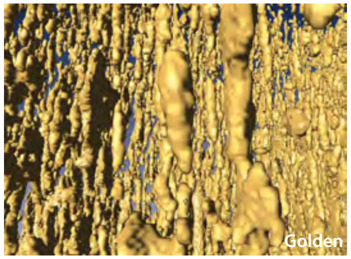
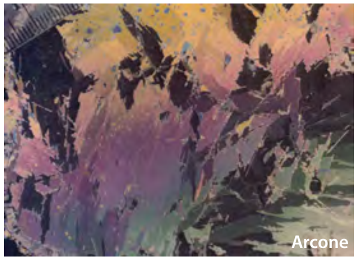
millimeters


Golden

centimeters


Arcone

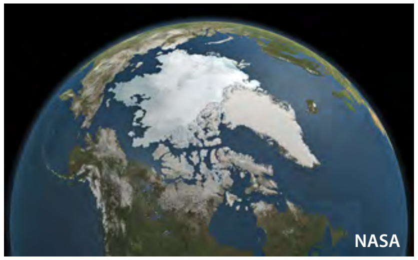
meters


Darelius

kilometers


NASA

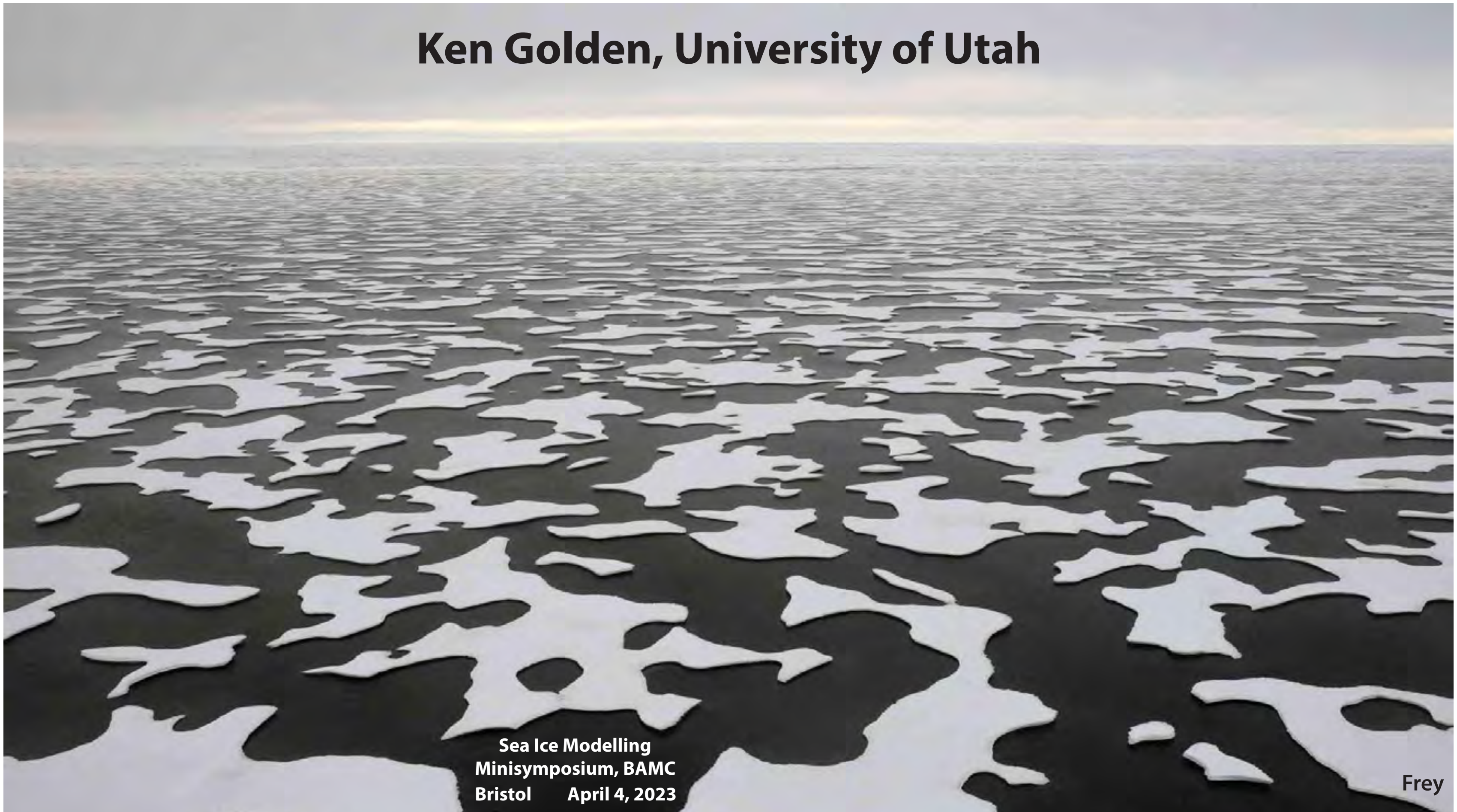
10^3 kilometers


NASA

From Micro to Macro in Sea Ice Modelling

What math tells us about sea ice and polar ecosystems in a warming climate

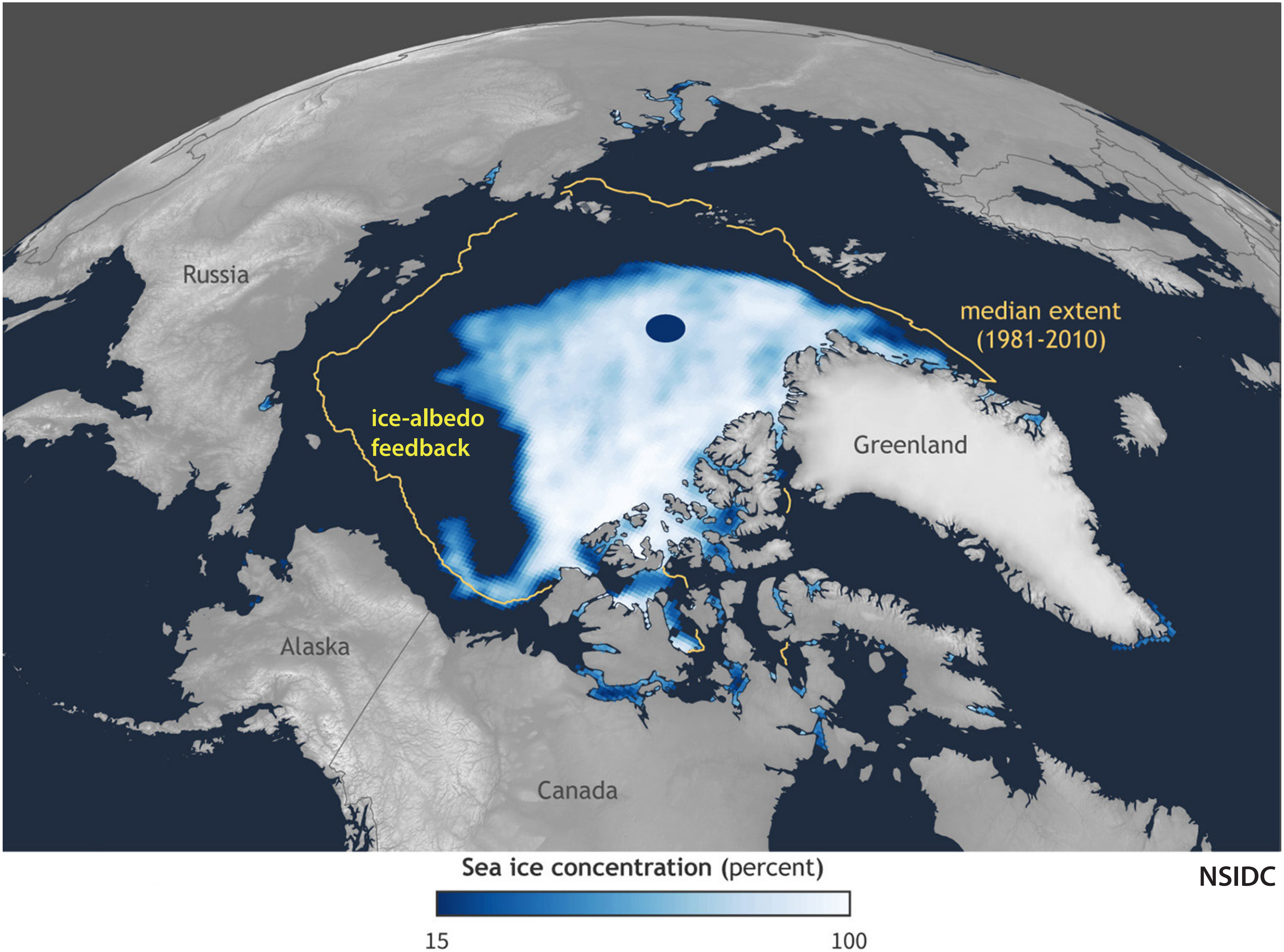
Ken Golden, University of Utah



Sea Ice Modelling
Minisymposium, BAMC
Bristol April 4, 2023

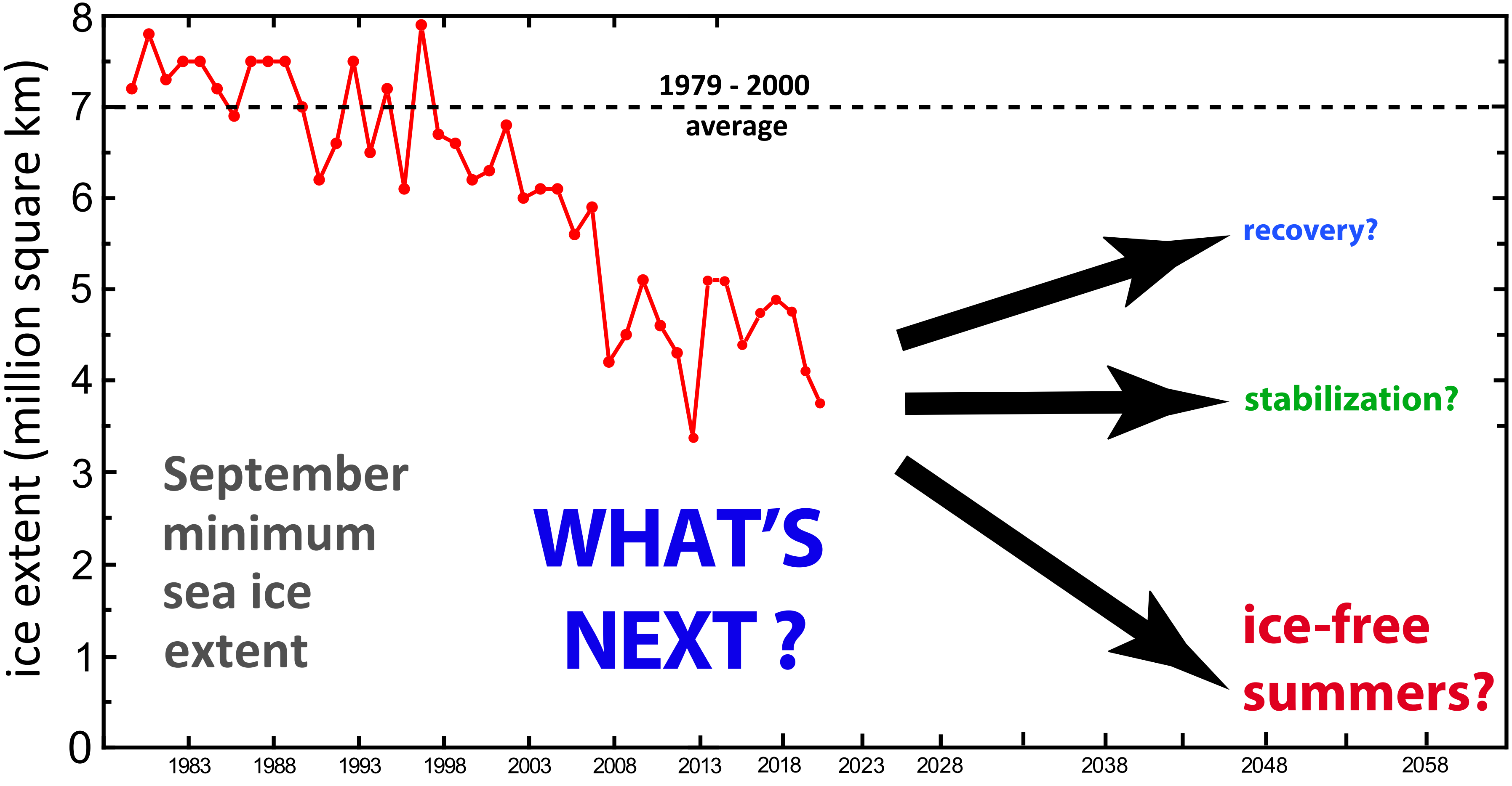
Frey

Arctic sea ice extent September 15, 2020



NSIDC

Predicting what may come next requires lots of math modeling.



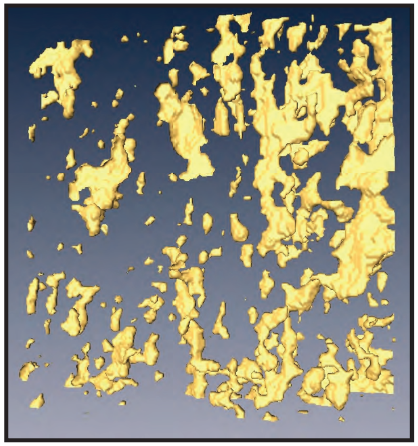
National Snow and Ice Data Center (NSIDC)

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

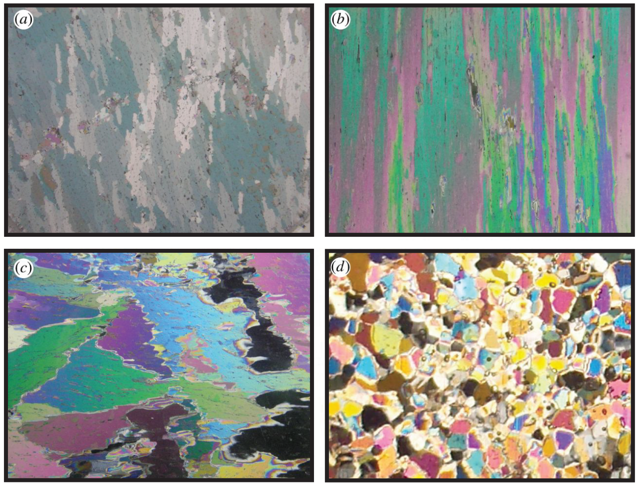
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

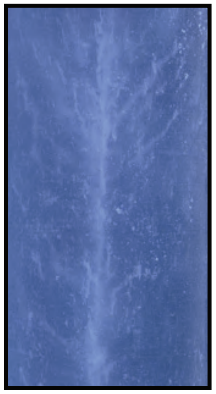
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

centimeters

---

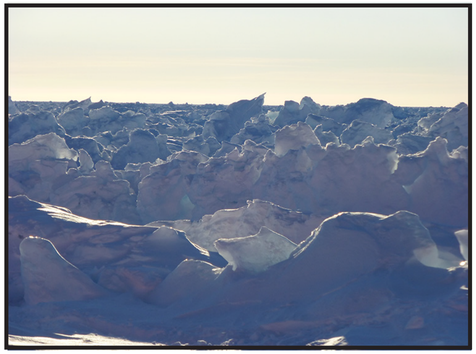
mesoscale

macroscale

Arctic melt ponds

K. Frey

Antarctic pressure ridges

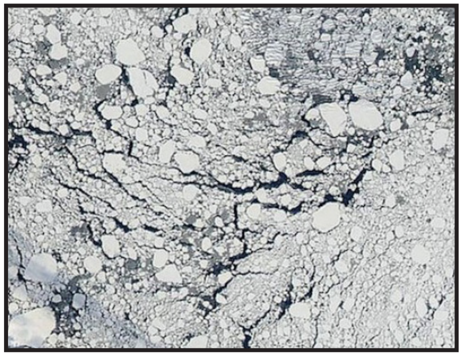


K. Golden

sea ice floes

J. Weller

sea ice pack

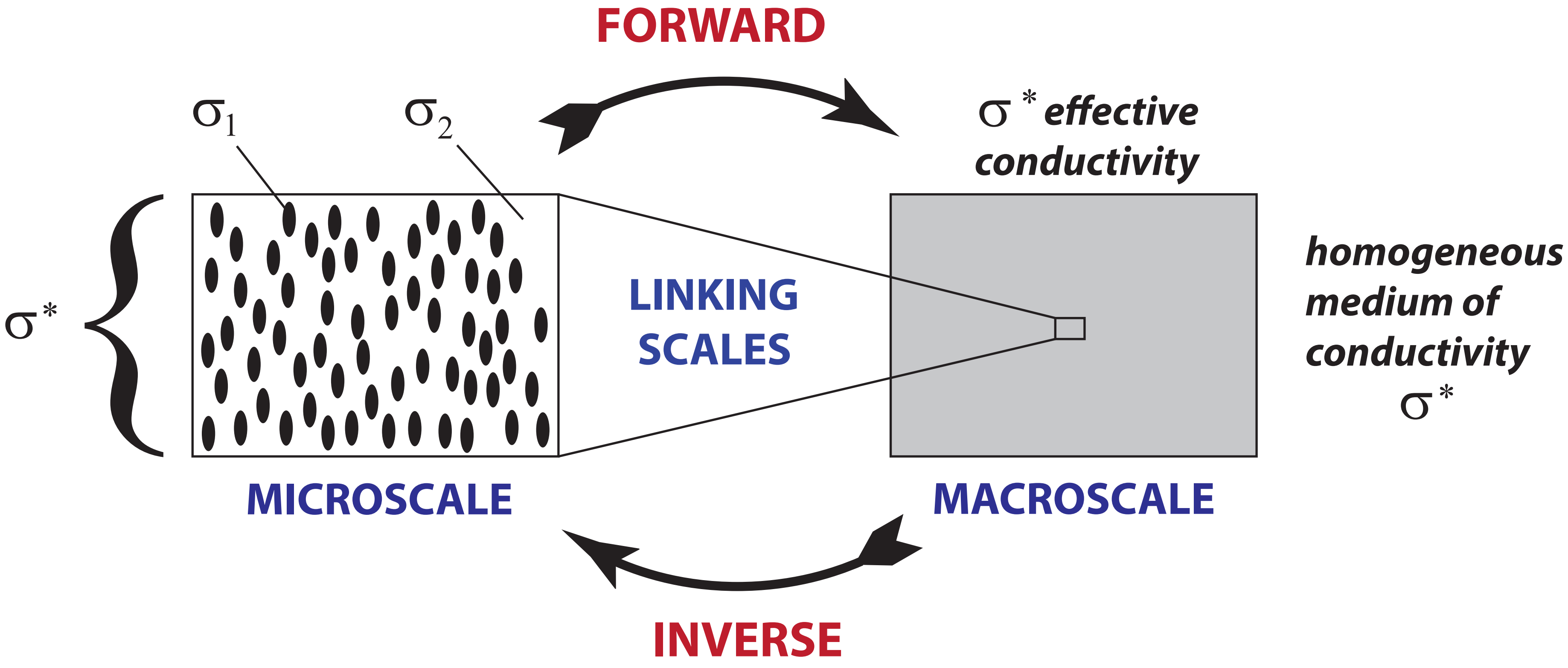


NASA

meters

kilometers

HOMOGENIZATION for Composite Materials



FORWARD

σ_1

σ_2

σ^* ***effective conductivity***

σ^*

LINKING SCALES

homogeneous medium of conductivity σ^*

MICROSCALE

MACROSCALE

INVERSE

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

Wiener 1912 : arithmetic and harmonic mean bounds on effective conductivity
Hashin and Shtrikman 1962 : variational bounds on effective conductivity

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

What is this talk about?

A tour of recent results on multiscale modeling of physical and ecological processes in the sea ice system, with a focus on novel mathematics.

microscale

mesoscale

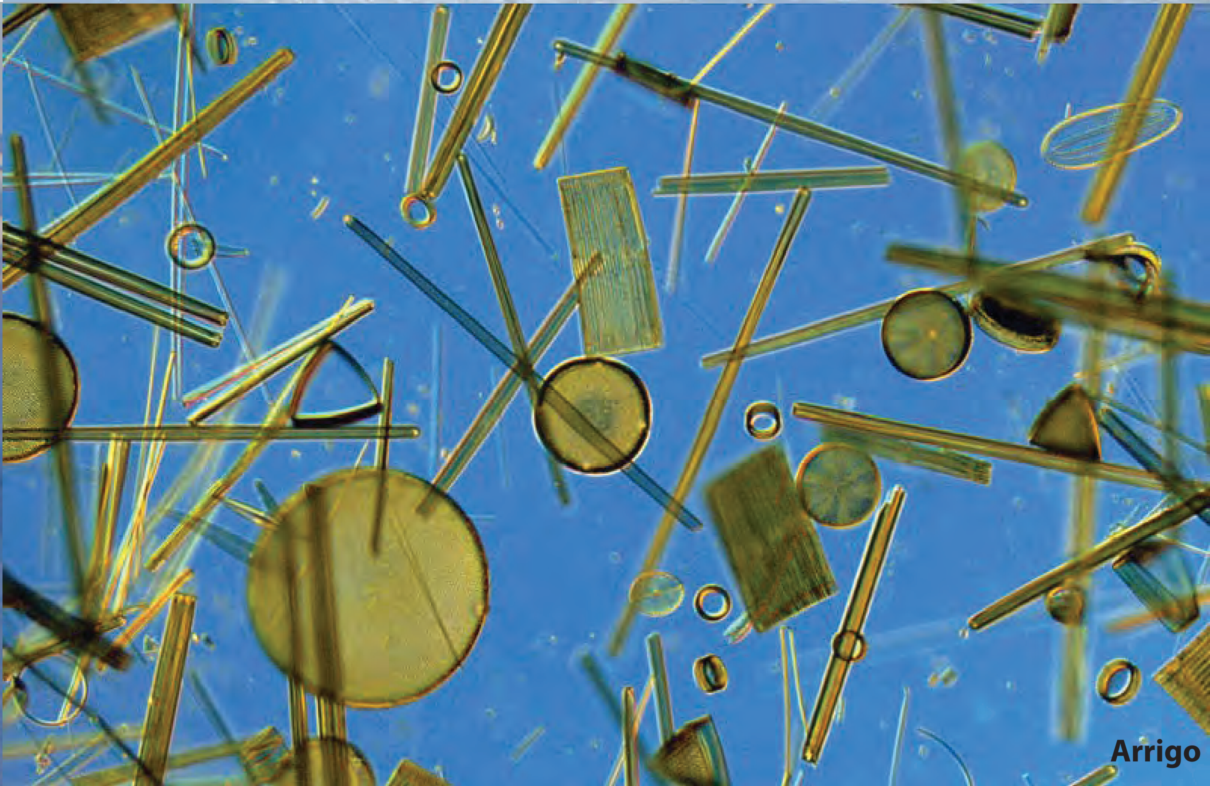
macroscale

From Microbes to Megafauna: How they impact and are impacted by the physics of sea ice

How do the physical properties of sea ice affect the communities it hosts?

How does the presence of life in and on sea ice affect its physical properties?

