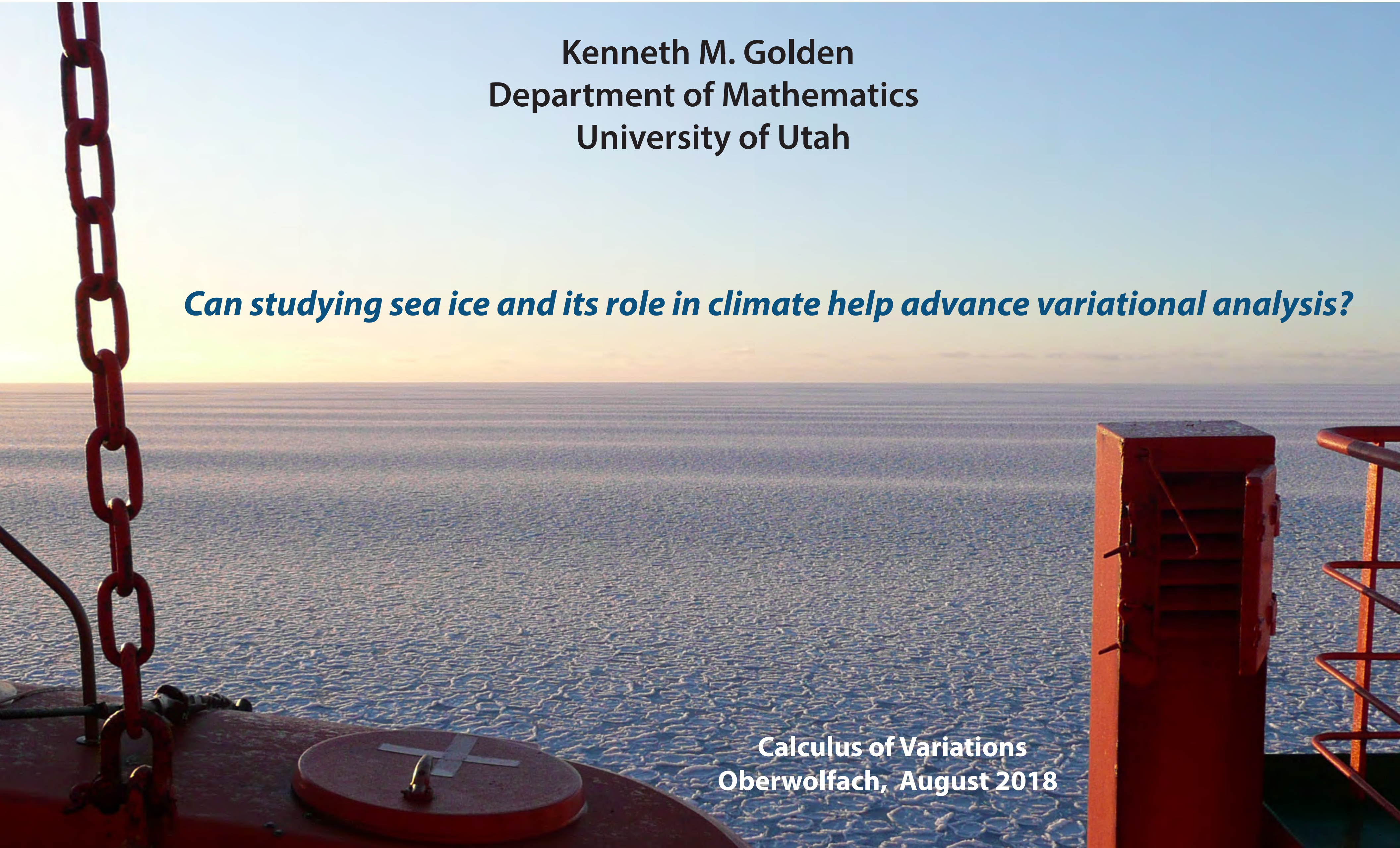
Homogenization, Random Matrices and Integral Representations for Transport in Composites

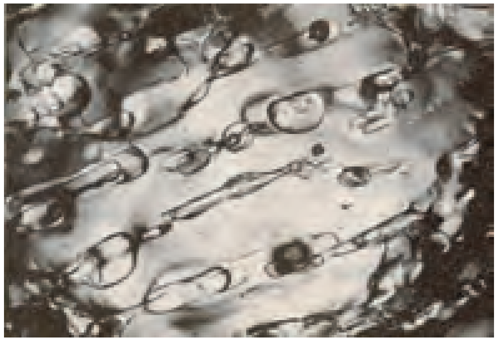


Kenneth M. Golden
Department of Mathematics
University of Utah

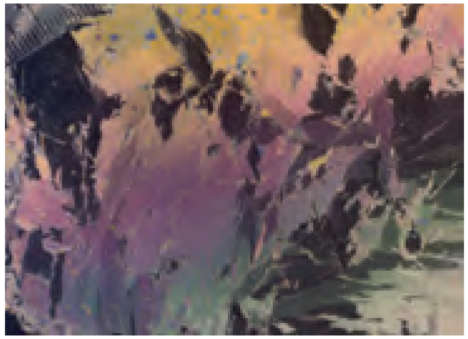
Can studying sea ice and its role in climate help advance variational analysis?

Calculus of Variations
Oberwolfach, August 2018

sea ice is a multiscale composite

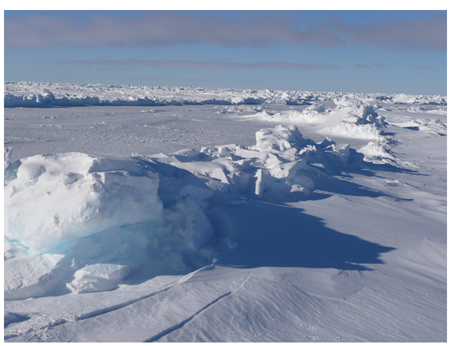


millimeters

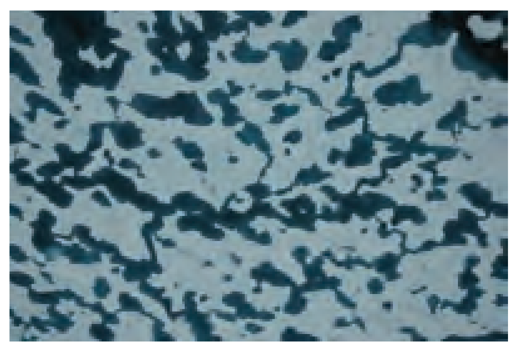


centimeters

meters



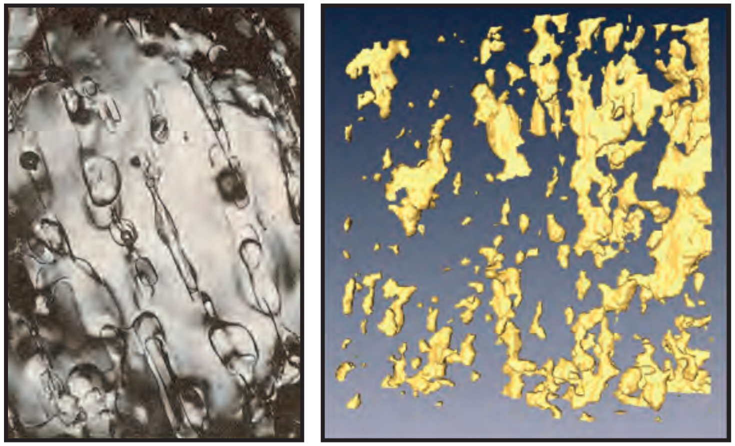
meters



kilometers

multiscale structure of sea ice

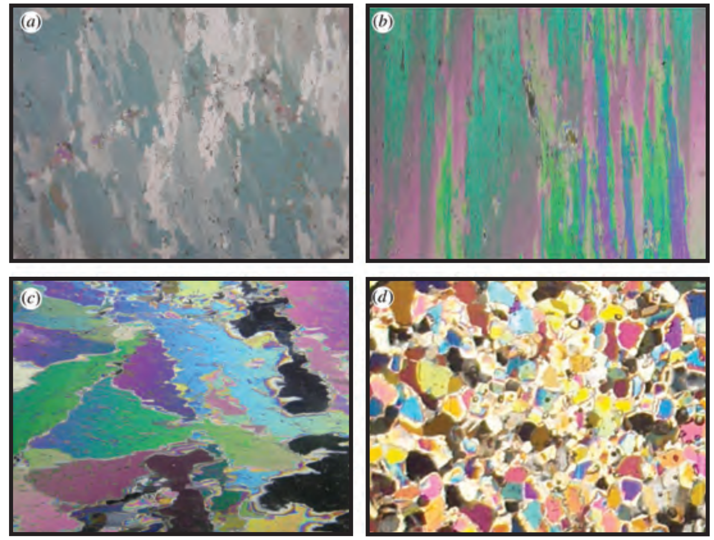
Brine Inclusions



mm

Golden et al GRL 2007

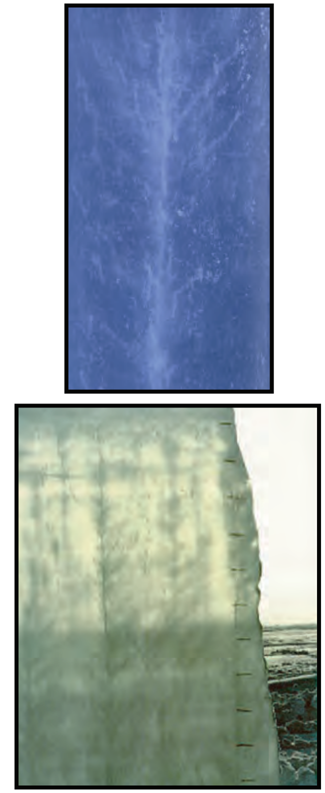
Polycrystals



cm

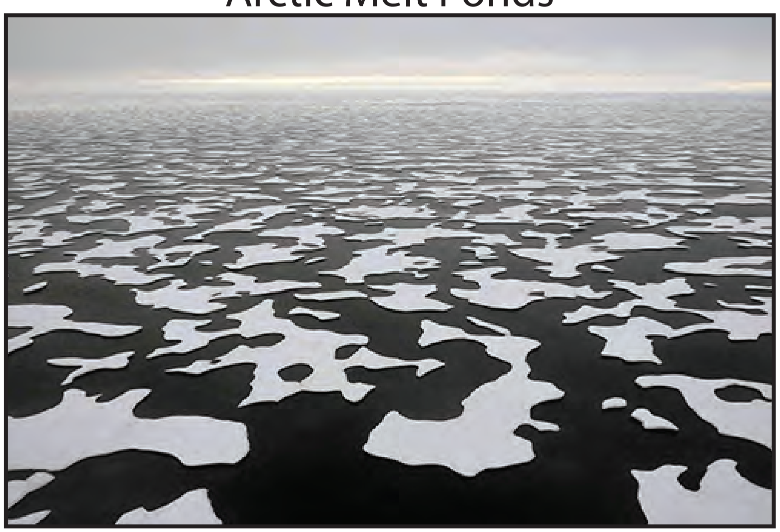
Gully et al. Proc. R. Soc. A 2015

Brine Channels



m

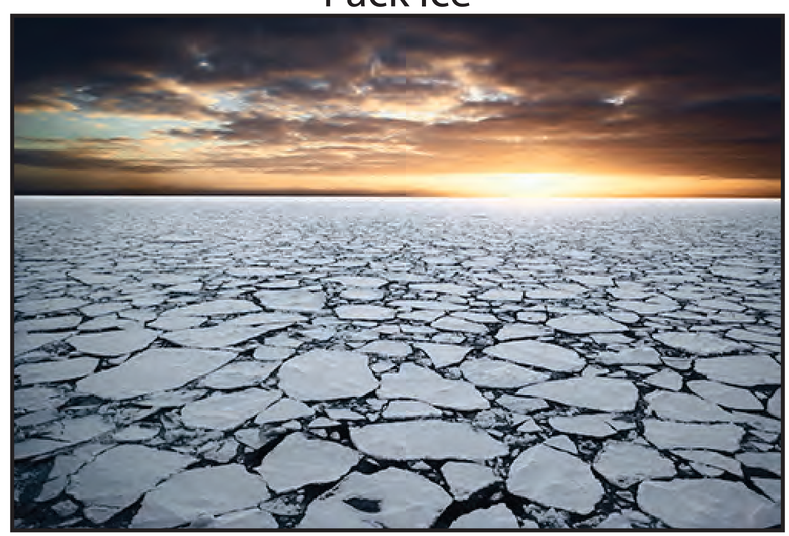
Arctic Melt Ponds



m

K. Frey

Pack Ice



km

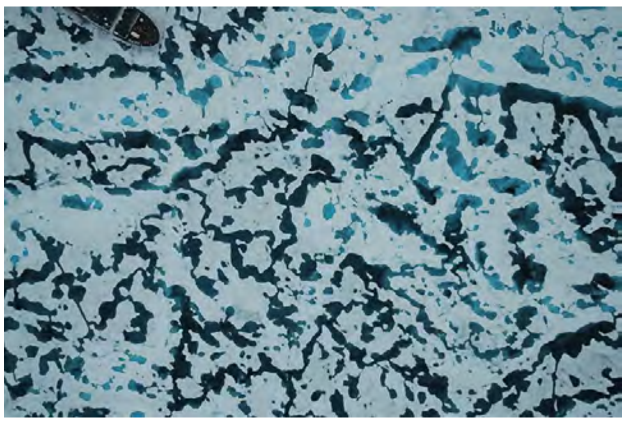
J. Weller

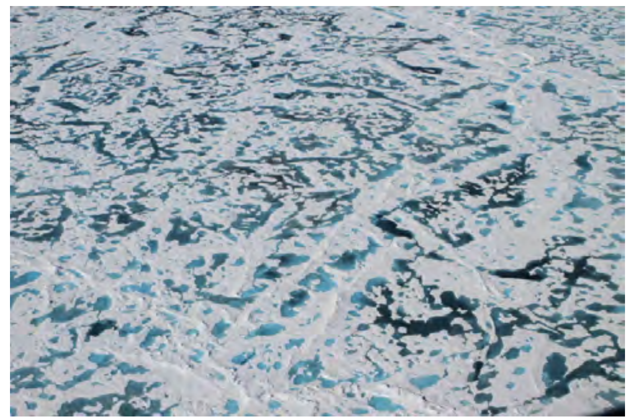
How do scales interact in the sea ice system?

basin scale - grid scale albedo

Linking Scales

km scale melt ponds

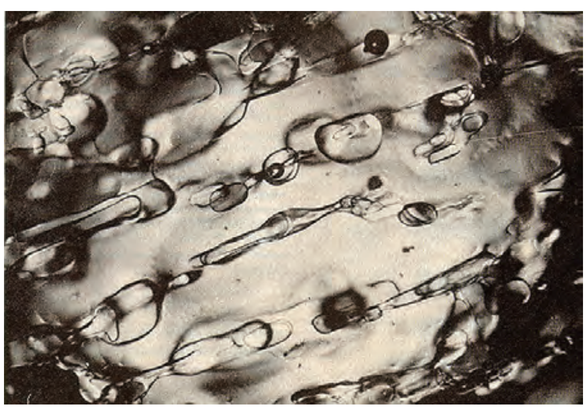




km scale melt ponds

Linking Scales

mm scale brine inclusions



meter scale snow topography

What is this talk about?

A tour of variational problems in the mathematical analysis of sea ice and its role in the climate system.

Addressing the problem of linking scales in Earth's sea ice system
MULTISCALE HOMOGENIZATION for SEA ICE
drives advances in theory of Stieltjes integrals for transport.

Find unexpected Anderson transition in composites along the way!

1. Fluid flow through sea ice, percolation

2. Analytic continuation for composites; BOUNDS
   remote sensing, inversion, spectral measures
   random matrix theory and Anderson transitions

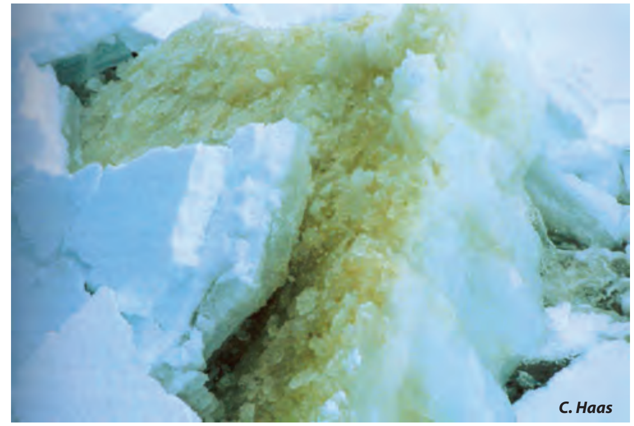
3. Stieltjes representations for advection diffusion, polycrystals, ocean waves in the sea ice pack

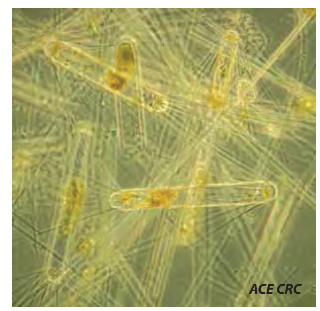
4. Arctic melt ponds, fractals, Ising model, Stieltjes integrals

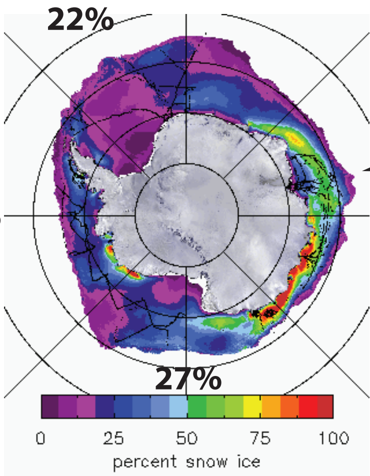
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities







T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

- evolution of salinity profiles

- ocean-ice-air exchanges of heat, CO_2

fluid permeability of a porous medium

how much water gets through the sample per unit time?

Darcy's Law

for slow viscous flow in a porous medium

averaged fluid velocity

pressure gradient

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\nabla p$$

viscosity

$\mathbf{k}$ = fluid permeability tensor

HOMOGENIZATION

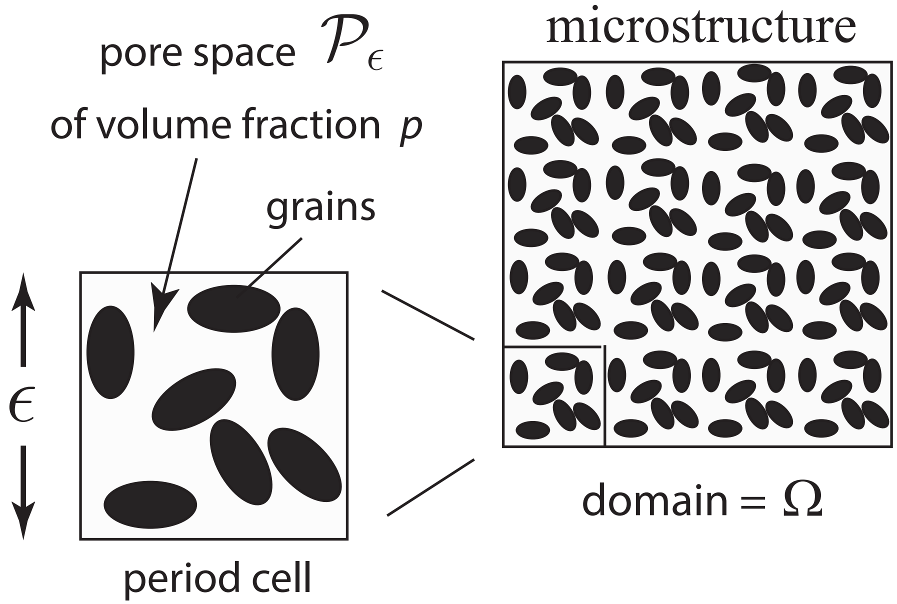
mathematics for analyzing effective behavior of heterogeneous systems

fluid permeability of a porous medium

HOMOGENIZE

how much fluid gets through the sample per unit time?

