Homogenization for Sea Ice

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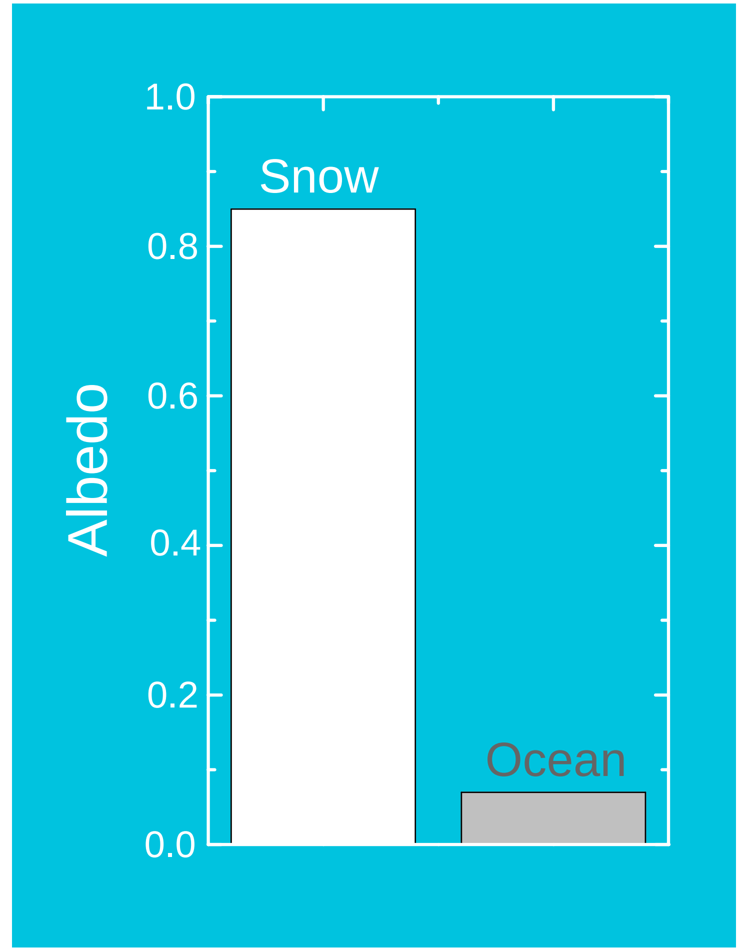
SEA ICE covers 7 - 10% of earth's ocean surface
• boundary between ocean and atmosphere
• mediates exchange of heat, gases, momentum
• global ocean circulation
• indicator and agent of climate change

polar ice caps critical to global climate in reflecting incoming solar radiation

white snow and ice reflect

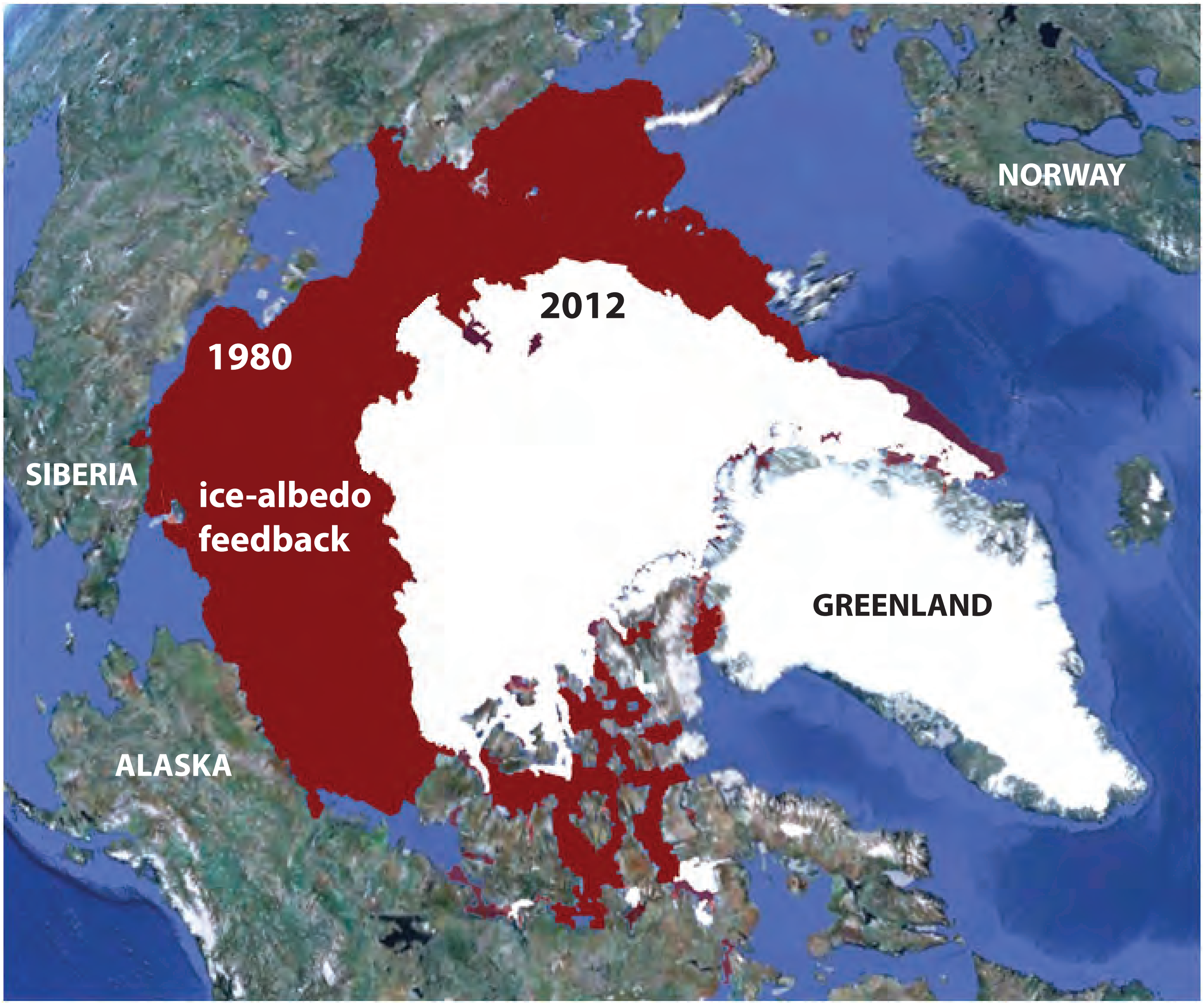


dark water and land absorb

$$\textbf{\textit{albedo}}\ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

Change in Arctic Sea Ice Extent

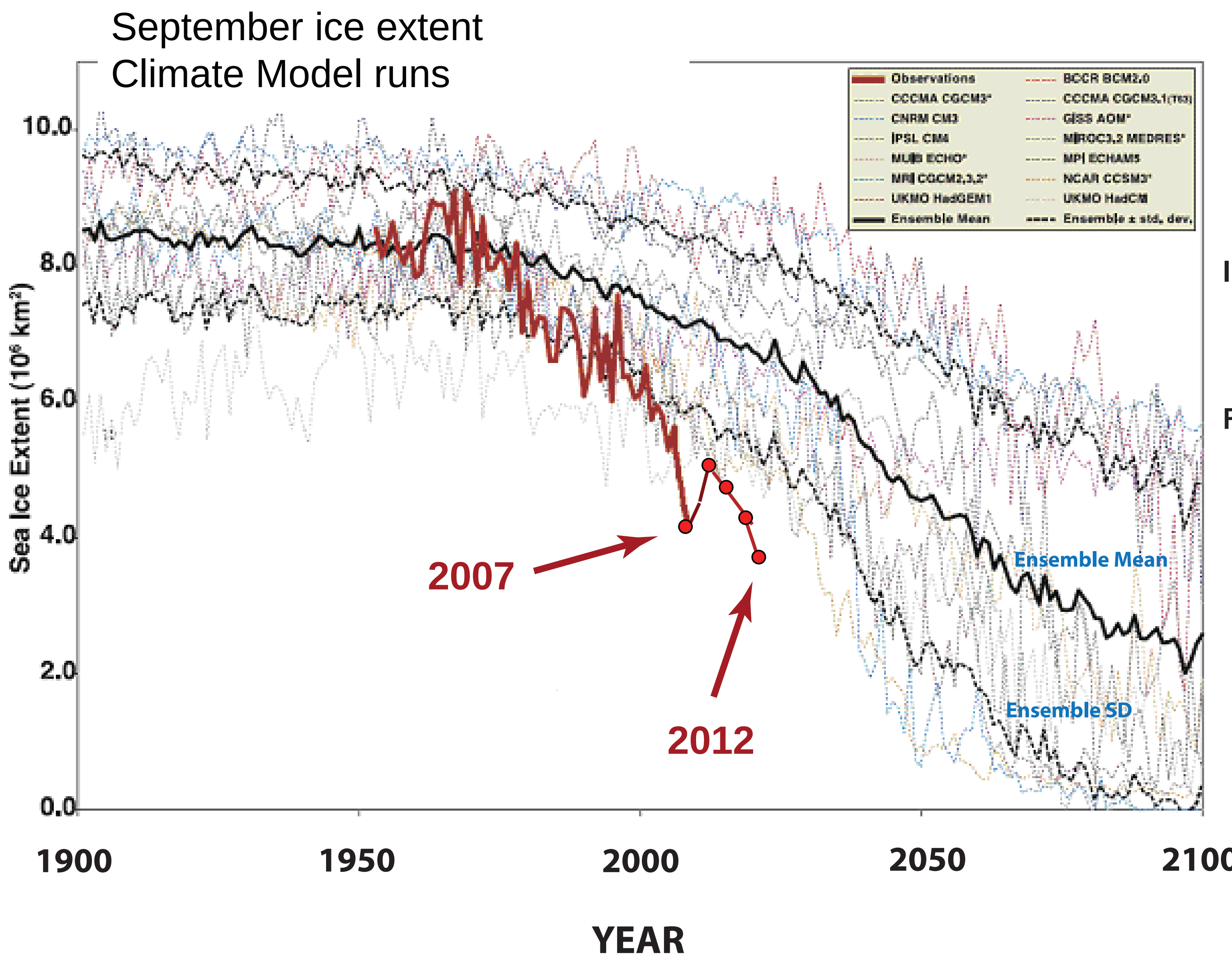
September 1980 -- 7.8 million square kilometers
September 2012 -- 3.4 million square kilometers



Perovich

Arctic sea ice decline - faster than predicted by climate models

Stroeve et al., GRL, 2007



IPCC AR4 Models

Intergovernmental Panel on Climate Change (IPCC)

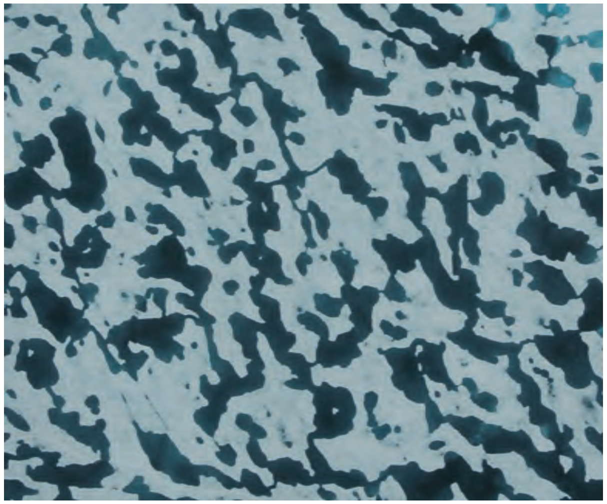
Fourth Assessment AR4, 2007

challenge

represent sea ice more rigorously in climate models

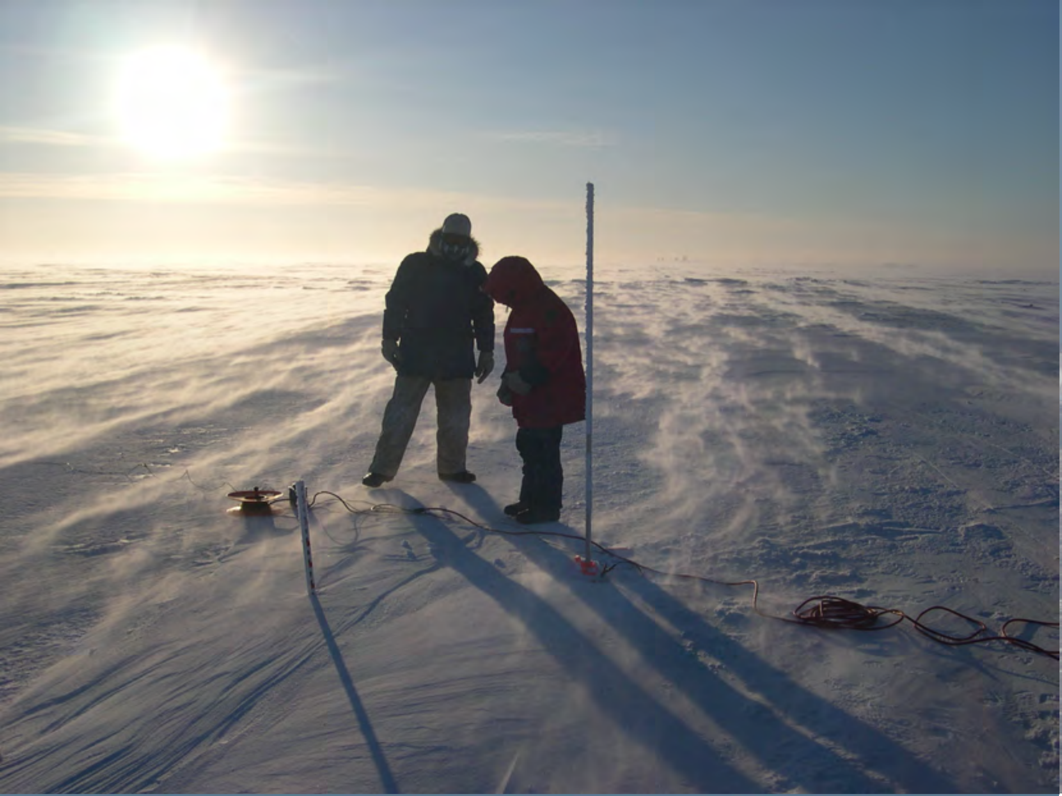
account for key processes

such as melt pond evolution

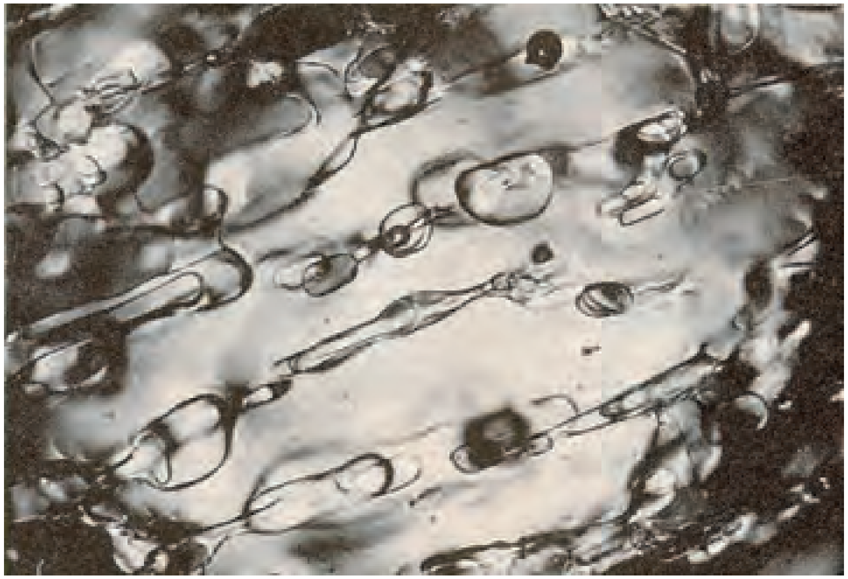


... and other sub-grid scale structures and processes

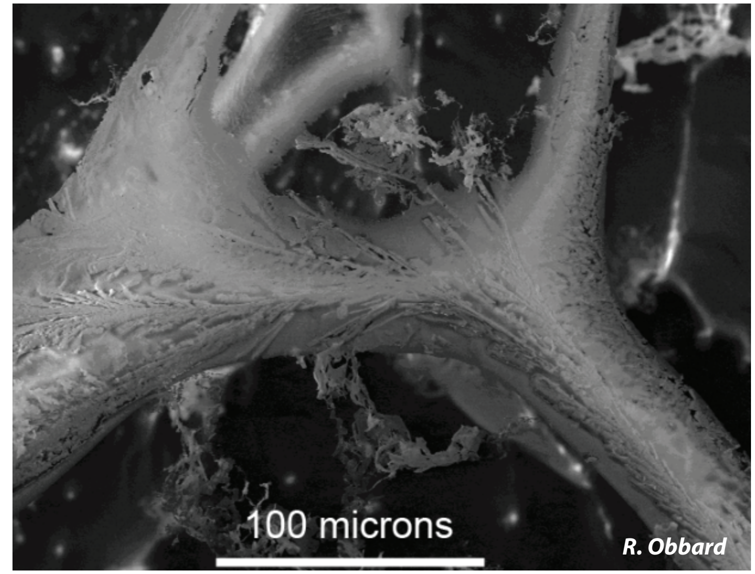
linkage of scales



sea ice may appear to be a barren, impermeable cap ...