Golden

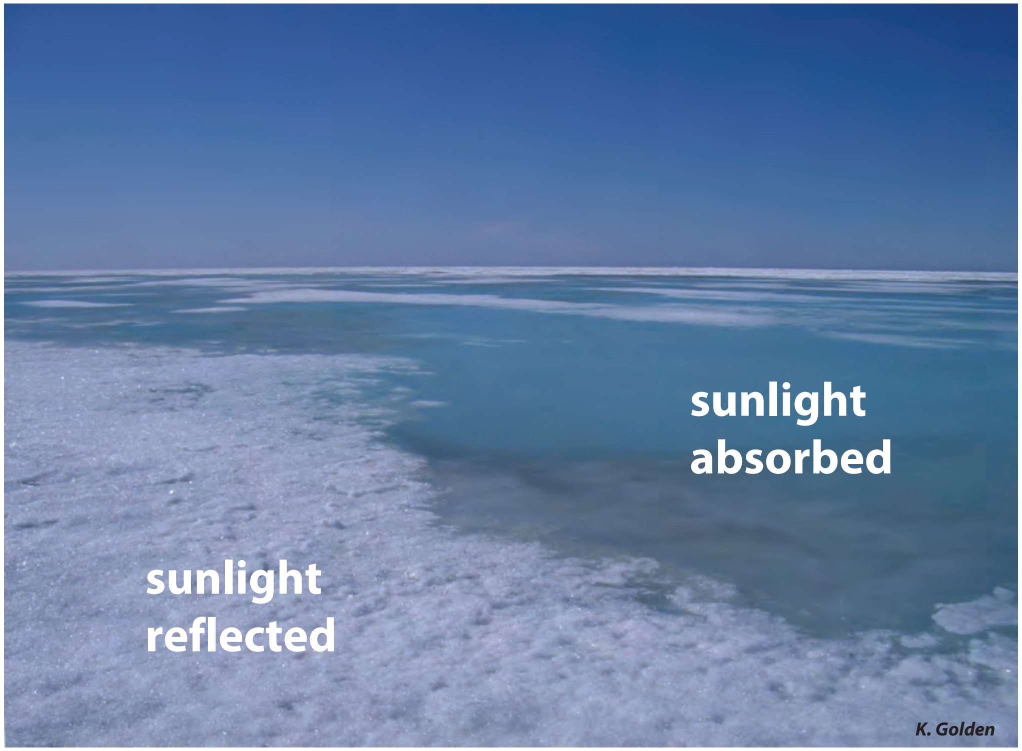


Arrigo

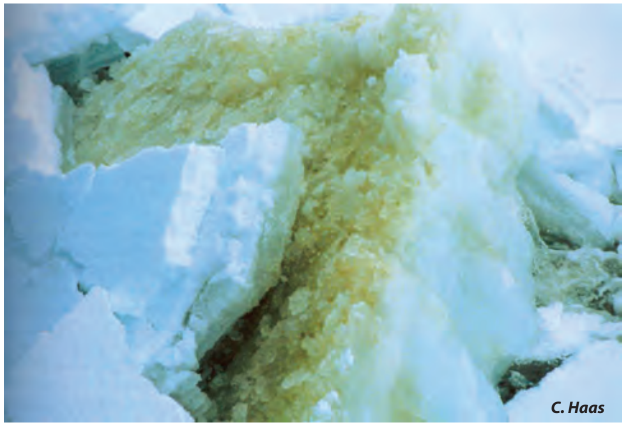
microscale

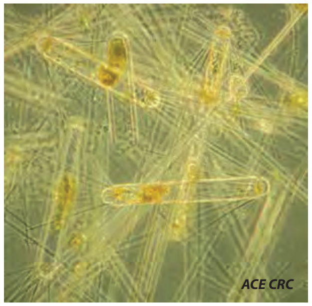
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

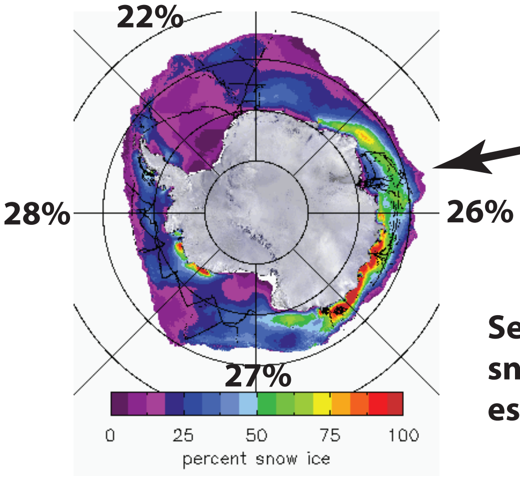
evolution of Arctic melt ponds and sea ice albedo



nutrient flux for algal communities







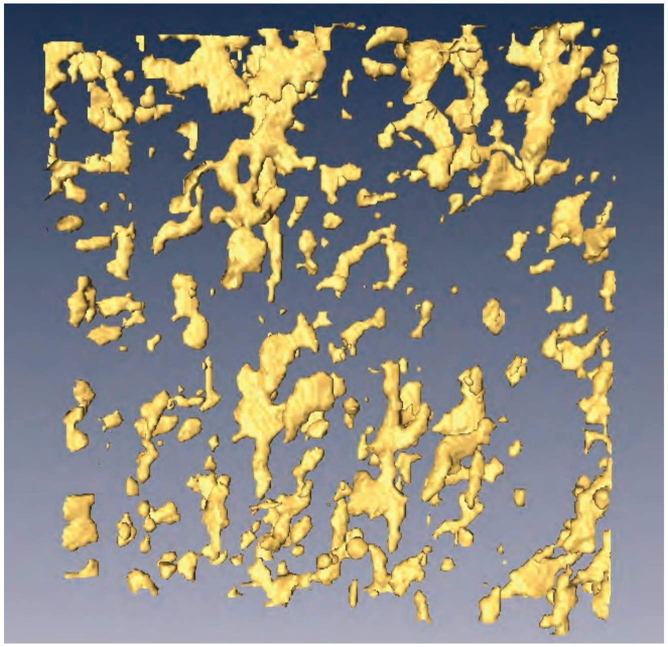
T. Maksym and T. Markus, 2008

September snow-ice estimates

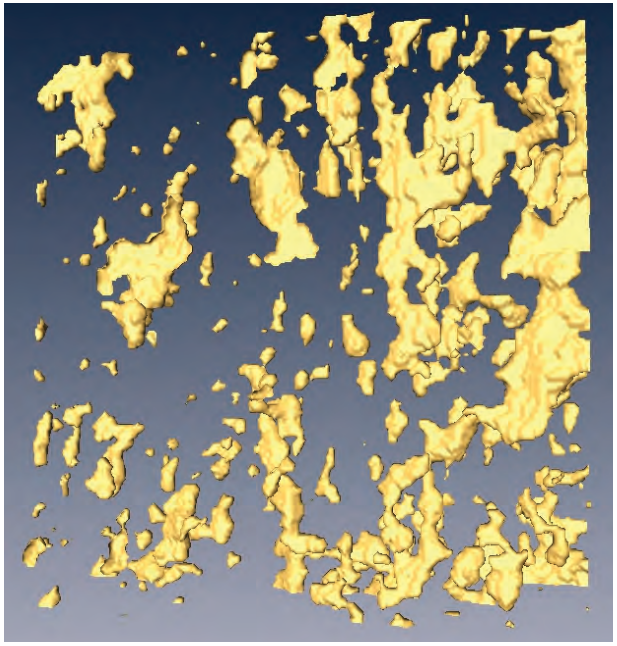
Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

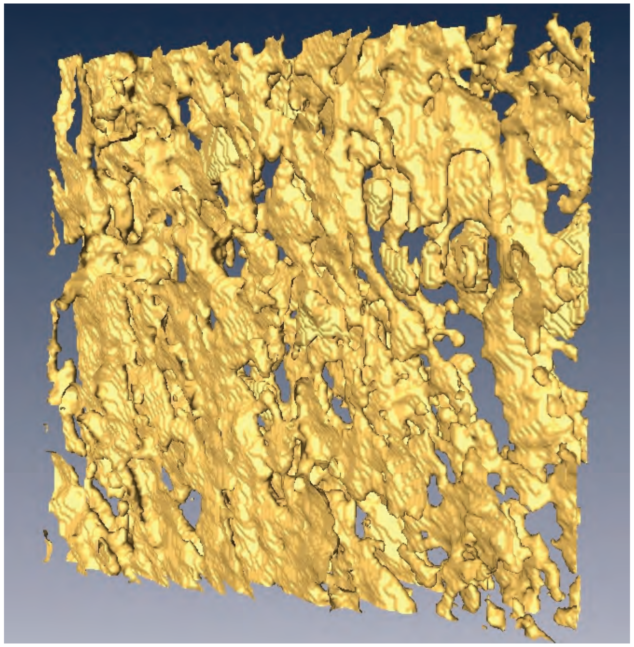
brine volume fraction and ***connectivity*** increase with temperature



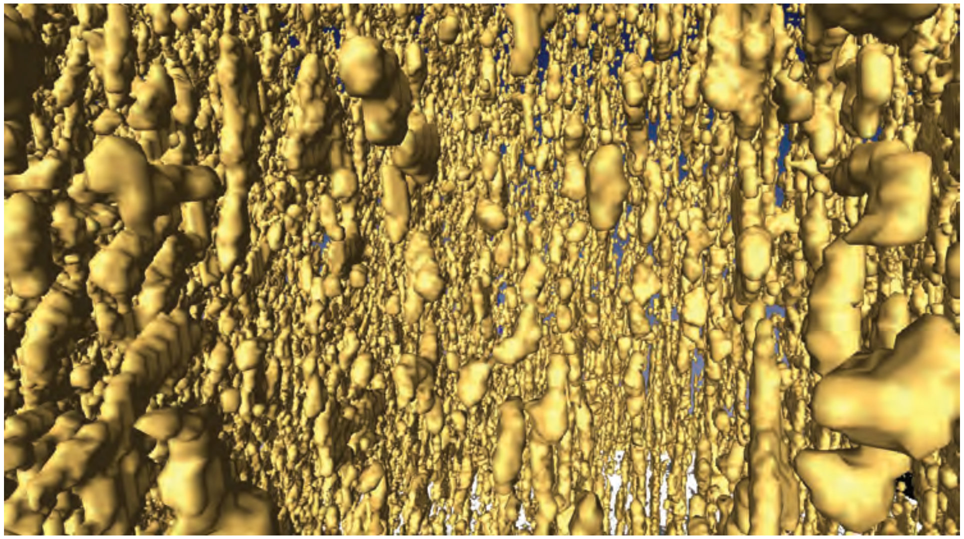
$T = -15\,^\circ\mathrm{C}$, $\phi = 0.033$



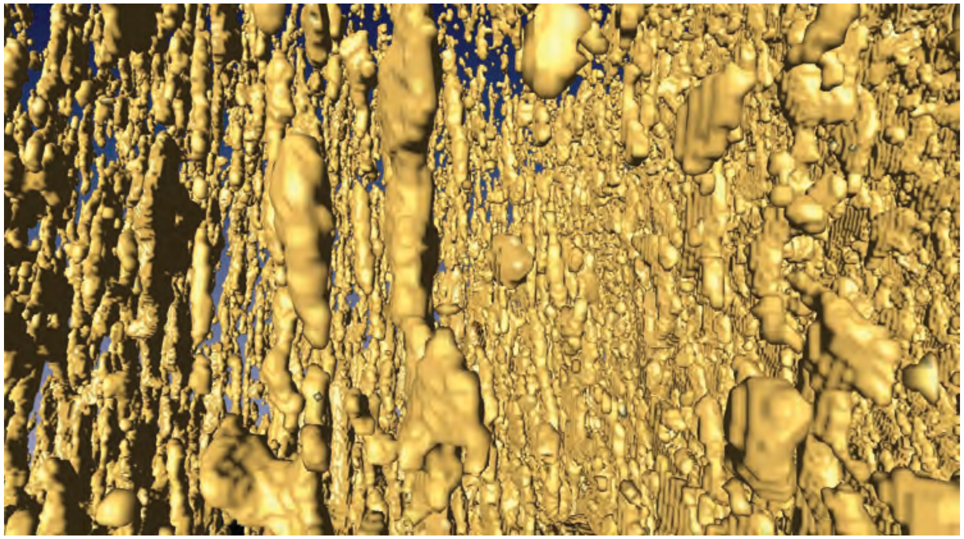
$T = -6\,^\circ\mathrm{C}$, $\phi = 0.075$



$T = -3\,^\circ\mathrm{C}$, $\phi = 0.143$



$T = -8\,^\circ\mathrm{C}$, $\phi = 0.057$



$T = -4\,^\circ\mathrm{C}$, $\phi = 0.113$

X-ray tomography for brine in sea ice

Golden *et al.*, *Geophysical Research Letters*, 2007

Critical behavior of fluid transport in sea ice

Arctic field data

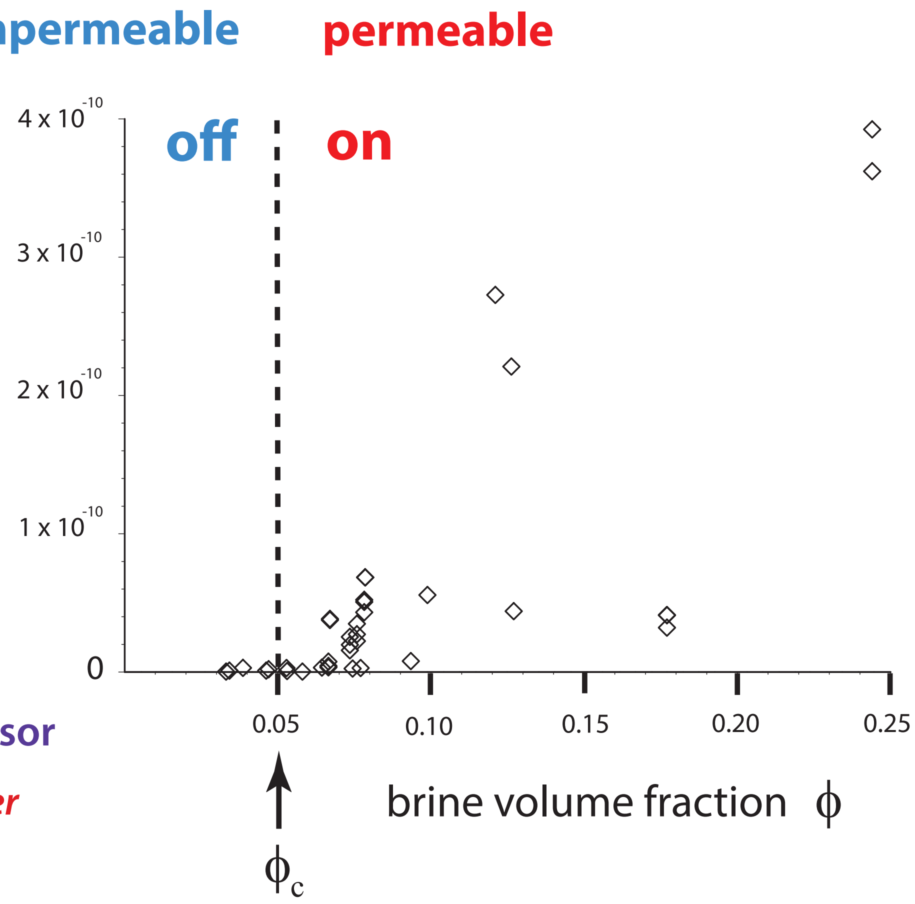
vertical fluid permeability k (m^2)

Darcy's Law

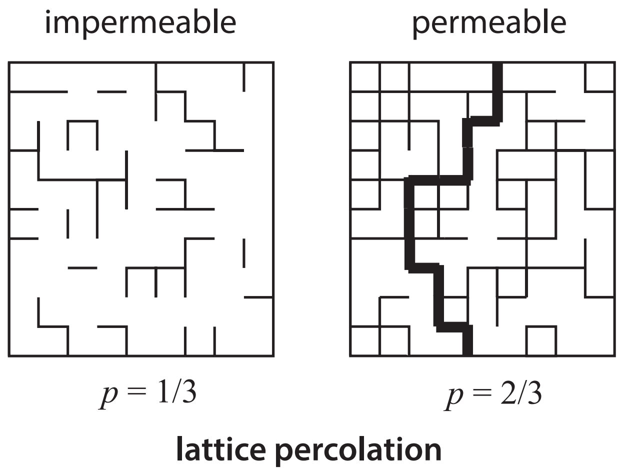
$$v = -\frac{\mathbf{k}}{\eta} \nabla p$$

$\mathbf{k}$ = fluid permeability tensor

homogenized parameter



"on - off" switch
for bulk fluid flow



lattice percolation

PERCOLATION THRESHOLD $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ C,\ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

sea ice algal communities

D. Thomas 2004

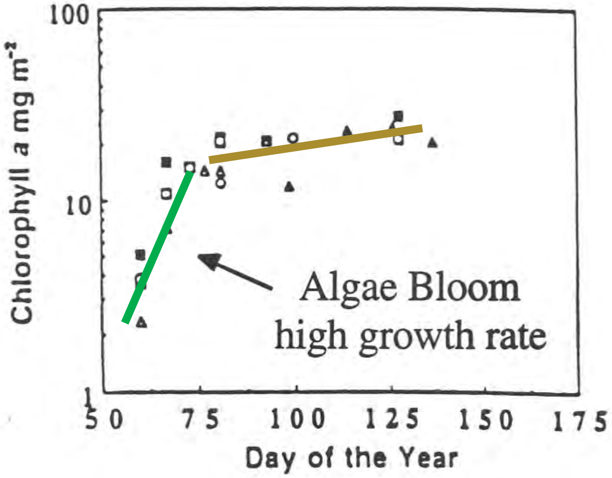
nutrient replenishment
controlled by ice permeability

biological activity turns on
or off according to
rule of fives

Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

critical behavior of microbial activity



Convection-fueled algae bloom
Ice Station Weddell

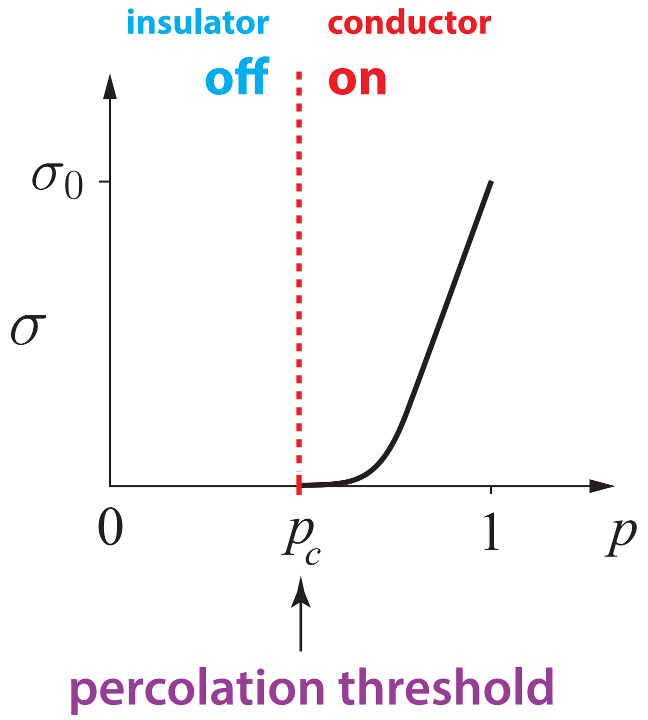
transport in percolation theory

MICRO $\xrightarrow{\text{lattice homogenization}}$ **MACRO**

local conductivity (electrical or fluid)

effective conductivity or fluid permeability

$$\textbf{bond} \longrightarrow \begin{cases} \sigma_0 & \text{probability } p \\ 0 & \text{probability } 1 - p \end{cases}$$



consider local conductivities

1 and *h* > 0

smooths, softens transition

$$\sigma(p) \sim \sigma_0 \, (p - p_c)^t \quad p \to p_c^+$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

1 ≤ ***t*** ≤ **2** (for idealized model), Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992

non-universal behavior in continuum

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007

Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

melt pond evolution

microscale

governs

mesoscale processes

percolation theory for fluid permeability

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

from critical path analysis in hopping conduction

hierarchical model

rock physics

network model

rigorous bounds

X-ray tomography for brine inclusions

confirms rule of fives

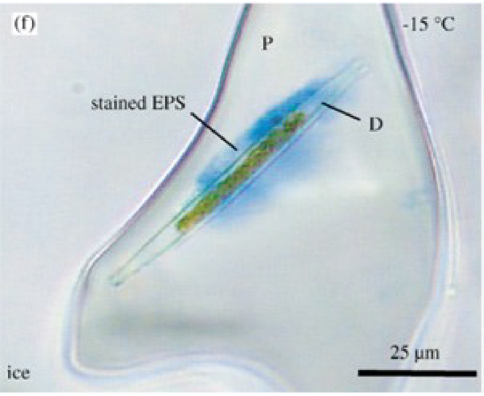
brine percolation threshold of $\phi = 5\%$ for bulk fluid flow

Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

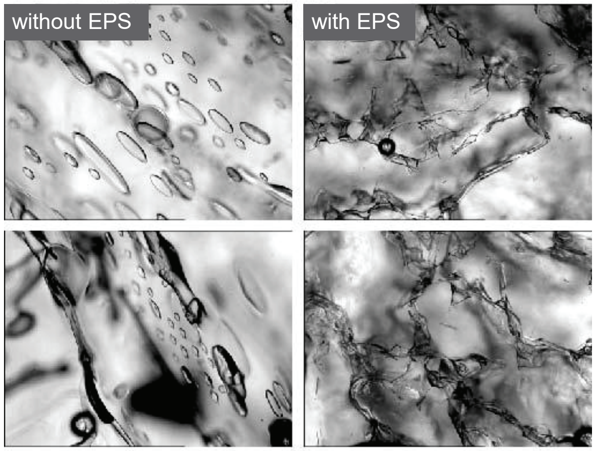
theories agree closely with field data

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

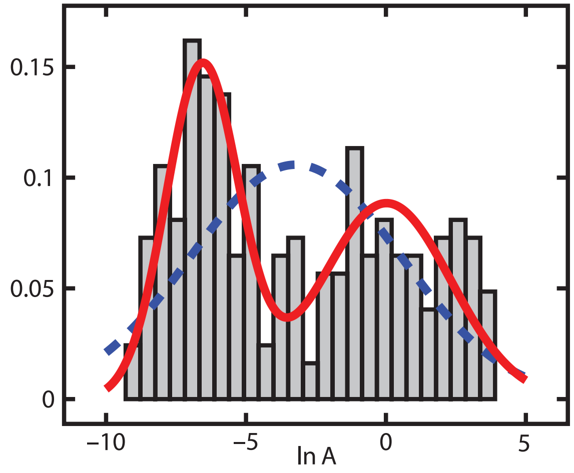
How does EPS affect fluid transport? **How does the biology affect the physics?**



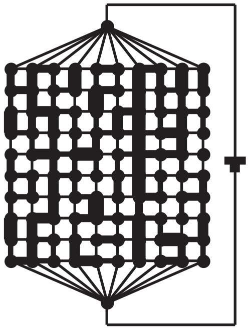
Krembs



Krembs, Eicken, Deming, PNAS 2011

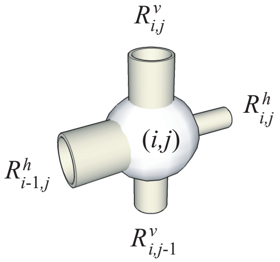


RANDOM PIPE MODEL



- 2D random pipe model with bimodal distribution of pipe radii

- Rigorous bound on permeability *k;* results predict observed drop in *k*

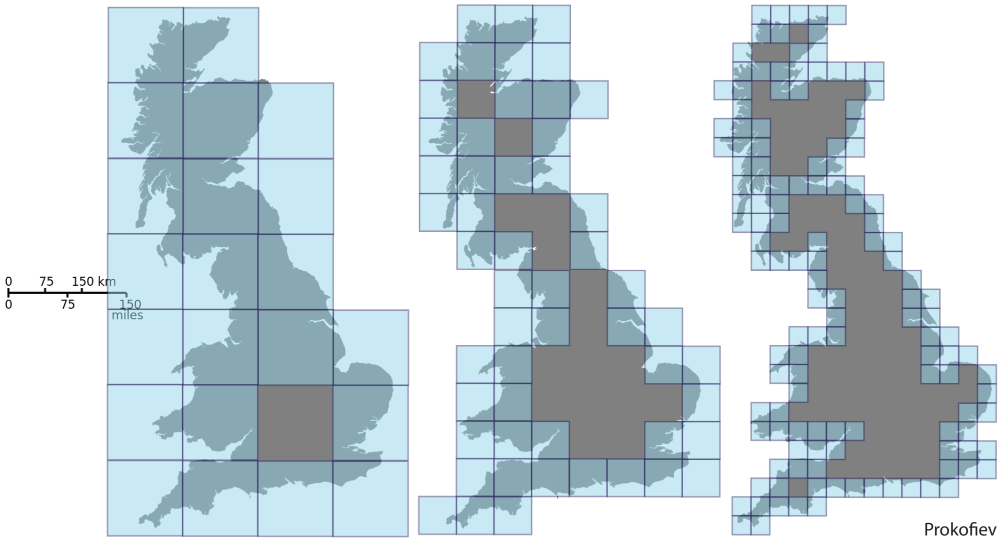


Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

Zhu, Jabini, Golden,
Eicken, Morris
***Ann. Glac.* 2006**

Thermal Evolution of Brine Fractal Geometry in Sea Ice

**Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O'Sadnick, Elena Cherkaev and Kenneth Golden, 2023**

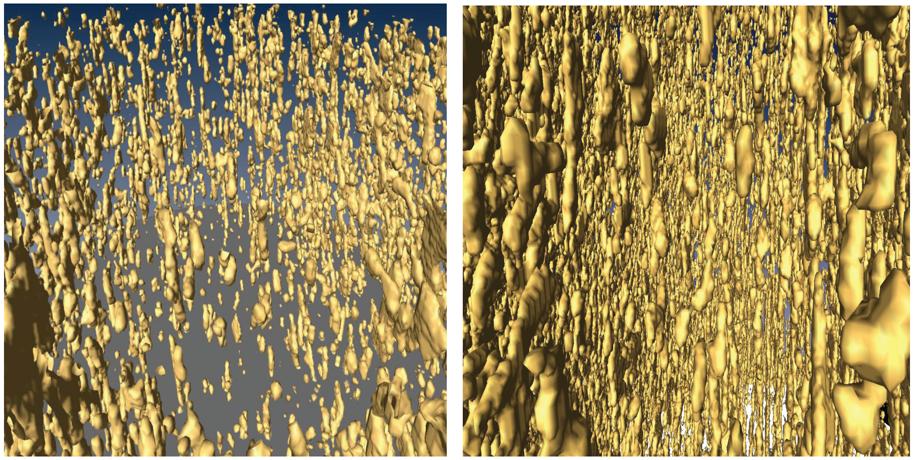


Prokofiev

fractal dimension of the British coastline by box counting

brine channels and inclusions "look" like fractals (from 30 yrs ago)

$T = -12°$ C, $\phi = 0.033$ $T = -8°$ C, $\phi = 0.057$

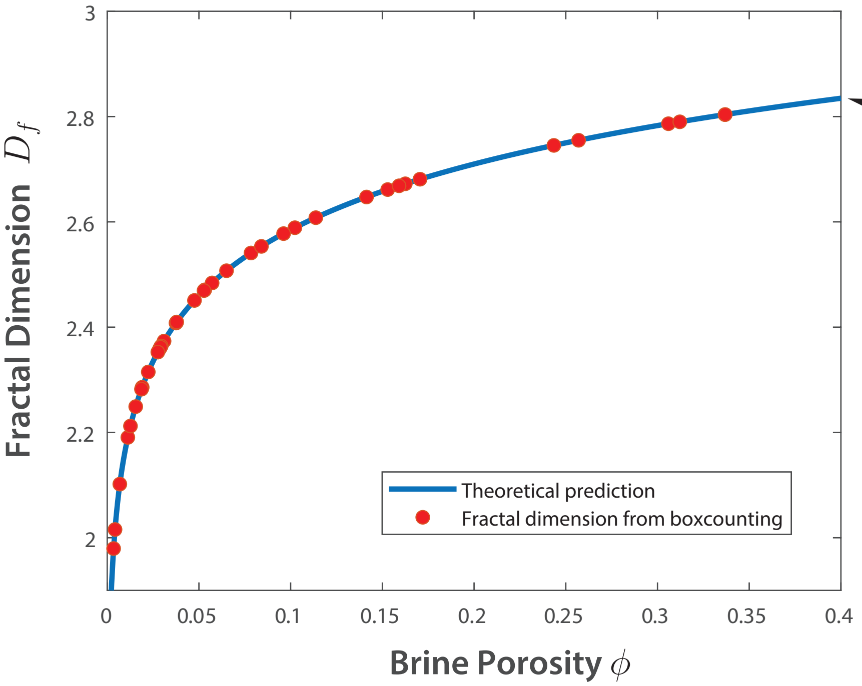


Golden, Eicken, et al. *GRL*, 2007

X-ray computed tomography of brine in sea ice

columnar and granular

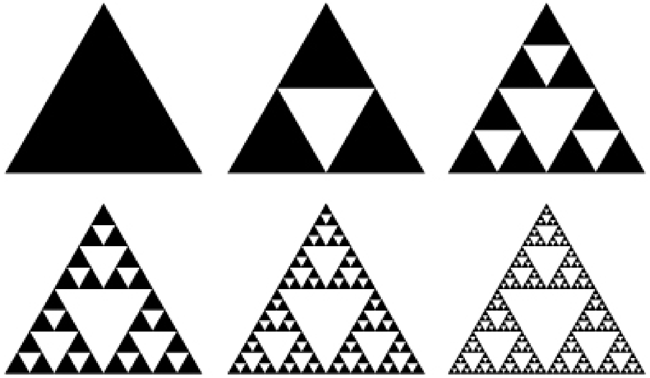
The first comprehensive, quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.