Stokes equations for fluid velocity $\mathbf{v}^\epsilon$, pressure p^ϵ, force $\mathbf{f}$:



$$\nabla p^\epsilon - \epsilon^2 \eta \Delta \mathbf{v}^\epsilon = \mathbf{f}, \qquad x \in \mathcal{P}_\epsilon$$

$$\nabla \cdot \mathbf{v}^\epsilon = 0, \qquad x \in \mathcal{P}_\epsilon$$

$$\mathbf{v}^\epsilon = 0, \qquad x \in \partial \mathcal{P}_\epsilon$$

$$\eta = \text{fluid viscosity}$$

via two-scale expansion

MACROSCOPIC EQUATIONS $\mathbf{v}^\epsilon \longrightarrow \mathbf{v}\,,\ p^\epsilon \longrightarrow p$ as $\epsilon \longrightarrow 0$

Darcy's law $(\mathrm{f} = 0)$

$$\mathbf{v} = -\frac{1}{\eta}\,\mathbf{k}\,\nabla p, \qquad x \in \Omega$$

$$\nabla \cdot \mathbf{v} = 0, \qquad x \in \Omega$$

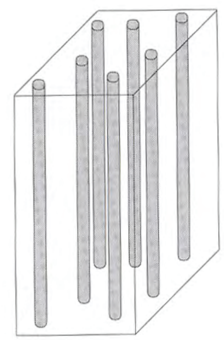
$\mathbf{k}(x) =$ **effective fluid permeability tensor**

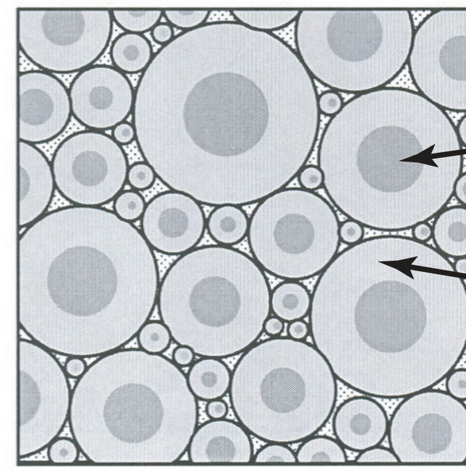
[Keller '80, Tartar '80, Sanchez-Palencia '80, J. L. Lions '81, Allaire '89, '91,'97]

PIPE BOUNDS on vertical fluid permeability k

Golden, Heaton, Eicken, Lytle, Mech. Materials 2006
Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophys. Res. Lett. 2007

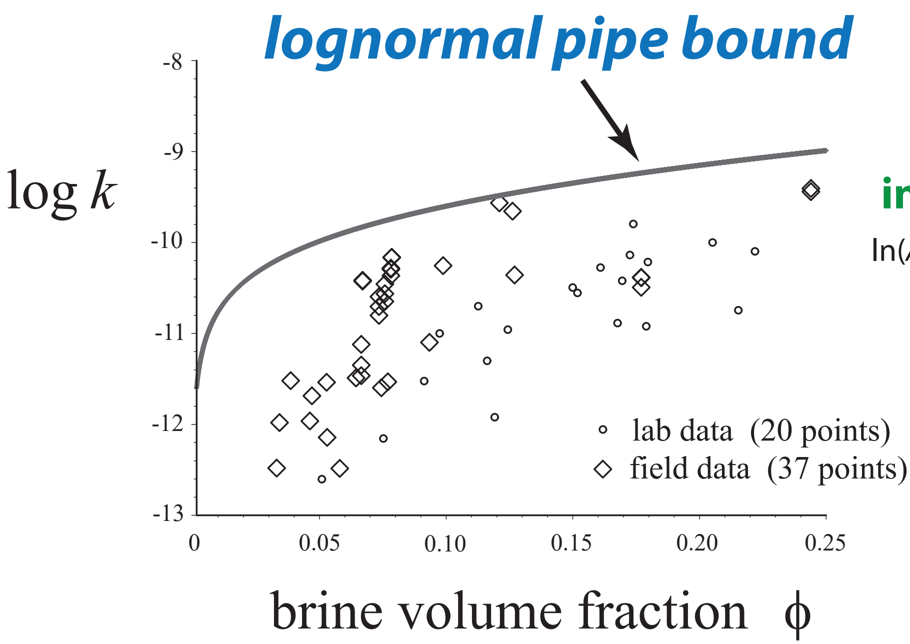
vertical pipes
with appropriate radii
maximize k





optimal coated cylinder geometry

fluid analog of arithmetic mean upper bound for effective conductivity of composites (Wiener 1912)



Golden et al., Geophys. Res. Lett. 2007

$$k \leq \frac{\phi \langle R^4 \rangle}{8 \langle R^2 \rangle} = \frac{\phi}{8} \langle R^2 \rangle e^{\sigma^2}$$

inclusion cross sectional areas A lognormally distributed

ln(A) normally distributed, mean μ (increases with T) variance σ^2 (Gow and Perovich 96)

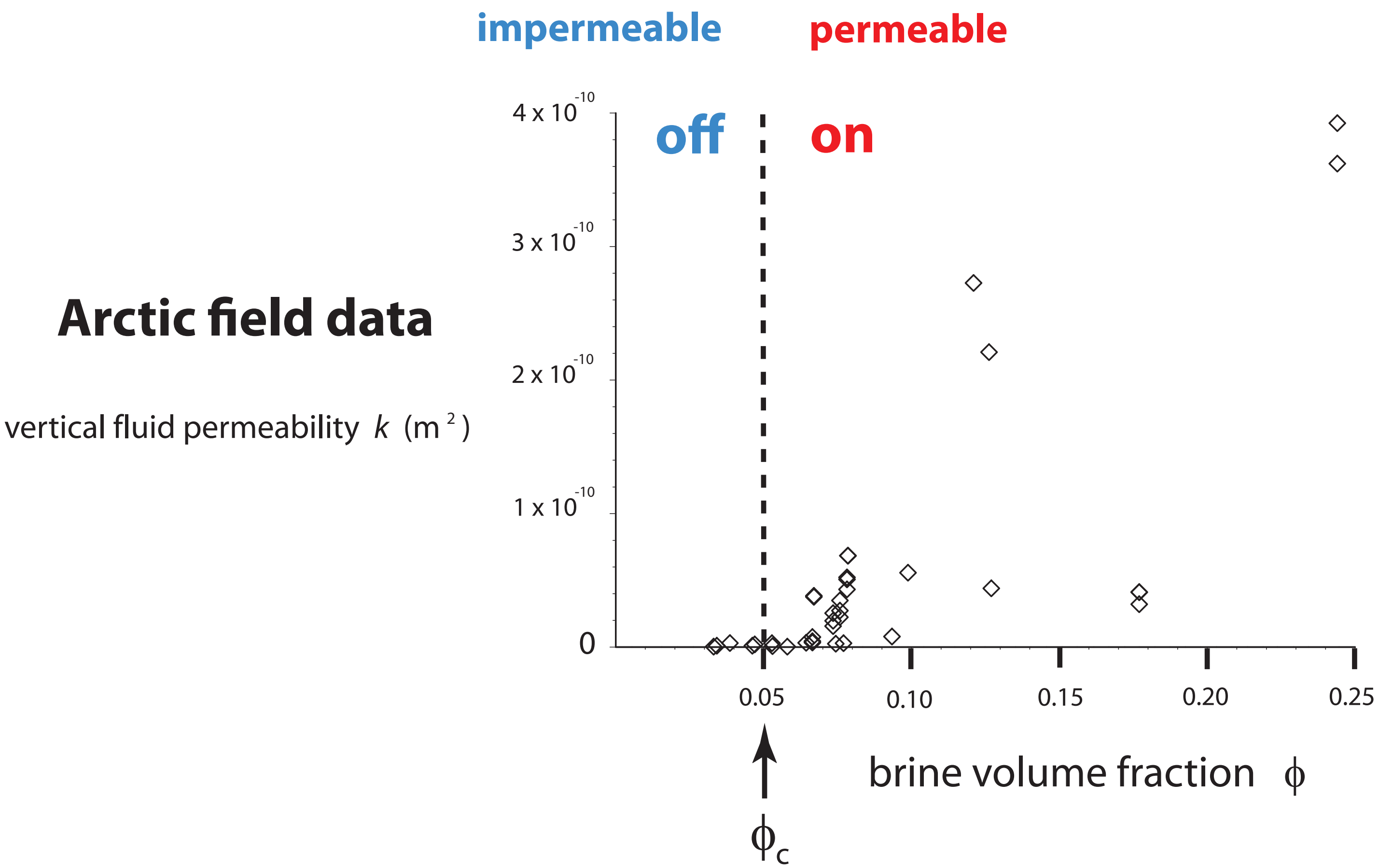
get bounds through variational analyis of ***trapping constant*** γ for diffusion process in pore space with absorbing BC

Torquato and Pham, *PRL* 2004

$$\mathbf{k} \leq \gamma^{-1} \mathbf{I}$$

for any ergodic porous medium (Torquato 2002, 2004)

Critical behavior of fluid transport in sea ice



on - off switch for fluid flow

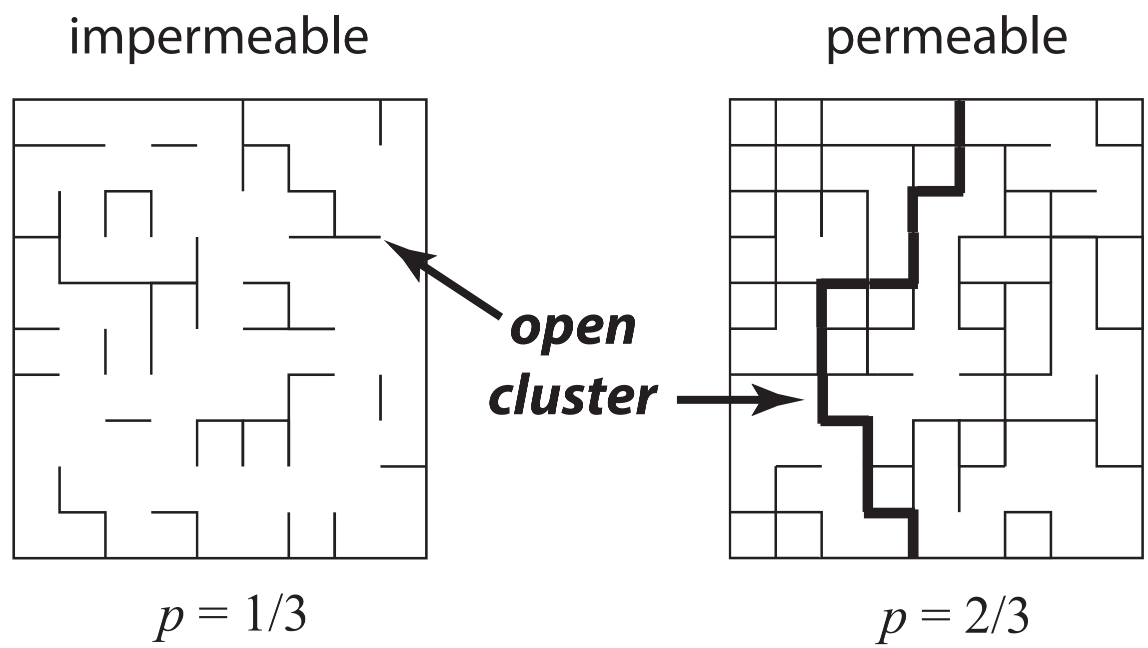
critical brine volume fraction $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ C,\ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007
Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

percolation theory

probabilistic theory of connectedness



bond → **open** *with probability* p
closed *with probability* $1-p$

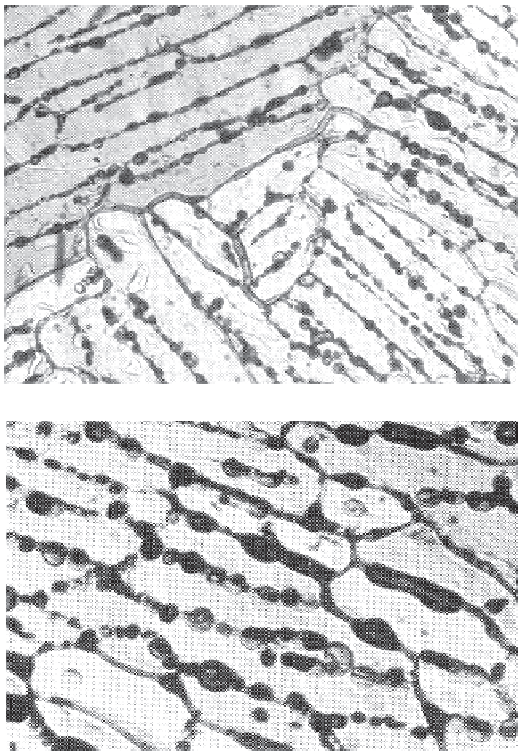
percolation threshold

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

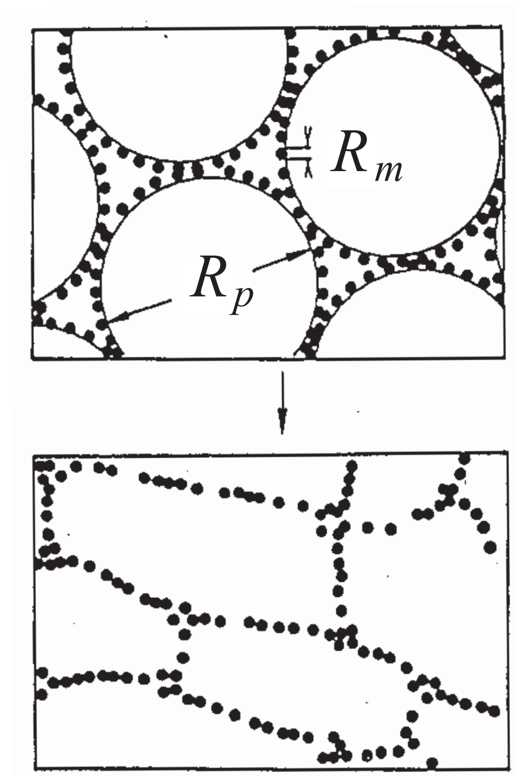
smallest *p* for which there is an infinite open cluster

Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

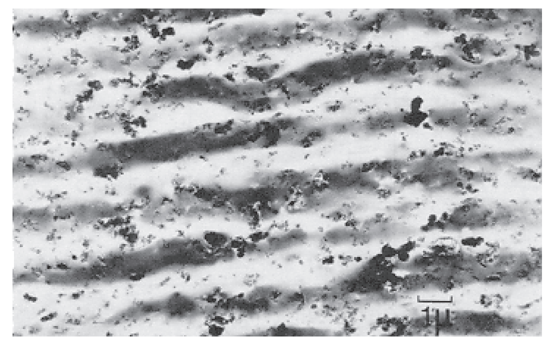
$\phi_c \approx 5\ \%$ Golden, Ackley, Lytle, *Science,* 1998



sea ice



compressed powder



radar absorbing composite

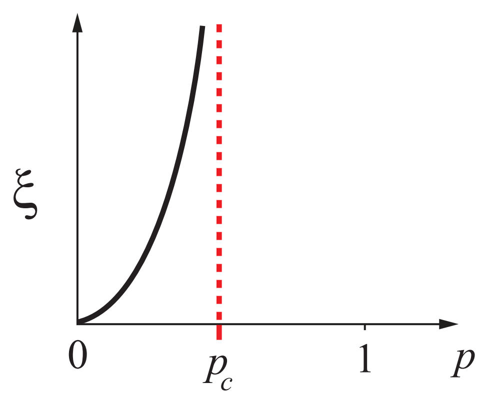
sea ice is radar absorbing

order parameters in percolation theory

geometry

correlation length

characteristic scale
of connectedness

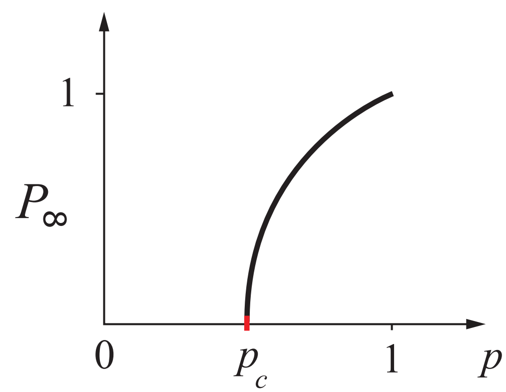


$$\xi(p) \sim |p - p_c|^{-\nu}$$

$$p \longrightarrow p_c$$

infinite cluster density

probability the origin
belongs to infinte cluster

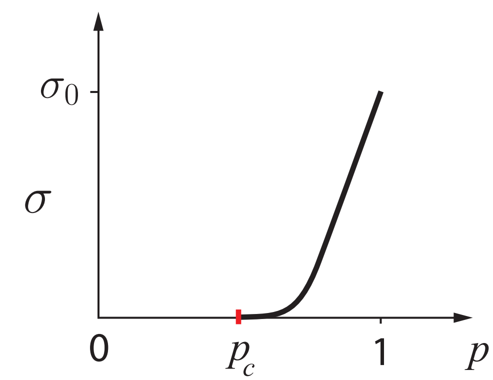


$$P_\infty(p) \sim (p - p_c)^{\beta}$$

$$p \longrightarrow p_c^+$$

transport

effective conductivity
or fluid permeability



$$\sigma(p) \sim \sigma_0 \, (p - p_c)^t$$

$$p \longrightarrow p_c^+$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

$1 \le t \le 2$ (for idealized model), Golden, *Phys. Rev. Lett.* 1990 **;** *Comm. Math. Phys.* 1992

