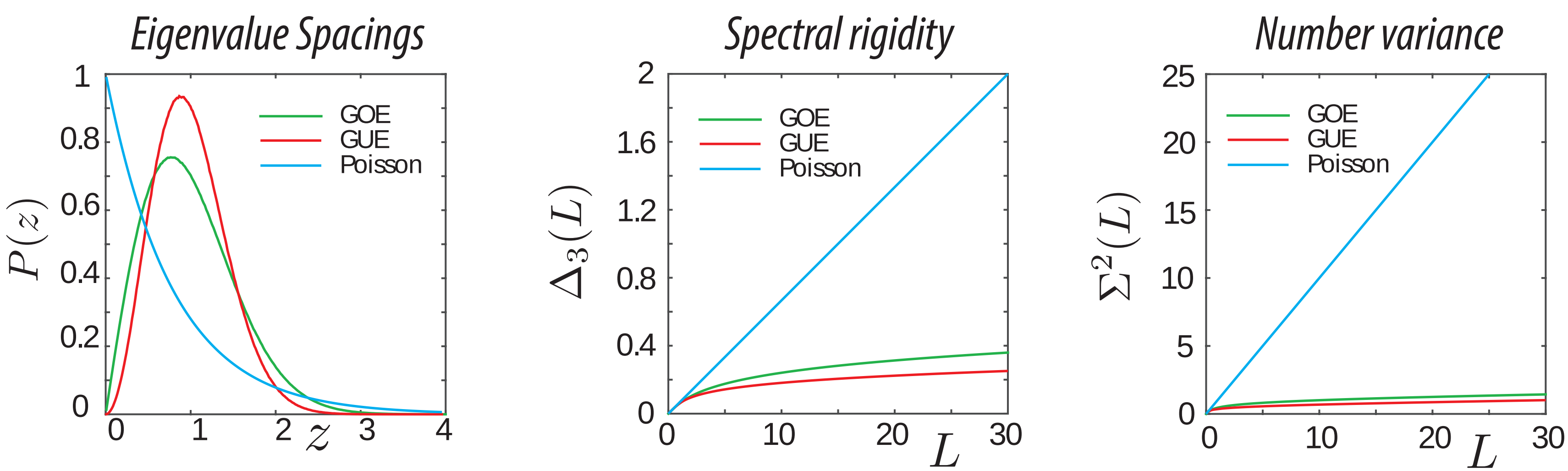
# Eigenvalue Statistics of Random Matrix Theory

***Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.***

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^T)/2 \quad$ ***Gaussian orthogonal ensemble (GOE)***

$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^\dagger)/2 \quad$ ***Gaussian unitary ensemble (GUE)***

*Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics*




***Short range correlations***

*eigenvalue repulsion*

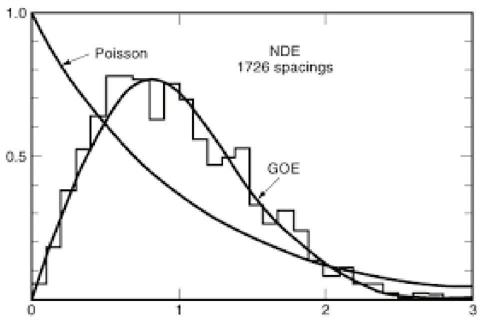
***Long range correlations measure the stiffness of the spectrum***

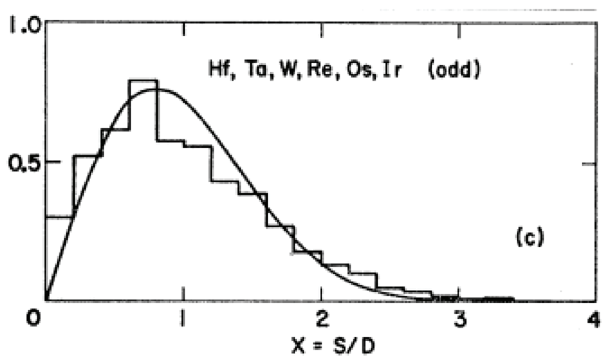
**RMT has since been used to characterize disorder driven transitions in mesoscopic conductors, neural networks, random graph theory, etc.**

**Phase transitions of such a physical system may be characterized by transitions in universal eigenvalue statistics.**

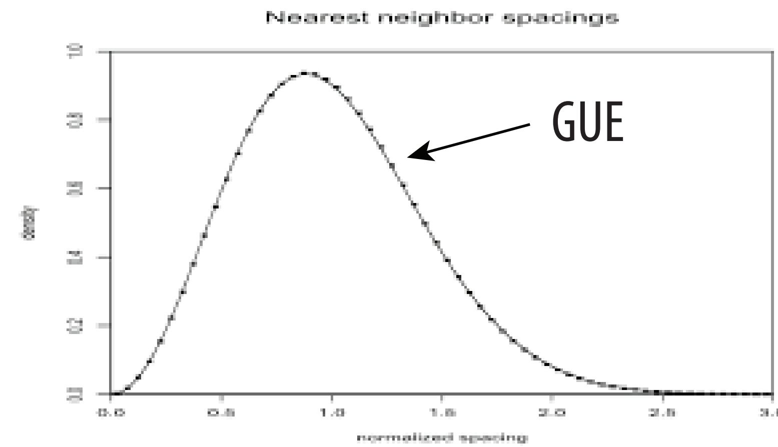
# Random matrix theory has been successful in describing universal features of many disordered systems