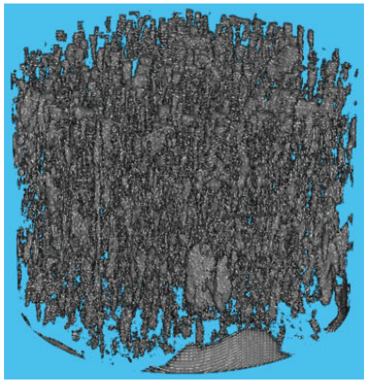


$$D_f = 3 - \frac{\ln\phi}{\ln(\lambda_{min}/\lambda_{max})}$$

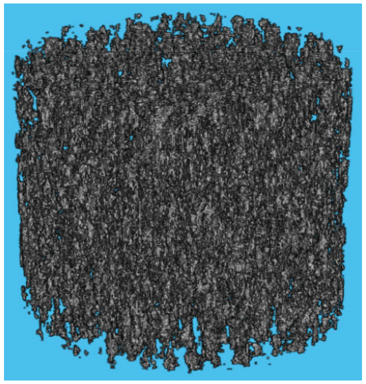
The blue curve is exact for the Sierpinski gasket (an exactly self-similar geometry); discovered for sandstones - statistically self-similar porous media.

Katz and Thompson, 1985
Yu and Li, 2001

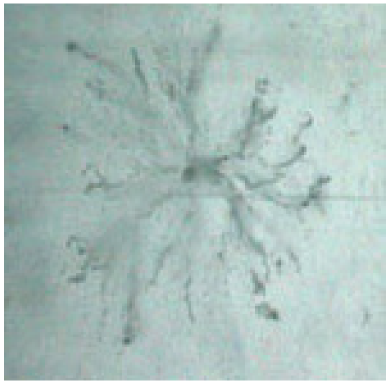




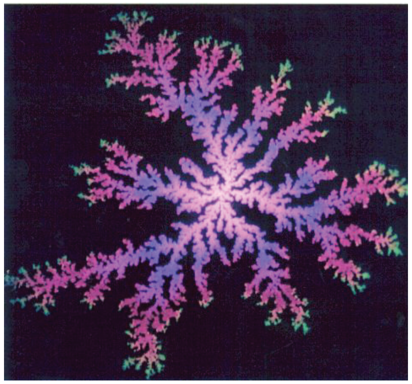
X-ray tomography



DLA model

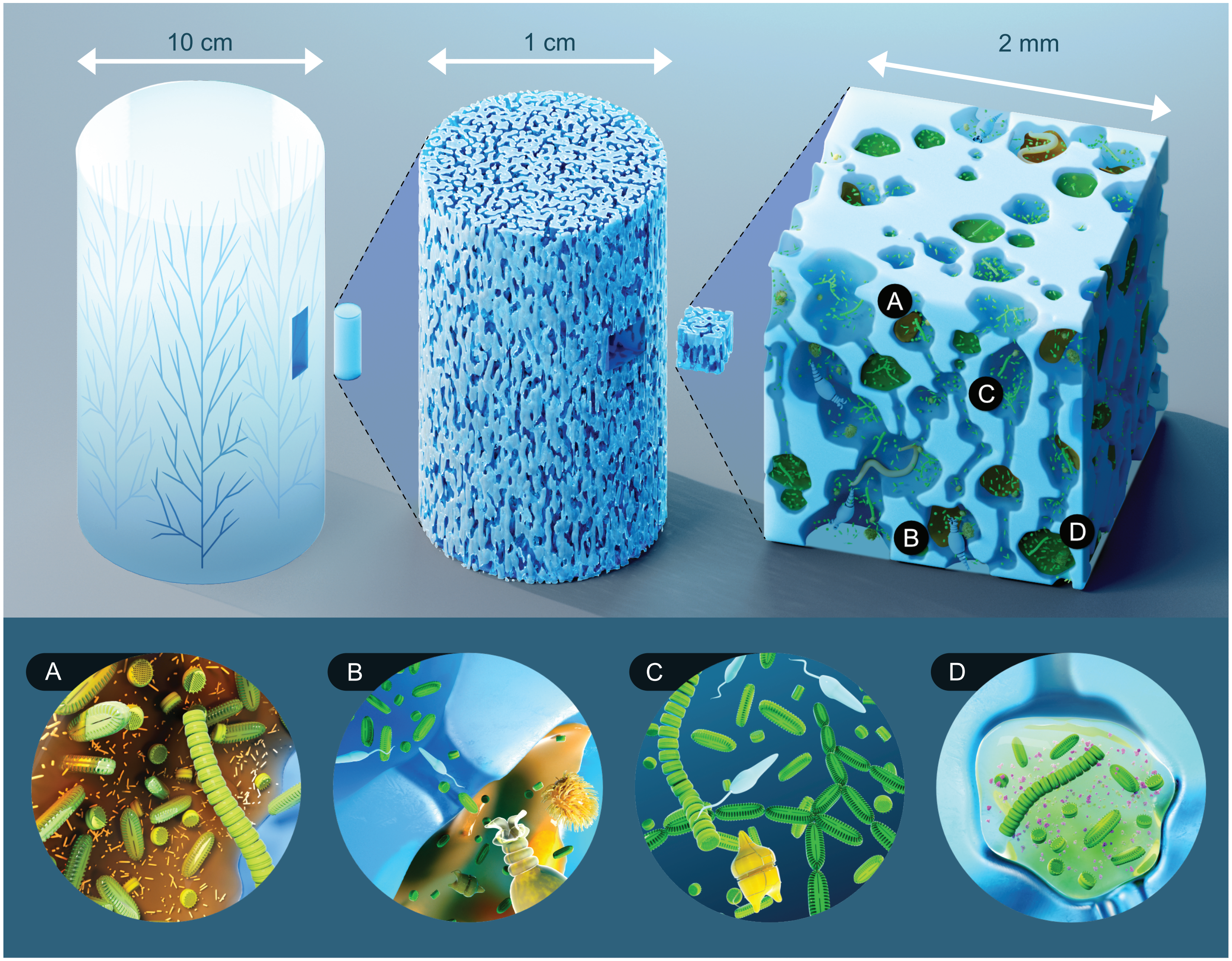


brine channel
in sea ice



diffusion limited
aggregation

Implications of brine fractal geometry on sea ice ecology and biogeochemistry



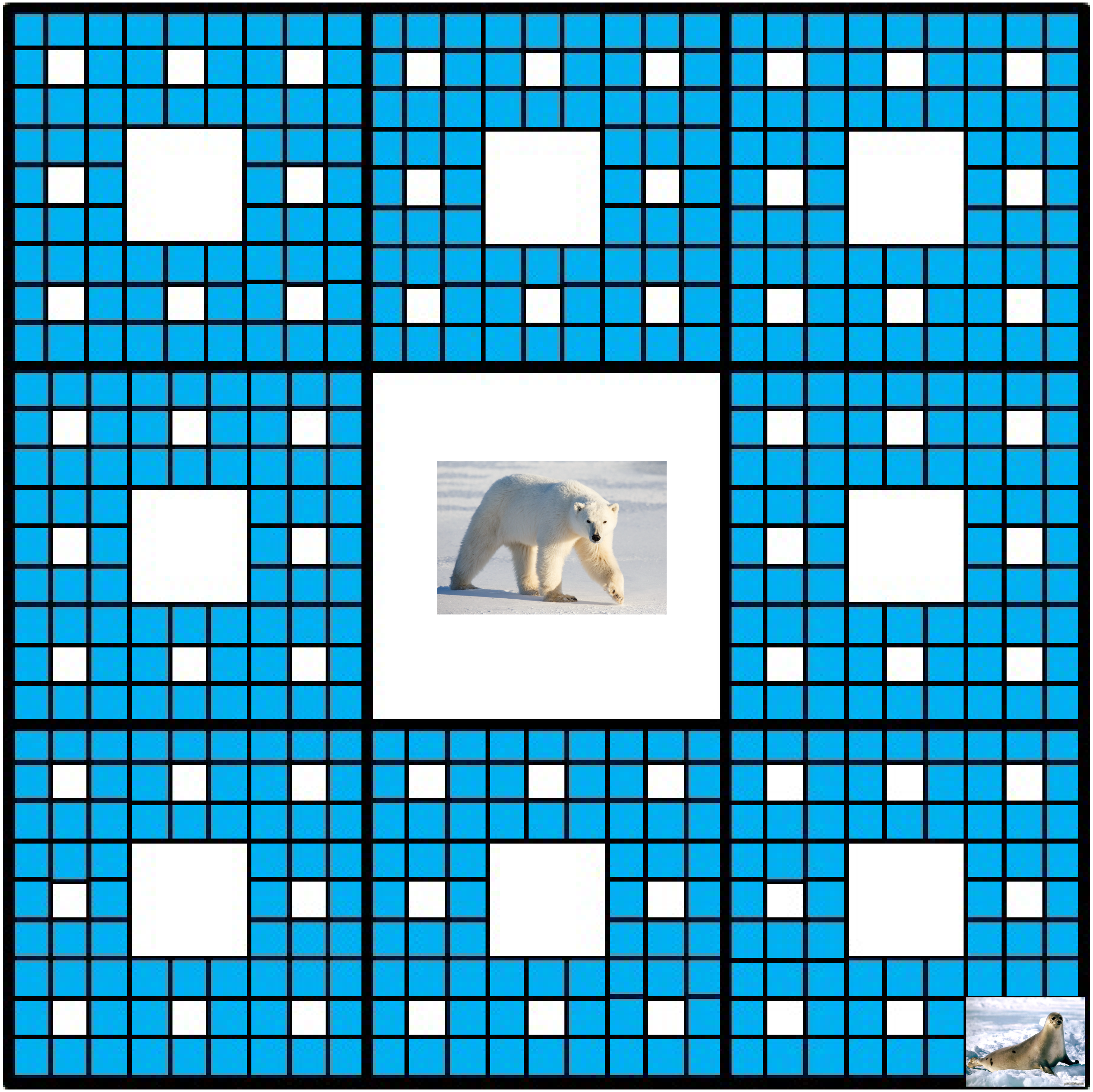
Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

(A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.

(B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).

(D) Ice algae secrete extracellular polymeric substances (EPS) which alter incusion geometry and may further increase the fractal dimension.



polar bear foraging in a fractal icescape

Nicole Forrester
Jody Reimer
Ken Golden

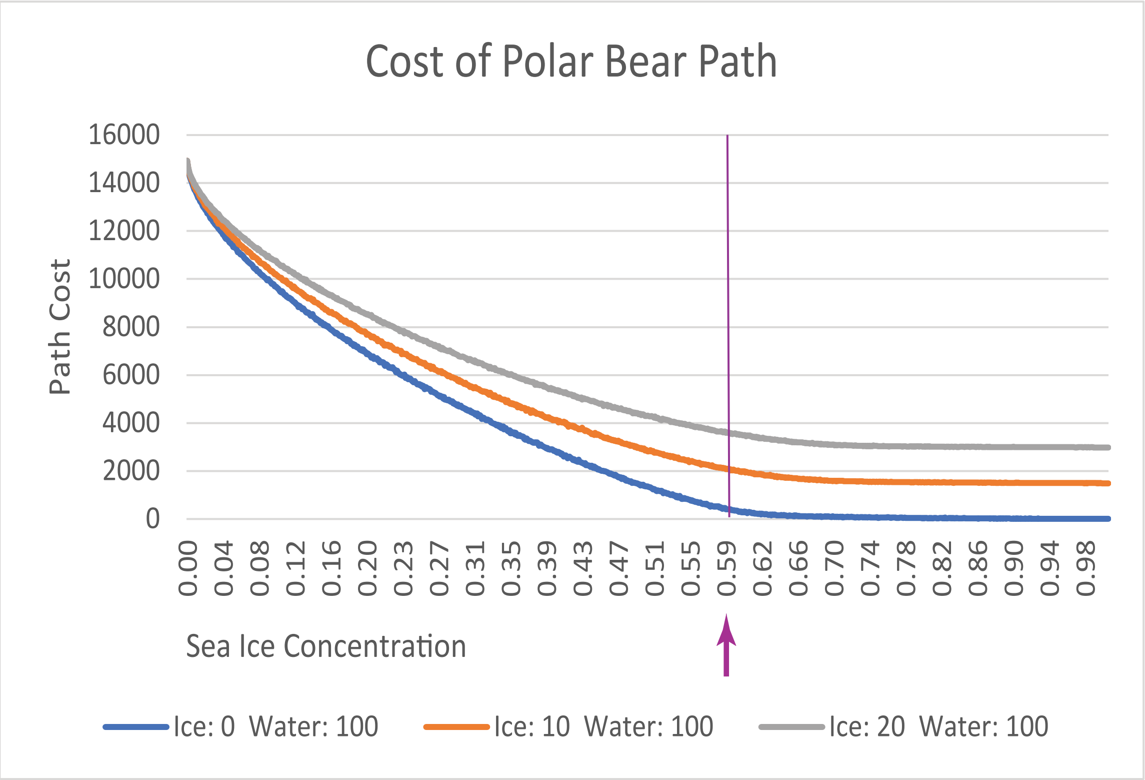
It costs the polar bear 5 times the energy to swim through water than to walk on sea ice.

What pathway to a seal minimizes energy spent?

Polar Bear Percolation

Optimal Movement of a Polar Bear in a Heterogenous Icescape

$C(p)$

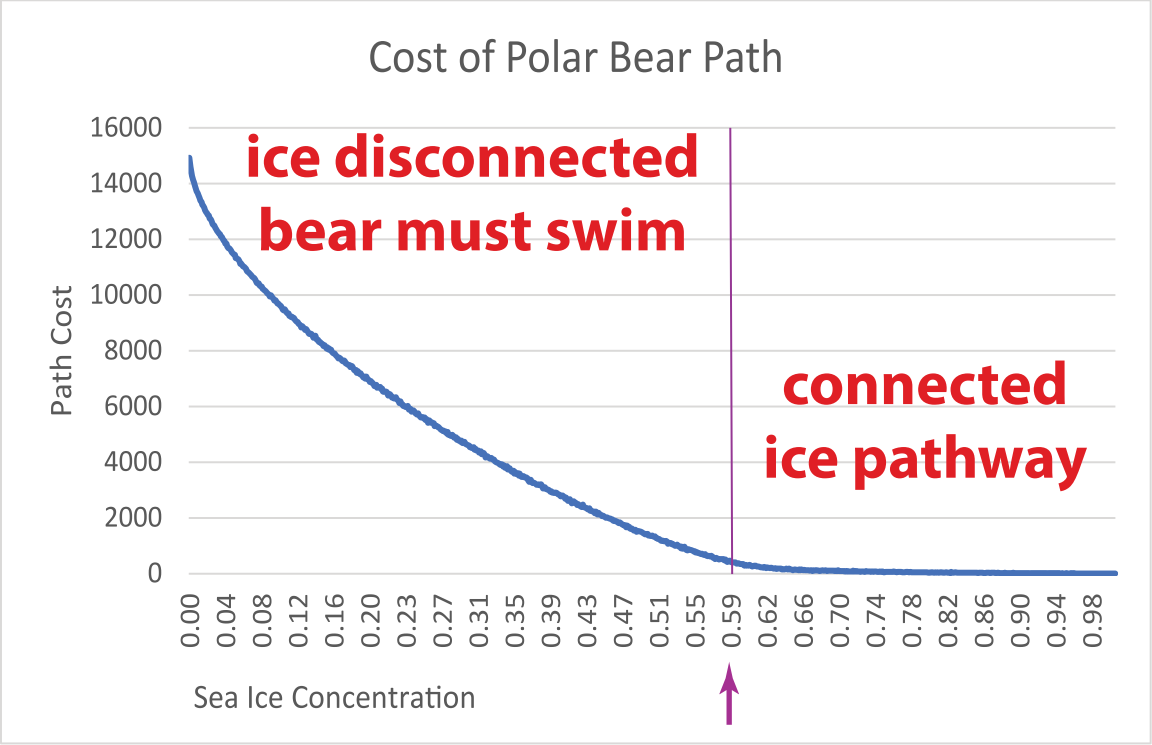


← $h = 0.2$

← $h = 0.1$

← $h = 0$

$C(p)$



ice disconnected bear must swim

connected ice pathway

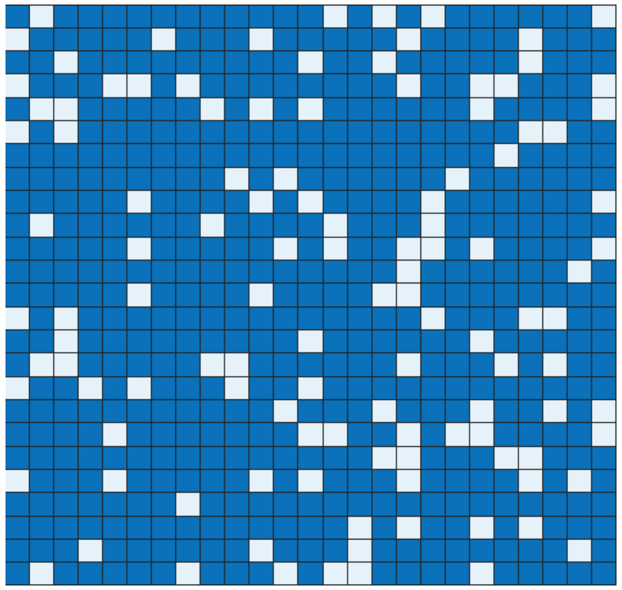
← $h = 0$

$$h = \frac{c_i}{c_w}$$

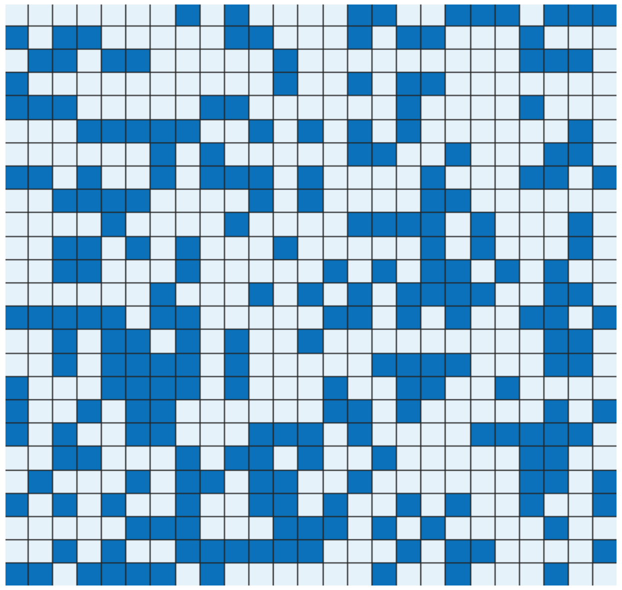
ratio of local "conductivities"

site percolation threshold

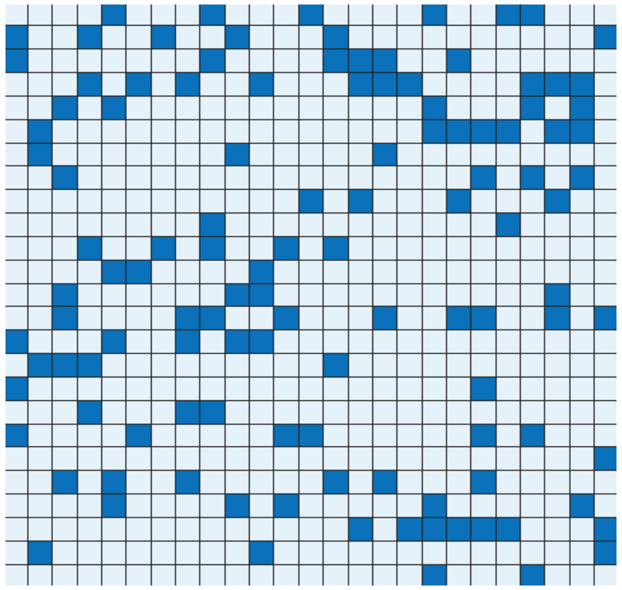
$p_c = 0.59 \;\; \text{for} \;\; d = 2$



20% Ice



60% Ice



80% Ice

Golden

Remote sensing of sea ice

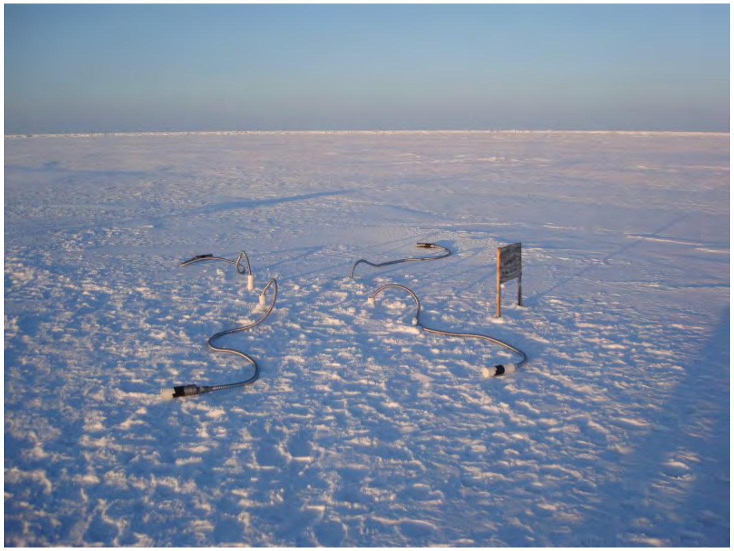
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice
properties from
electromagnetic
(EM) data

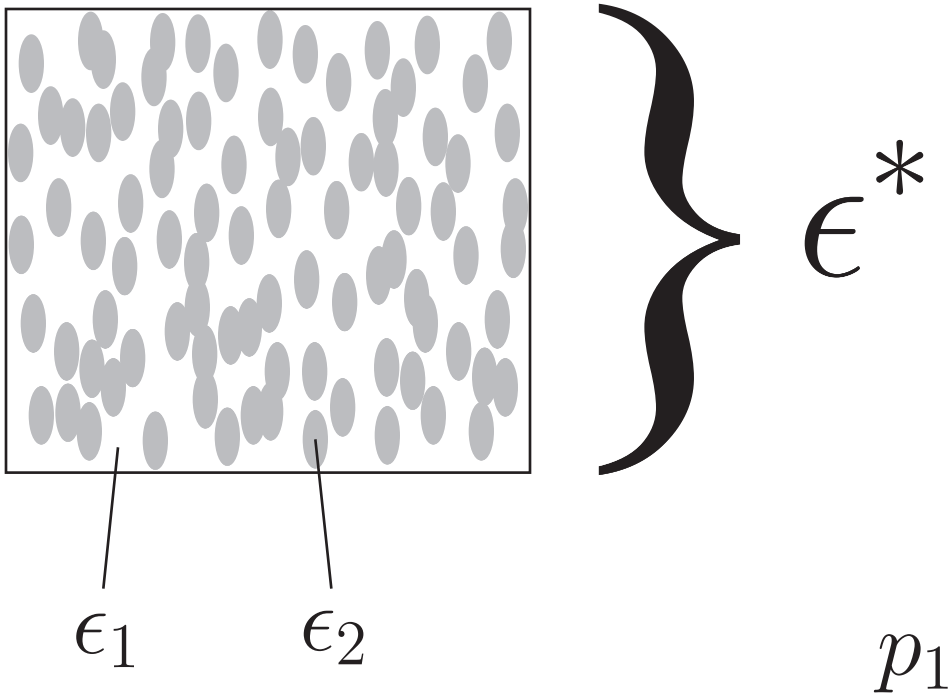
ε^*

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry} \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

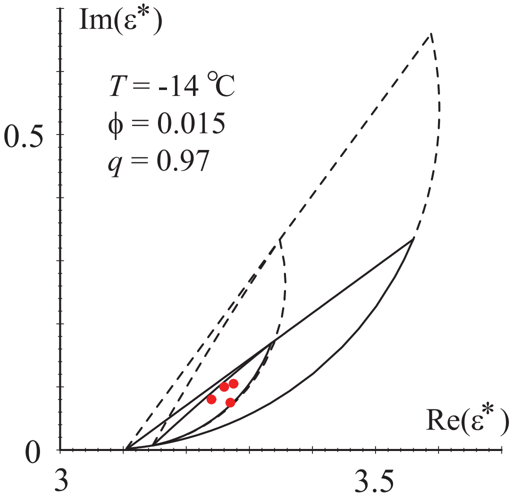
$\Gamma\chi$: microscale → macroscale

$\Gamma\chi$ *links scales*

Golden and Papanicolaou, Comm. Math. Phys. 1983

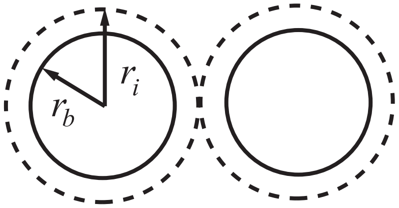
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle

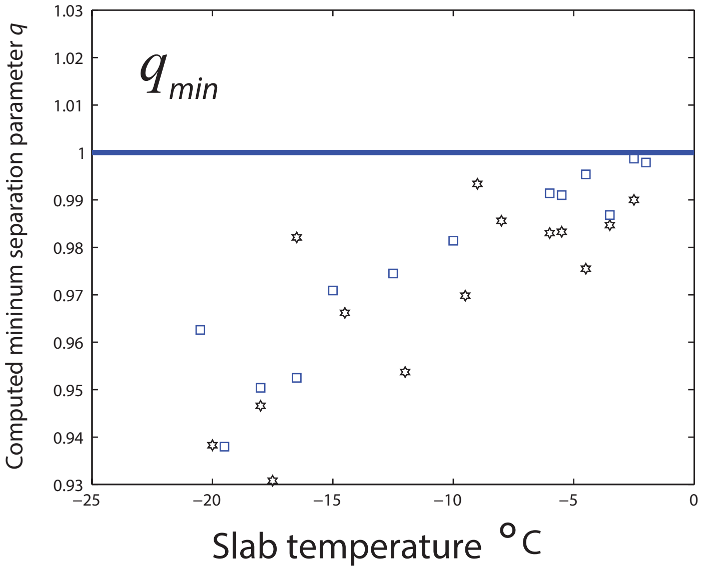


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



Inverse Homogenization

Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001), McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* $\longrightarrow$ ***composite geometry***
(spectral measure μ)

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

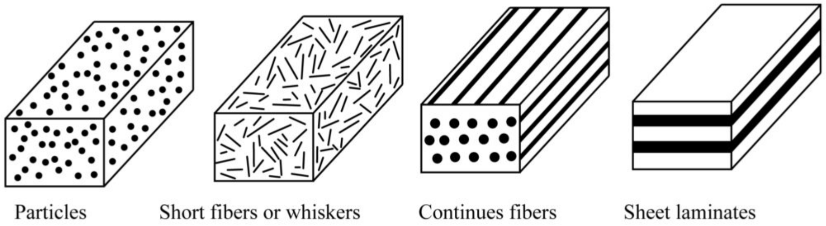
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

rigorous inverse bound
on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

complexities of mixture geometry



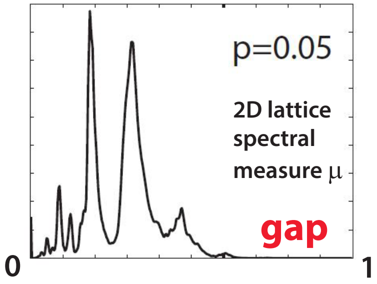
Bartleby.com

distilled **distilled**

Analytic Continuation Method

Stieltjes Integral Representations
for Homogenized Parameters

Bergman 1978, Milton 1979
Golden & Papanicolaou 1983



spectral properties of operator (matrix)
~ quantum states, energy levels for atoms

