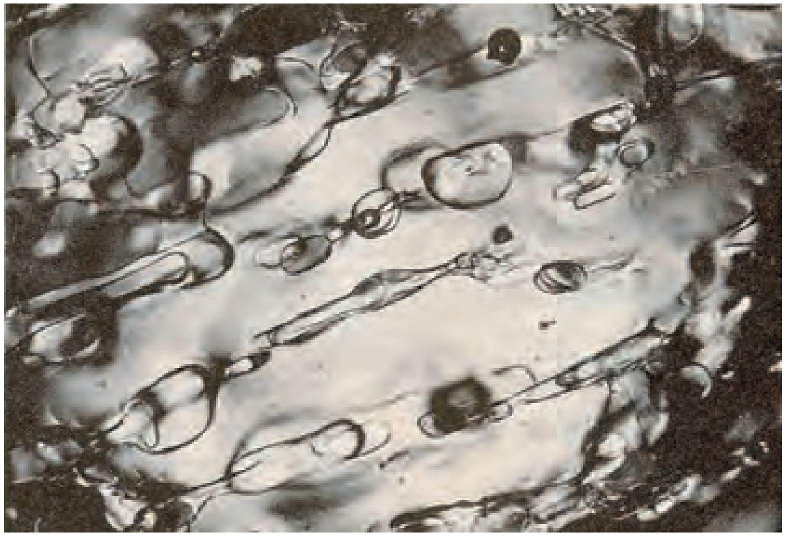
Anderson Transition in Metamaterials

Kenneth M. Golden
Department of Mathematics
University of Utah

sea ice is a multiscale composite

structured on many length scales - from tenths of mm's to tens of km's

brine inclusions

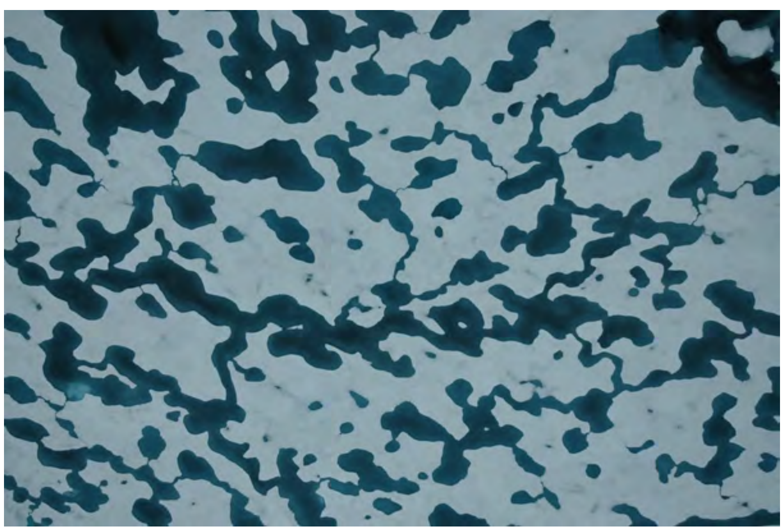


millimeters

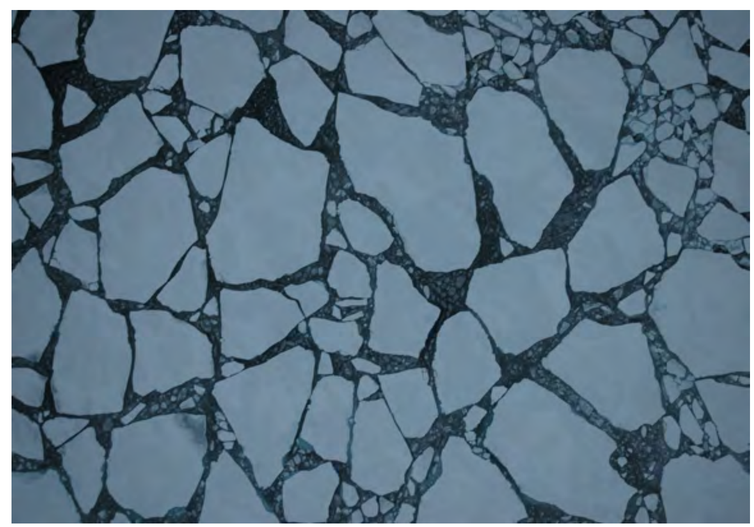
pancakes

centimeters

melt ponds



meters



ice floes

kilometers

What is this talk about?

Using methods of statistical physics and composite materials to LINK SCALES in the sea ice system ... compute effective behavior.

Take a tour of our sea ice methods relevant to optics and metamaterials find unexpected Anderson transition in composites along the way!

HOMOGENIZATION

1. Sea ice microphysics and fluid transport

percolation theory

2. EM transport, waves

Stieltjes integrals, spectral measures random matrices, Anderson transitions

3. Fractals and Arctic melt ponds

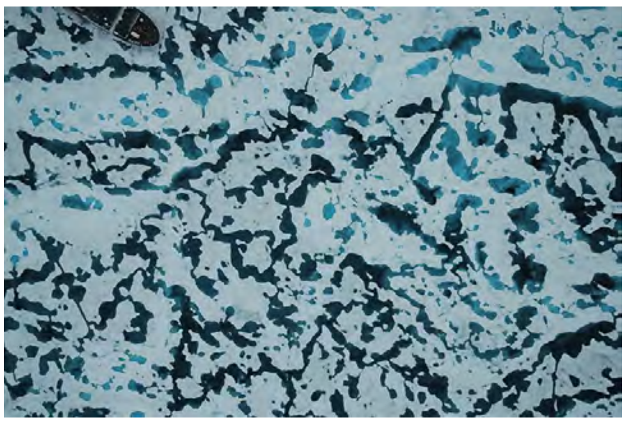
continuum percolation and the Ising model

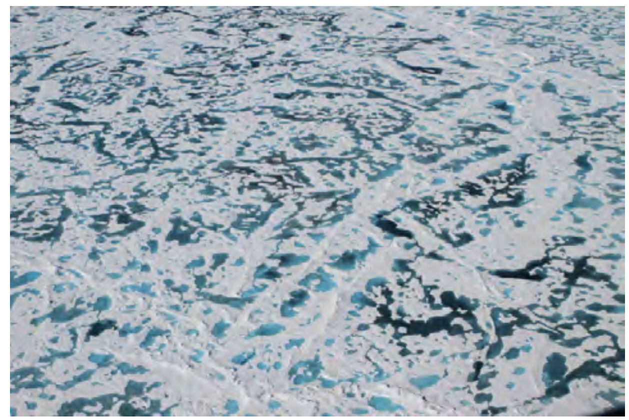
How do scales interact in the sea ice system?