Spacing distributions of energy levels for heavy atomic nuclei

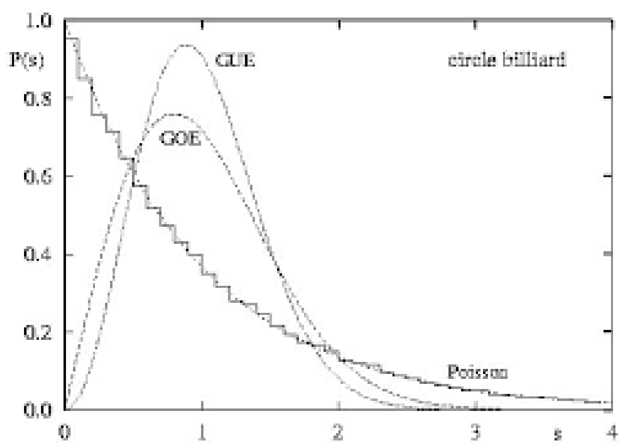


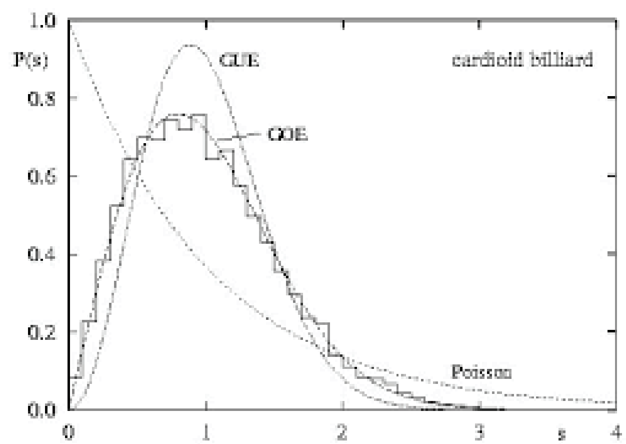

Spacing distributions of the first billion zeros of the Riemann zeta function




Spacing distributions of energy levels for quantum chaos

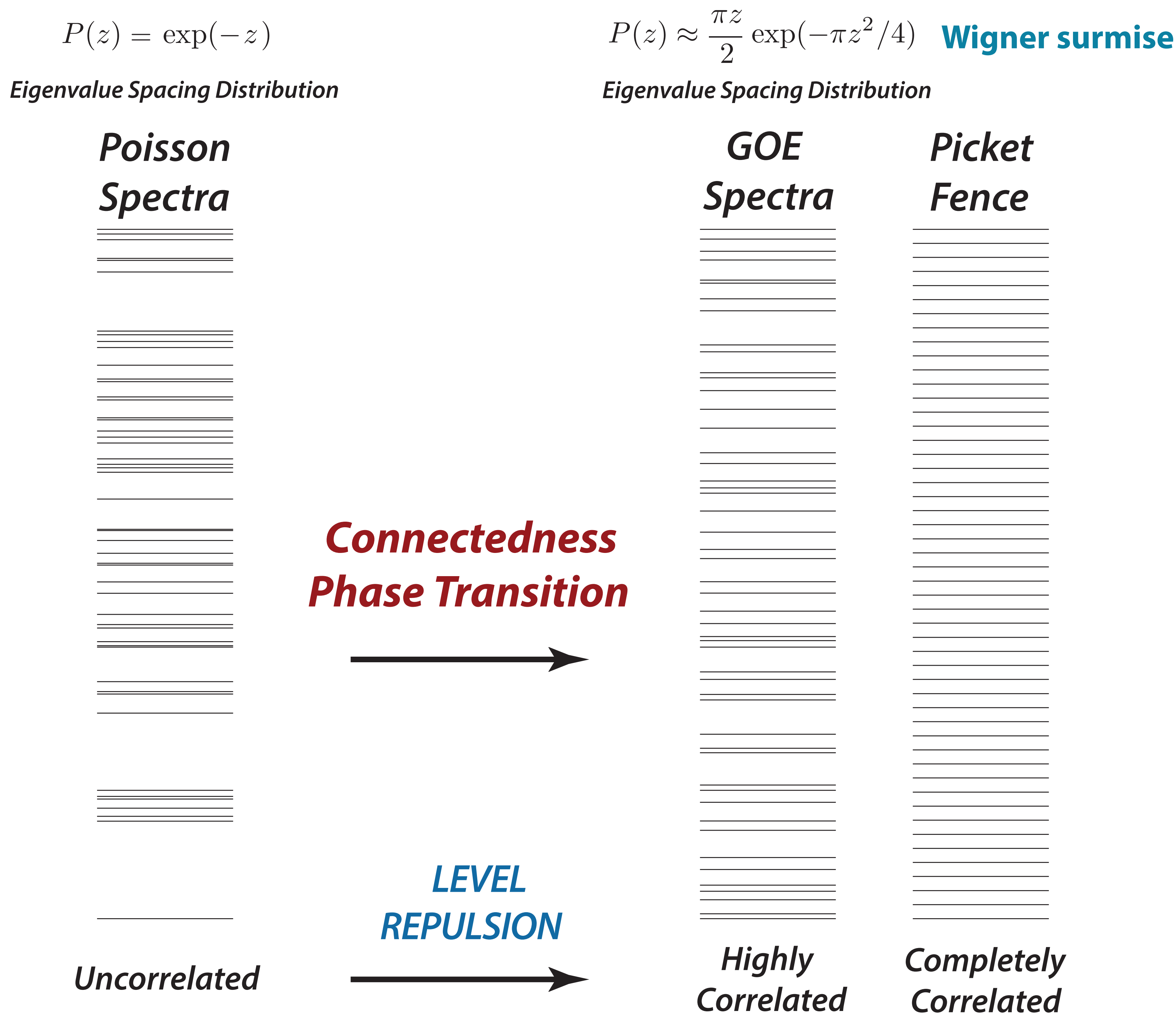



non-chaotic

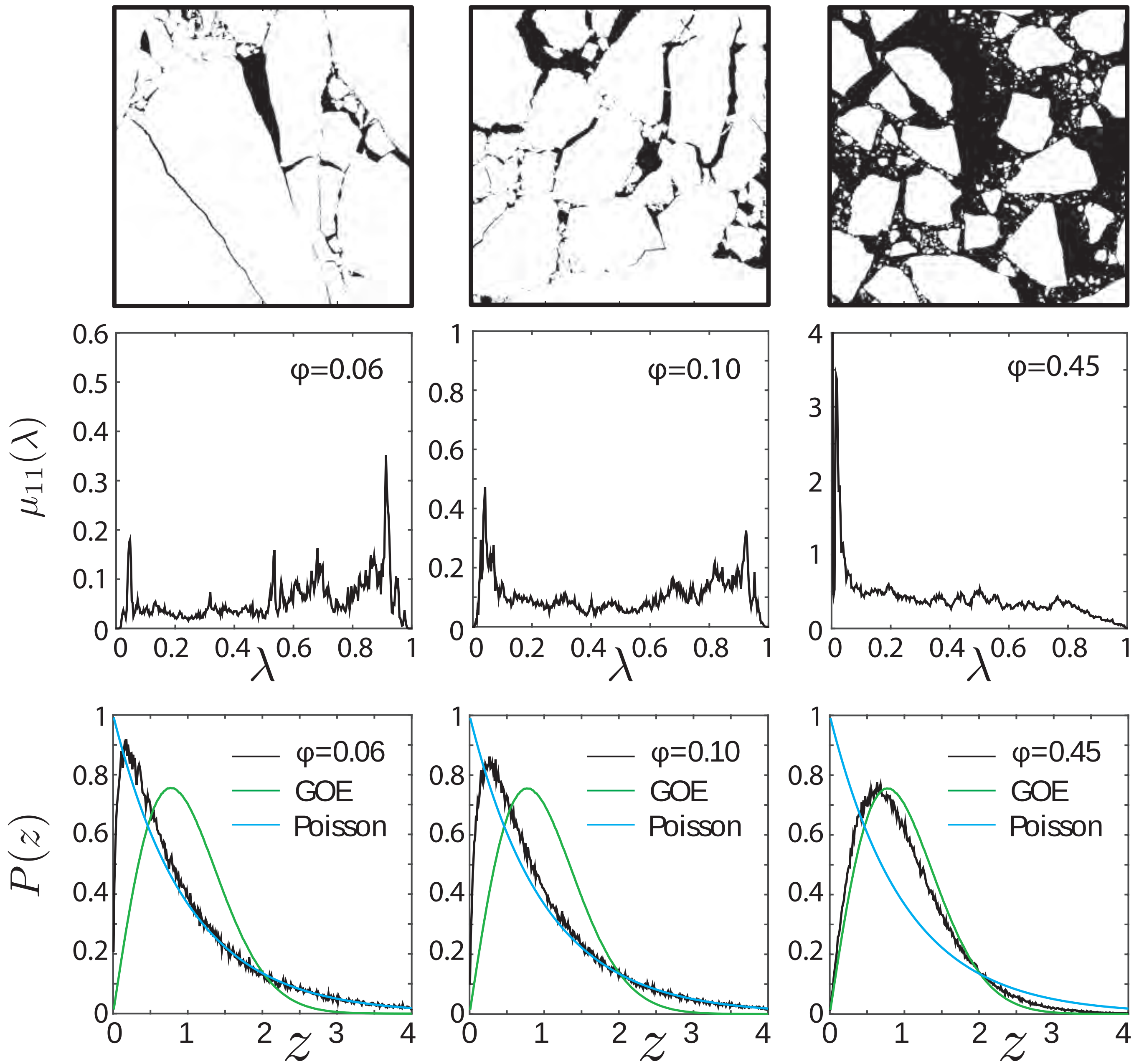



chaotic

# Transition in Eigenvalue Correlations



# Spectral computations for Arctic sea ice pack



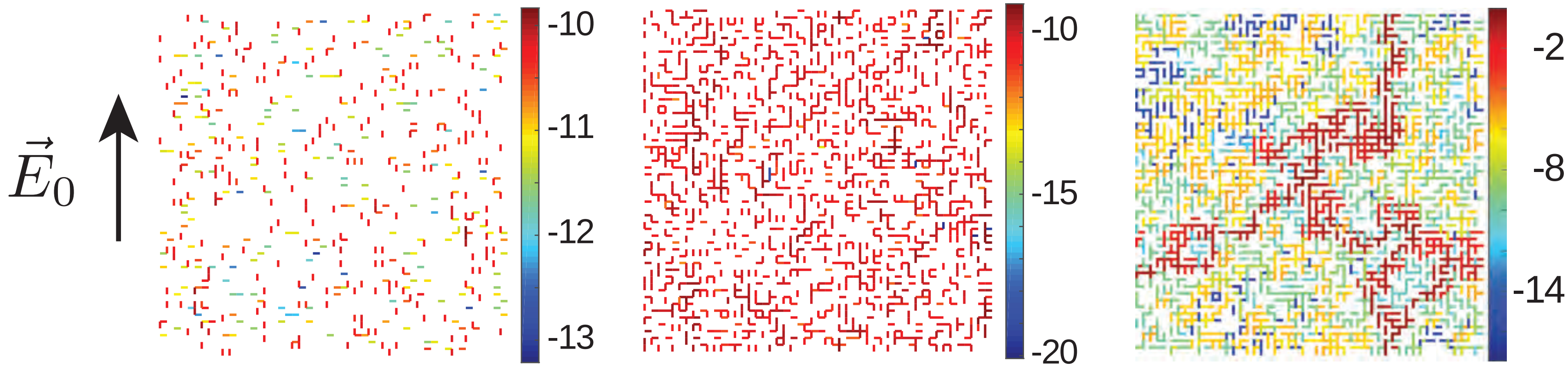

**spectral measures**

**eigenvalue spacing distributions**

**TRANSITION**

# Spectral statistics for 2D random resistor network

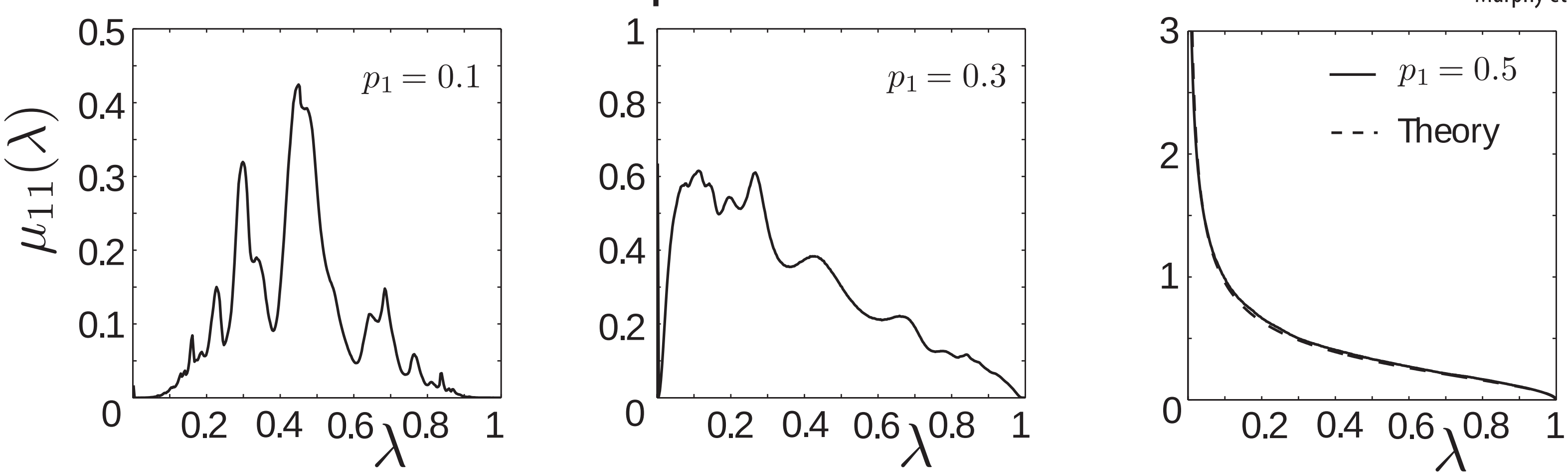
## Log Electric Field




Murphy and Golden, J. Math. Phys., 2012

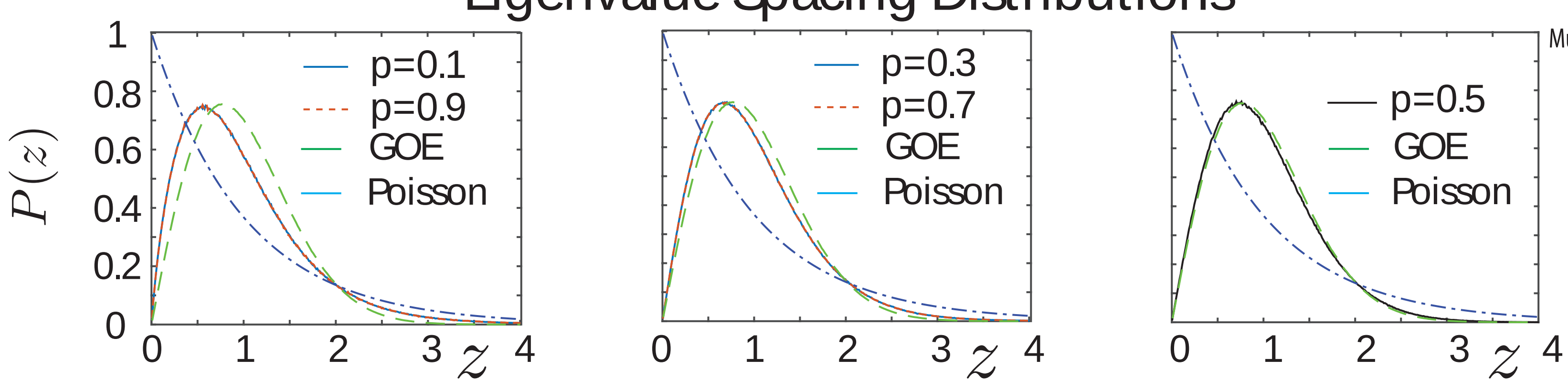
Murphy et al. Comm. Math. Sci., 2015

## Spectral Measures




## Eigenvalue Spacing Distributions

Murphy, Cherkaev, Golden, 2016



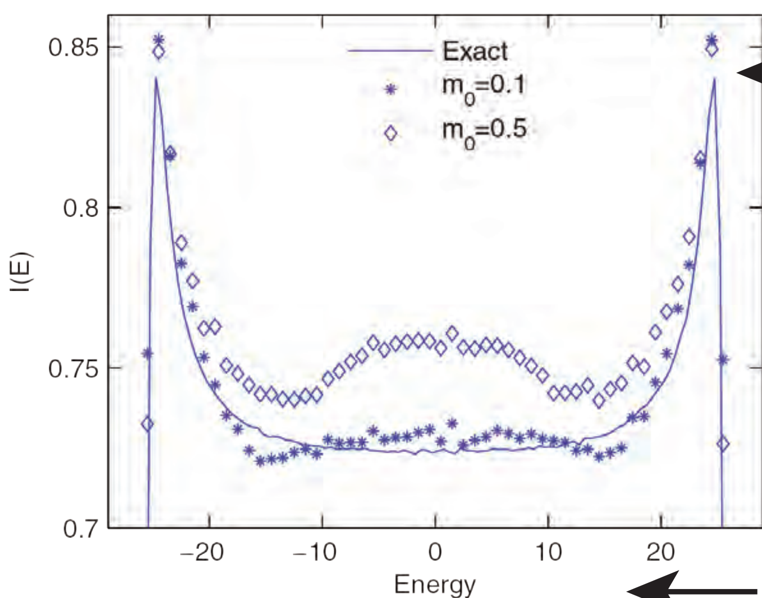
# Eigenvector Localization and Random Matrix Theory

Inverse Participation Ratio: $I(\vec{v}_n) = \sum_{i=1}^{N} |(\vec{v}_n)_i|^4$

Completely Localized: $I(\vec{e}_n) = 1$

Completely Extended: $I\left(\frac{1}{\sqrt{N}}\vec{1}\right) = \frac{1}{N}$

Anderson Model



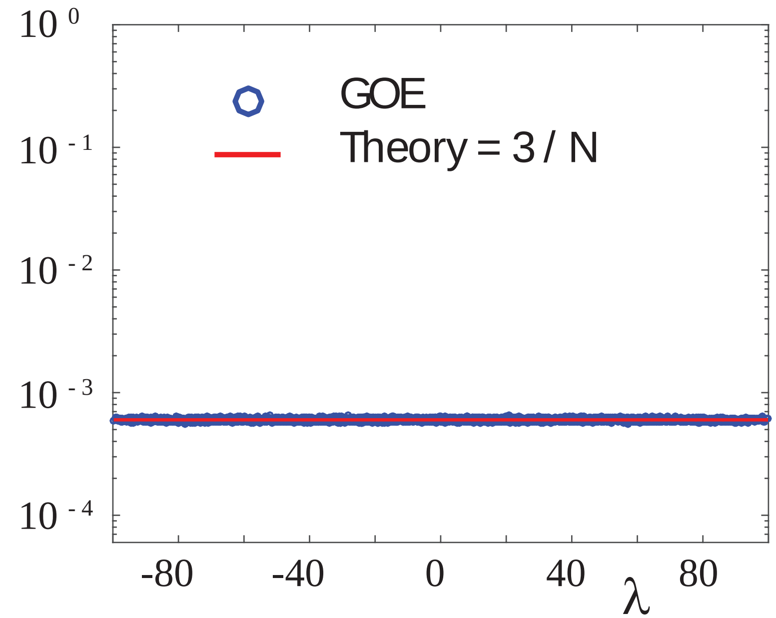

**Mobility Edge**

**“Eigenvalues”**

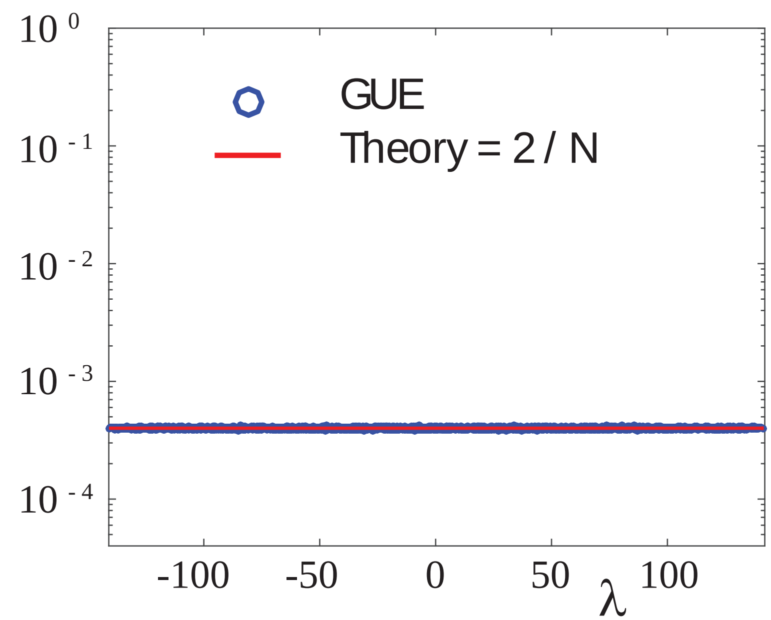
FIG. 4. (Color online) IPR for Anderson model in two dimensions with $x = 6.25$ ($w = 50$) from exact diagonalization (solid line) and from LDRG with different values of the cutoff $m_0$. LDRG data are averaged over 100 runs of systems with $100 \times 100$ sites.

PHYSICAL REVIEW B 90, 060205(R) (2014)

N = 5000




N = 5000



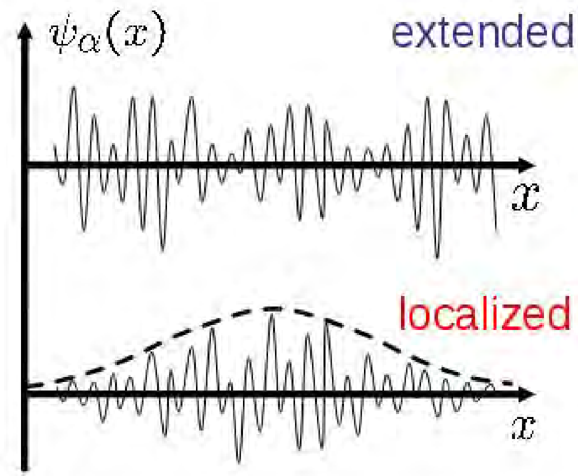
# Anderson localization in quantum systems

## metal / insulator transition at critical disorder

potential $V(x)$ — wavefunctions — energy spacings

low disorder

high disorder

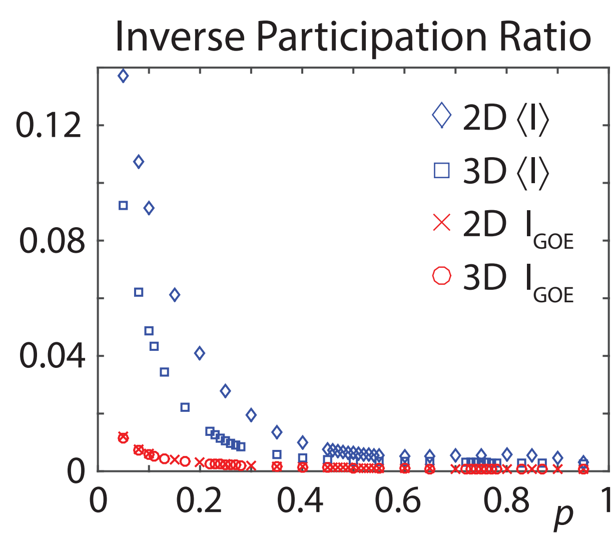



GOE

Poisson

*Anderson, 1958*
*Shklovshii et al, 1993*
*Evangelou, 1992*

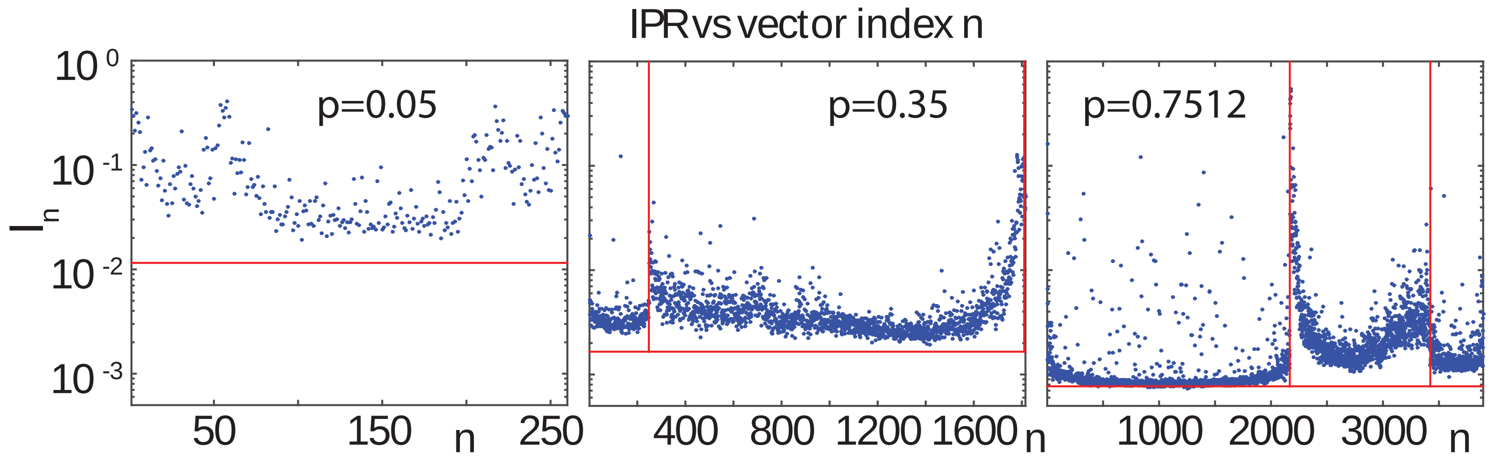
***classical analogue of Anderson localization***