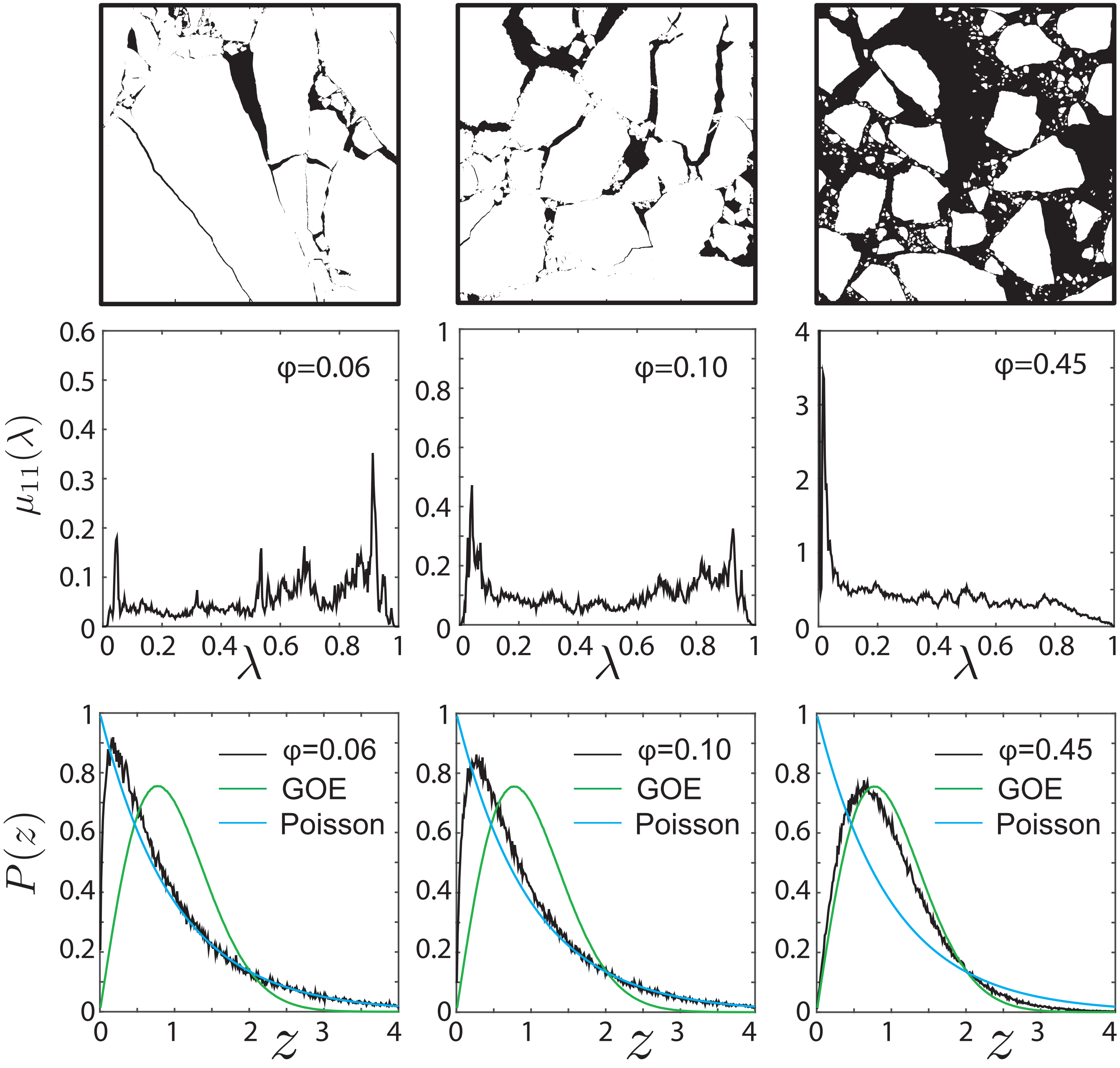
eigenvectors eigenvalues

EXTEND to: polycrystals, advection diffusion, waves through ice pack

Spectral computations for sea ice floe configurations

spectral measures

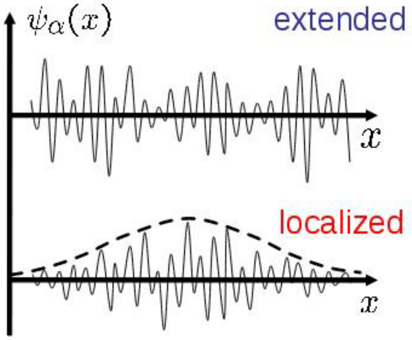
eigenvalue spacing distributions



uncorrelated → **level repulsion**

UNIVERSAL Wigner-Dyson distribution

Murphy, Cherkaev, Golden, *Phys. Rev. Lett.* 2017



Anderson localization

disorder-driven

metal / insulator transition

Anderson 1958
Mott 1949
Evangelou 1992
Shklovshii et al 1993

Wave equations

propagation vs. localization in wave physics:
quantum, optics, acoustics, water waves

Laplace + Diffusion
equations

we find percolation-driven

Anderson transition for classical transport in composites

mobility edges, localization, universal spectral statistics

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

but no wave interference or scattering effects at play!

Order to disorder in quasiperiodic composites

Morison, Murphy, Cherkaev, Golden, Comm. Phys. 2022

sea ice inspired - high tech spin off

tunable quasiperiodic composites with exotic properties

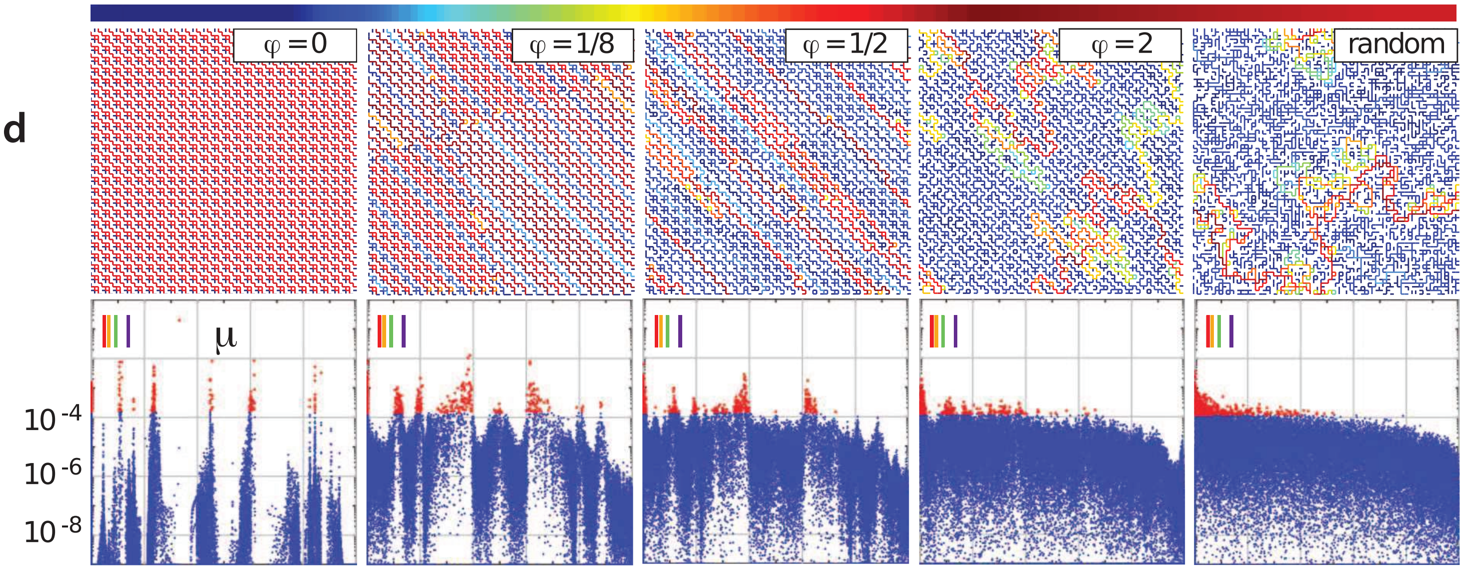
(optical, electrical, thermal, ...), Anderson localization; our Moiré patterned geometries are similar to twisted bilayer graphene

increasing twist angle between two lattices

periodic → quasiperiodic

electric field strength

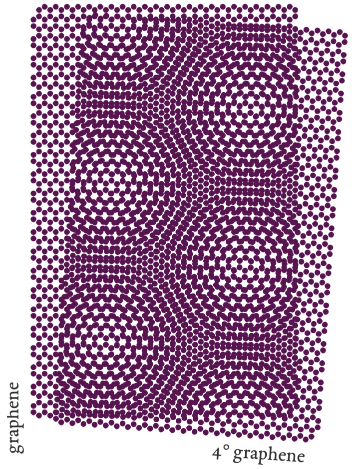
spectral measure

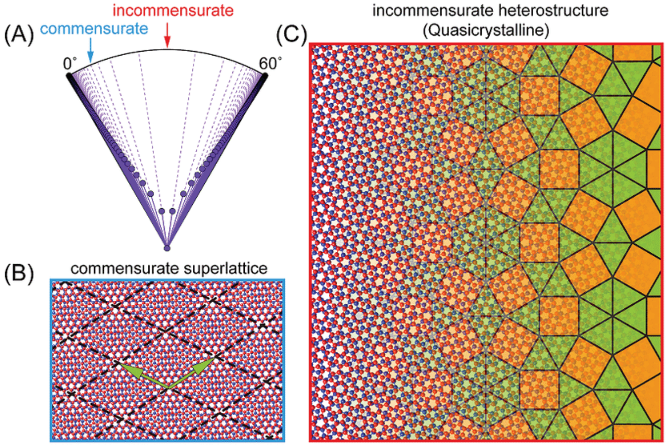


RRN at percolation threshold

twisted bilayer graphene

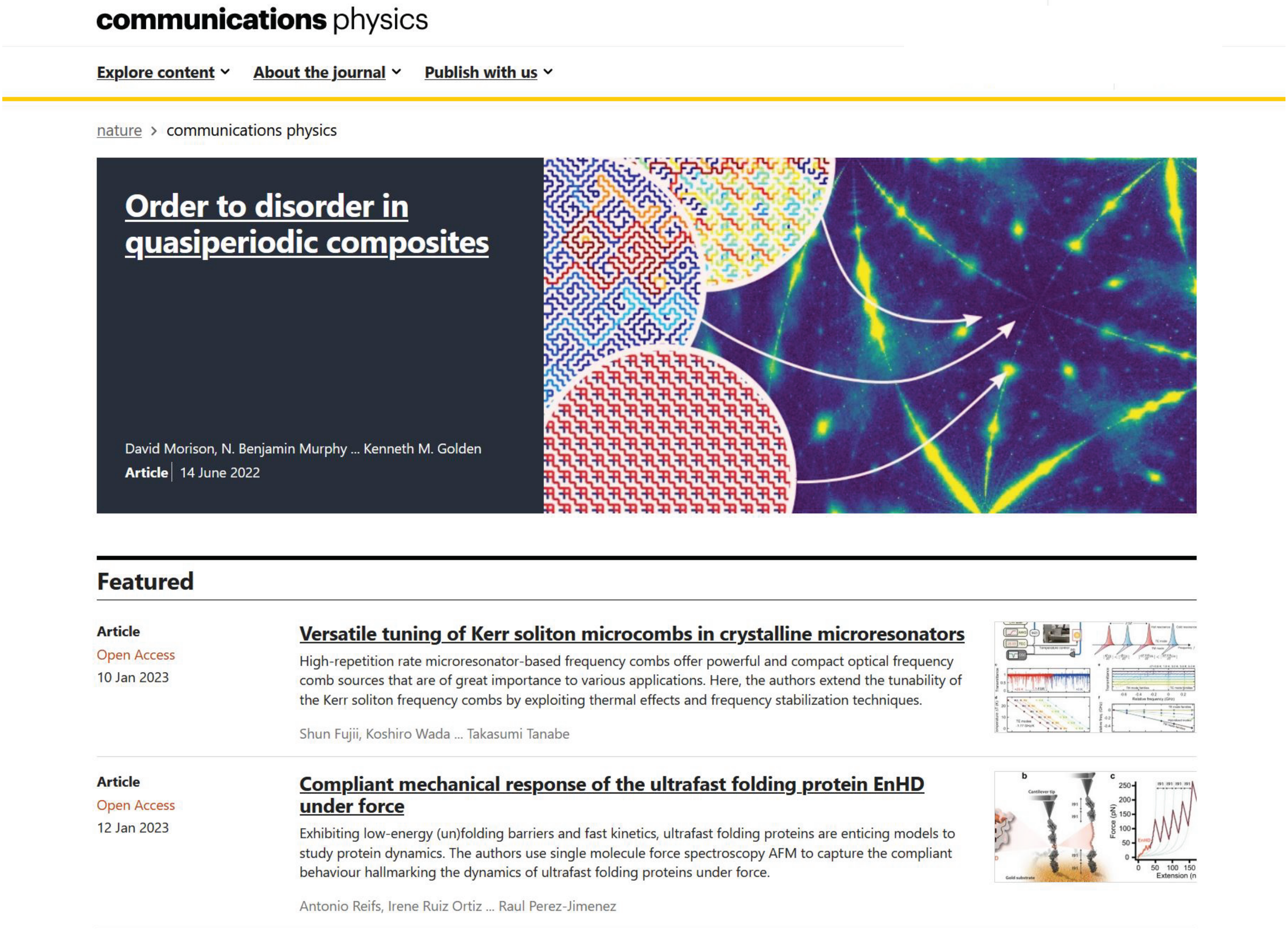
superconducting magic twist angle





Yao et al., 2018

constellation of periodic systems in a sea of randomness



communications physics

Order to disorder in quasiperiodic composites

David Morison, N. Benjamin Murphy ... Kenneth M. Golden

Article | 14 June 2022

Featured

Article
Open Access
10 Jan 2023

Versatile tuning of Kerr soliton microcombs in crystalline microresonators

High-repetition rate microresonator-based frequency combs offer powerful and compact optical frequency comb sources that are of great importance to various applications. Here, the authors extend the tunability of the Kerr soliton frequency combs by exploiting thermal effects and frequency stabilization techniques.

Shun Fujii, Koshiro Wada ... Takasumi Tanabe

Article
Open Access
12 Jan 2023

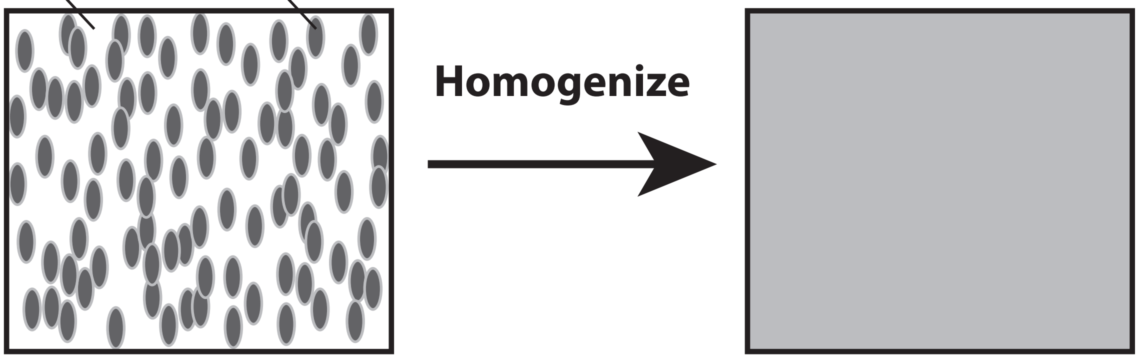
Compliant mechanical response of the ultrafast folding protein EnHD under force

Exhibiting low-energy (un)folding barriers and fast kinetics, ultrafast folding proteins are enticing models to study protein dynamics. The authors use single molecule force spectroscopy AFM to capture the compliant behaviour hallmarking the dynamics of ultrafast folding proteins under force.

Antonio Reifs, Irene Ruiz Ortiz ... Raul Perez-Jimenez

The math doesn't care if it's two phase composites or polycrystals!

Two-component composites



Local conductivity

σ_1 σ_2

inhomogeneous medium

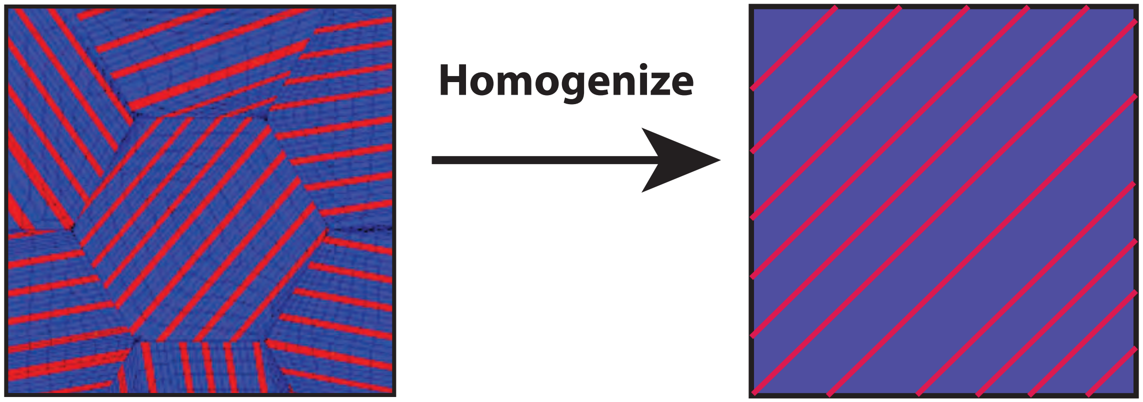
Homogenize

σ^* effective conductivity

homogeneous medium

Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Polycrystalline media



Conductivity of crystal directions

σ_1 σ_2

inhomogeneous medium

Homogenize

σ^* effective conductivity

homogeneous medium

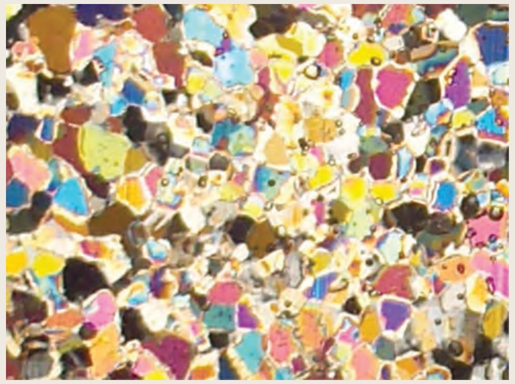
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**

  orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

THE
ROYAL
SOCIETY
PUBLISHING

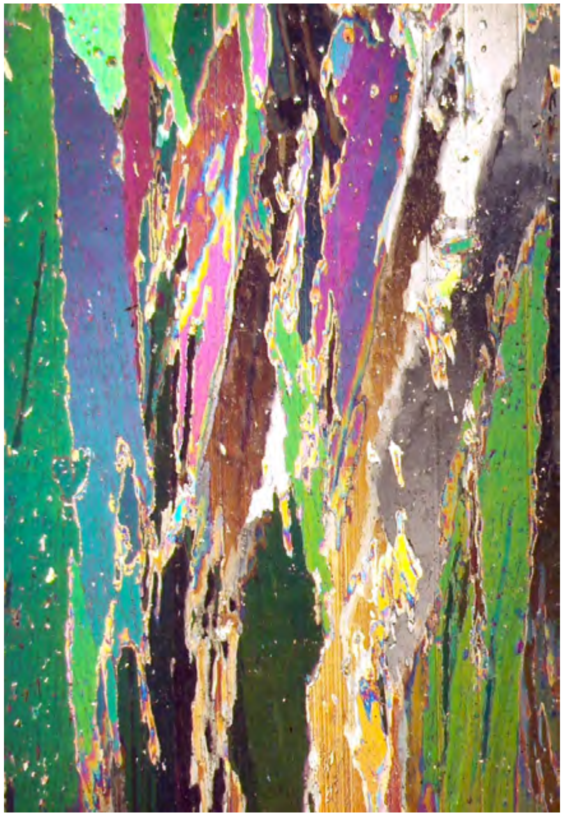
21/01/15 1:47 PM

higher threshold for fluid flow in granular sea ice

microscale details impact "mesoscale" processes

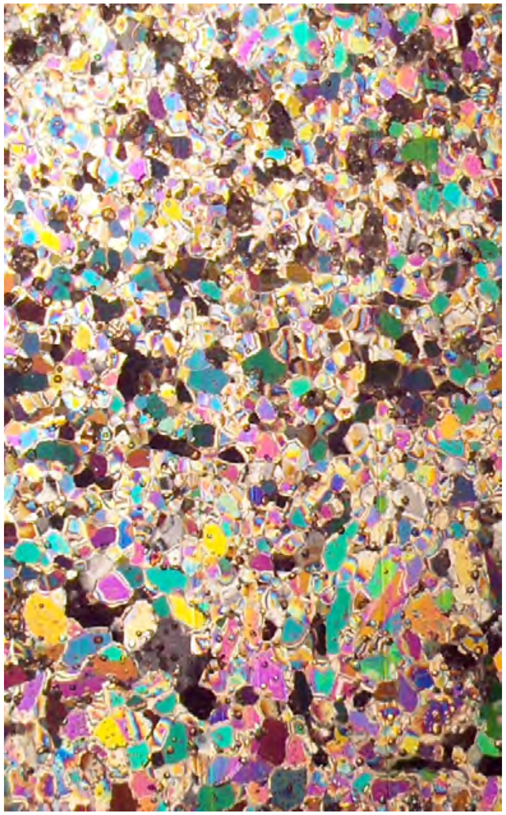
nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar



5%

granular



10%

Golden, Sampson, Gully, Lubbers, Tison 2023

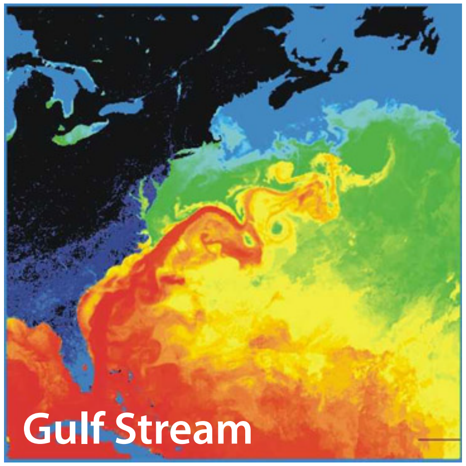
electromagnetically distinguishing ice types
Kitsel Lusted, Elena Cherkaev, Ken Golden

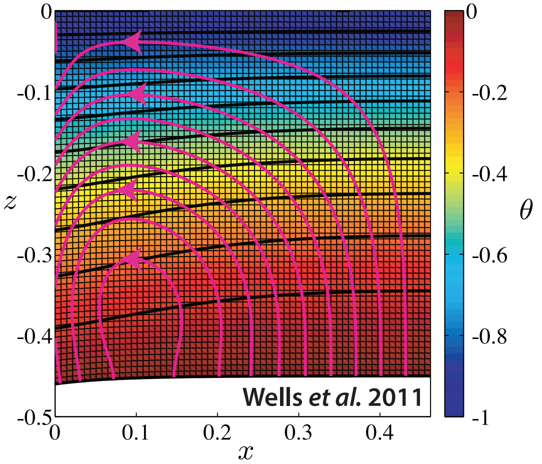
mesoscale

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\Big\downarrow \quad \textit{homogenize}$$

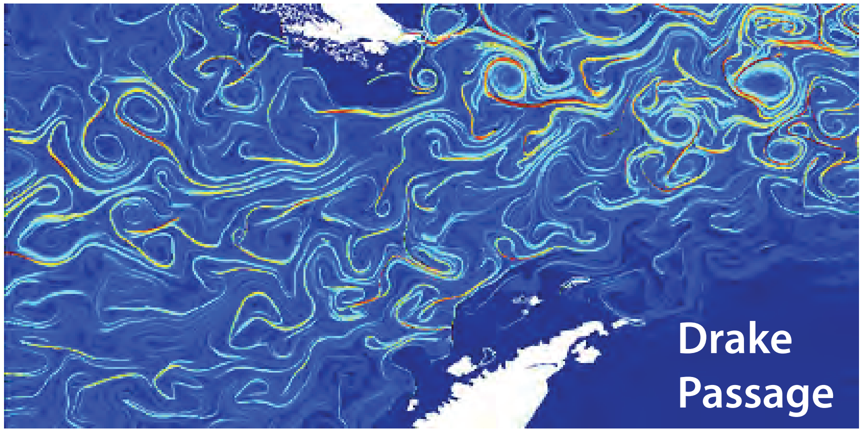
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

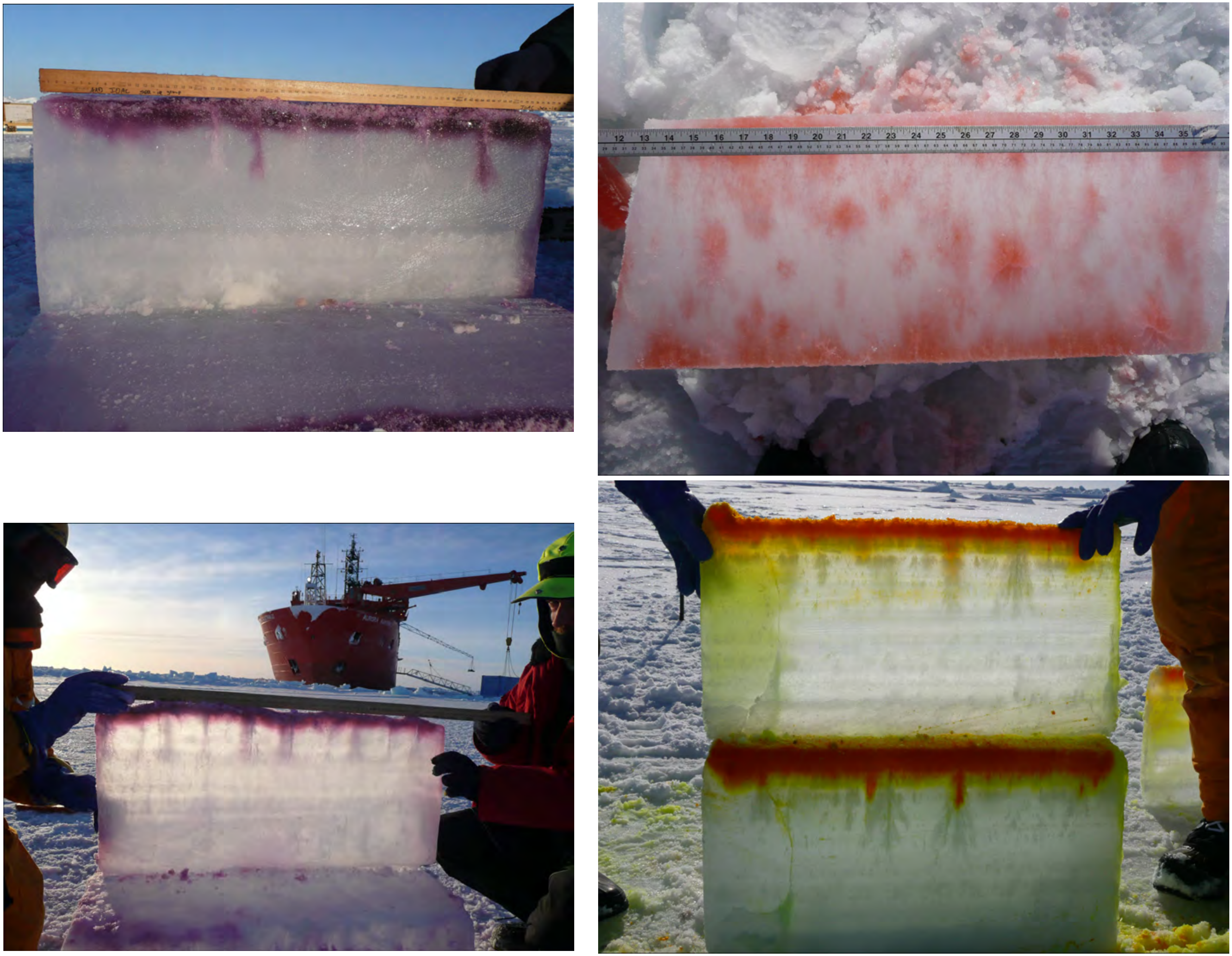
Stieltjes integral for κ^* ***with spectral measure***