brine inclusions in sea ice (mm)



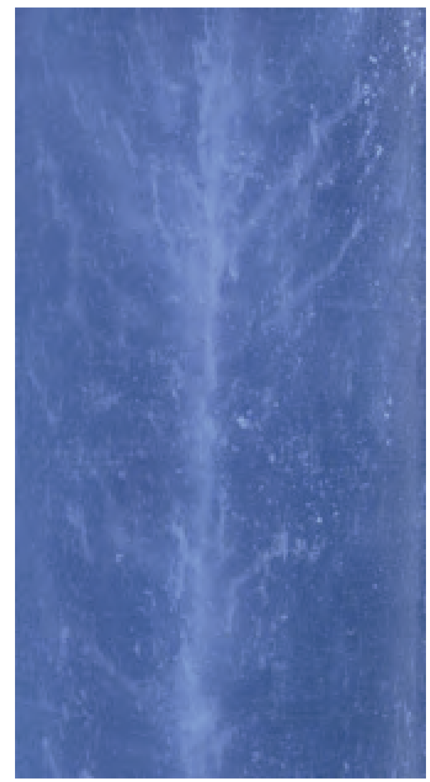
micro - brine channel (SEM)

brine channels (cm)

sea ice is a porous composite

pure ice with brine, air, and salt inclusions

horizontal section

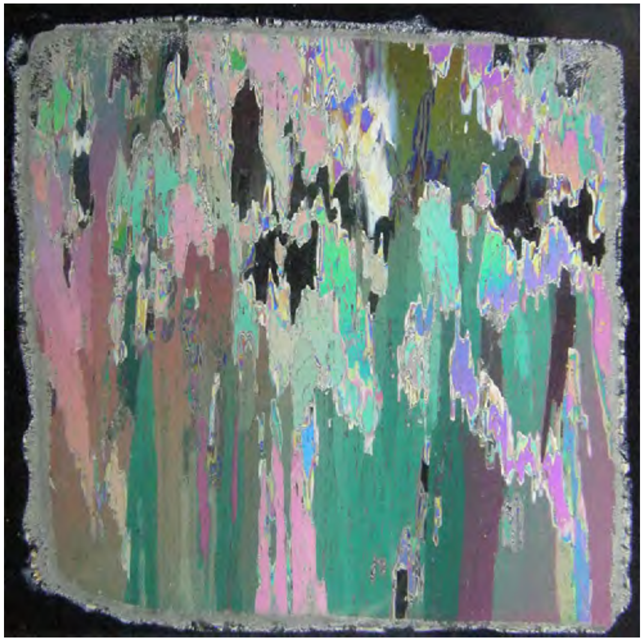


vertical section

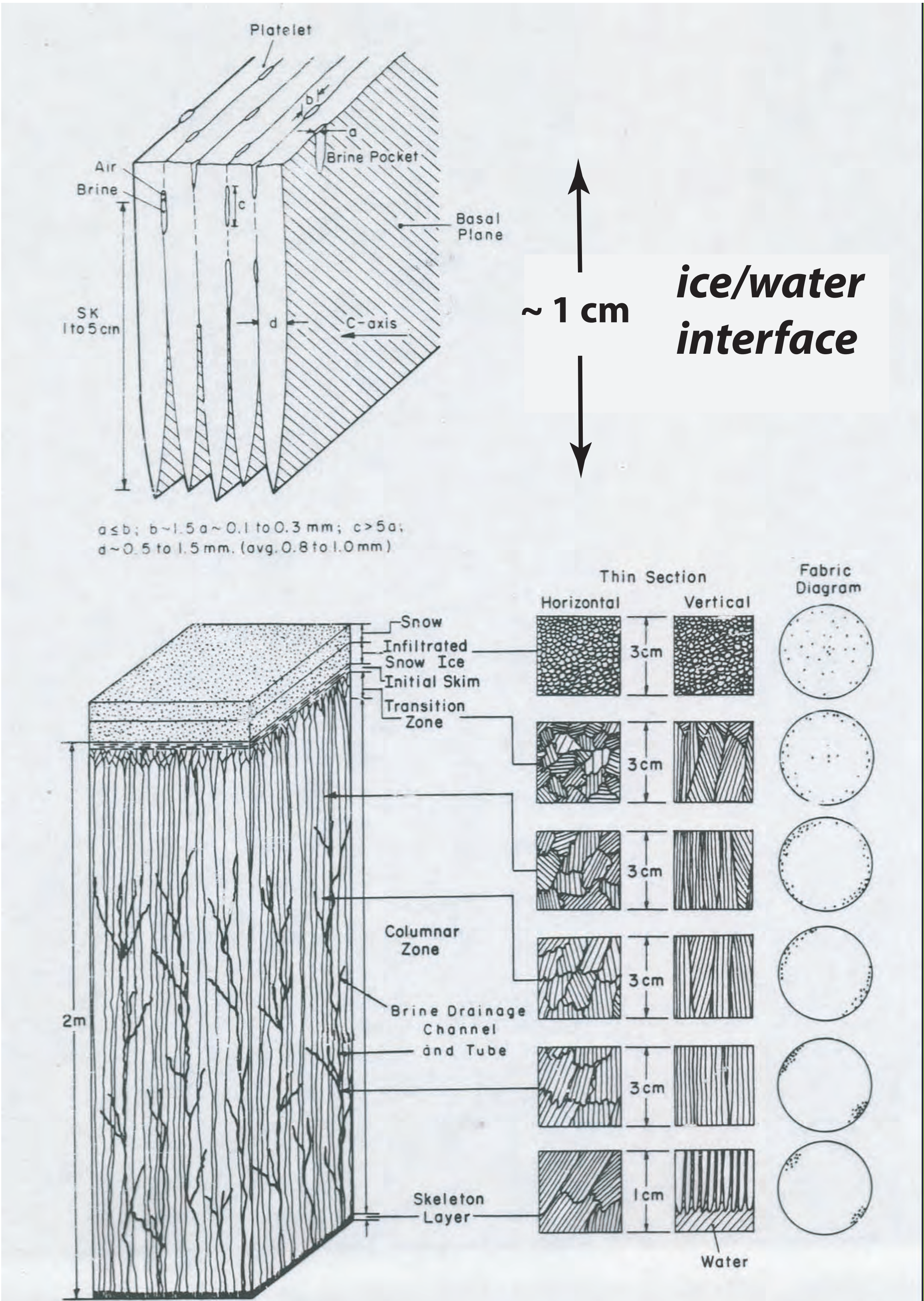
cross-sections of sea ice structure

$$T_{freeze} = -1.8^{\circ}\,\mathrm{C}$$

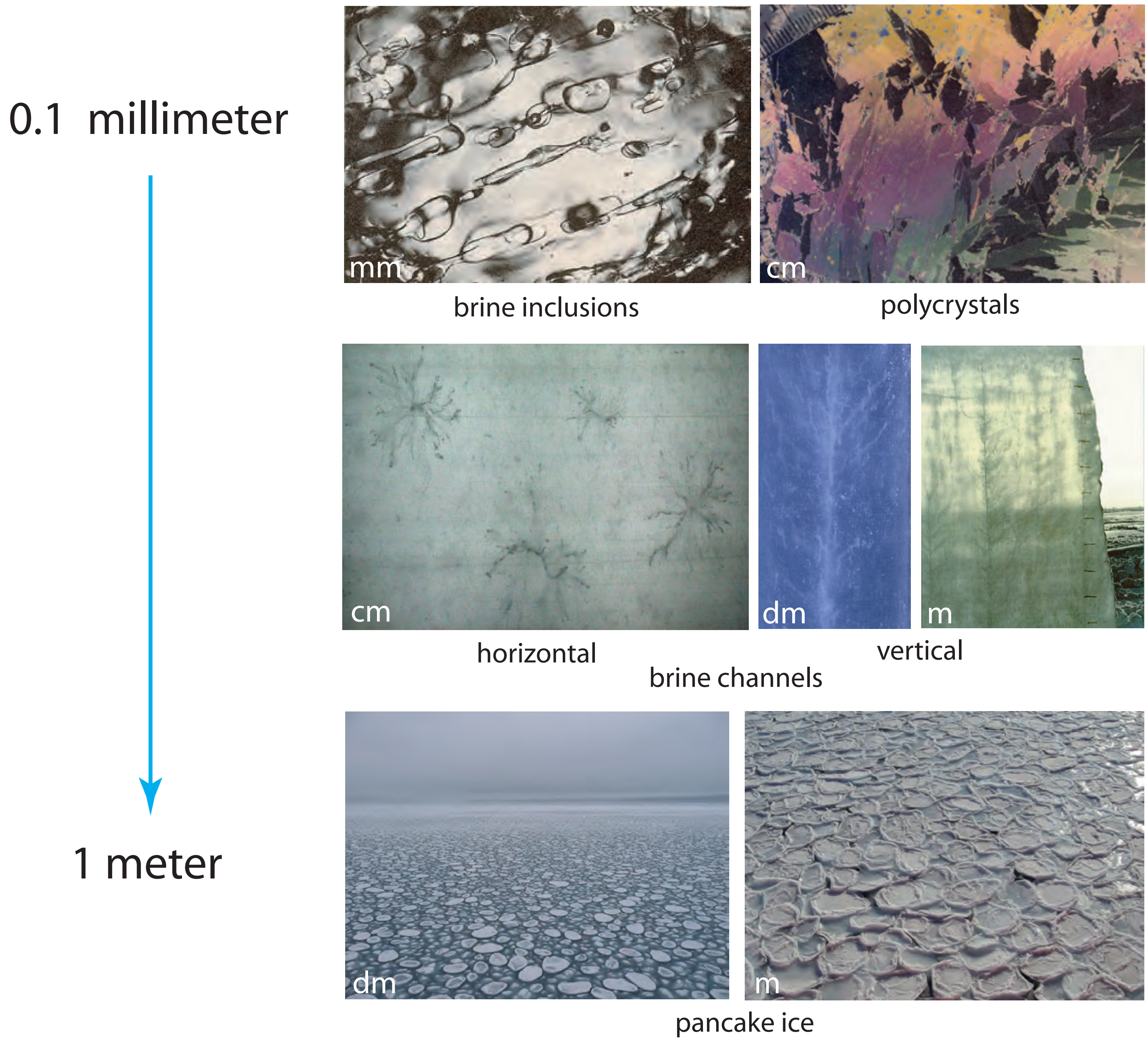
crystallographic texture

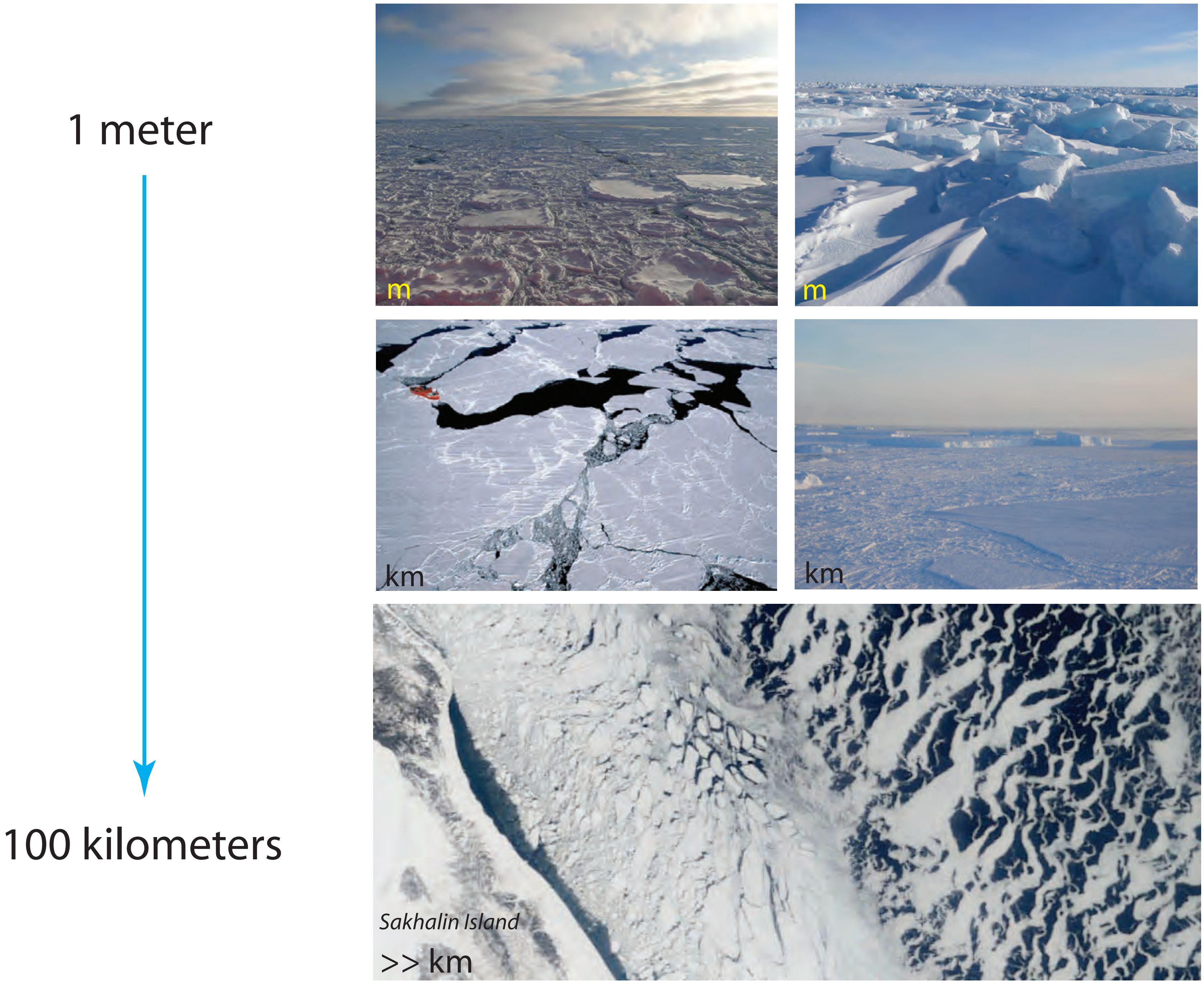


vertical thin section



sea ice displays ***multiscale*** structure over 10 orders of magnitude




1 meter
100 kilometers
m
m
km
km
Sakhalin Island
>> km

What is this talk about?