basin scale -
grid scale
albedo

Linking Scales

km
scale
melt
ponds

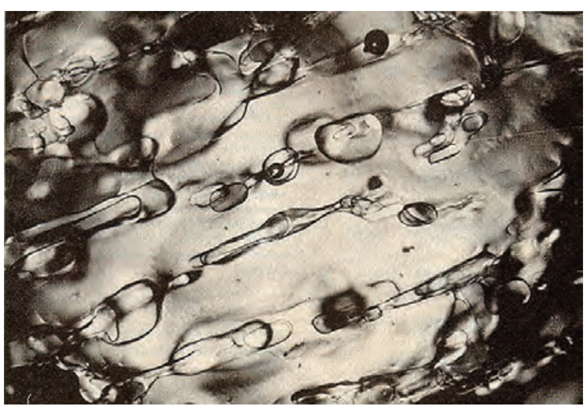




km
scale
melt
ponds

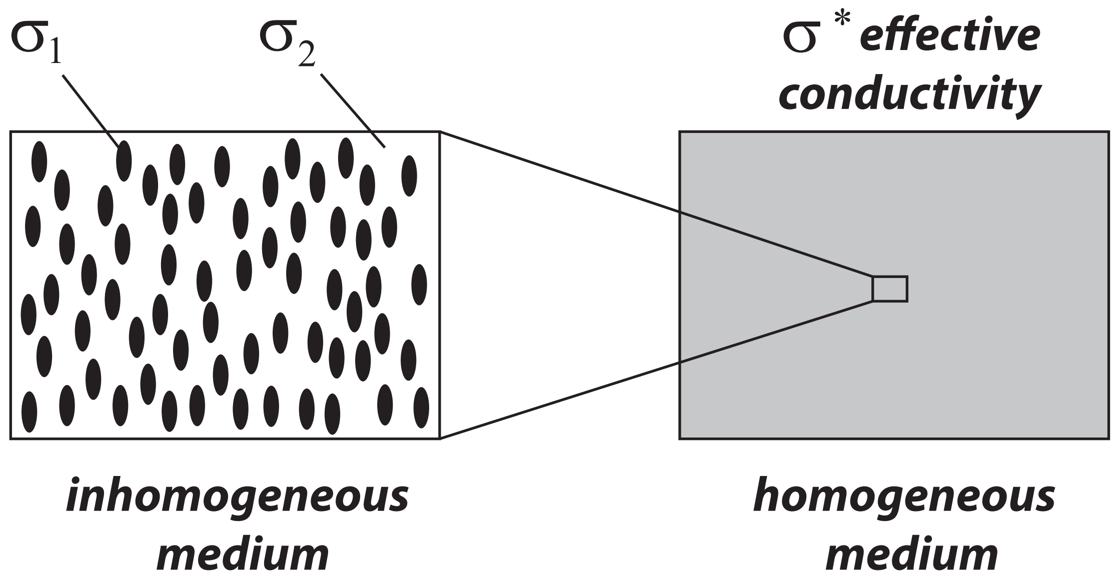
Linking Scales

mm
scale
brine
inclusions



meter
scale
snow
topography

HOMOGENIZATION - Linking Scales in Composites



find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

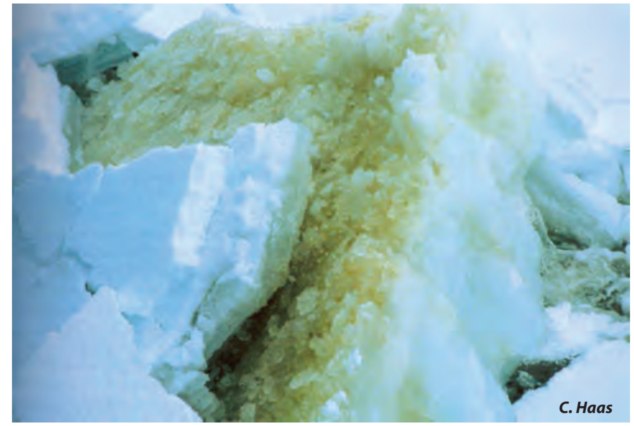
*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*
*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

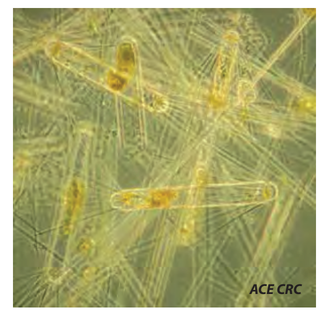
widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

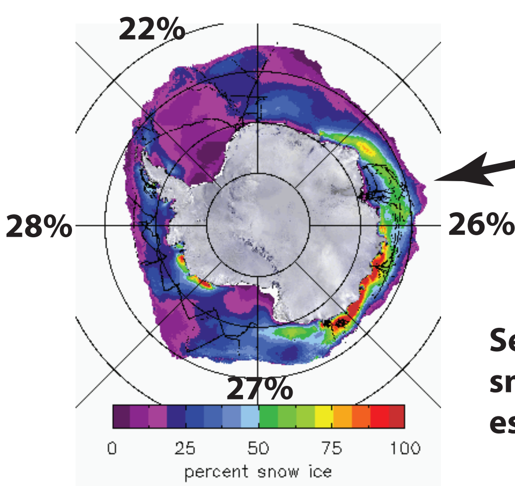
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities







T. Maksym and T. Markus, 2008

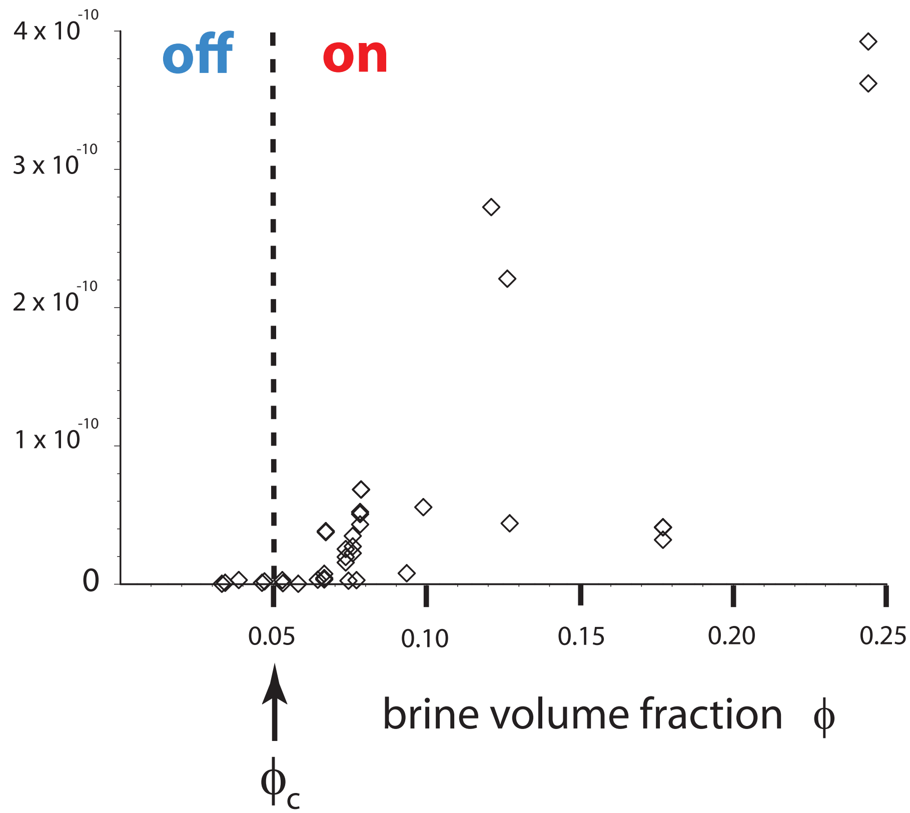
September snow-ice estimates

Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

Critical behavior of fluid transport in sea ice

impermeable permeable



Arctic field data

vertical fluid permeability k (m ^{2})

on - off switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ C,\ S \approx 5$ ppt

RULE OF FIVES

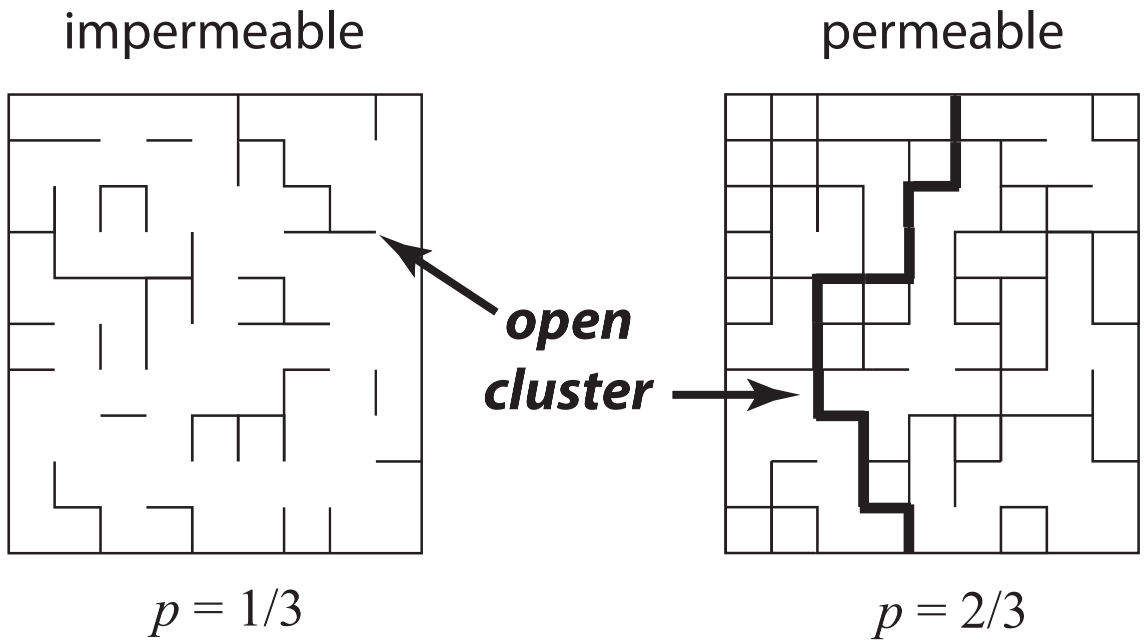
Golden, Ackley, Lytle *Science* 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

percolation theory

probabilistic theory of connectedness



bond $\longrightarrow$ **open** *with probability* p
closed *with probability* $1-p$