non-universal behavior in continuum

Thermal evolution of permeability and microstructure in sea ice

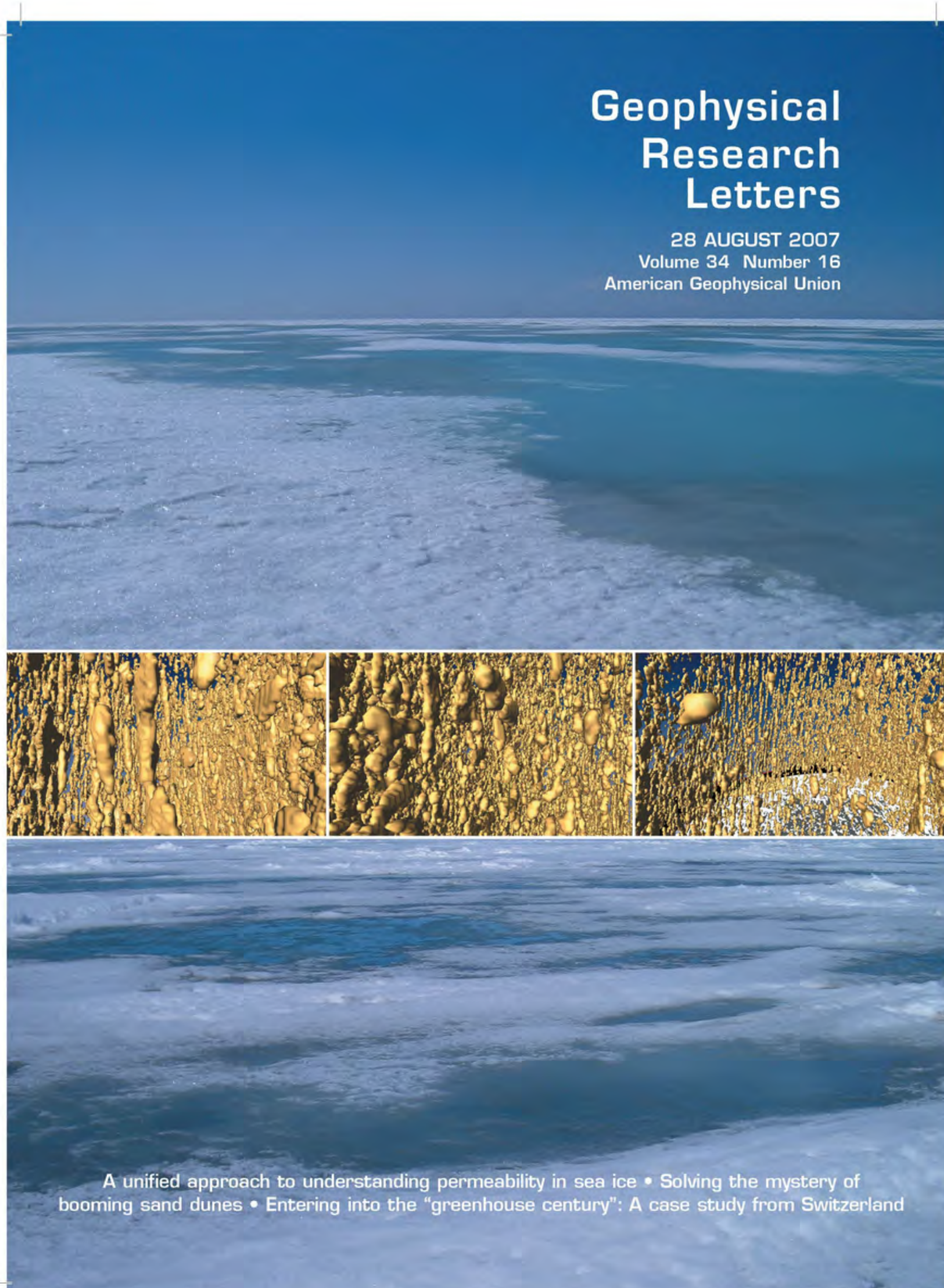
Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007

micro-scale

controls

macro-scale

processes



percolation theory

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

hierarchical model
network model
rigorous bounds

agree closely
with field data

X-ray tomography for brine inclusions

unprecedented look at thermal evolution of brine phase and its connectivity

confirms rule of fives

Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

Remote sensing of sea ice

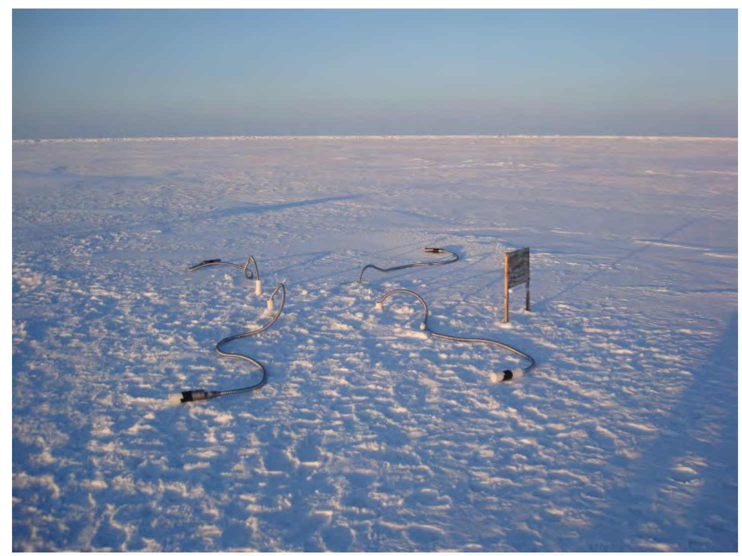
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

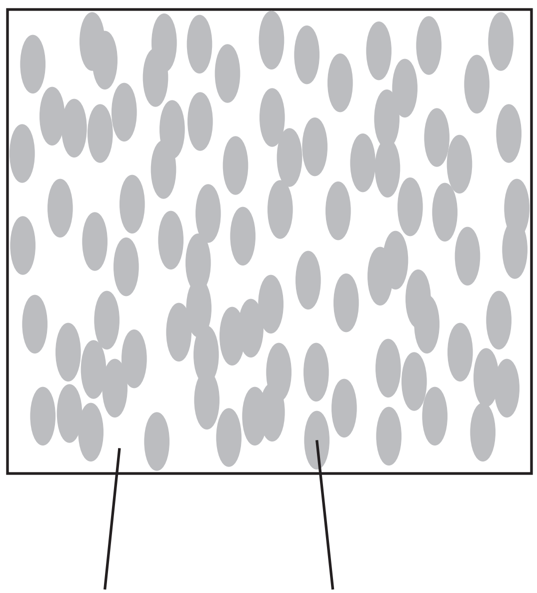
$$\varepsilon^*$$

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry} \right)$$

Herglotz function

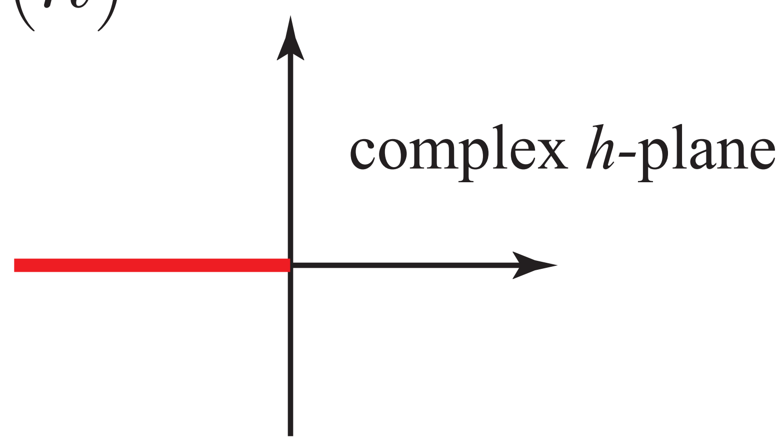
Analytic continuation method for bounding complex ϵ^*

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

$$m(h) = \frac{\epsilon^*}{\epsilon_2}\left(\frac{\epsilon_1}{\epsilon_2}\right) \qquad h = \frac{\epsilon_1}{\epsilon_2}$$

Exploit analytic properties of $m(h)$

- $m(h)$ analytic off negative real axis



- $m(h)$: UHP $\longrightarrow$ UHP

Theory of Effective Electromagnetic Behavior of Composites

analytic continuation method

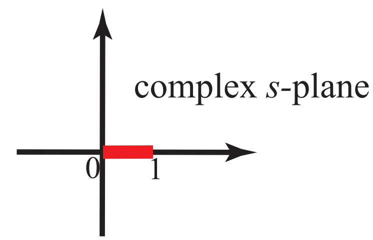
Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)
Theory of Composites, Milton (2002)

composite geometry
(spectral measure μ) $\longrightarrow$ ε^*

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* $\longrightarrow$ ***composite geometry***
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

Stieltjes integral representation

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

material parameters

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass $= p_1$
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = (s + \Gamma\chi)^{-1} e_k$$

$\Gamma\chi$ **: microscale $\longrightarrow$ macroscale**

$\Gamma\chi$ ***links scales***

Golden and Papanicolaou, Comm. Math. Phys. 1983

using the Stieltjes integral representation to obtain bounds

"linear programming" Golden and Papanicolaou, *CMP* 1983

M_1 = **the set of positive Borel measures on [0,1], compact, convex**

$F_s(\mu): \; M_1 \longrightarrow \mathbb{C}$ **linear functional**

extremal values (bounds) are images of extreme points of M_1

Dirac point measures $\dfrac{\mu_0}{s - z^*}$

---

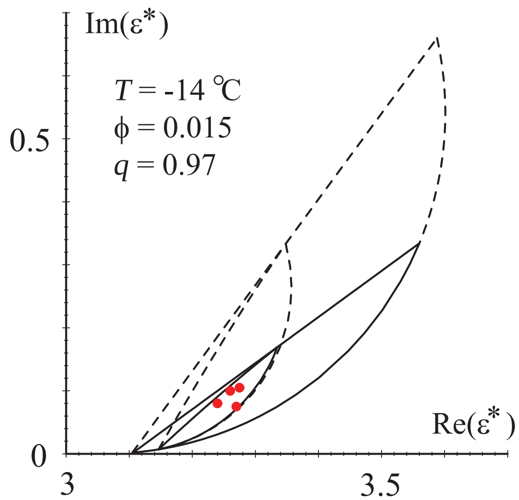
higher order bounds -- iterated fractional linear transformations

$$F_1(s) = \frac{1}{\mu_0} - \frac{1}{sF(s)}$$ ←

Baker 1969
Milton 1981
Bergman 1982
Felderhof 1984
Golden 1986

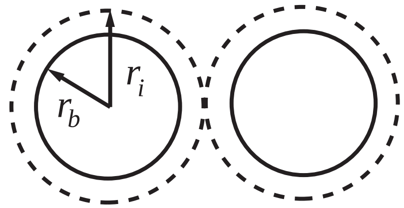
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle



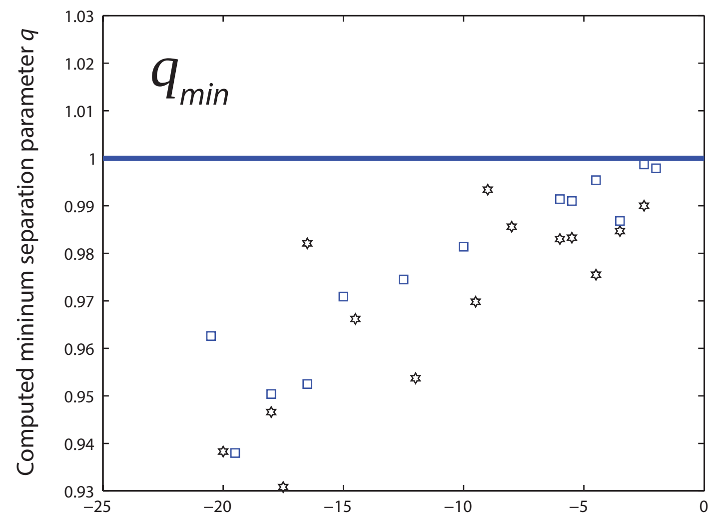
$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

Bruno 1991

inverse bounds



Slab temperature °C

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

direct calculation of spectral measure

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).

2. The fundamental operator $\chi\Gamma\chi$ becomes a random matrix depending only on the composite geometry.

3. Compute the eigenvalues λ_i and eigenvectors of $\chi\Gamma\chi$ with inner product weights α_i

$$\mu(\lambda) = \sum_i \alpha_i \, \delta(\lambda - \lambda_i)$$

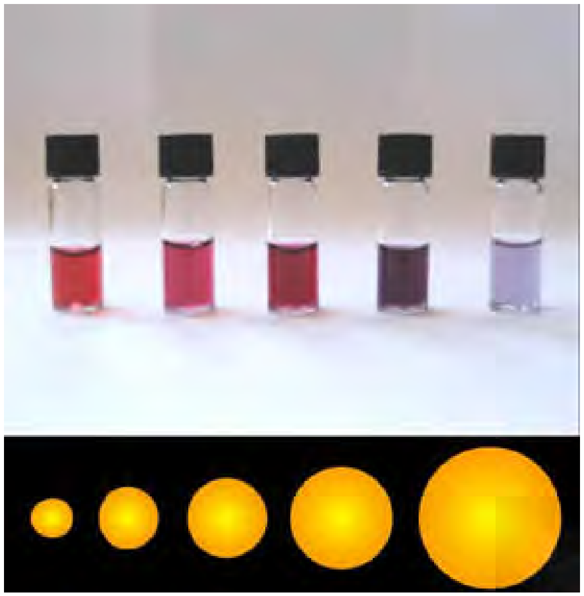
Dirac point measure (Dirac delta)

earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Surface Plasmon Resonances

collective oscillations of electrons on metal / dielectric interface

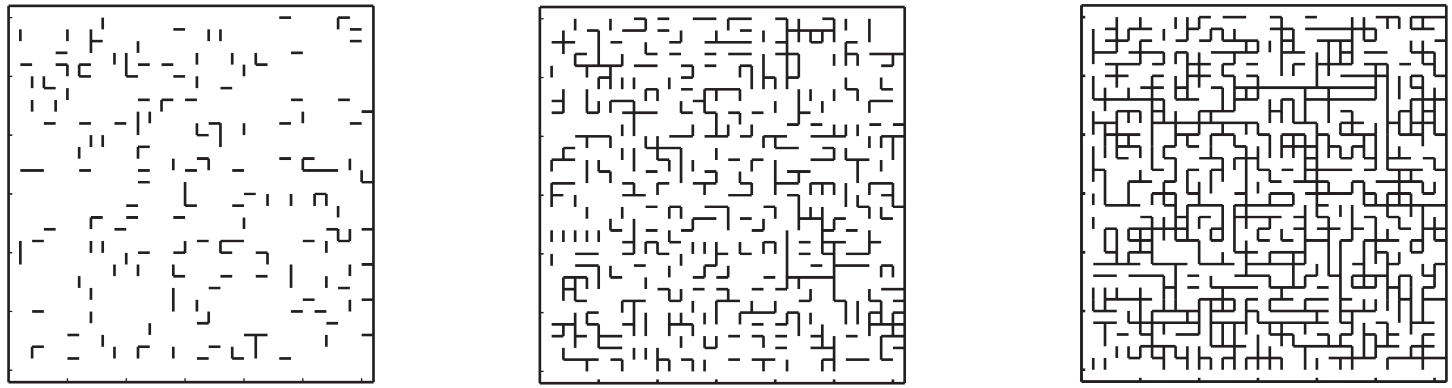


suspension of
gold nanoparticles
absorbs green
and blue light:

WE SEE RED

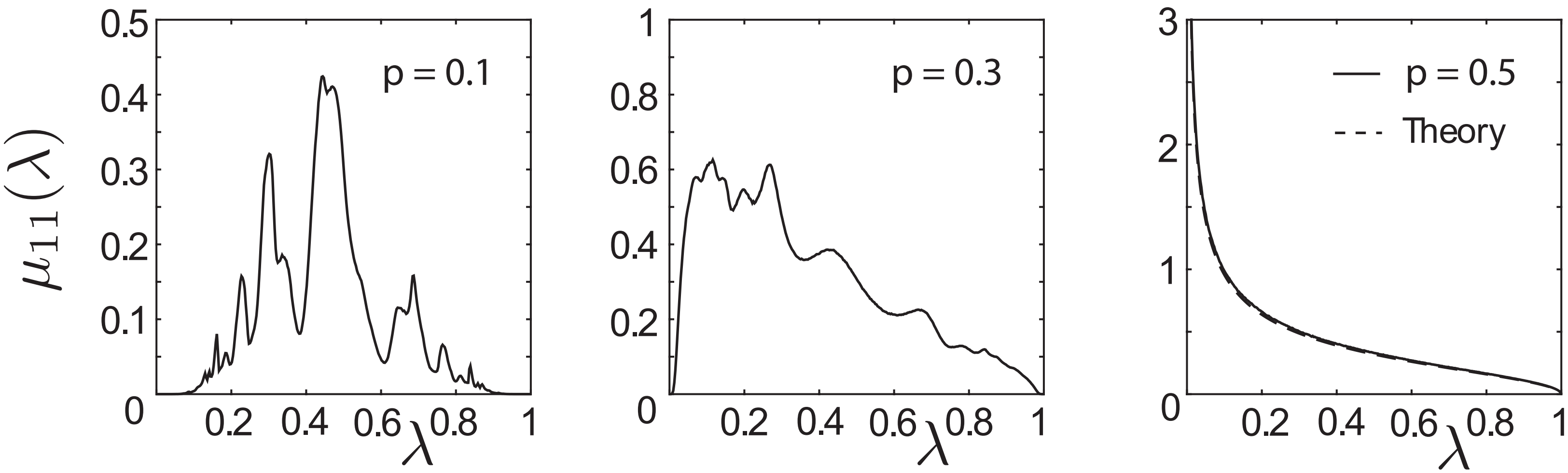
Michael Faraday's gold colloids - origins of nanoscience 1850s