An overview of mathematical models of composite materials and statistical physics used to study sea ice and improve climate models.

homogenization, multiscale analysis, phase transitions

1. **Opposite poles of climate modeling**

   PDE's, ODE's, SDE's, and dynamical systems

2. **Sea ice microphysics and composite structure**

   homogenization, fluid flow, diffusion processes, percolation theory

3. **Electromagnetic monitoring of sea ice; advection diffusion**

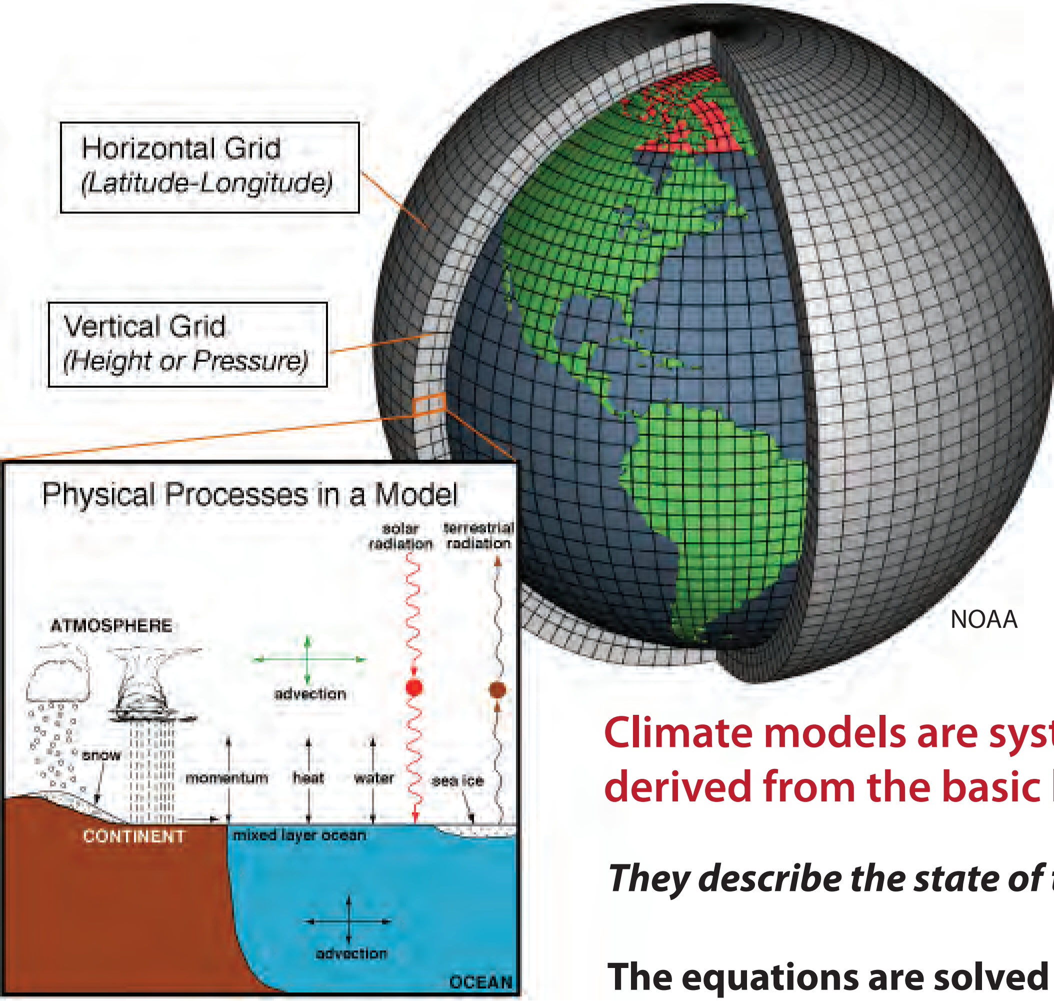
   integral representations, spectral measures, random matrix theory

4. **Fractal geometry of Arctic melt ponds**

   continuum percolation, network and voter models, Ising model

critical behavior **cross - pollination**

Global Climate Models



climate results from complex interactions between atmosphere, cryosphere, oceans, land, biosphere

fueled by the nonuniform spatial distribution of incoming solar radiation.

Stute, et al., PNAS 2001

Climate models are systems of partial differential equations (PDE) derived from the basic laws of physics, chemistry, and fluid motion.

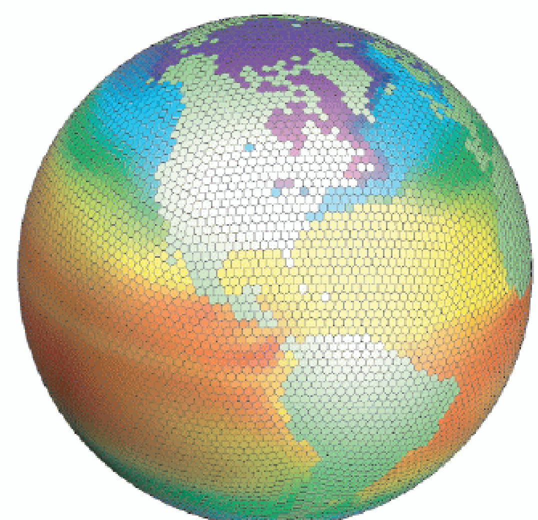
They describe the state of the ocean, ice, atmosphere, land, and their interactions.

The equations are solved on 3-dimensional grids of the air-ice-ocean-land system (with horizontal grid size ~ 100 km), using powerful computers.

sub - grid scale processes

General Circulation Models (GCM)

atmospheric GCM and ocean GCM
+ sea ice, ice sheet, land components } *coupling*



Randall *et al.*, 2002

sea ice components of GCM's

What are the key ingredients -- or ***governing equations*** that need to be solved on grids using powerful computers?

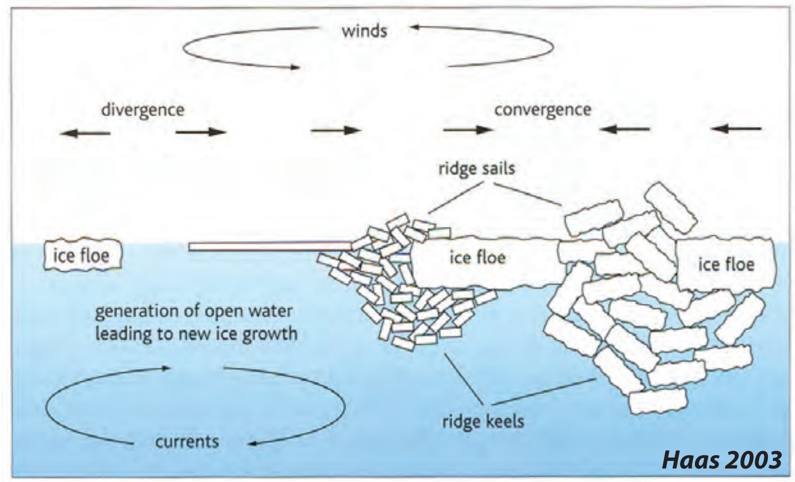
1. Ice thickness distribution $g(x, y, h, t)$ evolution equation

dynamics + *thermodynamics*

(Thorndike *et al.* 1975)

$$\frac{Dg}{Dt} = -g\nabla \cdot \mathbf{u} + \Psi(g) - \frac{\partial}{\partial h}(fg) + \mathcal{L}$$

nonlinear PDE incorporating ice velocity field
ice growth and melting
mechanical redistribution
- ridging and opening



2. Conservation of momentum, stress *vs.* strain relation (Hibler 1979)

$$m\frac{D\mathbf{u}}{Dt} = -mf\mathbf{k} \times \mathbf{u} + \boldsymbol{\tau}_a + \boldsymbol{\tau}_o - mg\nabla H + \mathbf{F}$$

F = ma for sea ice *dynamics*

3. Heat equation of sea ice and snow

(Maykut and Untersteiner 1971)

thermodynamics

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla \cdot k(T)\,\nabla T$$

+ ***balance of radiative and thermal fluxes on interfaces***

ice thickness distribution $g(x, y, h, t)$ evolution equation

$$\frac{Dg}{Dt} = -g\nabla \cdot \mathbf{u} + \Psi(g) - \frac{\partial}{\partial h}(fg) + \mathcal{L}$$

	$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \mathbf{u} \cdot \nabla g$	Lagrangian or convective derivative	
	$\mathbf{u}$	ice velocity field	
	h	ice thickness	
	$-g\nabla \cdot \mathbf{u}$	flux divergence	
depend on g	Ψ	mechanical redistribution opening and ridging	
	f	thermodynamic growth rate	
	$\frac{\partial}{\partial h}(fg)$	ice growth/melt	***ice floe area*** ***melt ponds***
	$\mathcal{L}$	lateral melting	***ice floe perimeter***

transform ice thickness distribution equation to Fokker-Planck type equation

Toppaladoddi and Wettlaufer, *PRL*, 2015

thickness h is a diffusion process with probability density g(h,t)

$$\Psi = \int_0^\infty [g(h+h')w(h+h',h') - g(h)w(h,h')]dh'$$

w = ***transition probability***

moments k_1 , k_2

Fokker-Planck

$$\frac{\partial g}{\partial t} = -\frac{\partial}{\partial h}\left[\left(\frac{\epsilon}{h} - k_1\right) g\right] + \frac{\partial^2}{\partial h^2}(k_2 g)$$

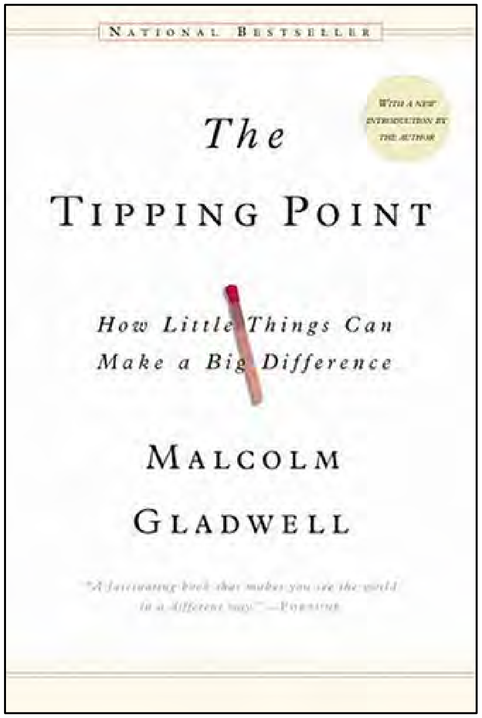
Langevin

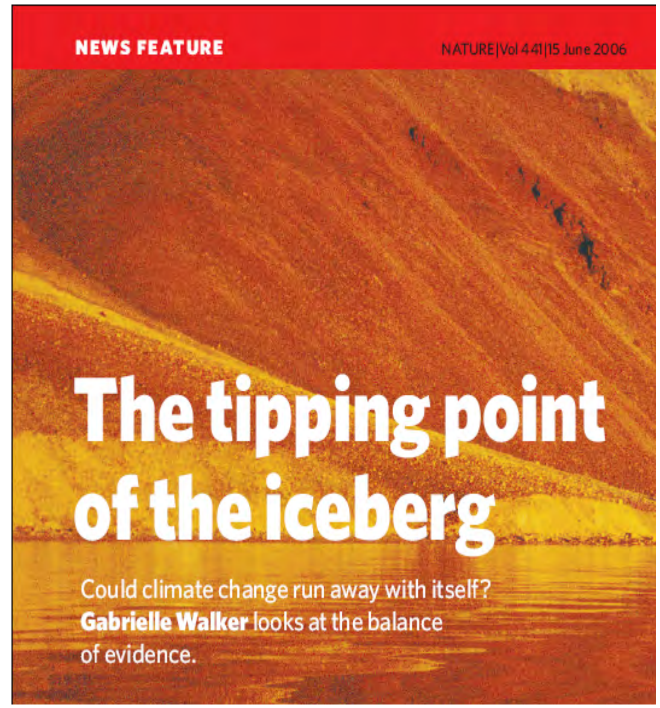
$$\frac{dh}{dt} = \left(\frac{\epsilon}{h} - k_1\right) + \sqrt{2k_2}\,\xi(t)$$

$\xi(t)$ = ***Gaussian white noise***

tipping points in the mainstream

climate tipping points – September Arctic sea ice cover





Melting of the Greenland ice sheet
Melting of the West Antarctic ice sheet
Permafrost and tundra loss, leading to the release of methane
Shutoff of N. Atlantic thermohaline conveyor (Gulf Stream) •••

Lenton, et al., PNAS 2008

Eisenman

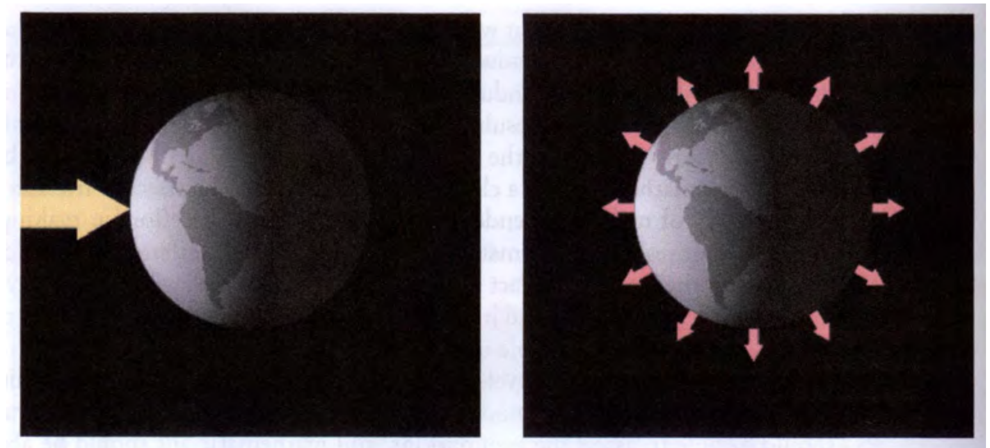
energy balance models

$S_0 = 1,368 \text{ Wm}^{-2}$

solar constant

total solar irradiance

$$Q = \frac{1}{4} S_0$$



Kaper and Engler 2013

T = global mean surface temperature

E_{in} = avg amt of solar energy reaching one square meter of Earth's surface per unit time

$$E_{\text{in}} = (1 - \alpha) Q$$

albedo

E_{out} = avg amt of energy emitted by one square meter of Earth's surface per unit time (and released)

$$E_{\text{out}} = \sigma T^4$$ Stefan-Boltzman

$$\boxed{C \frac{dT}{dt} = E_{\text{in}} - E_{\text{out}}} = 0 \quad \text{at equilibrium}$$

$$E_{\text{in}} = (1-\alpha)Q \qquad Q = \tfrac{1}{4}S_0$$

α **albedo**

$$E_{\text{out}} = \sigma T^4$$

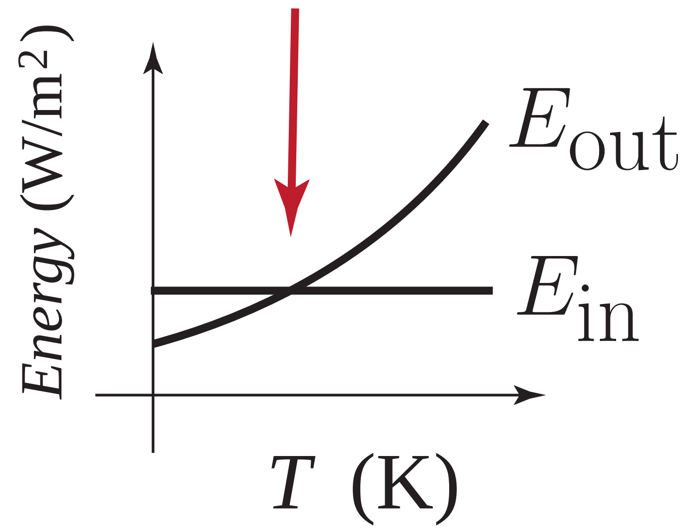
Stefan-Boltzman

$$\boxed{C\,\frac{dT}{dt} = E_{\text{in}} - E_{\text{out}}}$$

$$C\,\frac{dT}{dt} = (1-\alpha)Q - \sigma T^4$$

C = heat capacity

equilibrium



$$E_{\text{in}} = E_{\text{out}}$$

$$\frac{dT}{dt} = 0$$

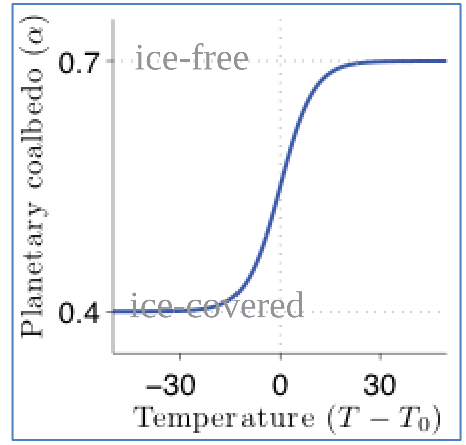
Equilibrium Temperature $T^* = 254.8$ K (–1° F)

