Herglotz Functions in the Mathematics of Sea Ice and Other Composites

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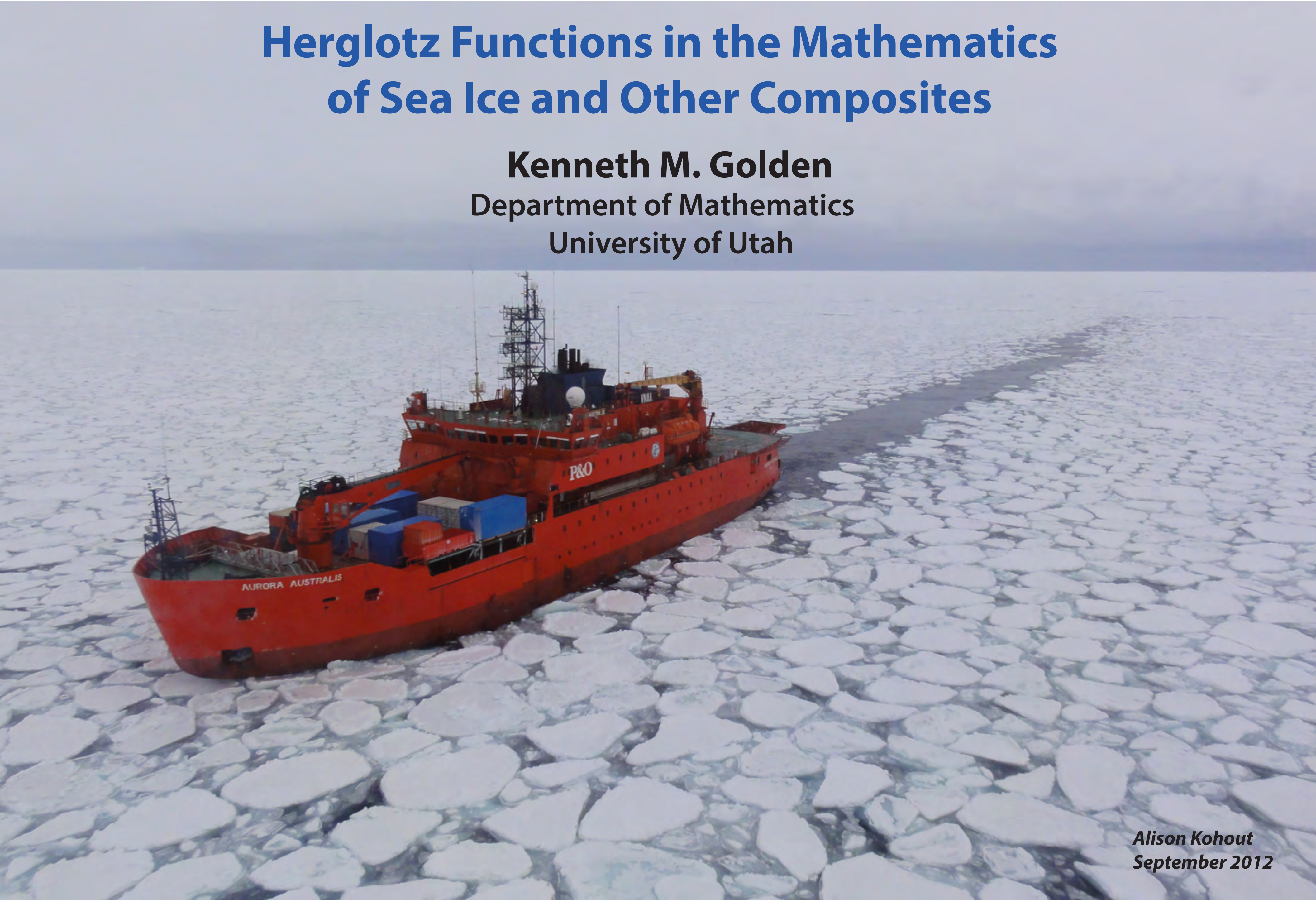


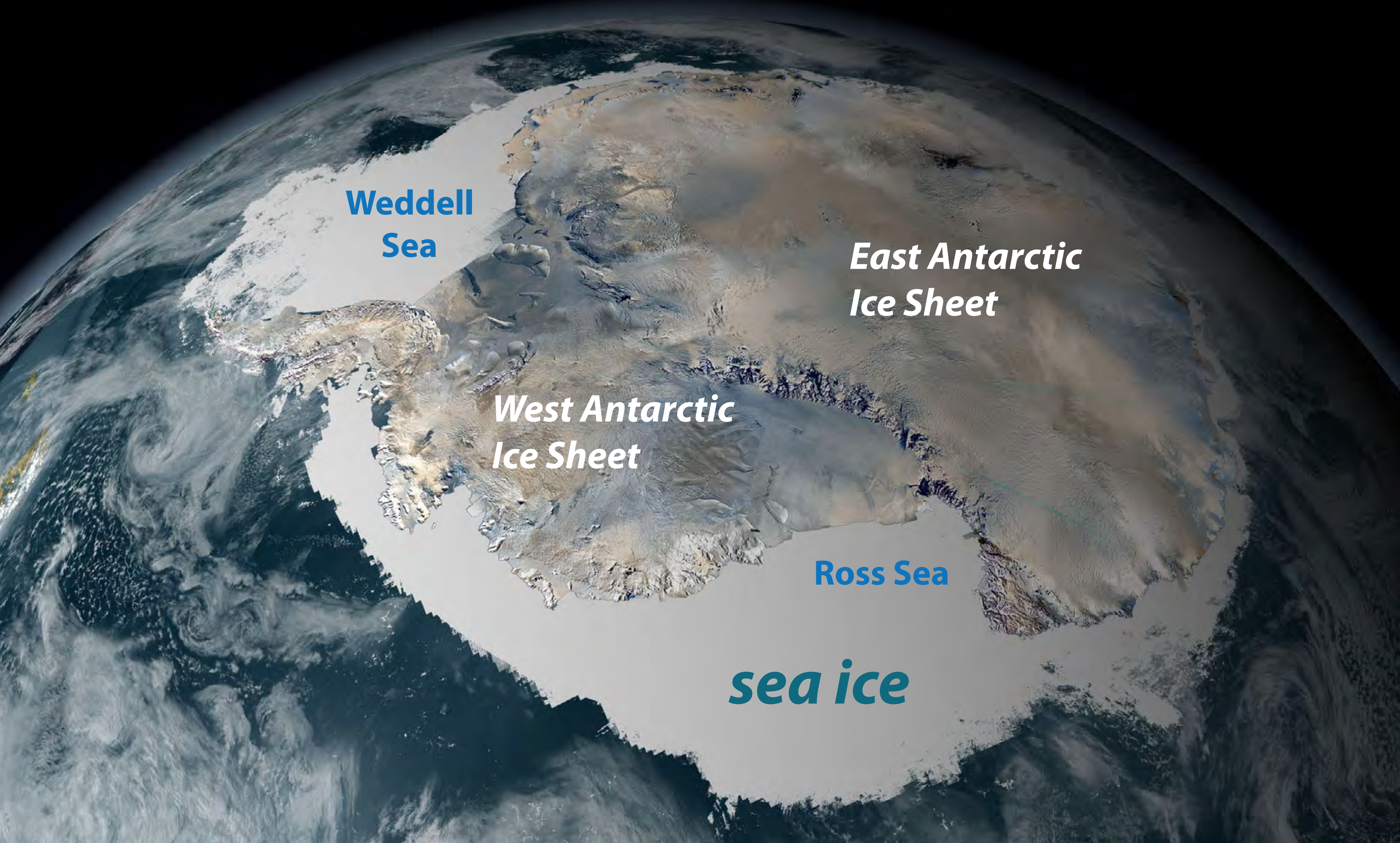
Alison Kohout September 2012

Institute Mittag Leffler Workshop

Herglotz-Nevanlinna functions and their applications 8 May 2017

ANTARCTICA

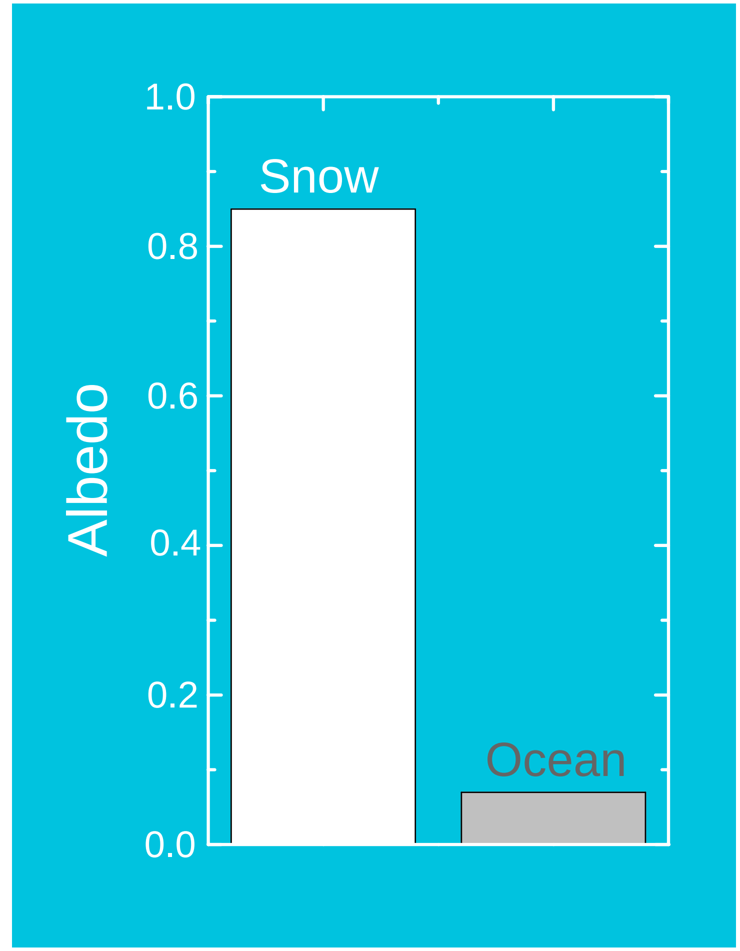
southern cryosphere




SEA ICE covers 7 - 10% of earth's ocean surface
• boundary between ocean and atmosphere
• mediates exchange of heat, gases, momentum
• global ocean circulation
• indicator and agent of climate change

polar ice caps critical to global climate in reflecting incoming solar radiation

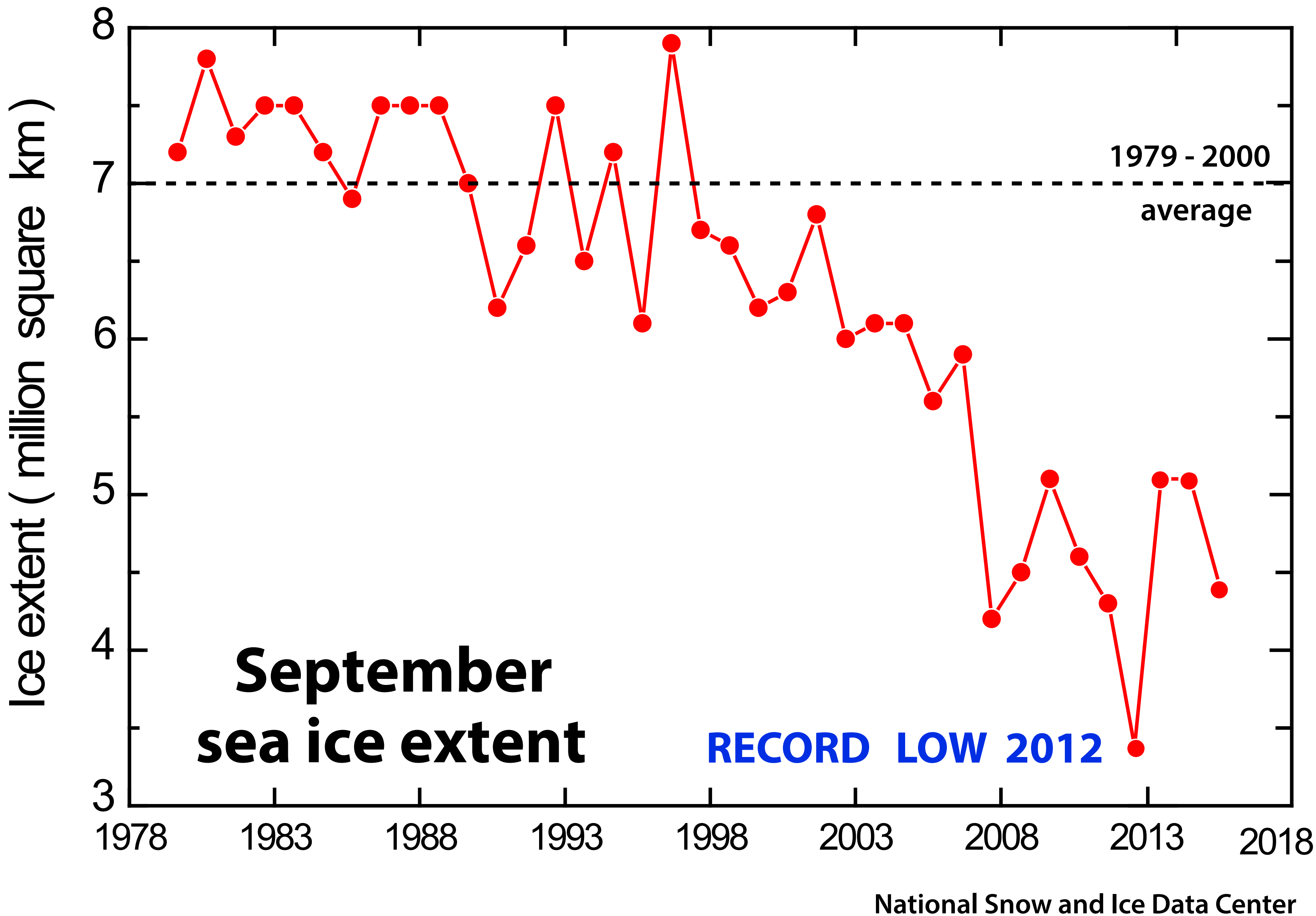
white snow and ice reflect



dark water and land absorb

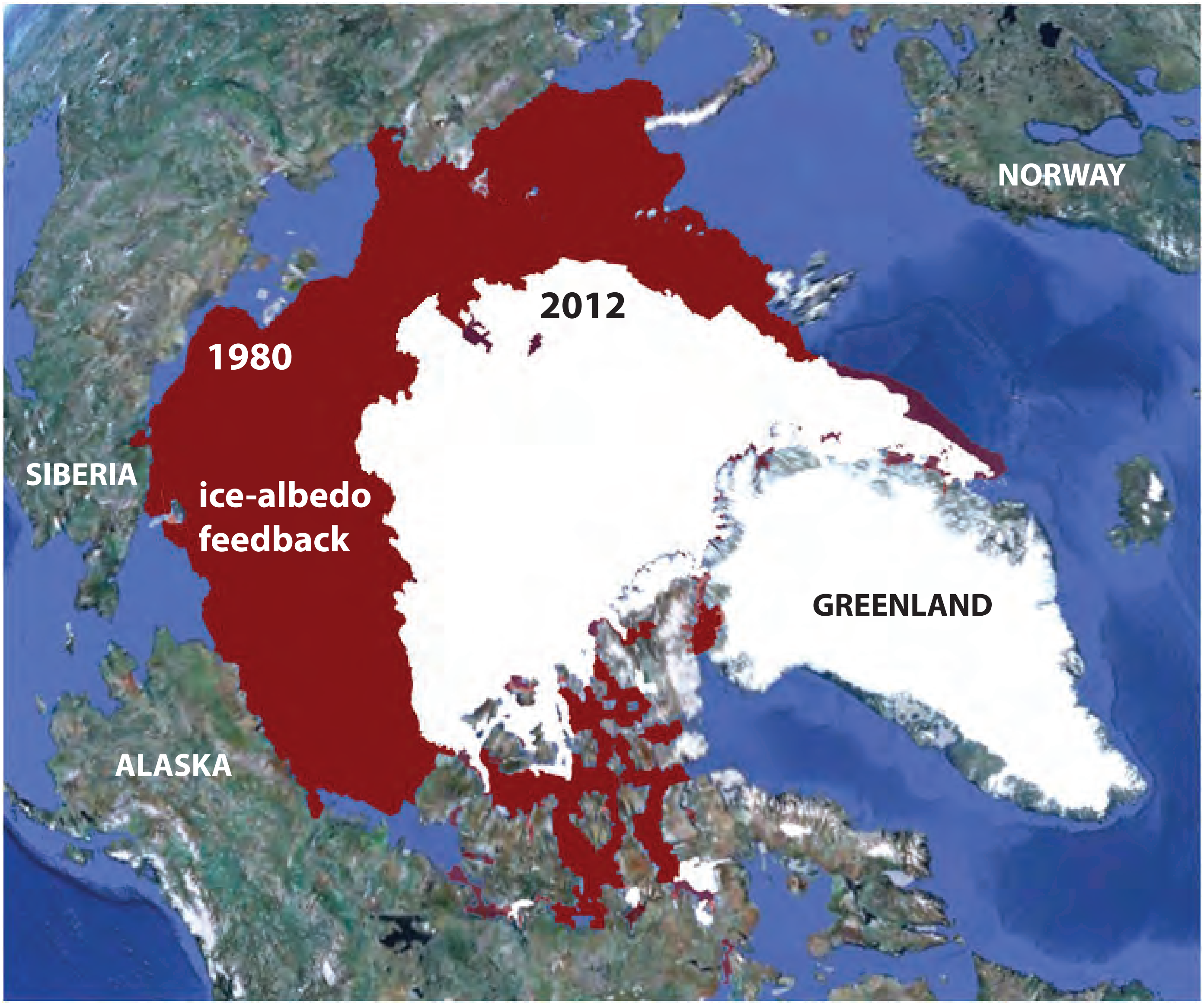
$$\textbf{\textit{albedo}}\ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

the summer Arctic sea ice pack is melting



Change in Arctic Sea Ice Extent

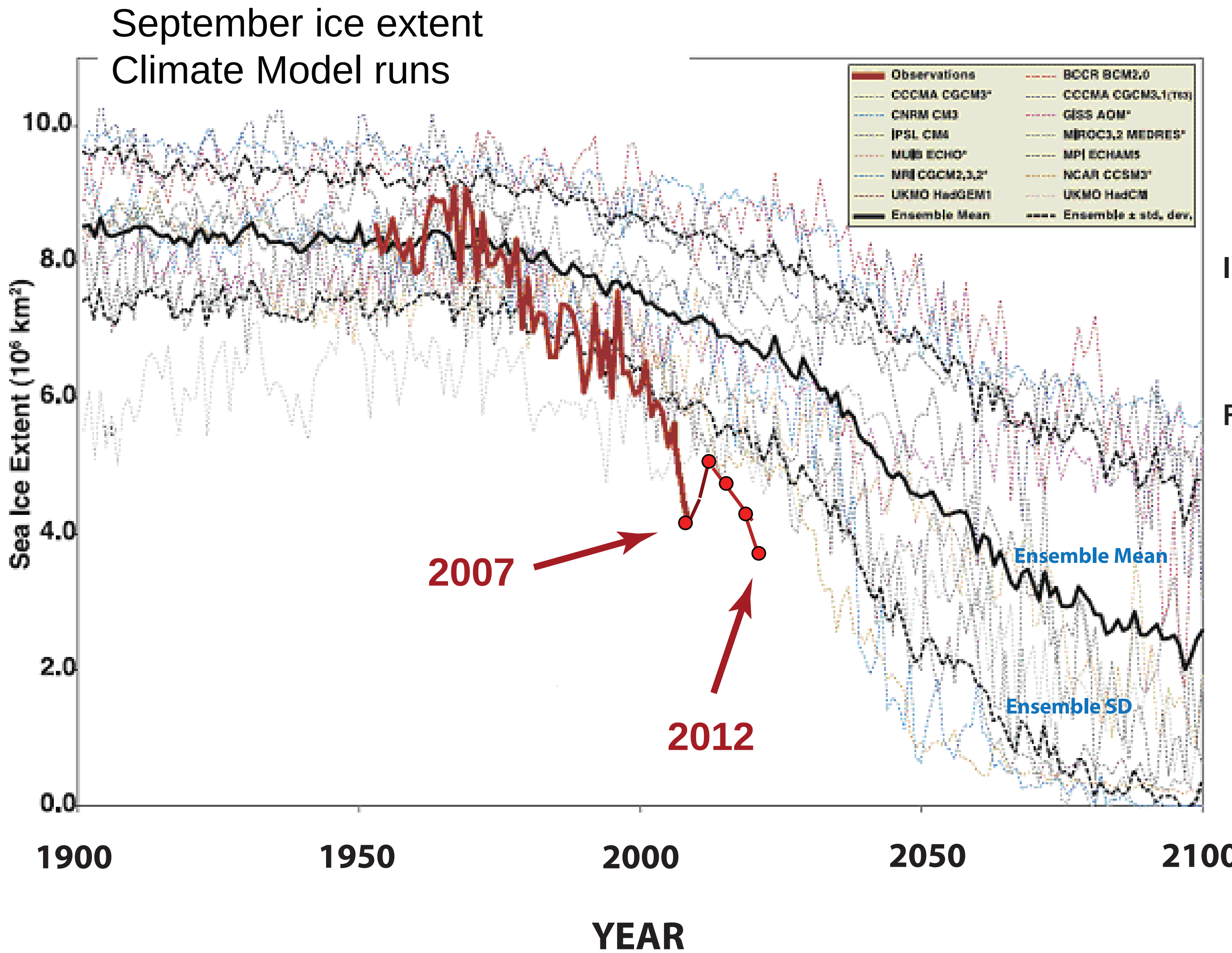
September 1980 -- 7.8 million square kilometers
September 2012 -- 3.4 million square kilometers



Perovich

Arctic sea ice decline - faster than predicted by climate models

Stroeve et al., GRL, 2007



IPCC AR4 Models

Intergovernmental Panel on Climate Change (IPCC)

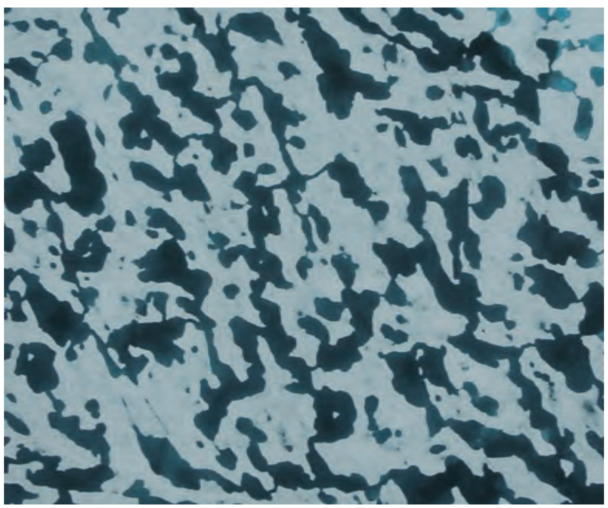
Fourth Assessment AR4, 2007

challenge

represent sea ice more rigorously in climate models

account for key processes

such as melt pond evolution



Impact of melt ponds on Arctic sea ice simulations from 1990 to 2007

Flocco, Schroeder, Feltham, Hunke, *JGR Oceans* 2012

For simulations with ponds September ice volume is nearly 40% lower.