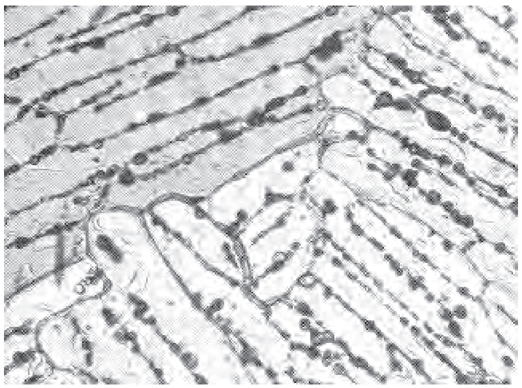
percolation threshold

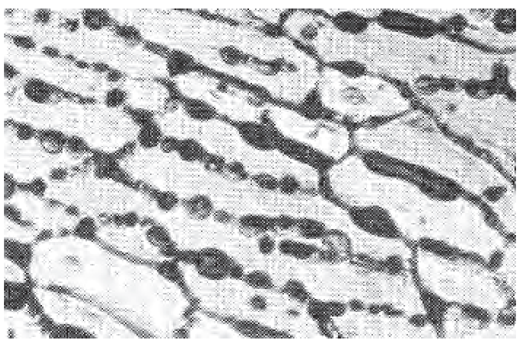
$$p_c = 1/2 \quad \text{for} \quad d = 2$$

smallest *p* for which there is an infinite open cluster

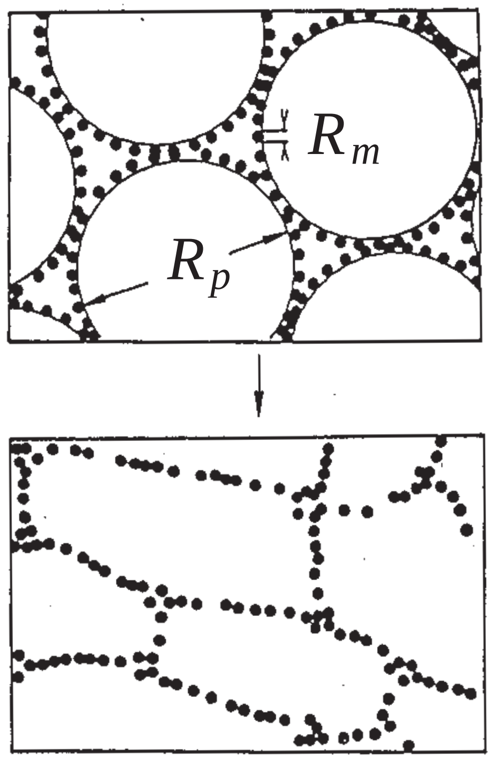
Continuum percolation model for *stealthy* materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

$\phi_c \approx 5\ \%$ Golden, Ackley, Lytle, *Science*, 1998

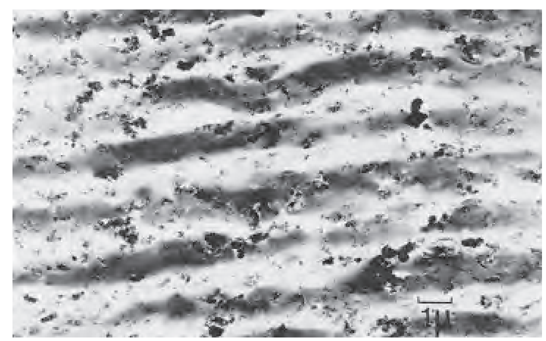




sea ice



compressed powder



radar absorbing composite

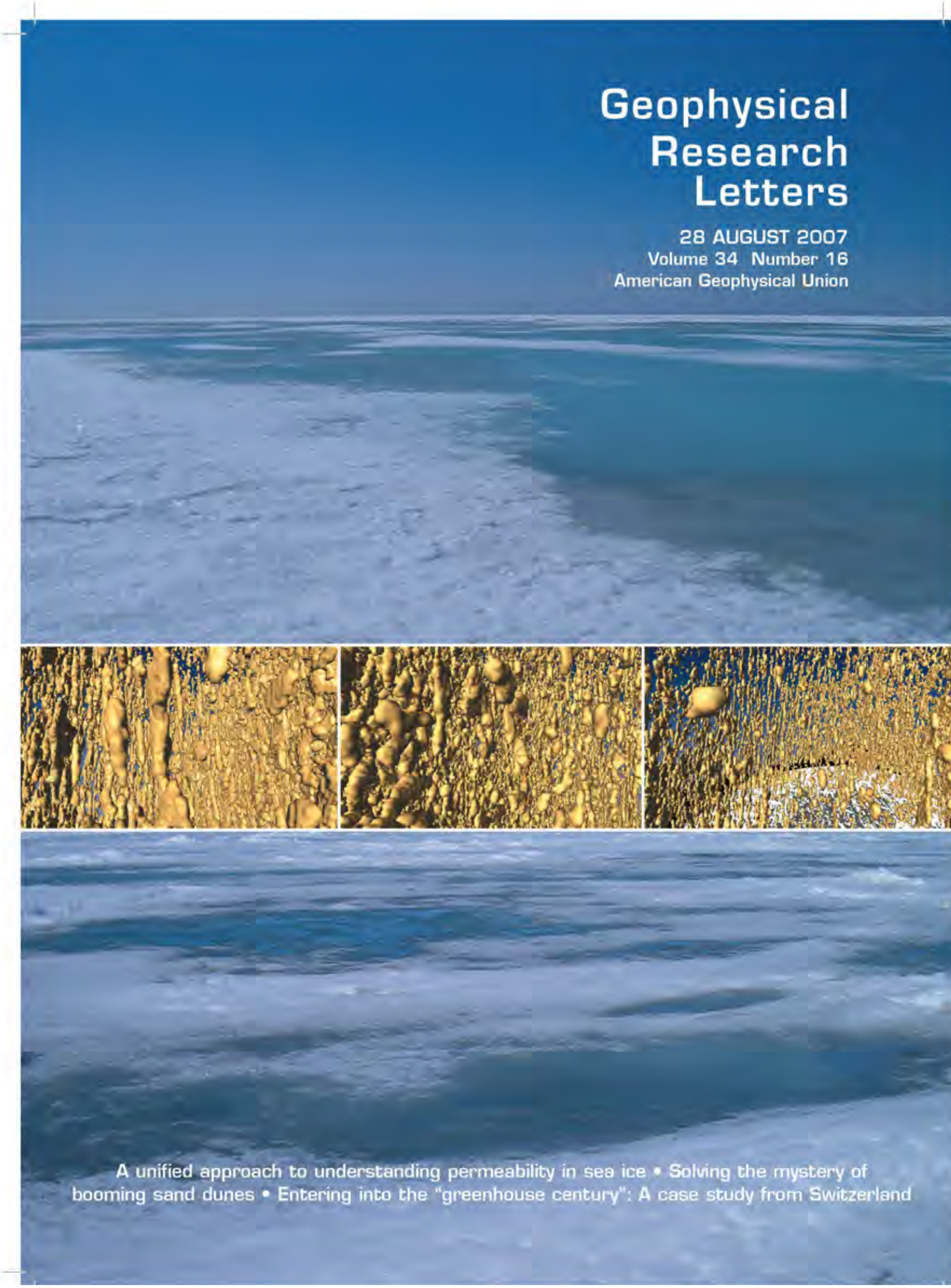
sea ice is radar absorbing

Thermal evolution of permeability and microstructure in sea ice **Golden, Eicken, Heaton, Miner, Pringle, Zhu**

micro-scale

controls

macro-scale processes



rigorous bounds
percolation theory
hierarchical model
network model

field data

X-ray tomography for brine inclusions

unprecedented look at thermal evolution of brine phase and its connectivity

Remote sensing of sea ice

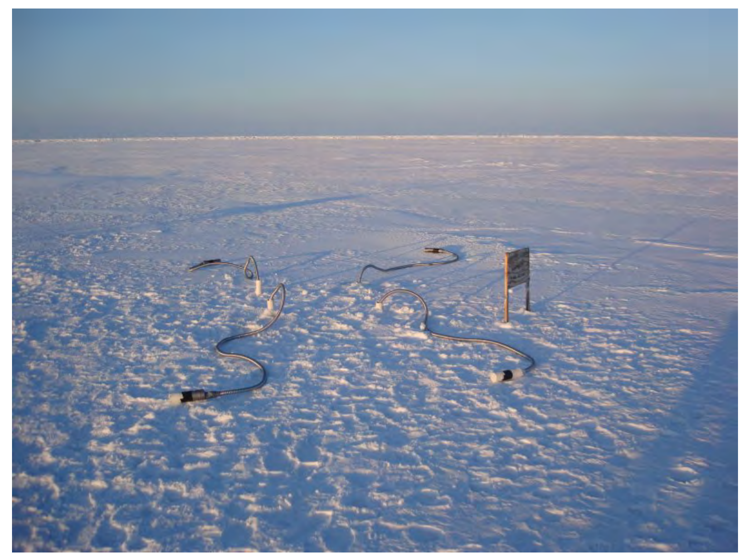
INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