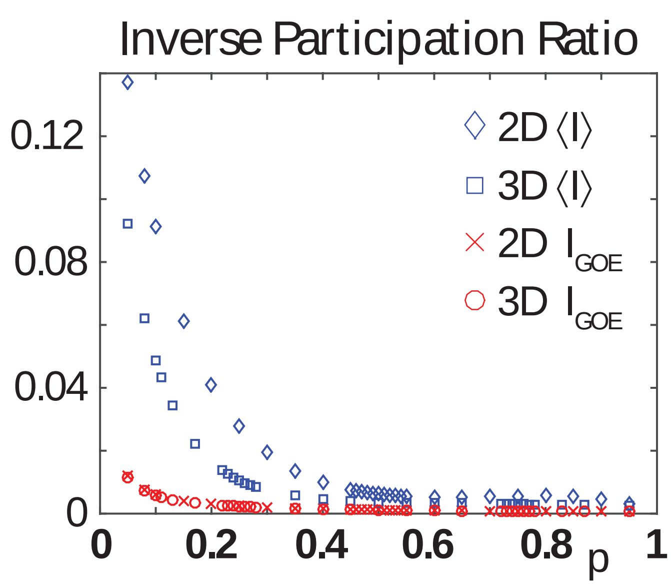



**transition to localized states**
**mobility edges**

***Murphy***
***Cherkaev***
***Golden 2016***

# Localization properties of eigenvectors in random resistor networks



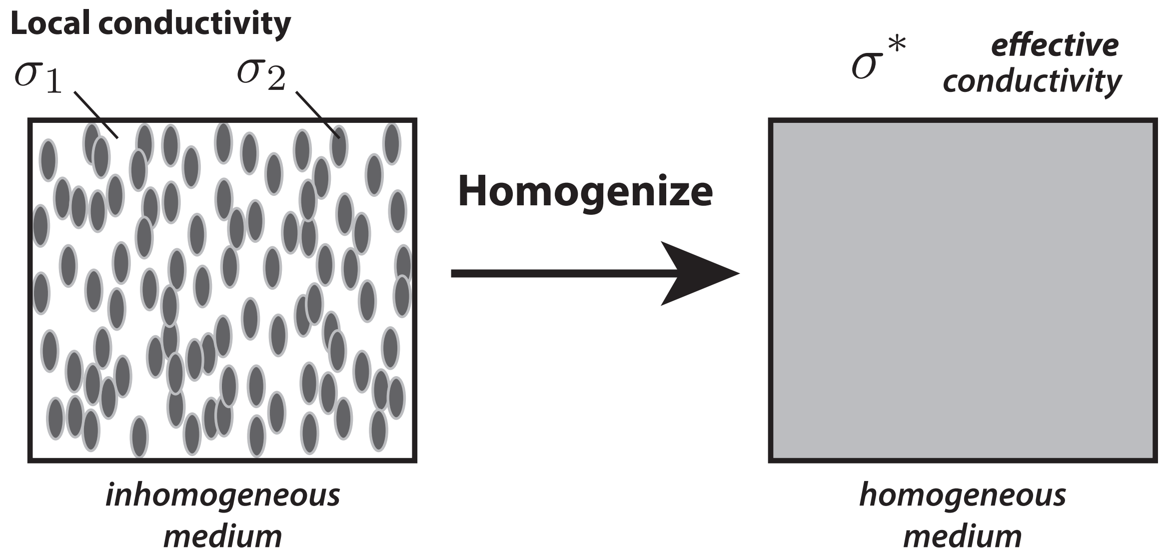




$$I_n = \sum_i (\vec{v}_n)_i^4$$

Murphy, Cherkaev, Golden, 2016

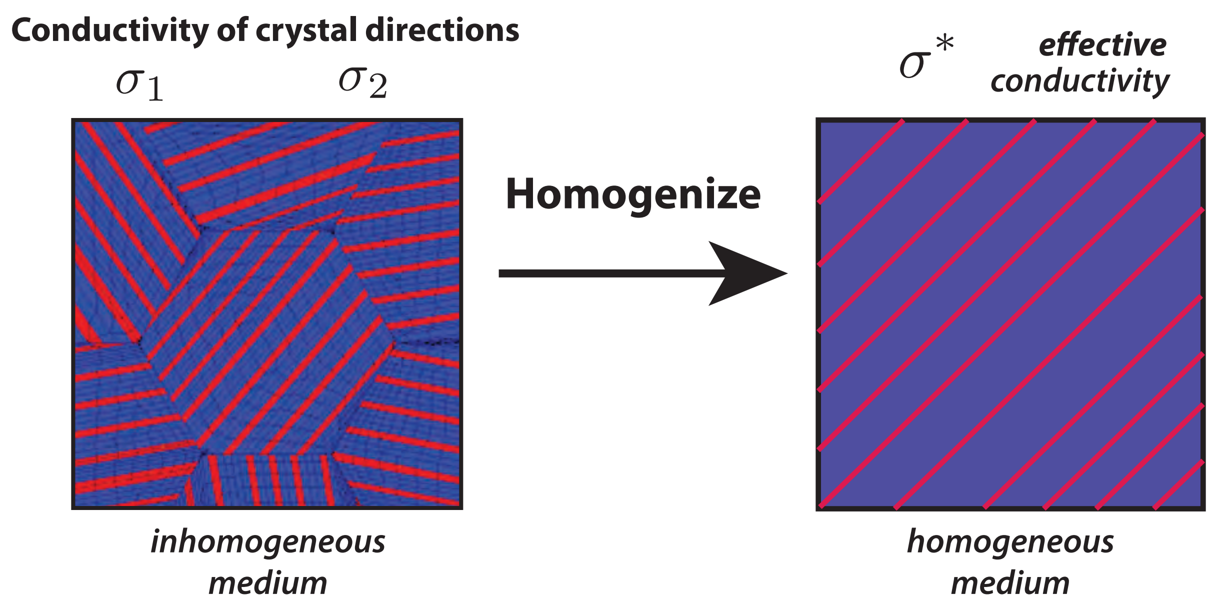
# Homogenization for composite materials

**Two-component composites**




**Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium**

**Polycrystalline media**



# *Mathematical formulation for composite materials*

**Two-component material**

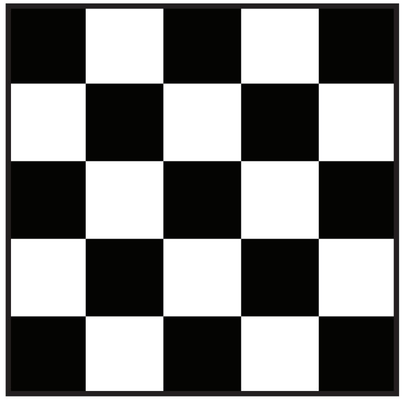
Local conductivity

$$\sigma = \sigma_1\chi_1 + \sigma_2\chi_2$$

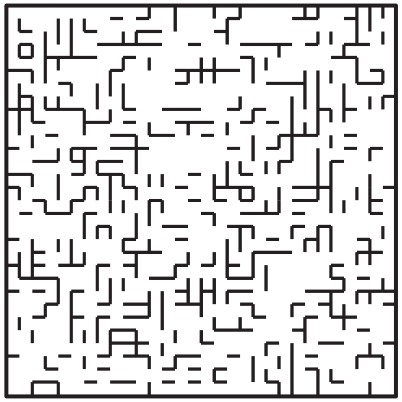
$$\chi_2 = 1 - \chi_1$$

$$\chi_i\chi_j = \delta_{ij}\chi_i$$

*Continuum composite*



*Discrete composite*



$$\vec{\nabla}\cdot\vec{J} = 0, \quad \vec{\nabla}\times\vec{E} = 0, \quad \vec{J} = \sigma\vec{E}, \quad \vec{E} = \vec{\nabla}\phi + \vec{e}_k, \quad \langle\vec{E}\rangle = \vec{e}_k$$

**Polycrystalline material**

Local conductivity

$$\sigma = R\,diag(\sigma_1,\sigma_2,\sigma_2)R^T$$
$$= \sigma_1X_1 + \sigma_2X_2$$

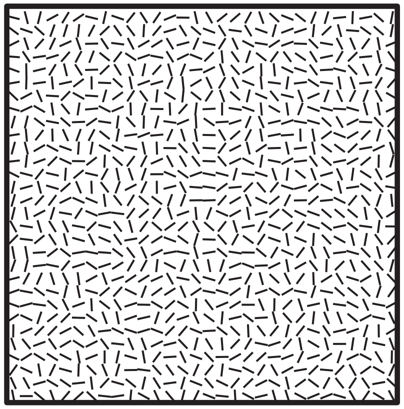
$$X_2 = I - X_1$$

$$X_iX_j = \delta_{ij}X_i$$

Random Rotation Matrix

*Continuum composite*

*Discrete composite*



# Integral representations for bulk transport coefficients

The ***effective conductivity*** is defined in terms of the ***system energy***

$$\langle \vec{J} \cdot \vec{E} \rangle = \sigma^* E_0^2 \,, \qquad \sigma^* = \sigma^*_{kk} \,, \qquad \langle \vec{E} \rangle = E_0 \vec{e}_k$$

- This defines a ***homogeneous medium*** which behaves ***macroscopically*** and ***energetically*** **just like the given inhomogeneous medium.**

---

***Resolvent formula***

$$\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{J} = 0, \qquad \longrightarrow \qquad X_1 \vec{E} = s E_0 (sI - X_1 \Gamma X_1)^{-1} X_1 \vec{e}_k$$

$$\vec{J} = \sigma \vec{E}, \qquad \sigma = \sigma_1 X_1 + \sigma_2 X_2$$

***Contrast parameter:*** $s = 1/(1 - \sigma_1/\sigma_2)$

***Projection onto curl-free fields:*** $\Gamma = -\vec{\nabla}(-\Delta)^{-1}\vec{\nabla}\cdot$

---

$$\sigma^*/\sigma_2 = 1 - \langle X_1 \vec{E} \cdot \vec{e}_k \rangle / (s E_0) = 1 - F(s)$$

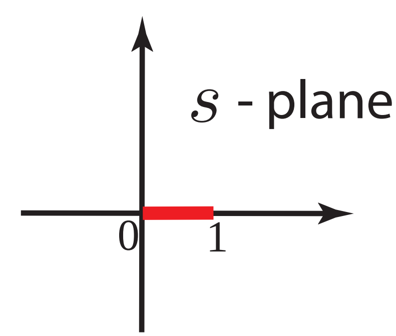
---

Gully et al., *Proc. Roy. Soc. A* 2015
Murphy, Cherkaev, Golden 2016

$$F(s) = \int_0^1 \frac{d\mu(\lambda)}{s - \lambda}$$

$\mu$

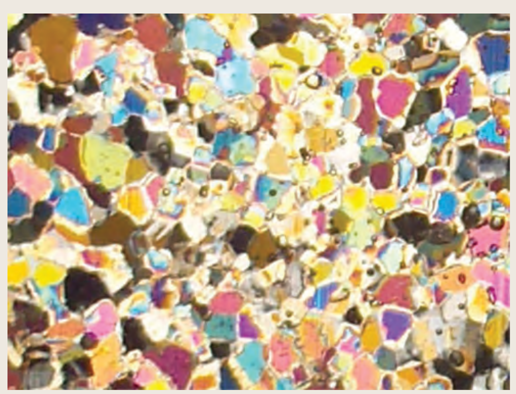
- spectral measure of the self-adjoint operator $X_1 \Gamma X_1$
- mass = average orientation
- higher moments depend on $n$-point correlations



# Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- Stieltjes integral representation for effective complex permittivity
- Forward and inverse bounds
- Applied to sea ice using two-scale homogenization
- Inverse bounds give method for distinguishing ice types using remote sensing techniques



**Proc. Roy. Soc. A    8 Feb 2015**

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

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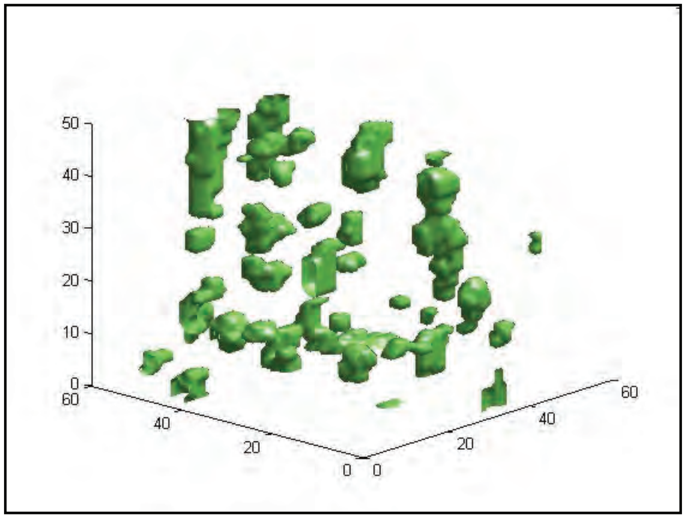
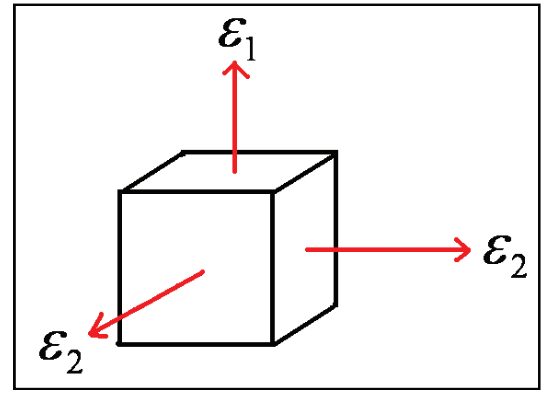
21/01/15 1:47 PM

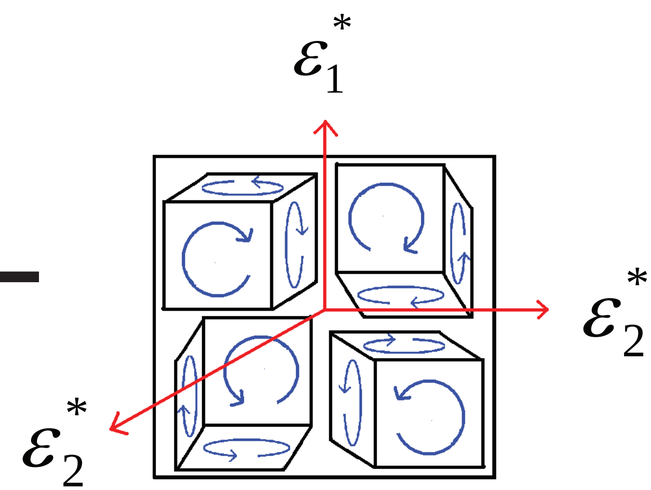
# *two scale homogenization for polycrystalline sea ice*



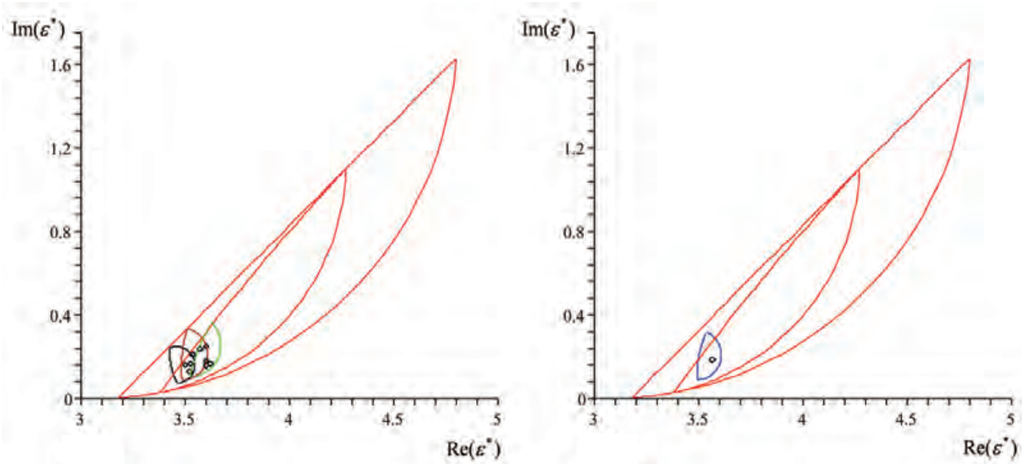
**numerical homogenization for single crystal**




**analytic continuation for polycrystals**

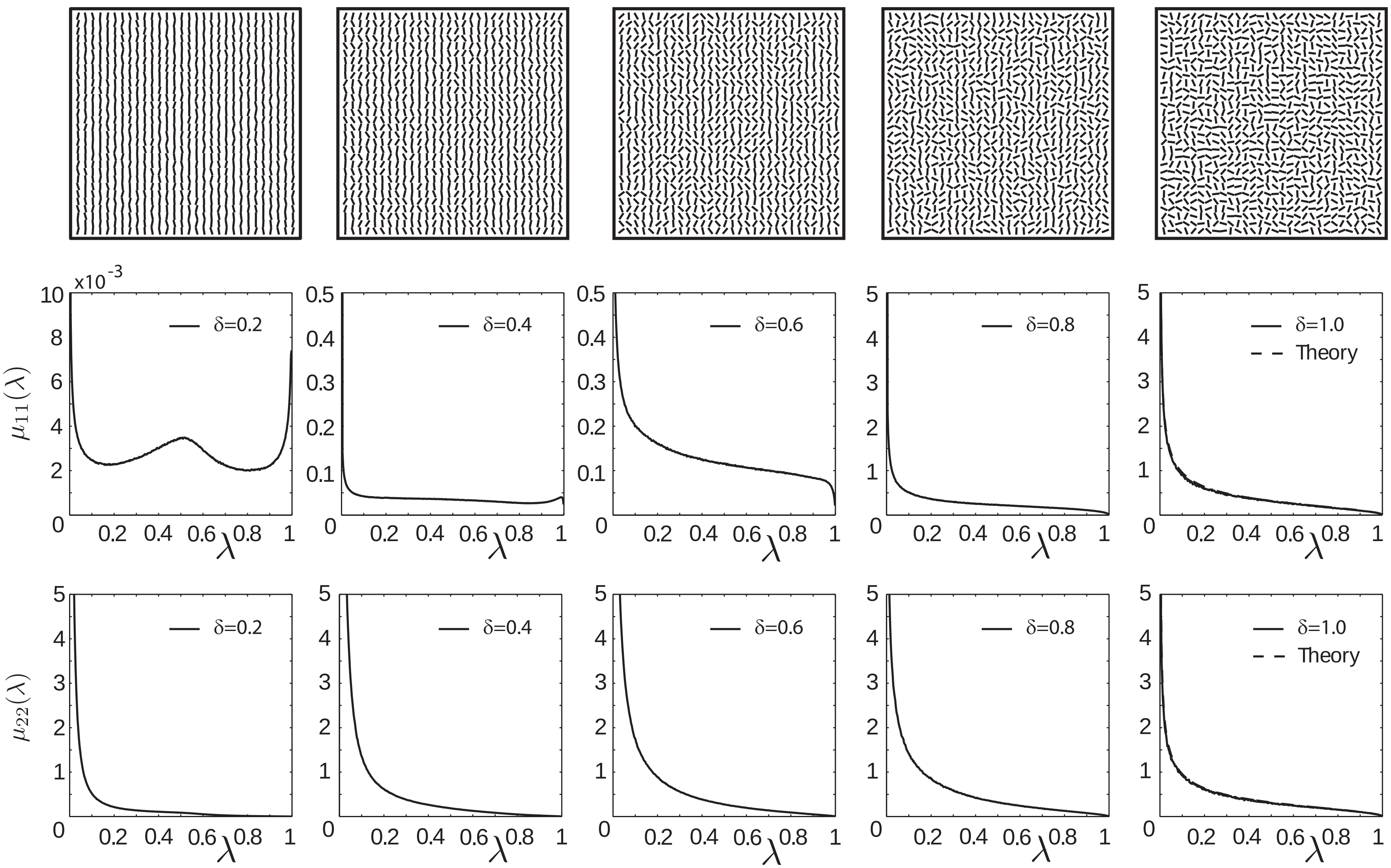



**bounds**




*Gully, Lin, Cherkaev, Golden, Proc. Roy. Soc. A (and cover) 2015*

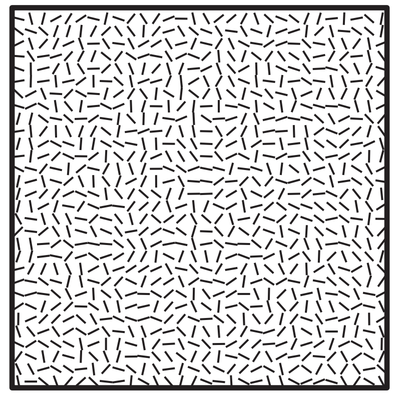
# Spectral measures for uniaxial polycrystalline media