... and other sub-grid scale structures and processes

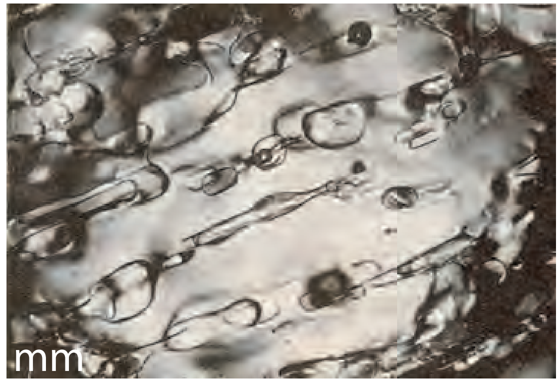
linkage of scales

sea ice is a multiscale composite

displaying structure over 10 orders of magnitude

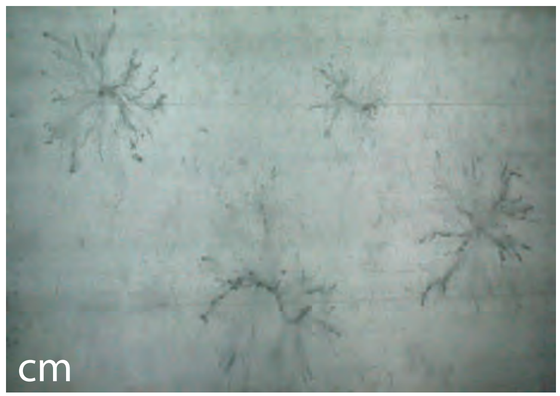
0.1 millimeter

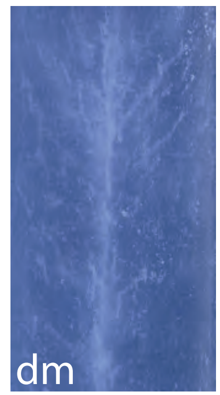
1 meter



brine inclusions

polycrystals

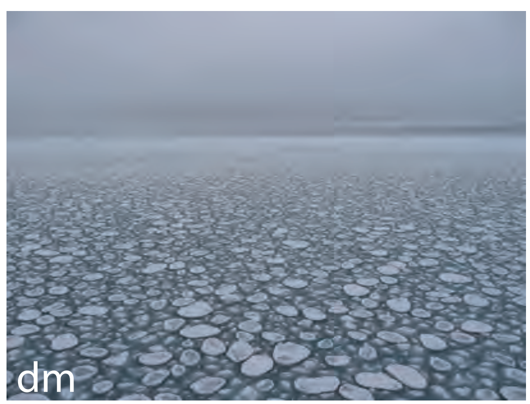


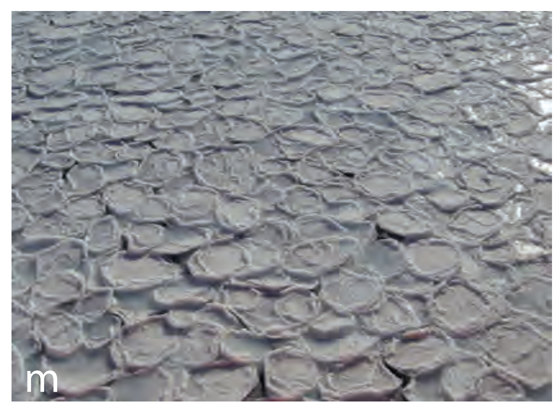


horizontal

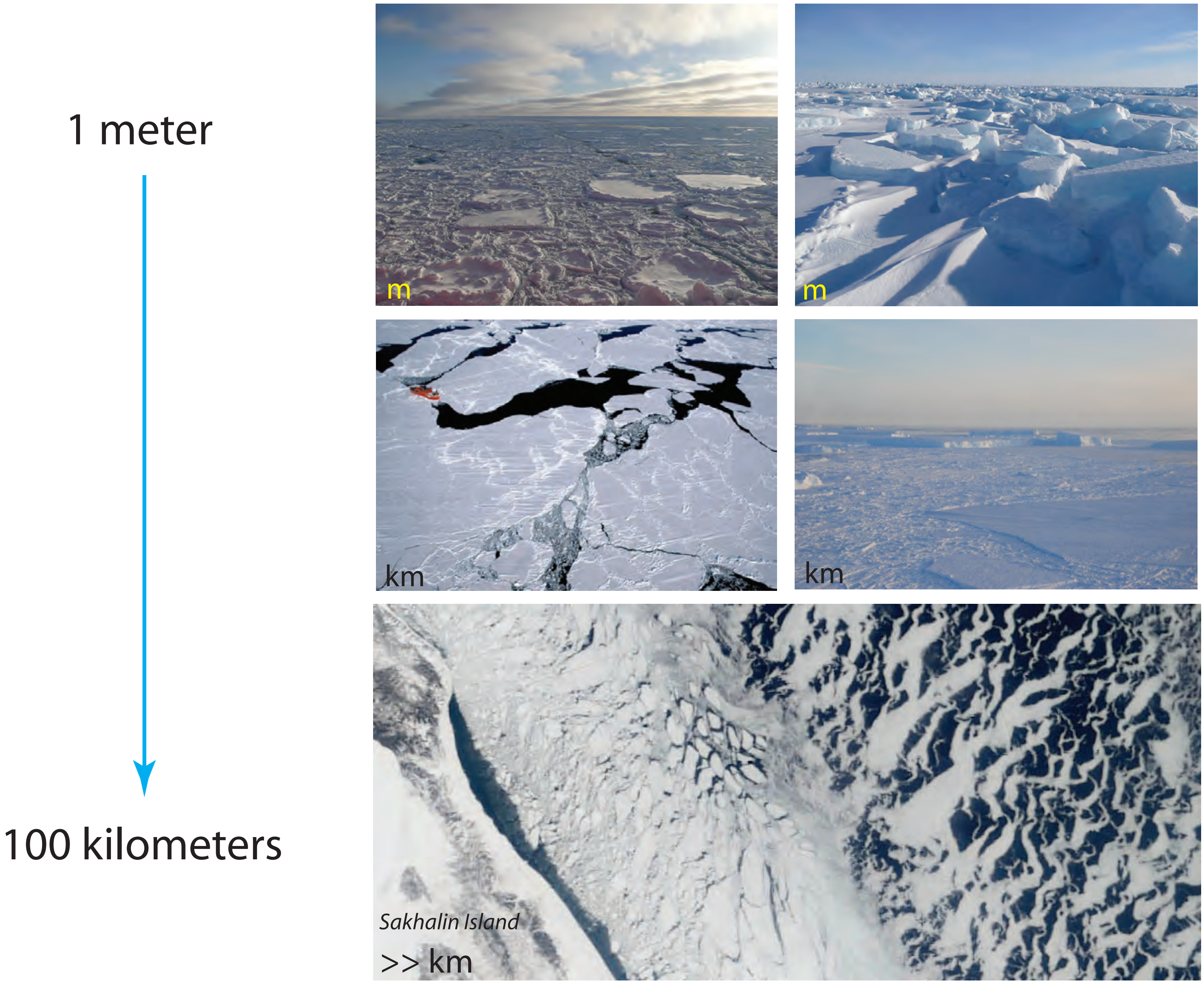
vertical

brine channels





pancake ice


1 meter
100 kilometers
m
m
km
km
Sakhalin Island
>> km

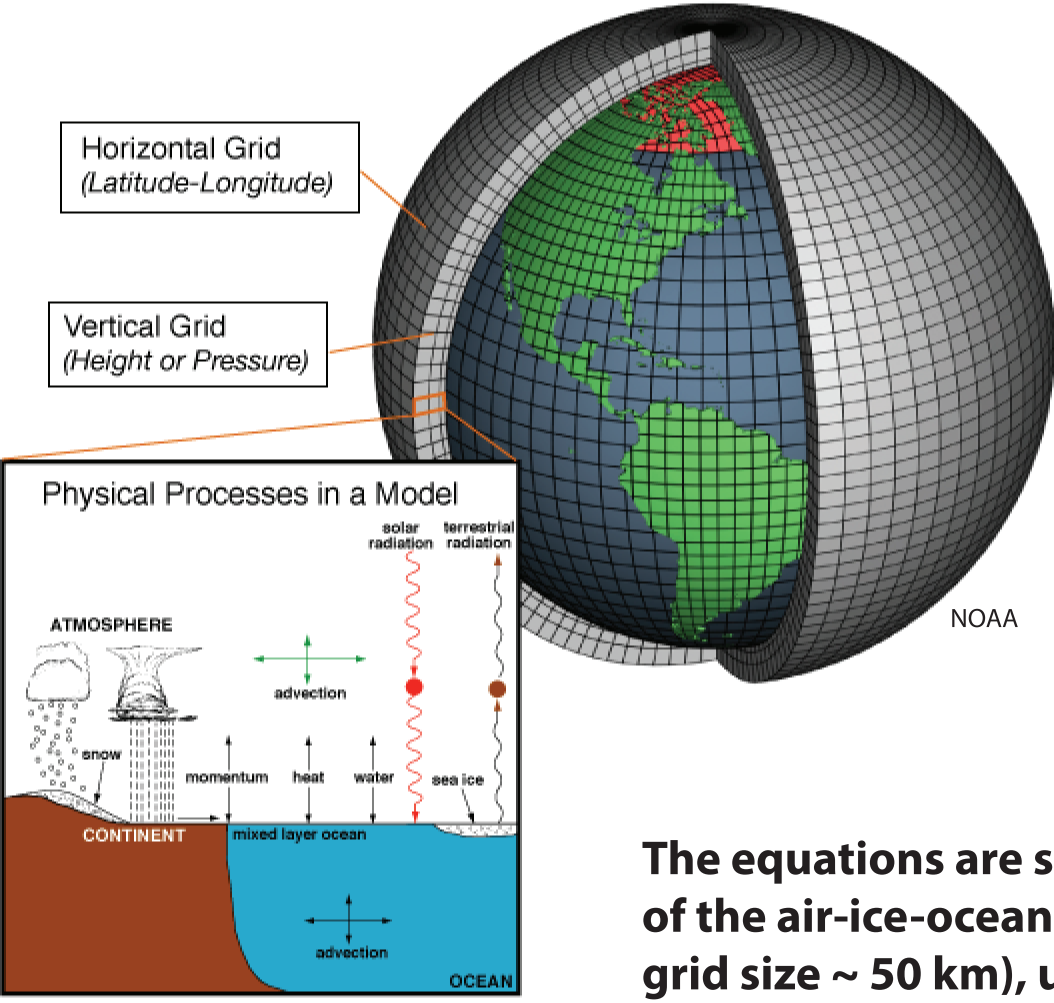
What is this talk about?

Using the mathematics of composite materials and statistical physics to study sea ice structures and processes ... to improve projections of climate change.

A tour of Herglotz functions and how they arise in the study of composites, and sea ice in particular

1. Fluid flow through sea ice, percolation
2. Homogenization for two phase composites
   remote sensing, inversion, spectral measures
3. Stieltjes representations for advection diffusion,
   polycrystals, ocean waves in the marginal ice zone
4. Herglotz functions and the Ising model
5. Multiphase media and the polydisc
6. Arctic and Antarctic field experiments

Global Climate Models



Climate models are systems of partial differential equations (PDE) derived from the basic laws of physics, chemistry, and fluid motion.

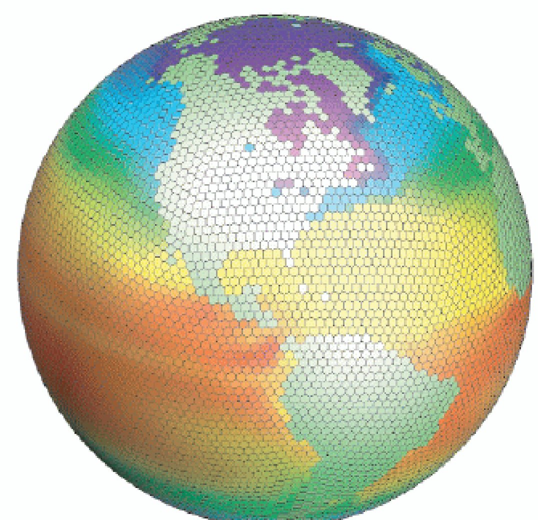
They describe the state of the ocean, ice, atmosphere, land, and their interactions.

The equations are solved on 3-dimensional grids of the air-ice-ocean-land system (with horizontal grid size ~ 50 km), using very powerful computers.

key challenge :

incorporating sub - grid scale processes

linkage of scales



Randall *et al.*, 2002

sea ice components of GCM's

What are the key ingredients -- or ***governing equations*** that need to be solved on grids using powerful computers?

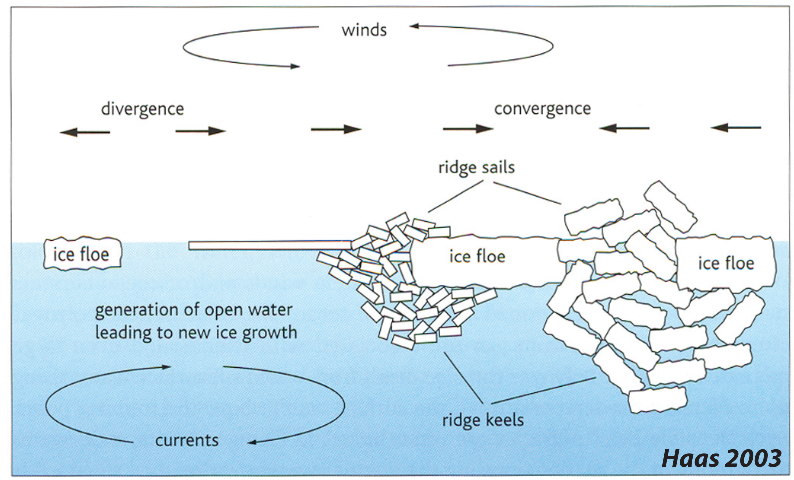
1. Ice thickness distribution $g(x, y, h, t)$ evolution equation

dynamics + ***thermodynamics***

(Thorndike *et al.* 1975)

$$\frac{Dg}{Dt} = -g\nabla \cdot \mathbf{u} + \Psi(g) - \frac{\partial}{\partial h}(fg) + \mathcal{L}$$

nonlinear PDE incorporating ice velocity field
ice growth and melting
mechanical redistribution
- ridging and opening



2. Conservation of momentum, stress *vs.* strain relation (Hibler 1979)

$$m\frac{D\mathbf{u}}{Dt} = -mf\mathbf{k}\times\mathbf{u} + \boldsymbol{\tau}_a + \boldsymbol{\tau}_o - mg\nabla H + \mathbf{F}$$

F = ma for sea ice ***dynamics***

3. Heat equation of sea ice and snow

(Maykut and Untersteiner 1971)

thermodynamics

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla\cdot k(T)\,\nabla T$$

+ *balance of radiative and thermal fluxes on interfaces*

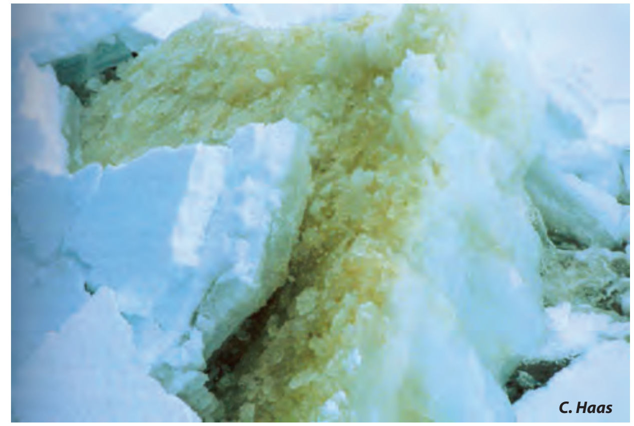
sea ice microphysics

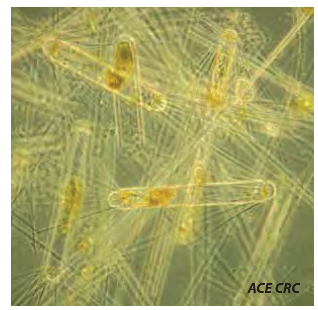
fluid transport

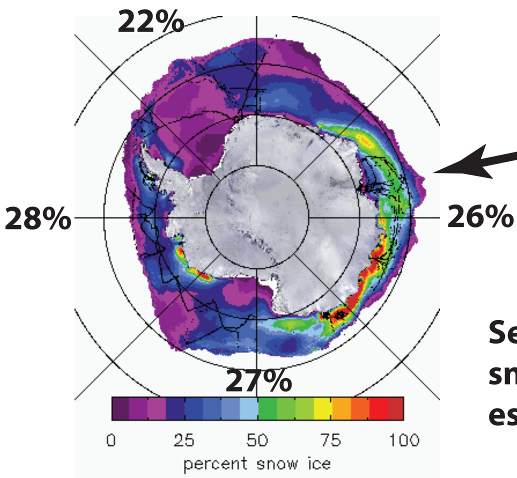
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities







T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

- evolution of salinity profiles

- ocean-ice-air exchanges of heat, CO_2

sea ice ecosystem

sea ice algae
support life in the polar oceans

fluid permeability *k* of a porous medium

porous concrete

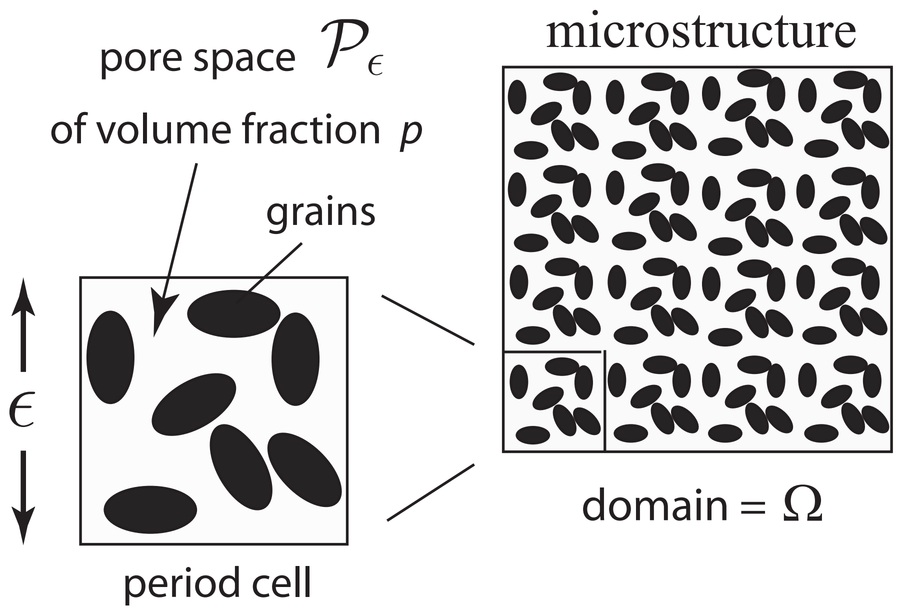
how much water gets through the sample per unit time?

HOMOGENIZATION

mathematics for analyzing effective behavior of heterogeneous systems

HOMOGENIZE as $\epsilon \to 0$

Stokes equations for fluid velocity $\mathbf{v}^\epsilon$, pressure p^ϵ, force $\mathbf{f}$:



$$\nabla p^\epsilon - \epsilon^2 \eta \Delta \mathbf{v}^\epsilon = \mathbf{f}, \qquad x \in \mathcal{P}_\epsilon$$

$$\nabla \cdot \mathbf{v}^\epsilon = 0, \qquad x \in \mathcal{P}_\epsilon$$

$$\mathbf{v}^\epsilon = 0, \qquad x \in \partial \mathcal{P}_\epsilon$$

$$\eta = \text{fluid viscosity}$$

via two-scale expansion

MACROSCOPIC EQUATIONS $\mathbf{v}^\epsilon \longrightarrow \mathbf{v}\,,\ p^\epsilon \longrightarrow p$ as $\epsilon \longrightarrow 0$

Darcy's law

($\mathrm{f} = 0$)

$$\mathbf{v} = -\frac{1}{\eta}\,\mathbf{k}\,\nabla p, \qquad x \in \Omega$$

$$\nabla \cdot \mathbf{v} = 0, \qquad x \in \Omega$$

$\mathbf{k}(x)$ = effective fluid permeability tensor

[Keller '80, Tartar '80, Sanchez-Palencia '80, J. L. Lions '81, Allaire '89, '91,'97]

Darcy's Law for slow viscous flow in a porous medium

averaged fluid velocity

pressure gradient

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\,\nabla p$$

viscosity

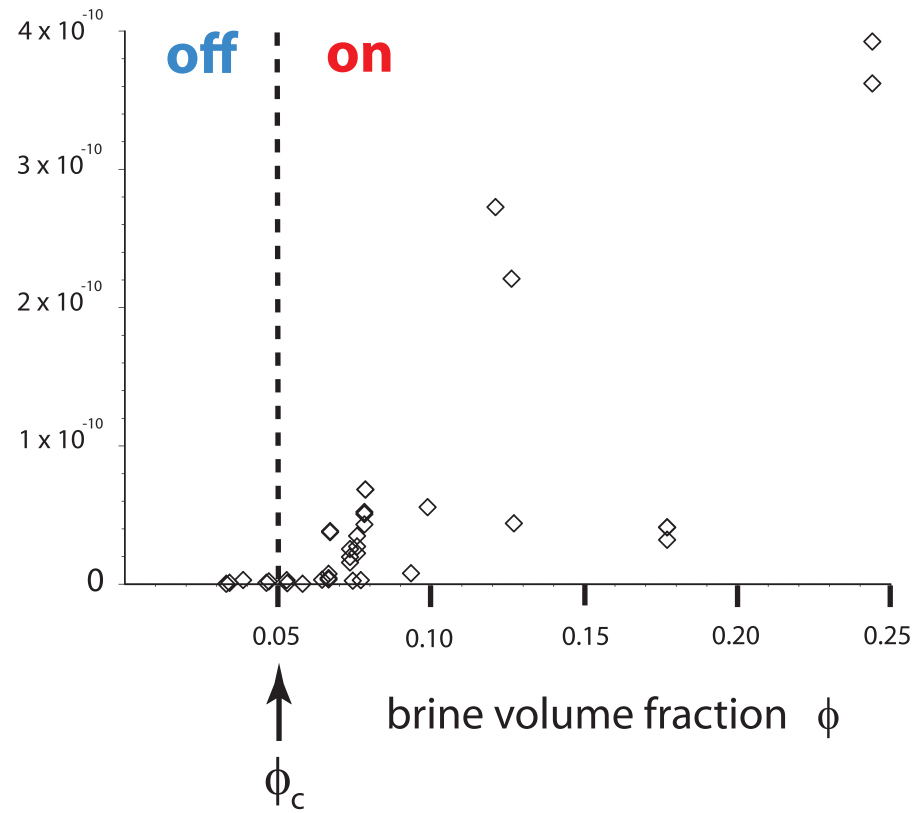
$\mathbf{k}$ = fluid permeability tensor

Critical behavior of fluid transport in sea ice

impermeable permeable

Arctic field data

vertical fluid permeability k (m^2)



on - off switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ \mathrm{C},\ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

sea ice algal communities

D. Thomas 2004

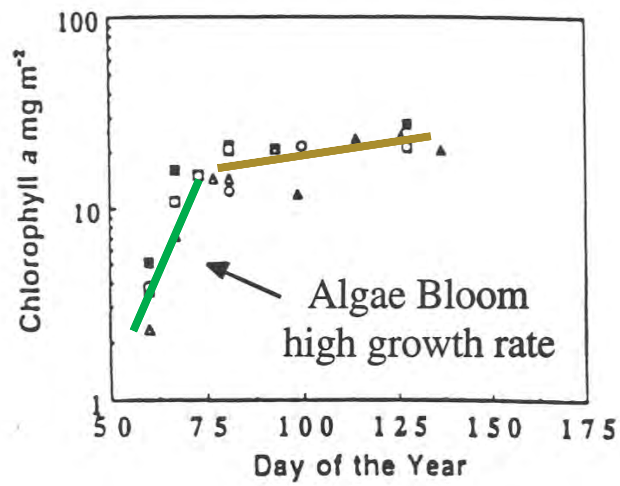
nutrient replenishment controlled by ice permeability

biological activity turns on or off according to
rule of fives

Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

critical behavior of microbial activity

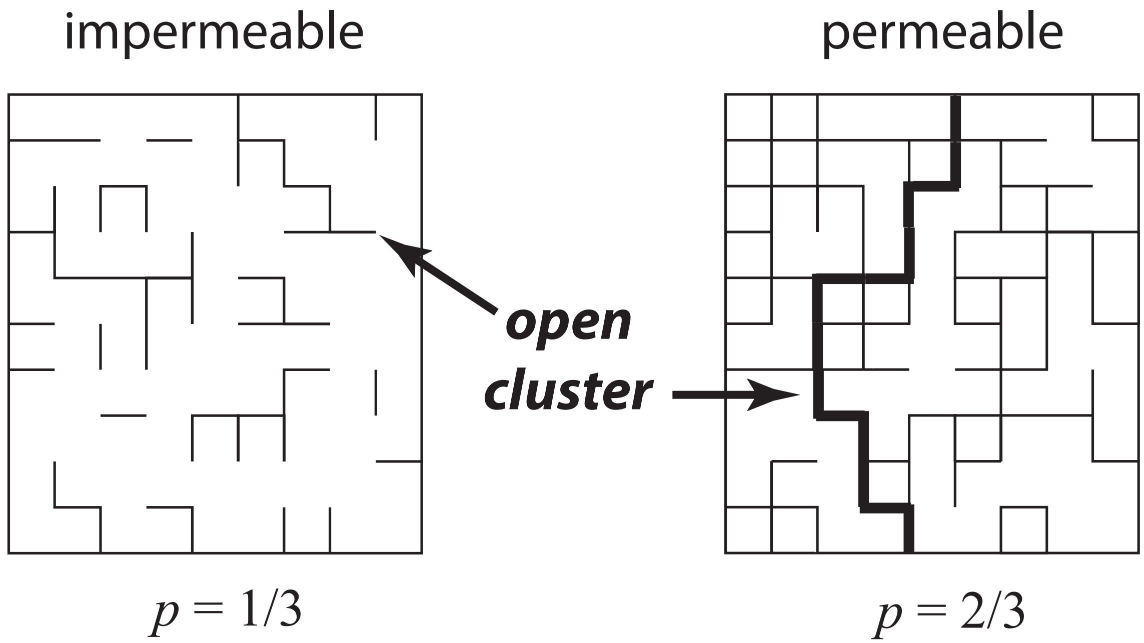


Convection-fueled algae bloom
Ice Station Weddell

Why is the rule of fives true?

percolation theory

probabilistic theory of connectedness



bond → **open** *with probability* p
closed *with probability* $1-p$

percolation threshold

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

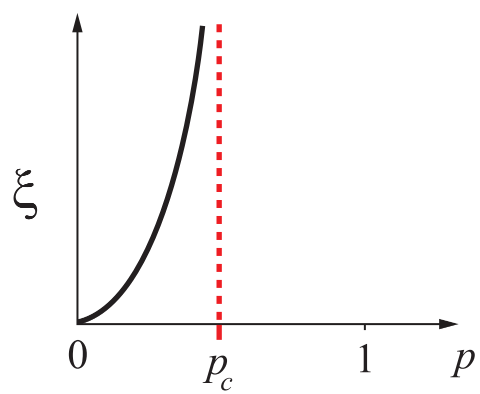
smallest p for which there is an infinite open cluster

order parameters in percolation theory

geometry

correlation length

characteristic scale
of connectedness

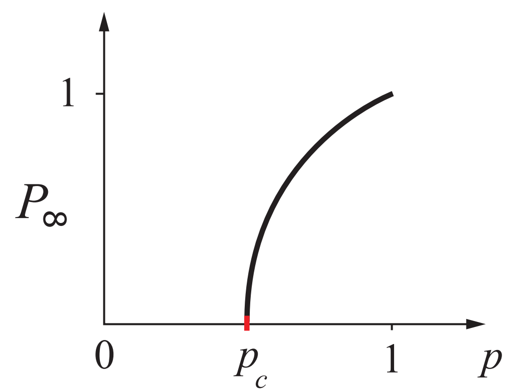


$$\xi(p) \sim |p - p_c|^{-\nu}$$

$$p \to p_c$$

infinite cluster density

probability the origin
belongs to infinte cluster

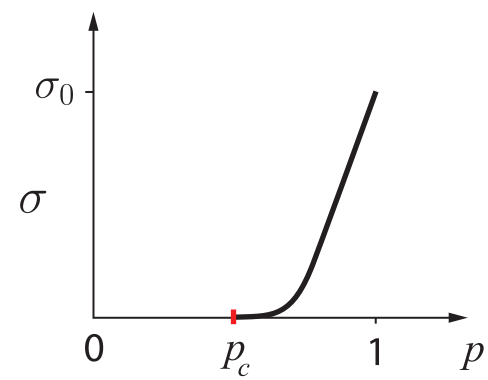


$$P_\infty(p) \sim (p - p_c)^{\beta}$$

$$p \to p_c^+$$

transport

effective conductivity
or fluid permeability



$$\sigma(p) \sim \sigma_0 \, (p - p_c)^t$$

$$p \to p_c^+$$

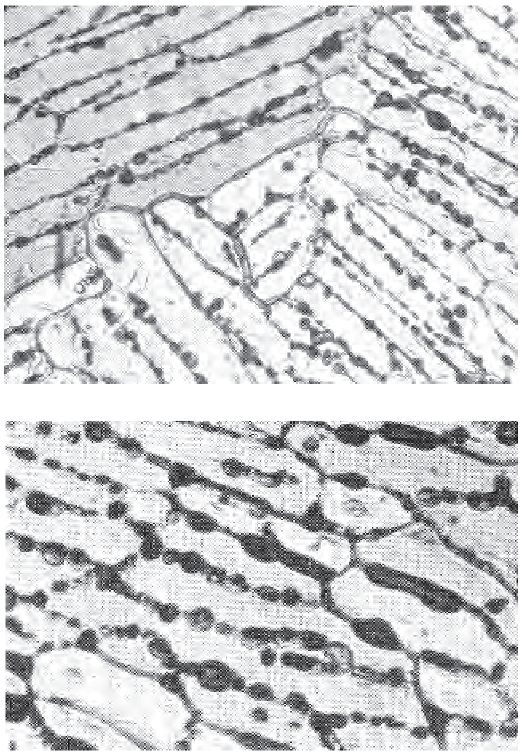
UNIVERSAL critical exponents for lattices -- depend only on dimension