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020



tracers flowing through inverted sea ice blocks



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

$$\kappa^* = \kappa \left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2} \right), \quad F(\kappa) \; = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H \Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H \Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

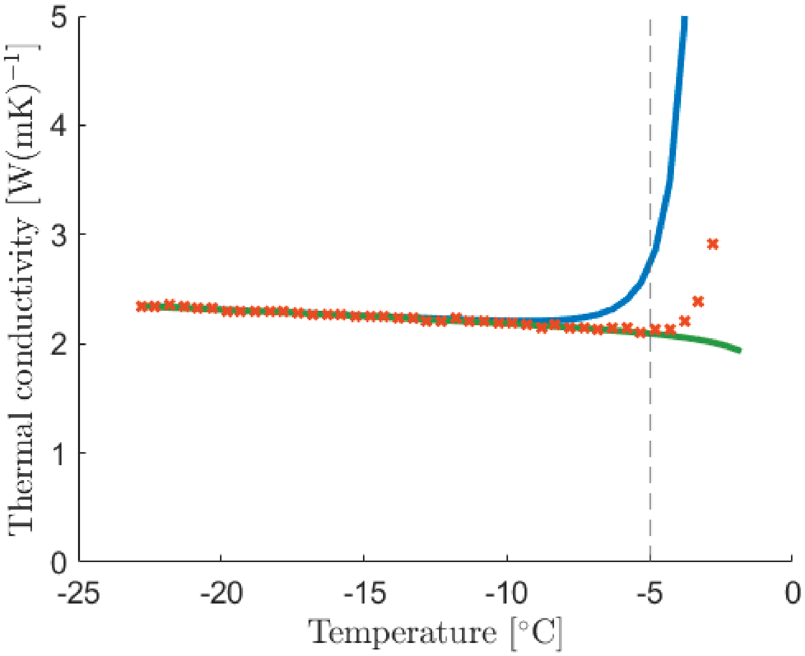
rigorous framework for numerical computations of
spectral measures and effective diffusivity for model flows

new integral representations, theory of moment calculations

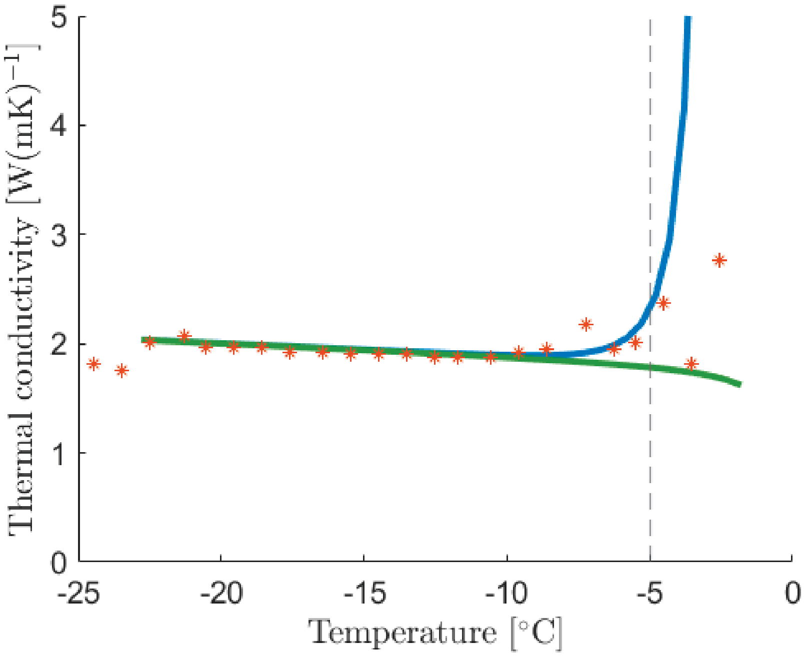
separation of material properties and flow field

Bounds on Convection Enhanced Thermal Transport

simulations



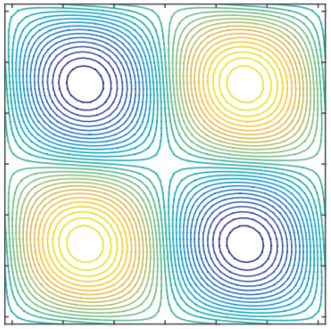
data [Trodahl et al., 2001]



Monte-Carlo simulations of SDE with temperature dependent Péclet number P

strength of advection B = κP/2π

Euler-Maruyama and subsampling methods for SDE



cat's eye flow model for brine convective flow

Rigorous Padé approximant bounds in terms of P using Stieltjes integral + analytic continuation method for the measure

Darcy velocity v = 0.5 [m/s]

Kraitzman, Hardenbrook, Dinh, Murphy, Cherkaev, Zhu, & Golden, 2023

wave propagation in the marginal ice zone (MIZ)

Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Cherkaev, Golden 2023

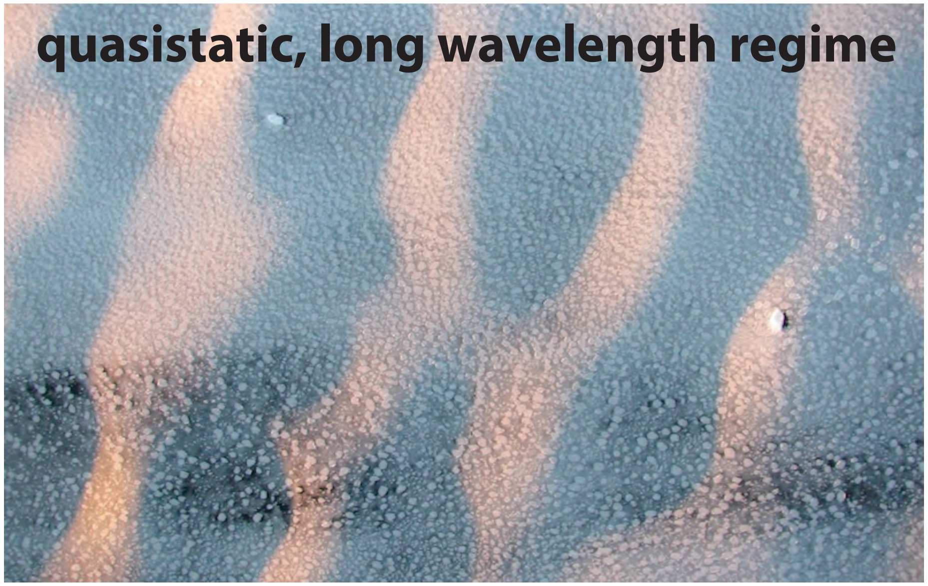
first theory of key parameter
in wave-ice interactions only
fitted to wave data before

Keller, 1998
Mosig, Montiel, Squire, 2015
Wang, Shen, 2012

Analytic Continuation Method

Bergman (78) - Milton (79)
integral representation for ε^*
Golden and Papanicolaou (83)

Milton, *Theory of Composites* (02)



quasistatic, long wavelength regime

homogenized
parameter
depends on
sea ice
concentration
and ice floe
geometry

like EM waves

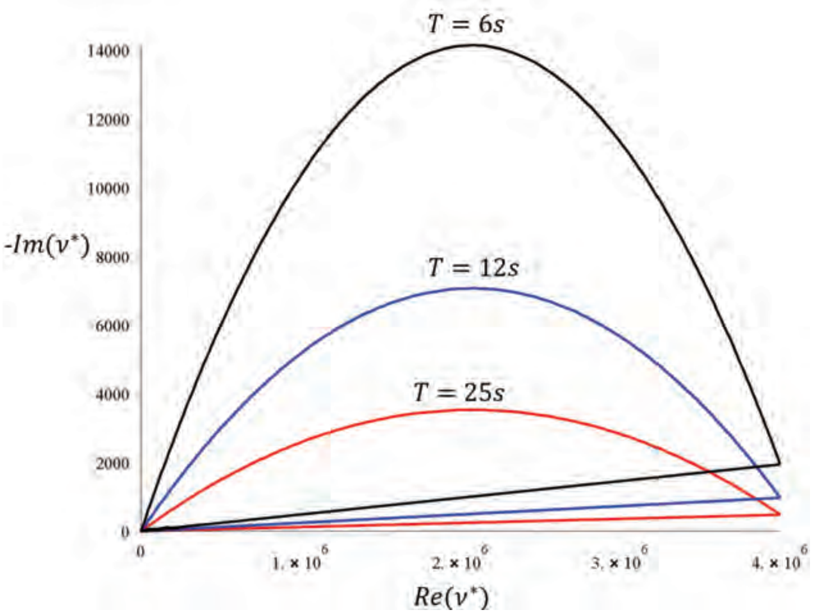
bounds on the effective complex viscoelasticity

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**

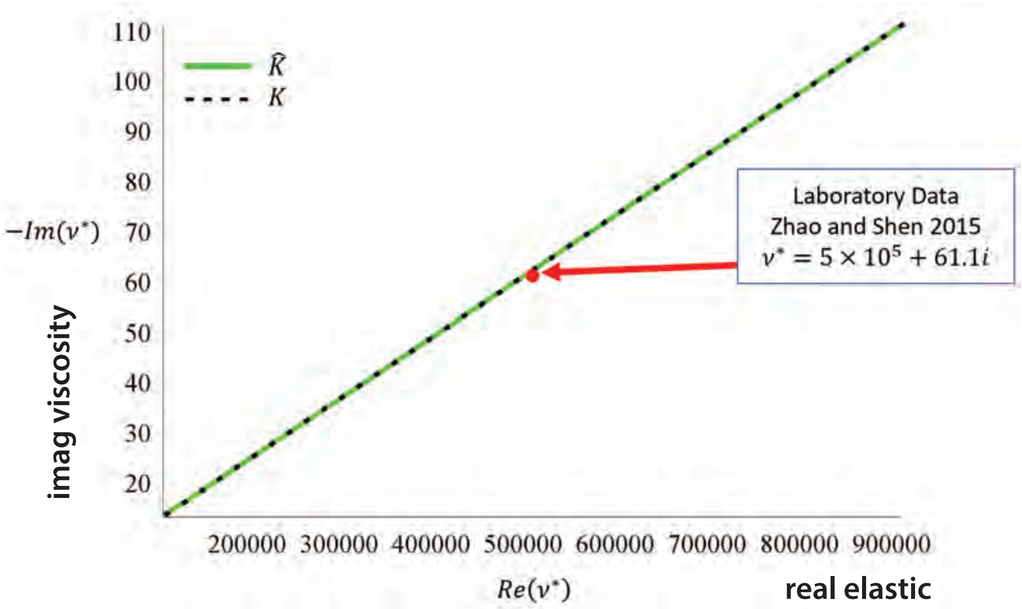
high contrast

complex elementary bounds (fixed area fraction of floes)



Elementary bounds for wave periods T.

matrix-particle bounds



Laboratory Data
Zhao and Shen 2015
$\nu^* = 5 \times 10^5 + 61.1i$

Golden

melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

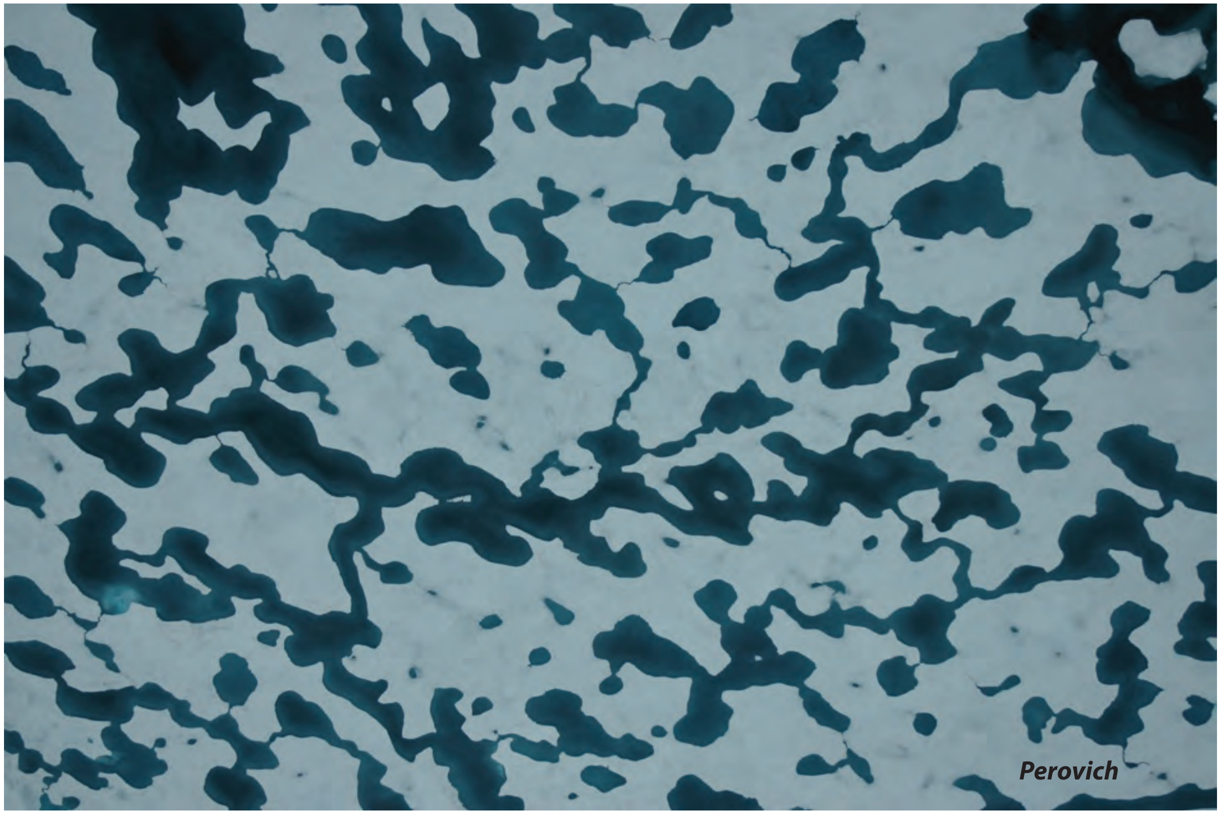
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012


Perovich

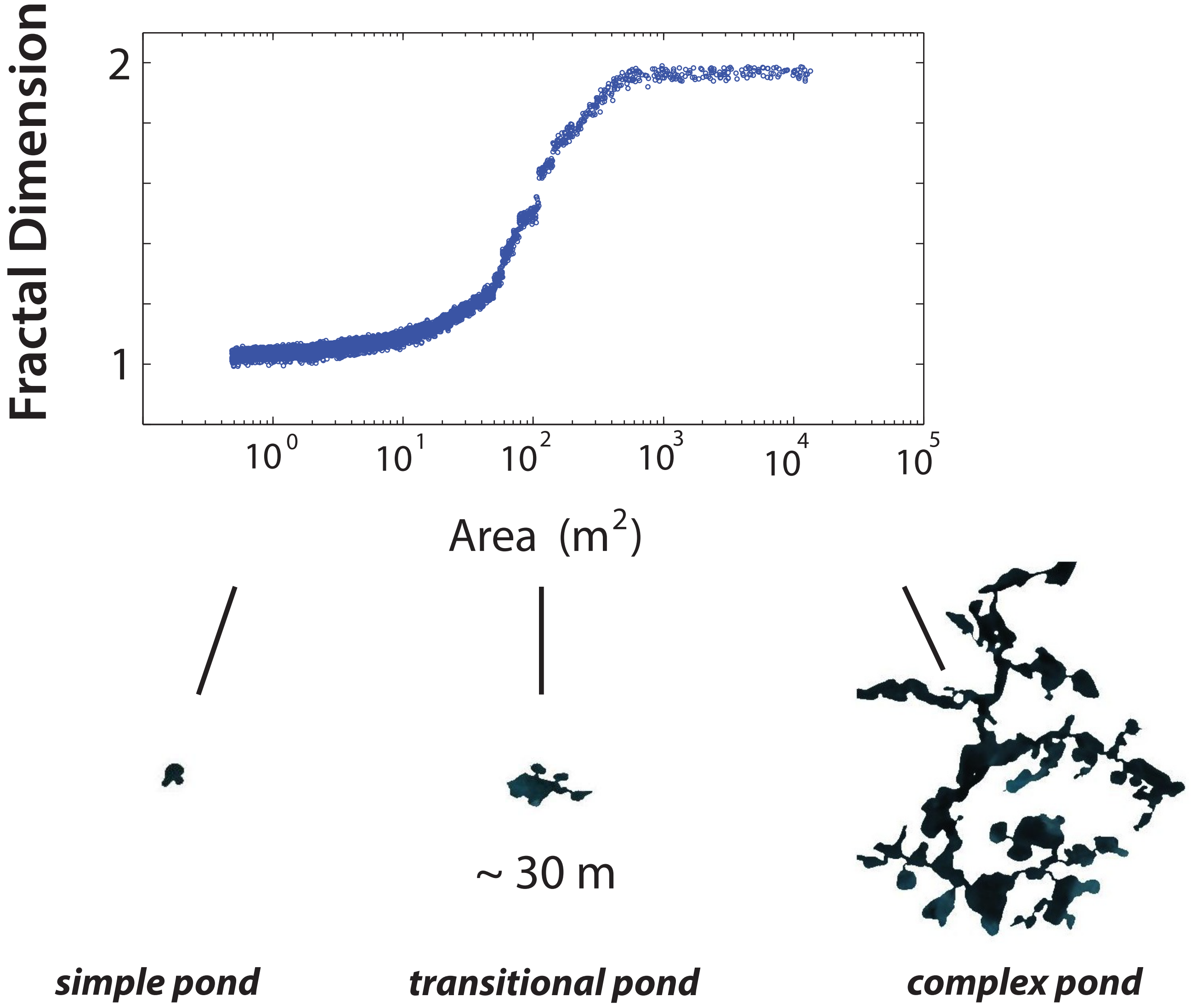
Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

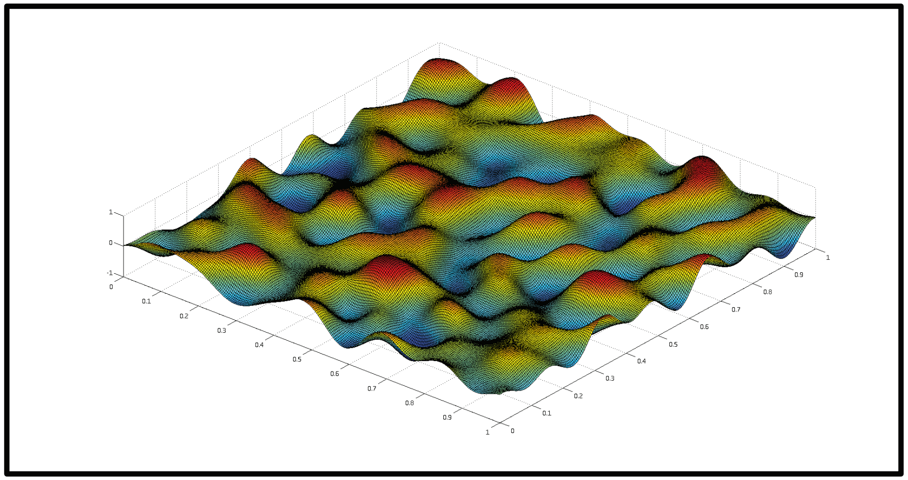
complexity grows with length scale



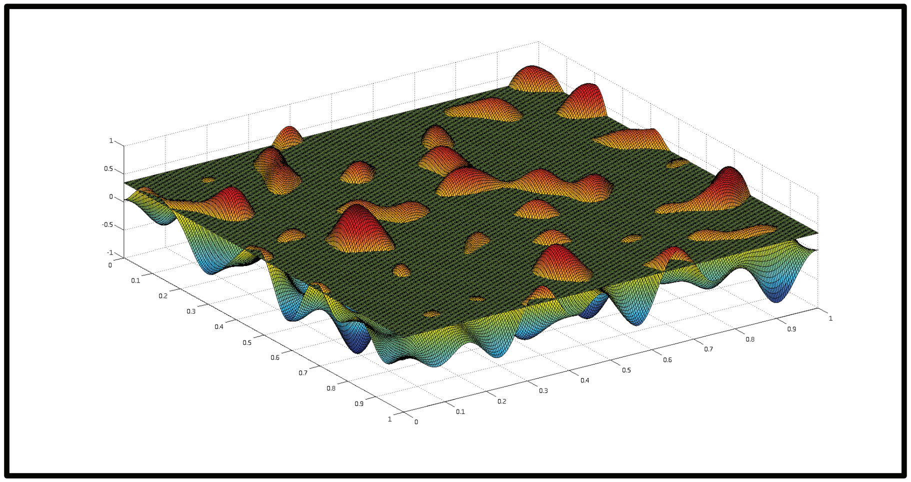
Continuum percolation model for melt pond evolution

level sets of random surfaces

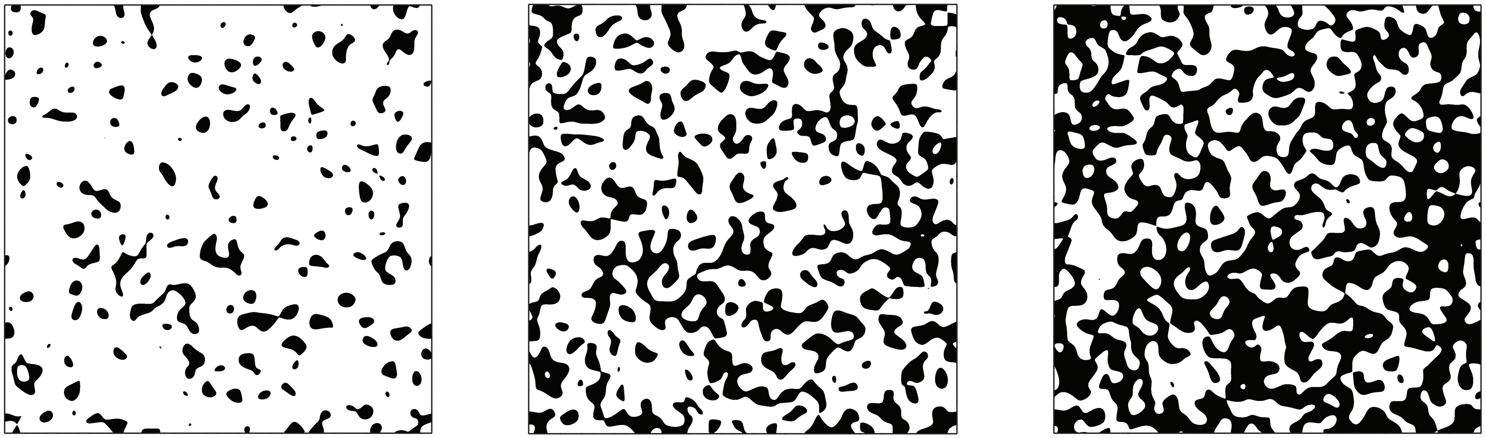
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography



intersections of a plane with the surface define melt ponds



electronic transport in disordered media

diffusion in turbulent plasmas

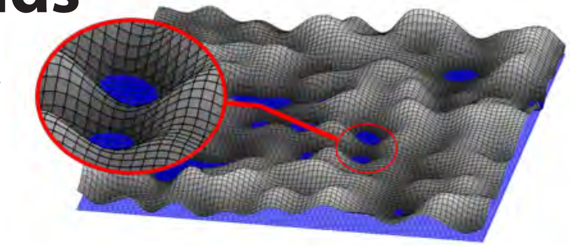
Isichenko, Rev. Mod. Phys., 1992

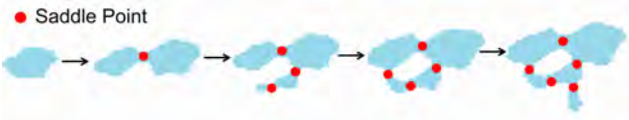
Saddle points of the sea ice surface and the fractal geometry of Arctic melt ponds

Physical Review Research (invited, in revision), Ryleigh Moore, Jacob Jones, Dane Gollero, Court Strong, Ken Golden

Morse Theory for Melt Ponds

ponds coalesce - change topology and complexify at saddle points; they drive the fractal transition