Actual Temperature $T = 287.7$ K (58° F)

temperature dependent albedo $\left\{ \begin{array}{l} \text{cold -- reflecting} \\ \text{warm -- absorbing} \end{array} \right.$

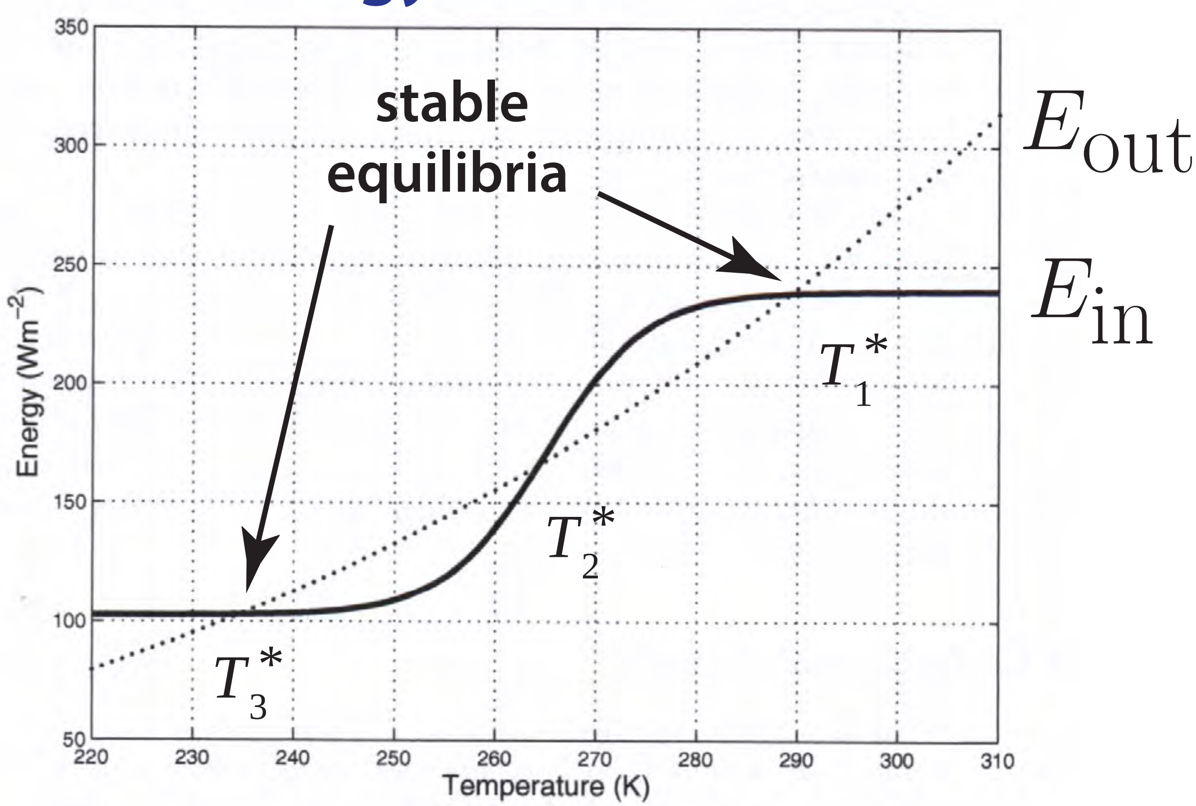
Planetary coalbedo (absorbed/incident radiation) depends on temperature, transitioning from ice-covered to ice-free values as temperature warms.

$$\alpha(T) = \alpha_i + (\alpha_o - \alpha_i)\,\frac{1}{2}\text{Tanh}\left(\frac{T - T_0}{\Delta T}\right)$$



Ian Eisenman

energy balance model



multiple equilibria

snowball Earth

bifurcation diagram

equilibrium surface temperature as a function of solar forcing

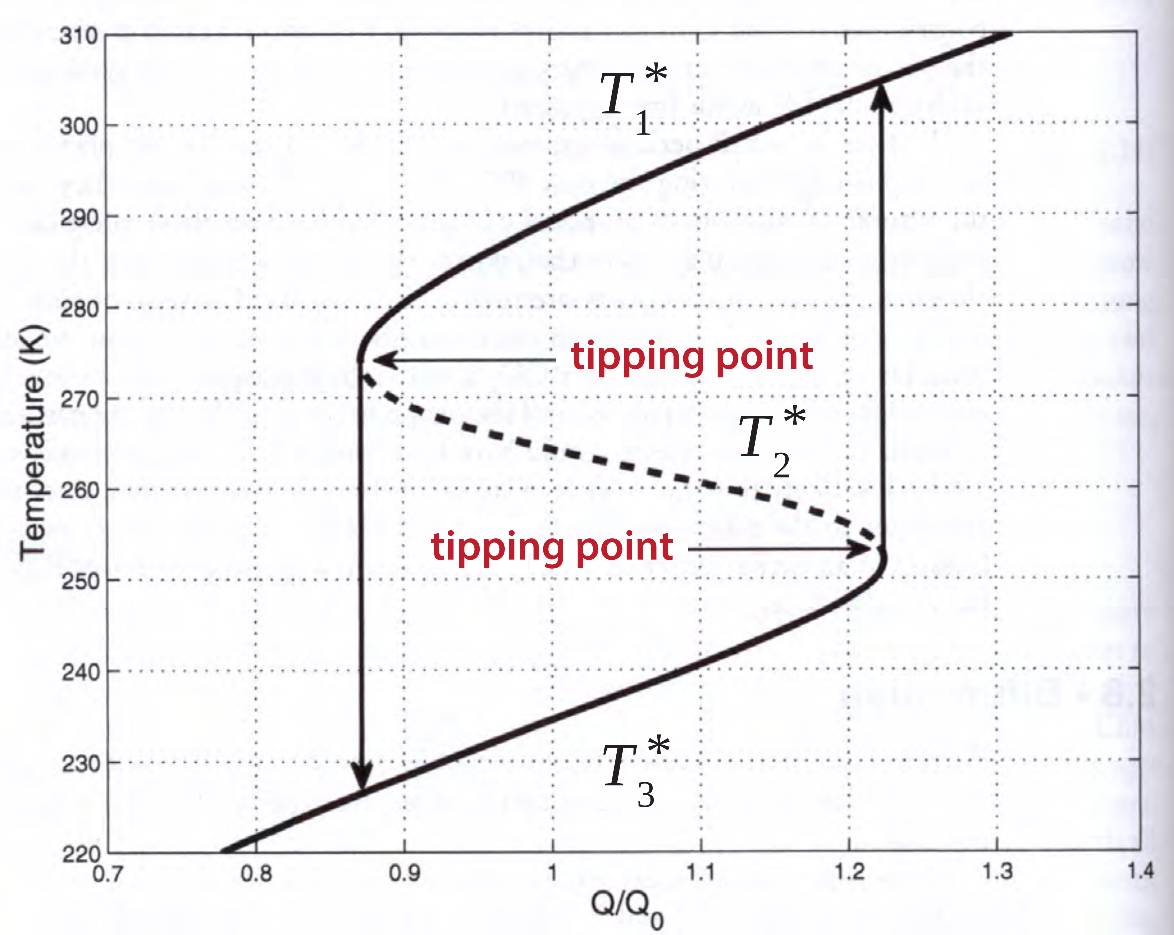
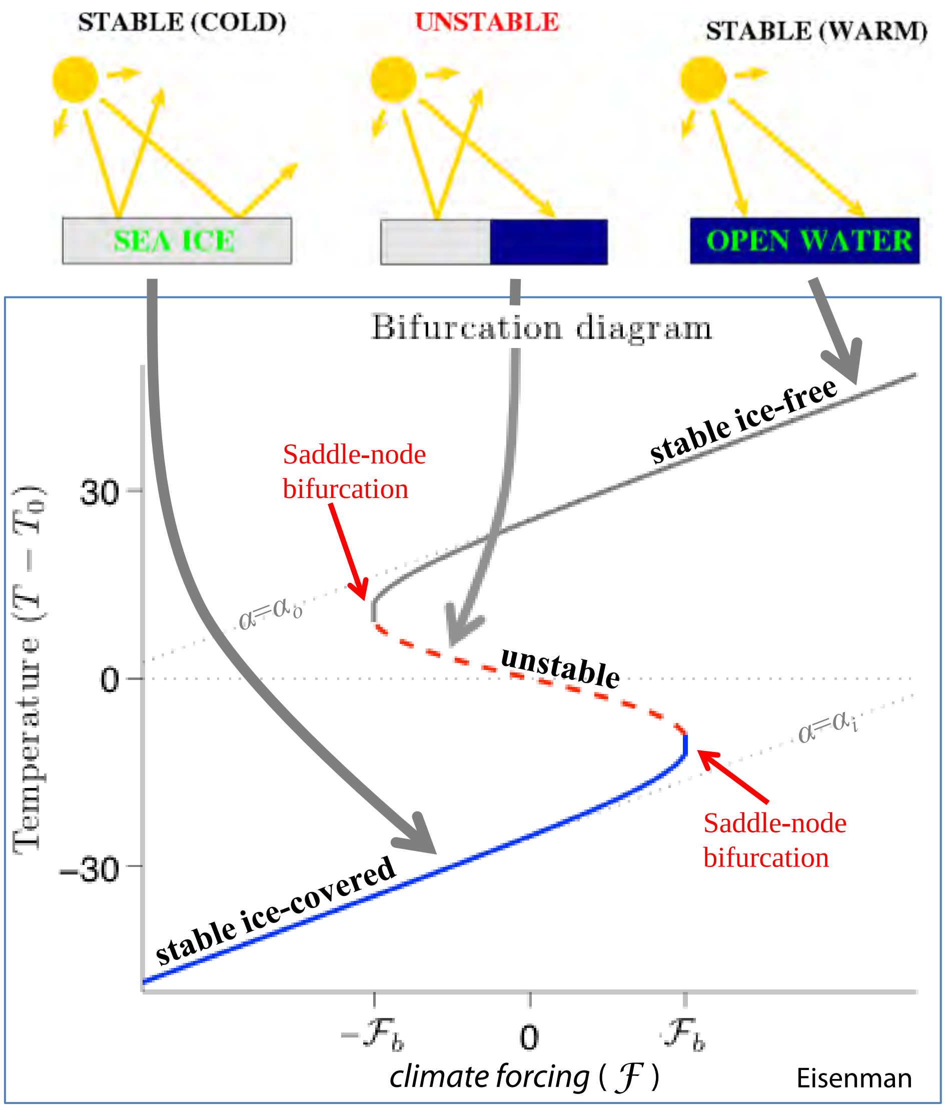


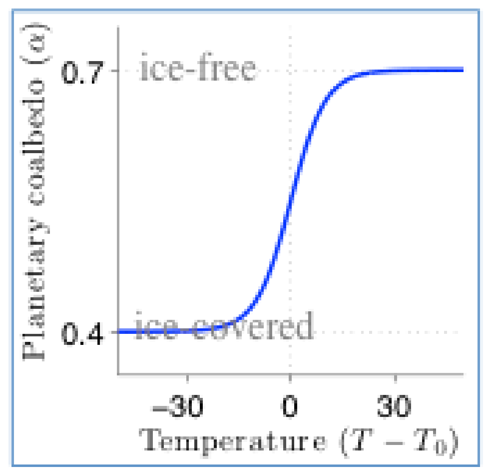
Figure 2.6. *Mean surface temperatures at equilibrium as a function of the solar constant (in units of its present value).*

Kaper and Engler 2013

Sea Ice Bifurcations



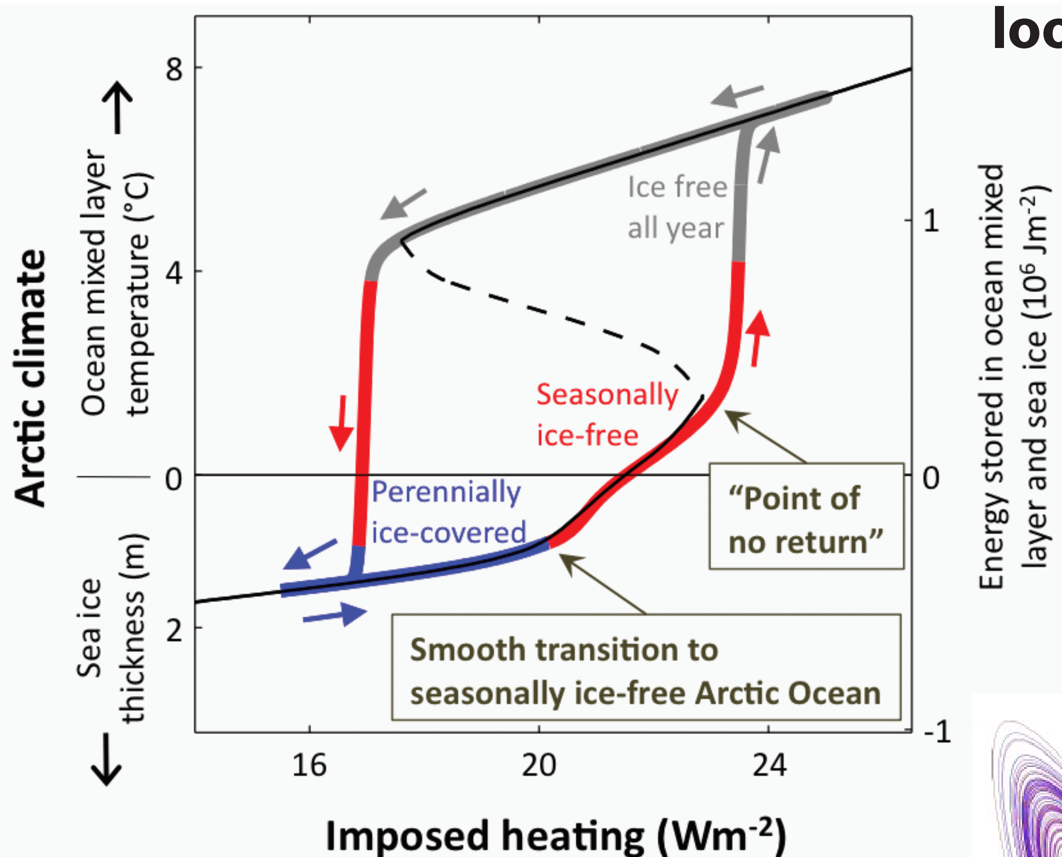
nonlinear ice-albedo feedback



Has Arctic sea ice loss passed through a "tipping point"?

opposite pole from GCM's: low order (toy) models of climate change

Eisenman, Wettlaufer, PNAS 2009: nonlinear ODE for energy in upper ocean



look for bifurcations, multiple equilibria

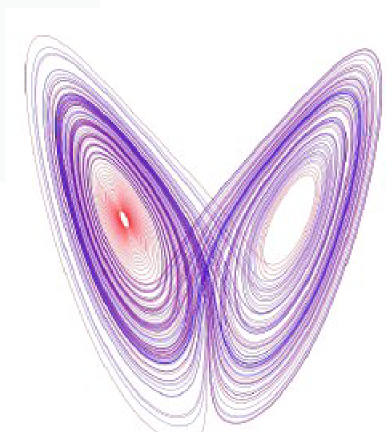
tipping point unlikely in loss of summer ice