$1 \leq t \leq 2$ (for idealized model), Golden, *Phys. Rev. Lett.* 1990 **;** *Comm. Math. Phys.* 1992

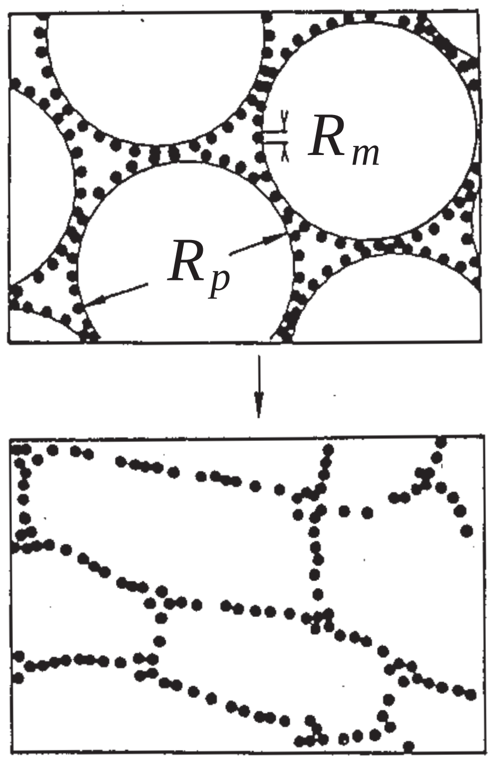
non-universal behavior in continuum

Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

$\phi_c \approx 5\ \%$ Golden, Ackley, Lytle, *Science*, 1998



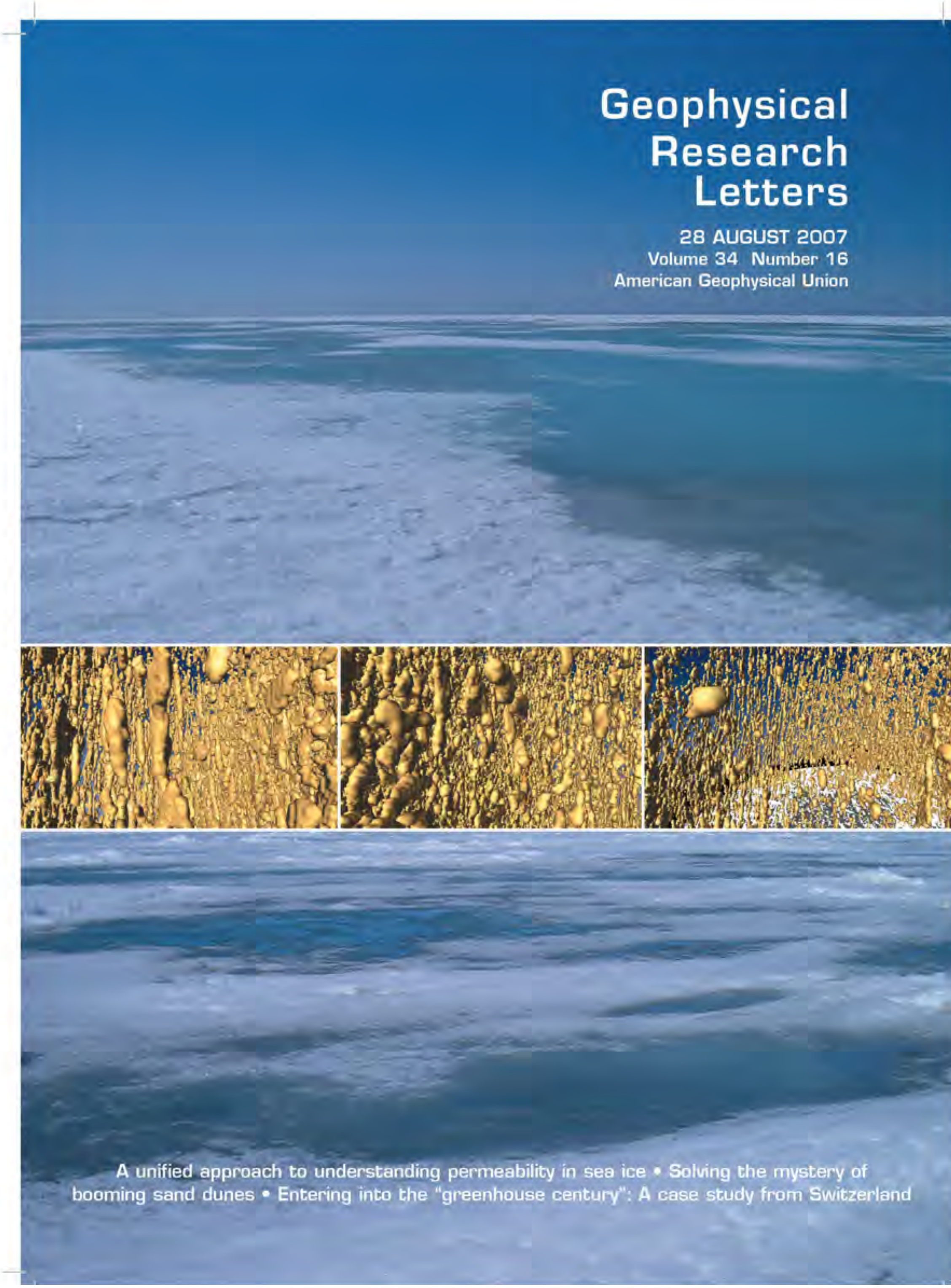
sea ice



compressed powder

radar absorbing composite

sea ice is radar absorbing



micro-scale

controls

macro-scale processes

rigorous bounds
percolation theory
hierarchical model
network model

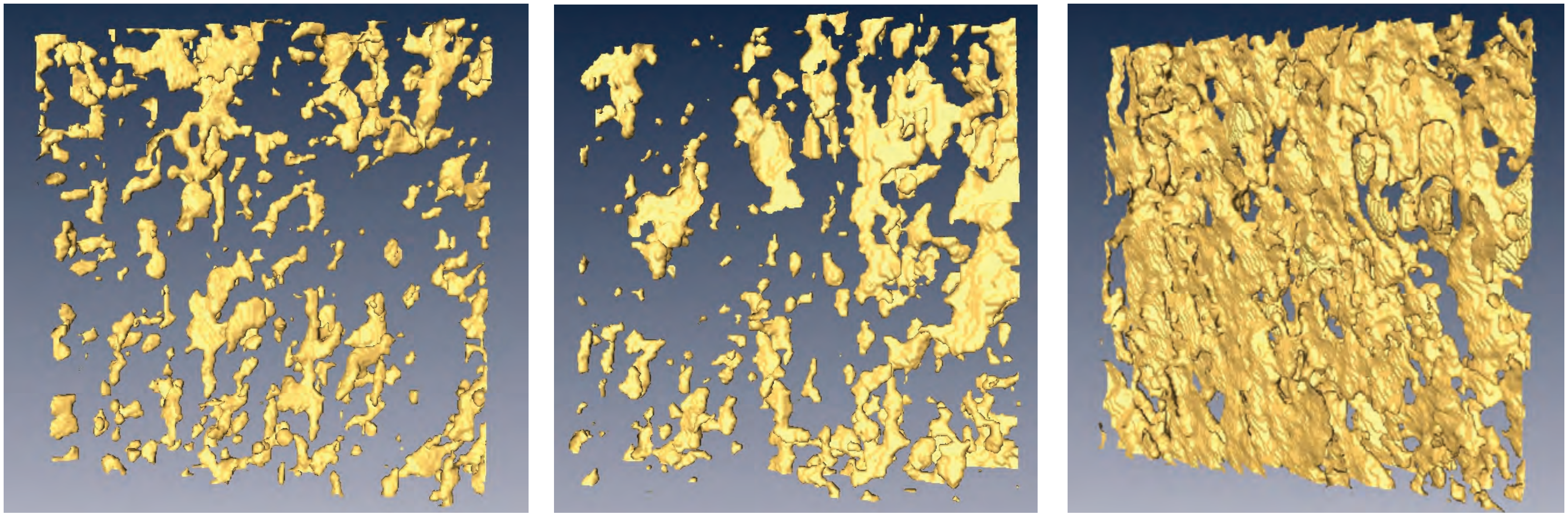
field data

X-ray tomography for brine inclusions

unprecedented look at thermal evolution of brine phase and its connectivity

brine connectivity (over cm scale)

8 x 8 x 2 mm



-15 °C, $\phi = 0.033$ -6 °C, $\phi = 0.075$ -3 °C, $\phi = 0.143$

X-ray tomography confirms percolation threshold

3-D images
pores and throats → 3-D graph
nodes and edges

analyze graph connectivity as function of temperature and sample size

- ***use finite size scaling techniques to confirm rule of fives***
- ***order parameter data from a natural material***

Pringle, Miner, Eicken, Golden, J. Geophys. Res. 2009

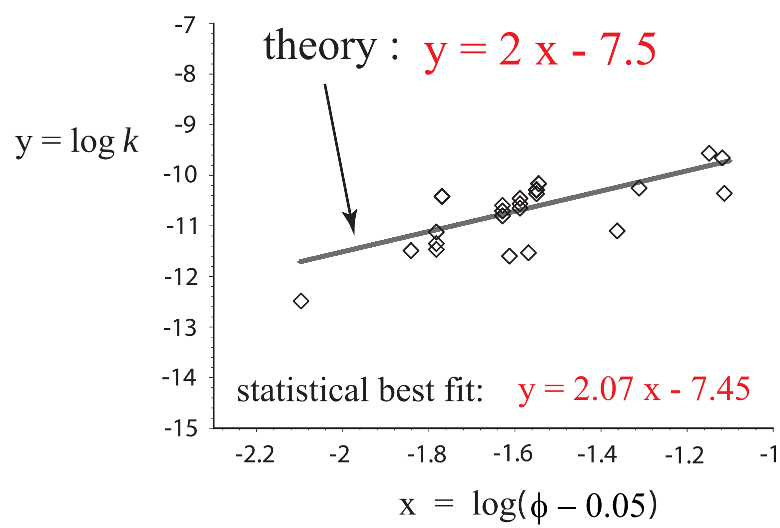
lattice and continuum percolation theories yield:

$$k(\phi) = k_0 (\phi - 0.05)^2$$

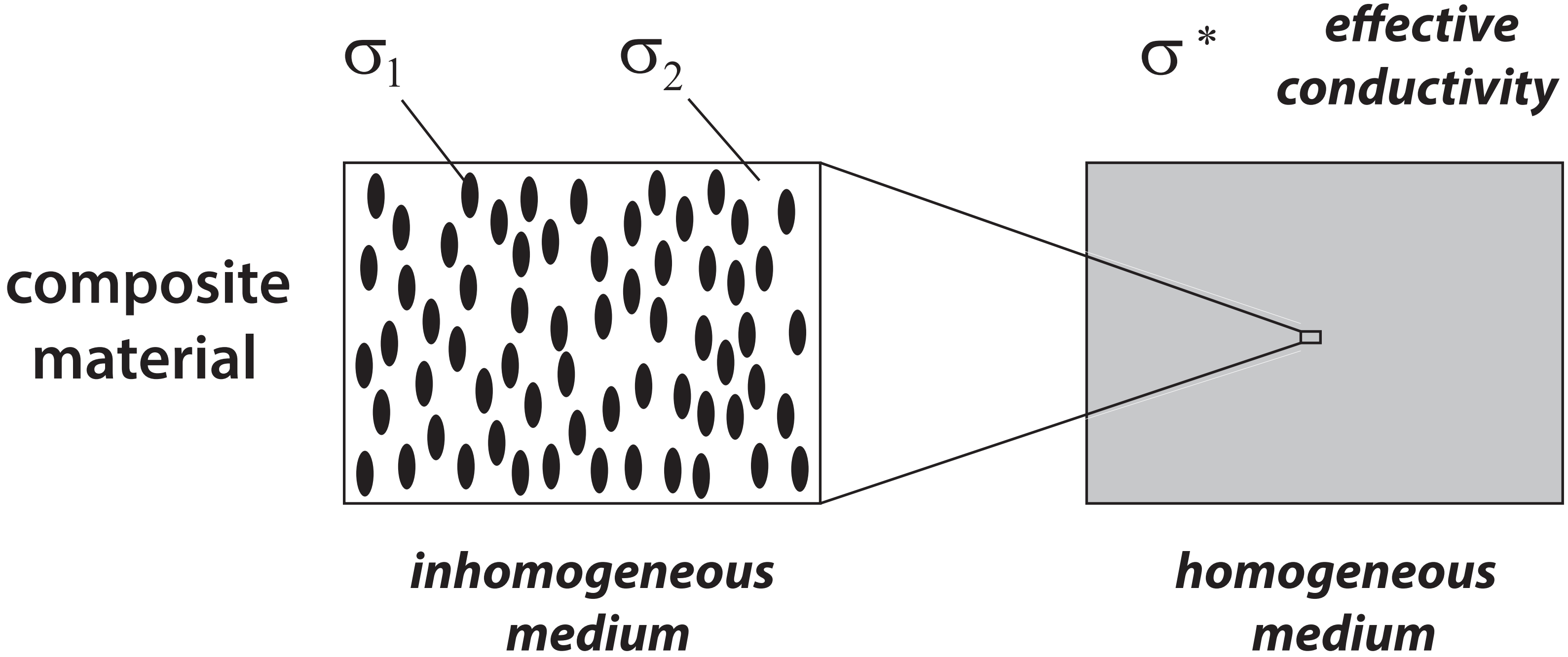
critical exponent t

$$k_0 = 3 \times 10^{-8}\ \mathrm{m}^2$$

- exponent is UNIVERSAL lattice value $t \approx 2.0$
- sedimentary rocks like sandstones also exhibit universality
- critical path analysis -- developed for electronic hopping conduction -- yields scaling factor k_0



HOMOGENIZATION



find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

Remote sensing of sea ice

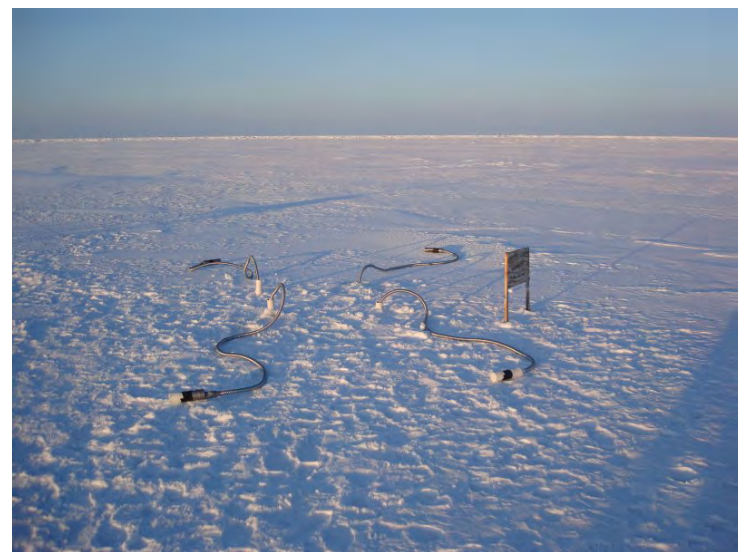
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

$$\varepsilon^*$$

effective complex permittivity (dielectric constant, conductivity)

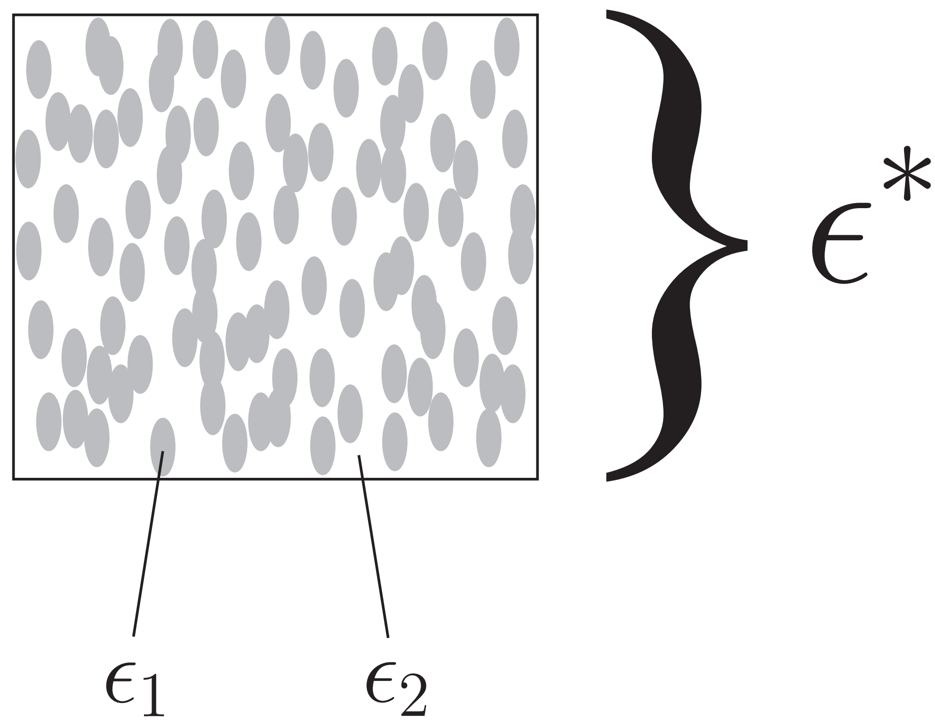


brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit

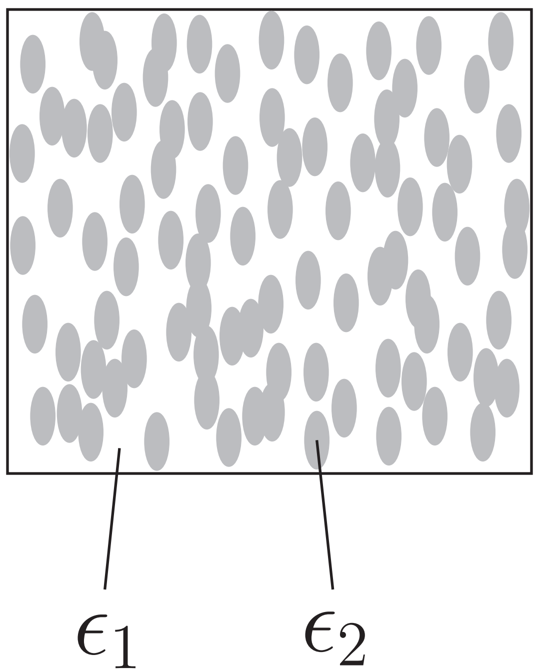
What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

HOMOGENIZATION



p_1 , p_2 = volume fractions of brine and ice

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry} \right)$$

ocean swells propagating through a vast field of pancake ice

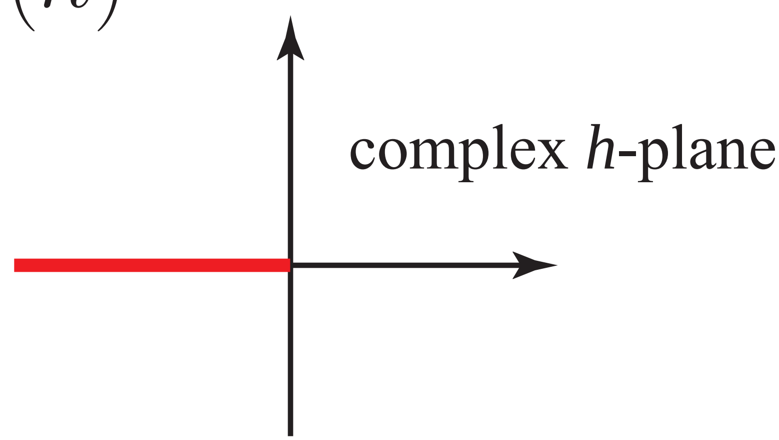
HOMOGENIZATION: long wave sees an effective medium, not individual floes

Analytic continuation method for bounding complex ϵ^*

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

$$m(h) = \frac{\epsilon^*}{\epsilon_2}\left(\frac{\epsilon_1}{\epsilon_2}\right) \qquad h = \frac{\epsilon_1}{\epsilon_2}$$

Exploit analytic properties of $m(h)$



- $m(h)$ analytic off negative real axis

- $m(h)$: UHP $\longrightarrow$ UHP

Theory of Effective Electromagnetic Behavior of Composites

analytic continuation method

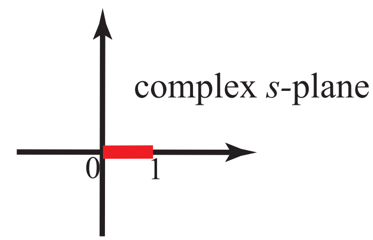
Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)
Theory of Composites, Milton (2002)

composite geometry
(spectral measure μ) → ε^*

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

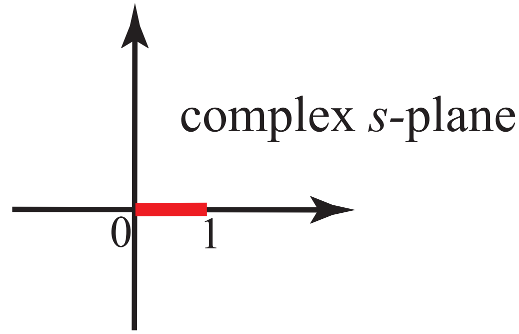
ε^* → ***composite geometry***
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

Stieltjes integral representation

separation of geometry from parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass $= p_1$
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = (s + \Gamma\chi)^{-1} e_k$$

Golden and Papanicolaou, Comm. Math. Phys. 1983

using the Stieltjes integral representation to obtain bounds

"linear programming" Golden and Papanicolaou, *CMP* 1983

M_1 = **the set of positive Borel measures on [0,1], compact, convex**

$$F_s(\mu):\ M_1 \longrightarrow \mathbb{C}$$ **linear functional**

extremal values (bounds) are images of extreme points of M_1

Dirac point measures $\dfrac{\mu_0}{s - z^*}$

---

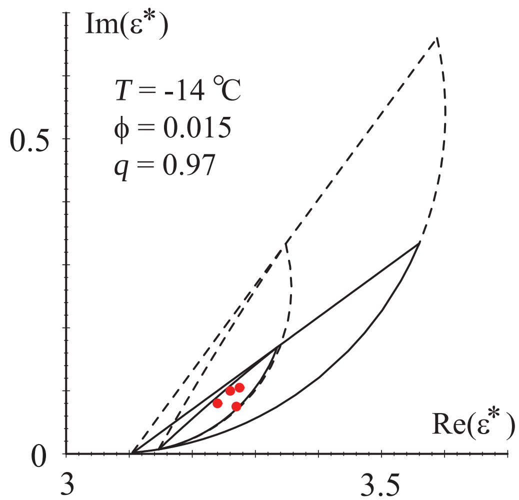
higher order bounds -- iterated fractional linear transformations

$$F_1(s) = \frac{1}{\mu_0} - \frac{1}{sF(s)}$$

Baker 1969
Milton 1981
Bergman 1982
Felderhof 1984
Golden 1986

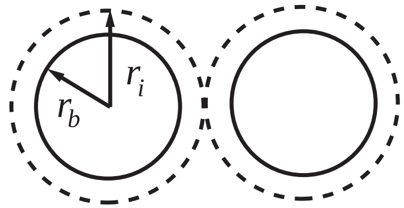
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle



$$q = r_b / r_i$$

$$0 < q < 1$$

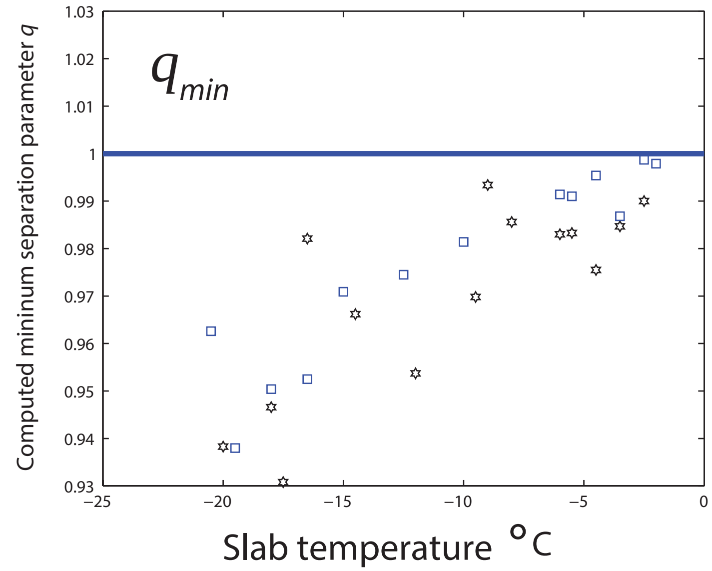
Golden 1995, 1997

Bruno 1991

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inverse bounds



inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε*

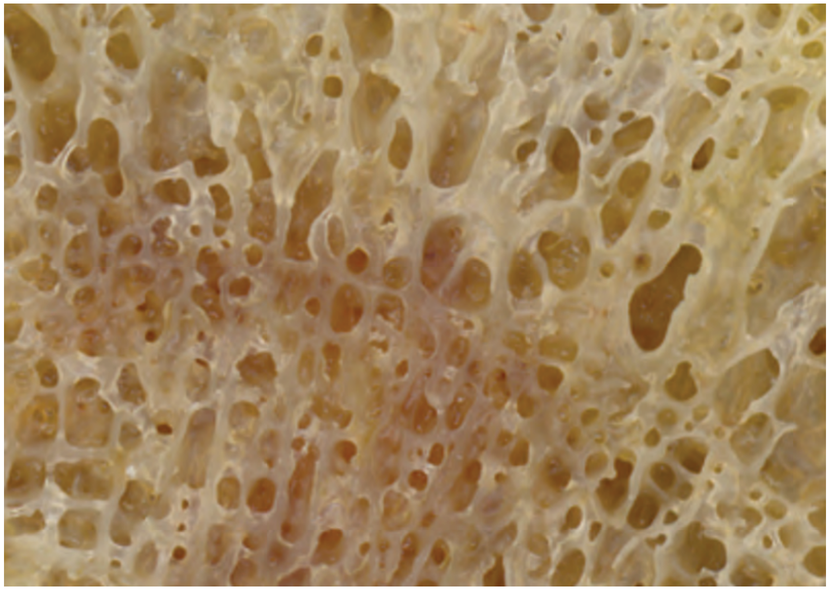
rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

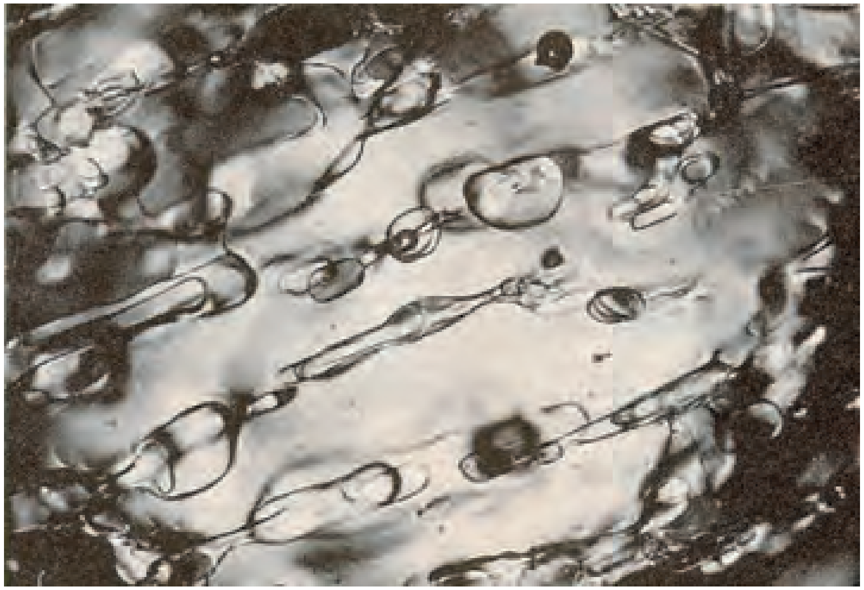
Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

the math doesn't care if it's sea ice or bone!