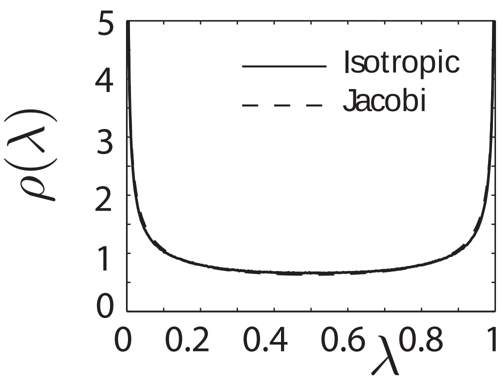

- Random crystallographic orientations angles θ measured from the vertical direction, uniformly distributed $\theta \sim U(-\delta\pi/2, \delta\pi/2)$

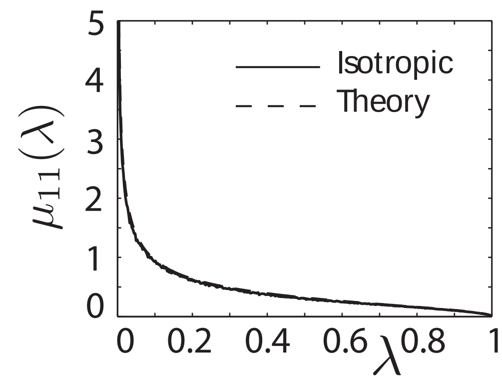
Murphy, Cherkaev, Golden, 2016

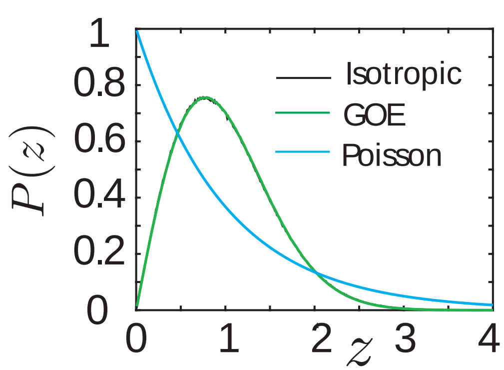
# *Isotropic polycrystalline media and the Jacobi random matrix ensemble*

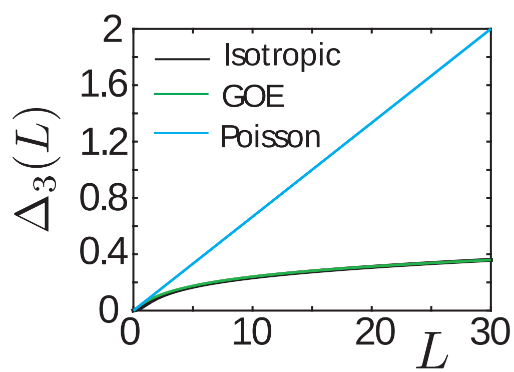


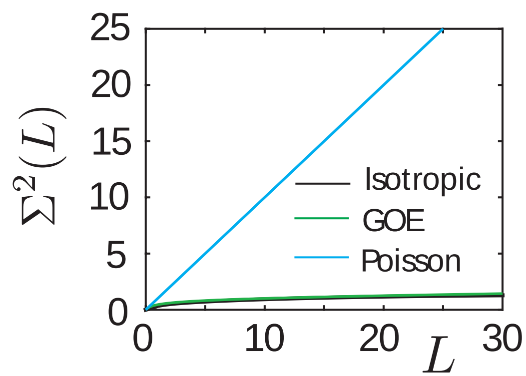
***Murphy, Cherkaev, Golden, 2016***



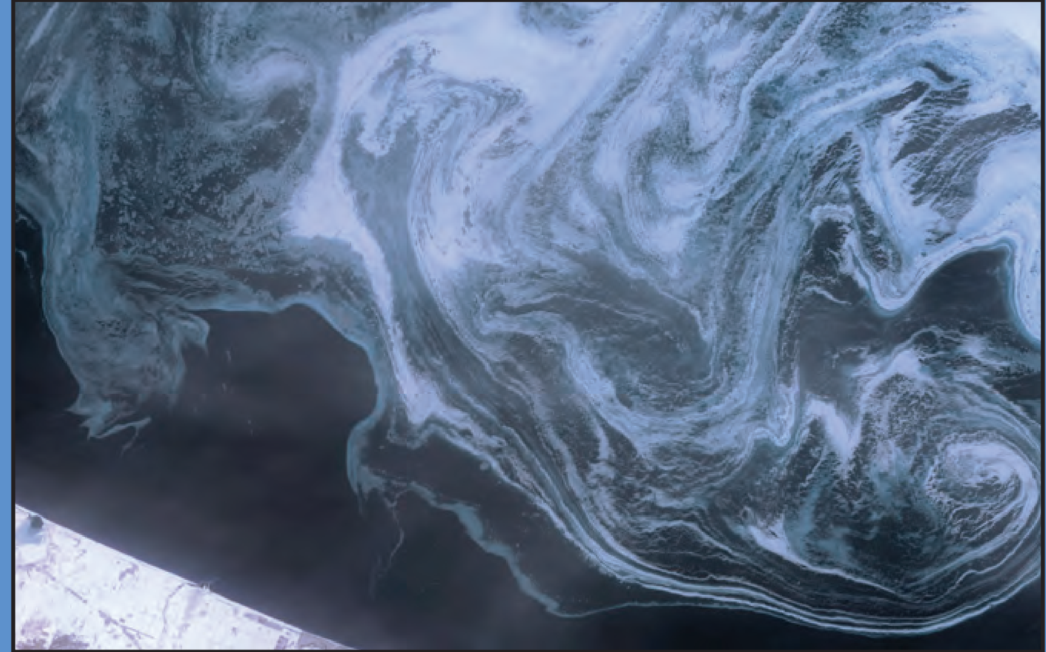



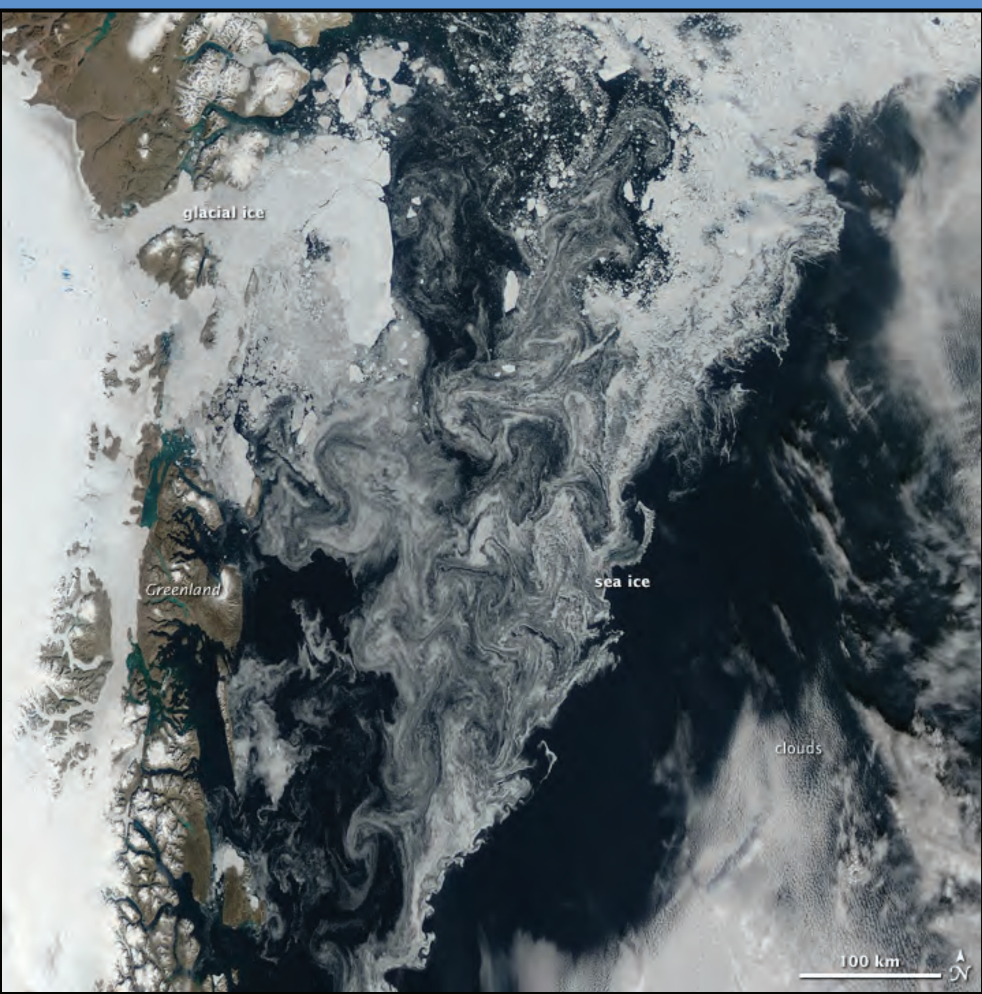



**Advection-diffusion** plays a key role in the transport of pack ice by atmospheric and oceanic flows.

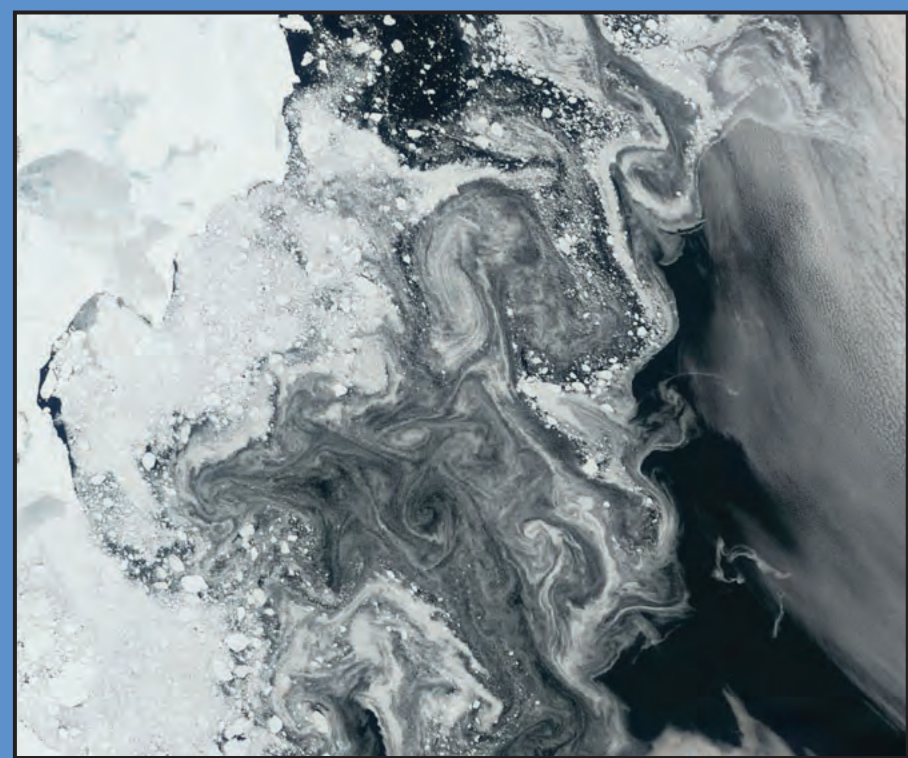


Sea of Okhotsk, NASA, 2009

Ice in the Greenland Sea (77.5° N, 9° W), NASA, 2014



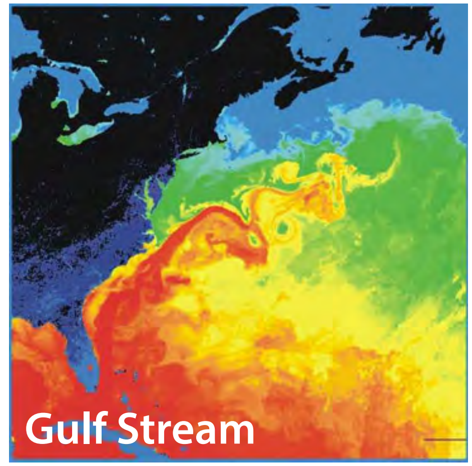

Off the northeastern coast of Greenland, NASA, 2006

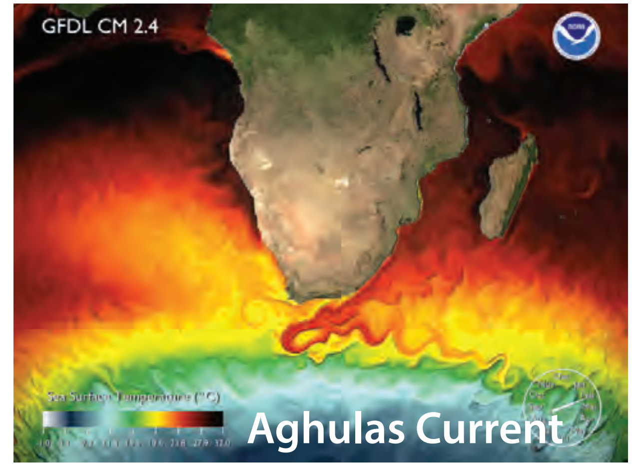


## *advection enhanced diffusion*

### effective diffusivity

tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere
salt and heat transport in ocean




**advection diffusion equation with a velocity field** $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

***homogenize***

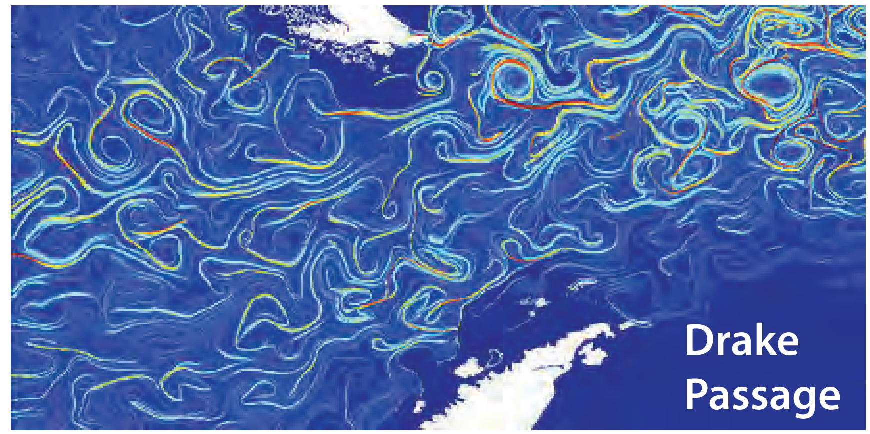
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

$\kappa^*$ ***effective diffusivity***

*analytic function of Péclet number*

***Stieltjes integral for*** $\kappa^*$ ***with spectral measure***

***Avellaneda and Majda, PRL 89, CMP 91***




$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\vec{u} = \vec{\nabla} \cdot H$$

---

$$\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{J} = 0,$$

$$\vec{J} = \sigma \vec{E}, \qquad \sigma = \kappa_0 I + S, \qquad \longrightarrow$$

***Resolvent formula***

$$\vec{\nabla}\phi = (\kappa_0 I - \imath \Gamma S \Gamma)^{-1}\, \Gamma H \vec{e}_k$$

***Projection onto curl-free fields:*** $\Gamma = -\vec{\nabla}(-\Delta)^{-1}\vec{\nabla}\cdot$

Steady flow

$$S = H$$

Dynamic flow

$$S = H + (-\Delta)^{-1} \frac{\partial}{\partial t}$$

---

$$\kappa^*/\kappa_0 = 1 + \langle \vec{\nabla}\phi \cdot \vec{\nabla}\phi \rangle = 1 + G(\kappa_0)$$