Spectral statistics for 2D random resistor network



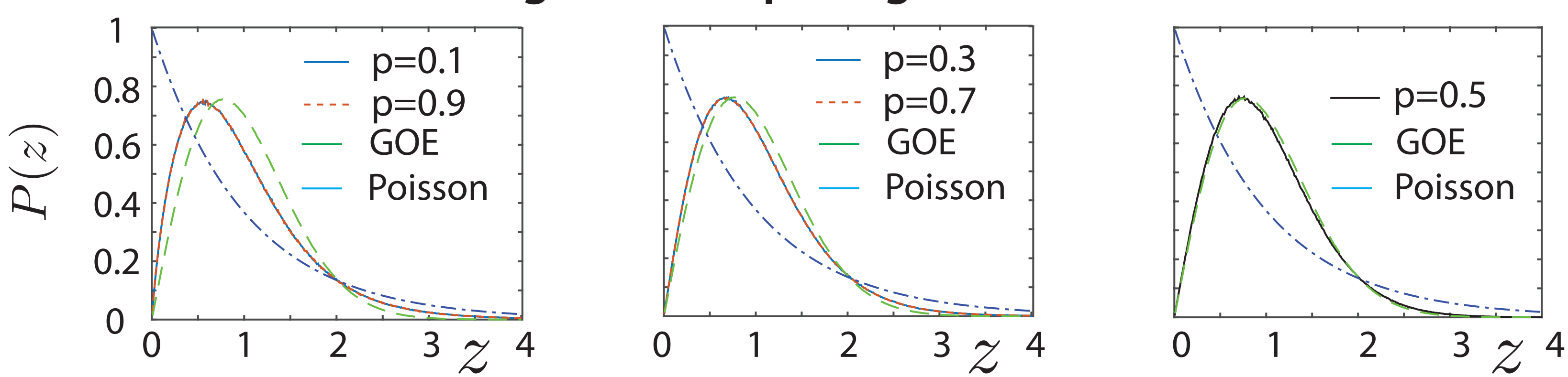
Spectral Measures

Murphy and Golden, *J. Math. Phys.*, 2012
Murphy et al. *Comm. Math. Sci.*, 2015



$p_c = 0.5$

Eigenvalue Spacing Distributions



Murphy, Cherkaev, Golden, *PRL*, 2017

Eigenvalue Statistics of Random Matrix Theory

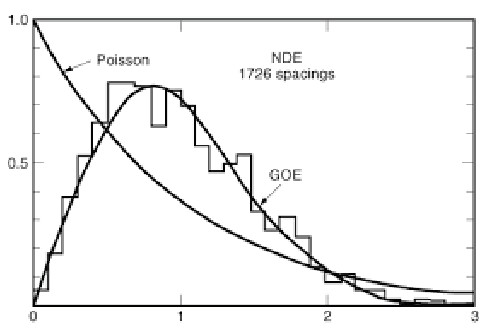
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

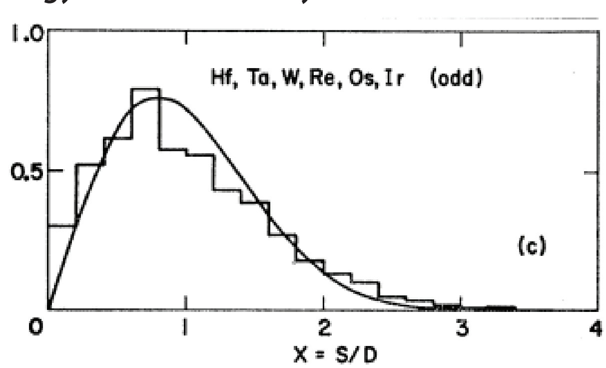
$[\mathbf{N}]_{ij} \sim N(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^T)/2$ $\quad$ ***Gaussian orthogonal ensemble (GOE)***

$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^\dagger)/2$ $\quad$ ***Gaussian unitary ensemble (GUE)***

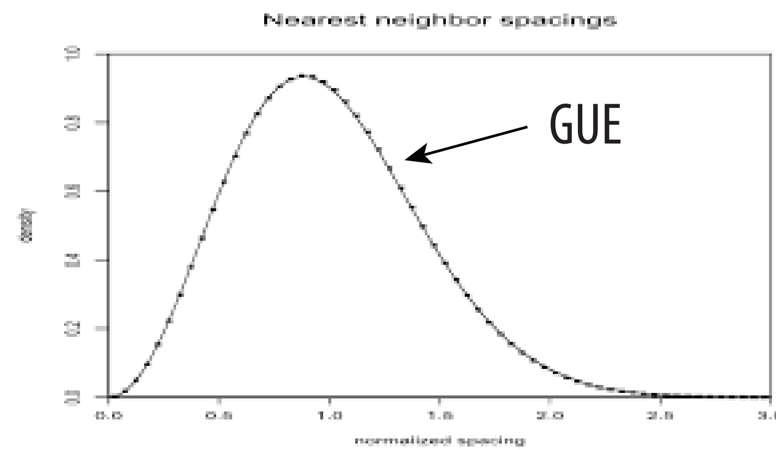
Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics

Spacing distributions of energy levels for heavy atomic nuclei





Spacing distributions of the first billion zeros of the Riemann zeta function



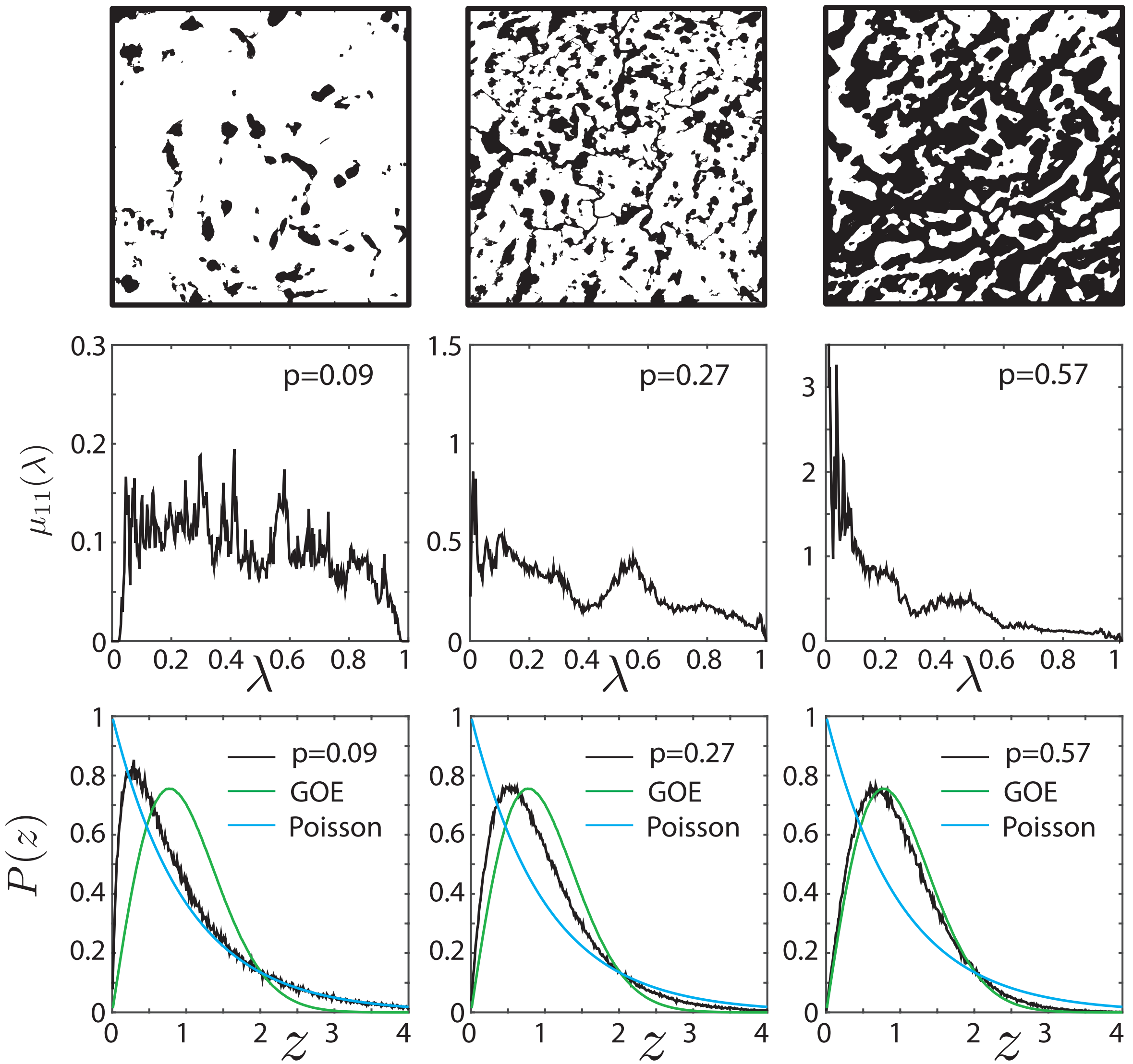
RMT used to characterize disorder-driven transitions in mesoscopic conductors, neural networks, random graph theory, etc.

Phase transitions ~ transitions in universal eigenvalue statistics.

Spectral computations for Arctic melt ponds

spectral measures

eigenvalue spacing distributions



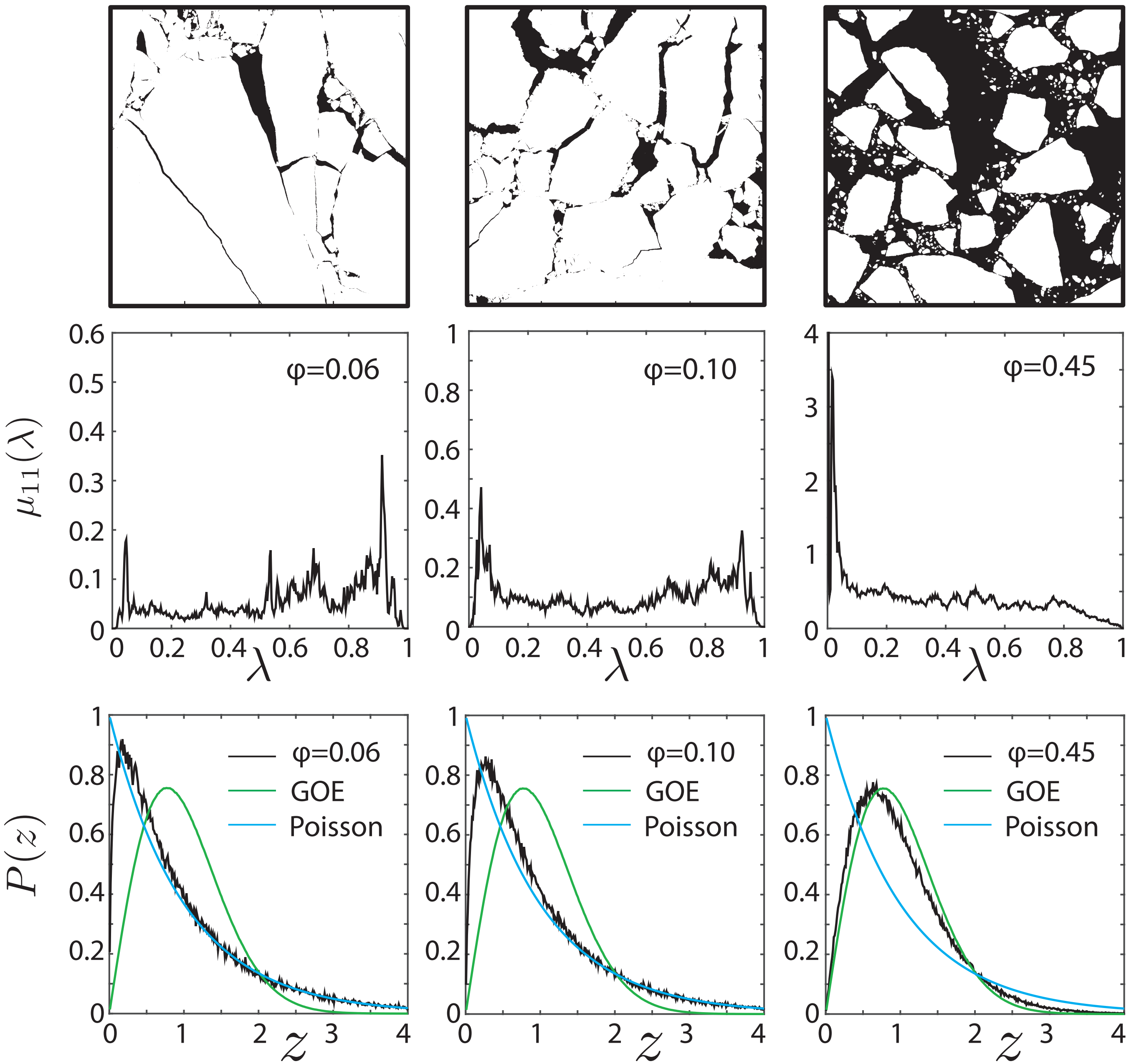
uncorrelated → **level repulsion**

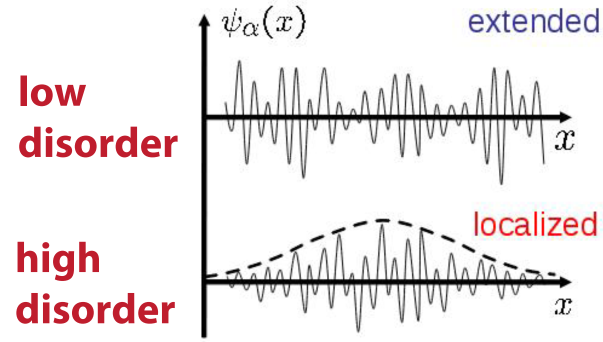
TRANSITION

Ben Murphy
Elena Cherkaev
Ken Golden
2017

eigenvalue statistics for transport tend toward the UNIVERSAL Wigner-Dyson distribution as the "conducting" phase percolates

Spectral computations for Arctic sea ice pack





metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

we find a surprising analog

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION TRANSITION → **transition to universal eigenvalue statistics (GOE)**
extended states, mobility edges

SPR

-- but without wave interference or scattering effects ! --

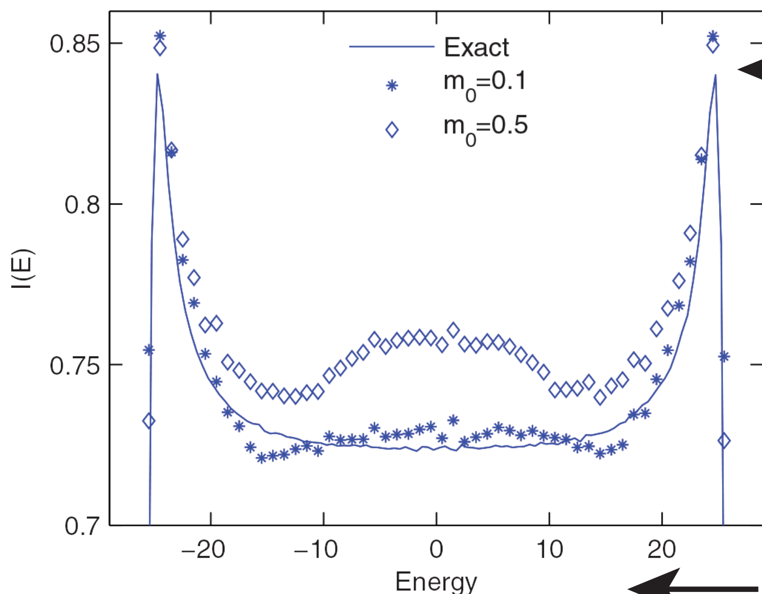
eigenvector localization and mobility edges

Inverse Participation Ratio: $$I(\vec{v}_n) = \sum_{i=1}^{N} |(\vec{v}_n)_i|^4$$

Completely Localized: $$I(\vec{e}_n) = 1$$

Completely Extended: $$I\left(\frac{1}{\sqrt{N}}\vec{1}\right) = \frac{1}{N}$$

Anderson Model



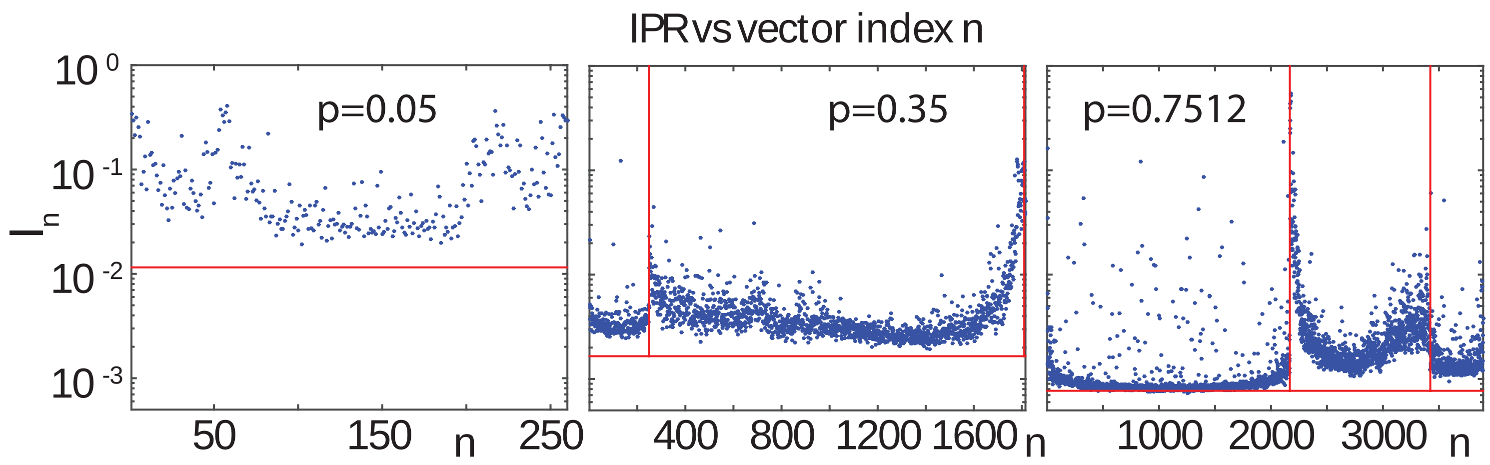
Mobility Edge

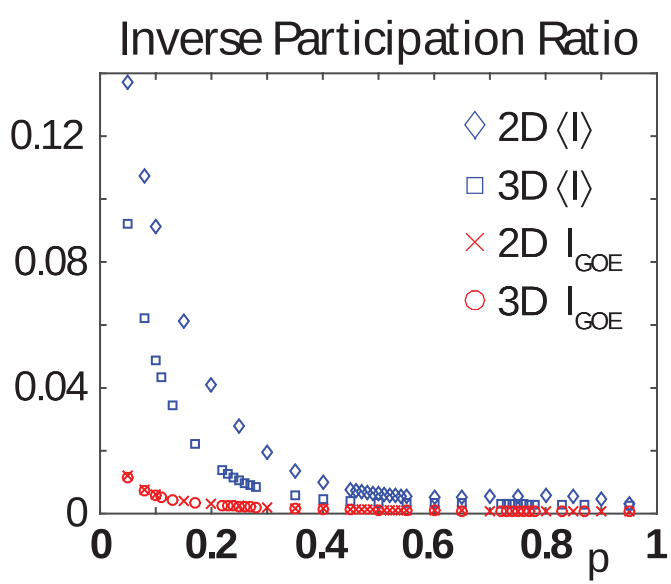
"Eigenvalues"

FIG. 4. (Color online) IPR for Anderson model in two dimensions with $x = 6.25$ ($w = 50$) from exact diagonalization (solid line) and from LDRG with different values of the cutoff m_0. LDRG data are averaged over 100 runs of systems with 100×100 sites.

PHYSICAL REVIEW B 90, 060205(R) (2014)

Localization properties of eigenvectors in random resistor networks





$$I_n = \sum_i (\vec{v}_n)_i^4$$

Murphy, Cherkaev, Golden, 2017

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

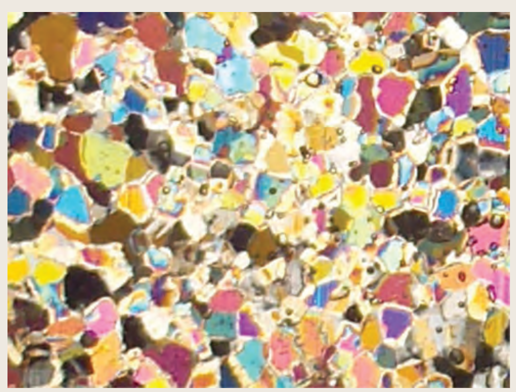
  Milton (1981, 2002), Barabash and Stroud (1999), ...