Abbot, Silber, Pierrehumbert, JGR 2011

bifurcations with clouds, ice loss

Sudakov, Vakulenko, Golden Comm. Nonlinear Sci. & Num. Sim., 2014

impact of melt ponds



Lorenz butterfly

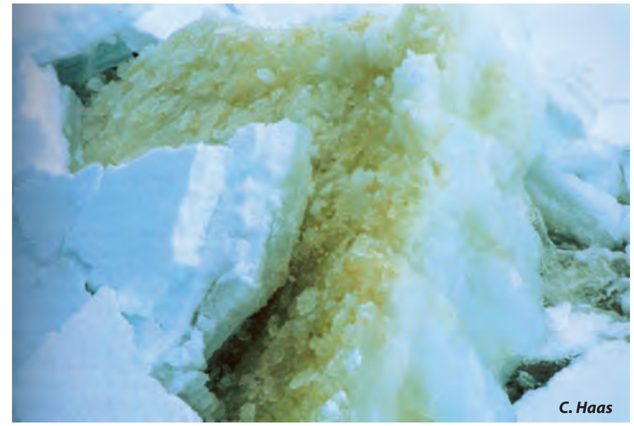
sea ice microphysics

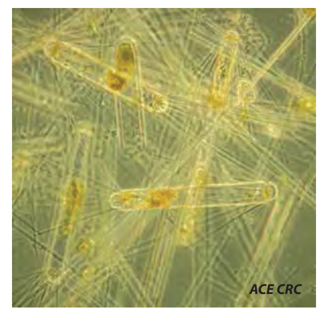
fluid transport

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities





- *drainage of brine and melt water*
- *ocean-ice-air exchanges of heat, CO_2*
- *Antarctic surface flooding and snow-ice formation*
- *evolution of salinity profiles*

linkage of scales

Darcy's Law for slow viscous flow in a porous medium

averaged fluid velocity

pressure gradient

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$

viscosity

$\mathbf{k}$ = fluid permeability tensor

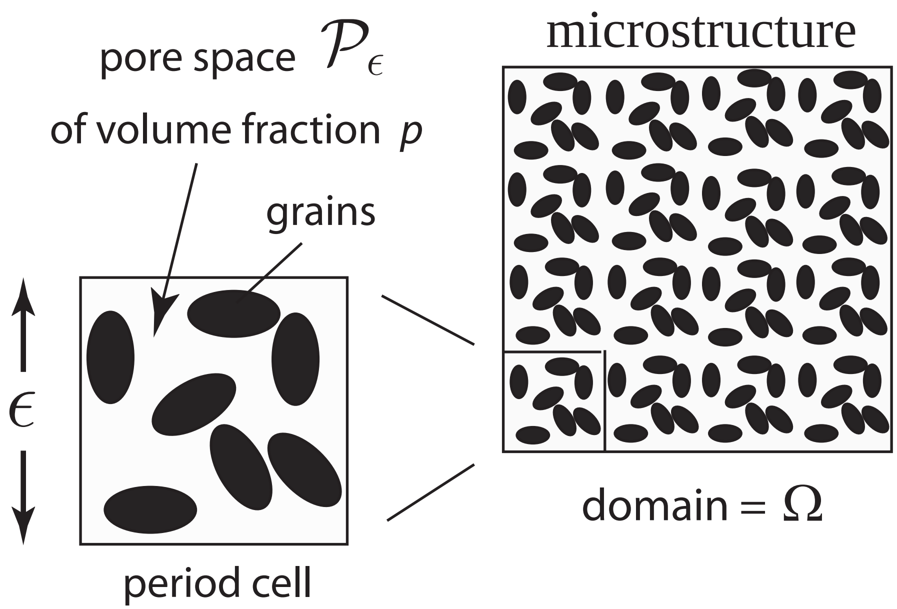
example of *homogenization*

mathematics for analyzing effective behavior of heterogeneous systems

e.g. transport properties of composites - electrical conductivity, thermal conductivity, etc.

HOMOGENIZE as $\epsilon \to 0$

Stokes equations for fluid velocity $\mathbf{v}^\epsilon$, pressure p^ϵ, force $\mathbf{f}$:



$$\nabla p^\epsilon - \epsilon^2 \eta \Delta \mathbf{v}^\epsilon = \mathbf{f}, \qquad x \in \mathcal{P}_\epsilon$$

$$\nabla \cdot \mathbf{v}^\epsilon = 0, \qquad x \in \mathcal{P}_\epsilon$$

$$\mathbf{v}^\epsilon = 0, \qquad x \in \partial \mathcal{P}_\epsilon$$

$\eta =$ fluid viscosity

via two-scale expansion

MACROSCOPIC EQUATIONS $\mathbf{v}^\epsilon \longrightarrow \mathbf{v}\,,\ p^\epsilon \longrightarrow p$ as $\epsilon \longrightarrow 0$

Darcy's law

(f = 0)

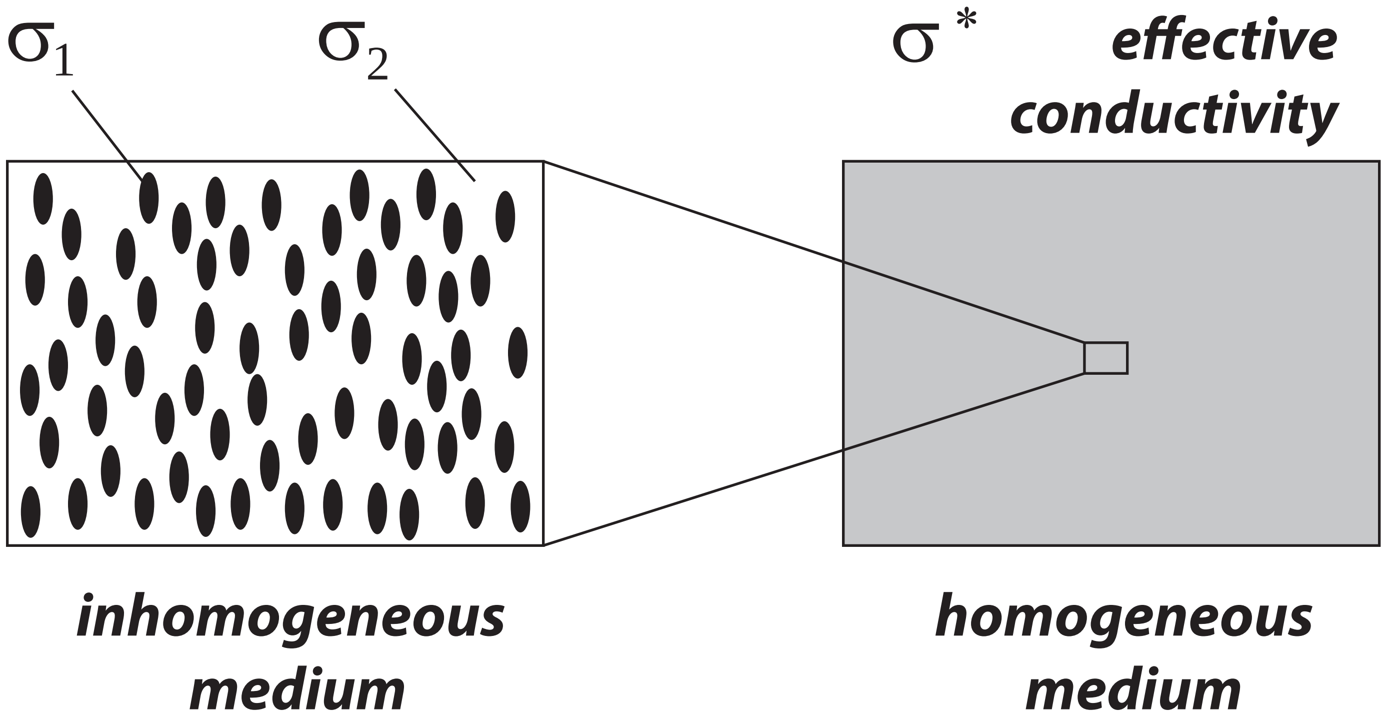
$$\mathbf{v} = -\frac{1}{\eta}\,\mathbf{k}\,\nabla p, \qquad x \in \Omega$$

$$\nabla \cdot \mathbf{v} = 0, \qquad x \in \Omega$$

$\mathbf{k}(x) =$ effective fluid permeability tensor

[Keller '80, Tartar '80, Sanchez-Palencia '80, J. L. Lions '81, Allaire '89, '91,'97]

HOMOGENIZATION

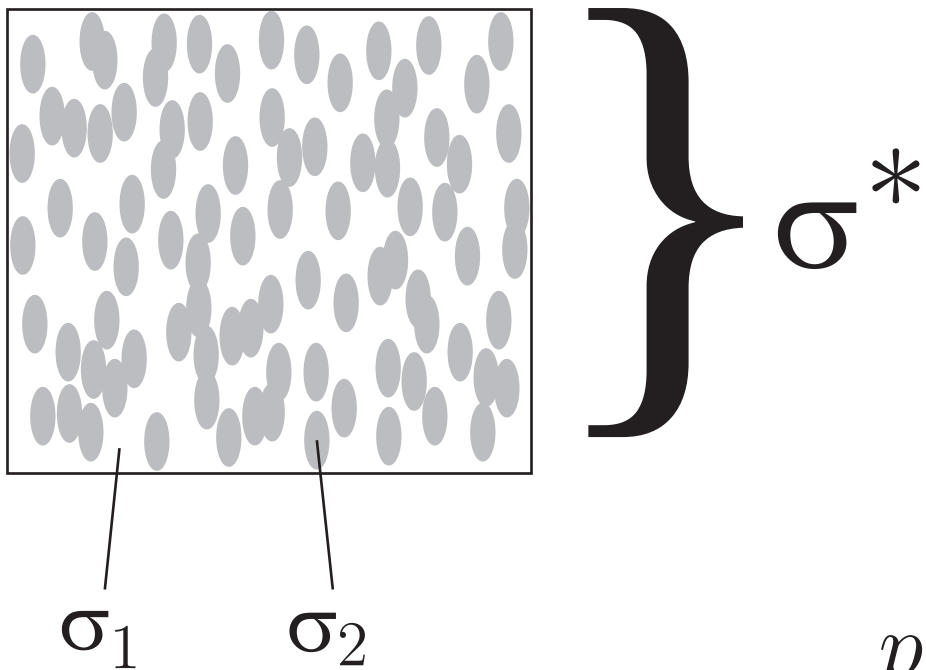


find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

Wiener 1912 : arithmetic and harmonic mean ***bounds*** *on effective conductivity*
Hashin and Shtrikman 1962 : variational ***bounds*** *on effective conductivity*

Effective electrical conductivity of a two phase composite



$$J = \sigma E$$

$$\nabla \cdot J = 0$$

$$\nabla \times E = 0$$

$$\langle J \rangle = \sigma^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

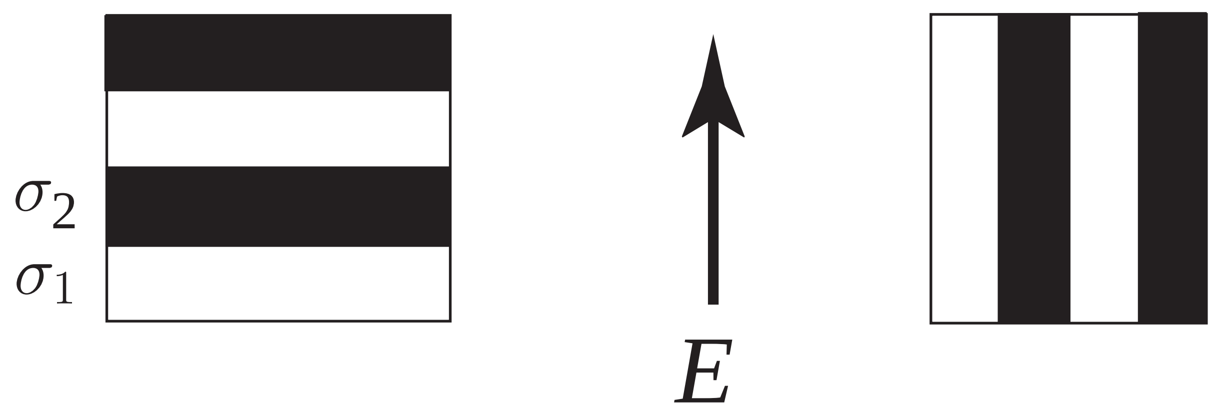
$$\sigma^* = \sigma^* \left(\frac{\sigma_1}{\sigma_2} , \text{ composite geometry} \right)$$

arithmetic and harmonic mean bounds on transport properties

effective electrical conductivity σ^* for two phase composite of σ_1 and σ_2

optimal bounds on σ^* for known volume fractions p_1 and p_2:

$$\frac{1}{\frac{p_1}{\sigma_1}+\frac{p_2}{\sigma_2}} \le \sigma^* \le p_1\sigma_1 + p_2\sigma_2$$



applied electric field

Wiener, 1912

electrical transport

fluid transport

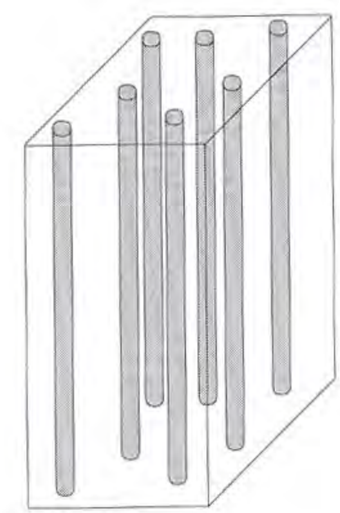
electrical conductance	$g_e = \pi r^2 \sigma$	fluid conductance	$g_f = \pi r^4 / 8\eta$
electrical conductivity	$\sigma_e = \sigma$	fluid conductivity	$\sigma_f = r^2 / 8\eta$

PIPE BOUNDS on vertical fluid permeability k :

vertical pipes

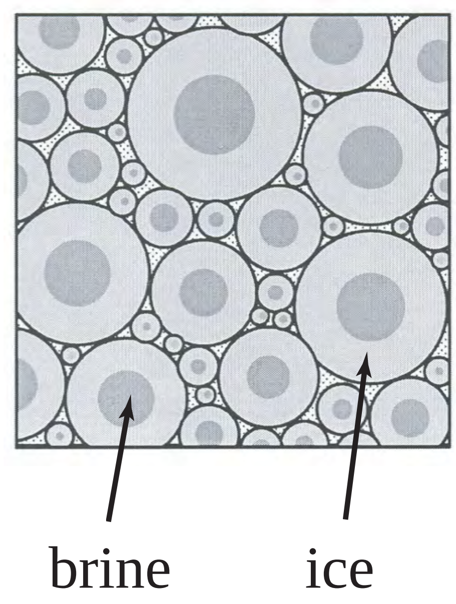
with appropriate radii

maximize k



fluid analog of arithmetic mean upper bound
on effective conductivity (Wiener 1912)

optimal coated
cylinder geometry



special case of optimal VOID BOUNDS

Torquato and Pham, *Phys. Rev. Lett.* 2004

analyze via **trapping constant** for
diffusion process in pore space

diffusing tracer with concentration *c*(*x*,*t*)

reacts with traps on pore boundaries

$$\frac{\partial c}{\partial t} = D\Delta c + G, \qquad x \in \Omega_b$$

D = diffusion constant

G = reactant source rate

κ = surface reaction rate constant

$$D\frac{\partial c}{\partial n} + \kappa c = 0, \qquad x \in \partial\Omega_b$$

boundary condition

trapping constant: $\gamma^{-1} = \langle u \rangle$

$u(x)$ is scaled concentration field in steady state with absorbing b.c. (diffusion-controlled limit)

$$\Delta u = -1, \;\; x \in \Omega_b \qquad u = 0, \;\; x \in \partial\Omega_b$$

variational inequality $\gamma \geq \langle \nabla v \cdot \nabla v \rangle^{-1}$

$$\forall v \in \{\text{ergodic } v(x) : \Delta v = -1, \;\; x \in \Omega_b\}$$

mean survival time: $\tau = \dfrac{1}{\gamma \phi D}$

use trial field in variational inequality

$$v(x) = \frac{1}{\phi_S} \int_\Omega g(x-y)[\chi_b(y) - \phi]dy, \qquad \phi_S = 1 - \phi$$

$$g(r) = \begin{cases} -\frac{1}{2\pi} \ln r, & d = 2 \\ \frac{1}{(d-2)\Omega(d)} r^{2-d}, & d \geq 3, \end{cases} \qquad \Omega(d) = 2\pi^{d/2}/\Gamma(d/2)$$

yields lower VOID BOUND: $\gamma \geq \dfrac{\phi_S^2}{\ell_P^2}$

where ℓ_P is a pore length scale defined by

$$\ell_P^2 = -\int_0^\infty (S_2(r) - \phi^2)\, r \ln r \, dr, \quad d = 2$$

$$\ell_P^2 = \frac{1}{d-2} \int_0^\infty (S_2(r) - \phi^2)\, r \, dr, \quad d \geq 3$$