---

**Steady flow:** **Murphy, Cherkaev, Zhu, Xin, Golden 2016**

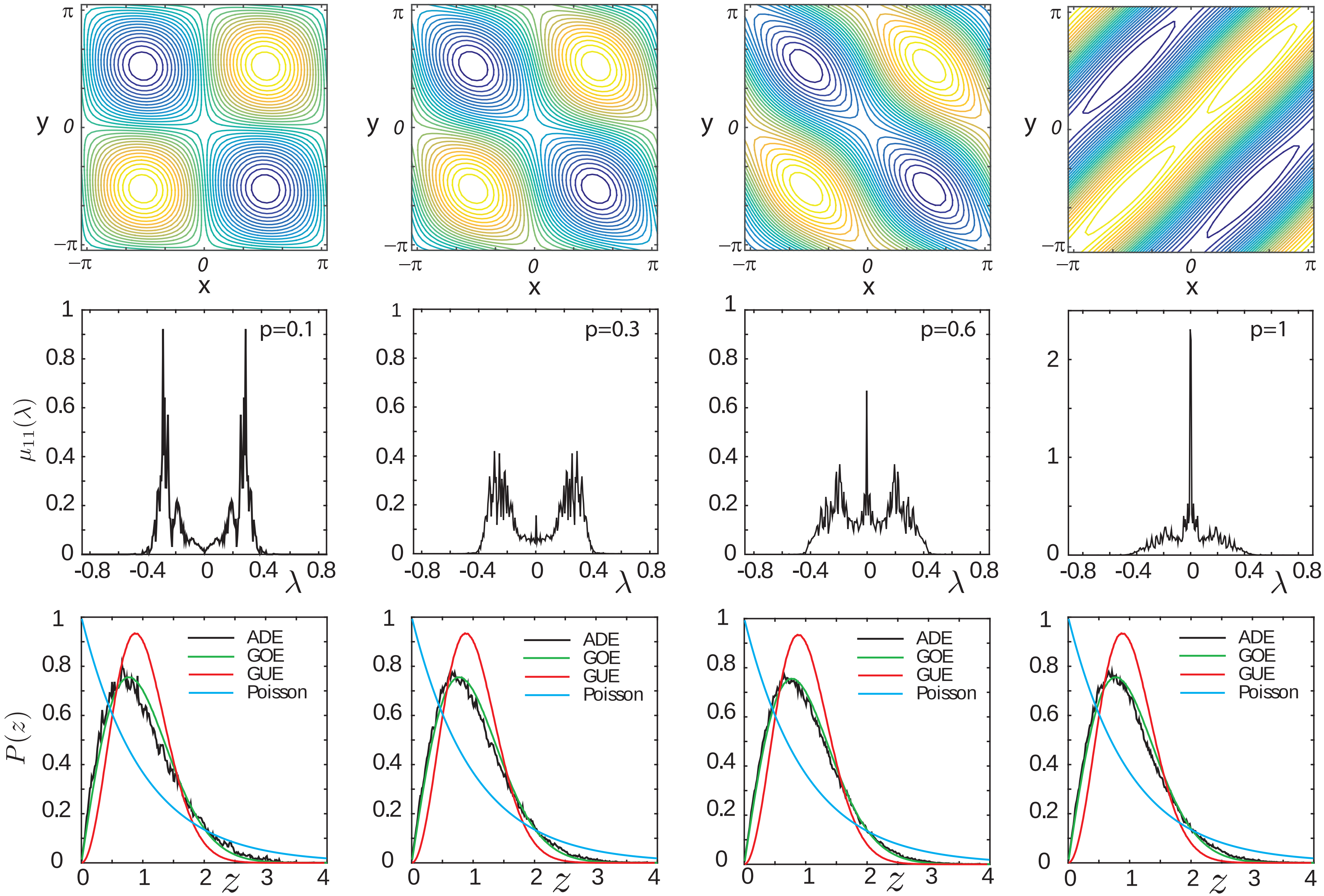
**Dynamic flow:** **Murphy, Cherkaev, Xin, Zhu, Golden 2016**

$$G(\kappa_0) = \int_{-\infty}^{\infty} \frac{d\nu(\lambda)}{\kappa_0^2 + \lambda^2}$$

$\nu$ — spectral measure of the self-adjoint operator $\imath \Gamma S \Gamma$

# Spectral measures and eigenvalue spacings for cat's eye flow

$$H(x,y) = \sin(x)\sin(y) + A\cos(x)\cos(y), \qquad A \sim U(-p,p)$$




Murphy, Cherkaev, Xin, Golden, 2016

## Thermal Conduction Enhanced with Convection

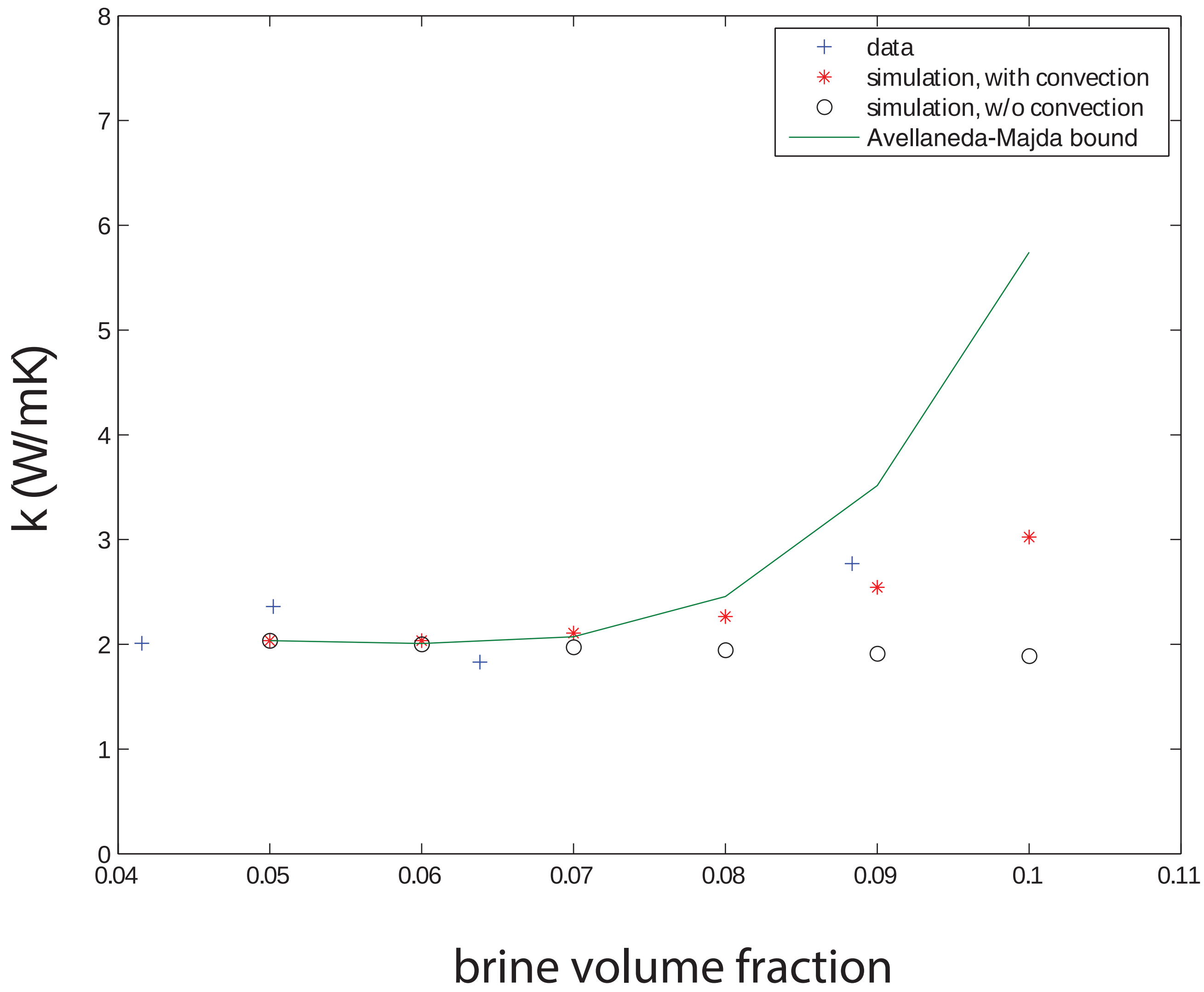
- Temperature on top surface driven by atmosphere conditions
- Bottom surface in contact with sea water
- Temperature field $T$ governed by a nonlinear convection-diffusion equation

$$\rho c \left( \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = \nabla \cdot (\kappa(T) \nabla T)$$

  with a Darcy velocity $\mathbf{u}$
- Parameters:
  - $\rho =$ bulk density
  - $c =$ specific heat
  - $\kappa(T) =$ temperature dependent thermal conductivity

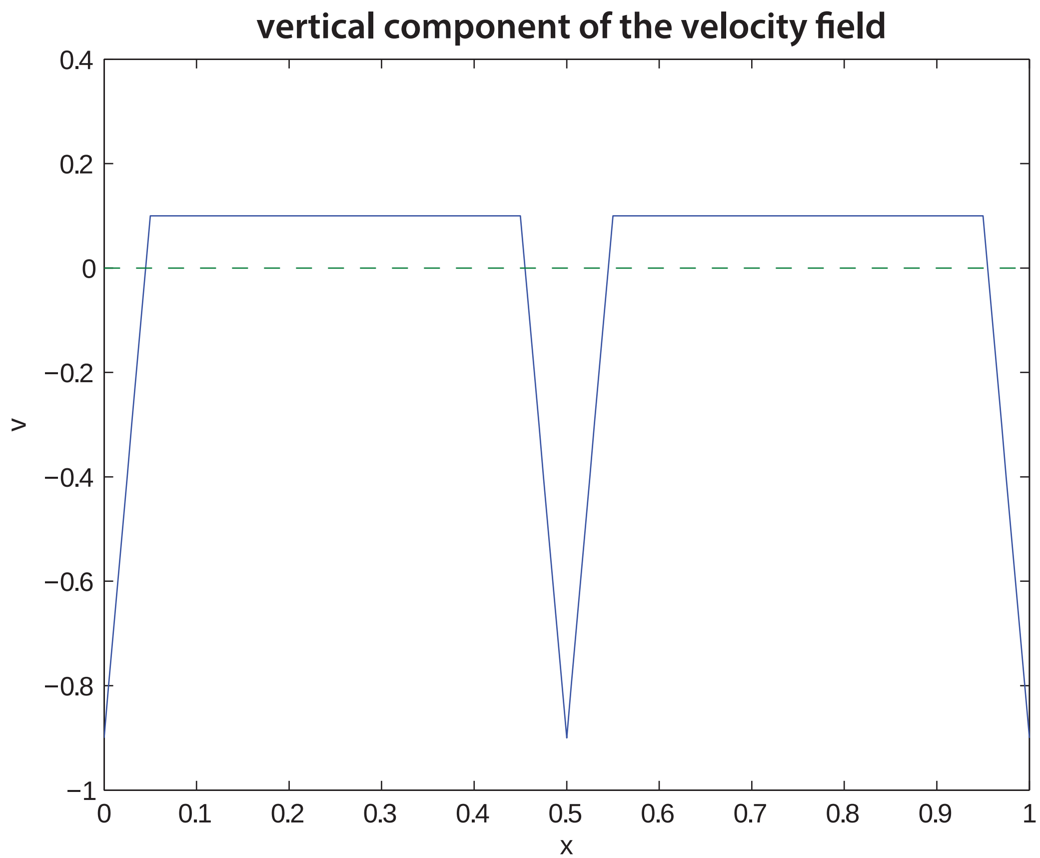
# convection enhanced thermal conductivity of sea ice for shear flow

## *numerical solution of advection diffusion equation, rigorous bounds*




***Wang, Zhu, Golden, 2016***

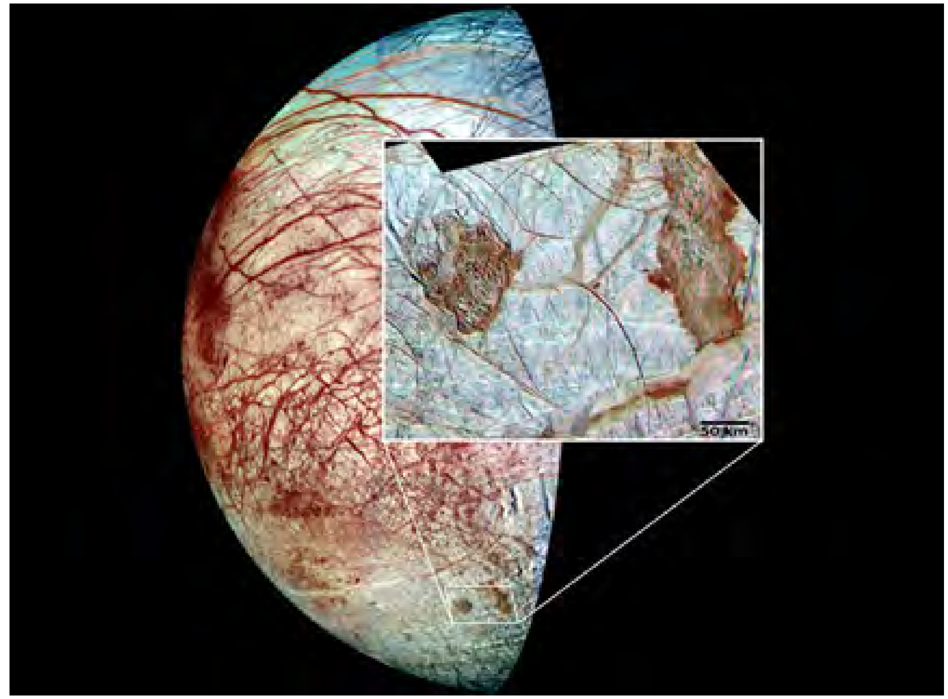
# Brine velocity field - slow uniform upward, fast plume downward



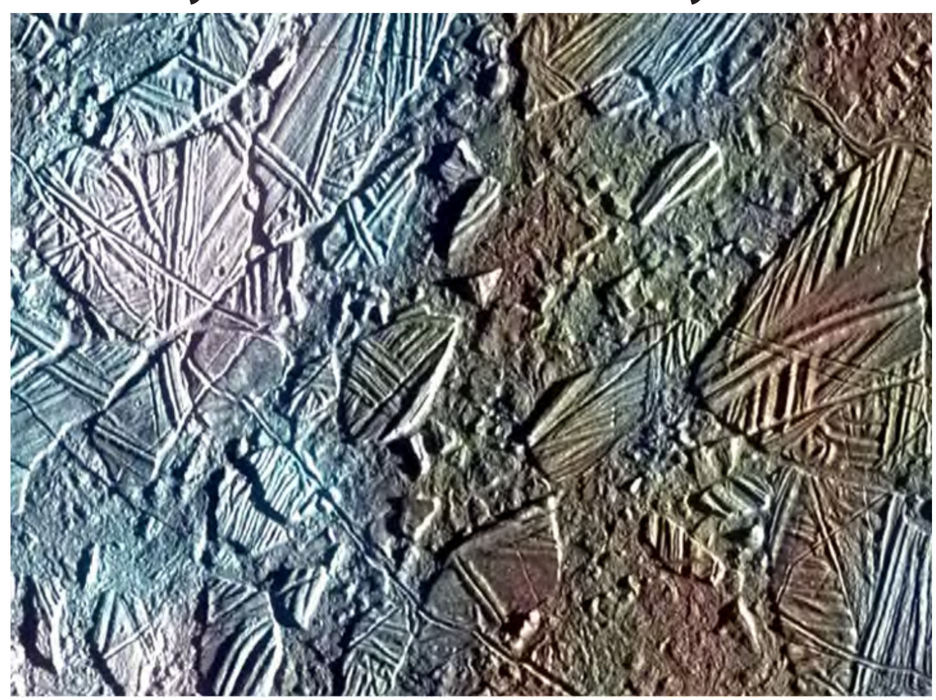
**What about fluid flow, thermal transport, and convective processes in the porous microstructure of "sea ice" on Europa?**

**Are sea ice algae and bacteria proxies for life on extraterrestrial, icy bodies?**

(Thomas, Dieckmann, Science, 2002)