- **Forward and inverse bounds**

  orientation statistics

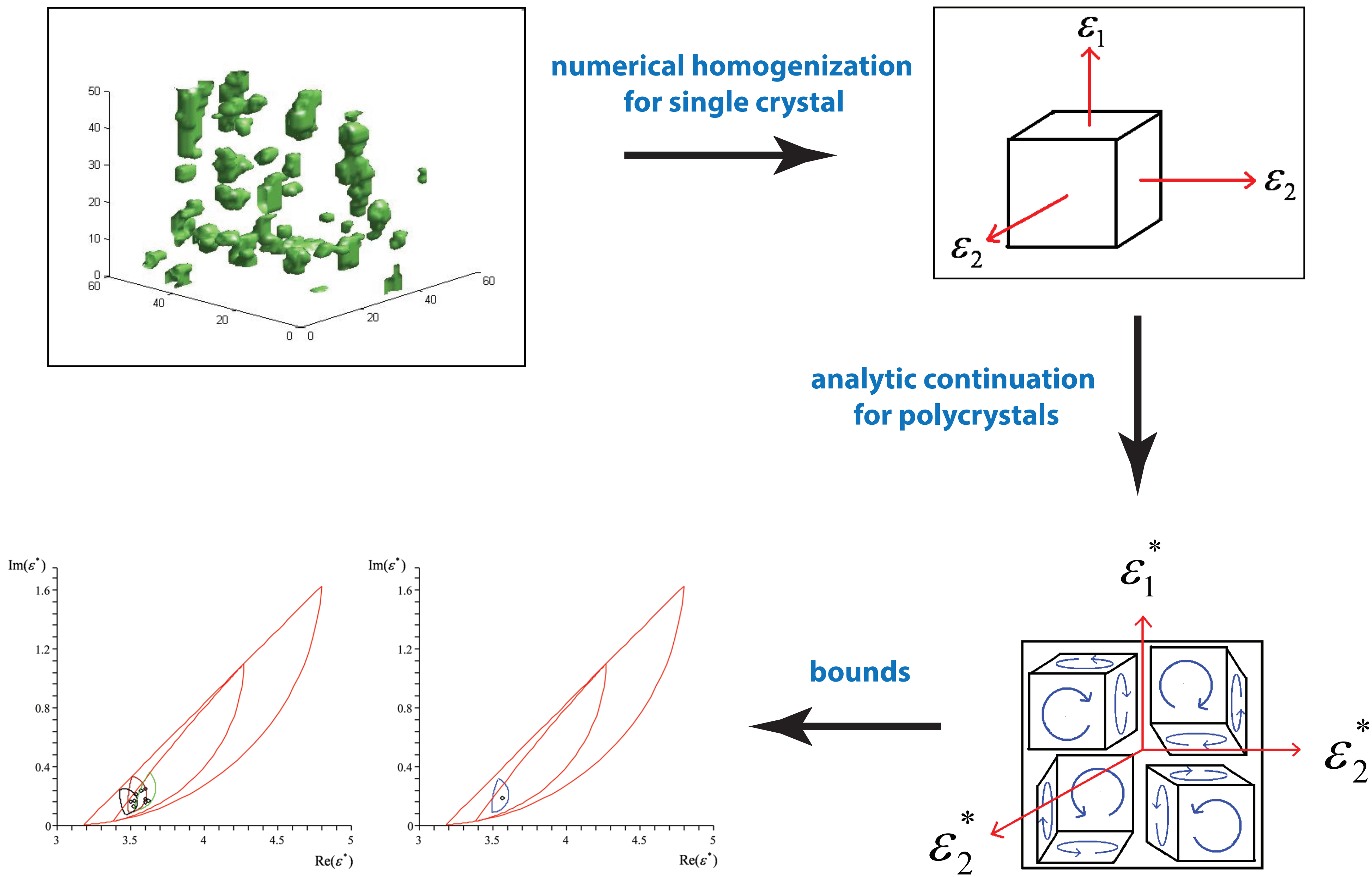
- **Applied to sea ice using two-scale homogenization**

- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

two scale homogenization for polycrystalline sea ice

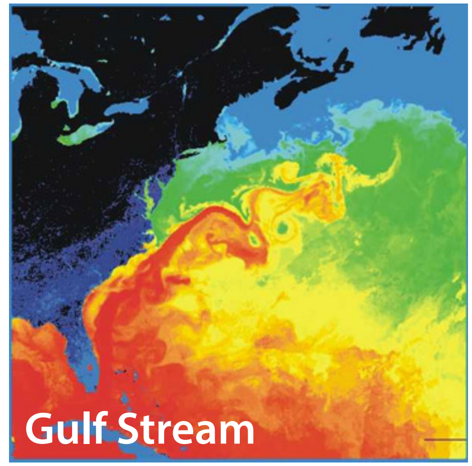


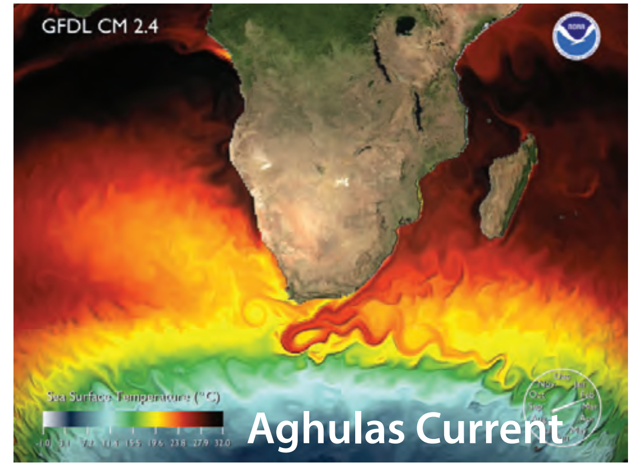
Gully, Lin, Cherkaev, Golden, Proc. Roy. Soc. A (and cover) 2015

advection enhanced diffusion

effective diffusivity

sea ice floes diffusing in ocean currents
diffusion of pollutants in atmosphere
salt and heat transport in ocean
heat transport in sea ice with convection





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\Big\downarrow \quad \textbf{\textit{homogenize}}$$

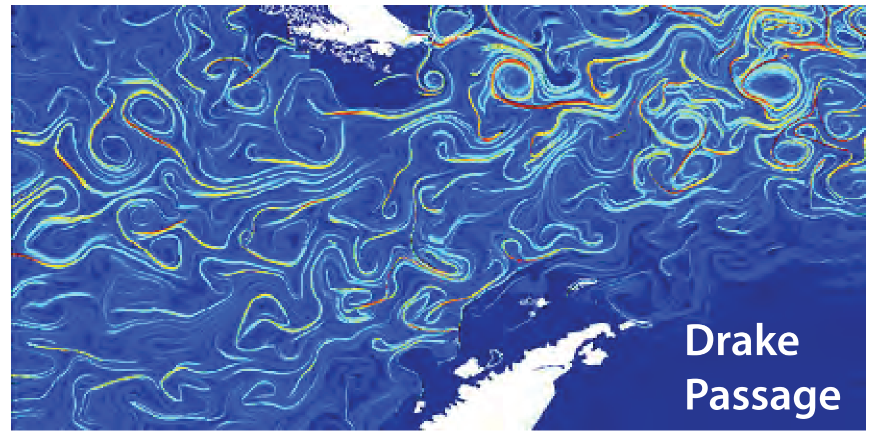
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

Stieltjes integral for κ^* ***with spectral measure***

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, 2018



Stieltjes Integral Representation for Advection Diffusion

[Murphy, Cherkaev, Zhu, Xin & Golden 2018]

[Murphy, Cherkaev, Xin, Zhu & Golden 2017]

$$\kappa^* = \kappa \left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2} \right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

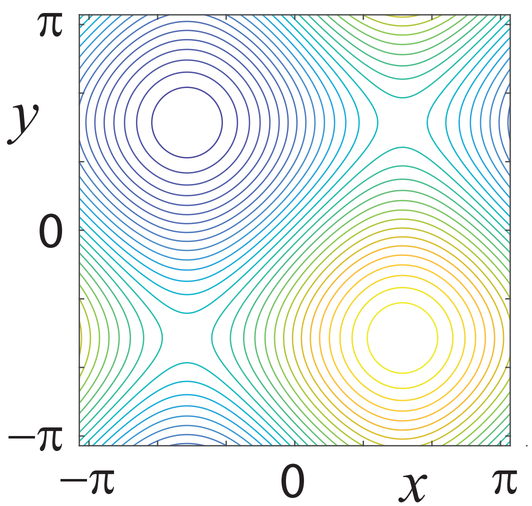
- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H \Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H \Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

separation of material properties and flow field

spectral measure calculations

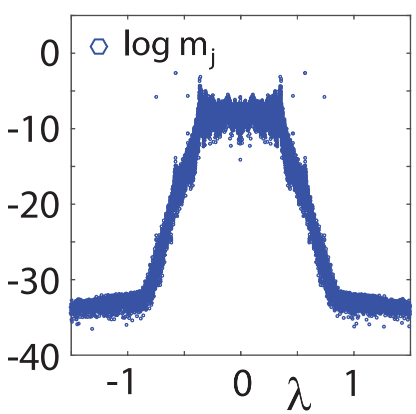
RIGOROUS BOUNDS on convection - enhanced thermal conductivity of sea ice

Kraitzman, Hardenbrook, Murphy, Zhu, Cherkaev, Golden 2018



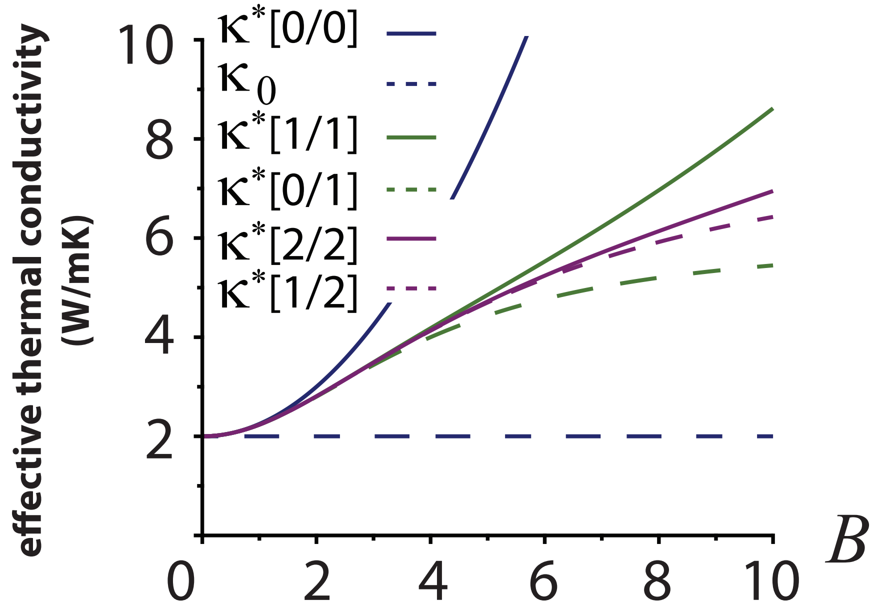
BC **- flow streamlines**

$$H = B\ \sin x - C\ \sin y \qquad B = C$$



spectral masses

Murphy, Cherkaev, Zhu, Xin, Golden 2017



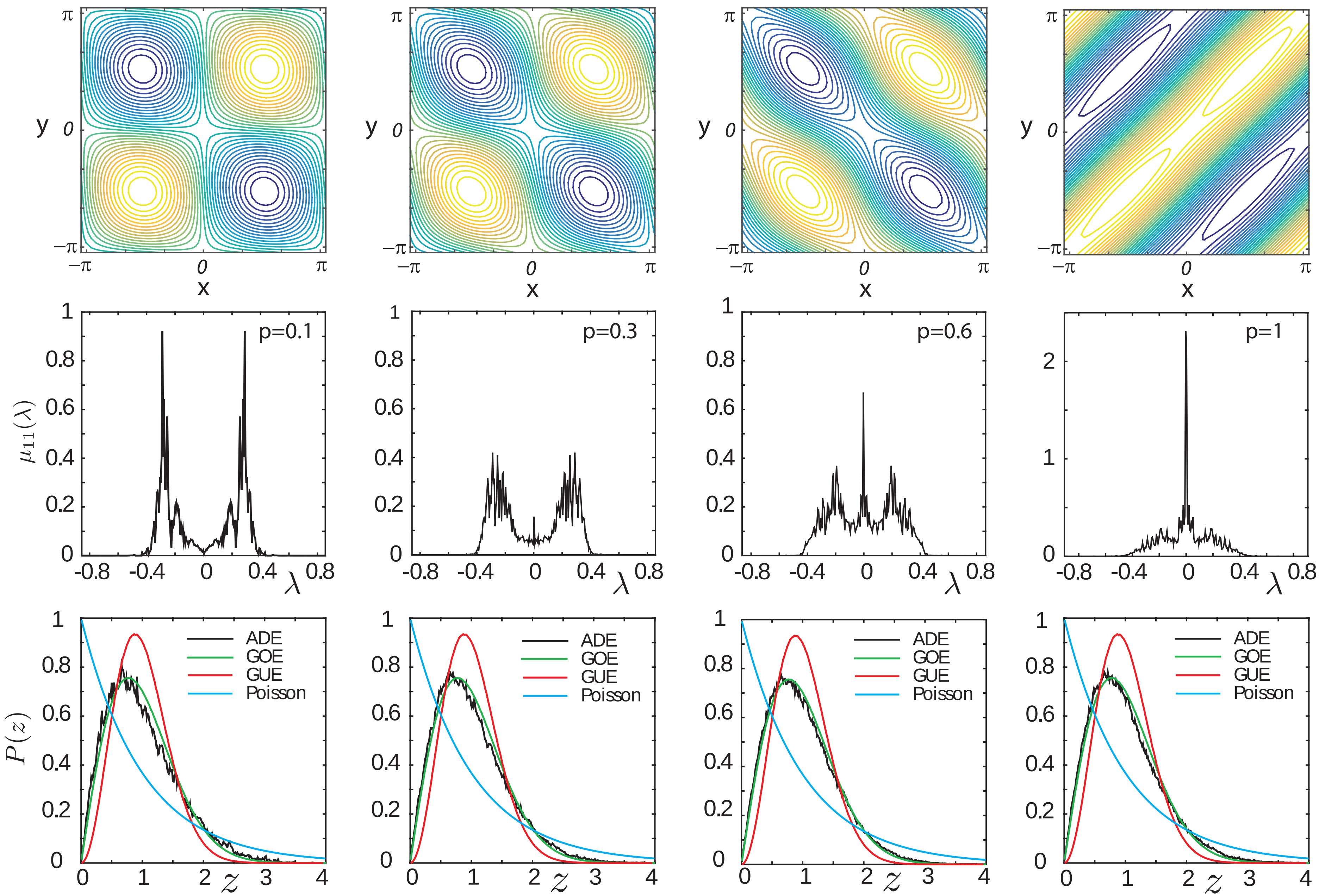
rigorous Padé bounds from Stieltjes integral + analytical calculations of moments of measure

Advection Enhanced Diffusion in a Porous Medium

Kraitzman, Cherkaev, Golden, 2018

Spectral measures and eigenvalue spacings for cat's eye flow

$$H(x,y) = \sin(x)\,\sin(y) + A\cos(x)\cos(y), \qquad A \sim U(-p,p)$$



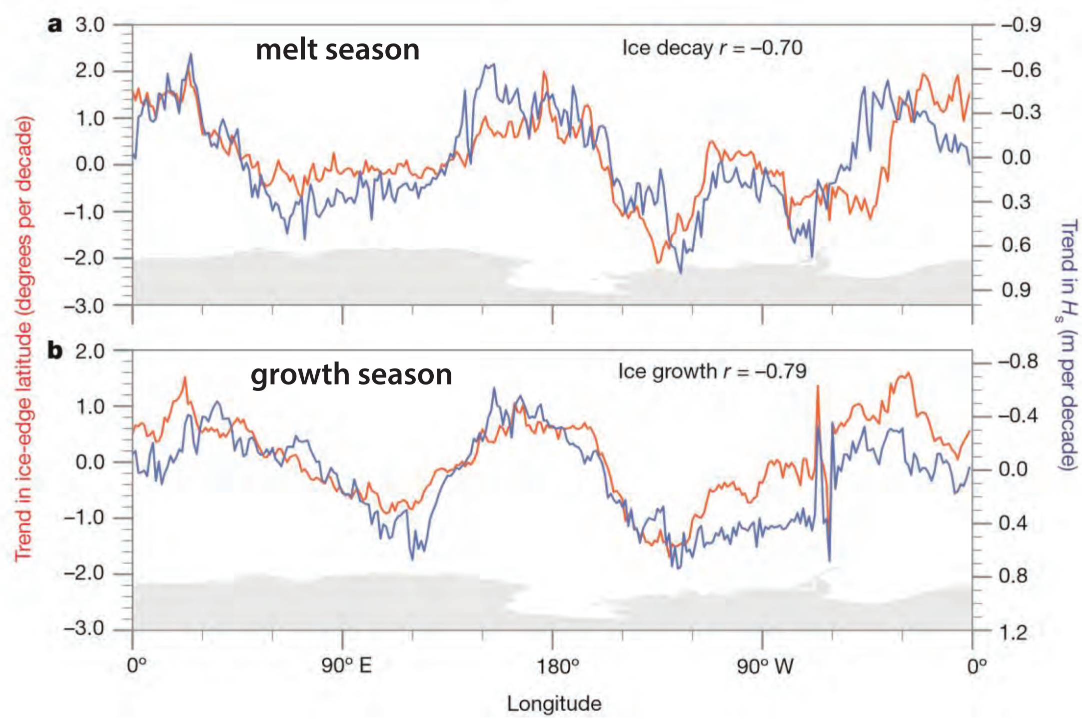
Murphy, Cherkaev, Zhu, Xin, Golden

Storm-induced sea-ice breakup and the implications for ice extent

Kohout et al., *Nature* 2014

- during three large-wave events, significant wave heights did not decay exponentially, enabling large waves to persist deep into the pack ice.