HUMAN BONE



SEA ICE



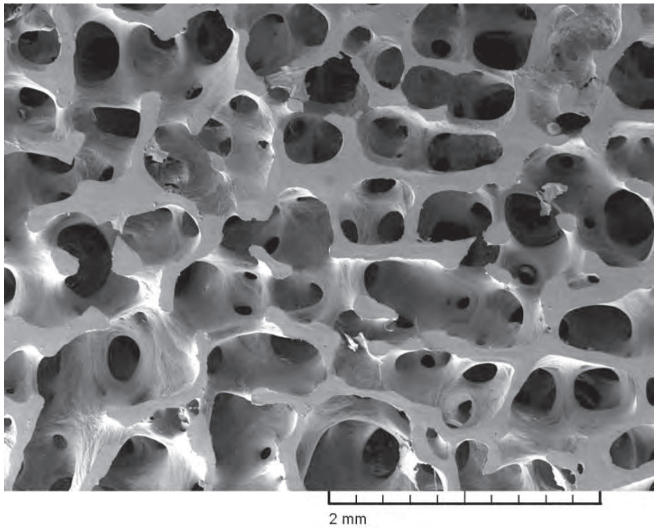
apply spectral measure analysis of brine connectivity and spectral inversion to electromagnetic monitoring osteoporosis

Golden, Murphy, Cherkaev, J. Biomechanics 2011

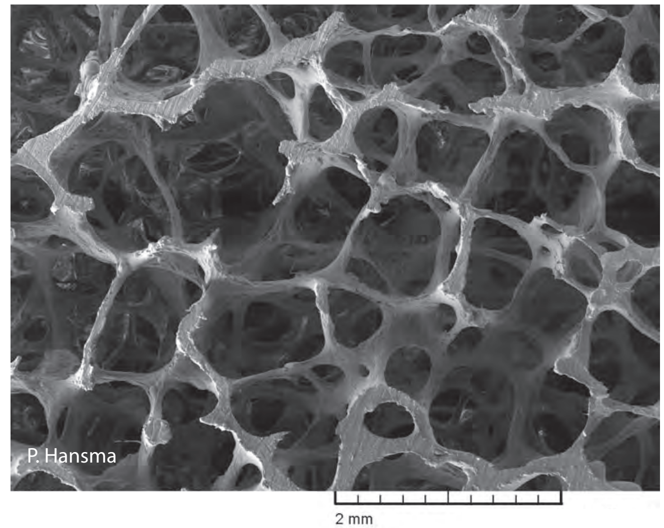
spectral characterization of porous microstructures in bone

Golden, Murphy, Cherkaev, J. Biomechanics 2011

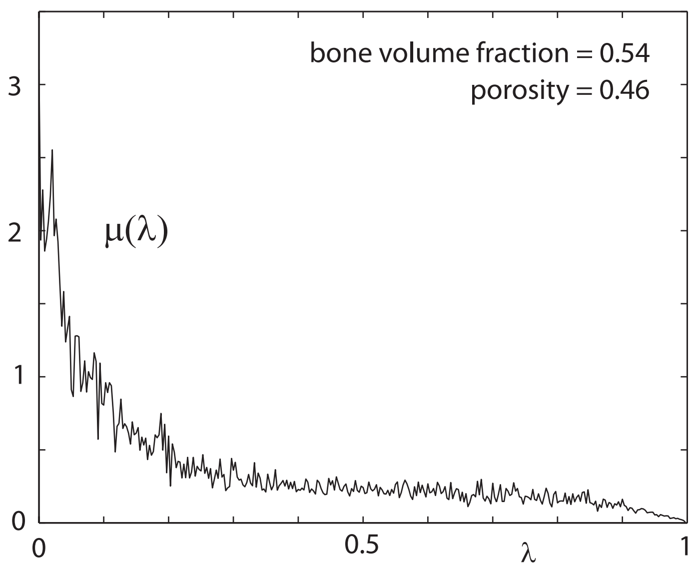
(a) young healthy trabecular bone



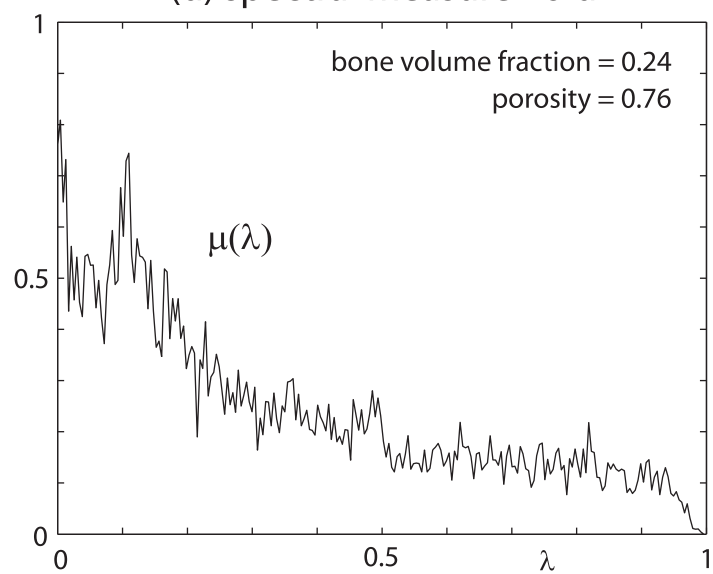
(b) old osteoporotic trabecular bone



(c) spectral measure - young



(d) spectral measure - old



+

reconstruction of spectral measures from complex permittivity data

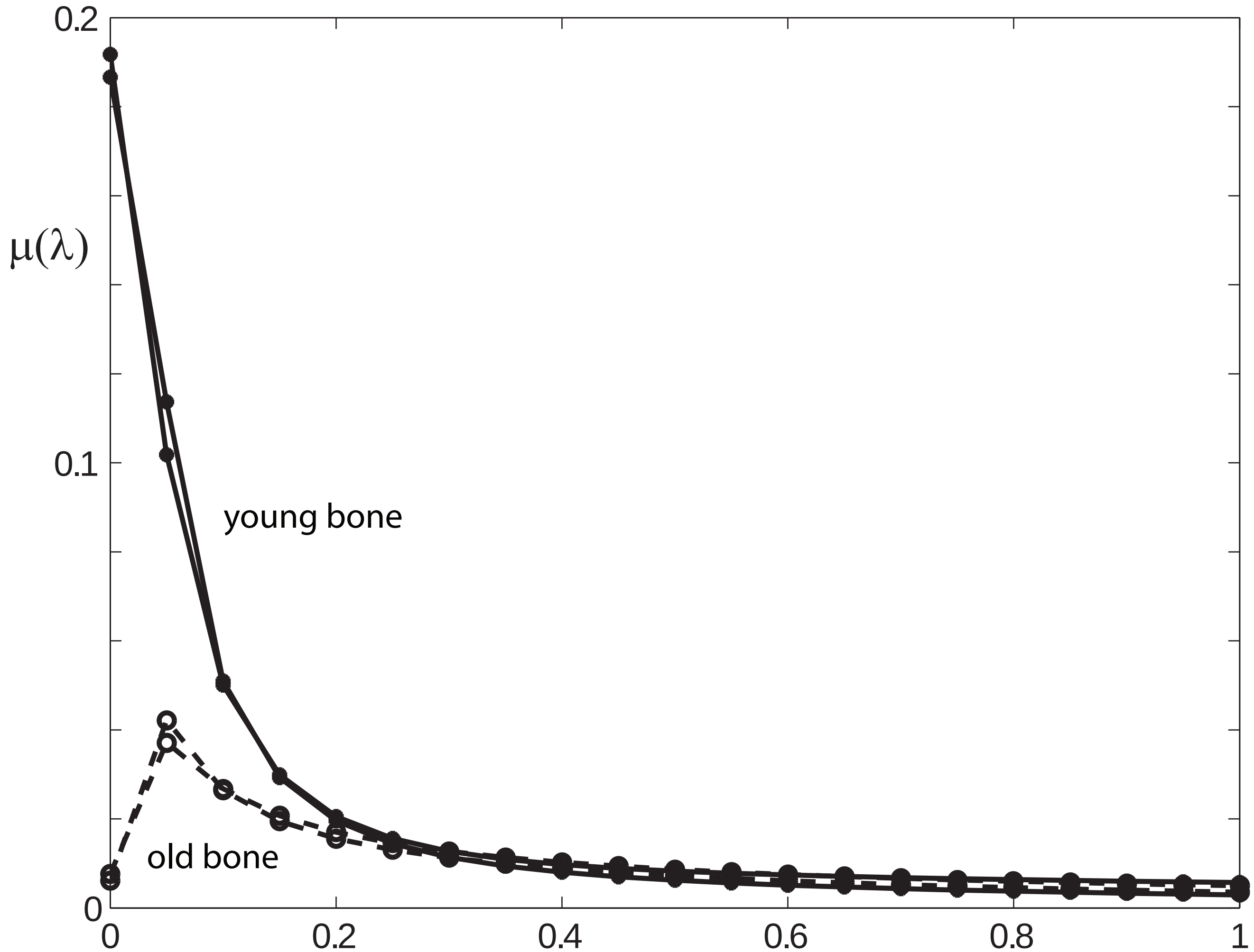
using regularized inversion scheme

EM monitoring of osteoporosis

loss of bone connectivity

the math doesn't care if it's sea ice or bone!

reconstruction of spectral measures from simulated complex permittivity data



regularized inversion scheme

Golden, Murphy, Cherkaev, J. Biomech. 2011

direct calculation of spectral measure

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).

2. The fundamental operator $\chi\Gamma\chi$ becomes a random matrix depending only on the composite geometry.

3. Compute the eigenvalues λ_i and eigenvectors of $\chi\Gamma\chi$ with inner product weights α_i

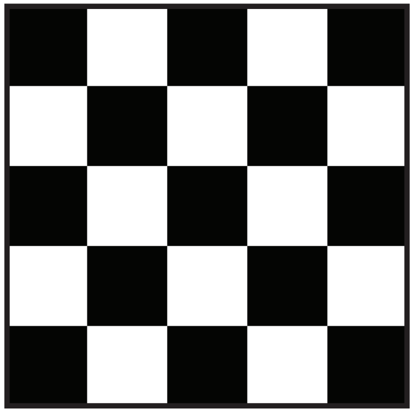
$$\mu(\lambda) = \sum_i \alpha_i \, \delta(\lambda - \lambda_i)$$

Dirac point measure (Dirac delta)

earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Continuum composite

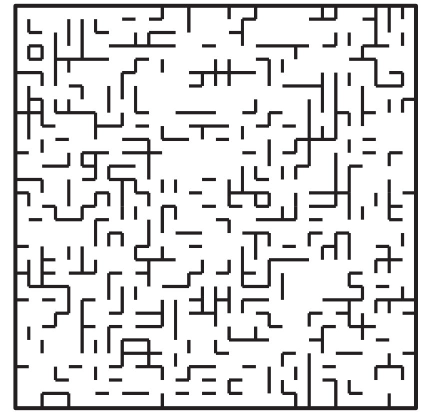


Spectral measures of

$$\chi_1 \Gamma \chi_1$$

Murphy, Hohenegger, Cherkaev, Golden
Comm. Math. Sci. 2015

Discrete composite



Integro-differential projection operator

$$\Gamma = \vec{\nabla}(\Delta^{-1})\vec{\nabla}\cdot$$

Projection matrix

$$\Gamma = \nabla(\nabla^{\mathrm{T}}\nabla)^{-1}\nabla^{\mathrm{T}}$$

Point-wise indicator function

$$\chi_1$$

Diagonal projection matrix

$$\chi_1$$

Resolvent representation of electric field

$$\chi_1 \vec{E} = sE_0(sI - \chi_1\Gamma\chi_1)^{-1}\chi_1\vec{e}_k$$

Series representation of electric field

$$\chi_1 \vec{E} = sE_0 \sum_j \frac{\vec{v}_j \cdot \chi_1 \vec{e}_k}{s - \lambda_j}\, \vec{v}_j$$

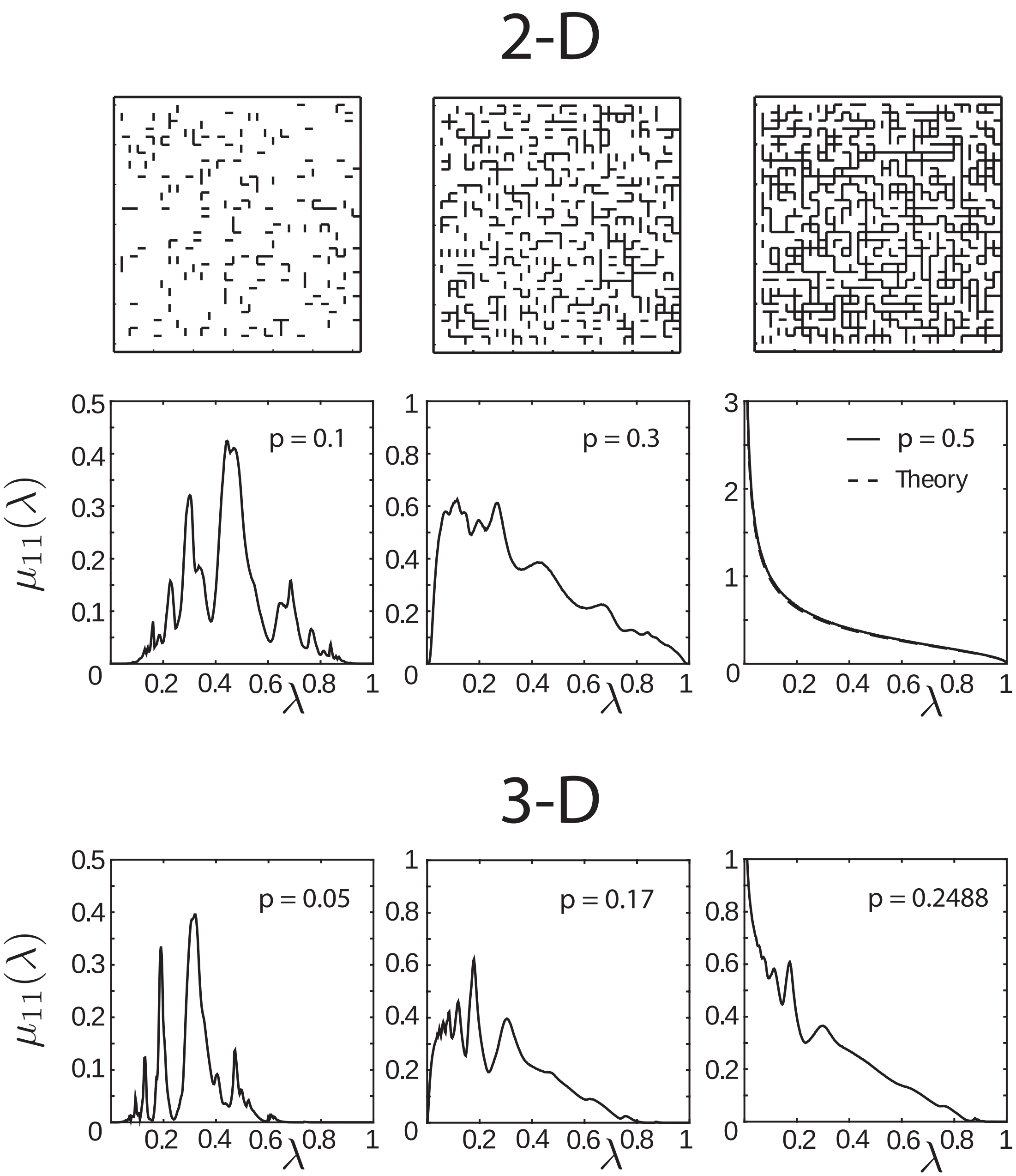
Integral representation

$$F(s) = \int_0^1 \frac{d\mu(\lambda)}{s - \lambda}$$

Series representation

$$F(s) = \sum_j \frac{(\vec{v}_j \cdot \chi_1 \vec{e}_k)^2}{s - \lambda_j}$$

Spectral Measures for Random Resistor Networks



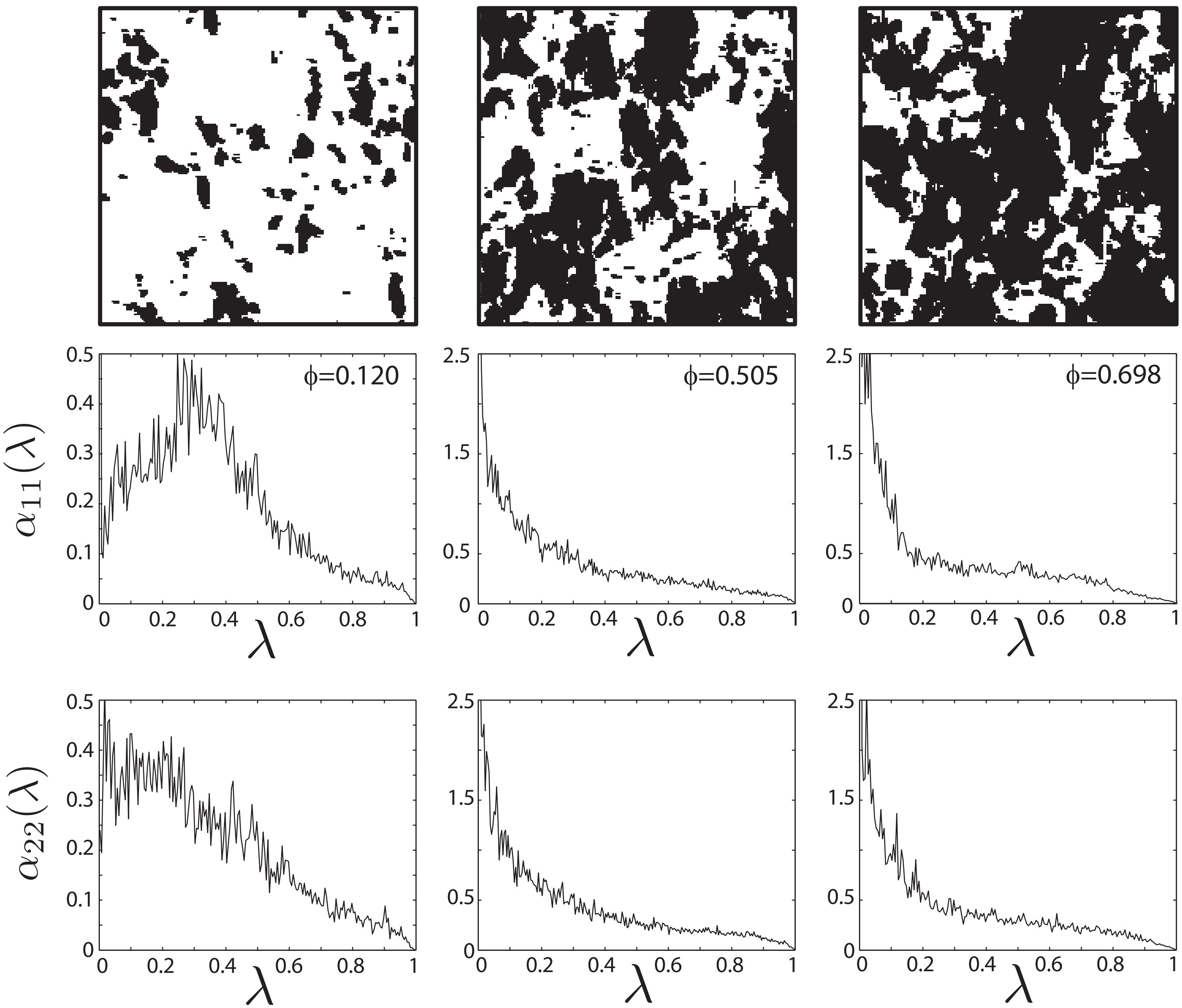
$p_c = 0.5$

$p_c \approx 0.2488$

spectral gaps collapse at the percolation transitions

Murphy and Golden, J. Math. Phys. (2012)

Spectral Measures for Sea Ice Structures: Brine Inclusions

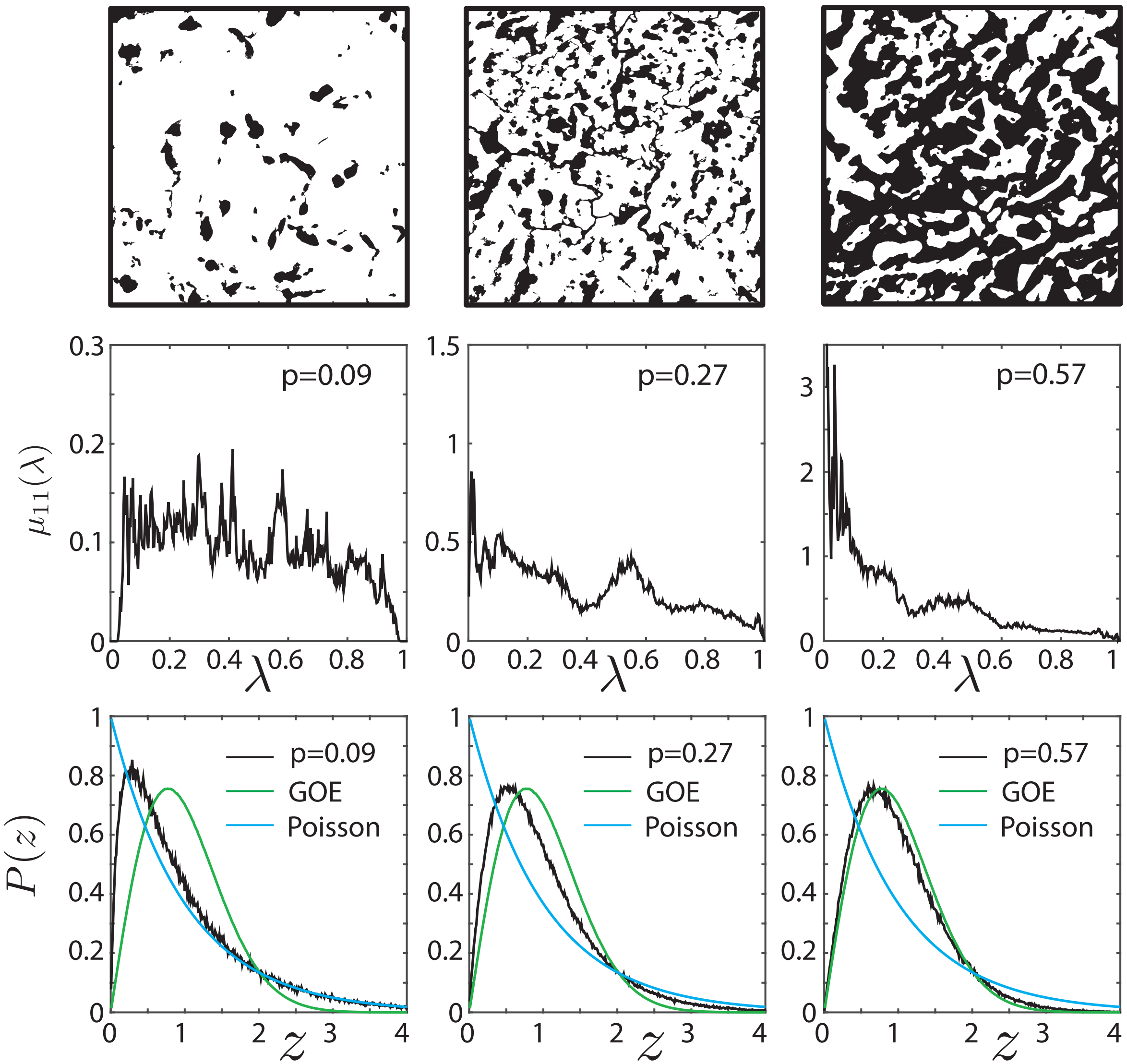


N. B. Murphy, C. Hohenegger, C. S. Sampson, D. K. Perovich, H. Eicken, E. Cherkaev, B. Alali, and K. M. Golden

Spectral computations for Arctic melt ponds

spectral measures

eigenvalue spacing distributions



uncorrelated → **level repulsion**

TRANSITION

Ben Murphy
Elena Cherkaev
Ken Golden
2017

eigenvalue statistics for transport tend toward the UNIVERSAL Wigner-Dyson distribution as the "conducting" phase percolates