**EUROPA - believed covered by deep briny ocean, with thick icy crust**



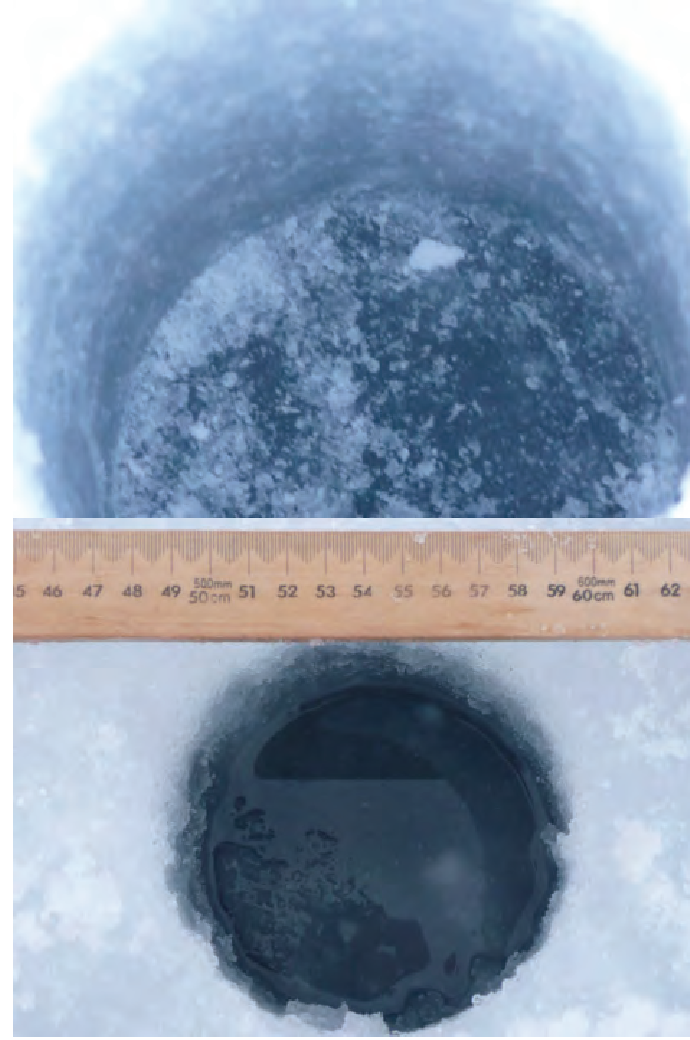
# Arctic and Antarctic field experiments

***develop electromagnetic methods of monitoring fluid transport and microstructural transitions***

extensive measurements of fluid and electrical transport properties of sea ice:

| | | |
|---|---|---|
| ***2007*** | ***Antarctic*** | ***SIPEX*** |
| ***2010*** | ***Antarctic*** | ***McMurdo Sound*** |
| ***2011*** | ***Arctic*** | ***Barrow AK*** |
| ***2012*** | ***Arctic*** | ***Barrow AK*** |
| ***2012*** | ***Antarctic*** | ***SIPEX II*** |
| ***2013*** | ***Arctic*** | ***Barrow AK*** |
| ***2014*** | ***Arctic*** | ***Chukchi Sea*** |


*photo by Jan Lieser*

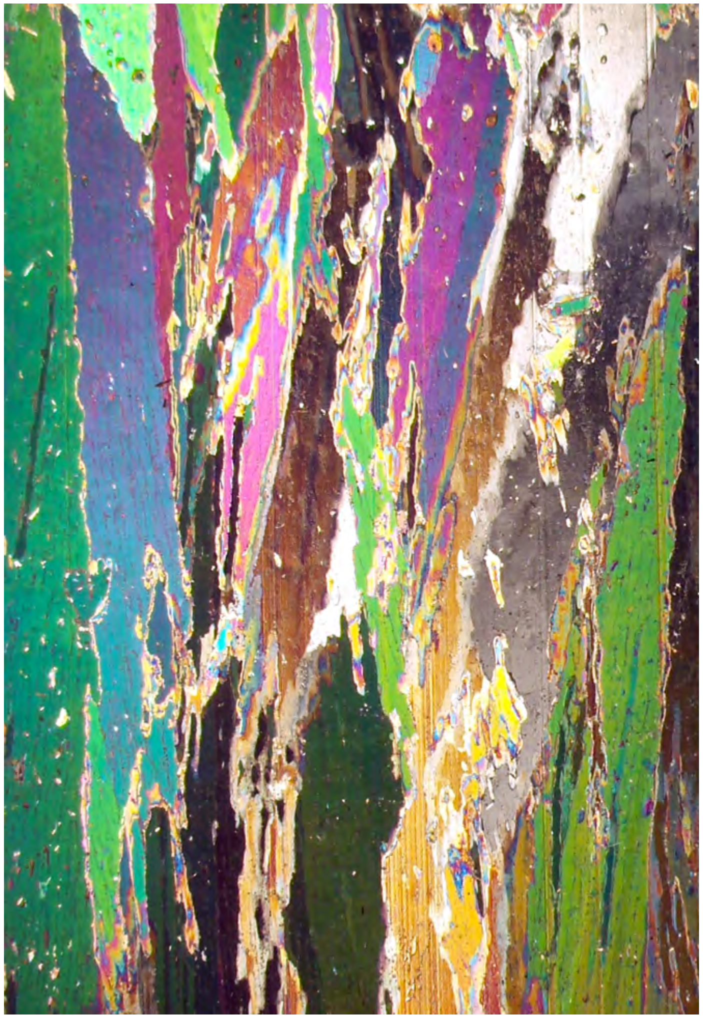


***measuring fluid permeability of Antarctic sea ice***

***SIPEX 2007***

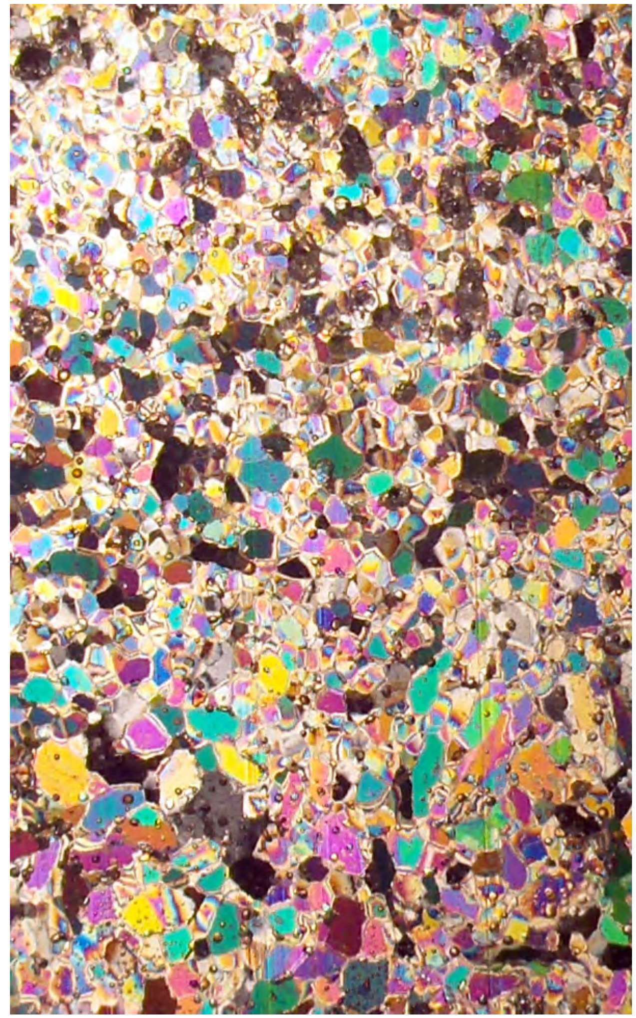
# higher threshold for fluid flow in Antarctic granular sea ice

columnar



**5%**

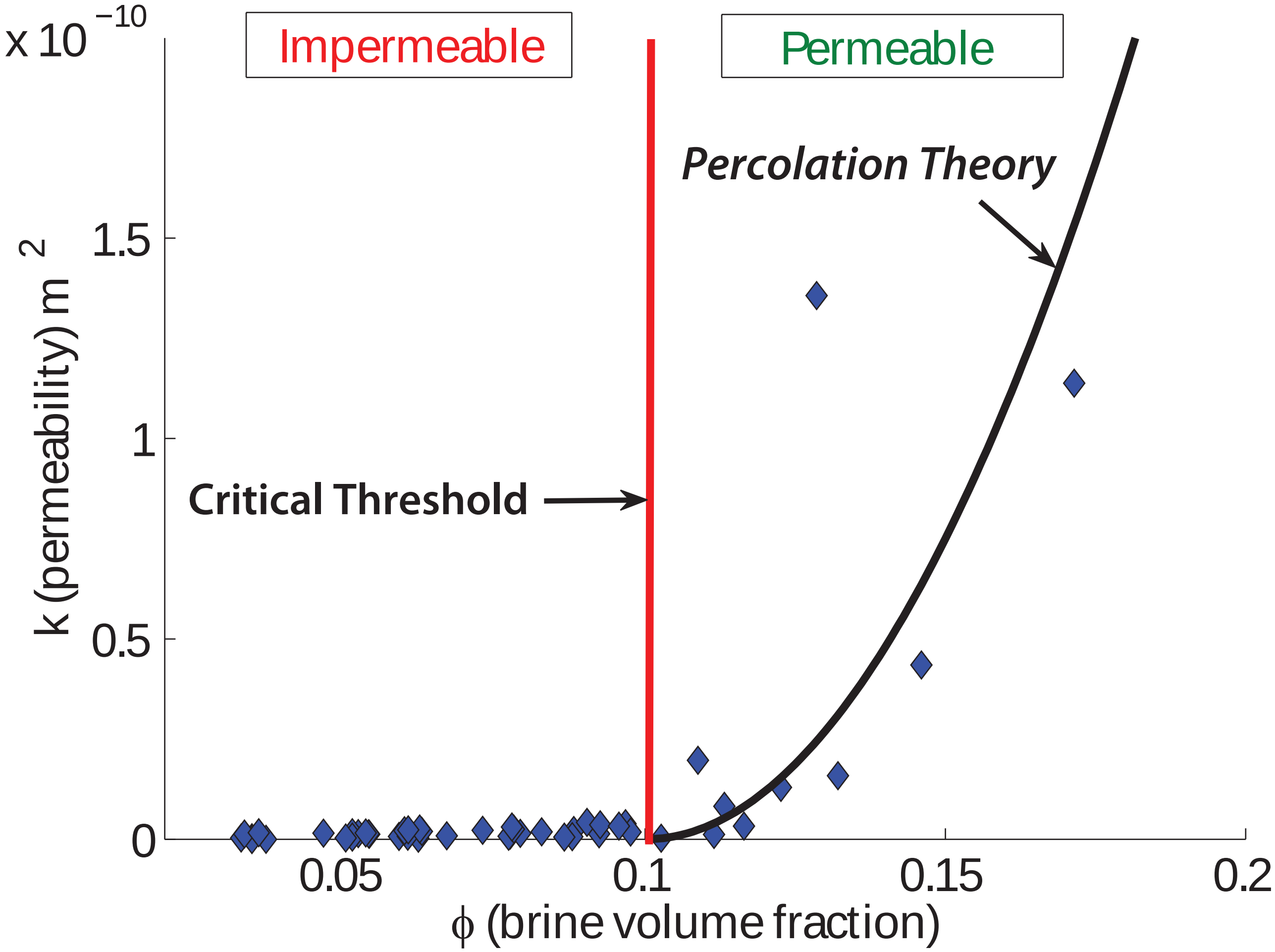
granular



**10%**

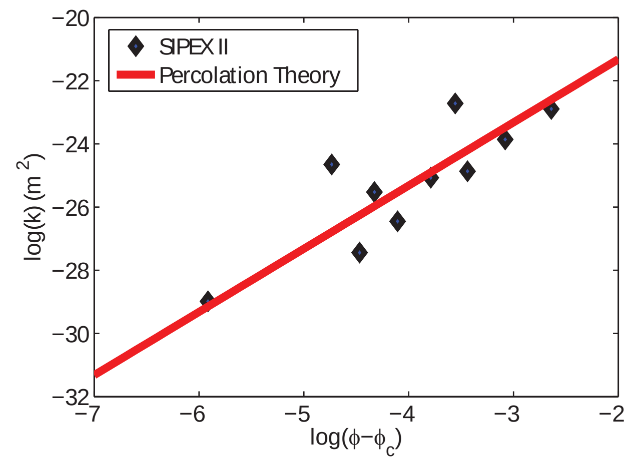
***Golden, Sampson, Gully, Lubbers, Tison 2016***

# SIPEX II vertical permeability data




*same universal critical exponent as lattice models*

**data above threshold**

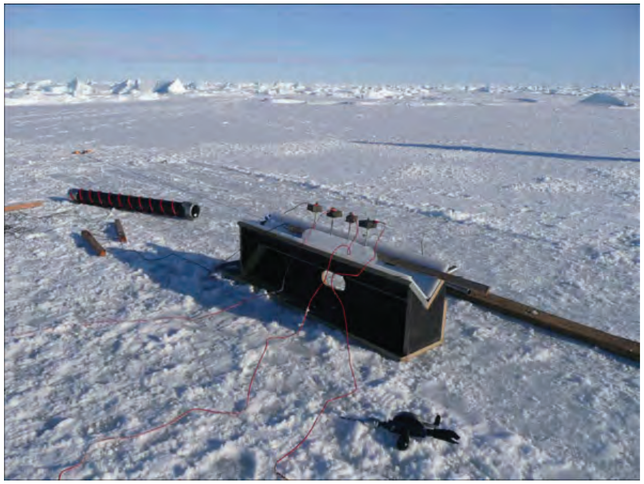


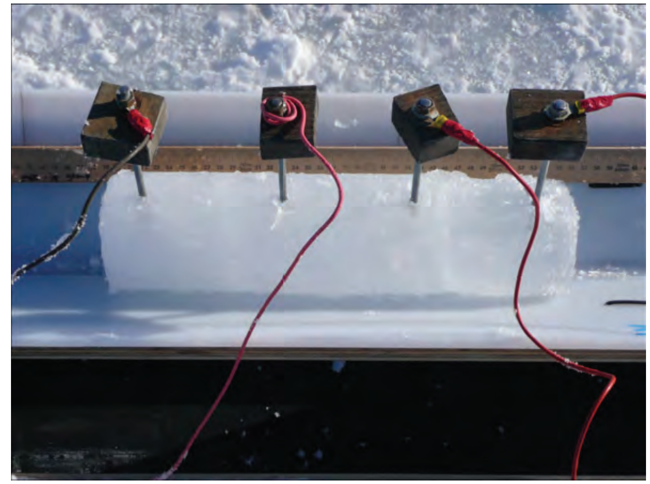

*higher threshold in granular ice predicted with percolation theory by Golden, et al. (Science, 1998)*

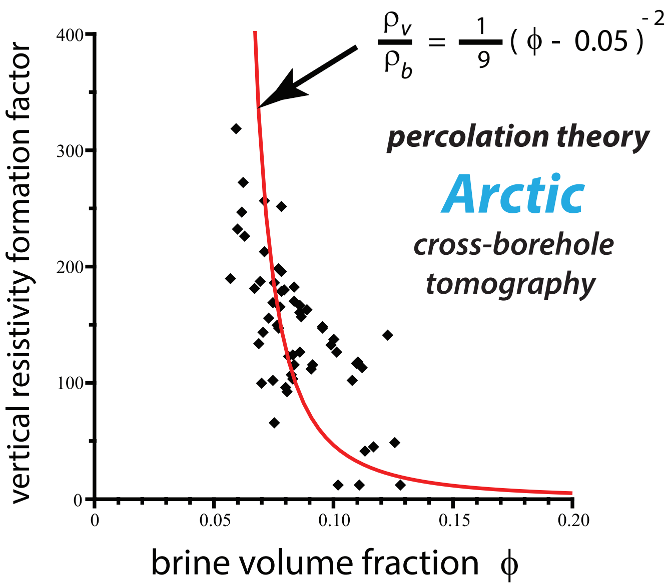
*not confirmed experimentally until SIPEX I (2007) and SIPEX II (2012)*

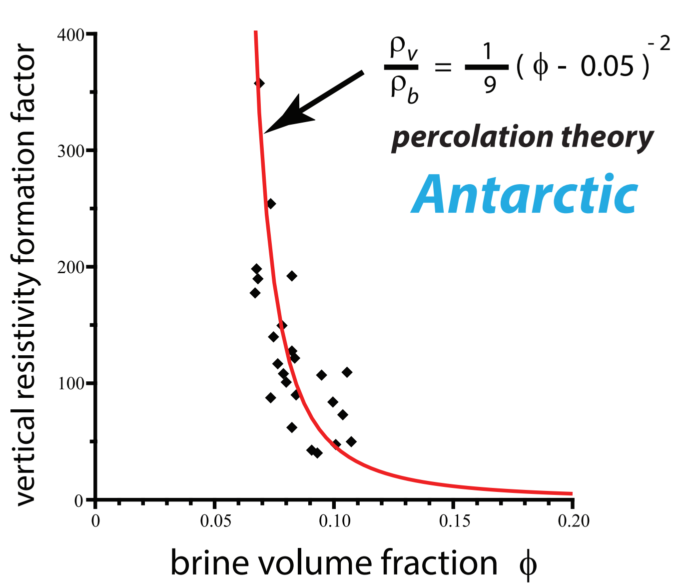
# critical behavior of electrical transport in sea ice