$$\varepsilon^*$$

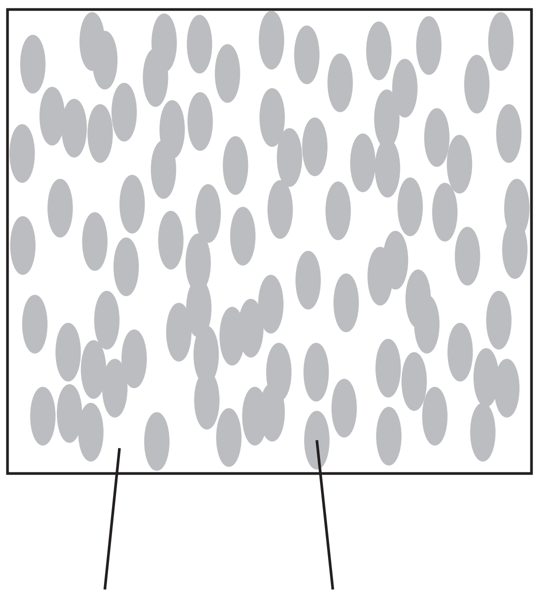
effective complex permittivity
(dielectric constant, conductivity)

sea ice thickness
ice concentration



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\Big\} \; \epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \; , \text{ composite geometry } \right)$$

Herglotz function

Analytic Continuation Method

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s-z}$$

material parameters

separates geometry from parameters

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass $= p_1$
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = (s + \Gamma\chi)^{-1} e_k$$

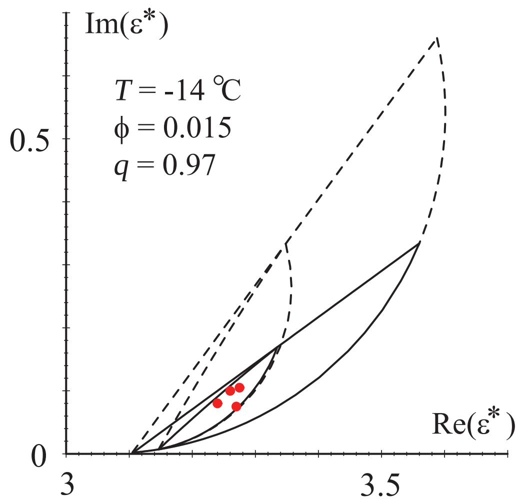
$\Gamma\chi$: microscale $\longrightarrow$ macroscale

$\Gamma\chi$ *links scales*

Golden and Papanicolaou, Comm. Math. Phys. 1983

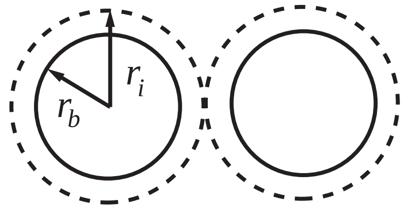
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle



$$q = r_b / r_i$$

$$0 < q < 1$$

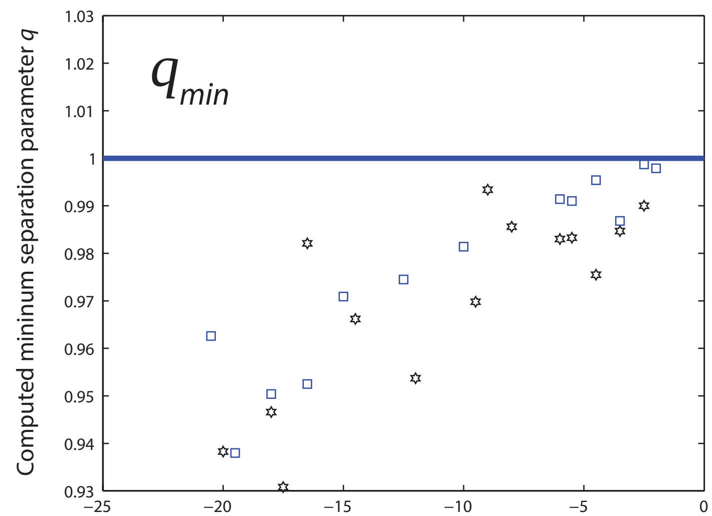
Golden 1995, 1997

Bruno 1991

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inverse bounds



inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

direct calculation of spectral measure

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).

2. The fundamental operator $\chi\Gamma\chi$ becomes a random matrix depending only on the composite geometry.

3. Compute the eigenvalues λ_i and eigenvectors of $\chi\Gamma\chi$ with inner product weights α_i

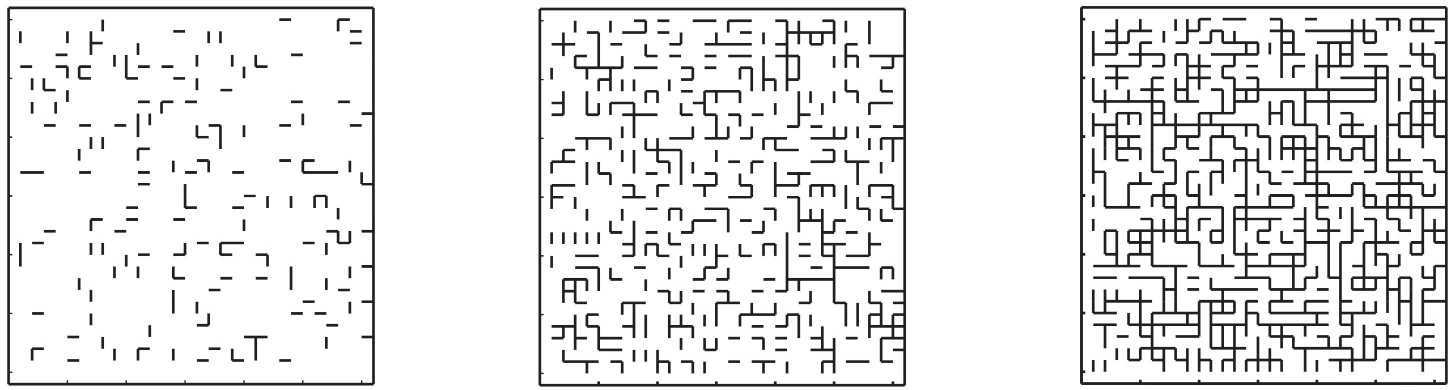
$$\mu(\lambda) = \sum_i \alpha_i \, \delta(\lambda - \lambda_i)$$

Dirac point measure (Dirac delta)

earlier studies of spectral measures

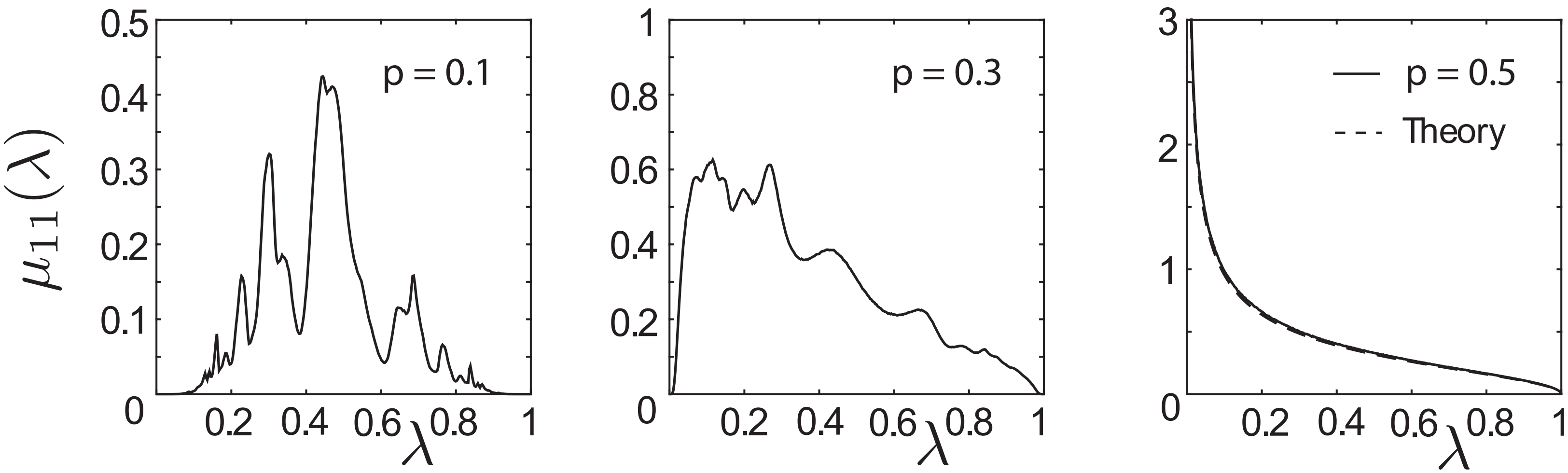
Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral statistics for 2D random resistor network