Eigenvalue Statistics of Random Matrix Theory

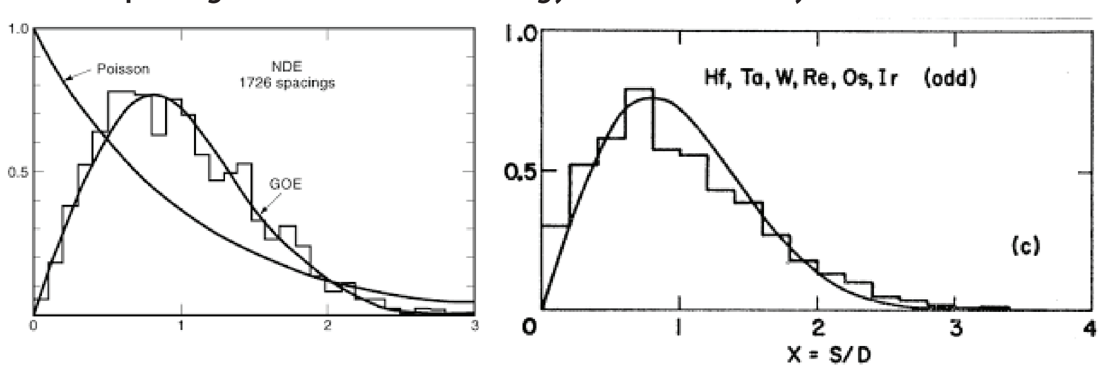
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^T)/2 \quad$ ***Gaussian orthogonal ensemble (GOE)***

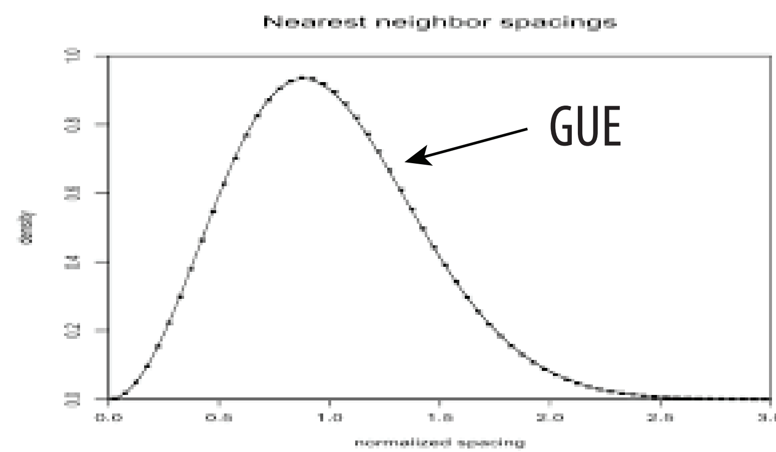
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^\dagger)/2 \quad$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics

Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



RMT used to characterize disorder-driven transitions in mesoscopic conductors, neural networks, random graph theory, etc.

Phase transitions ~ transitions in universal eigenvalue statistics.

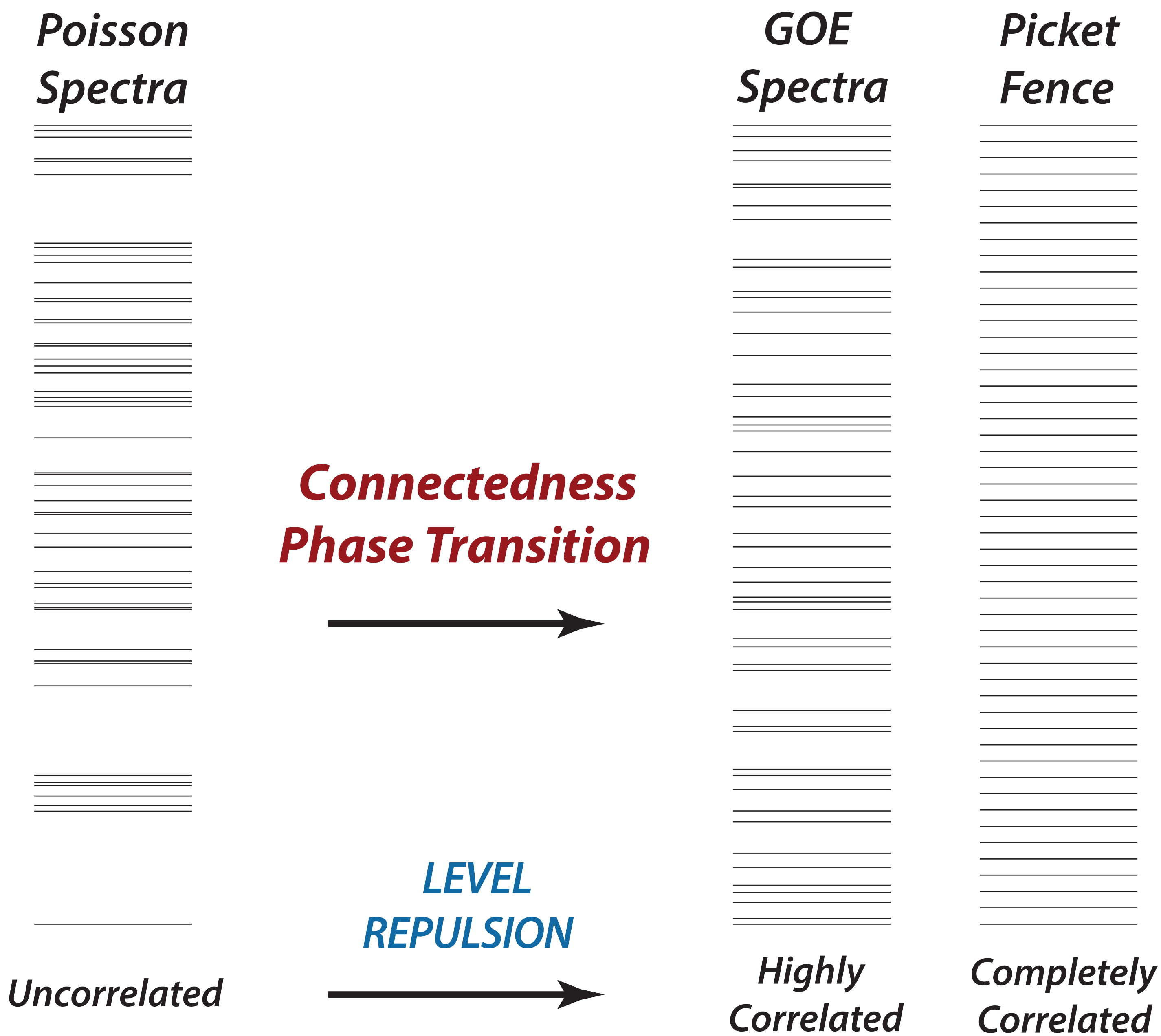
Transition in Eigenvalue Correlations

$$P(z) = \exp(-z)$$

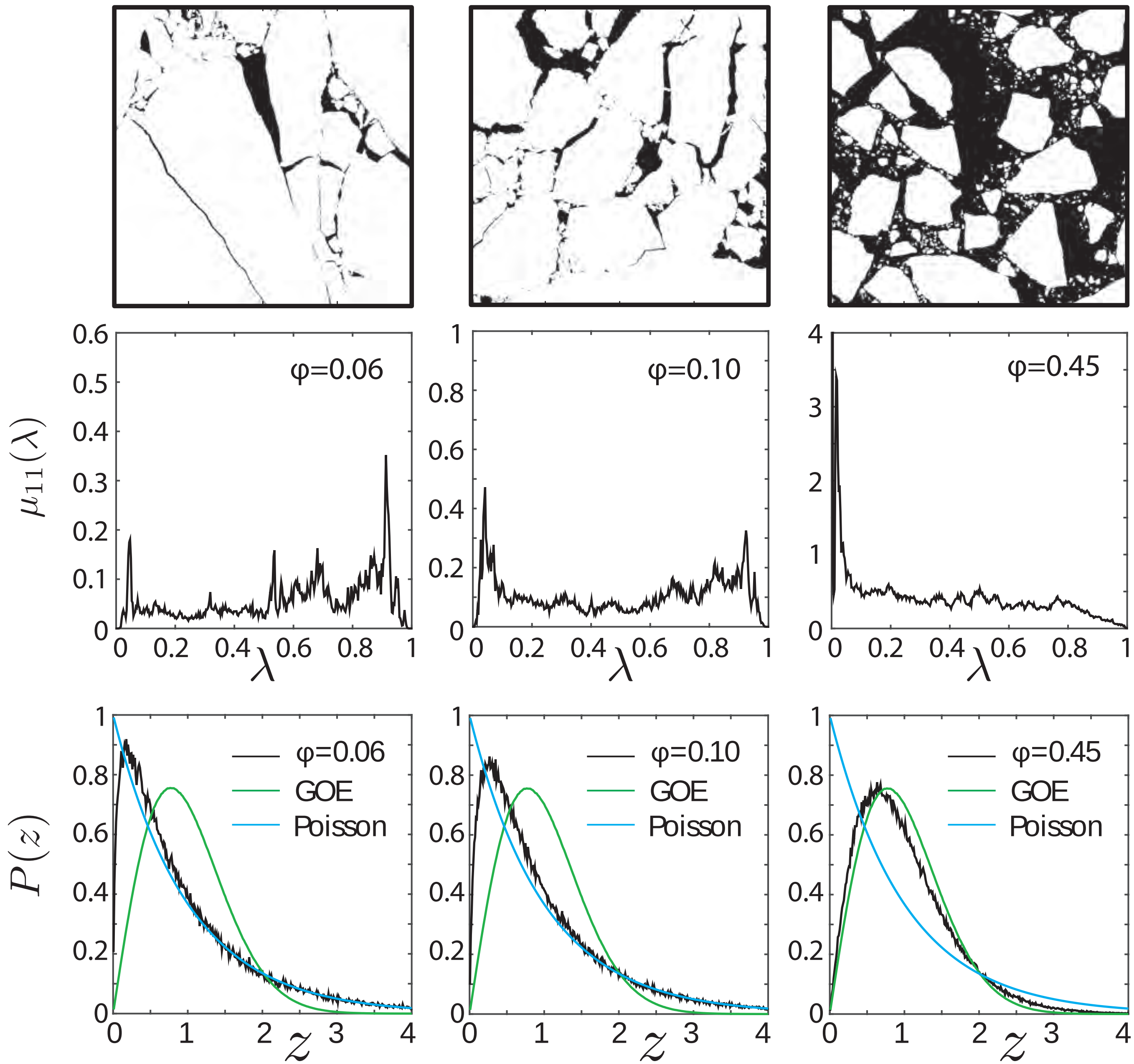
Eigenvalue Spacing Distribution

$$P(z) \approx \frac{\pi z}{2} \exp(-\pi z^2/4)$$ **Wigner surmise**

Eigenvalue Spacing Distribution



Spectral computations for Arctic sea ice pack



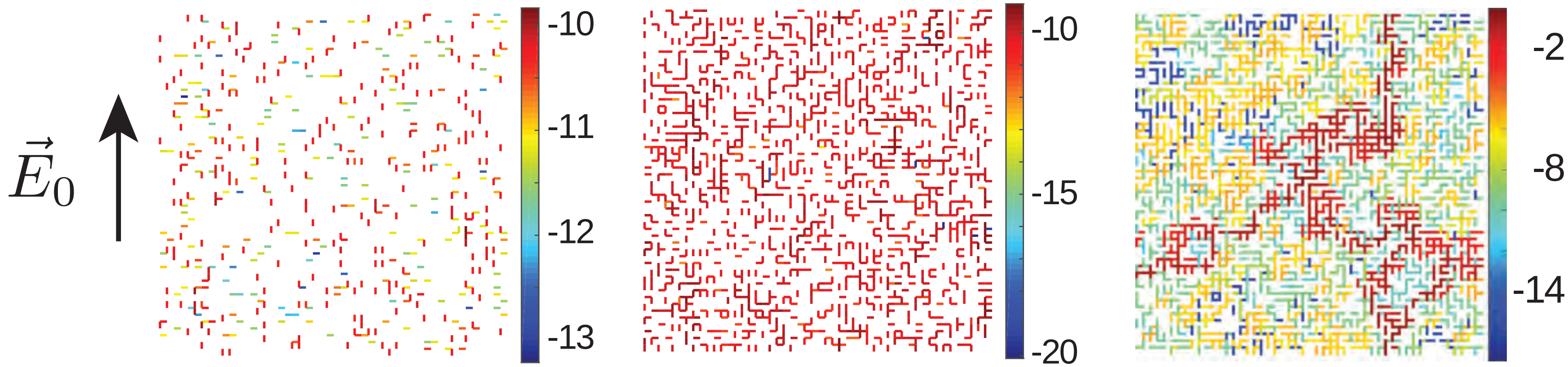
spectral measures

eigenvalue spacing distributions

TRANSITION

Spectral statistics for 2D random resistor network

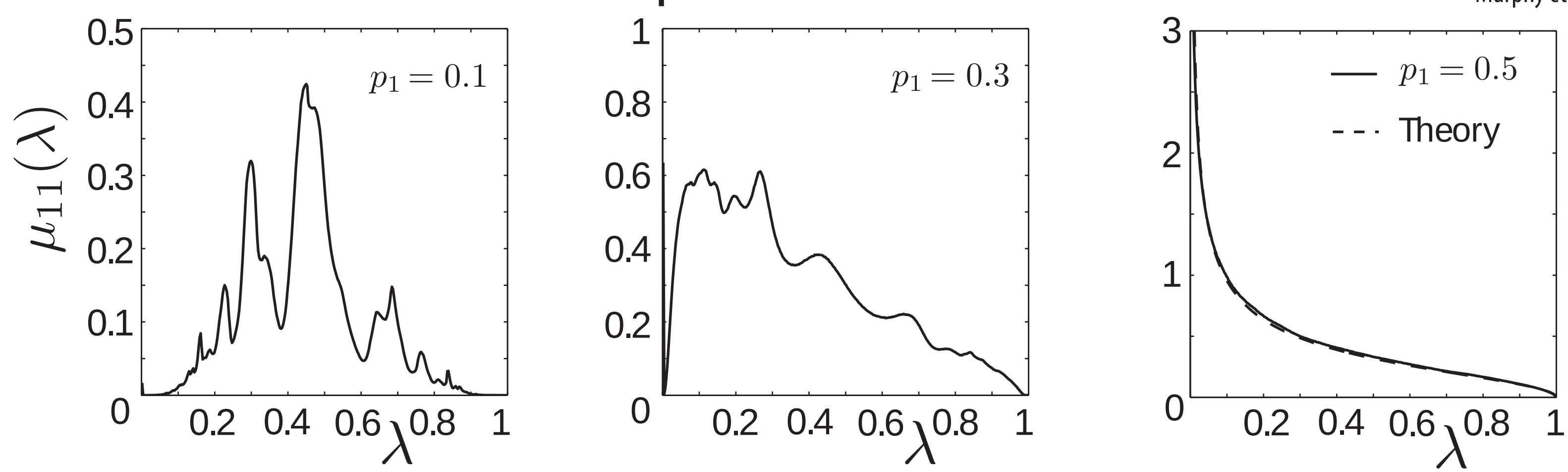
Log Electric Field



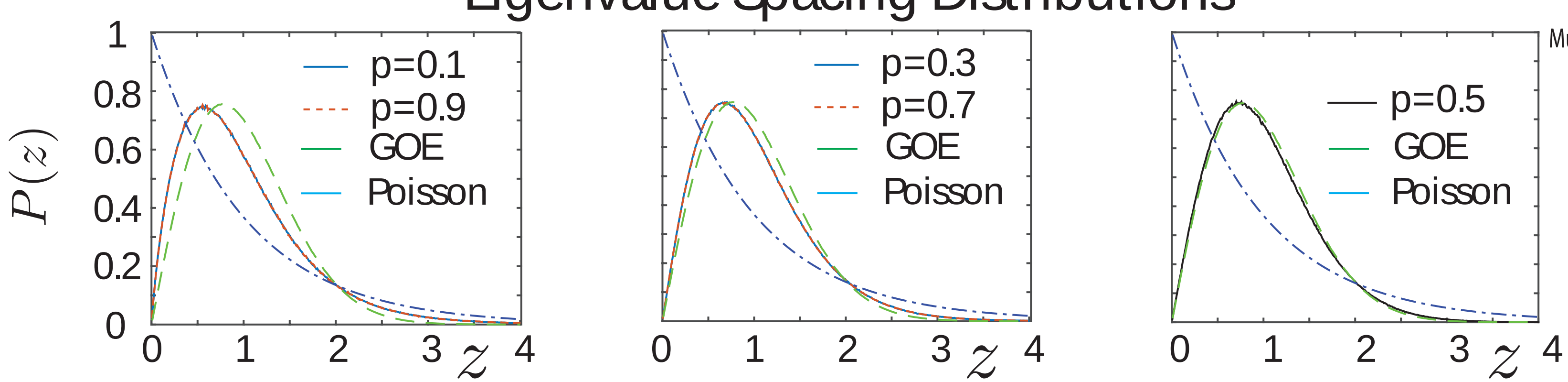
Murphy and Golden, J. Math. Phys., 2012

Murphy et al. Comm. Math. Sci., 2015

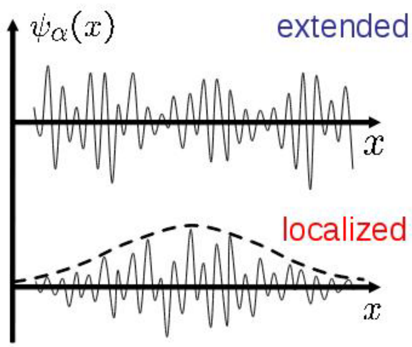
Spectral Measures



Eigenvalue Spacing Distributions



Murphy, Cherkaev, Golden, 2016



metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

we find a surprising analog

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION TRANSITION → **transition to universal eigenvalue statistics (GOE)**
extended states, mobility edges

-- but without wave interference or scattering effects ! --

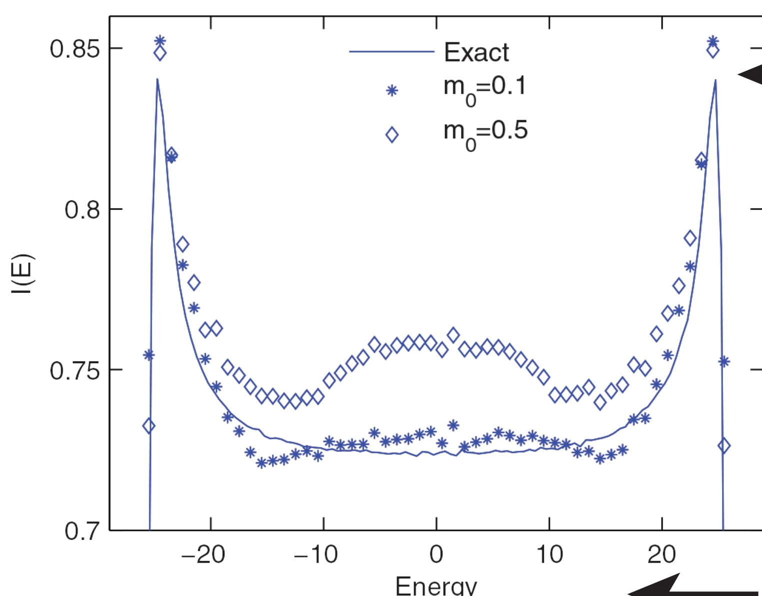
eigenvector localization and mobility edges

Inverse Participation Ratio: $$I(\vec{v}_n) = \sum_{i=1}^{N} |(\vec{v}_n)_i|^4$$

Completely Localized: $$I(\vec{e}_n) = 1$$

Completely Extended: $$I\left(\frac{1}{\sqrt{N}}\vec{1}\right) = \frac{1}{N}$$

Anderson Model



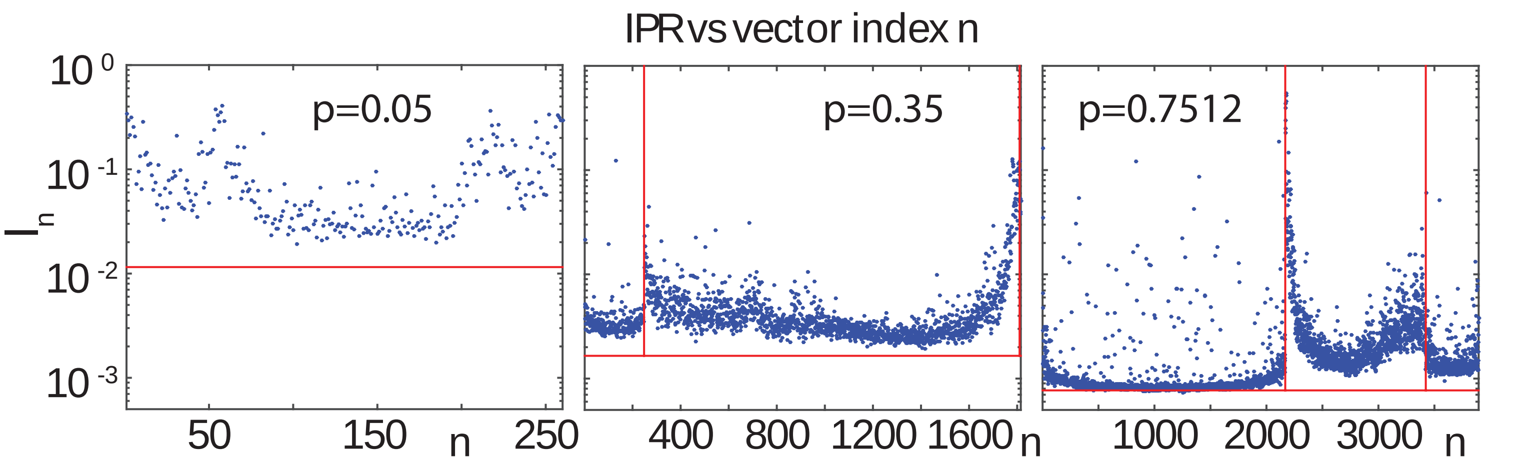
Mobility Edge

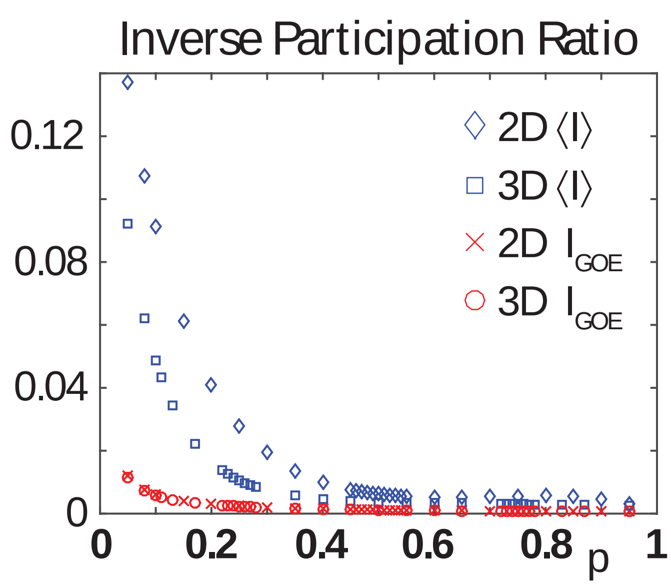
“Eigenvalues”

FIG. 4. (Color online) IPR for Anderson model in two dimensions with $x = 6.25$ ($w = 50$) from exact diagonalization (solid line) and from LDRG with different values of the cutoff m_0. LDRG data are averaged over 100 runs of systems with 100×100 sites.