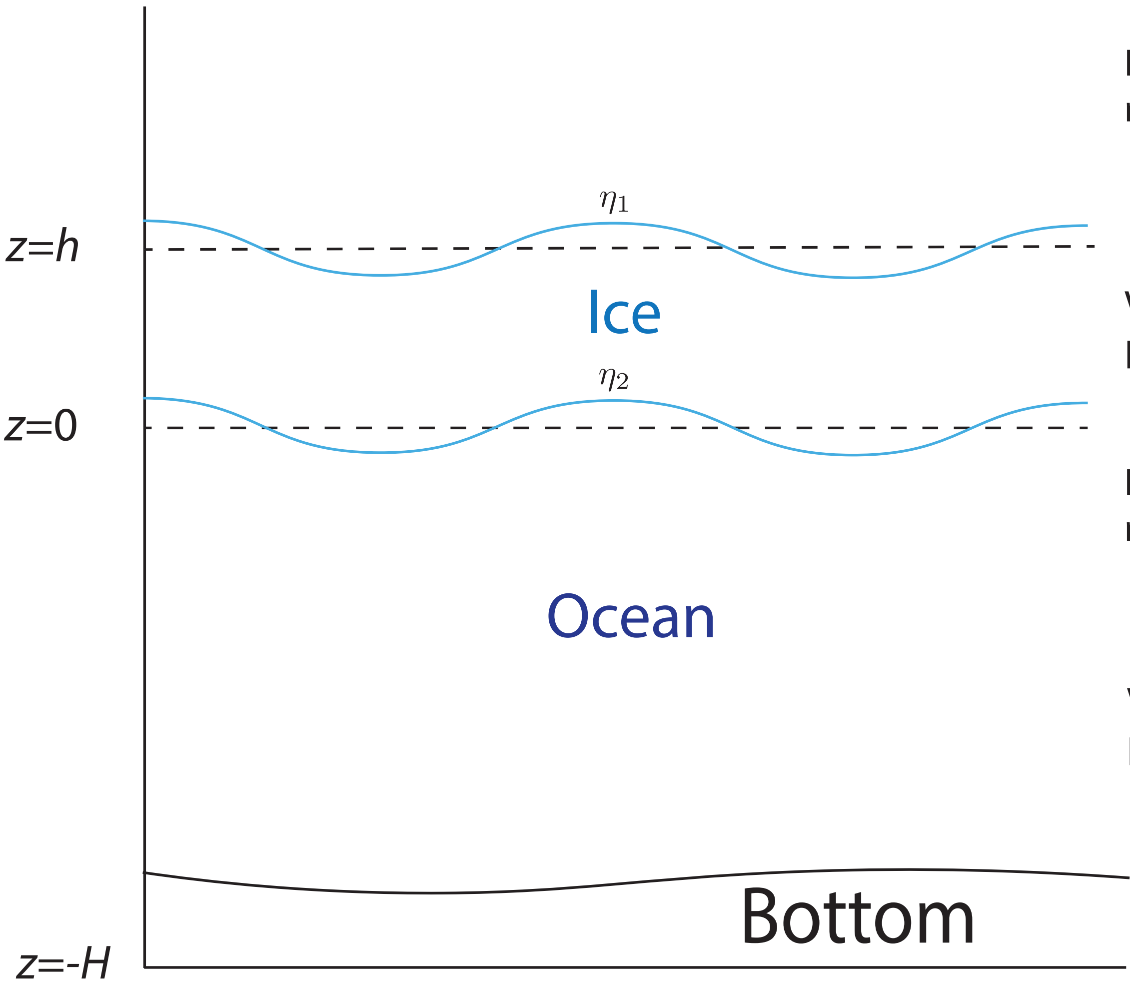
- large waves break sea ice much farther from the ice edge than would be predicted by the commonly assumed exponential decay



ice extent compared with significant wave height

Waves have strong influence on both the floe size distribution and ice extent.

Two Layer Models and Effective Parameters



Viscous fluid layer (Keller 1998)
Effective Viscosity $\boxed{\nu}$

Equations of motion:
$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 U + g$$

Viscoelastic fluid layer (Wang-Shen 2010)
Effective Complex Viscosity $\boxed{\nu_e = \nu + iG/\rho\omega}$

Equations of motion
$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu_e \nabla^2 U + g$$

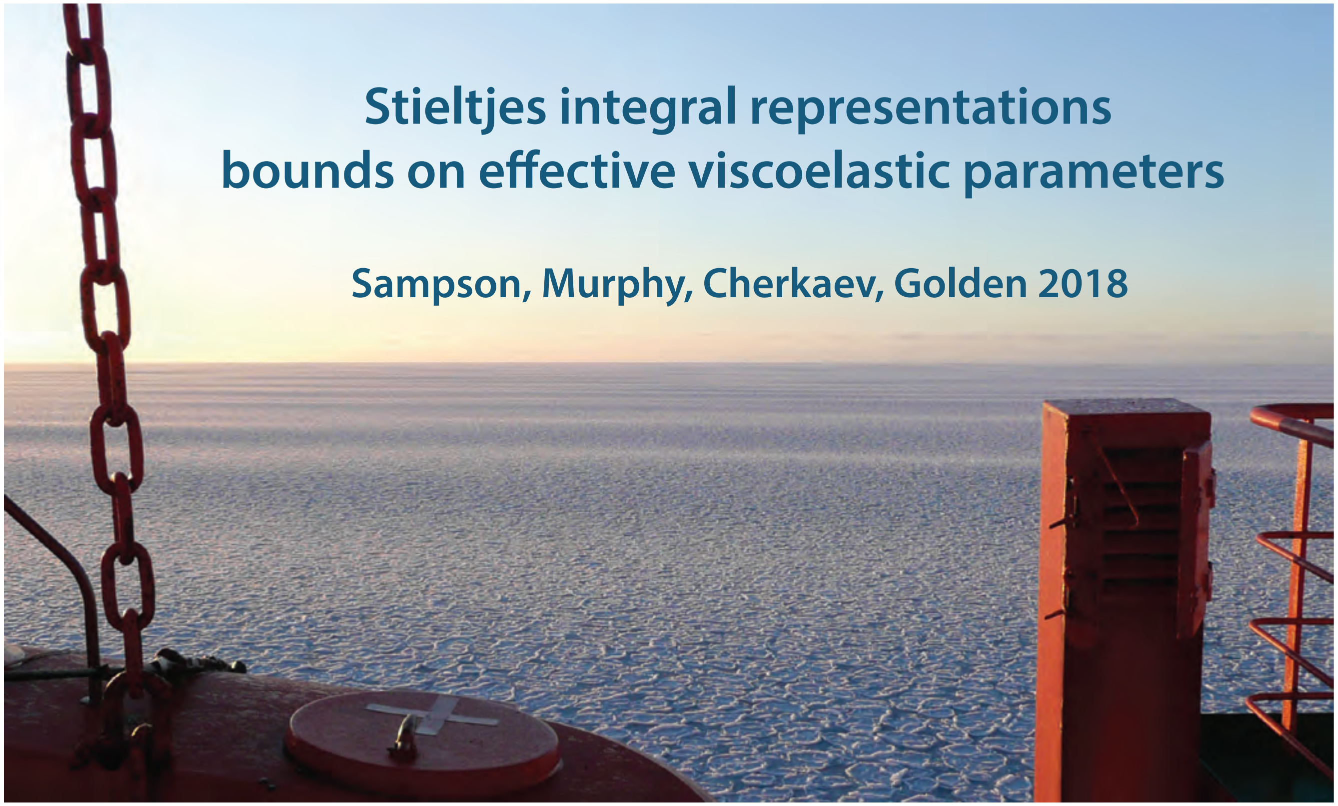
Viscoelastic thin beam (Mosig *et al.* 2015)
Effective Complex Shear Modulus $\boxed{G_v = G - i\omega\rho\nu}$

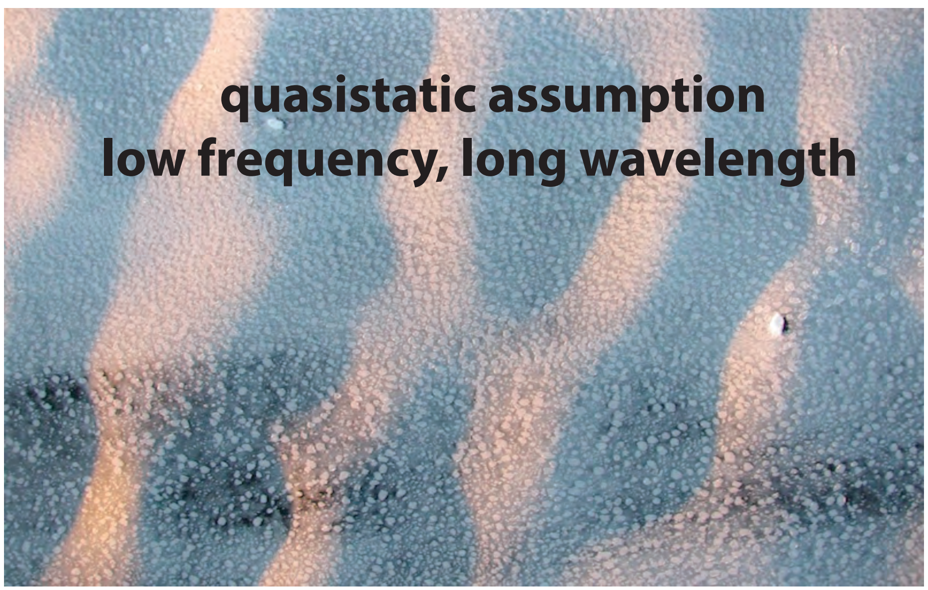
Stieltjes integral representation for effective complex viscoelastic parameter; bounds

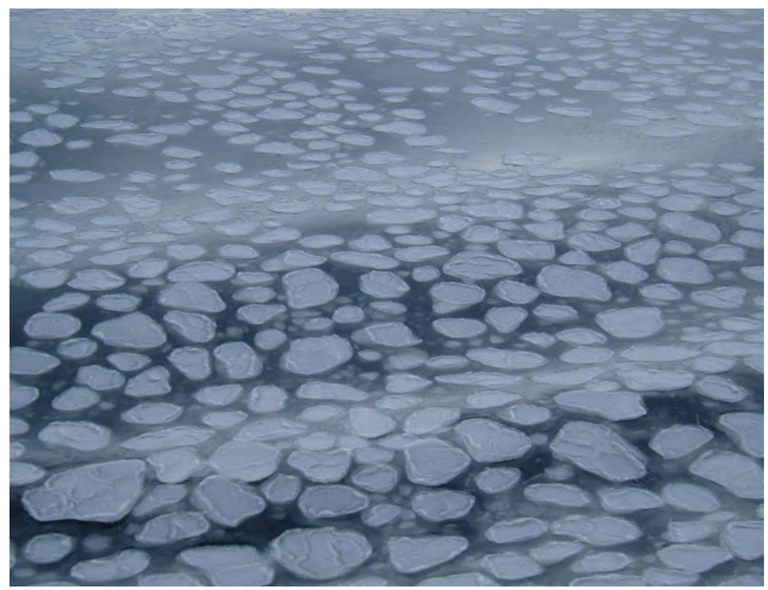
Sampson, Murphy, Cherkaev, Golden 2018

G shear modulus
P pressure
ω angular frequency
U velocity field
ν viscosity
λ Poission ratio
ρ density
g gravity

wave propagation in the marginal ice zone







Stieltjes Integral Representation for Complex Viscoelasticity

homogenized $\langle \sigma_{ij} \rangle = C^*_{ijkl} \langle \epsilon_{kl} \rangle$

local $\nabla \cdot \sigma = 0$ $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$

$$C_{ijkl} = (\nu_1 \chi + (1-\chi)\nu_2)\lambda_s$$

$$\nabla \cdot \big((\nu_1 \chi + (1-\chi)\nu_2)\lambda_s : \epsilon \big) = 0$$

$$s = \frac{1}{1 - \frac{\nu_1}{\nu_2}}$$

Strain Field

$$\epsilon = \frac{1}{2}[\nabla u + (\nabla u)^T] = \nabla^s u$$

$\epsilon = \epsilon_0 + \epsilon_f$ where $\epsilon_f = \nabla^s \phi$

Elasticity Tensor

$$C^*_{ijkl} = \nu^* \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl} \right) = \nu^* \lambda_s$$

RESOLVENT $\epsilon = \left(1 - \frac{1}{s}\Gamma\chi\right)^{-1} \epsilon_0$ $\Gamma = \nabla^s (\nabla \cdot \nabla^s)^{-1} \nabla \cdot$ ϵ_0 **avg strain**

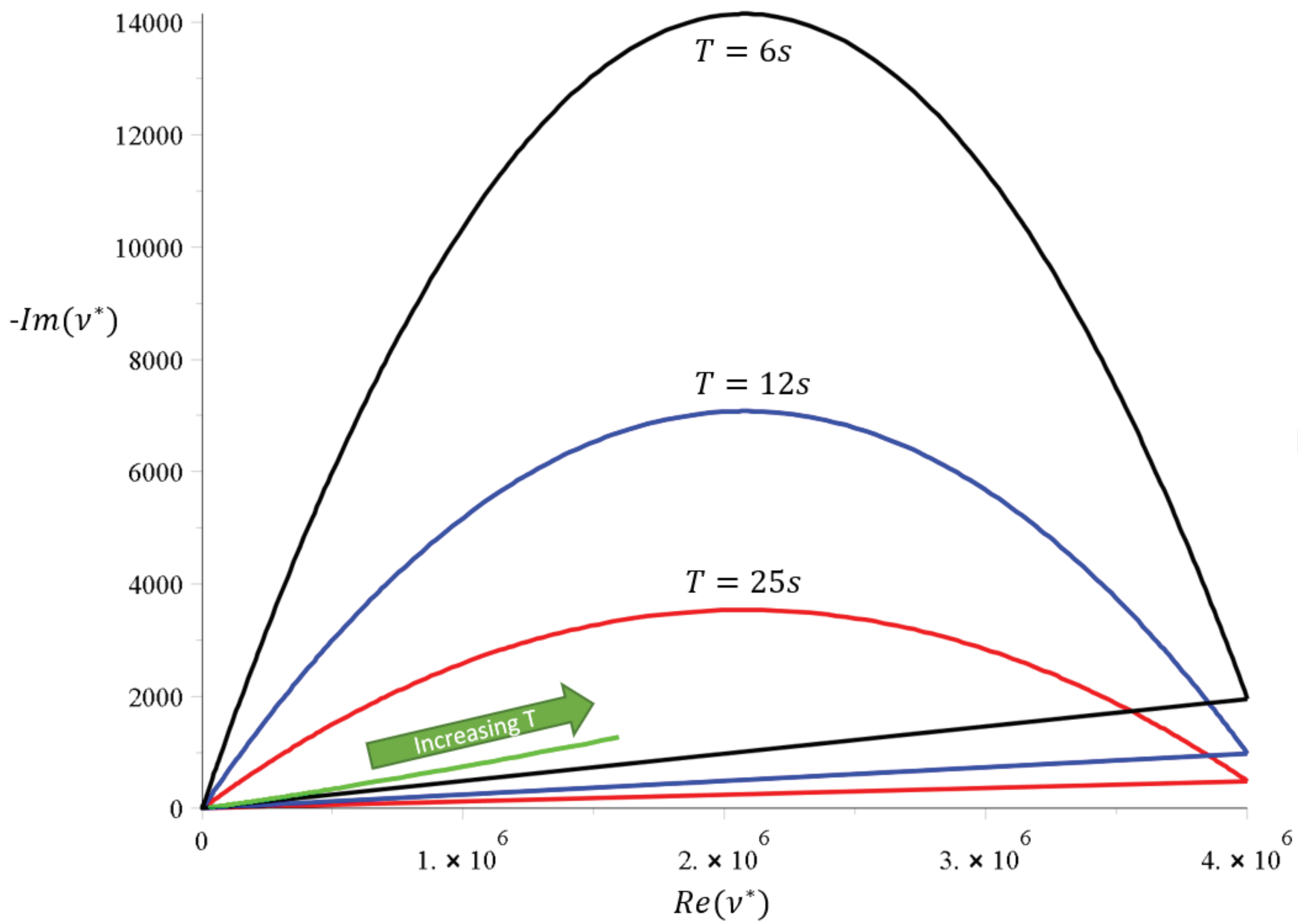
$$F(s) = 1 - \frac{\nu^*}{\nu_2} \qquad F(s) = ||\epsilon_0||^{-2} \int_\Sigma \frac{d\mu(\lambda)}{s - \lambda}$$

bounds on the effective complex viscoelasticity

complex elementary bounds
(fixed area fraction of floes)

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**



Sampson, Murphy, Cherkaev, Golden 2018

melt pond formation and albedo evolution:

- ***major drivers in polar climate***
- ***key challenge for global climate models***

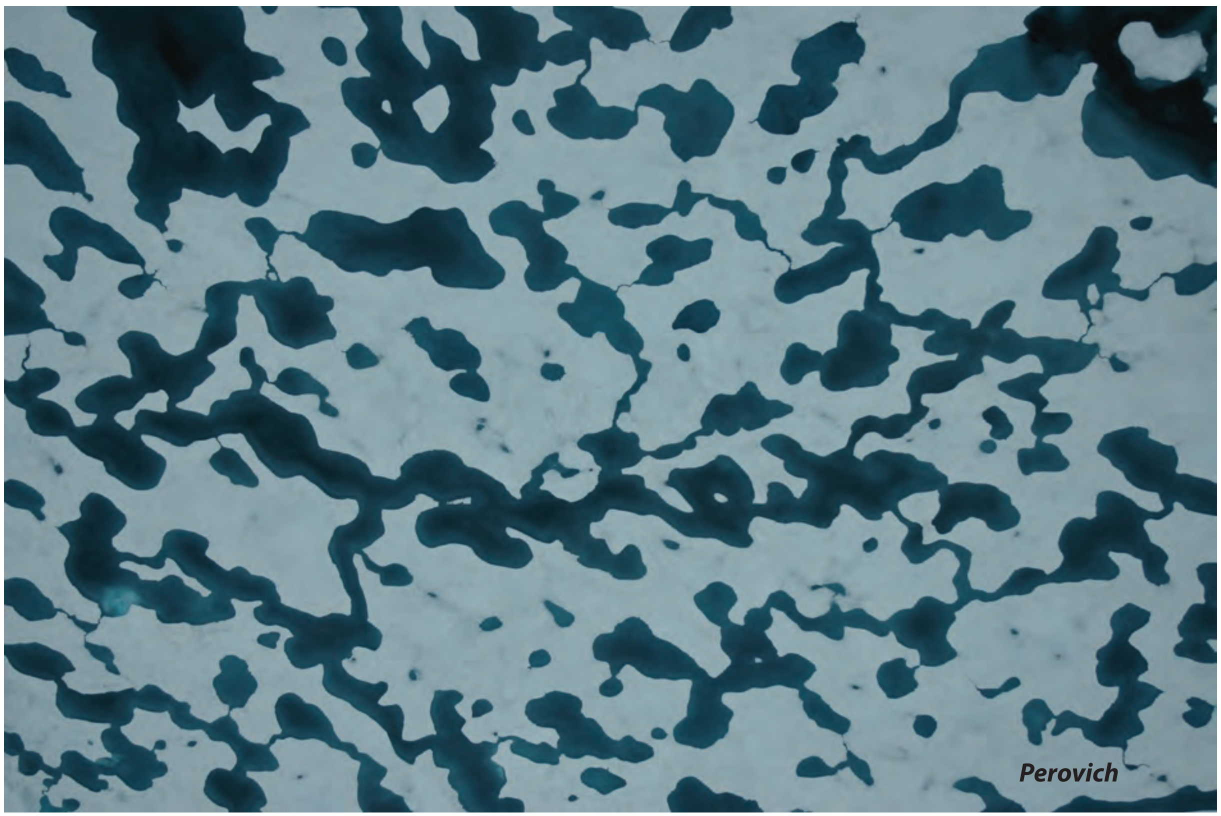
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012