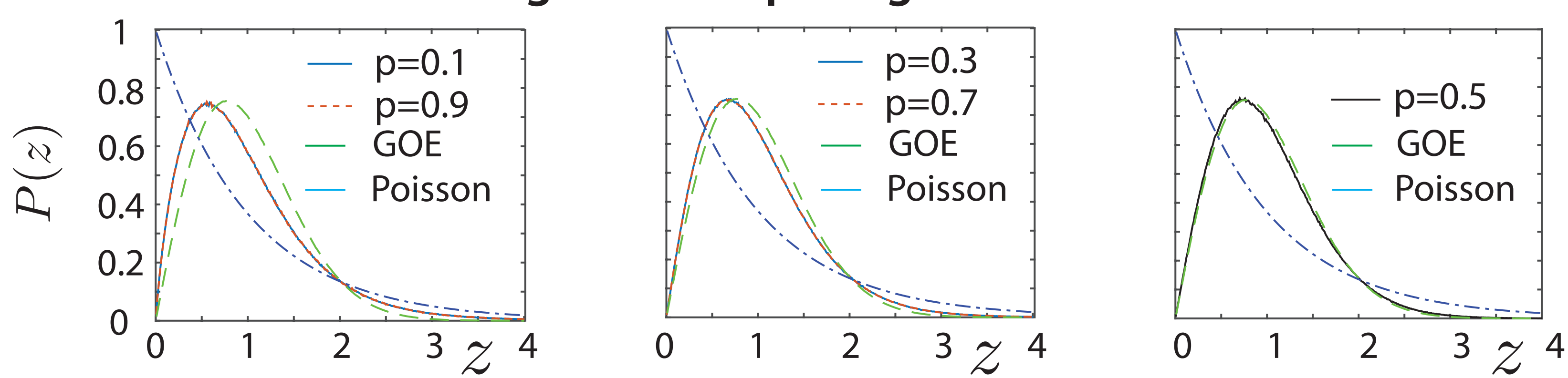
Spectral Measures

Murphy and Golden, *J. Math. Phys.*, 2012
Murphy et al. *Comm. Math. Sci.*, 2015



$p_c = 0.5$

Eigenvalue Spacing Distributions



Murphy, Cherkaev, Golden, *PRL*, 2017

Eigenvalue Statistics of Random Matrix Theory

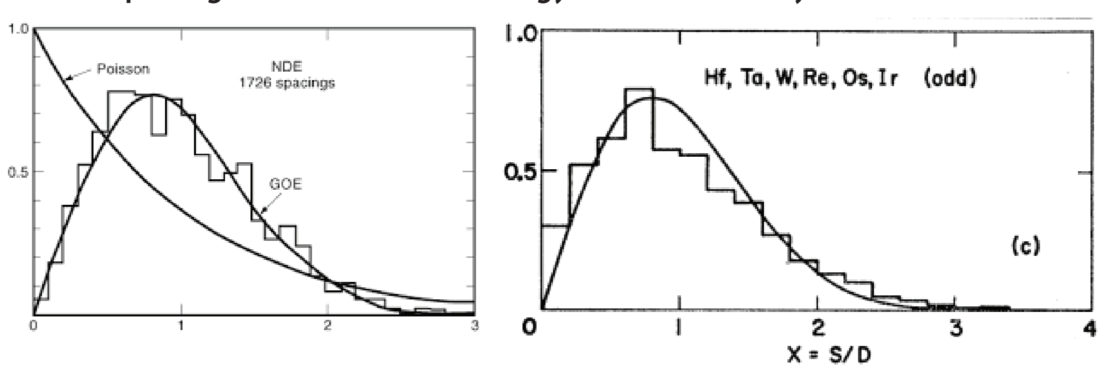
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^{T})/2 \quad$ ***Gaussian orthogonal ensemble (GOE)***

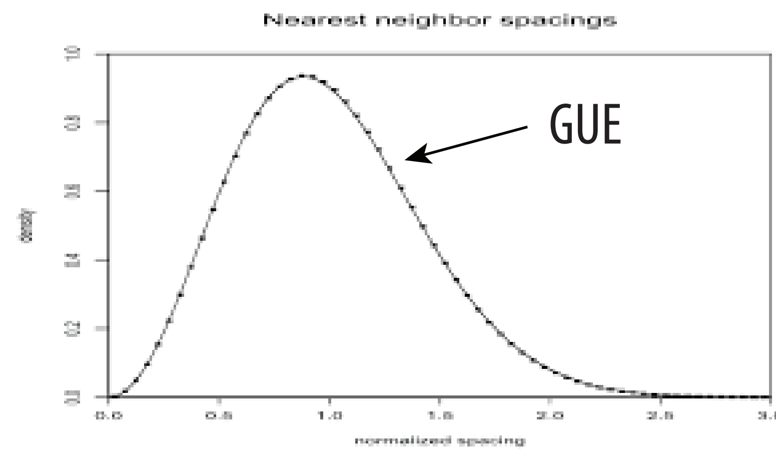
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^{\dagger})/2 \quad$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics

Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



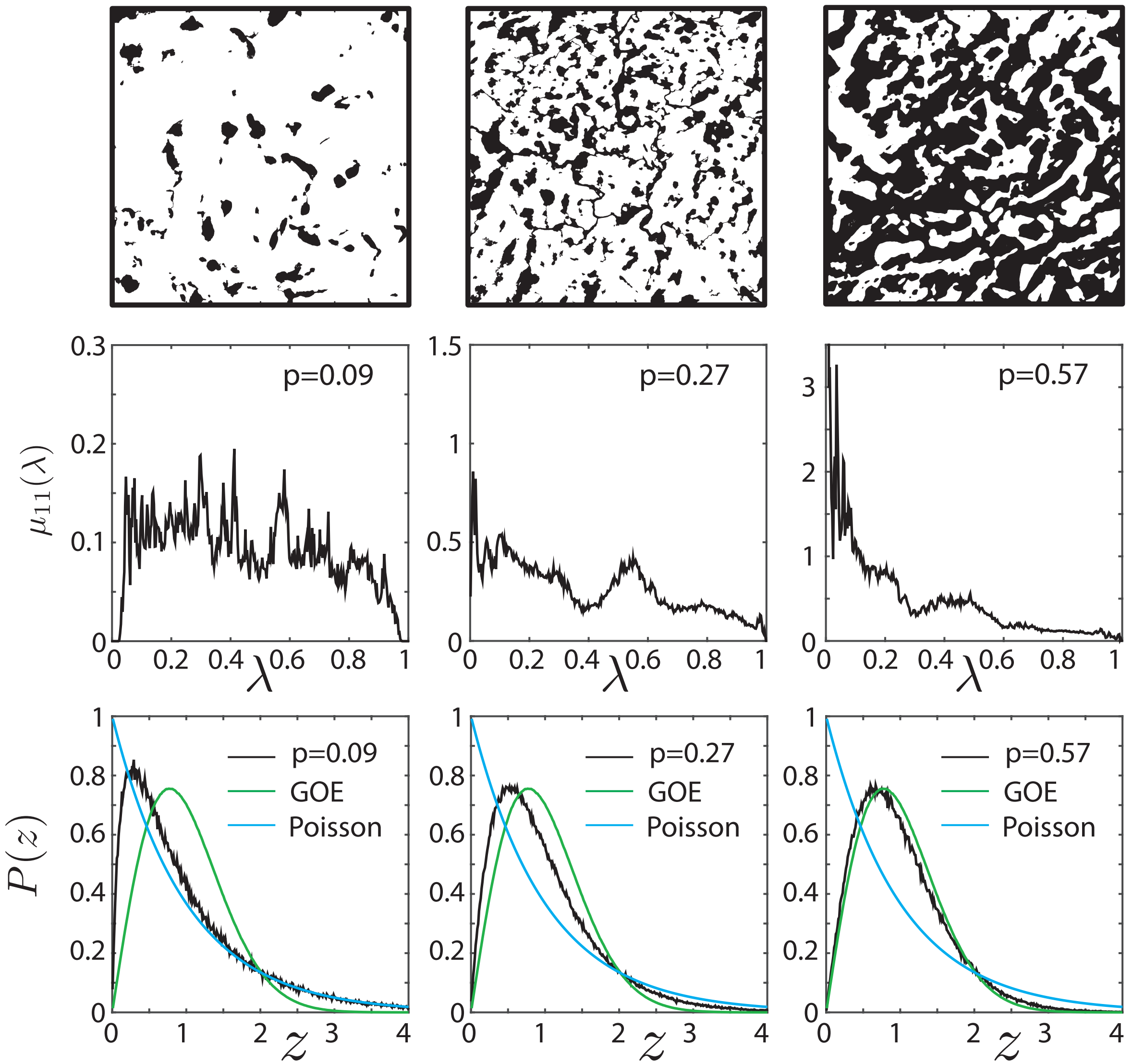
RMT used to characterize disorder-driven transitions in mesoscopic conductors, neural networks, random graph theory, etc.