PHYSICAL REVIEW B 90, 060205(R) (2014)

Localization properties of eigenvectors in random resistor networks





$$I_n = \sum_i (\vec{v}_n)_i^4$$

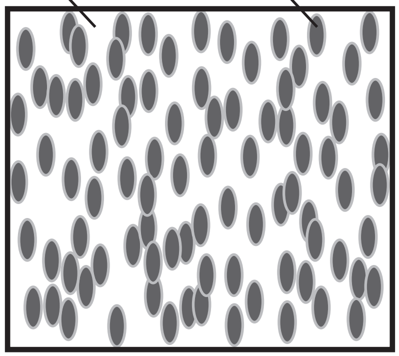
Murphy, Cherkaev, Golden, 2017

Homogenization for composite materials

Two-component composites

Local conductivity

σ_1 σ_2



inhomogeneous medium

Homogenize →

σ^* ***effective conductivity***

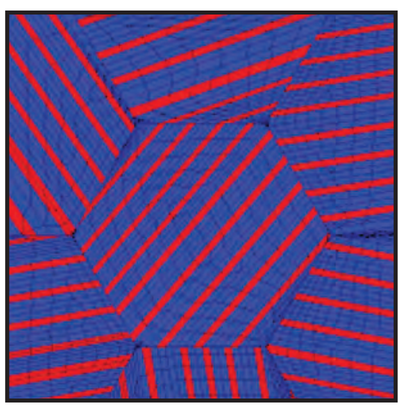
homogeneous medium

Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Polycrystalline media

Conductivity of crystal directions

σ_1 σ_2



inhomogeneous medium

Homogenize →

σ^* ***effective conductivity***

homogeneous medium

Mathematical formulation for composite materials

Two-component material

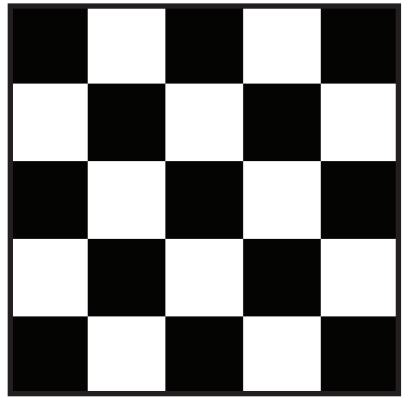
Local conductivity

$$\sigma = \sigma_1\chi_1 + \sigma_2\chi_2$$

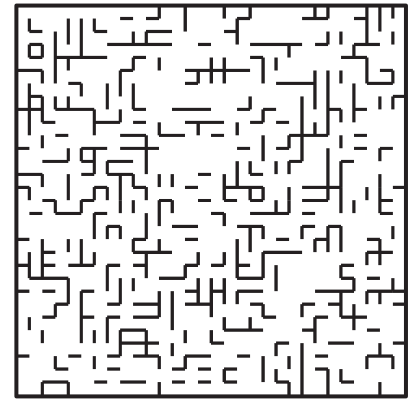
$$\chi_2 = 1 - \chi_1$$

$$\chi_i\chi_j = \delta_{ij}\chi_i$$

Continuum composite



Discrete composite



$$\vec{\nabla}\cdot\vec{J} = 0, \quad \vec{\nabla}\times\vec{E} = 0, \quad \vec{J} = \sigma\vec{E}, \quad \vec{E} = \vec{\nabla}\phi + \vec{e}_k, \quad \langle\vec{E}\rangle = \vec{e}_k$$

Polycrystalline material

Local conductivity

$$\sigma = R\,diag(\sigma_1, \sigma_2, \sigma_2)R^T$$
$$= \sigma_1 X_1 + \sigma_2 X_2$$

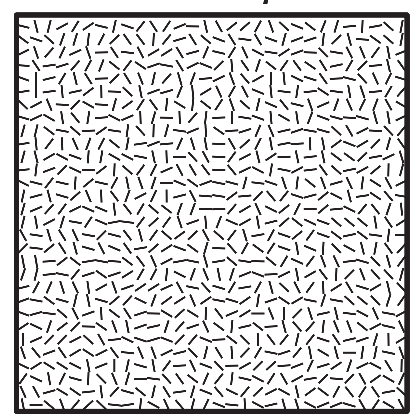
$$X_2 = I - X_1$$

$$X_i X_j = \delta_{ij} X_i$$

Random Rotation Matrix

Continuum composite

Discrete composite



Integral representations for bulk transport coefficients

The ***effective conductivity*** is defined in terms of the ***system energy***

$$\langle \vec{J} \cdot \vec{E} \rangle = \sigma^* E_0^2 \, , \qquad \sigma^* = \sigma^*_{kk} \, , \qquad \langle \vec{E} \rangle = E_0 \vec{e}_k$$

- This defines a ***homogeneous medium*** which behaves ***macroscopically*** and ***energetically*** **just like the given inhomogeneous medium.**

$$\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{J} = 0, \qquad \longrightarrow$$

$$\vec{J} = \sigma \vec{E}, \qquad \sigma = \sigma_1 X_1 + \sigma_2 X_2$$

Resolvent formula

$$X_1 \vec{E} = s E_0 (sI - X_1 \Gamma X_1)^{-1} X_1 \vec{e}_k$$

Contrast parameter: $s = 1/(1 - \sigma_1/\sigma_2)$

Projection onto curl-free fields: $\Gamma = -\vec{\nabla}(-\Delta)^{-1}\vec{\nabla}\cdot$

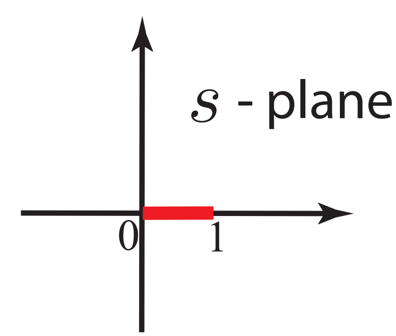
$$\sigma^*/\sigma_2 = 1 - \langle X_1 \vec{E} \cdot \vec{e}_k \rangle / (s E_0) = 1 - F(s)$$

Gully et al., *Proc. Roy. Soc. A* 2015
Murphy, Cherkaev, Golden 2016

$$F(s) = \int_0^1 \frac{d\mu(\lambda)}{s - \lambda}$$

μ

- spectral measure of the self-adjoint operator $X_1 \Gamma X_1$
- mass = average orientation
- higher moments depend on n-point correlations



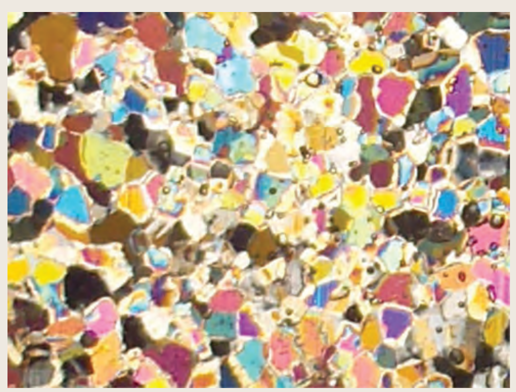
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...

- **Forward and inverse bounds**
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS OF SCIENTIFIC PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

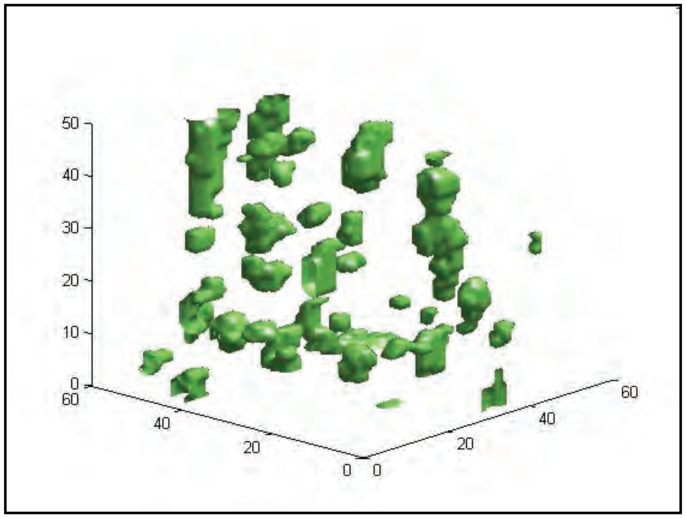
A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

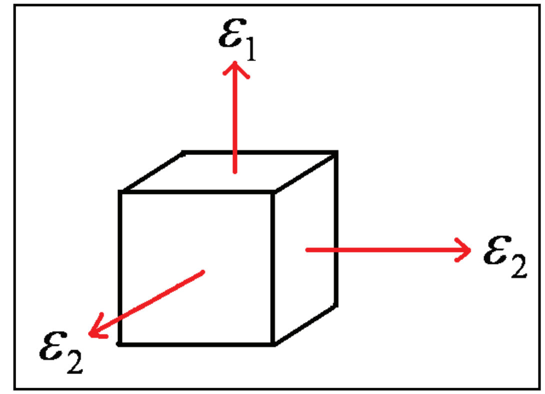
THE ROYAL SOCIETY PUBLISHING

21/01/15 1:47 PM

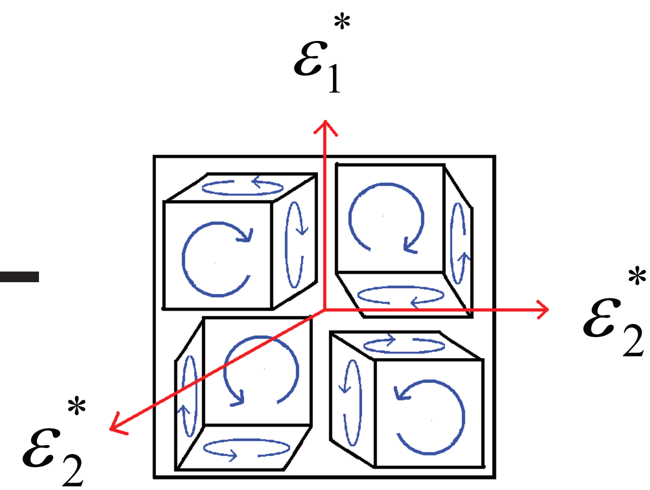
two scale homogenization for polycrystalline sea ice



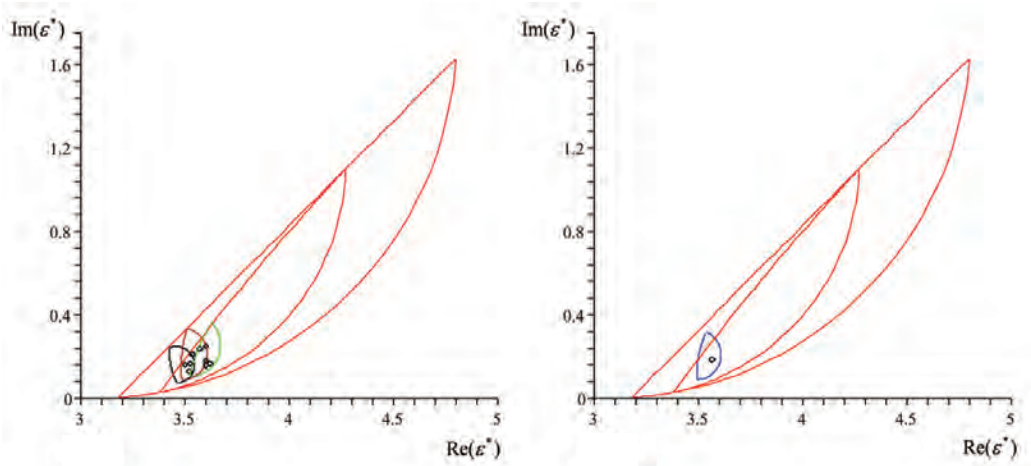
numerical homogenization
for single crystal



analytic continuation
for polycrystals

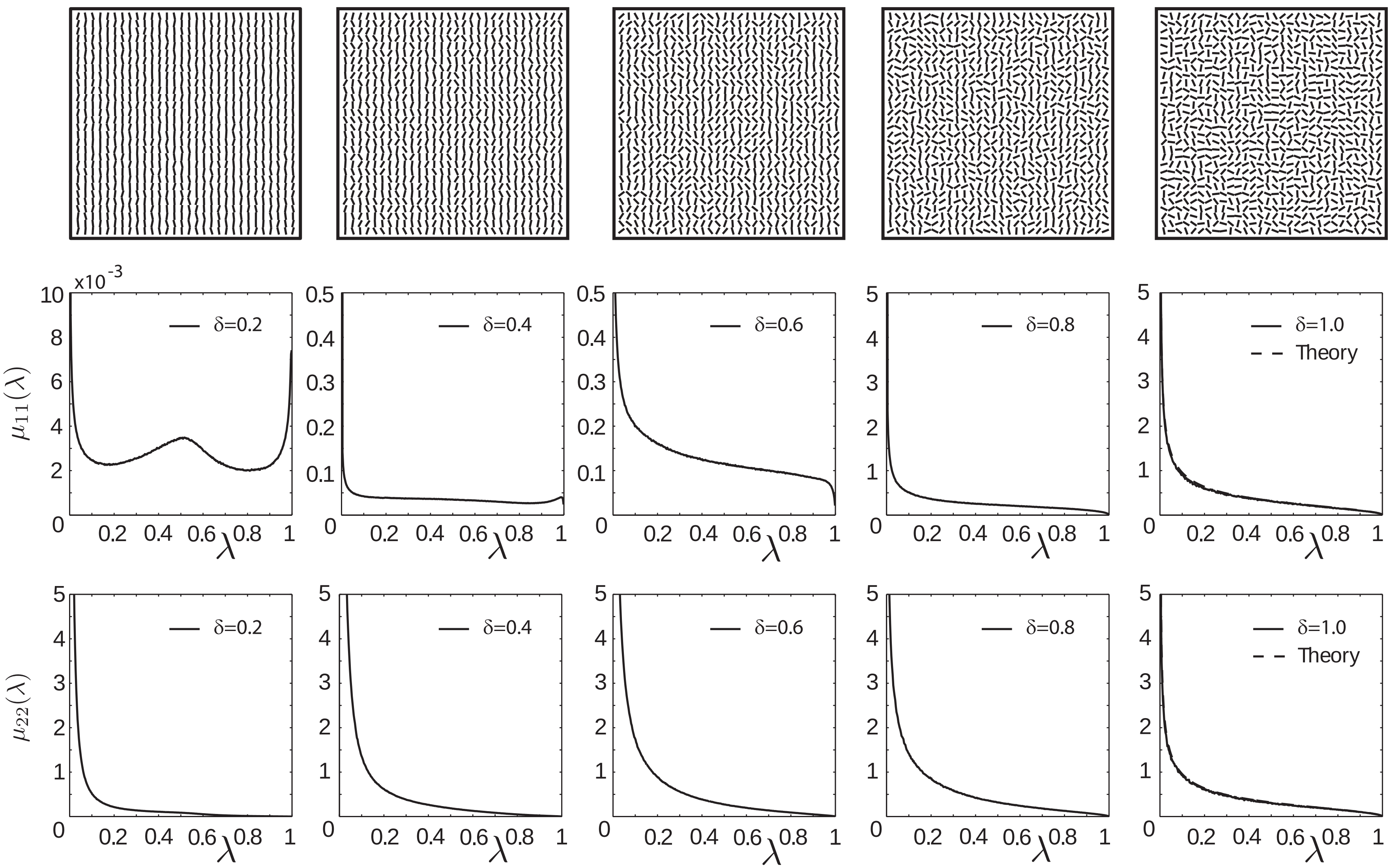


bounds



Gully, Lin, Cherkaev, Golden, Proc. Roy. Soc. A (and cover) 2015

Spectral measures for uniaxial polycrystalline media



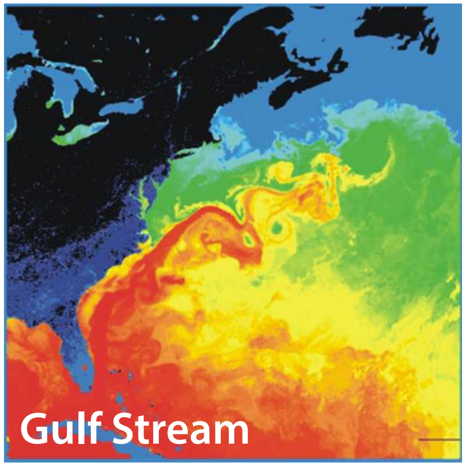
- Random crystallographic orientations angles θ measured from the vertical direction, uniformly distributed $\theta \sim U(-\delta\pi/2, \delta\pi/2)$

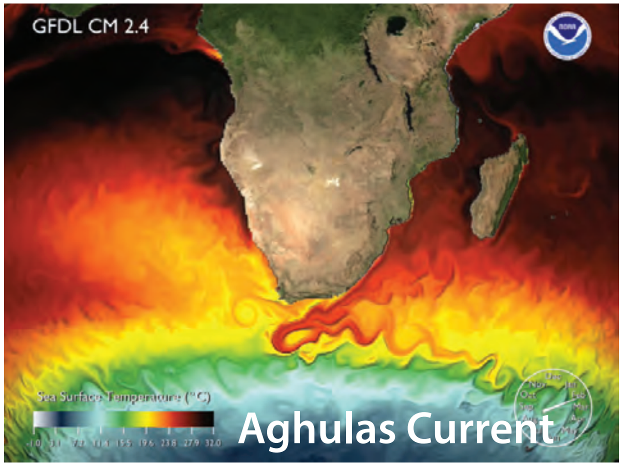
Murphy, Cherkaev, Golden, 2017

advection enhanced diffusion

effective diffusivity

tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere
salt and heat transport in ocean
heat transport in sea ice with convection





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

homogenize

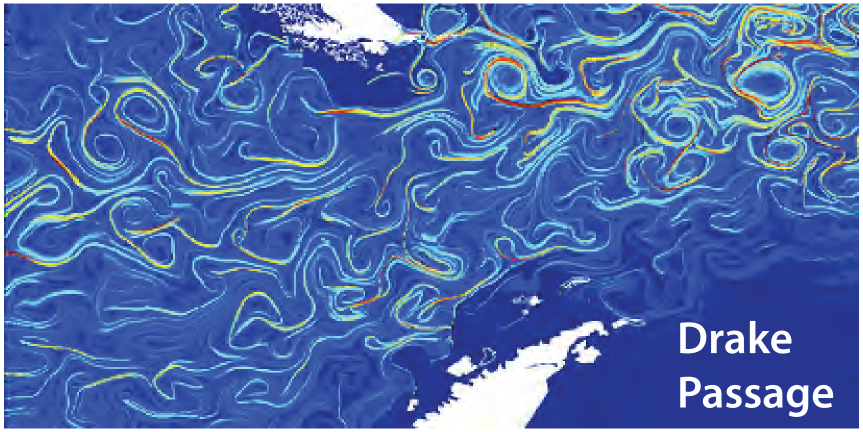
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* effective diffusivity

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, 2017



Stieltjes integral for κ^* with spectral measure

composites

Golden and Papanicolaou, CMP 1983

$$\frac{\epsilon^*}{\epsilon_2} = 1 - \int_0^1 \frac{d\mu(\lambda)}{s - \lambda}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

advection diffusion

Avellaneda and Majda, PRL 89, CMP 91

$$\frac{\kappa^*}{\kappa_0} = 1 - \int_0^\infty \frac{d\rho(z)}{t - z}$$

$t = -1/\xi^2, \;\; \xi$ = Péclet number

- computations of spectral measures and effective diffusivity for model flows

$$i\Gamma H \Gamma \qquad \vec{u} = \kappa_0 \, \xi \, \vec{\nabla} \cdot \mathbf{H}$$

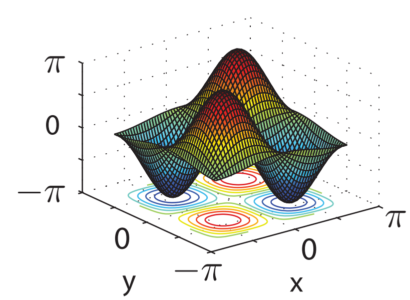
$\mathbf{H}$ antisymmetric vector potential

Murphy, Cherkaev, Zhu, Xin, Golden 2017

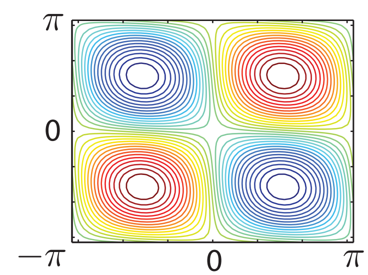
- rigorous bounds and computations on convection enhanced thermal conductivity of sea ice

Liu, Hardenbrook, Kraitzman, Zhu, Murphy, Cherkaev, Golden, 2017

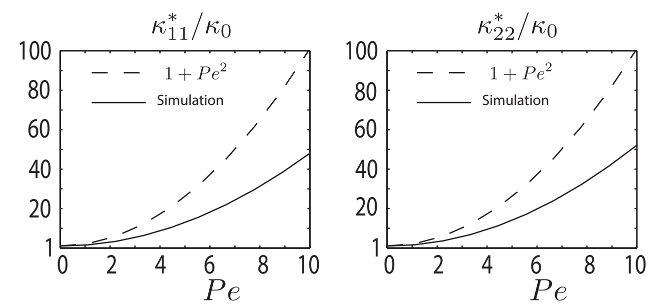
stream function



streamlines



effective diffusivities



$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\vec{u} = \vec{\nabla} \cdot H$$

$$\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{J} = 0,$$

$$\vec{J} = \sigma \vec{E}, \qquad \sigma = \kappa_0 I + S, \qquad \longrightarrow$$

Resolvent formula

$$\vec{\nabla}\phi = (\kappa_0 I - \imath \Gamma S \Gamma)^{-1} \, \Gamma H \vec{e}_k$$

Projection onto curl-free fields: $\Gamma = -\vec{\nabla}(-\Delta)^{-1}\vec{\nabla}\cdot$

Steady flow

$$S = H$$

Dynamic flow

$$S = H + (-\Delta)^{-1} \frac{\partial}{\partial t}$$

$$\kappa^* / \kappa_0 = 1 + \langle \vec{\nabla}\phi \cdot \vec{\nabla}\phi \rangle = 1 + G(\kappa_0)$$

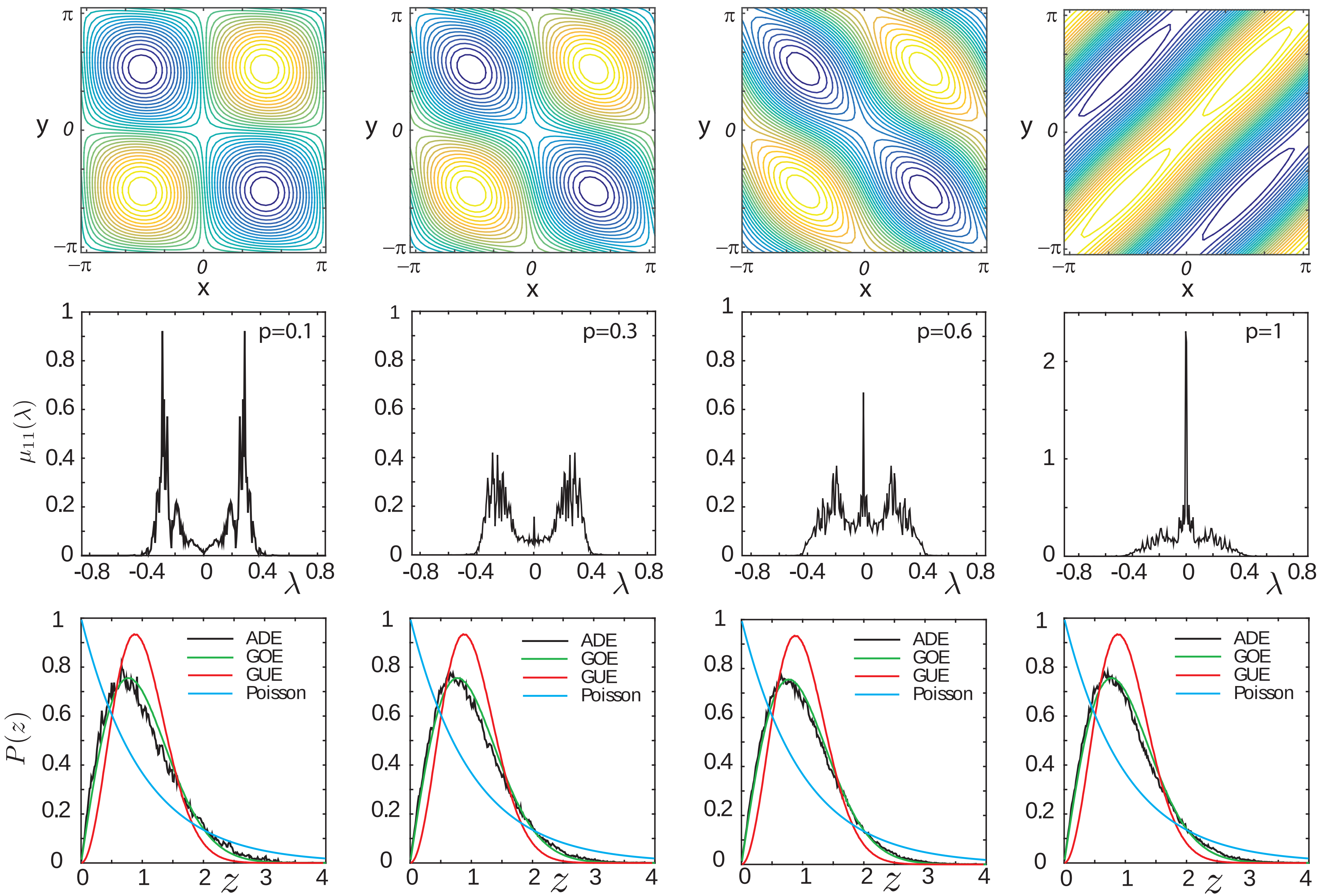
Steady flow: **Murphy, Cherkaev, Zhu, Xin, Golden 2017**
Dynamic flow: **Murphy, Cherkaev, Xin, Zhu, Golden 2017**

$$G(\kappa_0) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu(\lambda)}{\kappa_0^2 + \lambda^2}$$