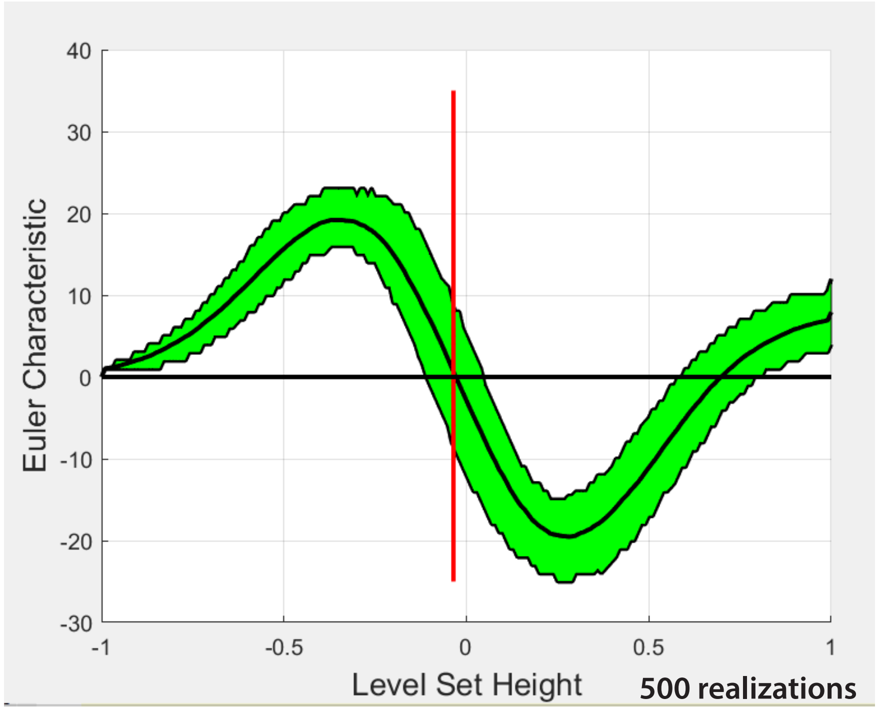


Topological Data Analysis (TDA) for Melt Ponds

persistent homology Euler characteristic = # maxima + # minima − # saddles

filtration - sequence of nested topological spaces, indexed by water level

Expected Euler Characteristic Curve (ECC)



tracks the evolution of the EC of the flooded surface as water rises

zero of ECC ~ percolation

percolation on a torus creates a giant cycle

Bobrowski & Skraba, 2020

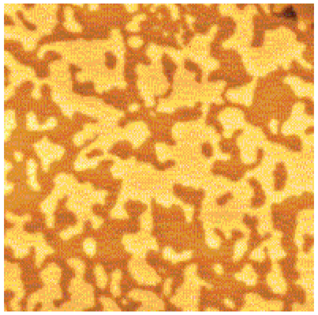
Carlsson, 2009

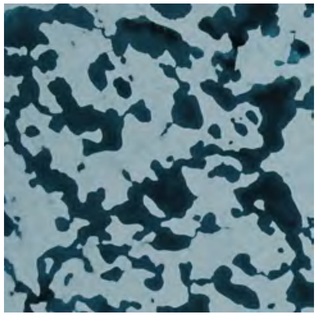
Vogel, 2002 GRF

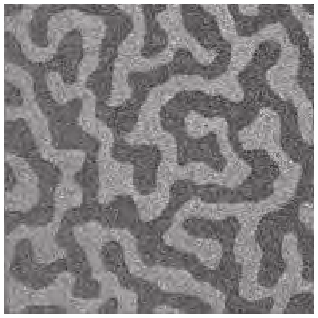
image analysis
porous media
cosmology
brain activity

From magnets to melt ponds

100 year old model for magnetic materials used to explain melt pond geometry

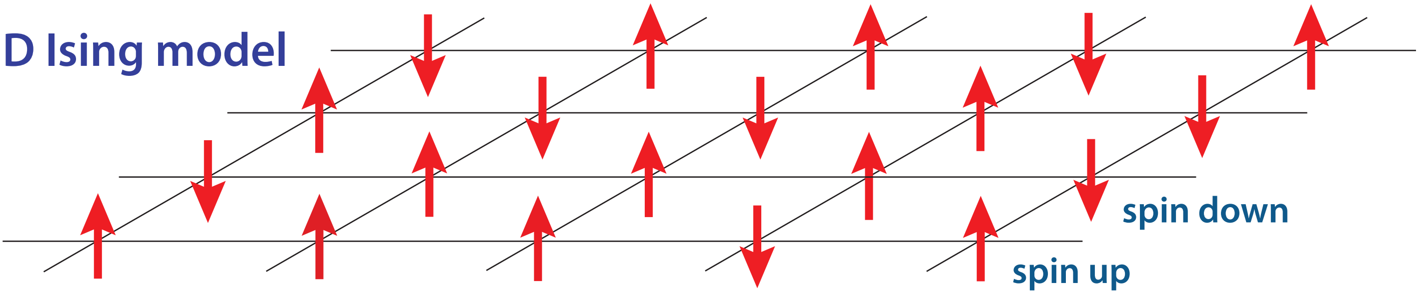

magnetic domains in cobalt

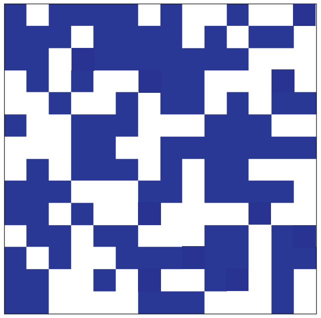

Arctic melt ponds


magnetic domains in cobalt-iron-boron

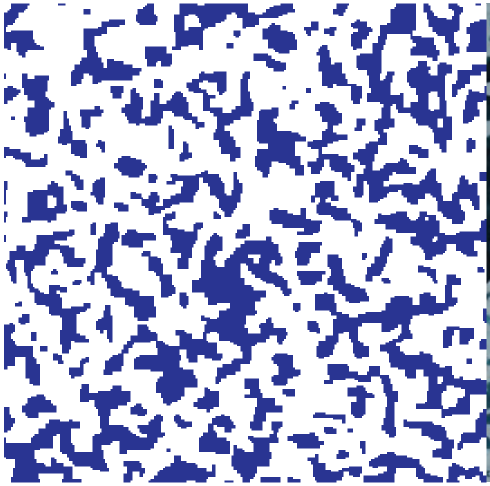

Arctic melt ponds

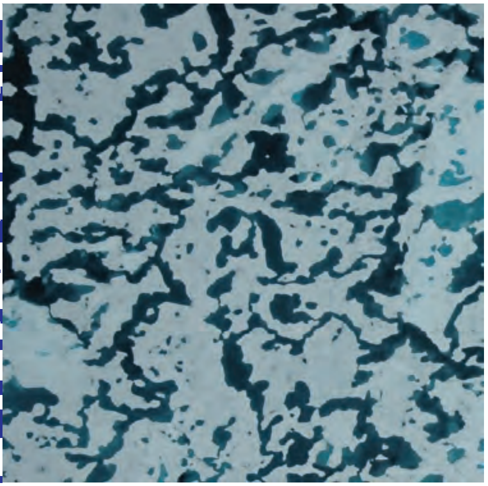
2D Ising model





model





real ponds (Perovich)

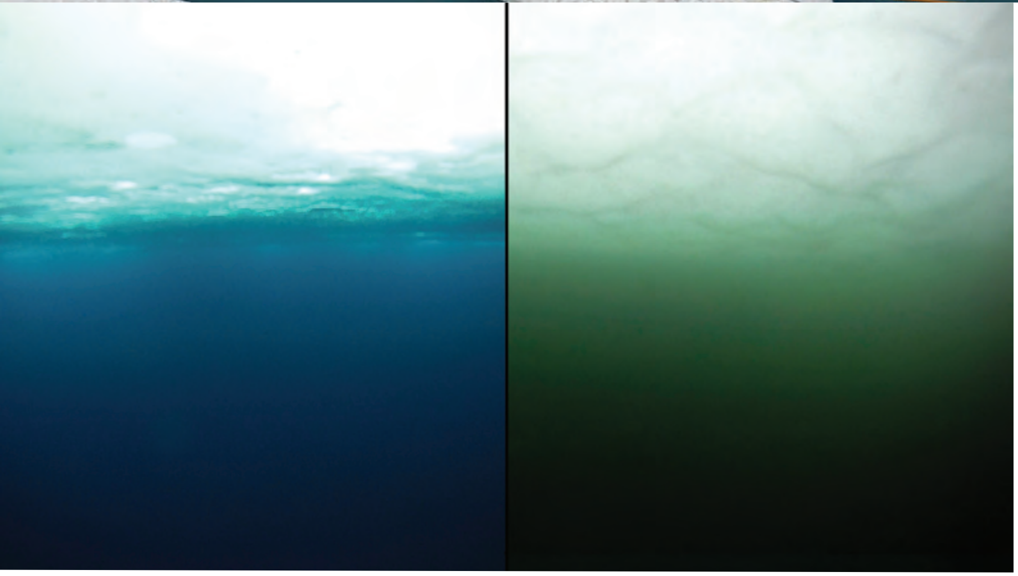
Ma, Sudakov, Strong, Golden, *New J. Phys.* 2019

Scientific American, EOS, PhysicsWorld, ...

Perovich

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS



no bloom **bloom**

massive under-ice algal bloom

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

Horvat, Rees Jones, Iams, Schroeder,
Flocco, Feltham, ***Science Advances*** 2017

The effect of melt pond geometry on the distribution of solar energy under first year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden
Geophys. Res. Lett. 2019

(2015 AMS MRC)

SEA ICE ALGAE

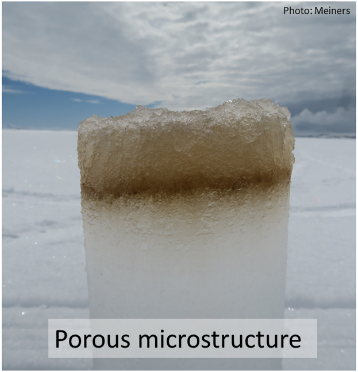


Photo: Meiners

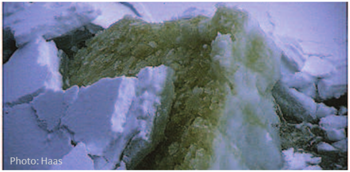
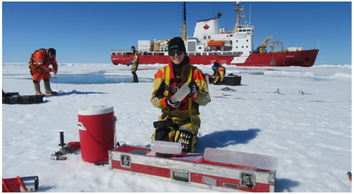


Photo: Haas

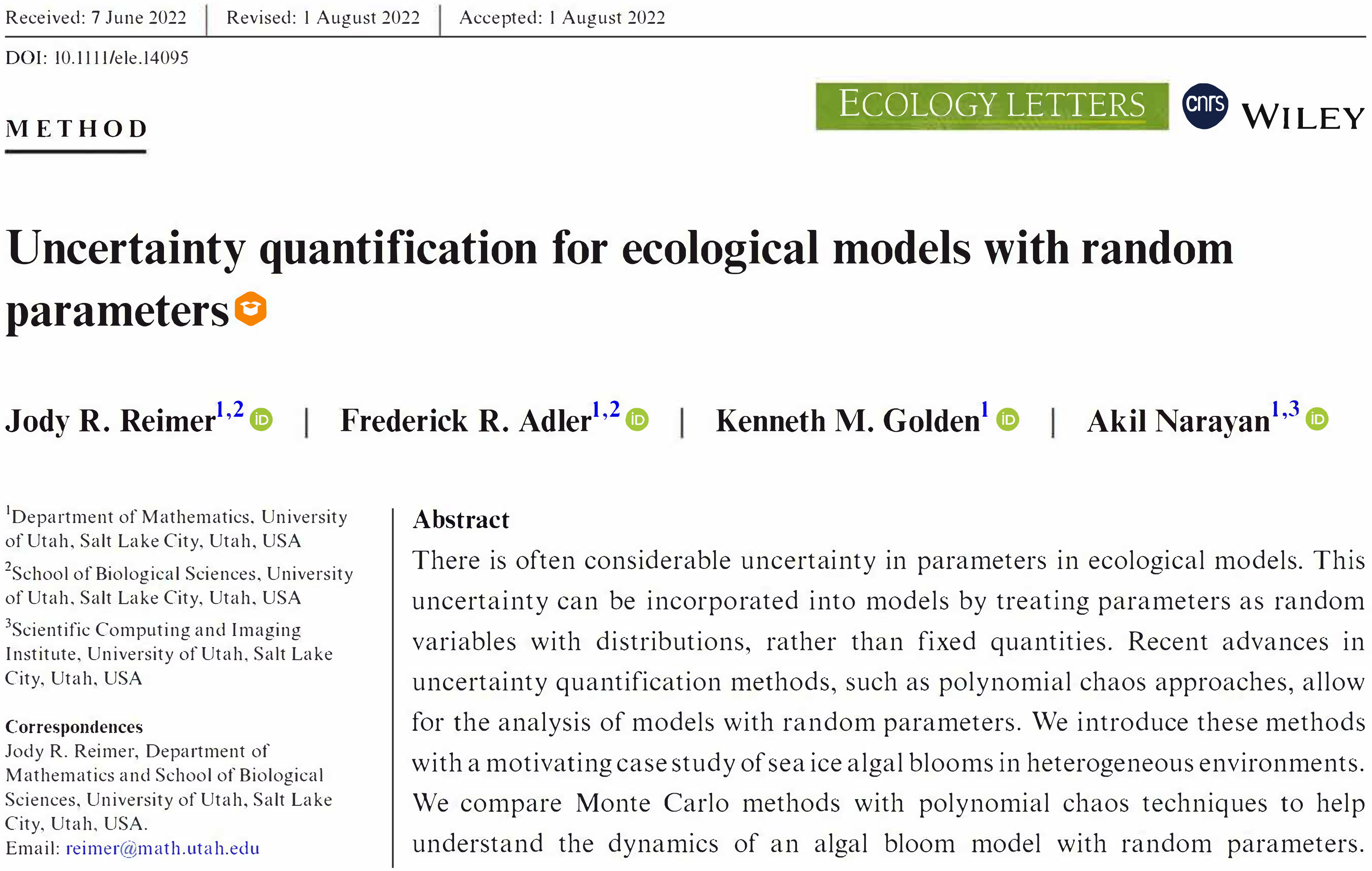


Can we improve agreement between algae models and data?

80% of polar bear diet can be traced to ice algae[*].

---

[*] Brown TA, et al. (2018). *PloS one*, 13(1), e0191631



Received: 7 June 2022 | Revised: 1 August 2022 | Accepted: 1 August 2022

DOI: 10.1111/ele.14095

ECOLOGY LETTERS WILEY

METHOD

Uncertainty quantification for ecological models with random parameters

Jody R. Reimer[1,2] | Frederick R. Adler[1,2] | Kenneth M. Golden[1] | Akil Narayan[1,3]

[1]Department of Mathematics, University of Utah, Salt Lake City, Utah, USA

[2]School of Biological Sciences, University of Utah, Salt Lake City, Utah, USA

[3]Scientific Computing and Imaging Institute, University of Utah, Salt Lake City, Utah, USA

Correspondences

Jody R. Reimer, Department of Mathematics and School of Biological Sciences, University of Utah, Salt Lake City, Utah, USA.

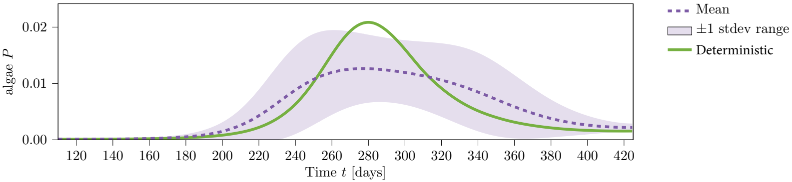
Email: reimer@math.utah.edu

Abstract

There is often considerable uncertainty in parameters in ecological models. This uncertainty can be incorporated into models by treating parameters as random variables with distributions, rather than fixed quantities. Recent advances in uncertainty quantification methods, such as polynomial chaos approaches, allow for the analysis of models with random parameters. We introduce these methods with a motivating case study of sea ice algal blooms in heterogeneous environments. We compare Monte Carlo methods with polynomial chaos techniques to help understand the dynamics of an algal bloom model with random parameters.

Introduce polynomial chaos approach to widely used ecological ODE models, but with random parameters.

ECOLOGICAL INSIGHTS



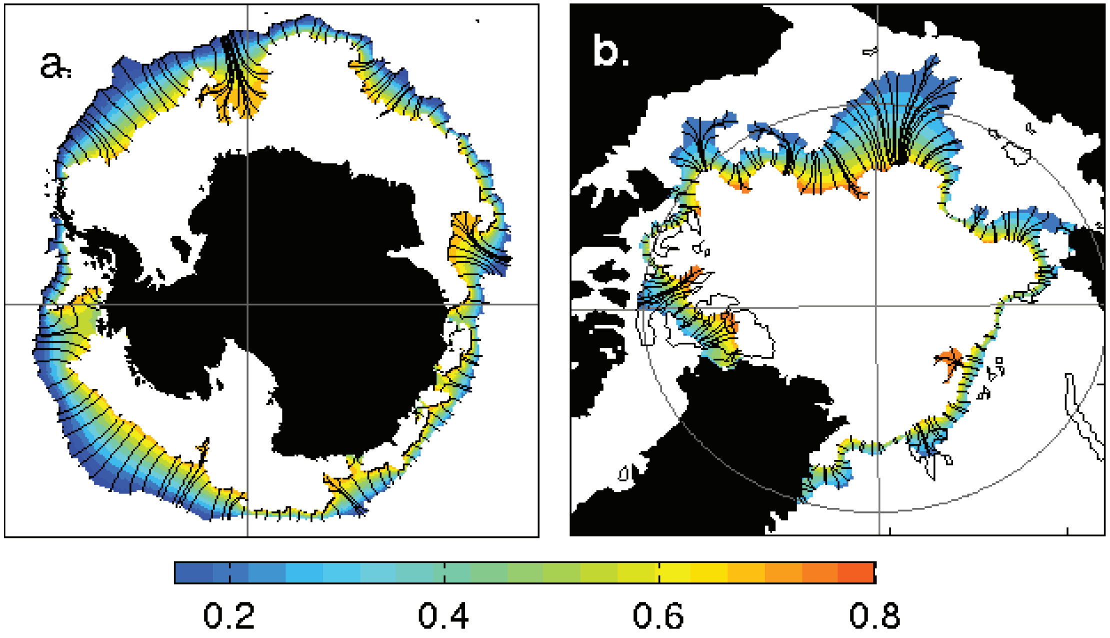
- lower peak bloom intensity
- longer bloom duration
- able to compare variance to data

macroscale

Marginal Ice Zone

MIZ

- biologically active region
- intense ocean-sea ice-atmosphere interactions
- region of significant wave-ice interactions



transitional region between dense interior pack ($c > 80\%$) sparse outer fringes ($c < 15\%$)

MIZ WIDTH
fundamental length scale of ecological and climate dynamics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

How to objectively measure the "width" of this complex, non-convex region?

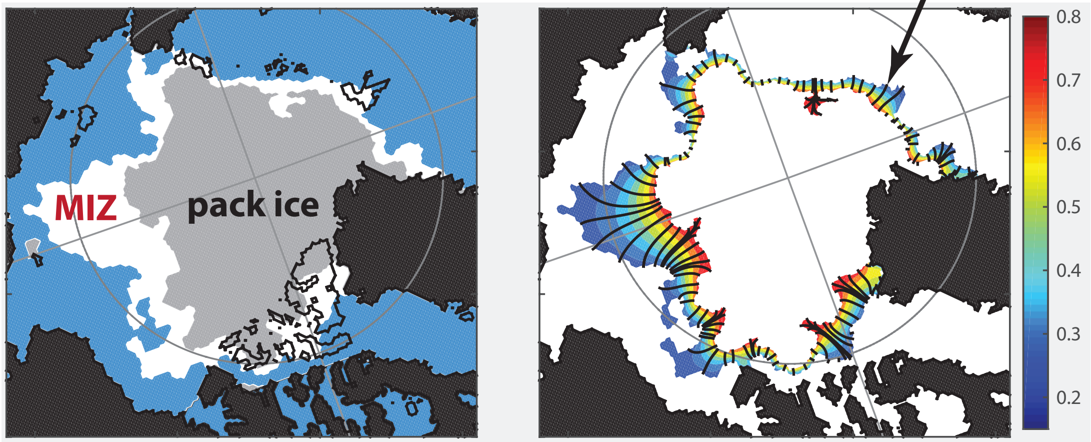
Objective method for measuring MIZ width motivated by medical imaging and diagnostics

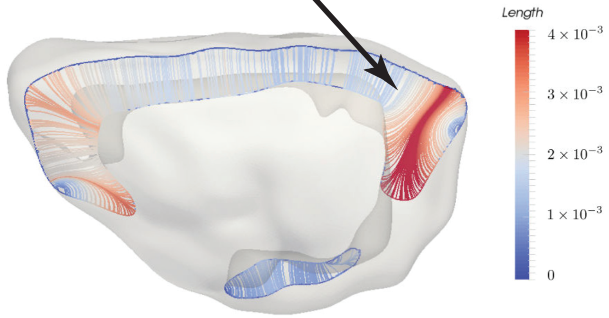
Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

39% widening
1979 - 2012

streamlines of a solution to Laplace's equation

"average" lengths of streamlines





Arctic Marginal Ice Zone

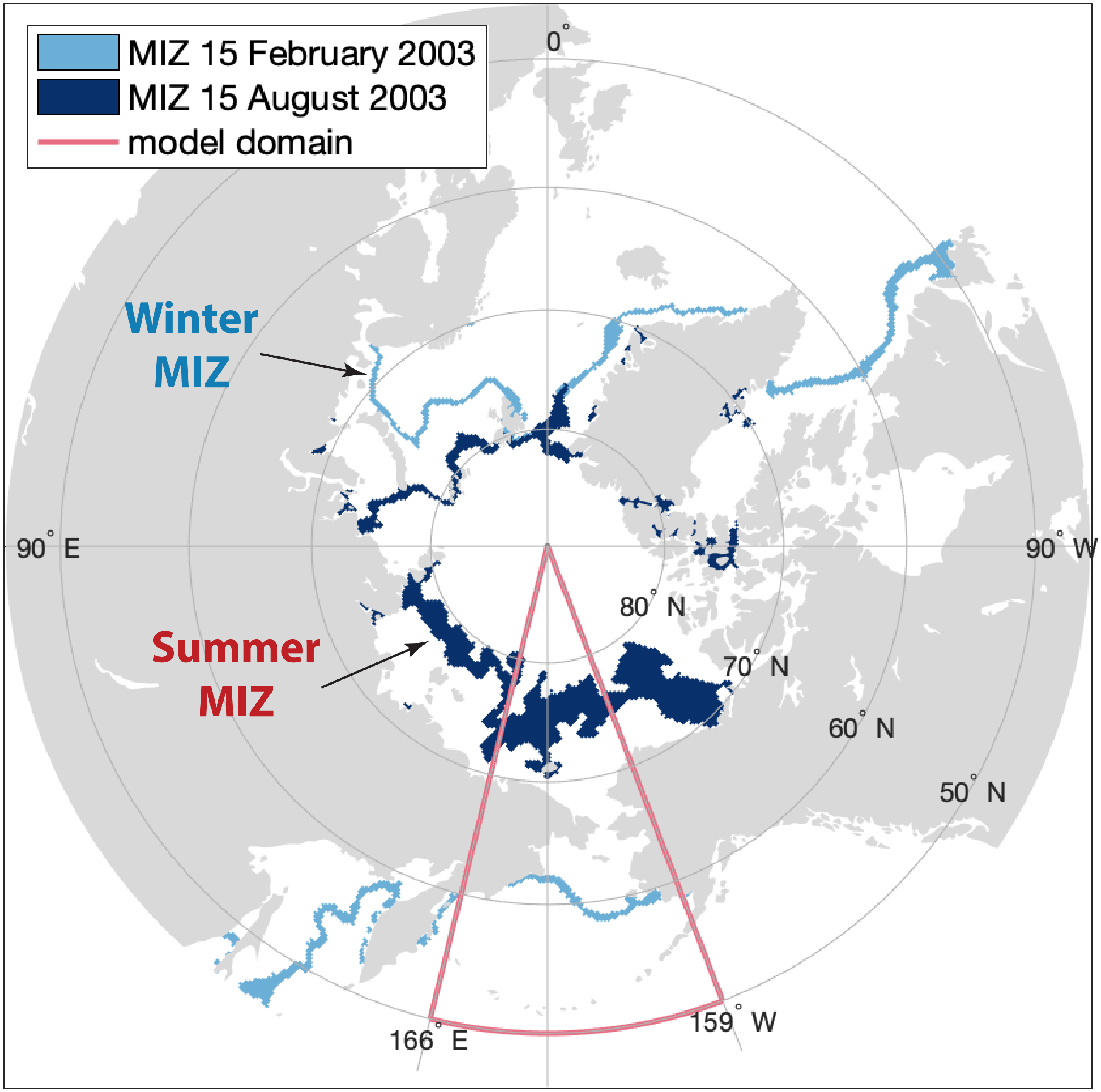
crossection of the cerebral cortex of a rodent brain

analysis of different MIZ WIDTH definitions

Strong, Foster, Cherkaev, Eisenman, Golden
J. Atmos. Oceanic Tech. 2017

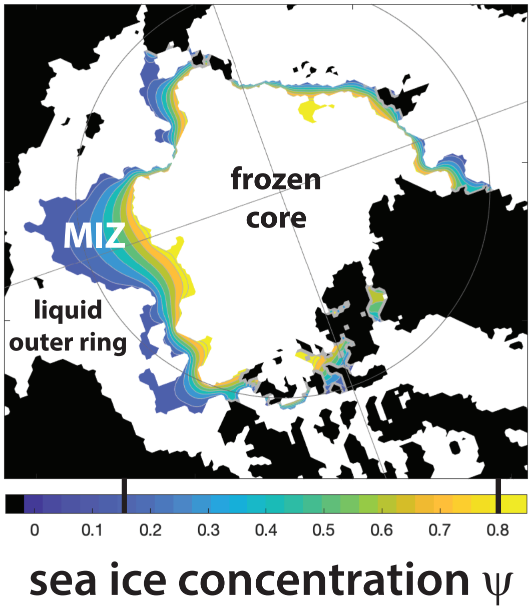
Strong and Golden
Society for Industrial and Applied Mathematics News, April 2017

Observed Arctic MIZ



Model larger scale effective behavior with partial differential equations that *homogenize* complex local structure and dynamics.

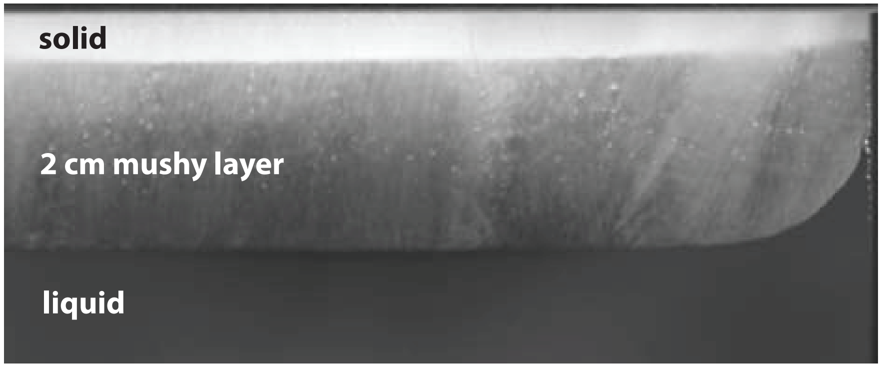
Arctic MIZ



sea ice concentration ψ

Predict MIZ width and location with basin-scale phase change model.

seasonal and long term trends



NaCl-H2O in lab
(Peppin et al., 2007;, J. Fluid Mech.)