Phase transitions ~ transitions in universal eigenvalue statistics.

Spectral computations for Arctic melt ponds

spectral measures

eigenvalue spacing distributions



uncorrelated → **level repulsion**

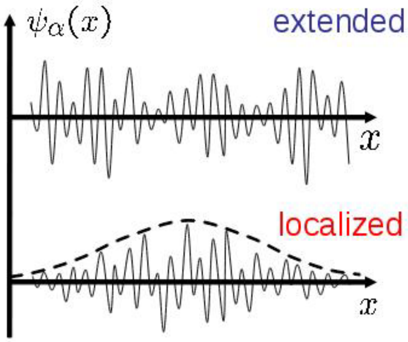
TRANSITION

Ben Murphy
Elena Cherkaev
Ken Golden
2017

eigenvalue statistics for transport tend toward the UNIVERSAL Wigner-Dyson distribution as the "conducting" phase percolates

low disorder

high disorder



metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

we find a surprising analog

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden   Phys. Rev. Lett.   2017

PERCOLATION TRANSITION

transition to universal eigenvalue statistics (GOE)
extended states, mobility edges

-- but without wave interference or scattering effects ! --

eigenvector localization and mobility edges

Inverse Participation Ratio: $$I(\vec{v}_n) = \sum_{i=1}^{N} |(\vec{v}_n)_i|^4$$

Completely Localized: $$I(\vec{e}_n) = 1$$

Completely Extended: $$I\left(\frac{1}{\sqrt{N}}\vec{1}\right) = \frac{1}{N}$$

Anderson Model

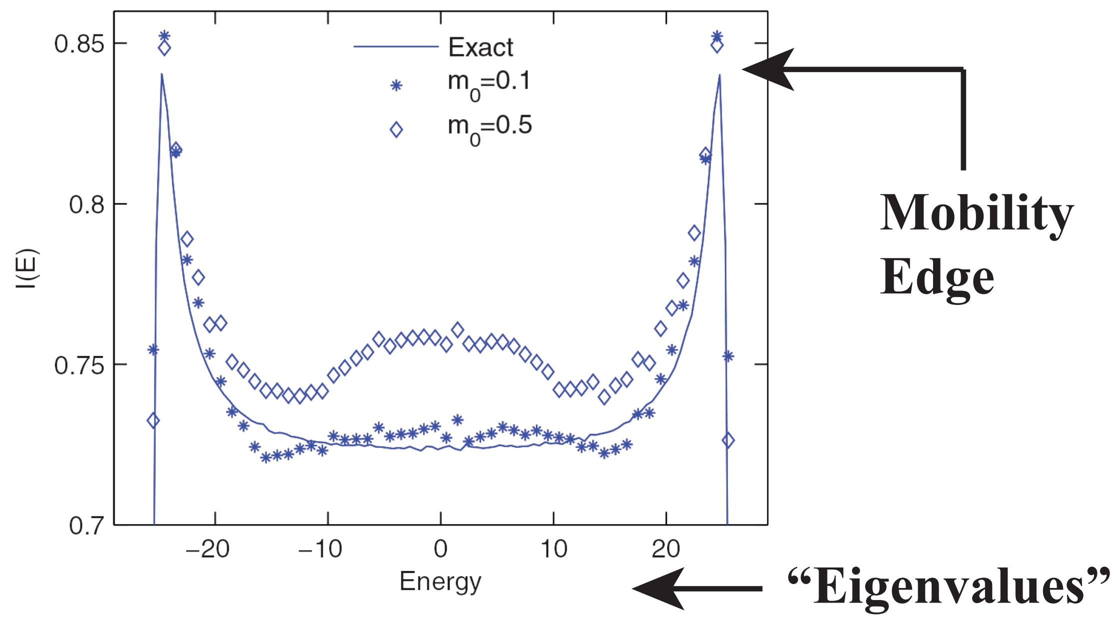
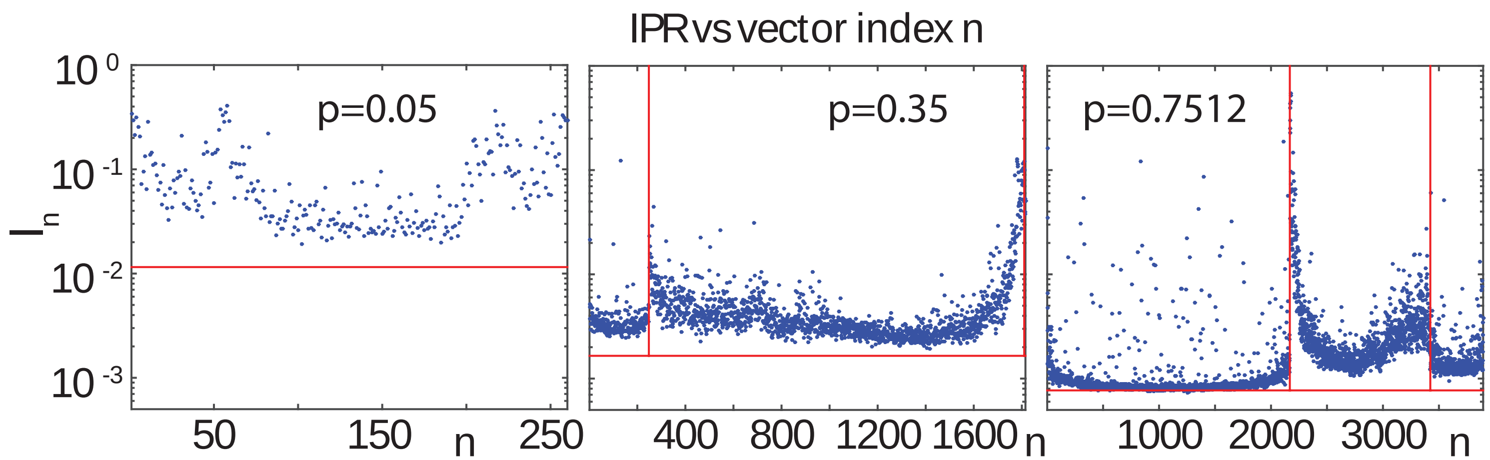
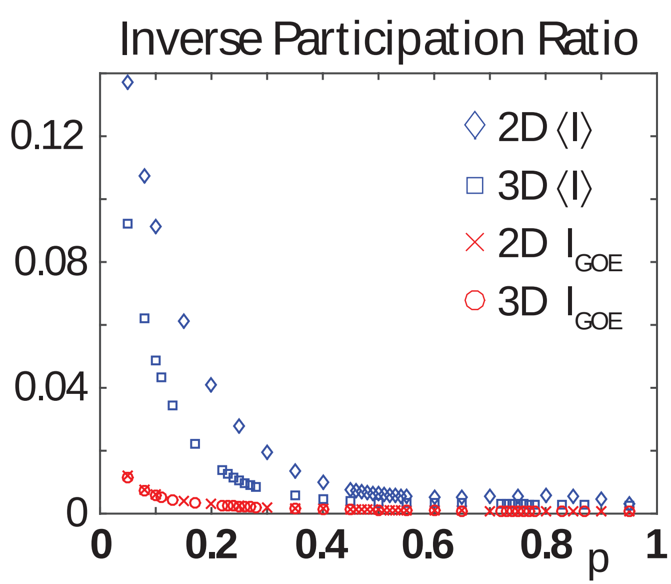


FIG. 4. (Color online) IPR for Anderson model in two dimensions with $x = 6.25$ ($w = 50$) from exact diagonalization (solid line) and from LDRG with different values of the cutoff m_0. LDRG data are averaged over 100 runs of systems with 100×100 sites.