ν — spectral measure of the self-adjoint operator $\imath \Gamma S \Gamma$

Spectral measures and eigenvalue spacings for cat's eye flow

$$H(x,y) = \sin(x)\sin(y) + A\cos(x)\cos(y), \qquad A \sim U(-p,p)$$



Murphy, Cherkaev, Xin, Golden, 2016

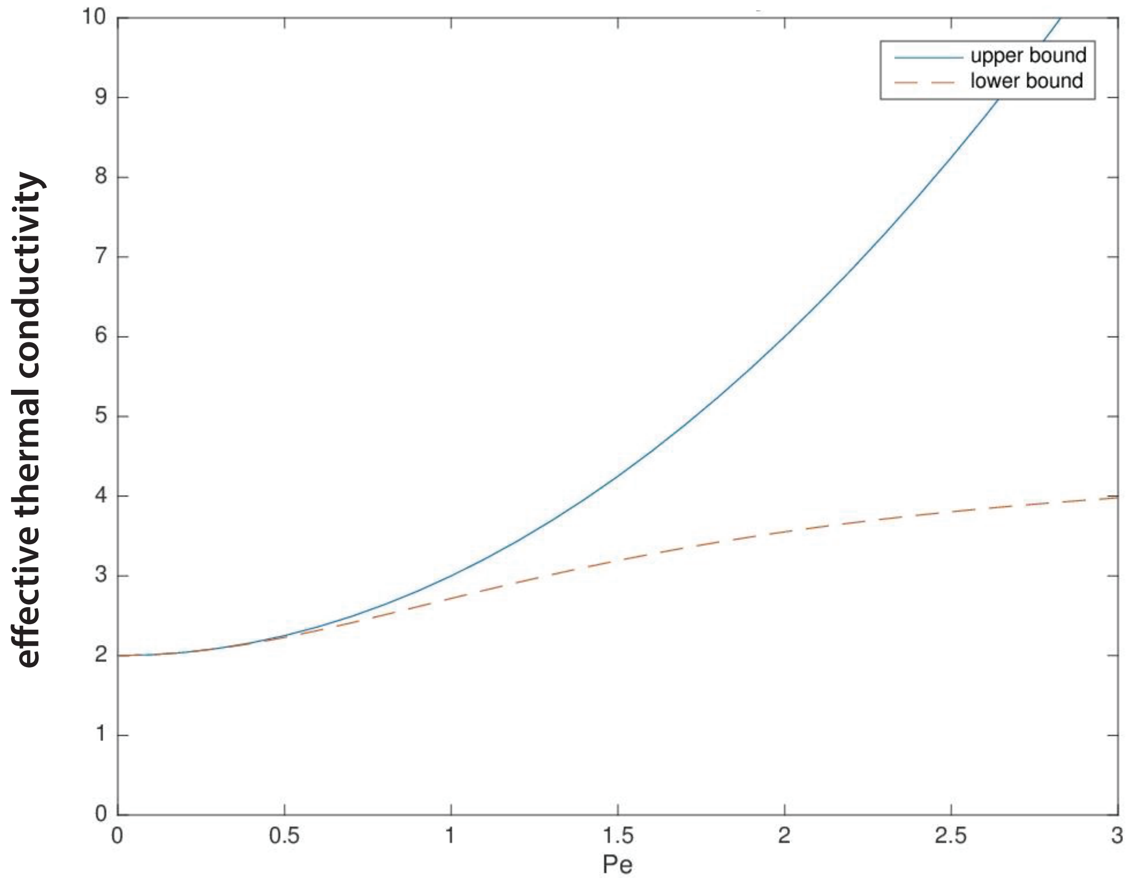
Thermal Conduction Enhanced with Convection

- Temperature on top surface driven by atmosphere conditions
- Bottom surface in contact with sea water
- Temperature field T governed by a nonlinear convection-diffusion equation

$$\rho c \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = \nabla \cdot (\kappa(T) \nabla T)$$

  with a Darcy velocity $\mathbf{u}$
- Parameters:
  - $\rho =$ bulk density
  - $c =$ specific heat
  - $\kappa(T) =$ temperature dependent thermal conductivity

bounds on the effective thermal conductivity of sea ice with convection (BC flow model)

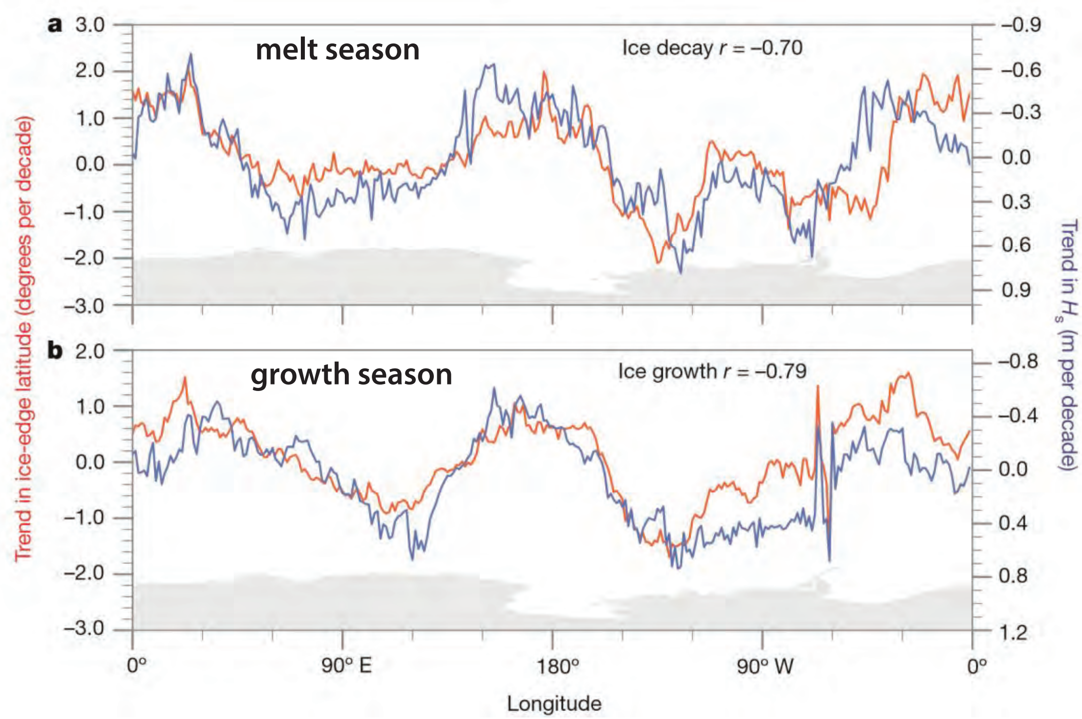


Hardenbrook, Kraitzman, Zhu, Murphy, Cherkaev, Golden

Storm-induced sea-ice breakup and the implications for ice extent

Kohout et al., *Nature* 2014

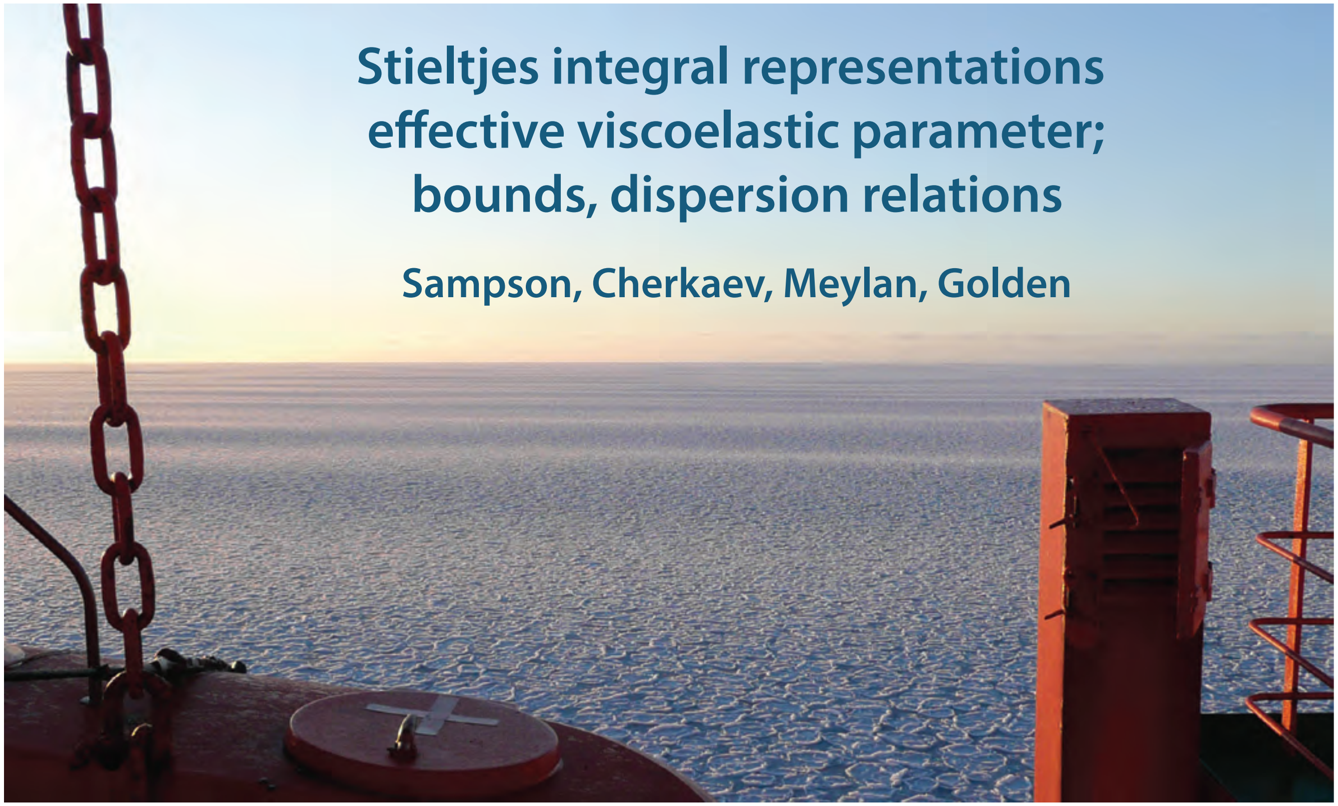
- during three large-wave events, significant wave heights did not decay exponentially, enabling large waves to persist deep into the pack ice.
- large waves break sea ice much farther from the ice edge than would be predicted by the commonly assumed exponential decay

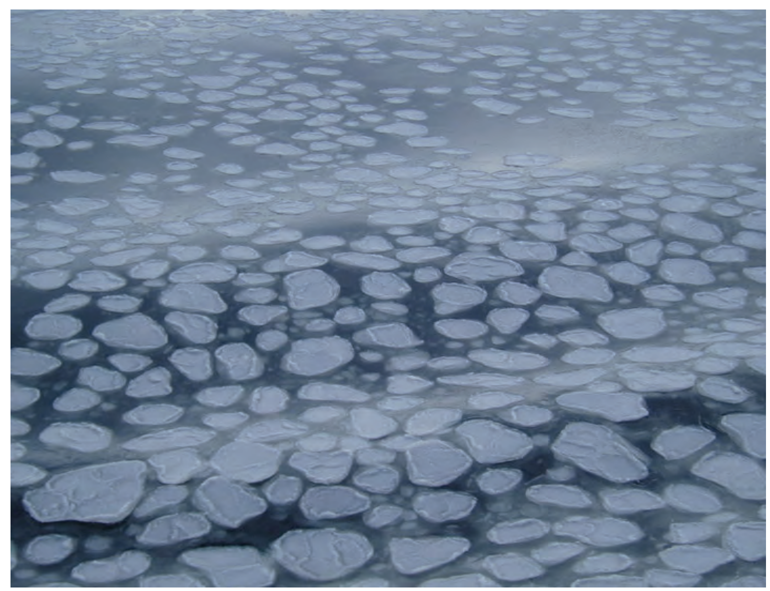


ice extent compared with significant wave height

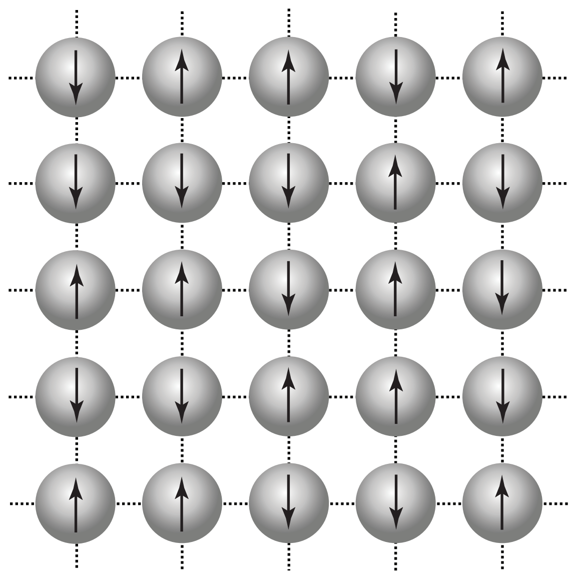
Waves have strong influence on both the floe size distribution and ice extent.

wave propagation in the marginal ice zone





Ising Model for a Ferromagnet



$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

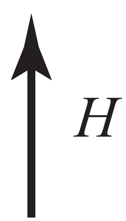
$$\mathcal{H}_\omega = -J \sum_{<i,j>} s_i s_j \; - \; H \sum_i s_i$$

nearest neighbor Ising Hamiltonian

for any configuration $\omega \in \Omega = \{-1, 1\}^N$ of the spins

$$J \geq 0$$

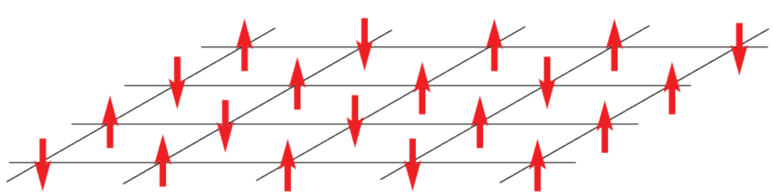
applied magnetic field



canonical partition function

$$Z_N(T, H) = \sum_{\omega \in \Omega} \exp(-\beta \mathcal{H}_\omega) = \exp(-\beta N f_N)$$

$$\beta = 1/kT$$



2–D Ising Model

free energy per site

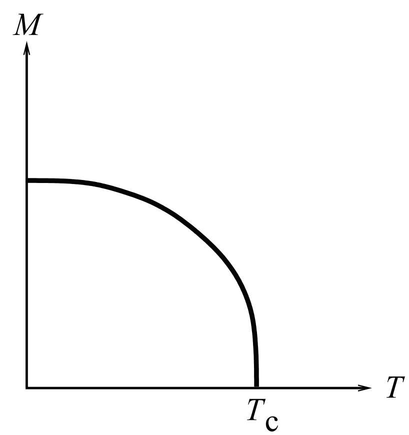
$$f_N(T, H) = \frac{-1}{\beta N} \log Z_N(T, H)$$

free energy $$f(T, H) = \lim_{N \to \infty} f_N(T, H)$$

magnetization

homogenized parameter like effective conductivity

$$M(T, H) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle = -\frac{\partial f}{\partial H}$$



Curie point
critical temperature

magnetic susceptibility $$\chi(T, H) = \frac{\partial M}{\partial H} = -\frac{\partial^2 f}{\partial H^2} \geq 0$$

partition function N^{th} **order polynomial** in the "activity" $z = exp(-2\beta H)$

$$Z_N(z) = \sum_{n=0}^{N} a_n z^n, \quad a_n \geq 0$$

with remarkable property:

Theorem (Lee -Yang 1952):

If $J \geq 0$, then $Z_N = 0$ implies z lies on the unit circle $|z| = 1$, or equivalently, H lies on the imaginary axis (with real β).

Then the partition function can be written as

$$Z_N(z) = a_N \prod_{n=1}^{N} (z - z_n), \qquad |z_n| = 1$$

Then $$f(T, H) = \frac{-1}{\beta} \int_{|t|=1} \log(z - t) d\nu(t) - 2d\beta J$$

Stieltjes integral representation for magnetization

and scaling relations for critical exponents

Baker *PRL* 1968

$$M(\tau) = \tau + \tau(1-\tau^2)G(\tau^2) \qquad \tau = \tanh(\beta H)$$

$$G(\tau^2) = \int_0^\infty \frac{d\psi(y)}{1+\tau^2 y}$$

Herglotz

$$M(T) = -\frac{\partial f}{\partial H} \sim (T_c - T)^\beta \qquad T \to {T_c}^-$$

$$\chi = -\frac{\partial^2 f}{\partial H^2} \sim (T - T_c)^{-\gamma} \qquad T \to {T_c}^+$$

Along the critical isotherm $T = T_c$

$$M(H) \sim H^{1/\delta} \qquad H \to 0^+$$

ψ supported in $[0, S(T)]$

$$S(T) \sim (T - T_c)^{-2\Delta}, T \to {T_c}^+$$

$$\beta = \Delta - \gamma$$

$$\delta = \Delta/(\Delta - \gamma)$$

Baker's inequalities

$$\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0$$

critical exponents γ_n of higher derivatives of f

$$\gamma_0 = \gamma$$

via analogous Herglotz structure for transport in composites, same critical analysis and scaling relations hold near percolation threshold

Golden, *J. Math. Phys.* 1995
Phys Rev. Lett. 1997

(Chuck Newman)

$$m(h) = \frac{\sigma^*}{\sigma_2} \qquad h = \frac{\sigma_1}{\sigma_2} \to 0 \qquad \sigma^*(p,h)$$

effective conductivity of two phase composite - lattice or continuum

$$F(s) = 1 - m(h) \qquad F(s) = \int_0^1 \frac{d\mu(w)}{s-w} \qquad w = \frac{y}{y+1}$$

$$m(h) = 1 + (h-1)g(h) \qquad g(h) = \int_0^\infty \frac{d\phi(y)}{1+hy}$$

Herglotz

$$\sigma^*(p,0) \sim (p-p_c)^t \qquad p \to p_c^+$$

$$\sigma^*(p_c,h) \sim h^{1/\delta} \qquad h \to 0^+$$

$$\chi(p) = \frac{\partial m}{\partial h} \sim (p_c - p)^{-\gamma} \qquad p \to p_c^-, \quad h = 0$$

$$\theta_h \sim (p_c - p)^\Delta \qquad p \to p_c^-$$

spectral gap

$$t = \Delta - \gamma$$

$$\delta = \frac{\Delta}{\Delta - \gamma}$$

lattices and continua obey same scaling relations as in stat mech

Baker's inequalities for transport

$$\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0, \quad n \geq 1$$

Ising model

partition function

$$Z_N(z) = a_N \prod_{n=1}^{N} (z - z_n), \qquad |z_n| = 1$$

free energy

$$f(T,H) = \frac{-1}{\beta} \int_{|t|=1} \log(z - t)\, d\nu(t)$$

order parameter

$$M(T) = -\frac{\partial f}{\partial H}$$

$$\frac{\partial^2 M}{\partial H^2} \le 0$$

G.H.S. inequality

Griffiths, Hurst, Sherman *JMP* 1970

transport in composites

$$\mathcal{Z}_N(s) = \prod_{n=1}^{N} (s - s_n), \qquad s_n \in [0,1]$$

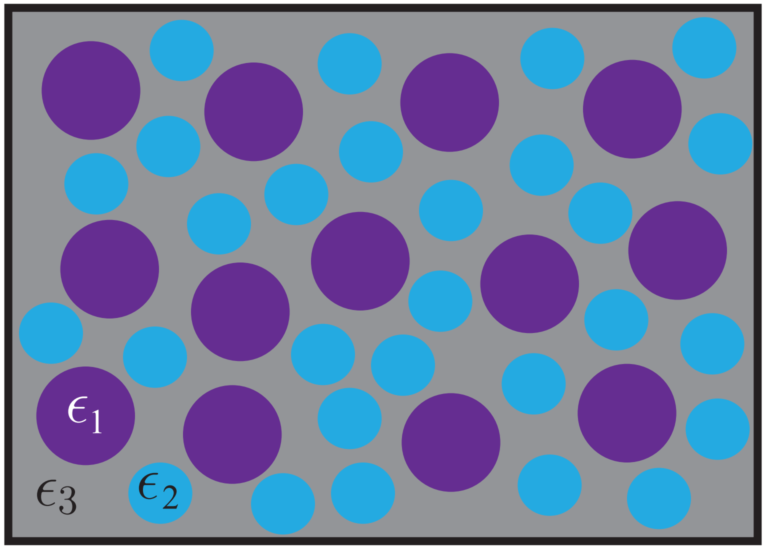
$$\Phi(p,s) = \int_0^1 \log(s - t)\, d\mu(t)$$

$$F(p,s) = \frac{\partial \Phi}{\partial s}$$

$$\frac{\partial^2 m}{\partial h^2} \le 0$$

Golden, *JMP* 1995; *PRL* 1997

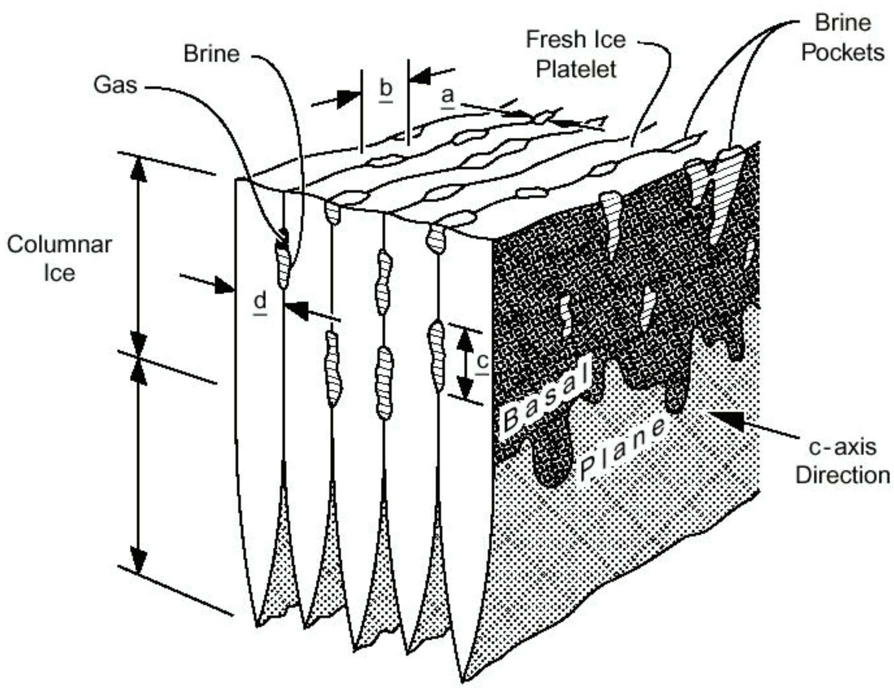
Multiphase Media

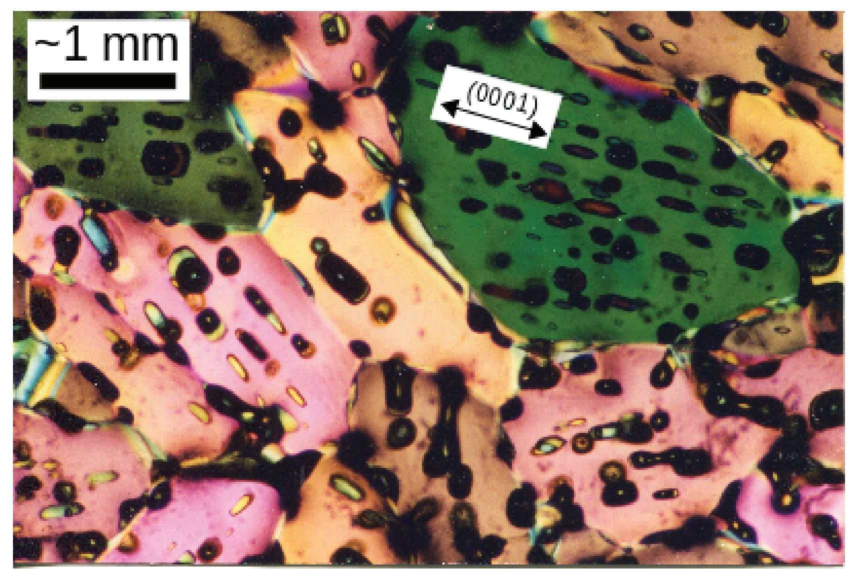


$$m(h_1, h_2) = \epsilon^*/\epsilon_3 \qquad h_i = \epsilon_i/\epsilon_3 \qquad i = 1, 2$$

$$F(s_1, s_2) = 1 - m(h_1, h_2)$$

$$s_i = 1/(1 - h_i)$$





Lusilier

Bounding the effective properties of multiphase composites

Milton, *Theory of Composites,* 2002

trajectory method (based on two phase, one variable theory)

Bergman, *Phys. Rep.* 1978, 1982, ...
Milton, 1981, ...

polydisc representation formula (Herglotz function in several variables)

Golden and Papanicolaou, *J. Stat. Phys.* 1985
Golden, *J. Mech. Phys. Solids* 1986

field equation recursion method

Milton, *Comm. Math. Phys. (I and II)* 1987

effective elasticity tensor

Ou, *Complex Vars. Elliptic. Eqs. 2011*

polydisc representation formula

$$f(\zeta_1, \zeta_2) = iF(s_1, s_2) \qquad f(\zeta_1, \zeta_2) : D^2 \to \{Ref > 0\}$$

$$D^2 = \{|\zeta_1| < 1\} \times \{|\zeta_2| < 1\} \qquad \zeta_j = \frac{s_j - i}{s_j + i}$$

$$f(\zeta_1, \zeta_2) = iImf(0,0) + \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}(H_1H_2 + H_1 + H_2 - 1)\mu(dt_1, dt_2)$$

$$H_1 = (e^{it_1} + \zeta_1)/(e^{it_1} - \zeta_1) \qquad H_2 = (e^{it_2} + \zeta_2)/(e^{it_2} - \zeta_2)$$

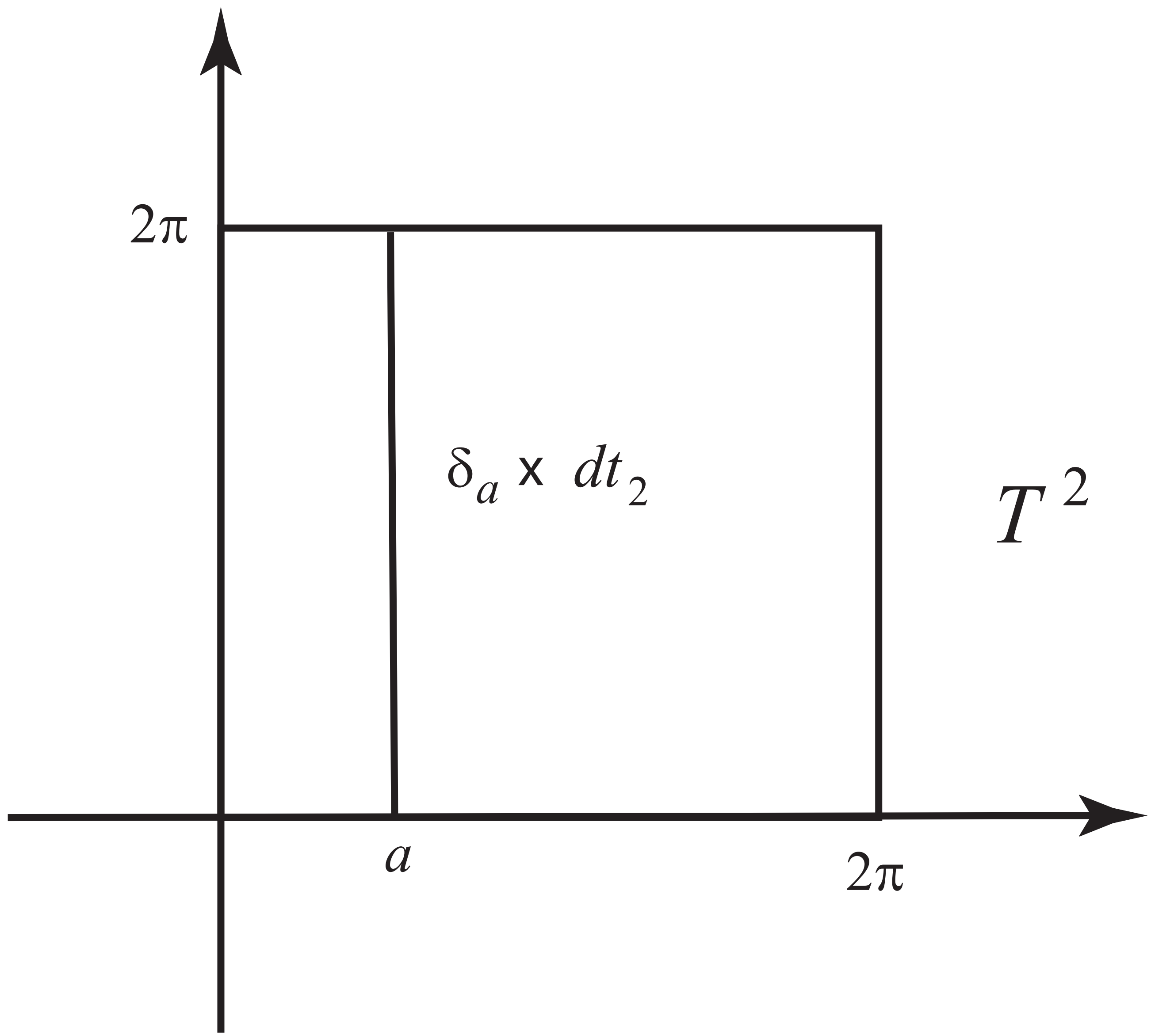
μ is a positive Borel measure on the torus T^2

satisfying a Fourier condition (excludes point measures)

SUNSPOT THEOREM:

Koranyi and Pukanszky, *Am. Math. Soc. Trans.* 1963
Vladimirov and Drozhzhinov, *Mat. Zametki* 1974
Golden and Papanicolaou, *J. Stat. Phys.* 1985

conjectured extremals yield bounds



set of extremal measures = ??