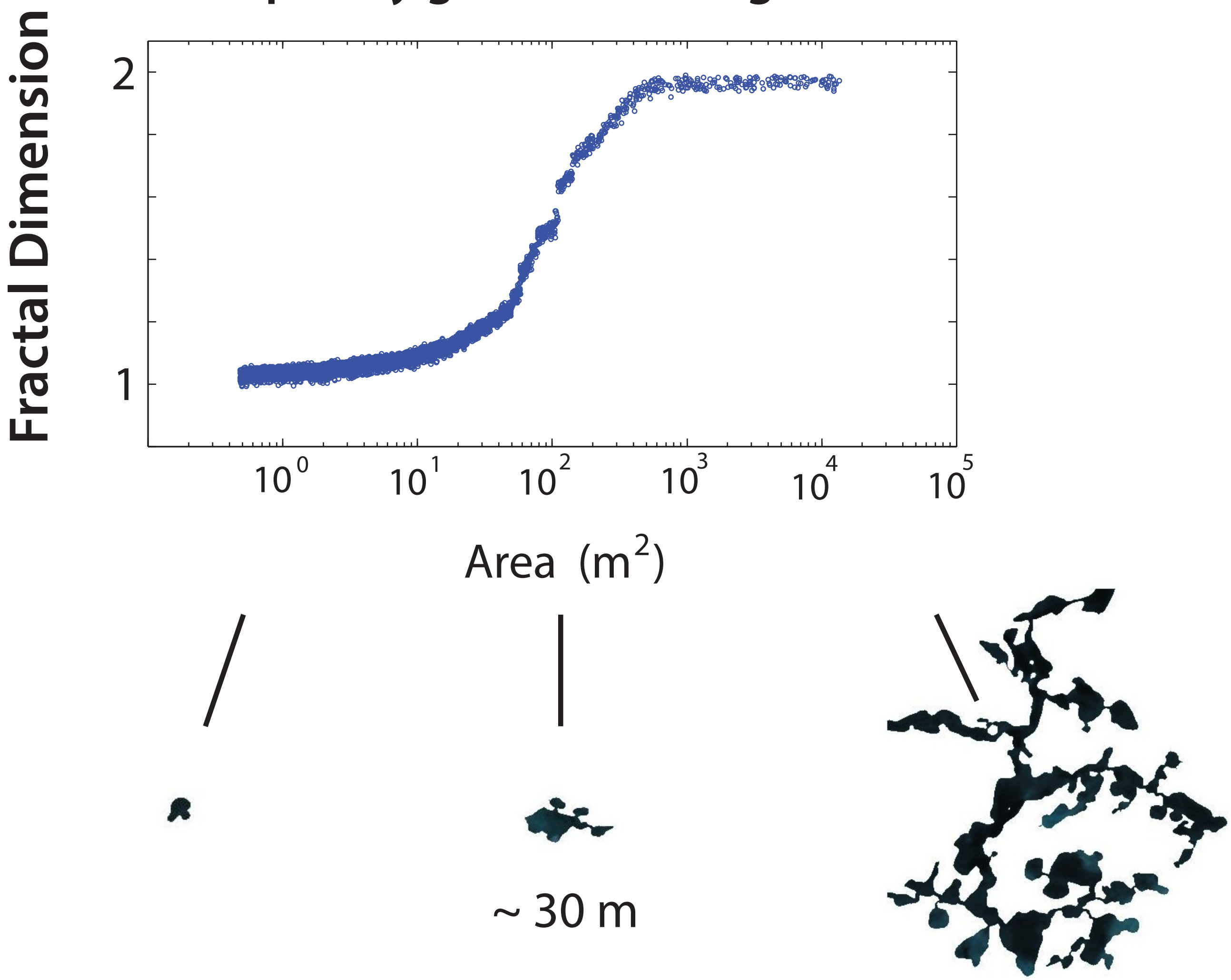
Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

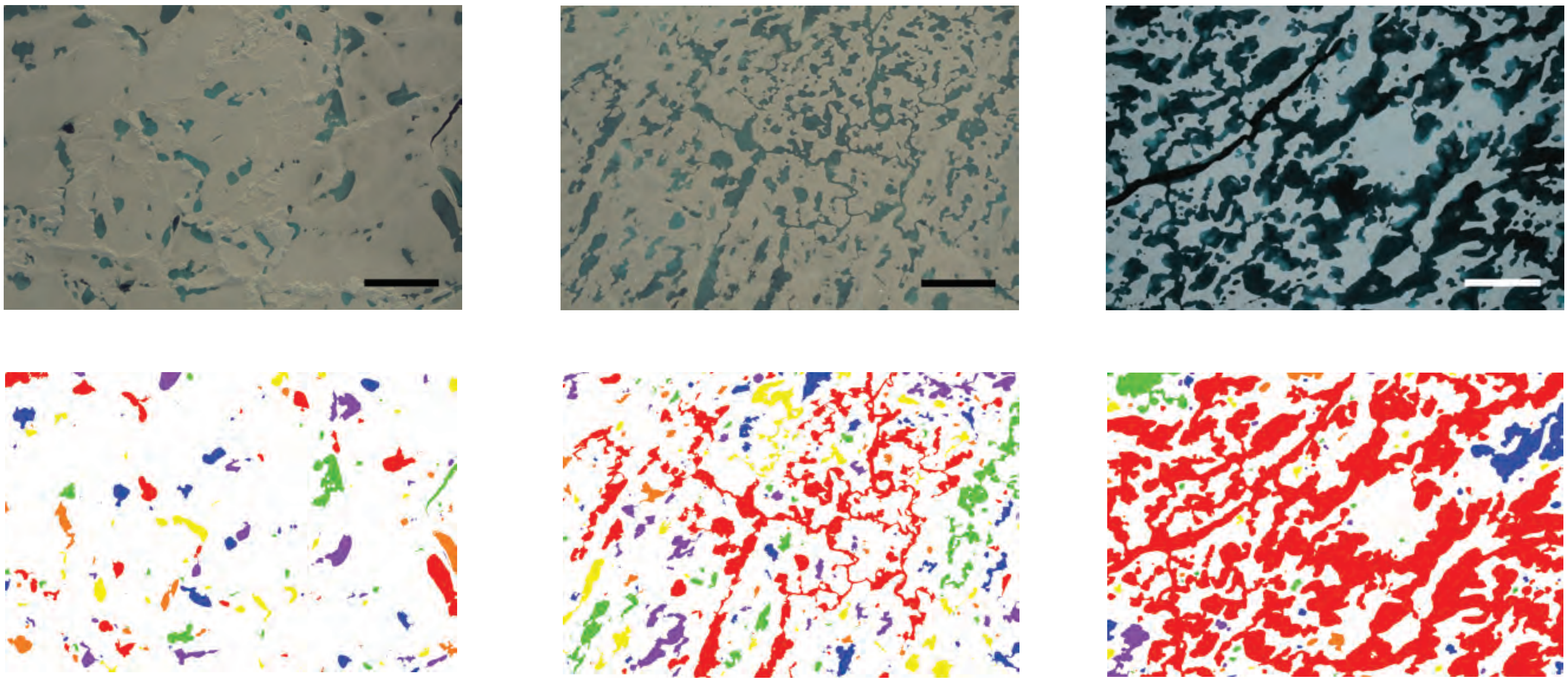
The Cryosphere, 2012

complexity grows with length scale



simple pond ***transitional pond*** ***complex pond***

small simple ponds coalesce to form large connected structures with complex boundaries



melt pond percolation

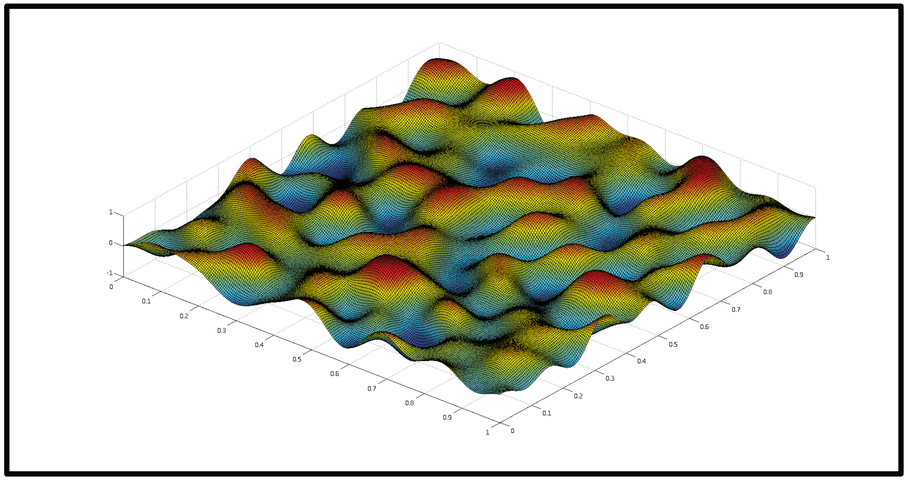
results on percolation threshold, correlation length, cluster behavior

Anthony Cheng (Hillcrest HS), Dylan Webb (Skyline HS), Court Strong, Ken Golden

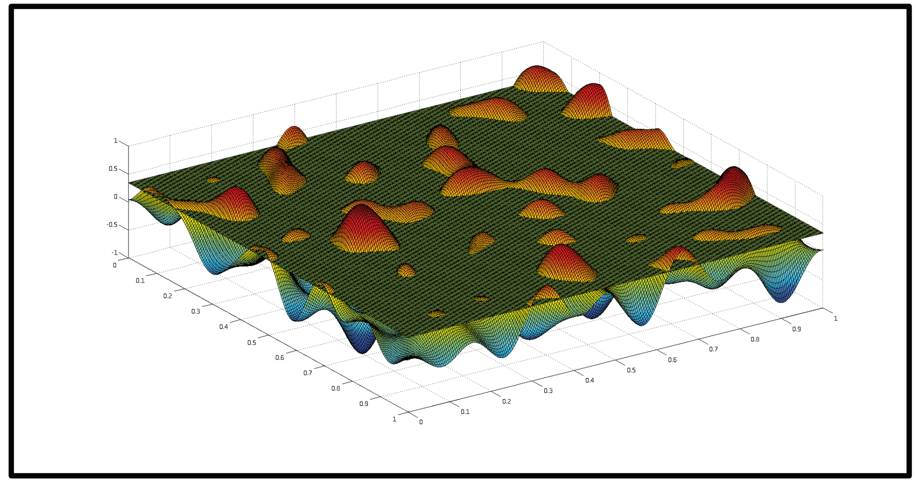
Continuum percolation model for melt pond evolution

level sets of random surfaces

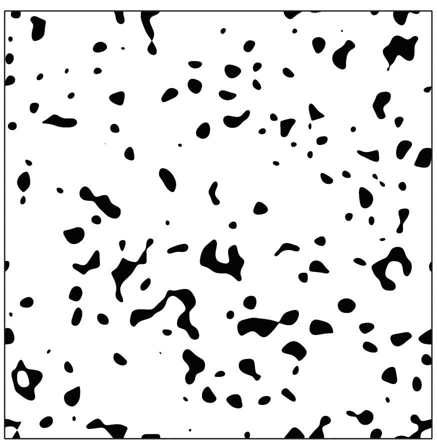
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018

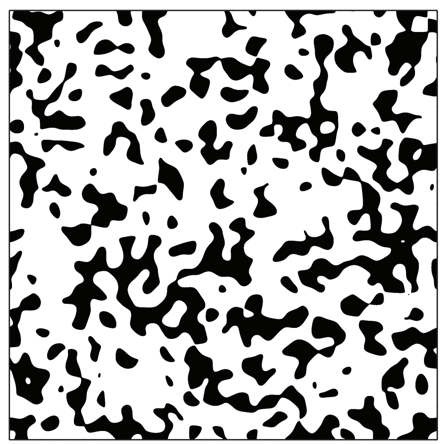


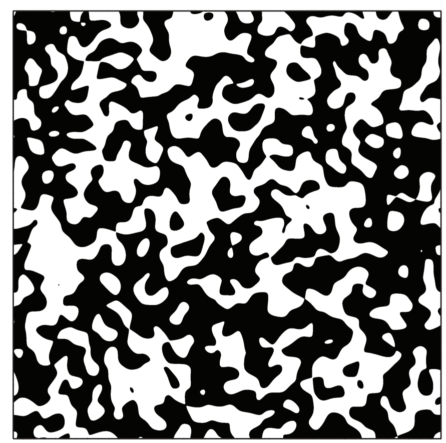
random Fourier series representation of surface topography



intersections of a plane with the surface define melt ponds

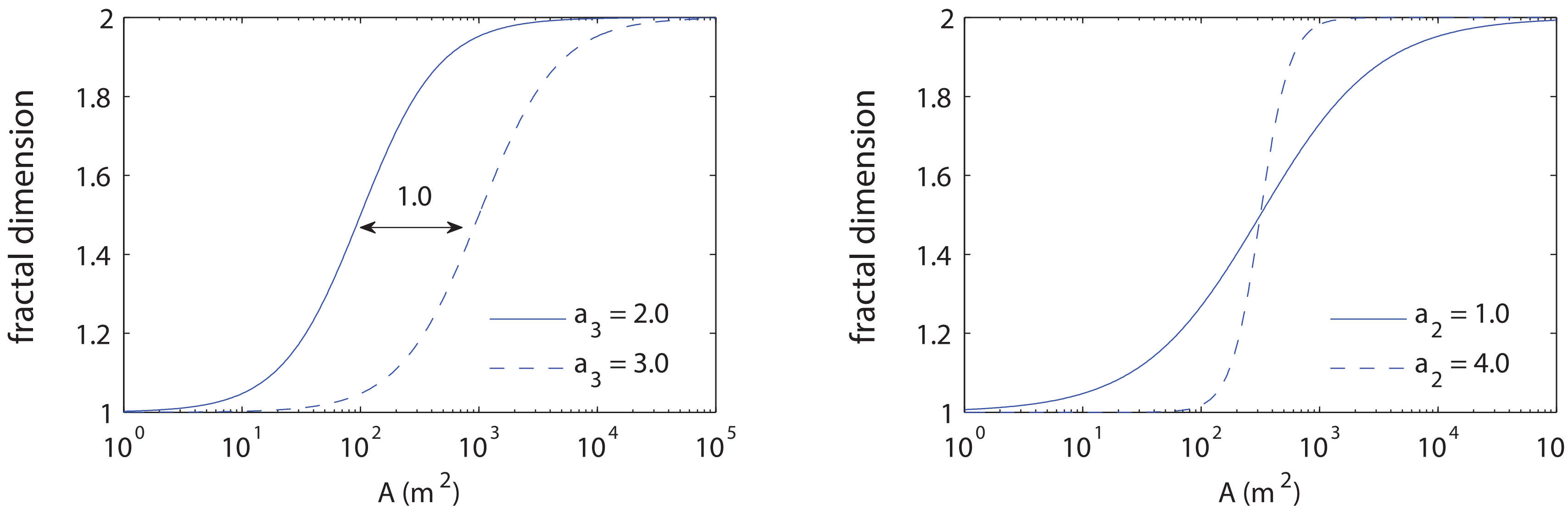




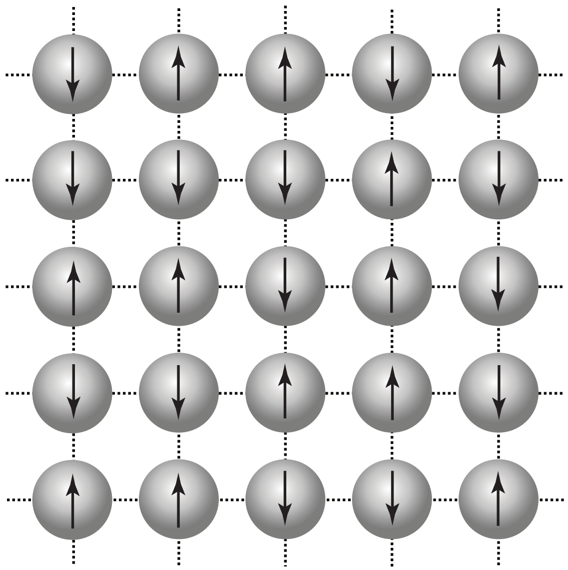


electronic transport in disordered media *diffusion in turbulent plasmas* *Isichenko, Rev. Mod. Phys., 1992*

fractal dimension curves depend on statistical parameters defining random surface



Ising Model for a Ferromagnet



$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

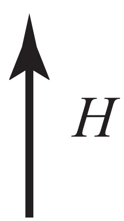
$$\mathcal{H}_\omega = -J \sum_{<i,j>} s_i s_j \; - \; H \sum_i s_i$$

nearest neighbor Ising Hamiltonian

for any configuration $\omega \in \Omega = \{-1, 1\}^N$ of the spins

$$J \geq 0$$

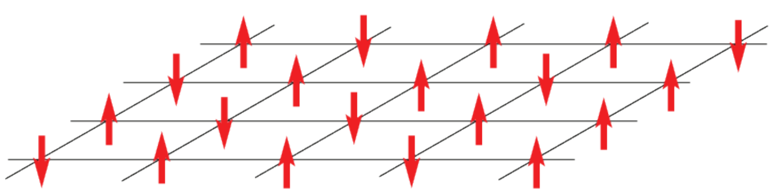
applied magnetic field



canonical partition function

$$Z_N(T, H) = \sum_{\omega \in \Omega} \exp(-\beta \mathcal{H}_\omega) = \exp(-\beta N f_N)$$

$$\beta = 1/kT$$



2–D Ising Model

free energy per site

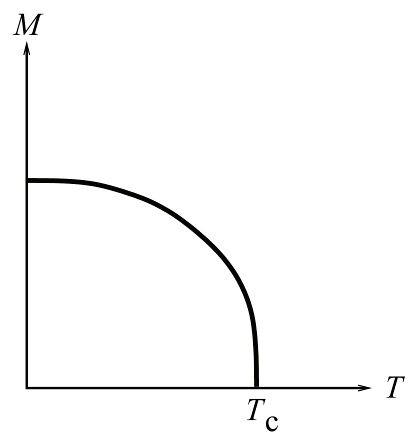
$$f_N(T, H) = \frac{-1}{\beta N} \log Z_N(T, H)$$

free energy $$f(T,H) = \lim_{N\to\infty} f_N(T,H)$$

magnetization

homogenized parameter like effective conductivity

$$M(T,H) = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle = -\frac{\partial f}{\partial H}$$