PHYSICAL REVIEW B 90, 060205(R) (2014)

Localization properties of eigenvectors in random resistor networks





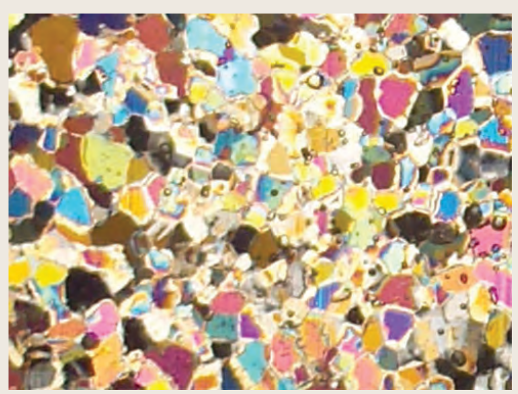
$$I_n = \sum_i (\vec{v}_n)_i^4$$

Murphy, Cherkaev, Golden, 2017

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A   8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS OF SCIENTIFIC PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

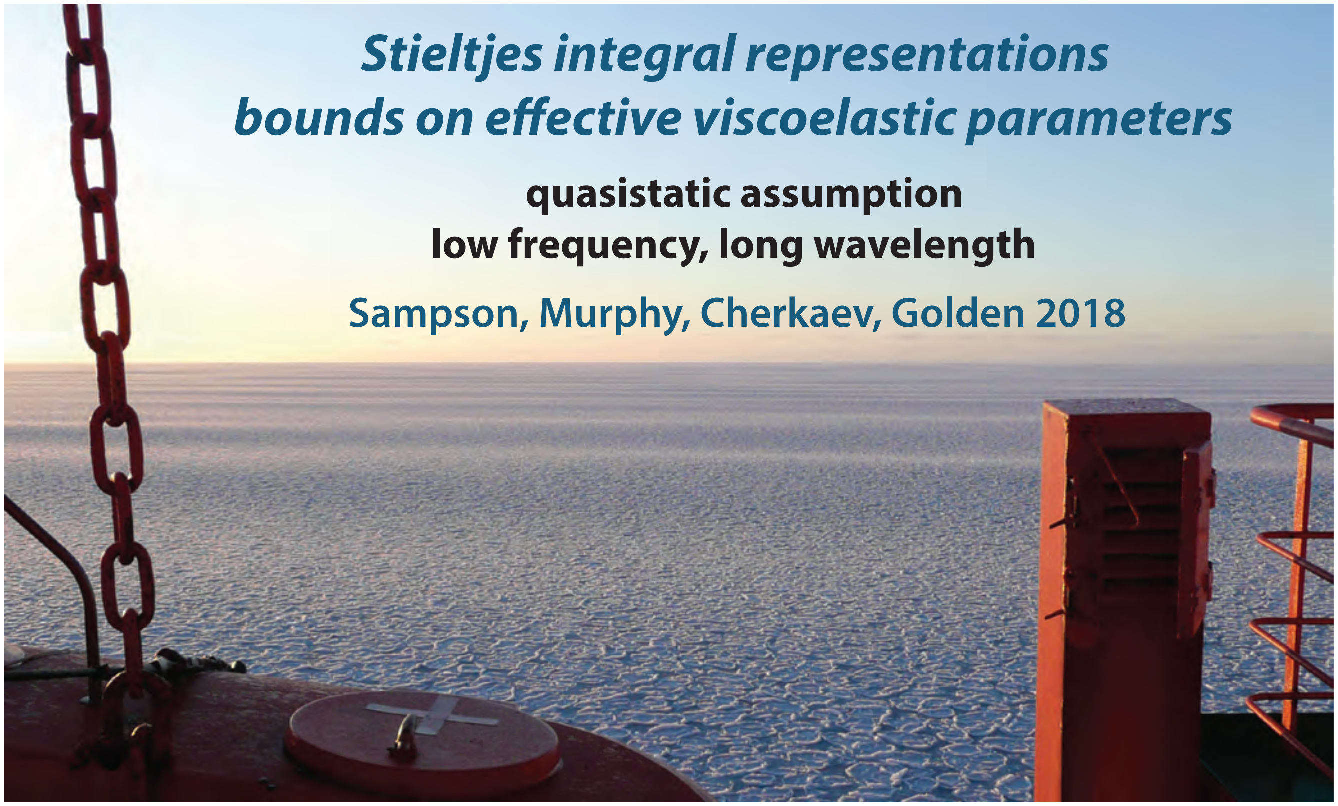
A method to distinguish between different types of sea ice using remote sensing techniques

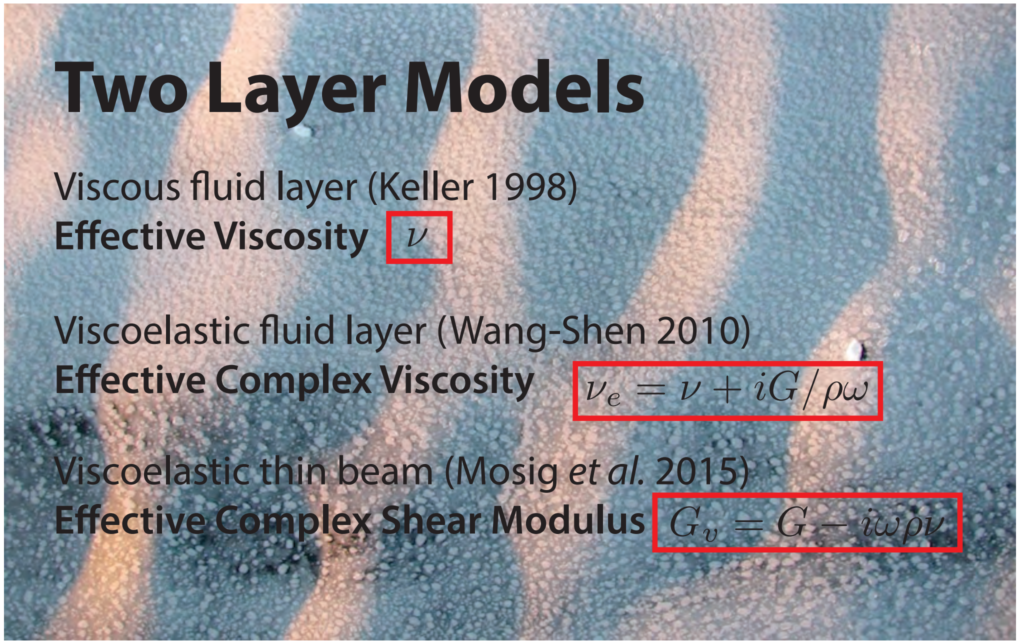
A computer model to determine how a human should walk so as to expend the least energy

THE ROYAL SOCIETY PUBLISHING

21/01/15 1:47 PM

wave propagation in the marginal ice zone





Stieltjes Integral Representation for Complex Viscoelasticity

homogenized $\langle \sigma_{ij} \rangle = C^*_{ijkl} \langle \epsilon_{kl} \rangle$

local $\nabla \cdot \sigma = 0$ $\sigma_{ij} = C_{ijkl} \epsilon_{kl}$

$$C_{ijkl} = (\nu_1 \chi + (1-\chi)\nu_2)\lambda_s$$

$$\nabla \cdot \big((\nu_1 \chi + (1-\chi)\nu_2)\lambda_s : \epsilon \big) = 0$$

$$s = \frac{1}{1 - \frac{\nu_1}{\nu_2}}$$

Strain Field

$$\epsilon = \frac{1}{2}[\nabla u + (\nabla u)^T] = \nabla^s u$$

$$\epsilon = \epsilon_0 + \epsilon_f \quad \text{where} \quad \epsilon_f = \nabla^s \phi$$

Elasticity Tensor

$$C^*_{ijkl} = \nu^* \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl} \right) = \nu^* \lambda_s$$

RESOLVENT $\epsilon = \left(1 - \frac{1}{s}\Gamma\chi \right)^{-1} \epsilon_0$ $\Gamma = \nabla^s (\nabla \cdot \nabla^s)^{-1} \nabla \cdot$ ϵ_0 **avg strain**