## electrical signature of the on-off switch for fluid flow








**cross-borehole tomography - electrical classification of sea ice layers**

***Golden, Eicken, Gully, Ingham, Jones, Lin, Reid, Sampson, Worby 2016***

## *melt pond formation and albedo evolution:*

- *major drivers in polar climate*
- *key challenge for global climate models*

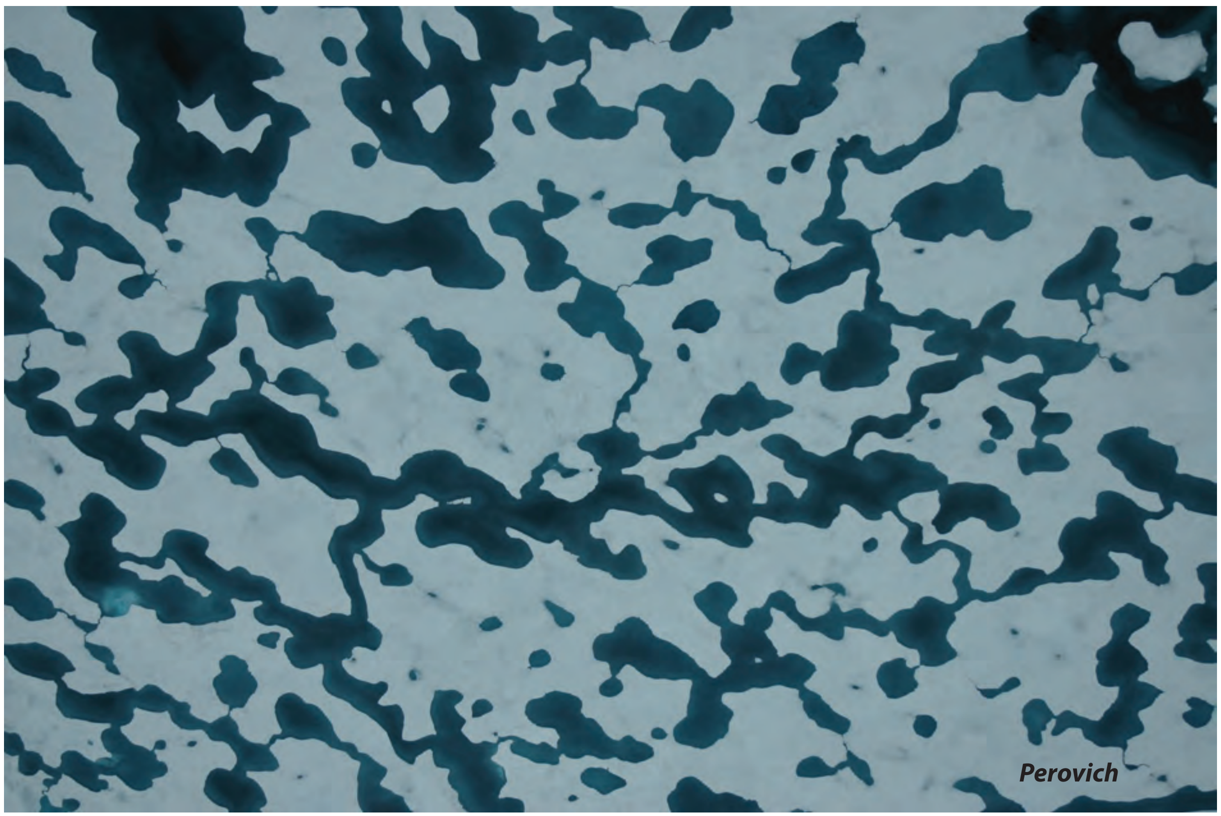
**numerical models of melt pond evolution, including topography, drainage (permeability), etc.**

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

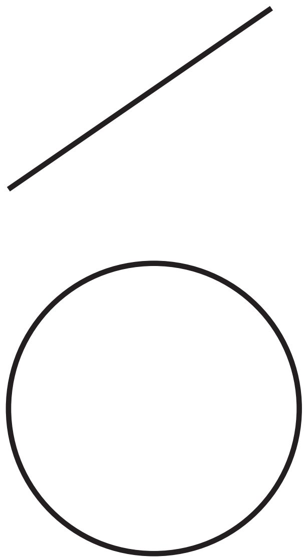
Flocco, Feltham, Hunke 2012

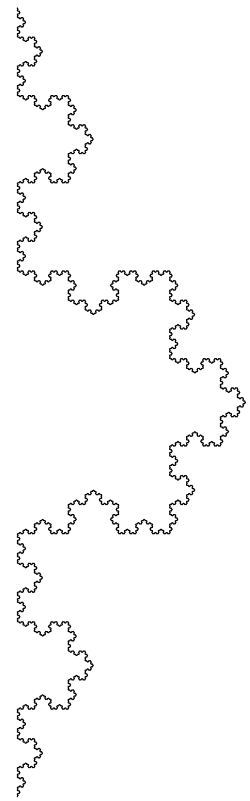


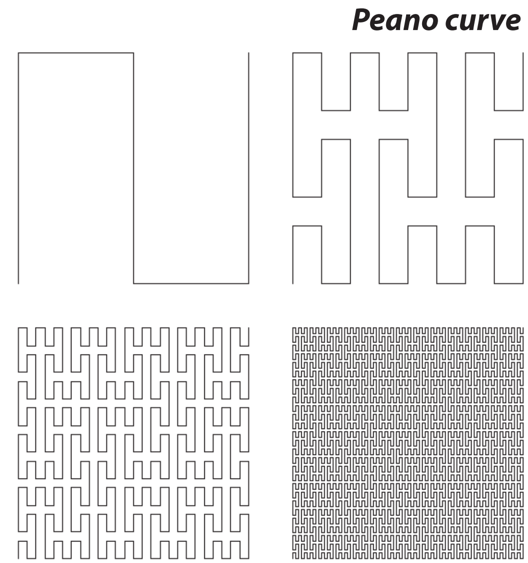

**Are there universal features of the evolution similar to phase transitions in statistical physics?**

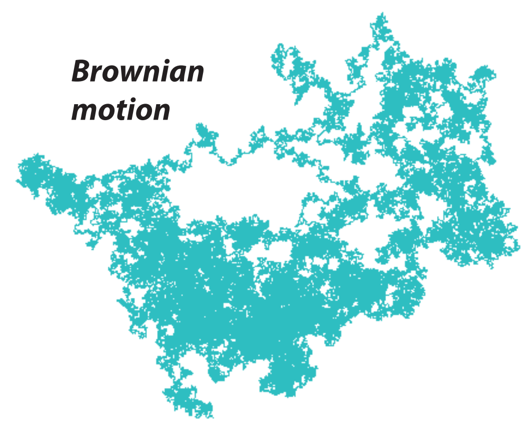
# fractal curves in the plane

## they wiggle so much that their dimension is >1








| simple curves | Koch snowflake | space filling curves |
|---|---|---|
| $D = 1$ | $D = 1.26$ | $D = 2$ |

# *clouds exhibit fractal behavior from 1 to 1000 km*

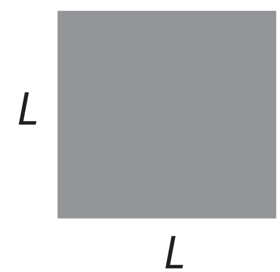
use ***perimeter-area*** data to find that cloud and rain boundaries are fractals

$$D \approx 1.35$$

***S. Lovejoy, Science, 1982***

$$P \sim \sqrt{A}$$

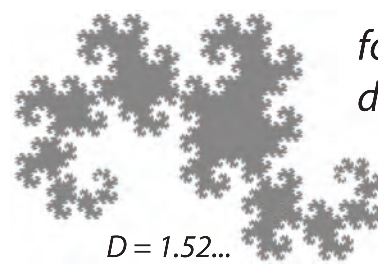
*simple shapes*




$A = L^2$

$P = 4L = 4\sqrt{A}$

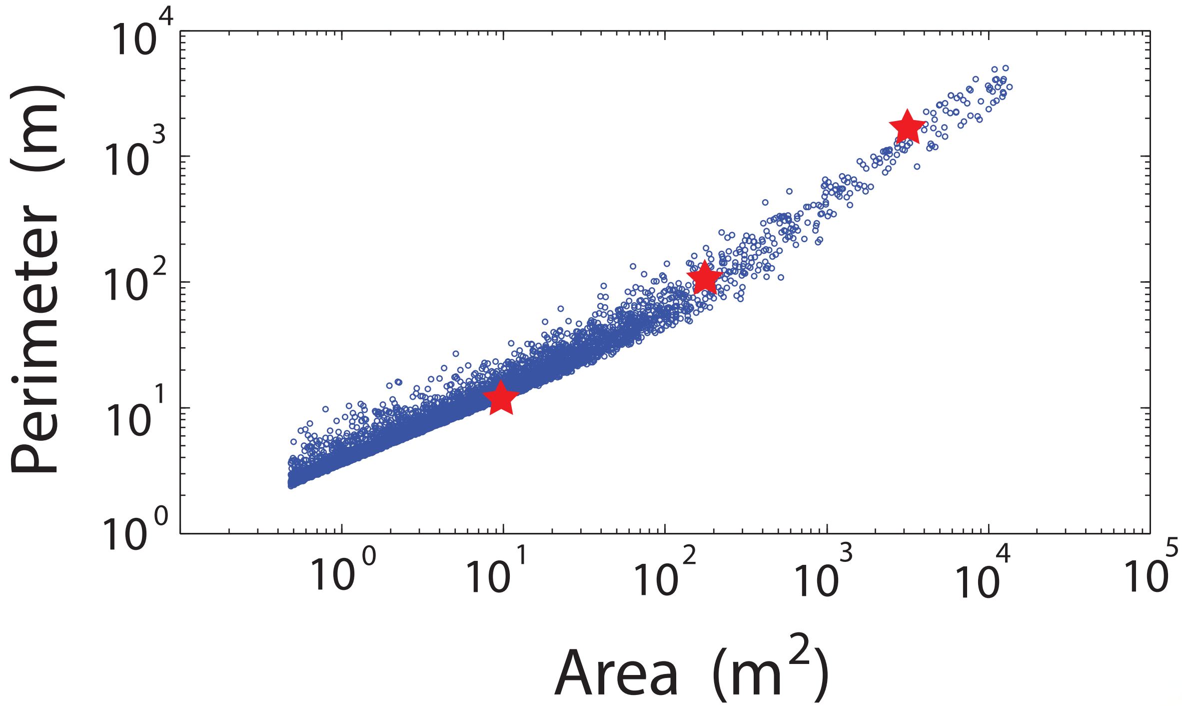
$$P \sim \sqrt{A}^{\,D}$$




*for fractals with dimension D*

## *Transition in the fractal geometry of Arctic melt ponds* *The Cryosphere, 2012*

**Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden**

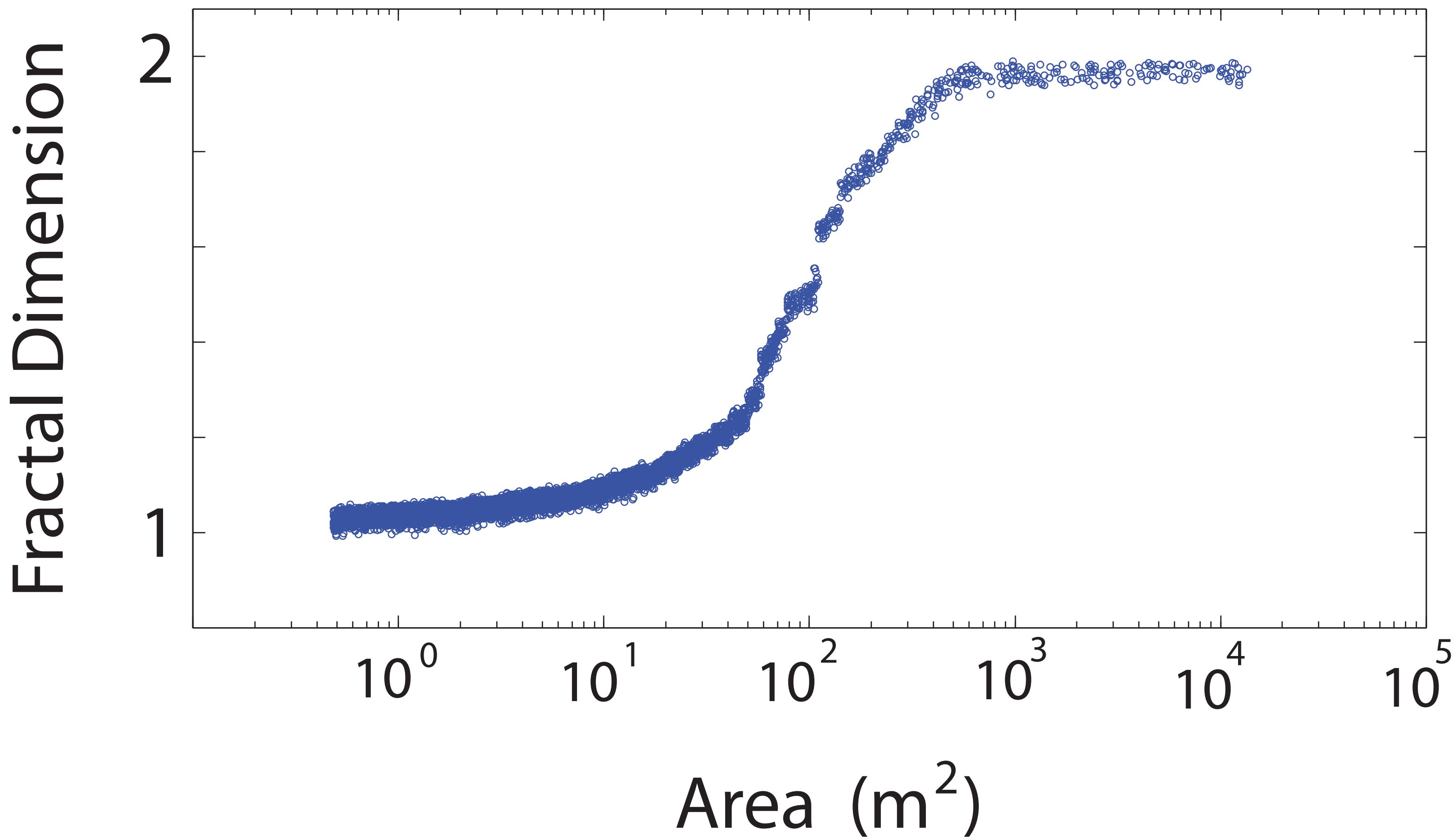



**data from 5269 Arctic melt ponds**


***simple pond*** ***transitional pond*** ***complex pond***

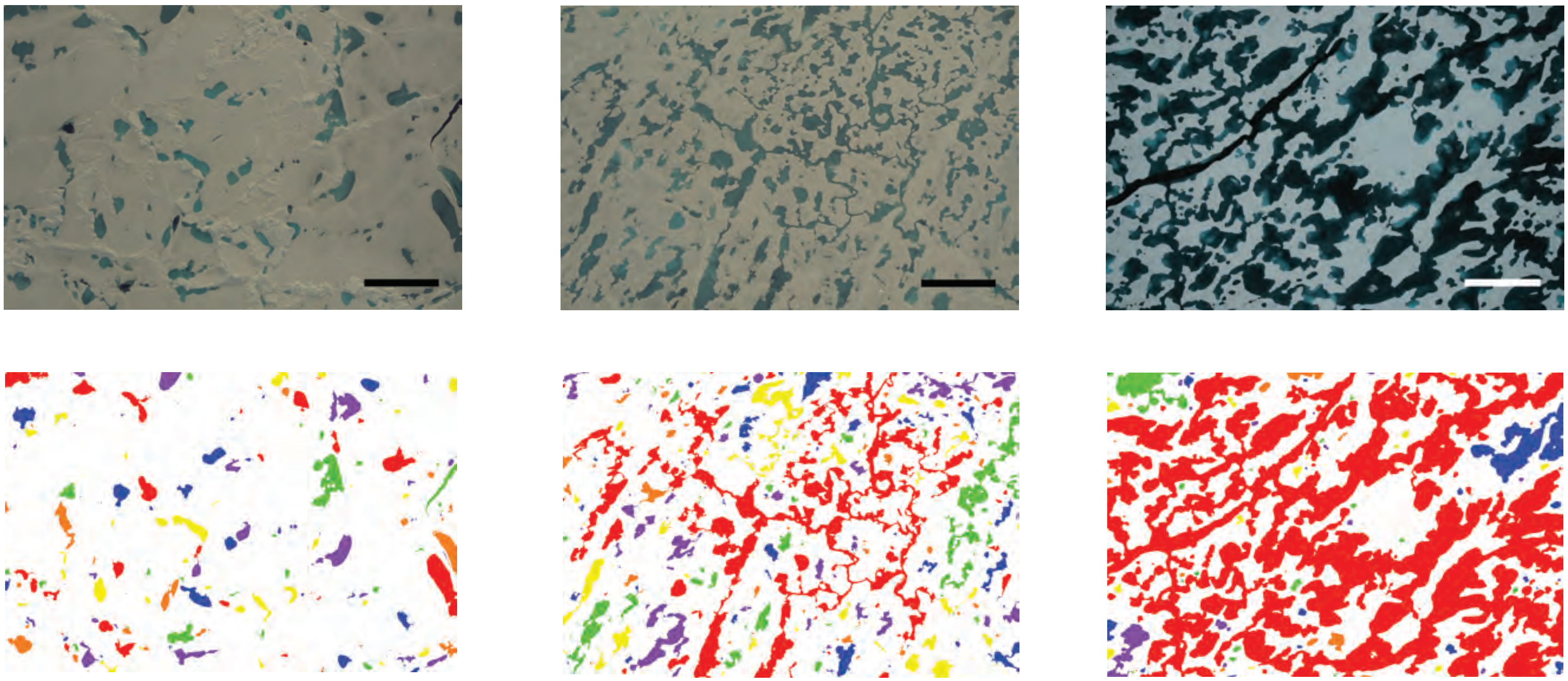
# *transition in the fractal dimension*

**complexity grows with length scale**




**compute “derivative” of area - perimeter data**

*small simple ponds coalesce to form*
*large connected structures with complex boundaries*