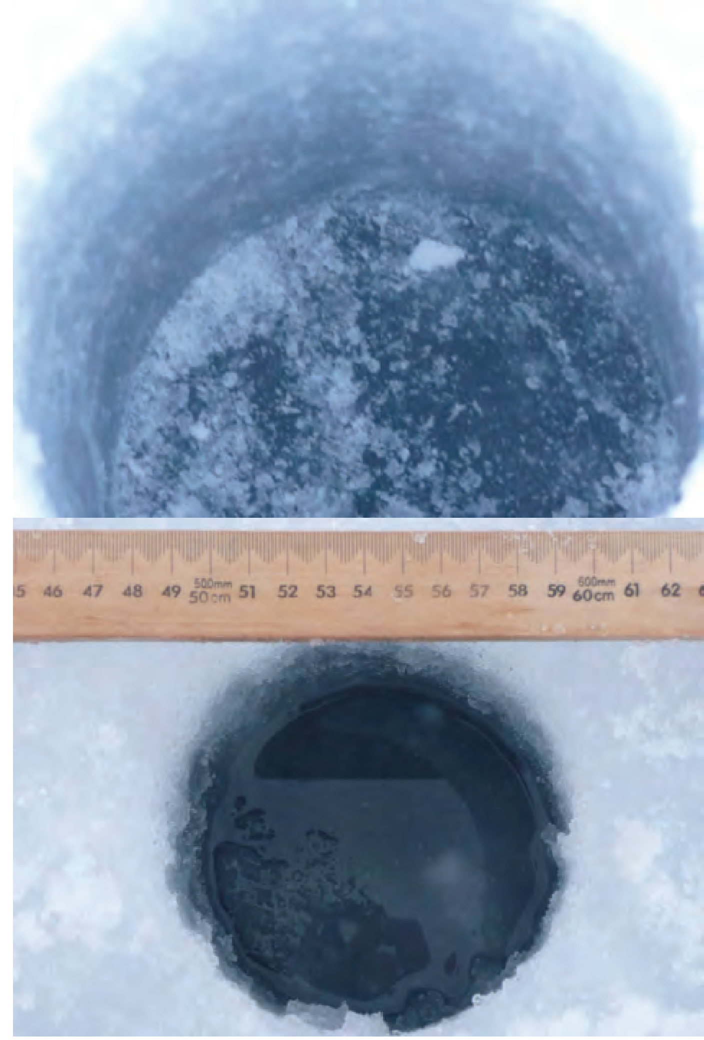
Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	*Antarctic*	*SIPEX*
2010	*Antarctic*	*McMurdo Sound*
2011	*Arctic*	*Barrow AK*
2012	*Arctic*	*Barrow AK*
2012	*Antarctic*	*SIPEX II*
2013	*Arctic*	*Barrow AK*
2014	*Arctic*	*Chukchi Sea*

photo by Jan Lieser

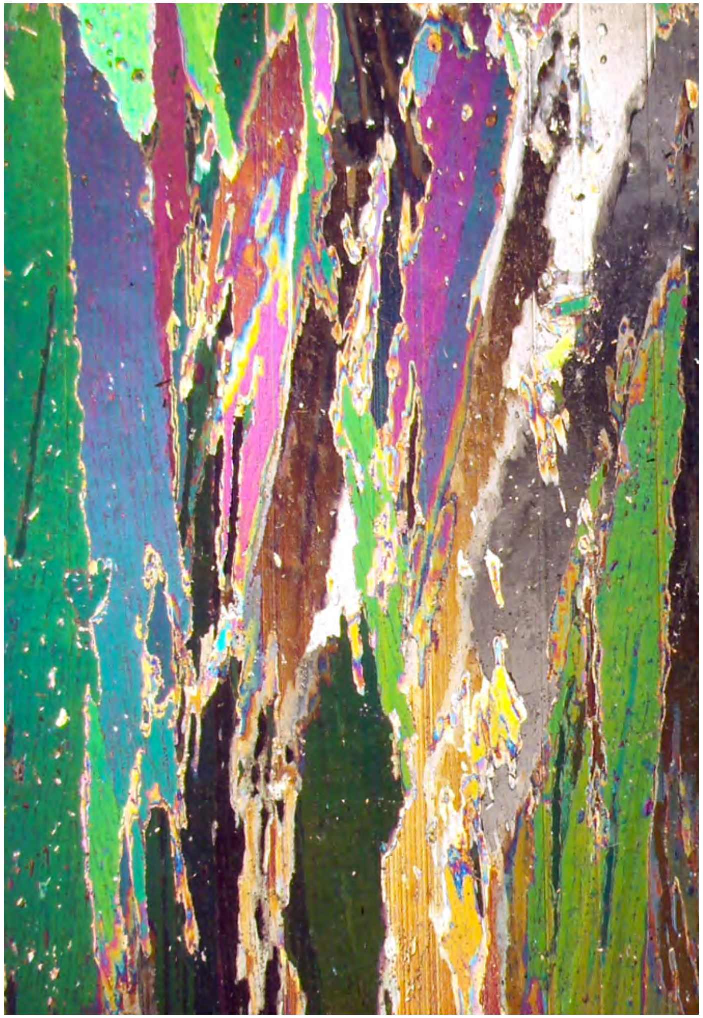


measuring
fluid permeability
of Antarctic sea ice

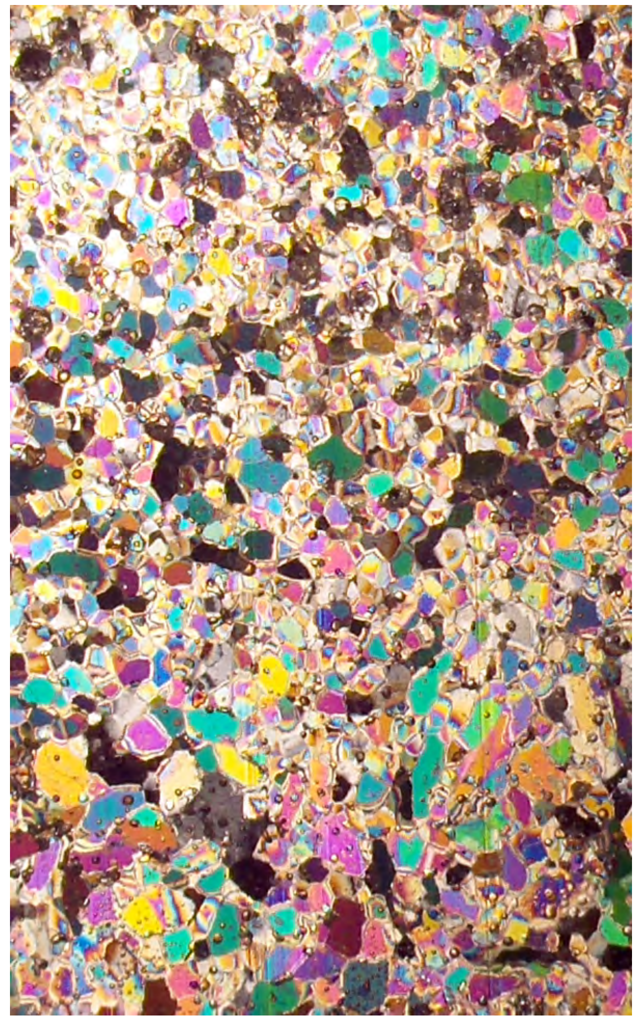
SIPEX 2007

higher threshold for fluid flow in Antarctic granular sea ice

columnar



granular

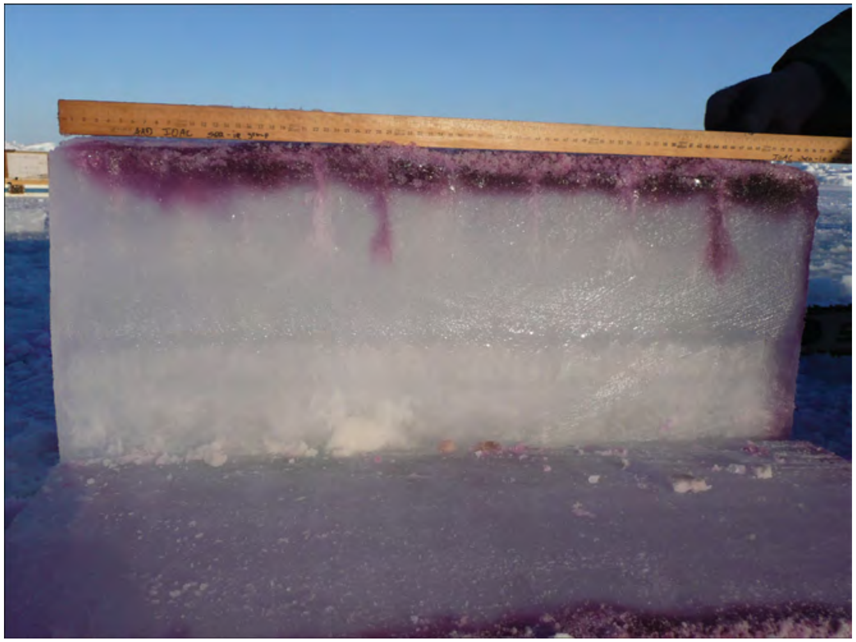


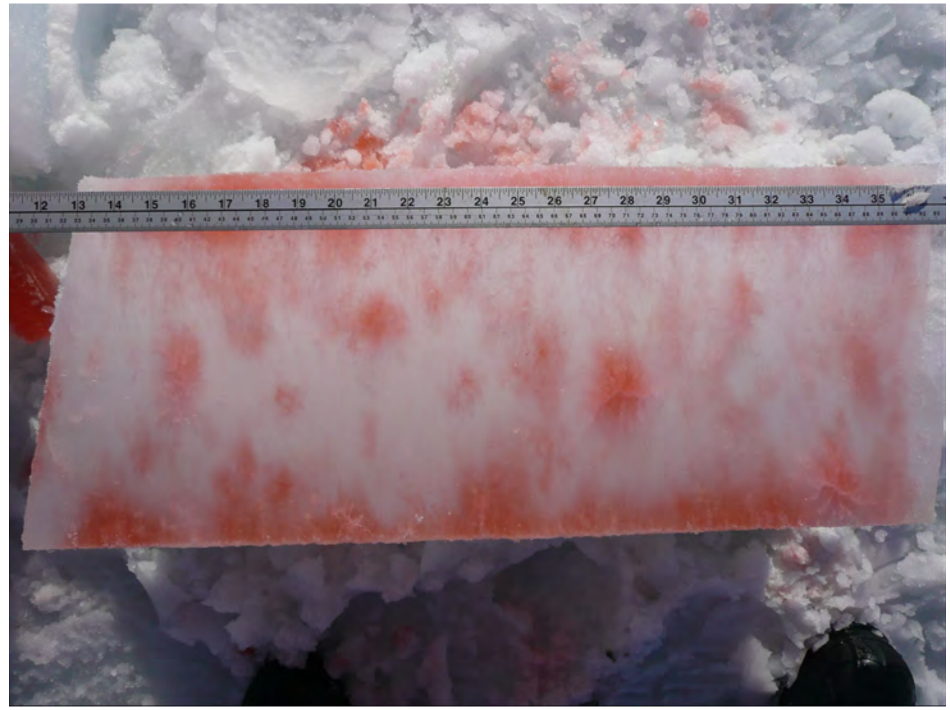
5%

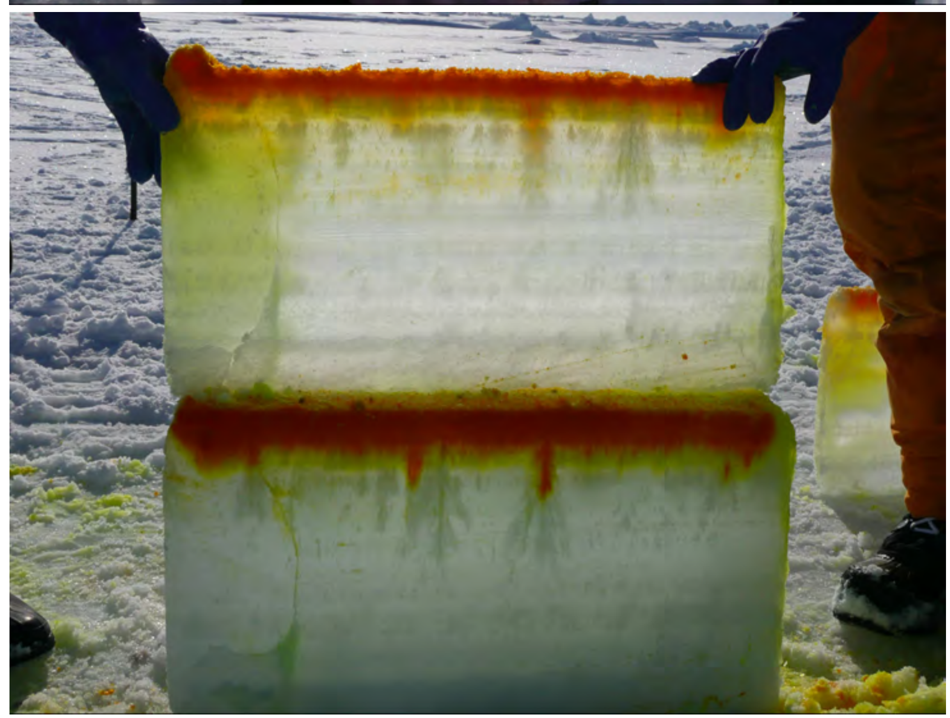
10%

Golden, Sampson, Gully, Lubbers, Tison 2016

tracers flowing through inverted sea ice blocks

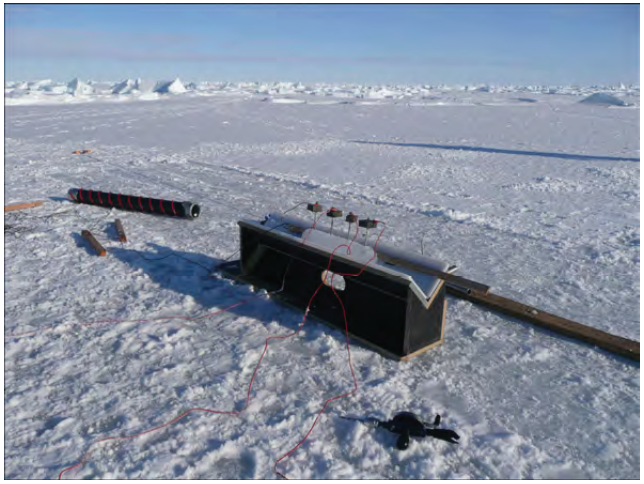


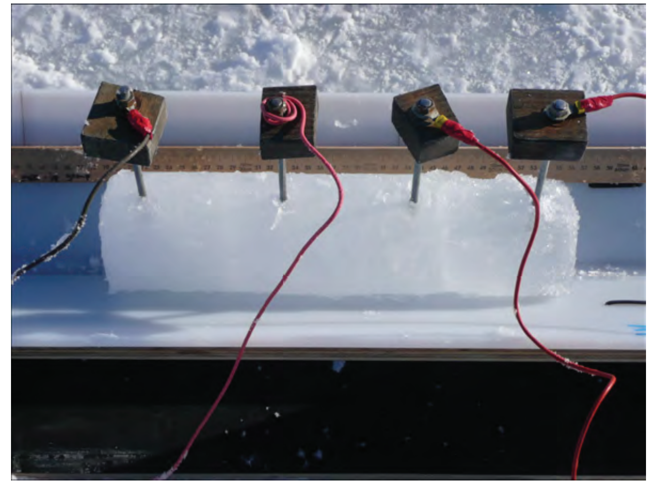


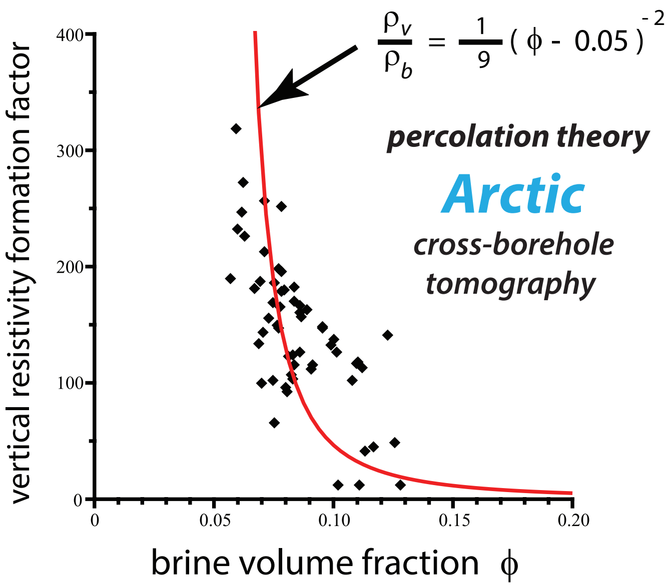


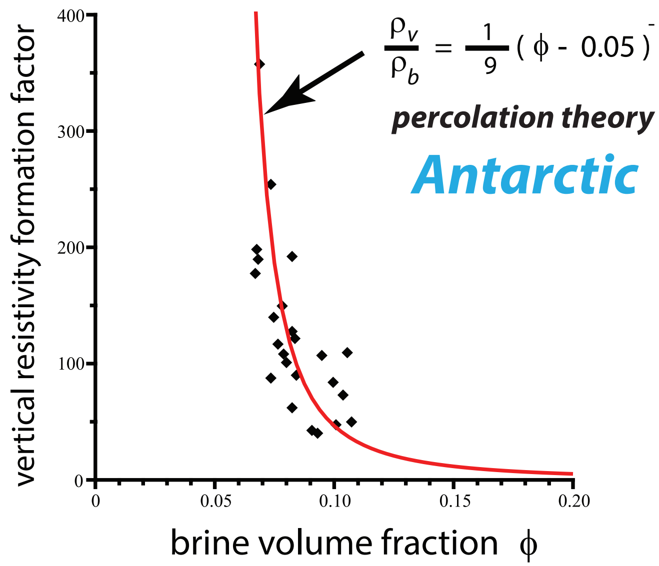
critical behavior of electrical transport in sea ice

electrical signature of the on-off switch for fluid flow









cross-borehole tomography - electrical classification of sea ice layers

Golden, Eicken, Gully, Ingham, Jones, Lin, Reid, Sampson, Worby 2017

Conclusions

1. Summer Arctic sea ice is melting rapidly.

2. Fluid flow through sea ice mediates many processes of importance to understanding climate change and the response of polar ecosystems.

3. Homogenization and statistical physics help *link scales* and provide rigorous methods for finding effective behavior, and advance how sea ice is represented in climate models.

4. Herglotz functions and Stieltjes integrals provide powerful methods of homogenization for sea ice structures and processes.

5. Our research will help to improve projections of climate change and the fate of the Earth sea ice packs.

THANK YOU

National Science Foundation

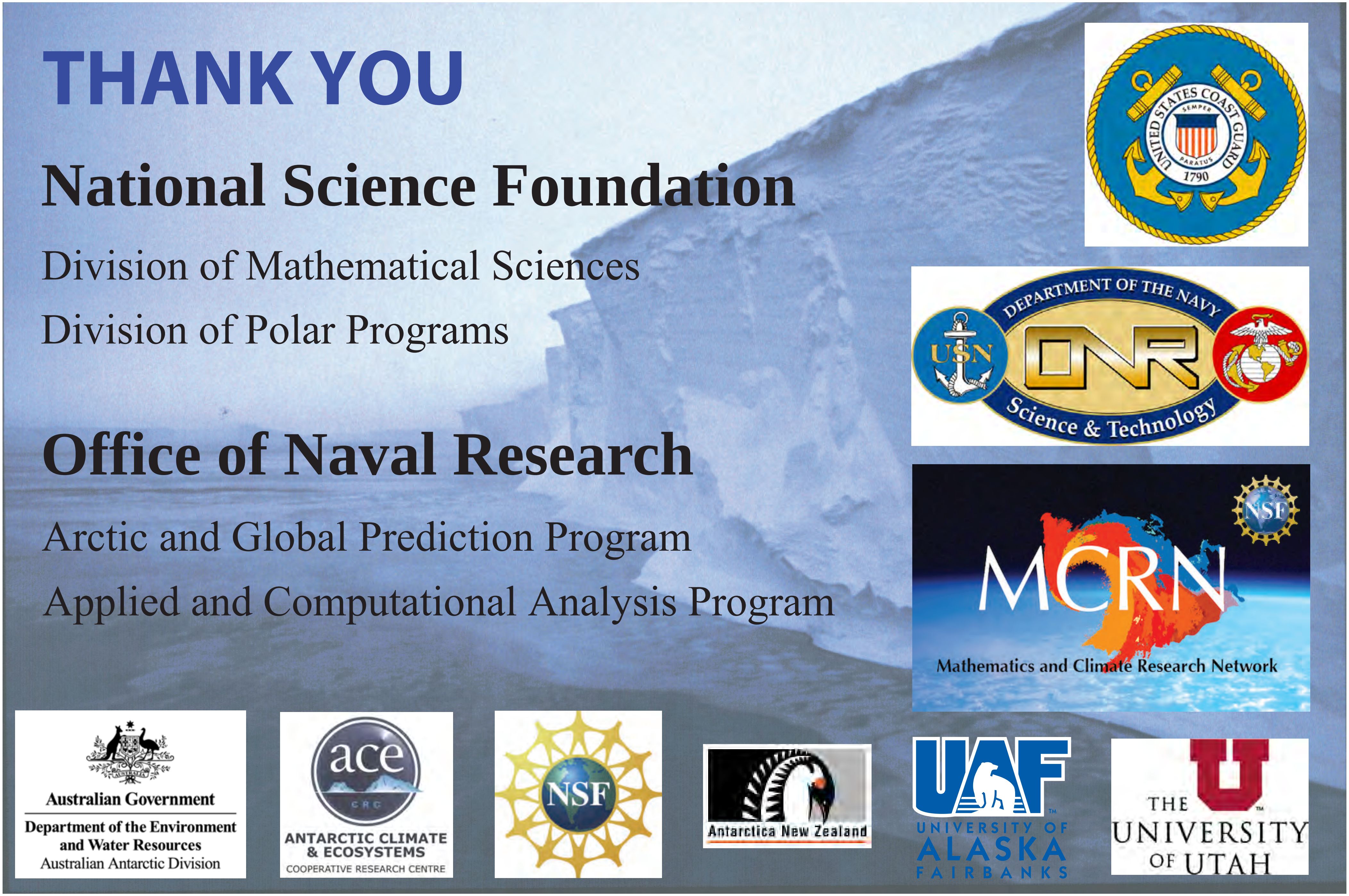
Division of Mathematical Sciences

Division of Polar Programs

Office of Naval Research

Arctic and Global Prediction Program

Applied and Computational Analysis Program



Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999