Partial differential equation models and deep learning for the sea ice concentration field, 2023

Delaney Mosier, Eric Brown, Court Strong,
Jingyi Zhu, Bao Wang, Ken Golden

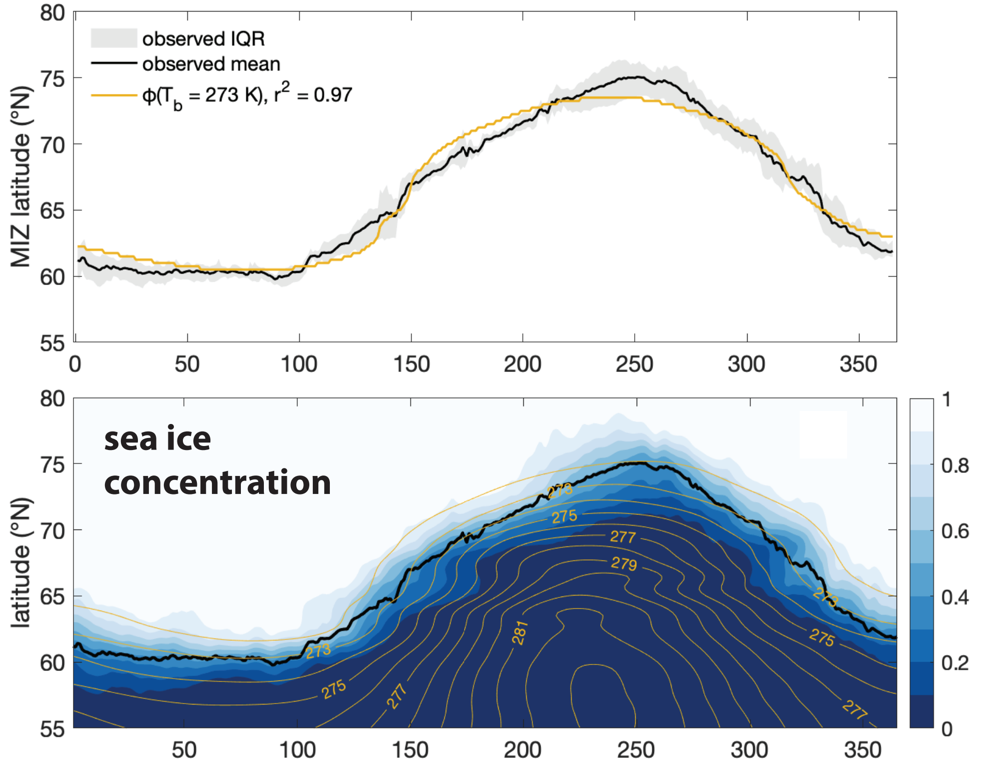
Annual cycle of Arctic marginal ice zone location and width explained by macroscale mushy layer model, 2023

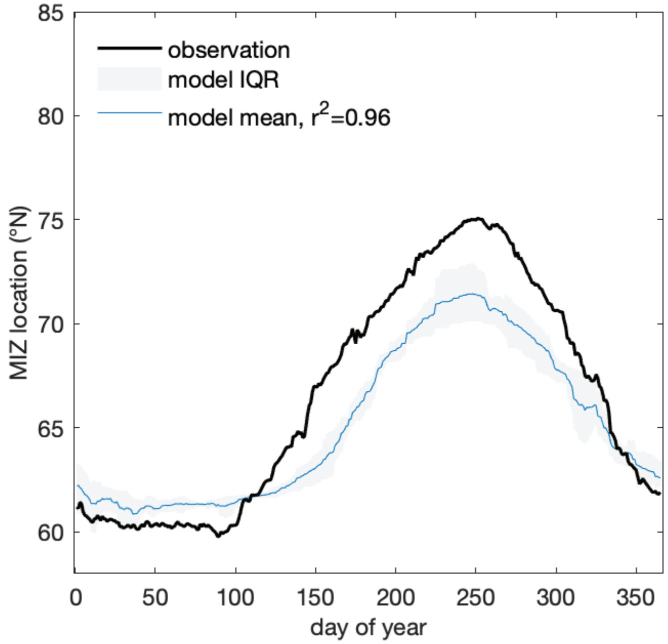
C. Strong, E. Cherkaev, and K. M. Golden

MIZ observations

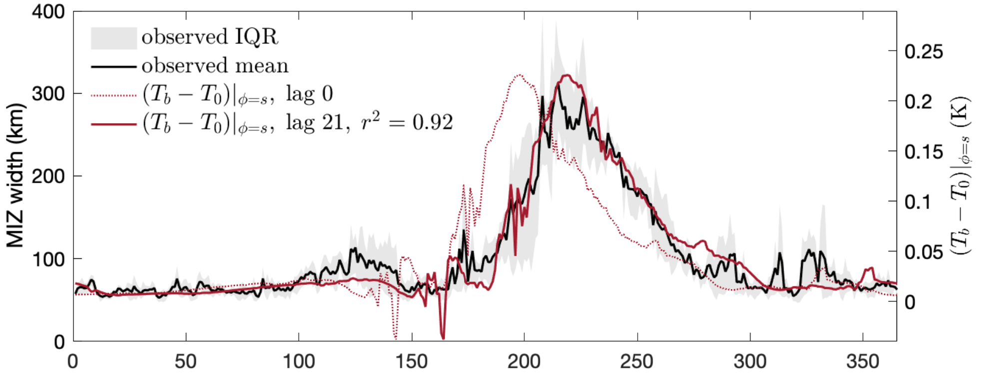
MIZ model vs. observations

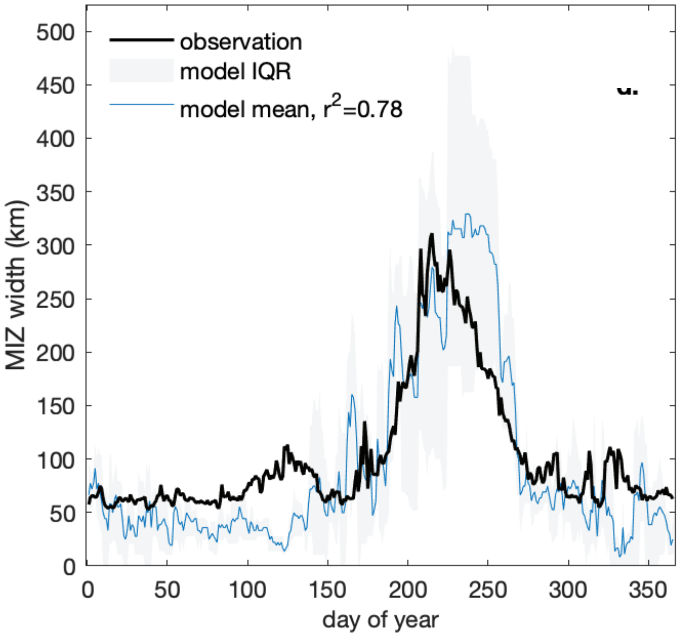
location





width



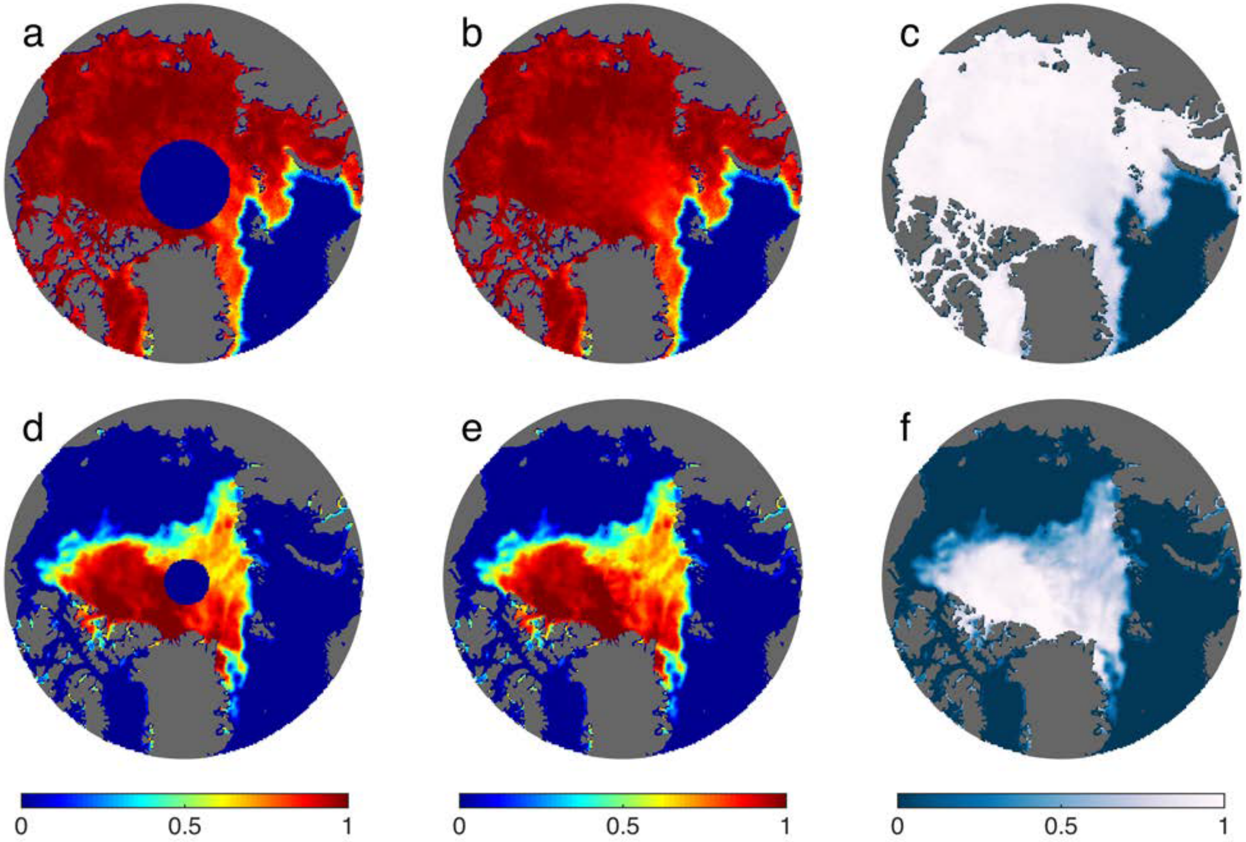


Model captures basic physics of MIZ dynamics.

Filling the polar data gap with partial differential equations

hole in satellite coverage of sea ice concentration field

previously assumed ice covered



Gap radius: 611 km
06 January 1985

Gap radius: 311 km
30 August 2007

$$\Delta\psi=0$$

fill = harmonic function with learned stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

NOAA/NSIDC Sea Ice Concentration CDR product update will use our PDE method.

Conclusions

Our research is helping to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

Mathematics for sea ice advances the theory of composites, inverse problems, and other areas of science and engineering.

Sea ice is really cool! Modeling it leads to unexpected areas of math and physics.

University of Utah Sea Ice Modeling Group (2017-2023)

Senior Personnel: Ken Golden, Distinguished Professor of Mathematics
Elena Cherkaev, Professor of Mathematics
Court Strong, Associate Professor of Atmospheric Sciences
Ben Murphy, Adjunct Assistant Professor of Mathematics

Postdoctoral Researchers: Noa Kraitzman, Jody Reimer, Bohyun Kim

Graduate Students: Kyle Steffen (now at UT Austin)
Christian Sampson (now at NCAR)
Huy Dinh (MURI sea ice Postdoc at NYU/Courant)
Rebecca Hardenbrook (-> Dartmouth Postdoc)
David Morison (Physics Department)
Ryleigh Moore
Delaney Mosier, Daniel Hallman, Julie Sherman

Undergraduate Students: Kenzie McLean, Jacqueline Cinella Rich,
Dane Gollero, Samir Suthar, Anna Hyde,
Kitsel Lusted, Ruby Bowers, Kimball Johnston,
Jerry Zhang, Nash Ward, David Gluckman,
Kayla Stewart, Nicole Forrester, Megan Long

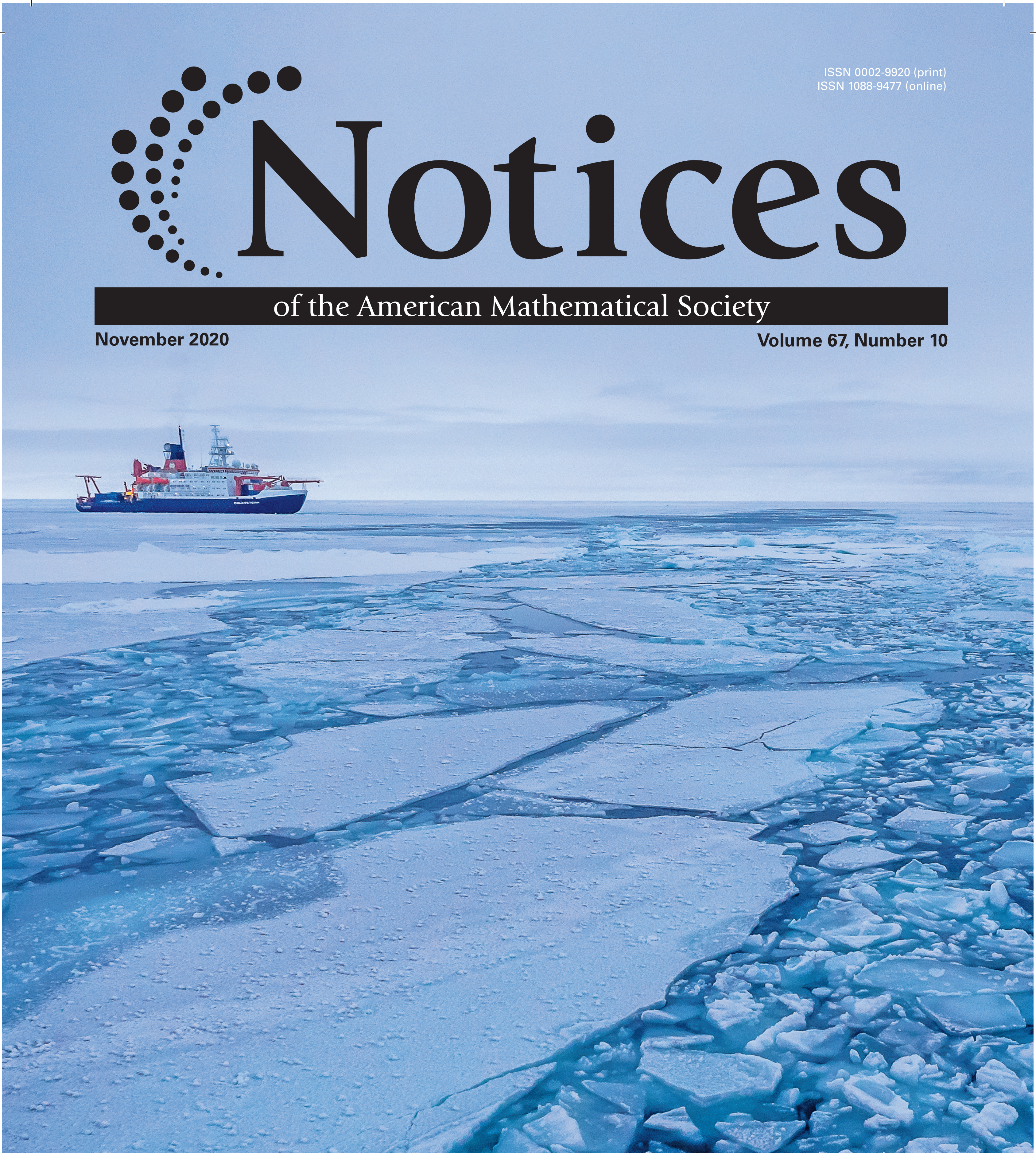
High School Students: J. Chapman, T. Quah, D. Webb, A. Lee, A. Dorsky

ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices

of the American Mathematical Society

November 2020
Volume 67, Number 10



AMERICAN
MATHEMATICAL
SOCIETY
Advancing research. Creating connections.

The cover is based on "Modeling Sea Ice," page 1535.

THANK YOU

Office of Naval Research

Applied and Computational Analysis Program

Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

UNITED STATES COAST GUARD
SEMPER PARATUS
1790

DEPARTMENT OF THE NAVY
USN ONR
Science & Technology

MCRN
Mathematics and Climate Research Network

Australian Government
Department of the Environment and Water Resources
Australian Antarctic Division

ace CRC
ANTARCTIC CLIMATE & ECOSYSTEMS
COOPERATIVE RESEARCH CENTRE

NSF

Antarctica New Zealand

UAF
UNIVERSITY OF ALASKA FAIRBANKS

THE UNIVERSITY OF UTAH

Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999

2—THE MERCURY   Thursday, July 23, 1998

Fire endangers Hobart's ice ship

By DAVID CARRIGG

AN engine-room fire has left the Hobart-based Antarctic research ship Aurora Australis without power in dangerous sea ice off the Antarctic coast.

None of the 79 people on board was injured in the blaze, which broke out early yesterday morning while the ship was in deep water 185km off the coast.

The extent of the damage is not known.

Australian Antarctic Division director Rex Moncur said the fire was extinguished by flooding the engine room with an inert gas.

The gas had to be cleared before crew wearing breathing apparatus could enter and assess the situation.

He said it could be some time before the extent of damage was known.

The 25 crew and 54 expeditioners, mostly from Hobart, would wear thermal clothing and stay below decks to keep warm.

"There is always a risk of becoming ice-bound in these waters at this time of the year but at this stage we don't expect to launch a rescue mission from Hobart," Mr Moncur said.

The ship was in regular radio contact with the Antarctic Division's Hobart office.

He expected the expeditioners and crew to abandon the pioneering winter voyage and return the ship to Hobart for repairs in about a week.

The Antarctic Division, which hires the ship from P&O Australia, would not be hiring another vessel for the expedition.

"It's a pretty specialist vessel so you couldn't get the sort of research capability that this ship has got readily available," Mr Moncur said.

"We hope the next voyage can still proceed on schedule, which is early September."

The Aurora Australis is owned by P&O Australia and charted by the Antarctic Division for about $11 million a year.

P&O Australia managing director Richard Hein said yesterday the company was assessing the situation and a number of rescue options were being considered.

It was too early to say whether P&O would be liable for the cost of the aborted mission.

The vessel left Hobart last Wednesday for a seven-week voyage mainly to study a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

The ship was nearing the polynya when the fire broke out.

Oceanographers believe a closer study of the phenomenon will lead to a better understanding of climate change.

CSIRO Marine Research oceanographer Steve Rintoul said the dense bottom water, created only in a few places in Antarctica and to a lesser extent in the North Atlantic, was critical to the chemistry and biology of the world's oceans.

A file photo of the Aurora Australis in Antarctica.

THE ADVERTISER (Adelaide)
Thurs 23 July 1998   Page 14

Fire strands Antarctic ship in sea ice

AN engine room fire has disabled the icebreaker Aurora Australis in sea ice, deep in Antarctic waters.

There were no injuries and the ship was not in danger after Tuesday night's fire, Australian Antarctic Division director Mr Rex Moncur said. But Mr Moncur said he expected it would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent and return to Hobart for repairs.

The cause of the fire was not known but the engines have been turned off, with the ship 100 nautical miles from the Antarctic coast.

THE CANBERRA TIMES
Thursday 23 July 1998   Page 4

Antarctic voyage stopped by fire

HOBART: An engine room fire has disabled the Australian icebreaker Aurora Australis in sea ice, deep in Antarctic waters.

Australian Antarctic Division director Rex Moncur said there were no injuries and the ship was not in danger after Tuesday night's fire.

But Mr Moncur said he expected Aurora Australis would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent to return to Hobart for repairs.

The fire had been extinguished and the engines were turned off, leaving the ship in sea ice about 100 nautical miles from the Antarctic coast, he said. The weather was good.

Crew had to wear breathing apparatus to enter the engine room and it was likely to be 24 hours before the damage could be fully assessed.

The Aurora, with 54 expeditioners and 25 crew, left Hobart last Wednesday for a seven-week voyage which was to have focused on a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

Mr Moncur said, the cause of the fire was not yet known.

Sydney Morning Herald
23 July, 1998

ICEBREAKER BURNS

A pioneering $2 million Australian scientific voyage to the mid-winter Antarctic pack ice is expected to be scrapped following an engine-room fire on the Aurora Australis yesterday.

The 54 people on board were forced on deck in the

2:45 am July 22, 1998

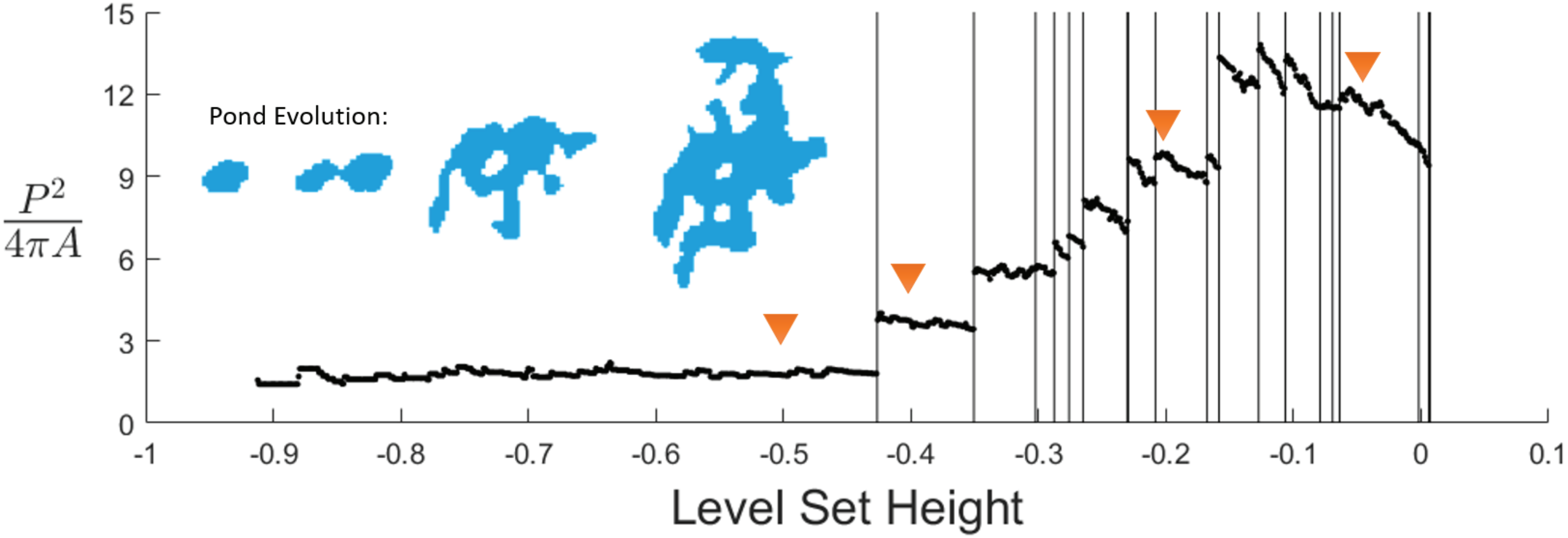
``Please don't be alarmed but we have an uncontrolled fire in the engine room ....''

about 10 minutes later ...

``Please don't be alarmed but we're lowering the lifeboats ....''

Main results

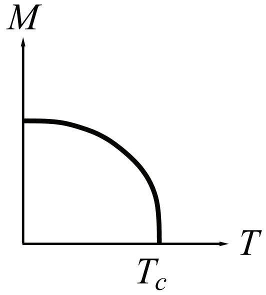
Isoperimetric quotient - as a proxy for fractal dimension - increases in discrete jumps when ponds coalesce at saddle points.



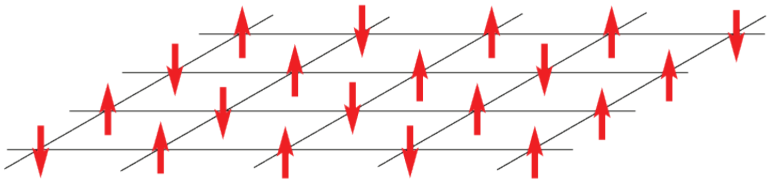
Horizontal fluid permeability "controlled" by saddles ~ electronic transport in 2D random potential.

drainage processes, seal holes

Ising Model for a Ferromagnet



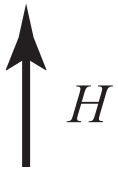
Curie point
critical temperature



$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

blue

white

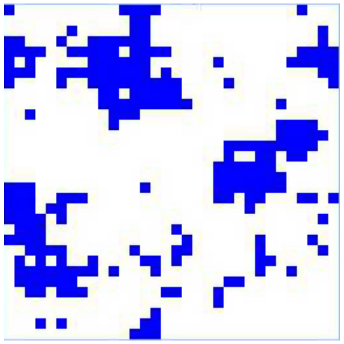


applied magnetic field

$$\mathcal{H} = -H \sum_i s_i - J \sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

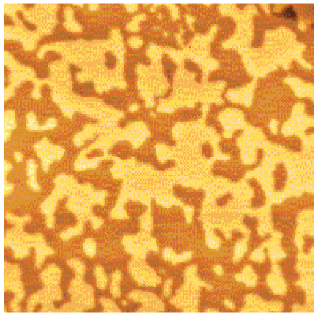
islands of like spins



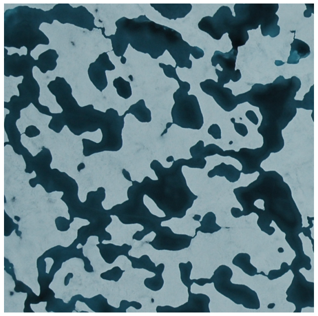
$$M(T,H) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$

effective magnetization

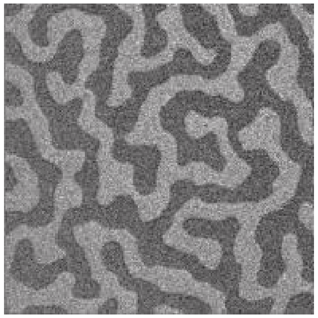
energy is lowered when nearby spins align with each other, forming **magnetic domains**



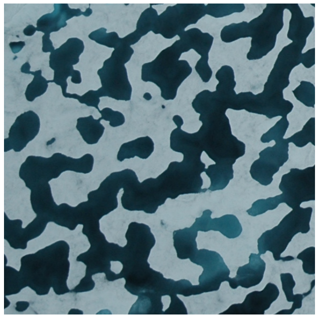
magnetic domains in cobalt



melt ponds (Perovich)



magnetic domains in cobalt-iron-boron



melt ponds (Perovich)

Ising model for ferromagnets ⟶ Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

$$\mathcal{H} = -\sum_{i}^{N} H_i \, s_i - J \sum_{<i,j>}^{N} s_i s_j \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

random magnetic field represents snow topography

magnetization M

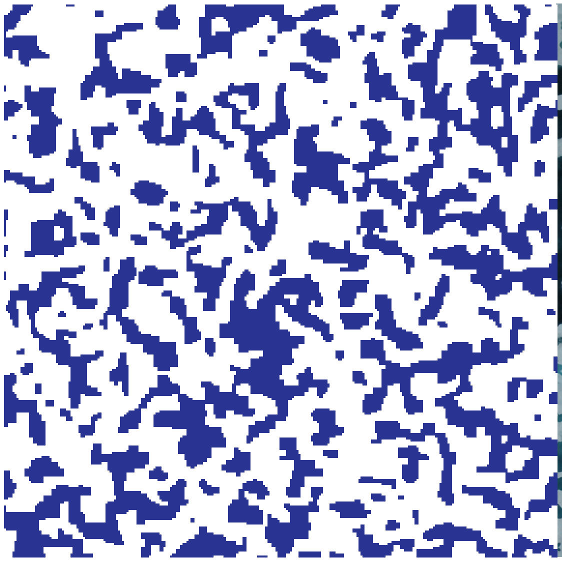
pond area fraction $F = \dfrac{(M+1)}{2}$

~ albedo

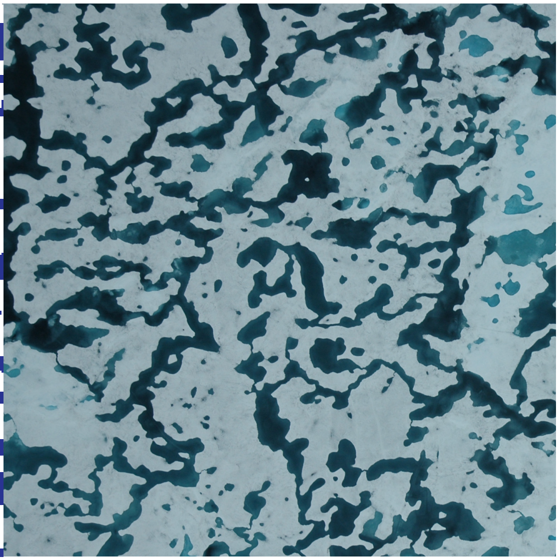
only nearest neighbor patches interact

Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system "flows" toward metastable equilibria.