$S_2(r)$

two-point correlation function

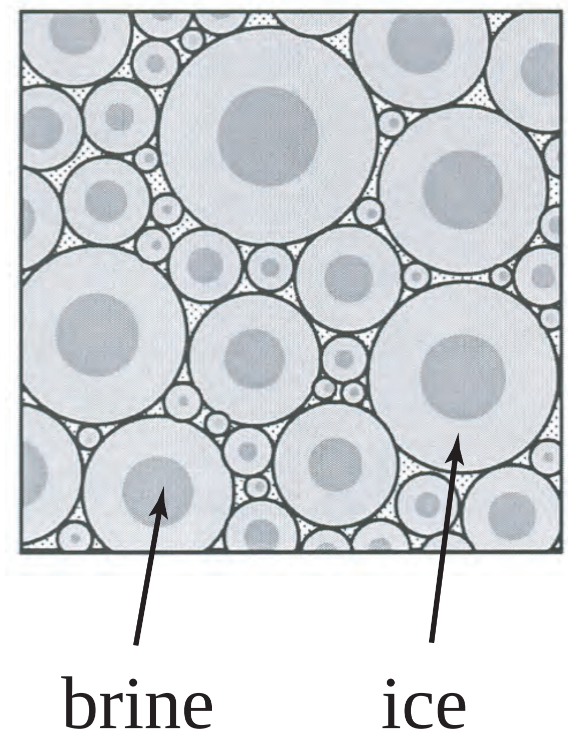
EVALUATE void bound for Hashin-Shtrikman coated cylinders (d=2) (Torquato and Pham, 2004)

$$\gamma \geq \frac{8\langle R_I^2\rangle}{\phi\langle R_I^4\rangle}\,, \quad d=2$$

$$\mathbf{k} \;\leq\; \gamma^{-1}\mathbf{I}$$

for any ergodic porous medium
(Torquato, 2002)

optimal coated
cylinder geometry



lognormal pipe bound: inclusion cross sectional areas A lognormally distributed

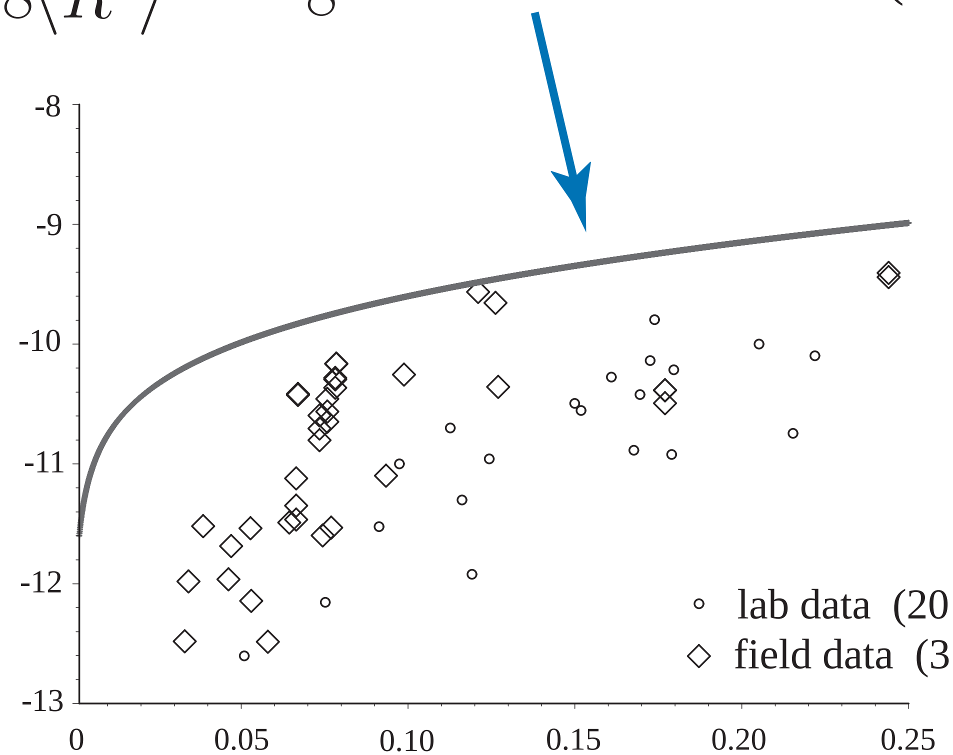
ln(A) is normally distributed with mean μ and variance σ ²

$$k \;\le\; \frac{\phi\langle R^4\rangle}{8\langle R^2\rangle} \;=\; \frac{\phi}{8}\langle R^2\rangle e^{\sigma^2}$$

μ increases with T and σ = 1
(Perovich and Gow, '96, ...)

log k

k = vertical fluid permeability of sea ice

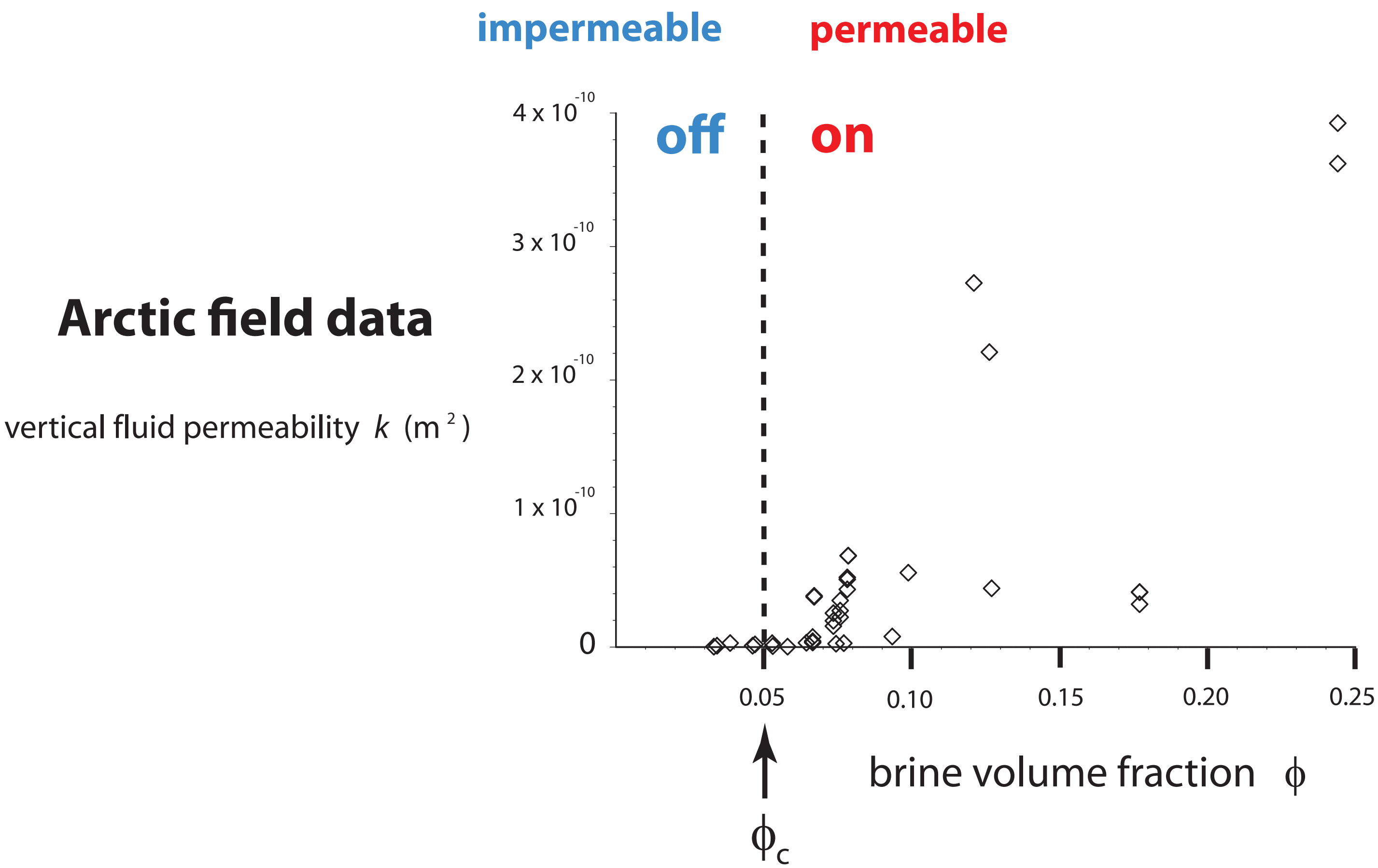


brine volume fraction ϕ

similar calculations
in math-finance
(stock prices tend to be
lognormally distributed)

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *GRL* 2007
Golden, Heaton, Eicken, Lytle, *Mech. Materials* 2006

Critical behavior of fluid transport in sea ice



on - off switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\%$ $\longleftrightarrow$ $T_c \approx -5^\circ C$, $S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

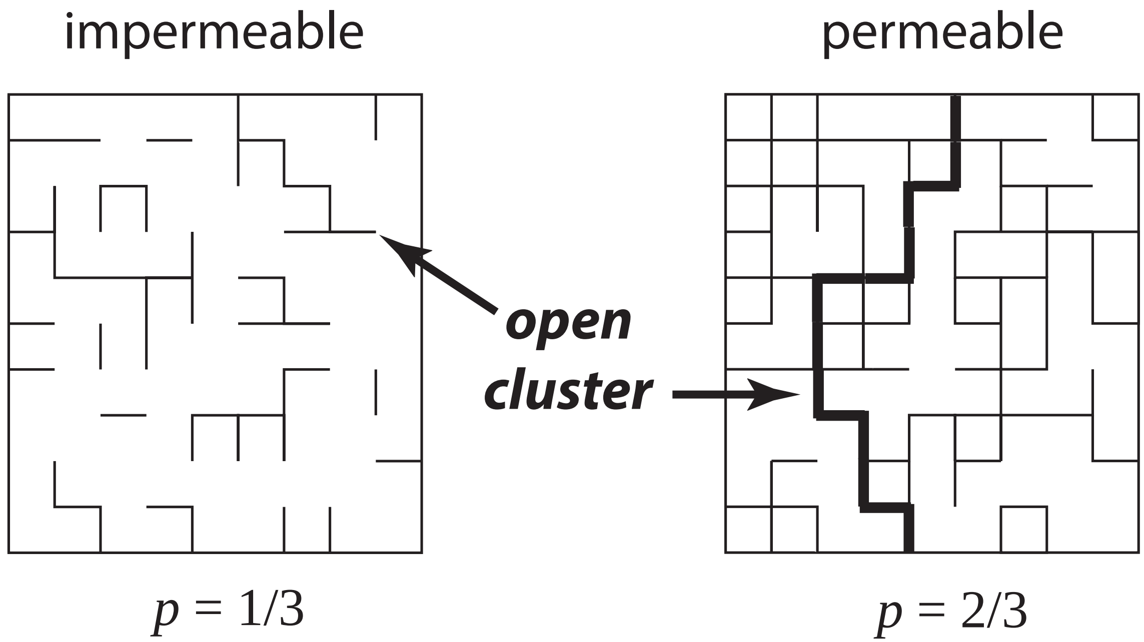
Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

Why is the rule of fives true?

percolation theory

probabilistic theory of connectedness



bond → **open** *with probability p*
closed *with probability $1\text{-}p$*

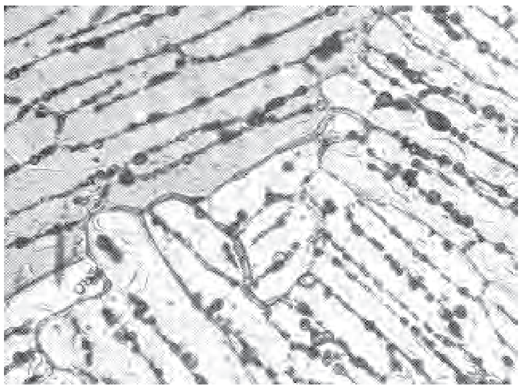
percolation threshold

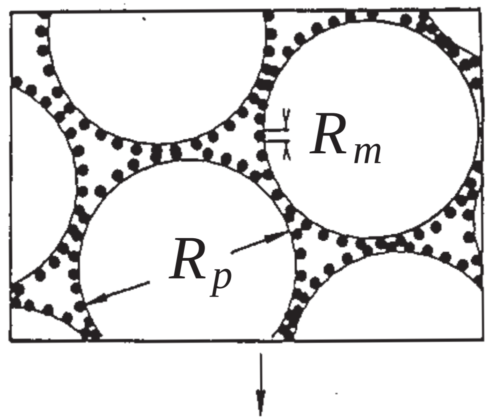
$$p_c = 1/2 \quad \text{for} \quad d = 2$$

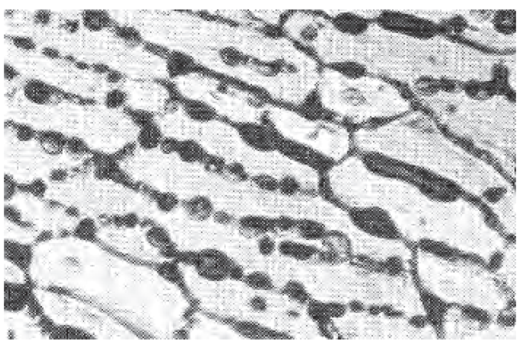
smallest p for which there is an infinite open cluster

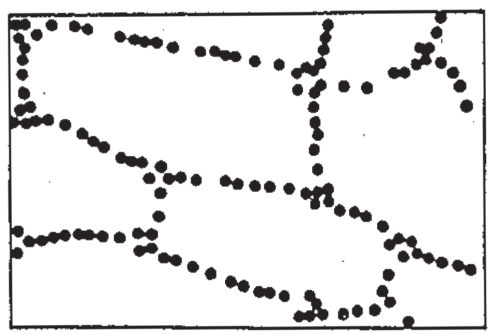
Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

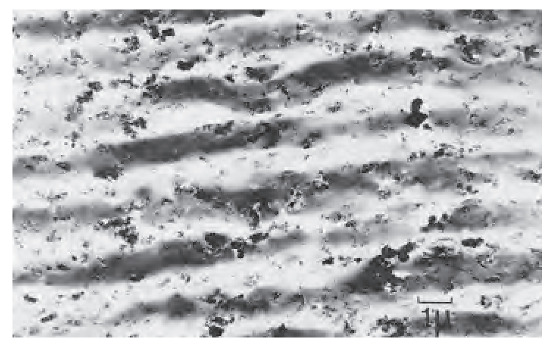
$\phi_c \approx 5\,\%$ Golden, Ackley, Lytle, *Science*, 1998











sea ice **compressed powder** **radar absorbing composite**

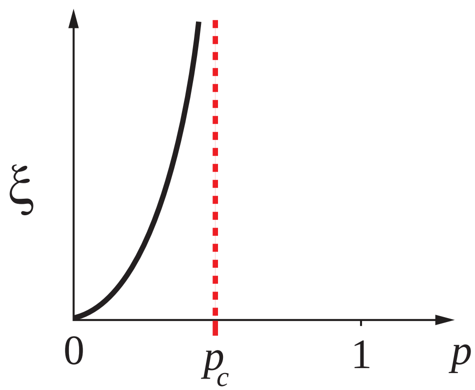
sea ice is radar absorbing

order parameters in percolation theory

geometry

correlation length

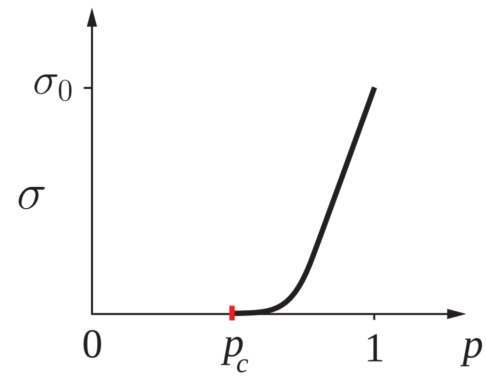
(characteristic scale of connectedness)



$$\xi(p) \sim |p - p_c|^{-\nu}$$

transport

effective conductivity
or fluid permeability



$$\sigma(p) \sim \sigma_0 \, (p - p_c)^t$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

1 $\leq$ *t* $\leq$ 2 (for idealized model) **Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992**