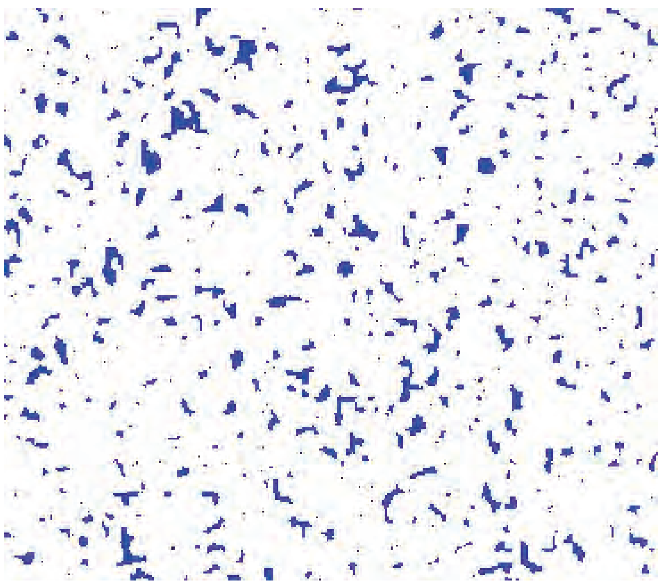
Curie point
critical temperature

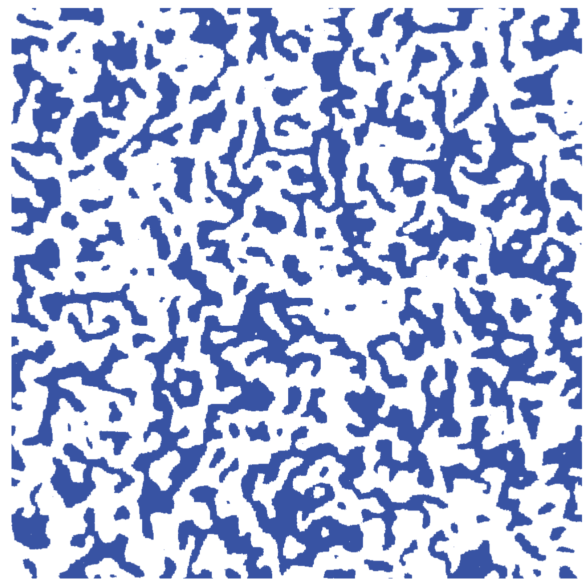
magnetic susceptibility $$\chi(T,H) = \frac{\partial M}{\partial H} = -\frac{\partial^2 f}{\partial H^2} \geq 0$$

Ising model for ferromagnets $\longrightarrow$ Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - H \sum_{i}^{N} s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

magnetization $M = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$ **pond coverage** $\frac{(M+1)}{2}$





"melt ponds" are clusters of magnetic spins that align with the applied field

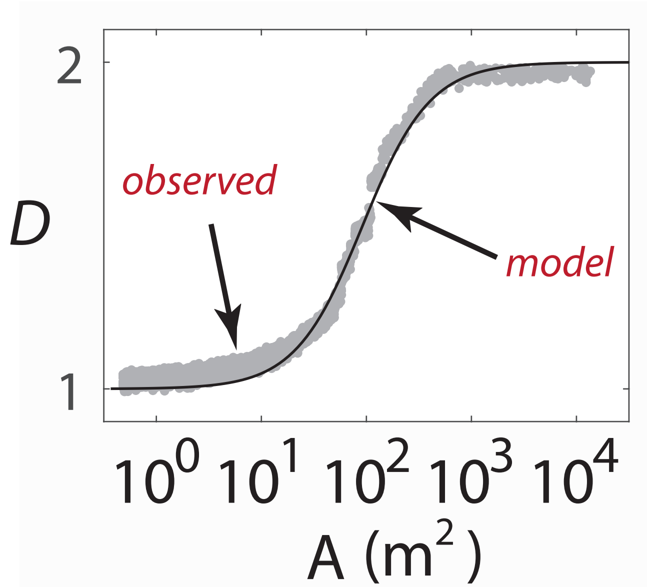
predictions of fractal transition, pond size exponent **Ma, Sudakov, Strong, Golden 2018**

Ising model results

Minimize Ising Hamiltonian energy

Random magnetic field represents snow topography; interaction term represents horizontal heat transfer.

Melt ponds —
metastable islands of like spins
in our random field Ising model.

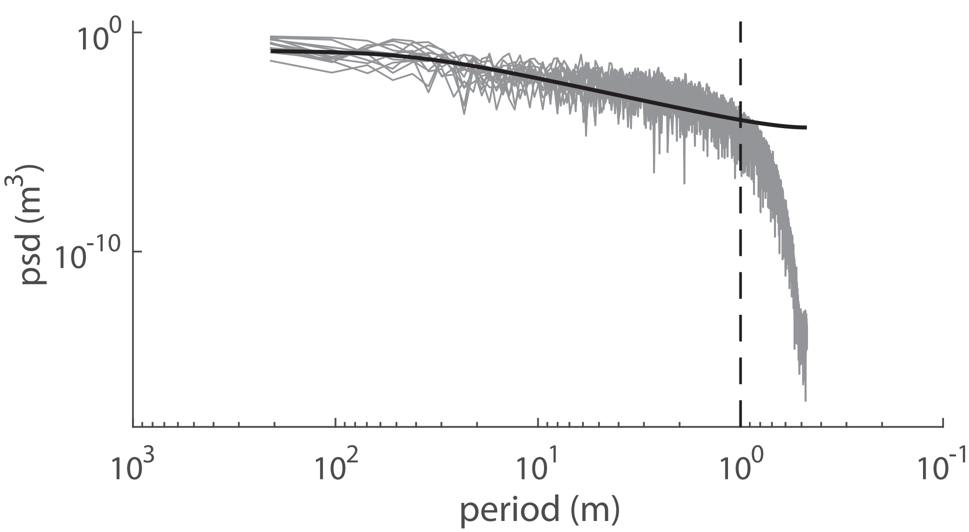


pond size distribution exponent

observed ***-1.5***
(Perovich, et al 2002)

model ***-1.58***

order out of disorder



The lattice constant *a* must be small relative to the 10-20 m length scales prominent in sea ice and snow topography. We set *a*=1 m as the length scale above which important spatially correlated fluctuations occur in the power spectrum of snow topography.

partition function N^{th} **order polynomial** in the "activity" $z = exp(-2\beta H)$

$$Z_N(z) = \sum_{n=0}^{N} a_n z^n, \quad a_n \geq 0$$

with remarkable property:

Theorem (Lee -Yang 1952):

If $J \geq 0$, then $Z_N = 0$ implies z lies on the unit circle $|z| = 1$, or equivalently, H lies on the imaginary axis (with real β).

Then the partition function can be written as

$$Z_N(z) = a_N \prod_{n=1}^{N} (z - z_n), \qquad |z_n| = 1$$

Then $$f(T, H) = \frac{-1}{\beta} \int_{|t|=1} \log(z - t) d\nu(t) - 2d\beta J$$

Stieltjes integral representation for magnetization

and scaling relations for critical exponents

Baker *PRL* 1968

$$M(\tau) = \tau + \tau(1-\tau^2)G(\tau^2) \qquad \tau = \tanh(\beta H)$$

$$G(\tau^2) = \int_0^\infty \frac{d\psi(y)}{1+\tau^2 y}$$

Herglotz

$$M(T) = -\frac{\partial f}{\partial H} \sim (T_c - T)^\beta \qquad T \to {T_c}^-$$

$$\chi = -\frac{\partial^2 f}{\partial H^2} \sim (T - T_c)^{-\gamma} \qquad T \to {T_c}^+$$

Along the critical isotherm $T = T_c$

$$M(H) \sim H^{1/\delta} \qquad H \to 0^+$$

ψ supported in $[0, S(T)]$

$$S(T) \sim (T - T_c)^{-2\Delta}, T \to {T_c}^+$$

$$\beta = \Delta - \gamma$$

$$\delta = \Delta/(\Delta - \gamma)$$

Baker's inequalities

$$\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0$$

critical exponents γ_n of higher derivatives of f

$$\gamma_0 = \gamma$$

via analogous Herglotz structure for transport in composites, same critical analysis and scaling relations hold near percolation threshold

Golden, *J. Math. Phys.* 1995
Phys Rev. Lett. 1997

(Chuck Newman)

$$m(h) = \frac{\sigma^*}{\sigma_2} \qquad h = \frac{\sigma_1}{\sigma_2} \to 0 \qquad \sigma^*(p, h)$$

effective conductivity of two phase composite - lattice or continuum

$$F(s) = 1 - m(h) \qquad F(s) = \int_0^1 \frac{d\mu(w)}{s - w} \qquad w = \frac{y}{y+1}$$

$$m(h) = 1 + (h-1)g(h) \qquad g(h) = \int_0^\infty \frac{d\phi(y)}{1 + hy}$$

Herglotz

$$\sigma^*(p, 0) \sim (p - p_c)^t \qquad p \to p_c^+$$

$$\sigma^*(p_c, h) \sim h^{1/\delta} \qquad h \to 0^+$$

$$\chi(p) = \frac{\partial m}{\partial h} \sim (p_c - p)^{-\gamma} \qquad p \to p_c^-, \quad h = 0$$

$$\theta_h \sim (p_c - p)^\Delta \qquad p \to p_c^-$$

spectral gap

$$t = \Delta - \gamma$$

$$\delta = \frac{\Delta}{\Delta - \gamma}$$

lattices and continua obey same scaling relations as in stat mech

Baker's inequalities for transport

$$\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0, \quad n \geq 1$$

Ising model

partition function

$$Z_N(z) = a_N \prod_{n=1}^{N} (z - z_n), \qquad |z_n| = 1$$

free energy

$$f(T,H) = \frac{-1}{\beta} \int_{|t|=1} \log(z - t)\, d\nu(t)$$

order parameter

$$M(T) = -\frac{\partial f}{\partial H}$$

$$\frac{\partial^2 M}{\partial H^2} \leq 0$$

G.H.S. inequality

Griffiths, Hurst, Sherman *JMP* 1970

transport in composites

$$\mathcal{Z}_N(s) = \prod_{n=1}^{N} (s - s_n), \qquad s_n \in [0,1]$$

$$\Phi(p,s) = \int_0^1 \log(s - t)\, d\mu(t)$$

$$F(p,s) = \frac{\partial \Phi}{\partial s}$$

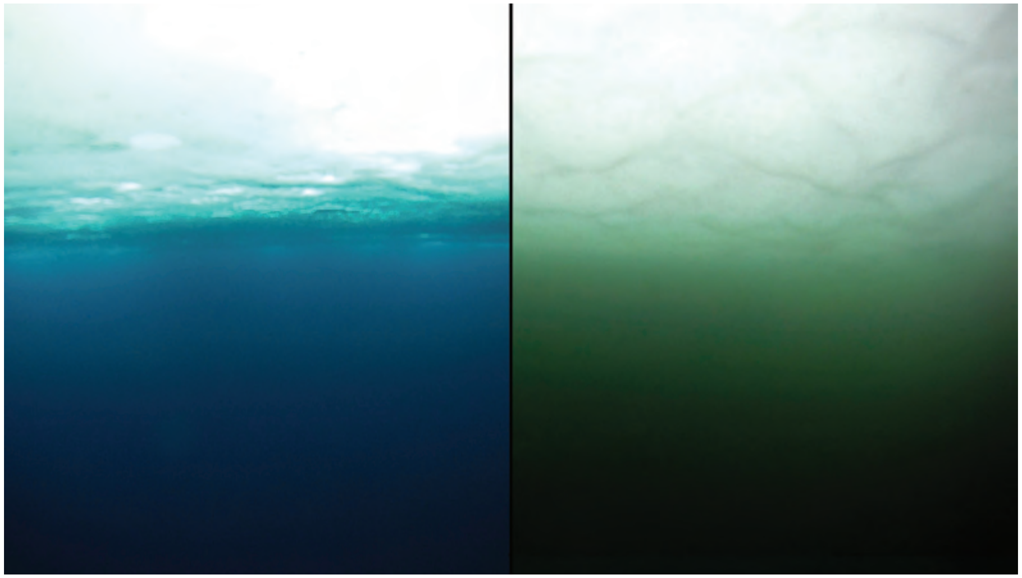
$$\frac{\partial^2 m}{\partial h^2} \leq 0$$

Golden, *JMP* 1995; *PRL* 1997

2011 massive under-ice algal bloom

Arrigo et al., *Science* 2012

melt ponds act as *WINDOWS* allowing light through sea ice



no bloom **bloom**

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

Horvat, Rees Jones, Iams, Schroeder, Flocco, Feltham, ***Science Advances***, 2017

The distribution of solar energy under ponded sea ice

Horvat, Flocco, Rees Jones, Roach, Golden, 2018

(2015 AMS MRC)

The Melt Pond Conundrum:

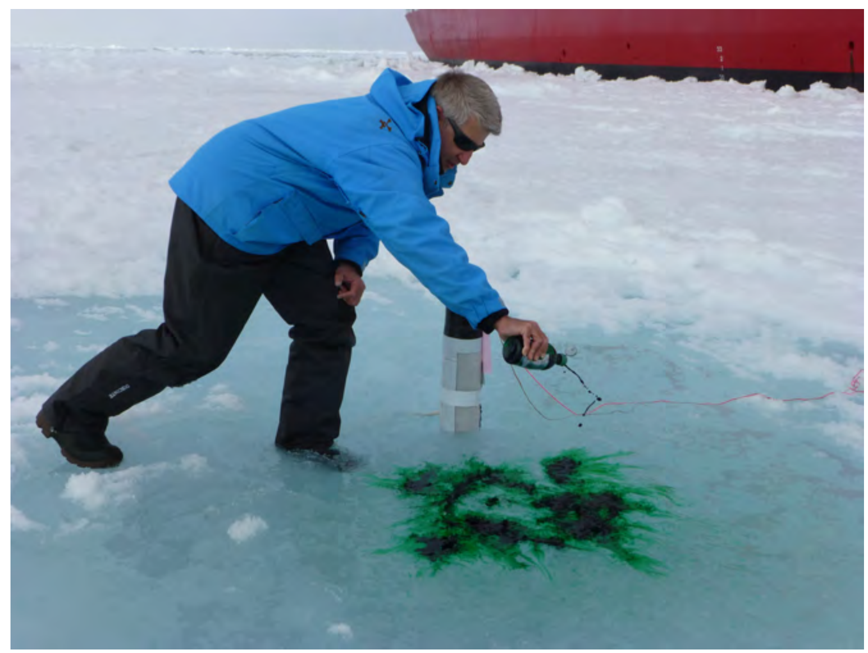
How can ponds form on top of sea ice that is highly permeable?

C. Polashenski, K. M. Golden, D. K. Perovich, E. Skyllingstad, A. Arnsten, C. Stwertka, N. Wright

Percolation Blockage: A Process that Enables Melt Pond Formation on First Year Arctic Sea Ice

J. Geophys. Res. Oceans 2017

2014 Study of Under Ice Blooms in the Chuckchi Ecosystem (SUBICE) aboard USCGC Healy



Conclusions

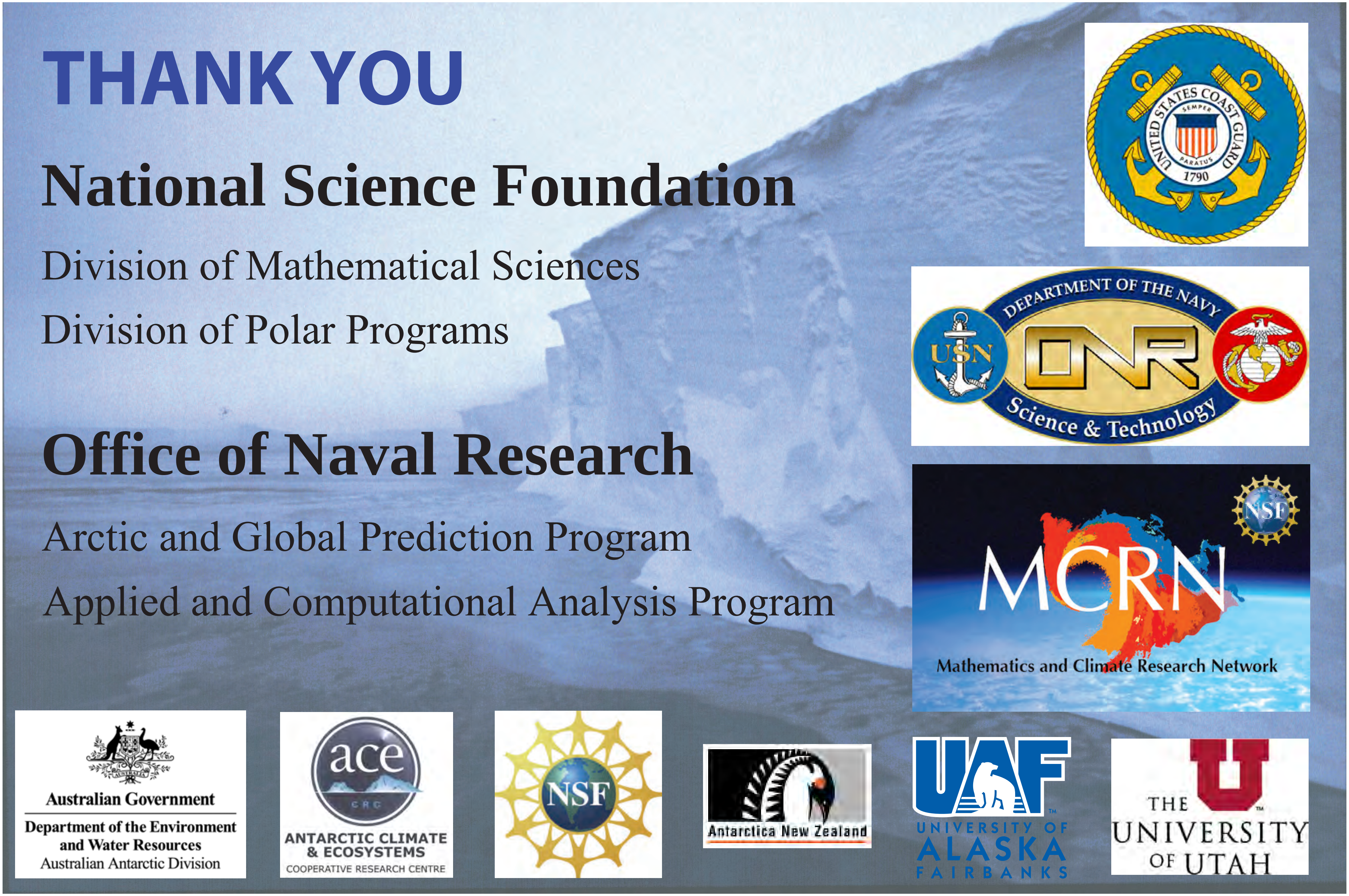
1. Sea ice is a fascinating multiscale composite with structure similar to many other natural and man-made materials.

2. Variational methods, Stieltjes integrals, bounds developed for sea ice advance transport theory and variational analysis.

3. Homogenization and statistical physics help *link scales in sea ice and composites*; provide rigorous methods for finding effective behavior; advance sea ice representations in climate models.

4. Sea ice modeling led to unexpected connections with random matrix theory and Anderson transitions.

5. Our research will help to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.



Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999