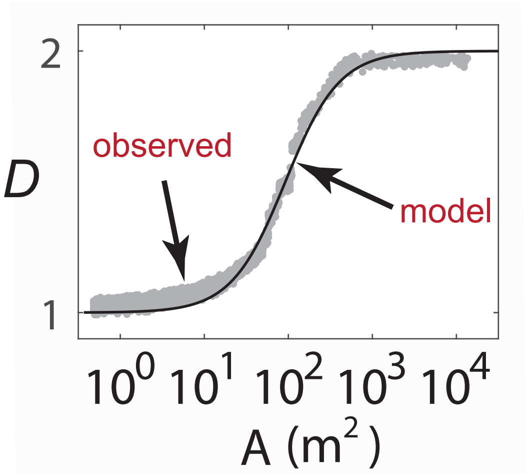
Order from Disorder



Ising model



melt pond photo (Perovich)



pond size distribution exponent

observed -1.5

(Perovich, *et al.* 2002)

model -1.58

Scientific American
EOS, PhysicsWorld, ...

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

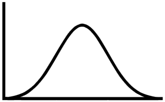
HETEROGENEITY IN INITIAL CONDITIONS

At each location within a larger region, we could consider

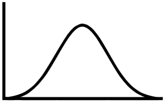
Nutrients $$\frac{dN}{dt} = \alpha - BNP - \eta N$$

Algae $$\frac{dP}{dt} = \gamma BNP - \delta P$$

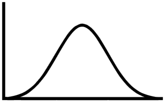
$$N(0) = N_0, \qquad P(0) = P_0$$



growth rate, B

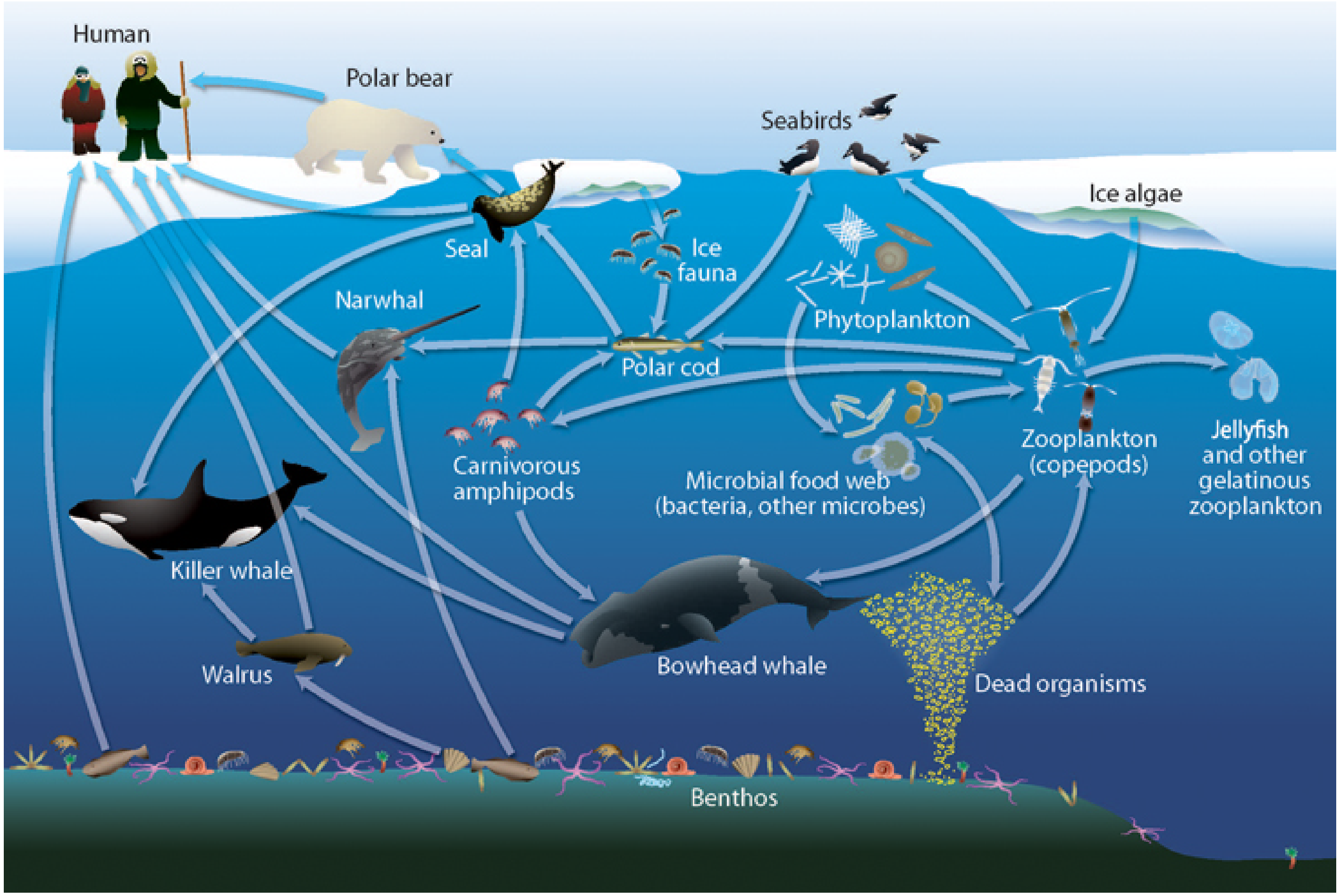


Initial nutrients, N_0



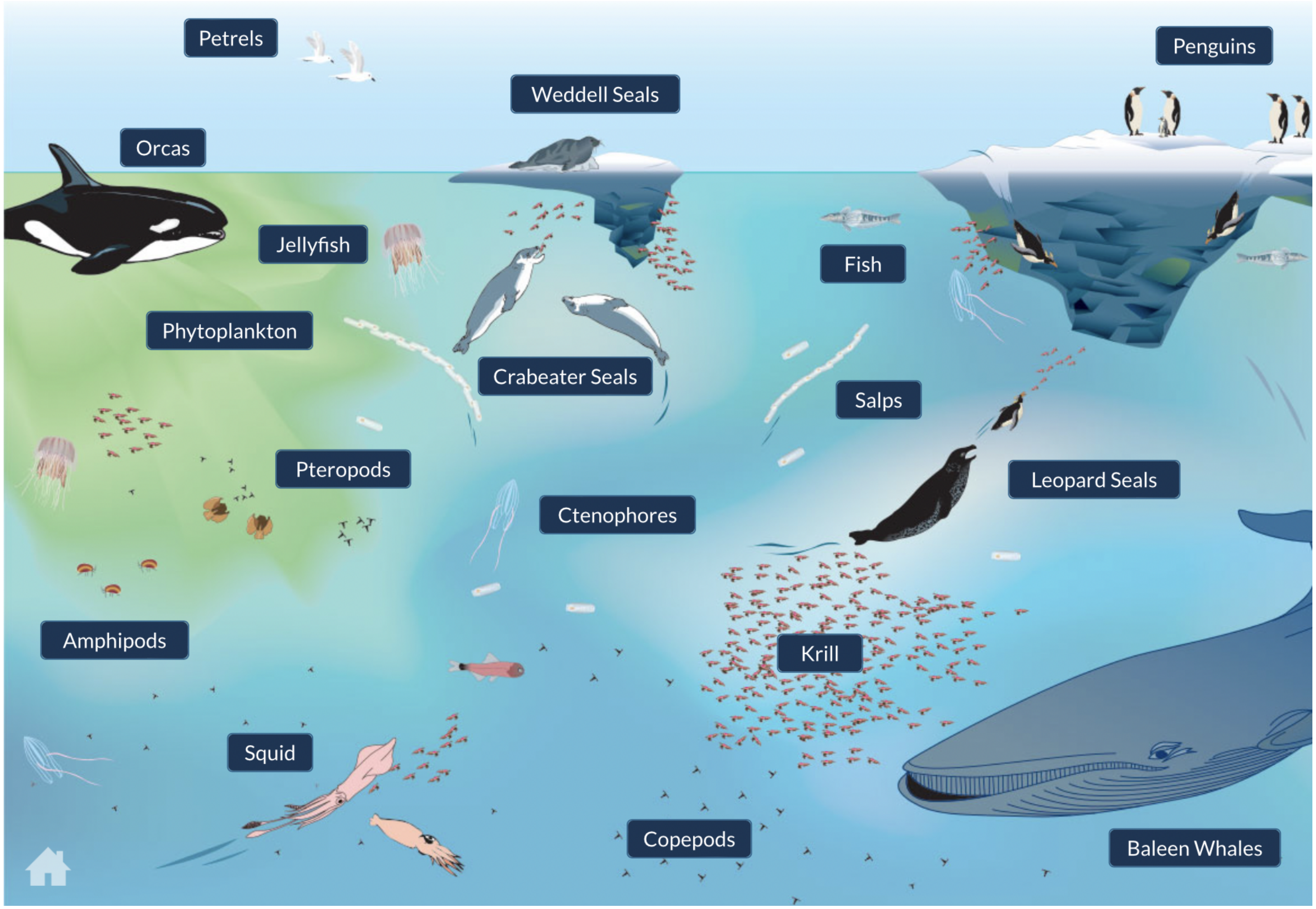
Initial algae, P_0

Arctic marine ecosystem



Darnis et al.

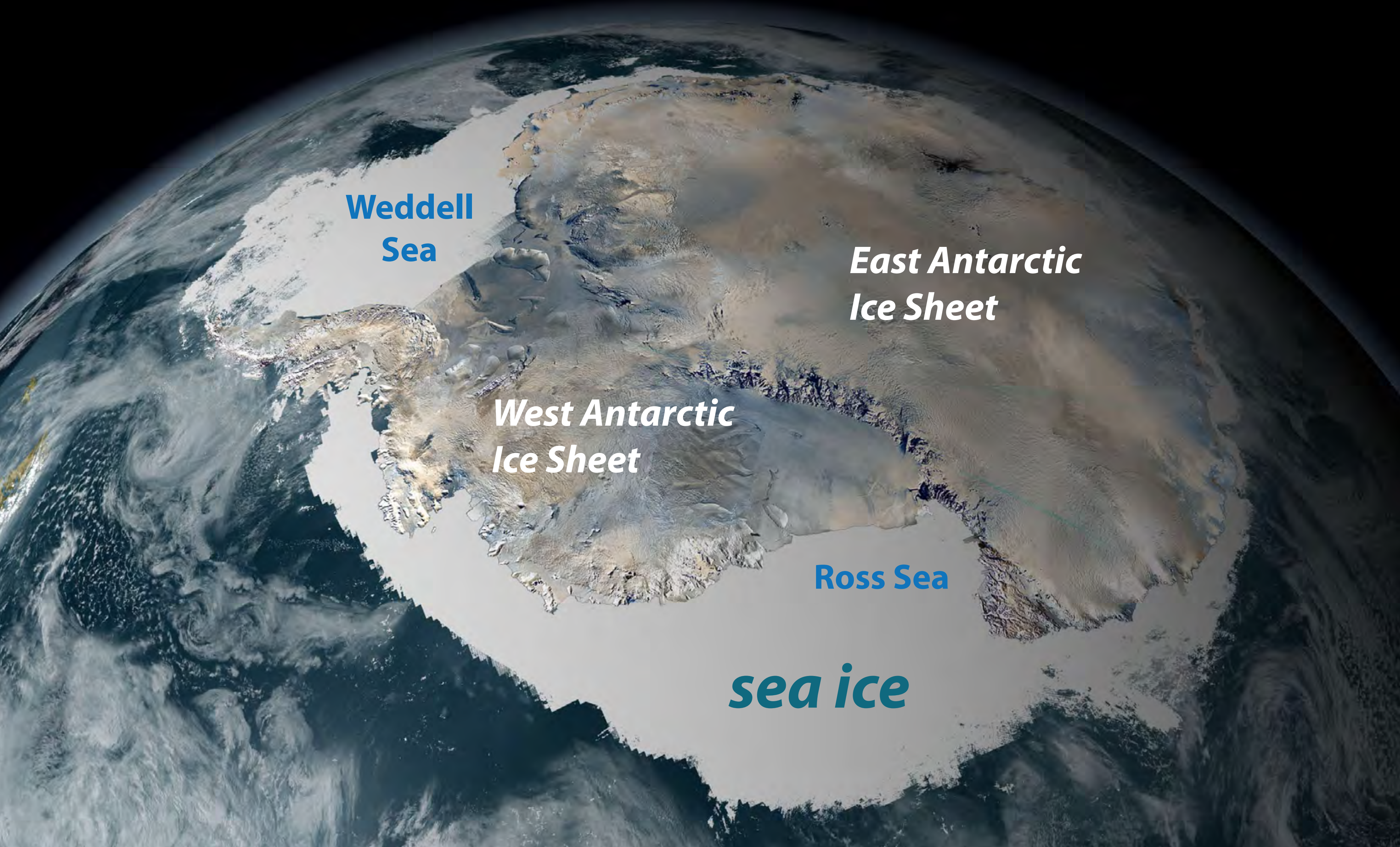
Antarctic marine ecosystem



WHOI

ANTARCTICA

southern cryosphere

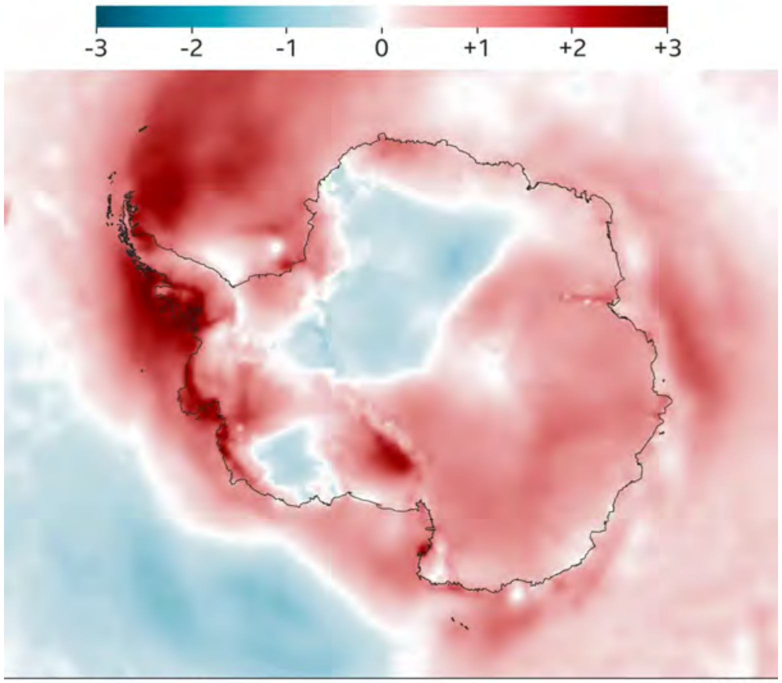


New Record Low for Antarctic Sea Ice

February 13, 2023

Much of Antarctica warmer than average

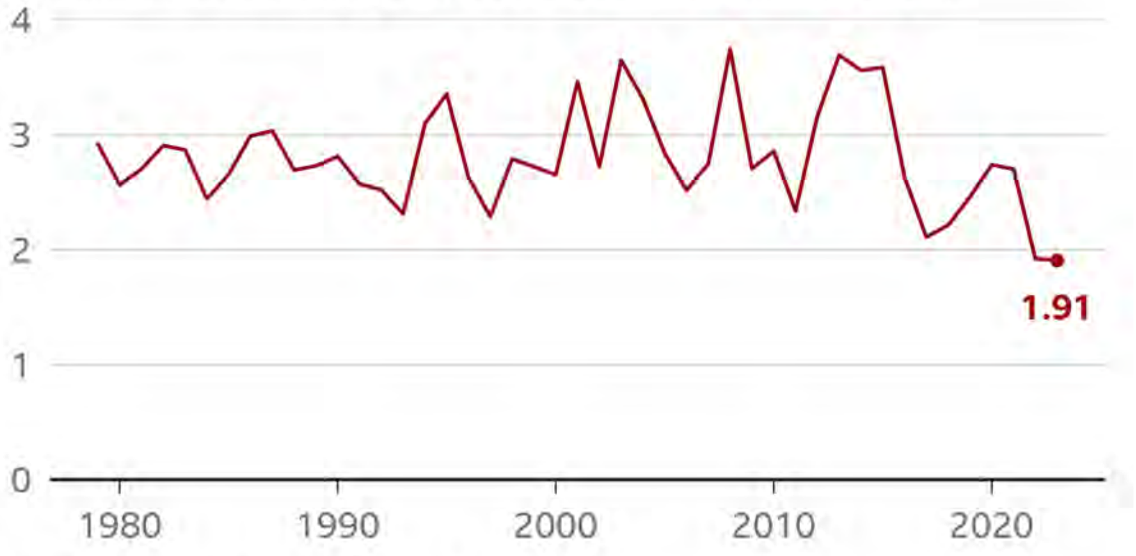
Mean 2022 surface air temp compared with 1991-2022 (°C)



Source: ECMWF ERA5

BBC

Minimum extent 1979-2023 (million sq km)



Five-day rolling average of sea-ice extent

Source: National Snow and Ice Data Center (NSIDC)

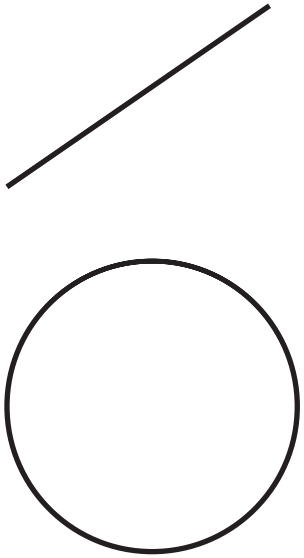
BBC

recent losses in comparison to the United States

Perovich

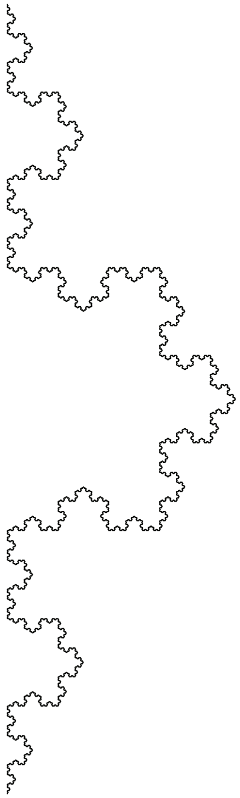
fractal curves in the plane

they wiggle so much that their dimension is >1



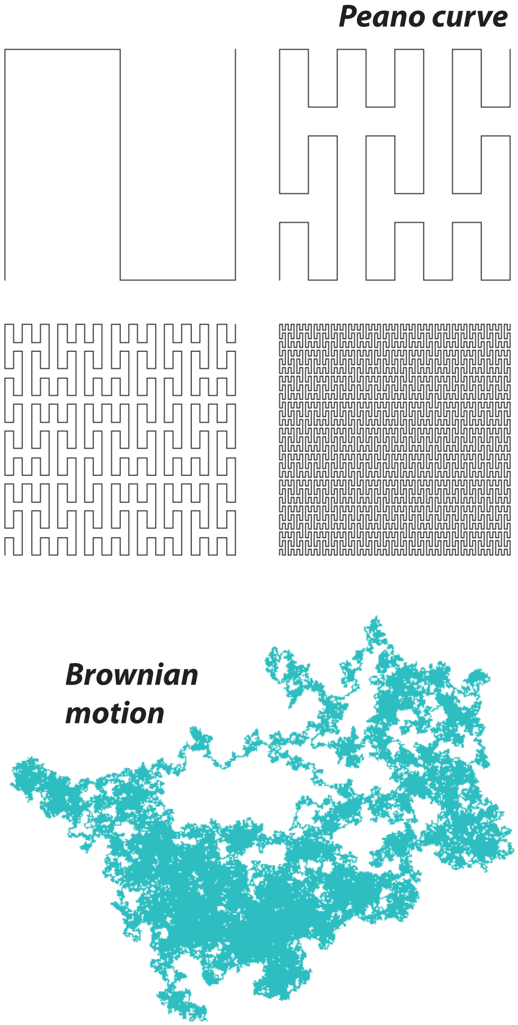
simple curves

***D* = 1**



Koch snowflake

***D* = 1.26**



Peano curve

Brownian motion

space filling curves

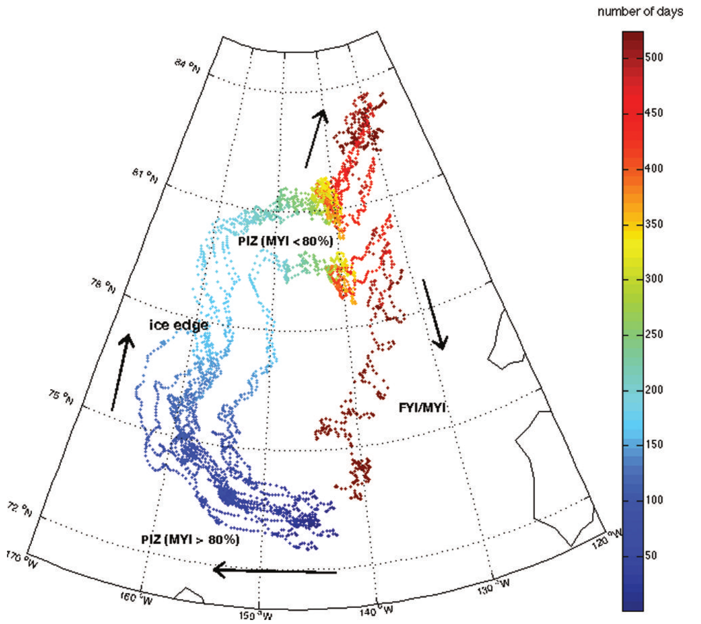
***D* = 2**

Anomalous diffusion in sea ice dynamics

Ice floe diffusion in winds and currents

observations from GPS data:

Jennifer Lukovich, Jennifer Hutchings,
David Barber, *Ann. Glac.* 2015



- On short time scales floes observed (buoy data) to exhibit Brownian-like behavior, but they are also being advected by winds and currents.

- Effective behavior is purely diffusive, sub-diffusive or super-diffusive depending on ice pack and advective conditions - Hurst exponent.

modeling:

Huy Dinh, Ben Murphy, Elena Cherkaev,
Court Strong, Ken Golden 2022

floe scale model to analyze transport regimes in terms of ice pack crowding, advective conditions

Delaney Mosier, Jennifer Hutchings, Jennifer Lukovich,
Marta D'Elia, George Karniadakis, Ken Golden 2022

learning fractional PDE governing diffusion from data

MIZ as a moving phase transition region

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + S$$

$$S = [\rho(c_l - c_s)T + \rho L]\frac{\partial \psi}{\partial t}$$

$$\psi = 1 - \left(\frac{T - T_s}{T_l - T_s}\right)^{\alpha}$$

$$k_x = \left(\frac{\psi}{k_s} + \frac{1-\psi}{k_l}\right)^{-1}$$

$$k_z = \psi k_s + (1-\psi)k_l$$

homogenization

ρ effective density

T temperature

c specific heat

L latent heat of fusion

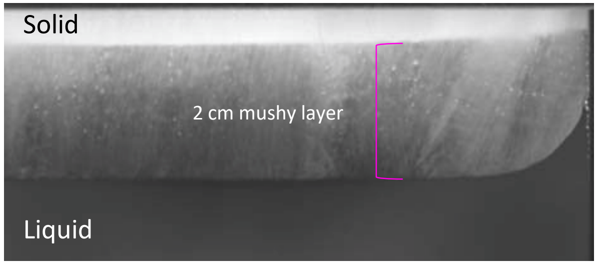
S models nonlinear phase change

ψ sea ice concentration

k effective diffusivity

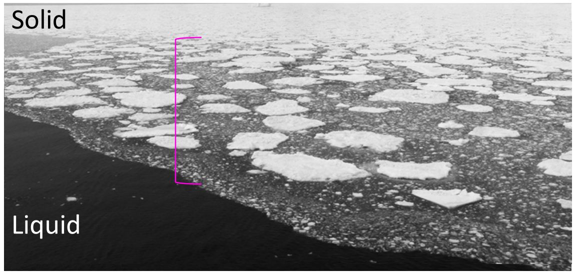
l liquid, s solid

Classical small-scale application



NaCl-H$_2$O in lab
(Peppin et al., 2007;, J. Fluid Mech.)

Macroscale application



- Develop multiscale PDE model for simulating phase transition fronts to predict MIZ seasonal cycles and decadal trends
- Model simulates MIZ as a large-scale mushy layer with effective thermal conductivity derived from physics of composite materials