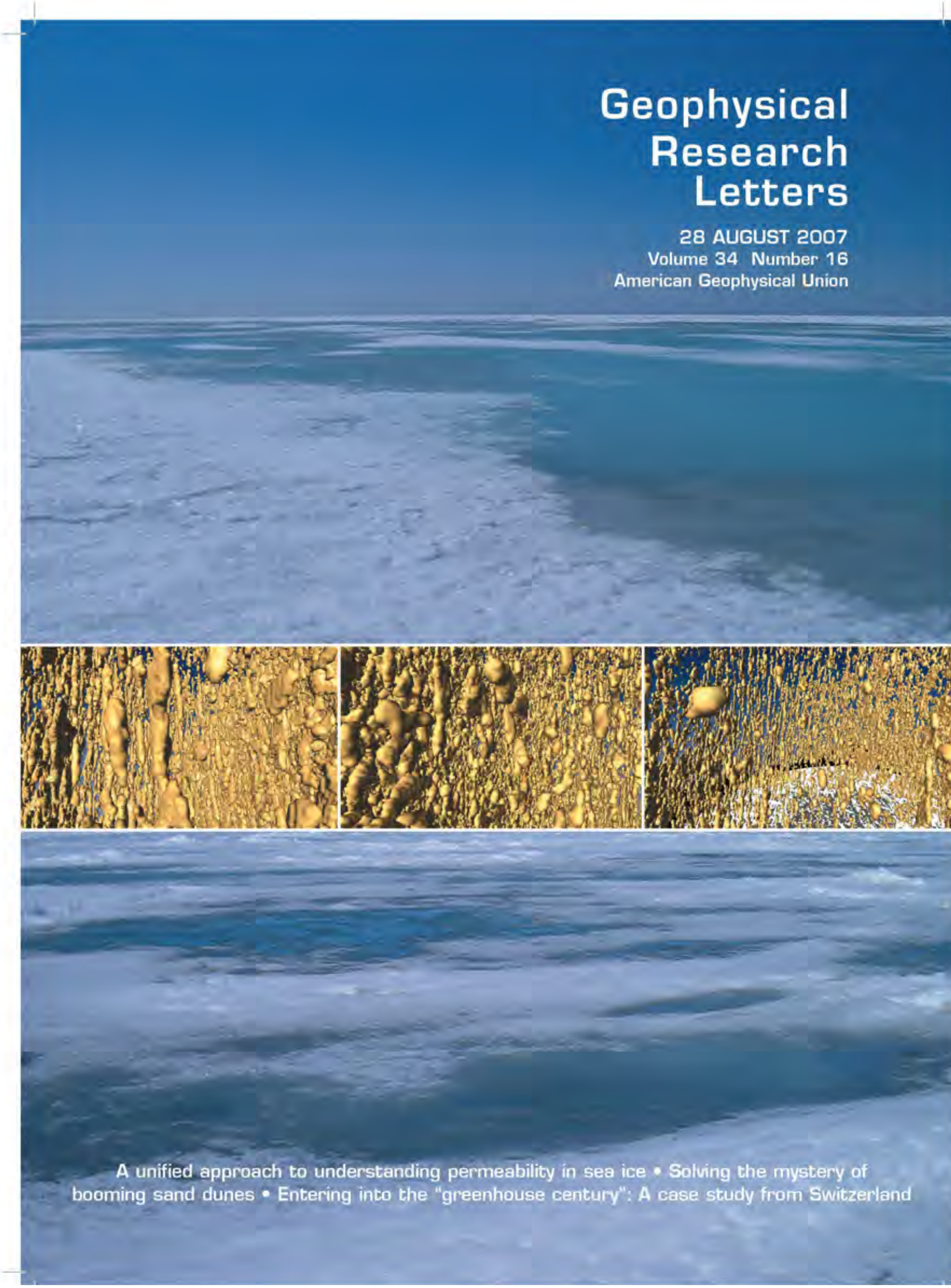
non-universal behavior in continuum

Thermal evolution of permeability and microstructure in sea ice **Golden, Eicken, Heaton, Miner, Pringle, Zhu**

micro-scale

controls

macro-scale processes



rigorous bounds
percolation theory
hierarchical model
network model

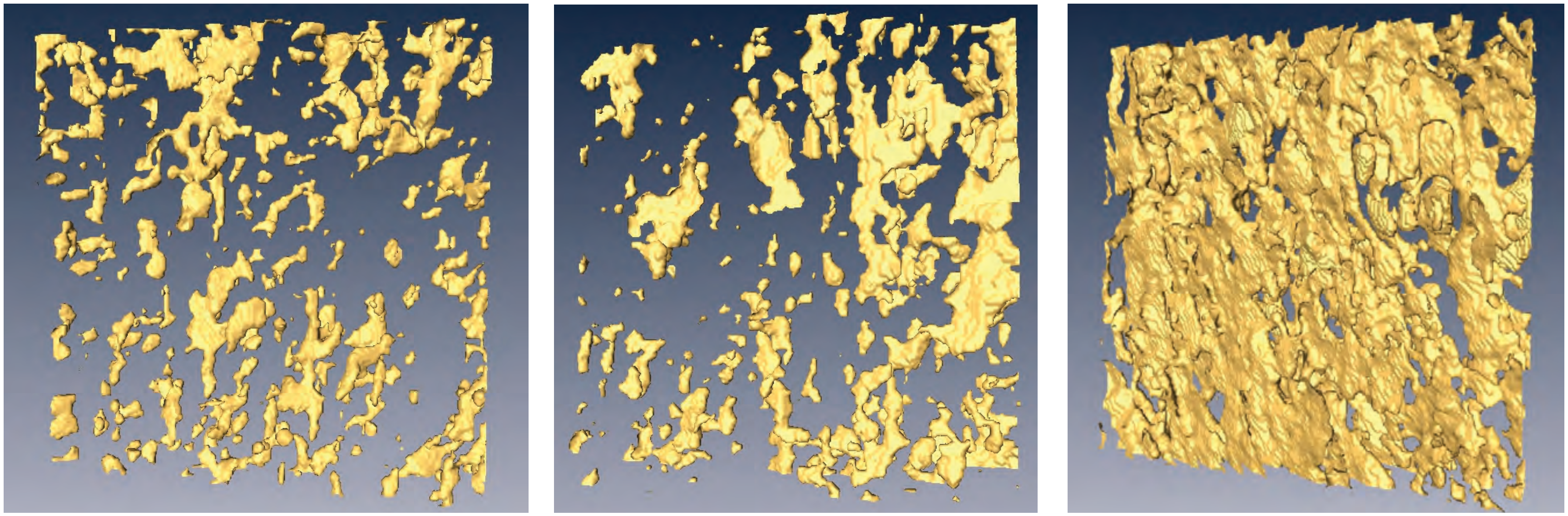
field data

X-ray tomography for brine inclusions

unprecedented look at thermal evolution of brine phase and its connectivity

brine connectivity (over cm scale)

8 x 8 x 2 mm



-15 °C, $\phi = 0.033$ -6 °C, $\phi = 0.075$ -3 °C, $\phi = 0.143$

X-ray tomography confirms percolation threshold

3-D images
pores and throats → 3-D graph
nodes and edges

analyze graph connectivity as function of temperature and sample size

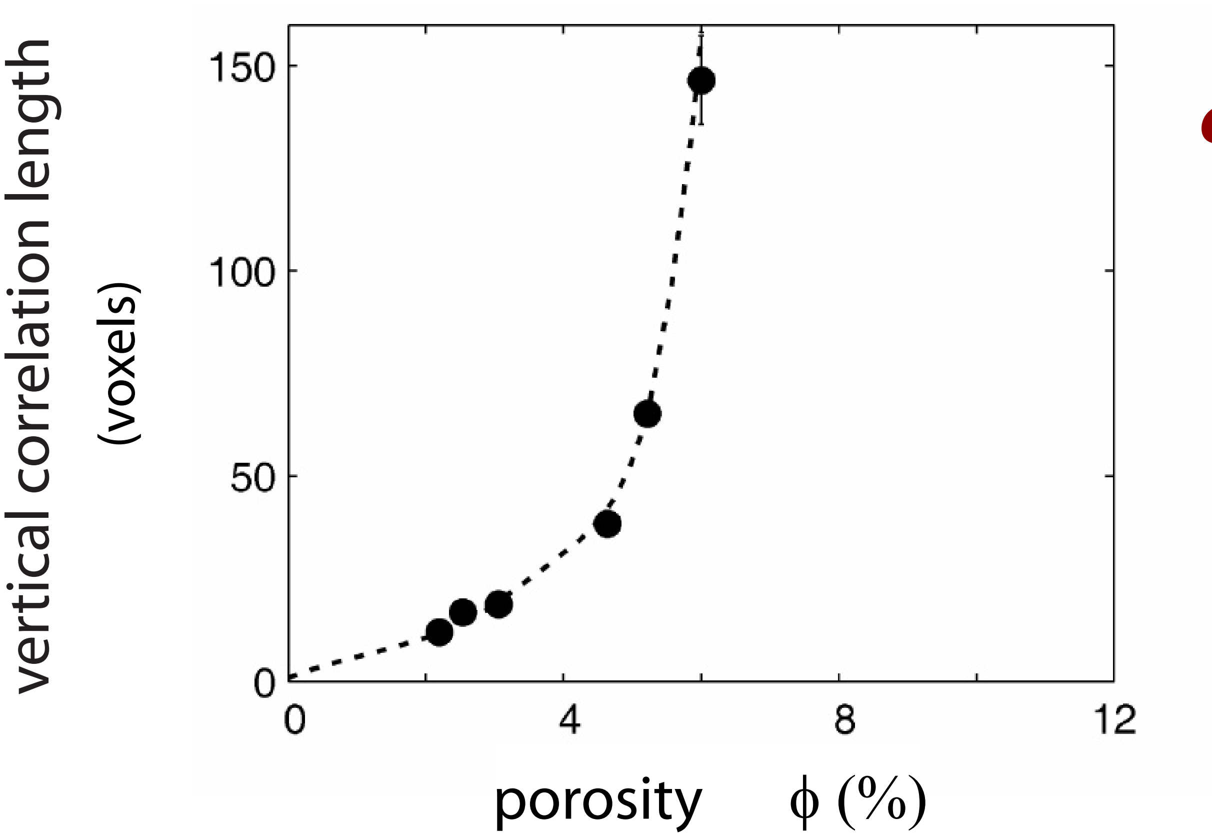
- ***use finite size scaling techniques to confirm rule of fives***
- ***order parameter data from a natural material***

Pringle, Miner, Eicken, Golden, J. Geophys. Res. 2009

The key connectivity functions of percolation theory have been computed extensively for many lattice models, but ***NOT*** for natural materials.

We have calculated them for sea ice single crystals and estimated anisotropic percolation thresholds.

Pringle, Miner, Eicken, Golden, JGR (Oceans) 2009



correlation length

characteristic scale of connectedness

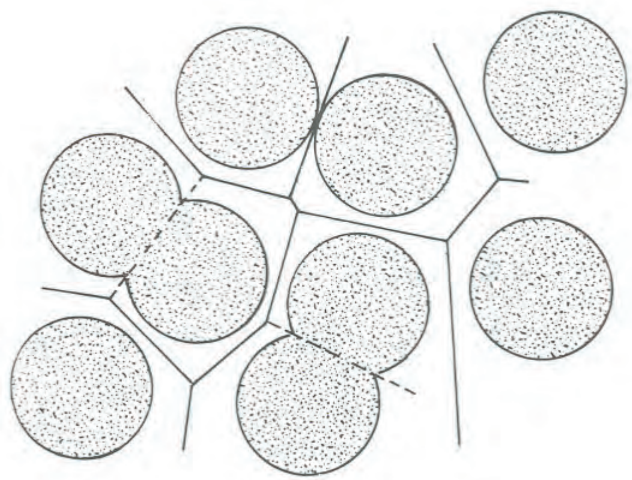
divergence of vertical ***correlation length*** for single crystal data

Non-universal behavior in the continuum:

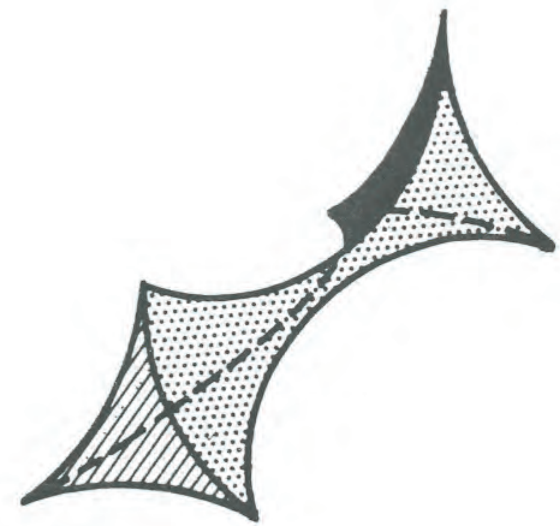
critical exponents for transport in Swiss cheese model take values different than for lattices, e.g. $t > 2$

$e \neq t$

Halperin, Feng, Sen, *Phys. Rev. Lett.* 1985



Swiss cheese model
$d = 2$



conducting neck in $d = 3$
Swiss cheese model

in general, non-universal exponents arise from a singular distribution of local conductances

In sea ice, this distribution is lognormal.

(excluding inclusions below cutoff)

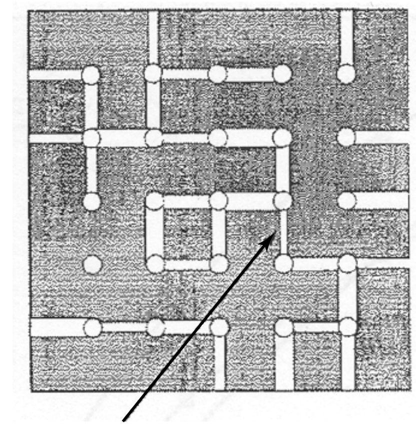
Thus, the permeability exponent for sea ice is **2**, the universal lattice value.