## melt pond percolation

results on percolation threshold, cluster behavior

*Anthony Cheng (Hillcrest HS), Bacim Alali, Ken Golden*

# High connectivity of meltpond networks allows vast expanses of meltwater to drain down seal holes, thaw holes, and into leads in the ice

P. Nicklen

NPEO

C. Lydersen

P. McGowan

meted.ucar.edu

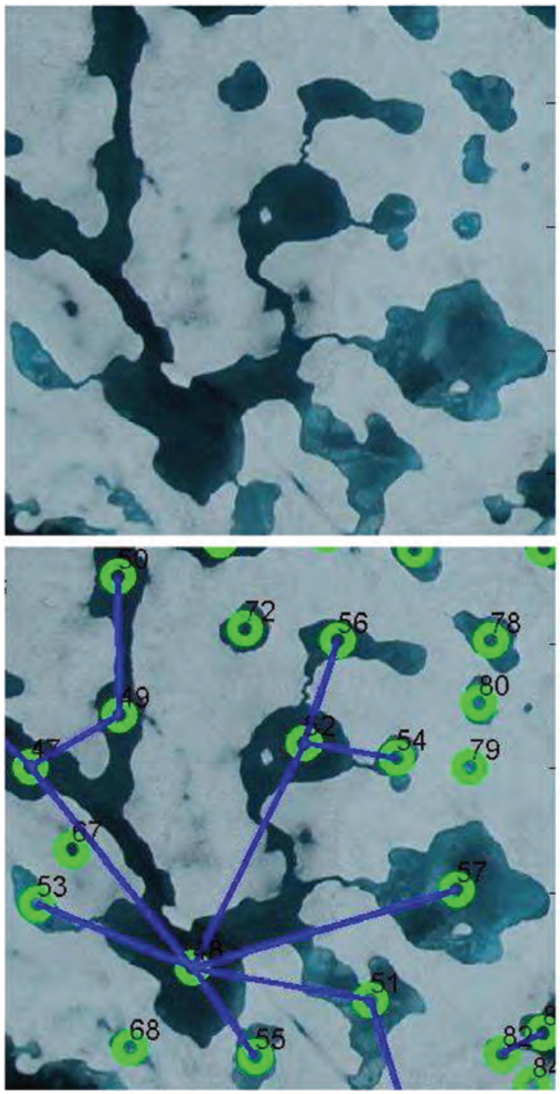

***melt pond evolution depends also on large-scale "pores" in ice cover***

*photos courtesy of C. Polashenski and D. Perovich*

# Network modeling of Arctic melt ponds

Barjatia, Tasdizen, Song, Sampson, Golden

*Cold Regions Science and Tecnology*, 2016




**develop algorithms to map images of melt ponds onto**

**random resistor networks**

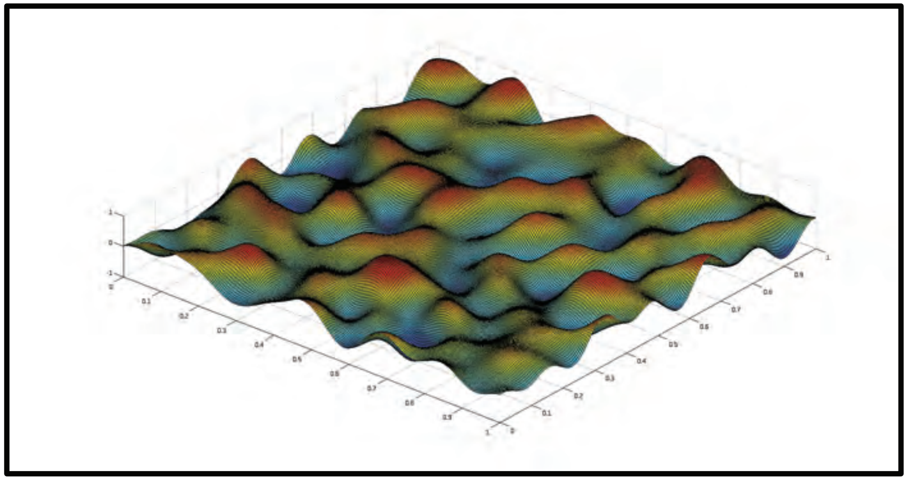
**graphs of nodes and edges with edge conductances**

edge conductance ~ neck width

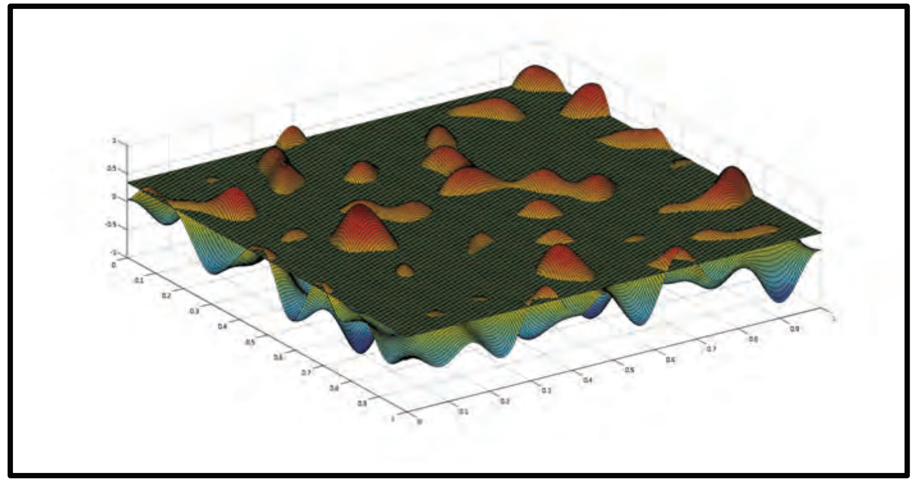
***compute effective horizontal fluid conductivity***

# Continuum percolation model for melt pond evolution

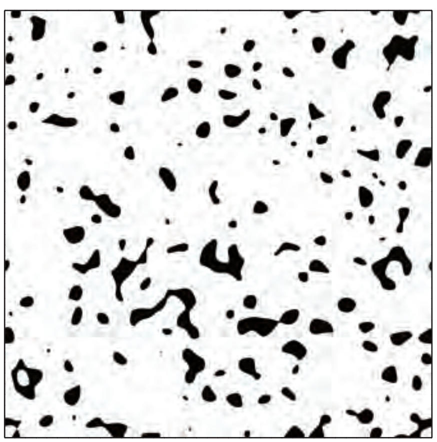
*Brady Bowen, Court Strong, Ken Golden, 2016*



random Fourier series representation of surface topography

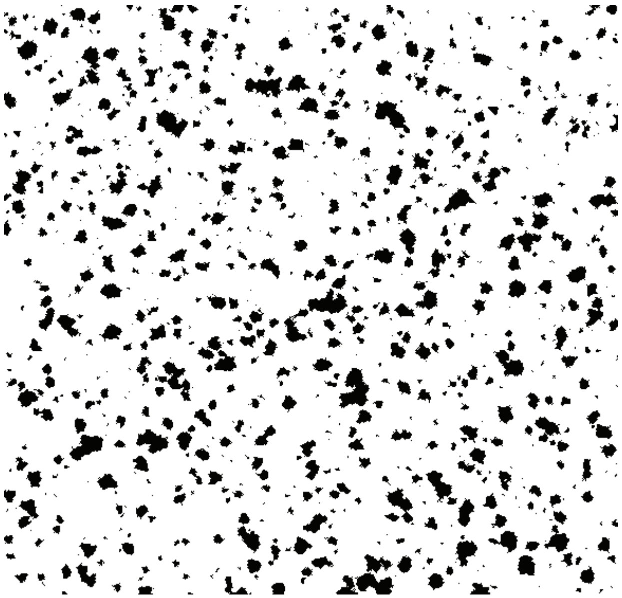


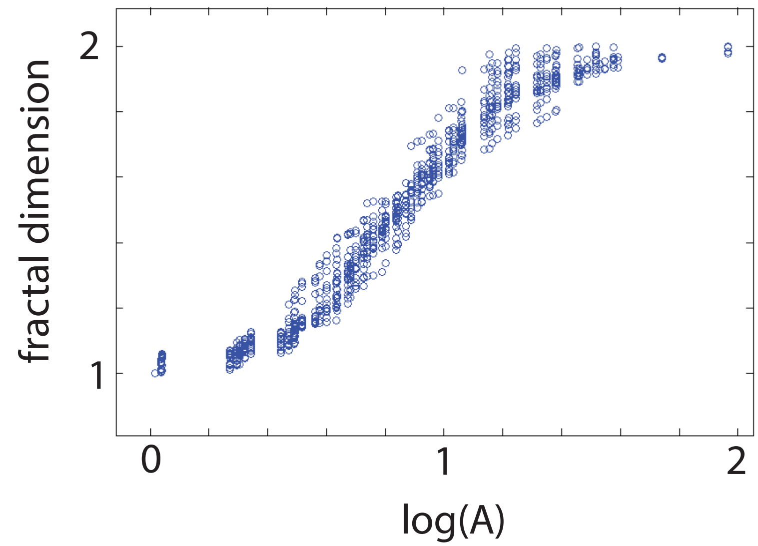
## intersections of a plane with the surface define melt ponds

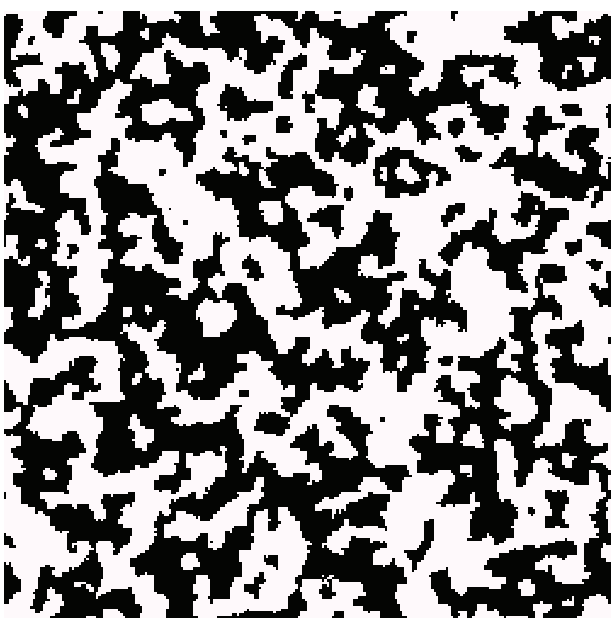


*electronic transport in disordered media* *diffusion in turbulent plasmas* *(Isichenko, Rev. Mod. Phys., 1992)*

# simple stochastic growth model of melt pond evolution





**voter model**

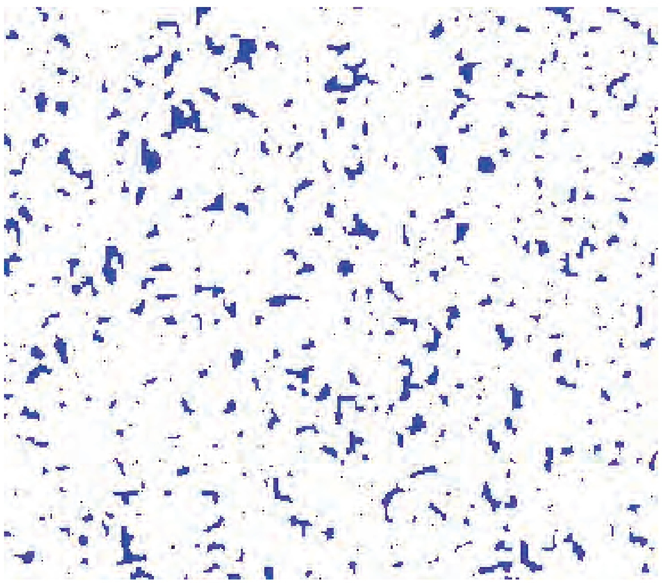
***a square is more likely to melt if its neighbors have melted***

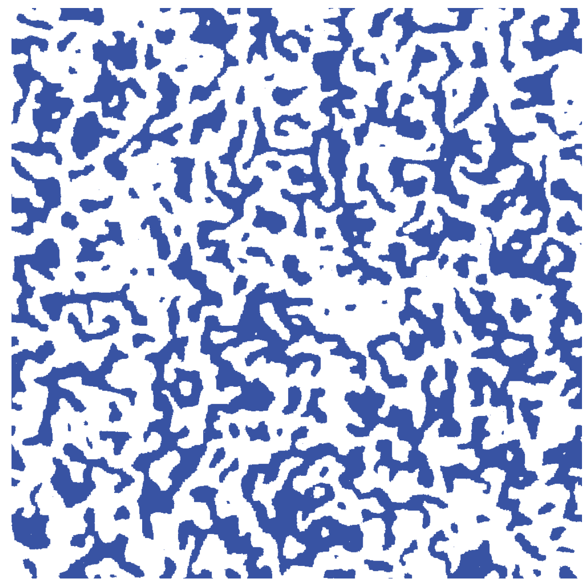
*Rebecca Nickerson (West HS, Salt Lake City) and Ken Golden*

## Ising model for ferromagnets $\longrightarrow$ Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - H \sum_{i}^{N} s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

**magnetization** $M = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$ **pond coverage** $\frac{(M+1)}{2}$





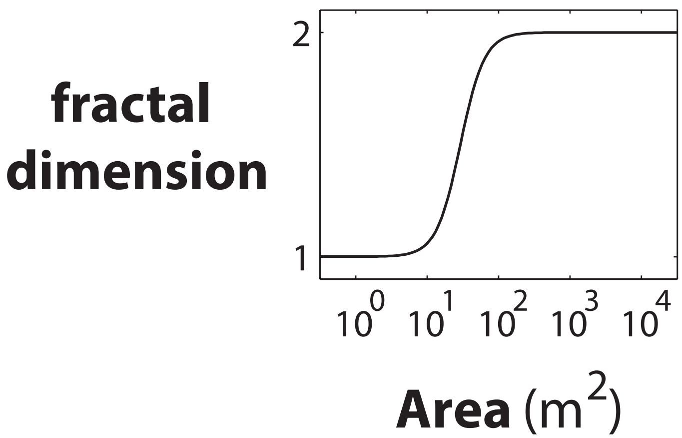
***"melt ponds" are clusters of magnetic spins that align with the applied field***

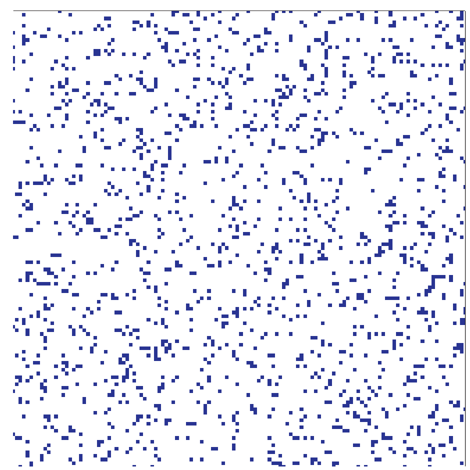
**Ma, Sudakov, Strong, Golden 2016**

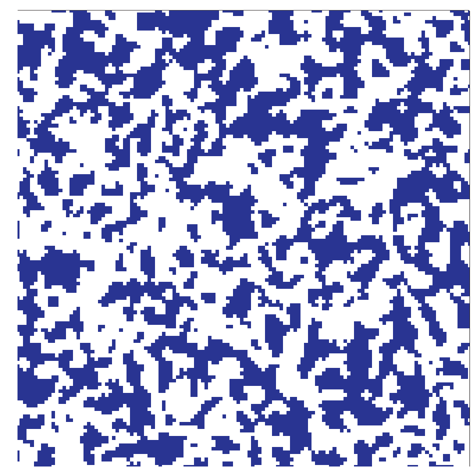
# Melt Pond Ising Model

- Minimize an Ising Hamiltonian
  random magnetic field represents the initial ice topography
  interaction term represents horizontal heat transfer

- Ice-albedo feedback incorporated by taking coupling constant in interaction term to depend on the pond coverage





- predicted fractal transition **70 m²** vs. **86 m²** observed

- predicted pond size distribution exponent **1.75**
  vs. **1.75** observed

# *Conclusions*

1. Summer Arctic sea ice is melting rapidly.

2. Low order (toy) models help us understand tipping point phenomena.

3. Fluid flow through sea ice mediates many processes of importance to understanding climate change and the response of polar ecosystems.

4. Homogenization and statistical physics help *link scales* and provide rigorous methods for finding effective behavior, and advance how sea ice is represented in climate models.

5. Critical behavior (in many forms) is inherent in the climate system.

6. Field experiments are essential to developing relevant mathematics.

7. Our research will help to improve projections of climate change and the fate of the Earth sea ice packs.

# THANK YOU

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