ESTIMATE fluid conductivity scaling factor $k_0 = r^2 / 8$

for media with broad range of conductances

CRITICAL PATH ANALYSIS

bottlenecks control flow



critical pore

Ambegaokar, Halperin, Langer 1971: CPA in electronic hopping conduction
Friedman, Seaton 1998: CPA in fluid and electrical networks
Golden, Kozlov 1999: rigorous CPA on long-range checkerboard model

$k_0 \approx r_c^2 / 8$ critical fluid conductivity

Microstructural analyses yield $r_c \approx$ 0.5 mm

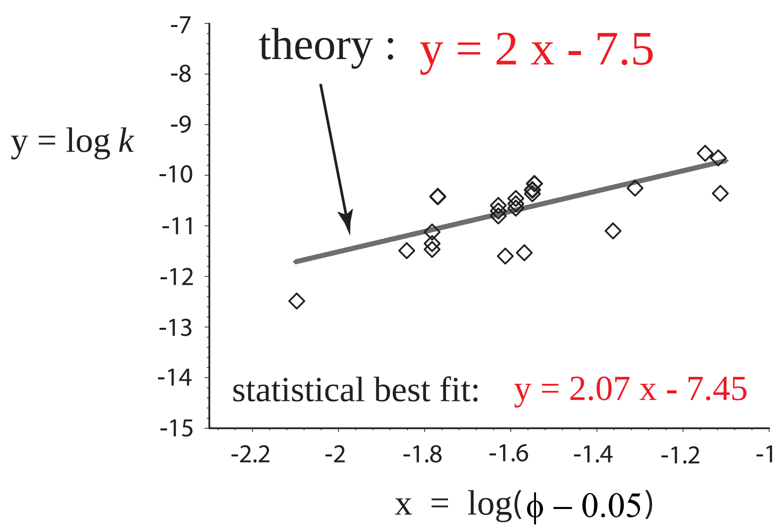
lattice and continuum percolation theories yield:

$$k(\phi) = k_0 \, (\phi - 0.05)^2$$

critical exponent t

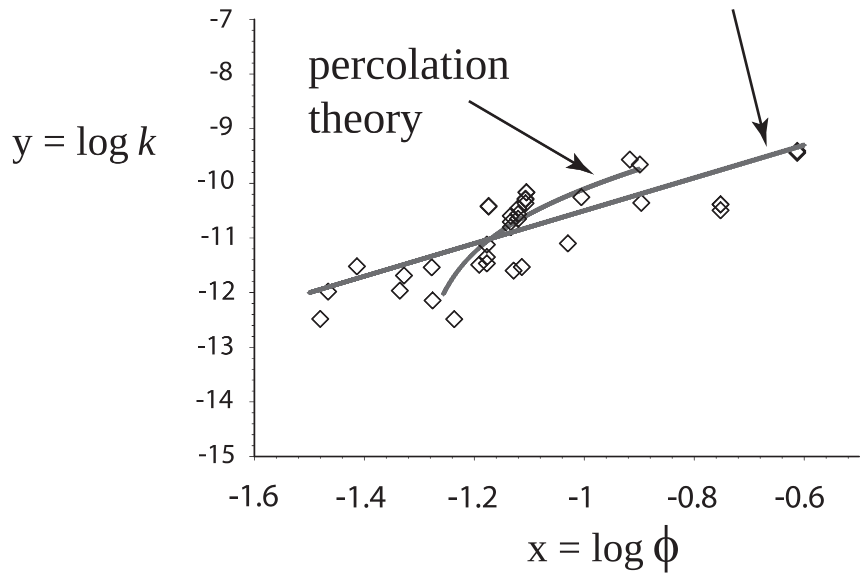
$$k_0 = 3 \times 10^{-8} \, \mathrm{m}^2$$

- exponent is UNIVERSAL lattice value $t \approx 2.0$
- sedimentary rocks like sandstones also exhibit universality
- critical path analysis -- developed for electronic hopping conduction -- yields scaling factor k_0



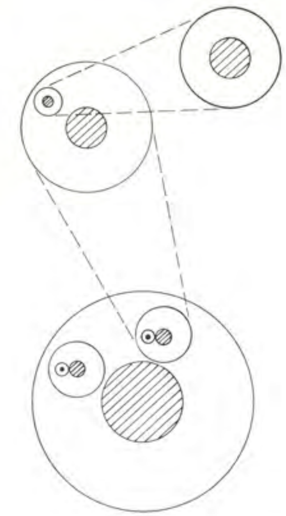
hierarchical and network models

hierarchical model : $y = 3x - 7.5$



statistical best fit of data: $y = 3.05\,x - 7.50$

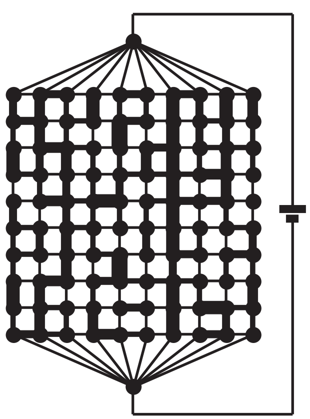
brine-coated spherical ice grains

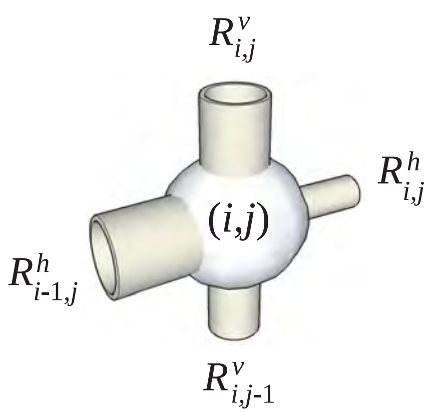


$$k(\phi) = k_0\, \phi^3$$

self-similar model used for porous rocks

Sen, Scala, Cohen 1981
Sheng 1990
Wong, Koplick, Tomanic 1984





random pipe network with radii chosen from measured inclusion distributions, solved with fast multigrid method

Zhu, Jabini, Golden, Eicken, Morris, Annals of Glaciology, 2006
Golden et al., Geophysical Research Letters, 2007
Zhu, Golden, Gully and Sampson, Physica B, 2010

diatoms in EPS-filled pores in natural sea ice

protects microorganisms against osmotic shock:

- highly concentrated brine
- fresh water from melt ponds

antifreeze, cryoprotectant

depresses freezing point

physical barrier from ice crystals

Transmitted light with Alcian Blue stain for EPS

Krembs, Eicken, Deming, PNAS 2011

Extracellular Polymeric Substances (EPS)

gelatinous ***exopolymers*** ***algal mucous***

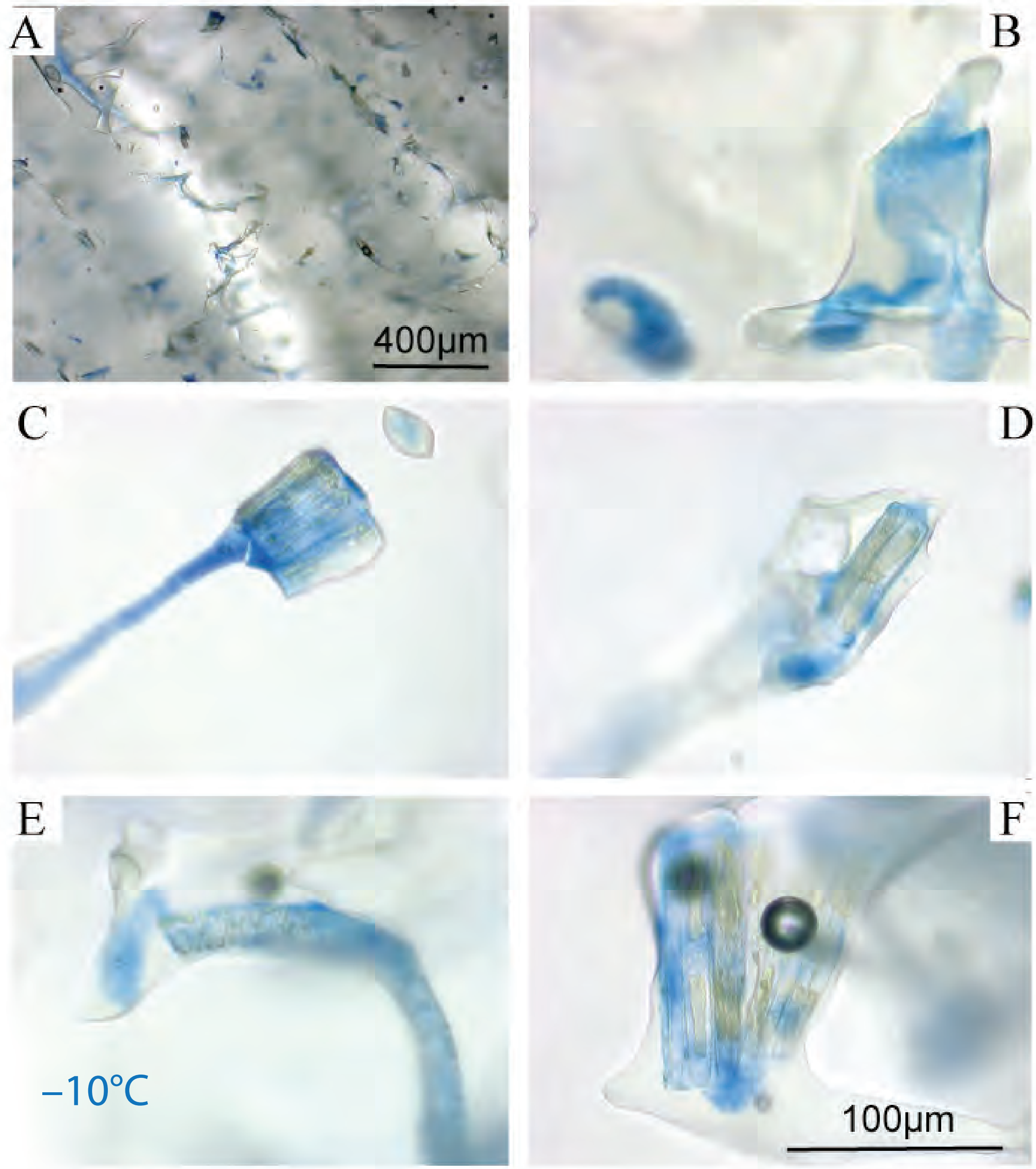
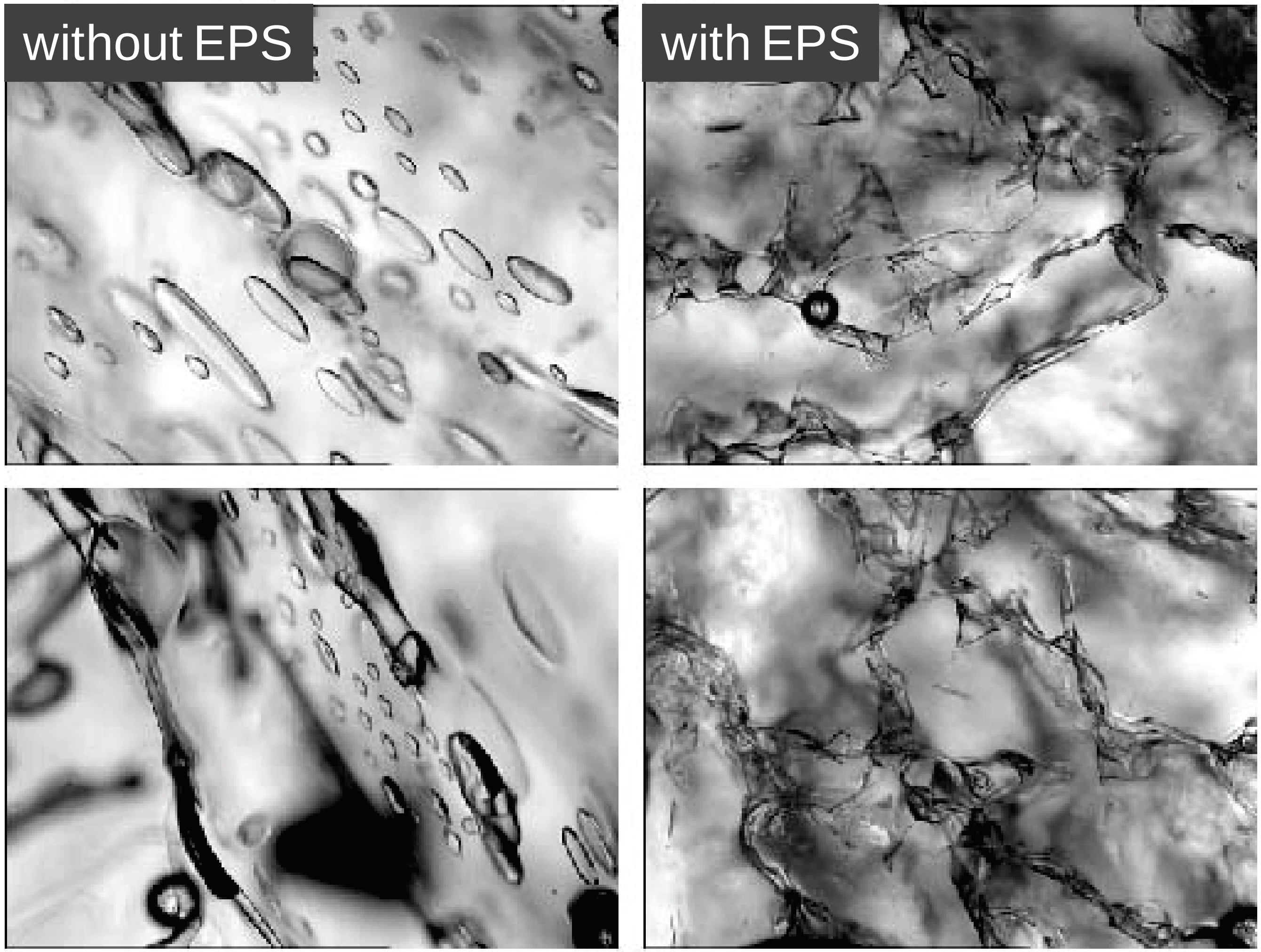


Fig. 1 Krembs et al.

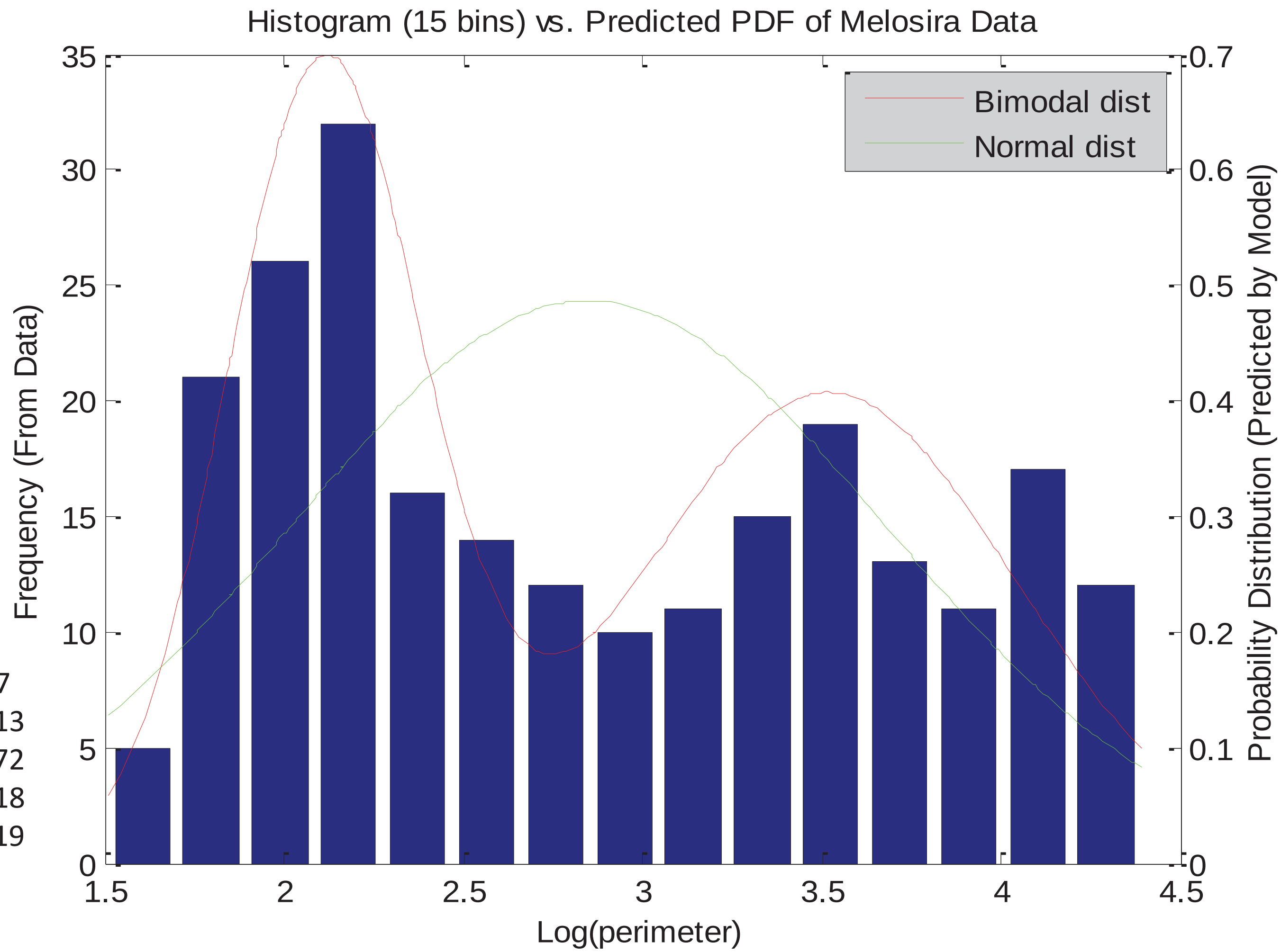
EPS changes sea ice microstructure



ellipsoidal inclusions **fractal inclusions**

Krembs, Eicken, Deming PNAS 2011

EPS appears to change brine inclusion size distribution from lognormal to ***bimodal*** lognormal



c = 0.47
μ1 = 2.113
σ1 = 0.272
μ2 = 3.518
σ2 = 0.519

A New Distribution

Zhu, Morris, Deming, Golden

How does EPS impact fluid transport?

random pipe network with new distribution

Steffen, Zhu, Epshteyn, Deming, Golden