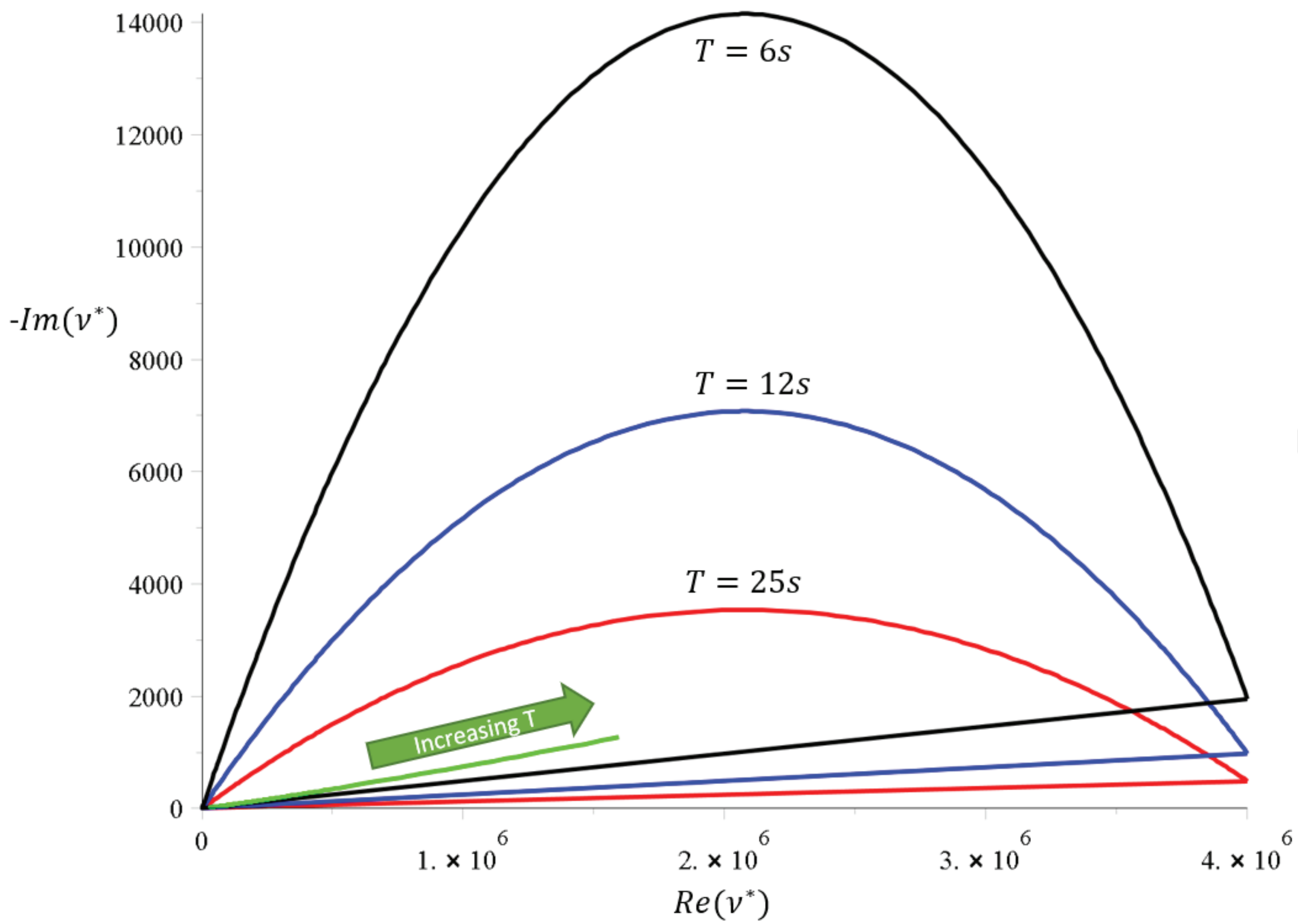
$$F(s) = 1 - \frac{\nu^*}{\nu_2} \qquad F(s) = ||\epsilon_0||^{-2} \int_\Sigma \frac{d\mu(\lambda)}{s - \lambda}$$

bounds on the effective complex viscoelasticity

complex elementary bounds
(fixed area fraction of floes)

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**



Sampson, Murphy, Cherkaev, Golden 2018

melt pond formation and albedo evolution:

- ***major drivers in polar climate***
- ***key challenge for global climate models***

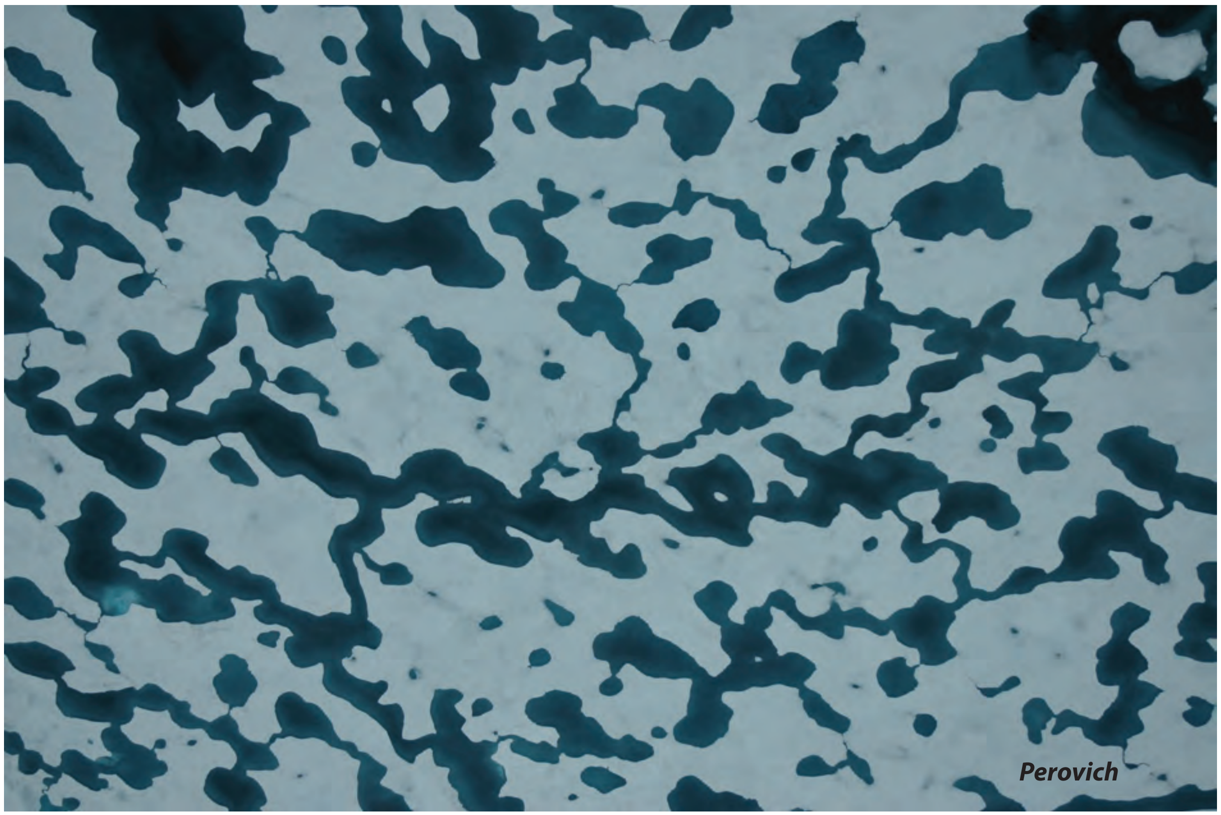
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

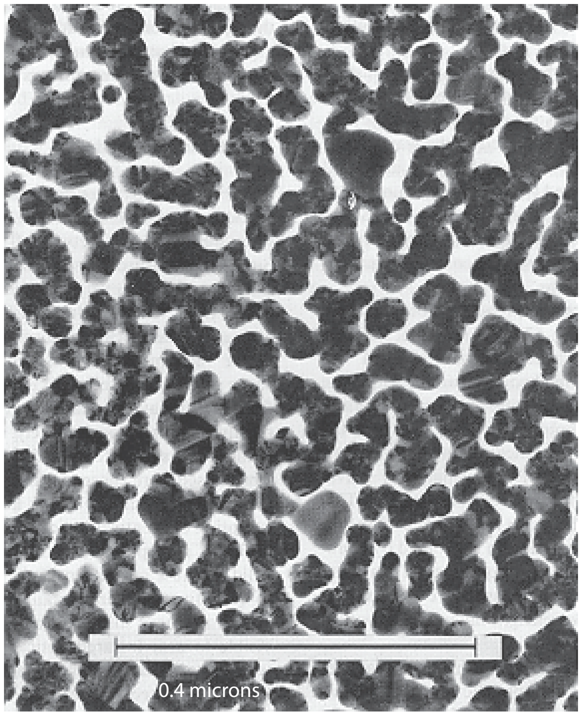
Flocco, Feltham, Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

thin silver film

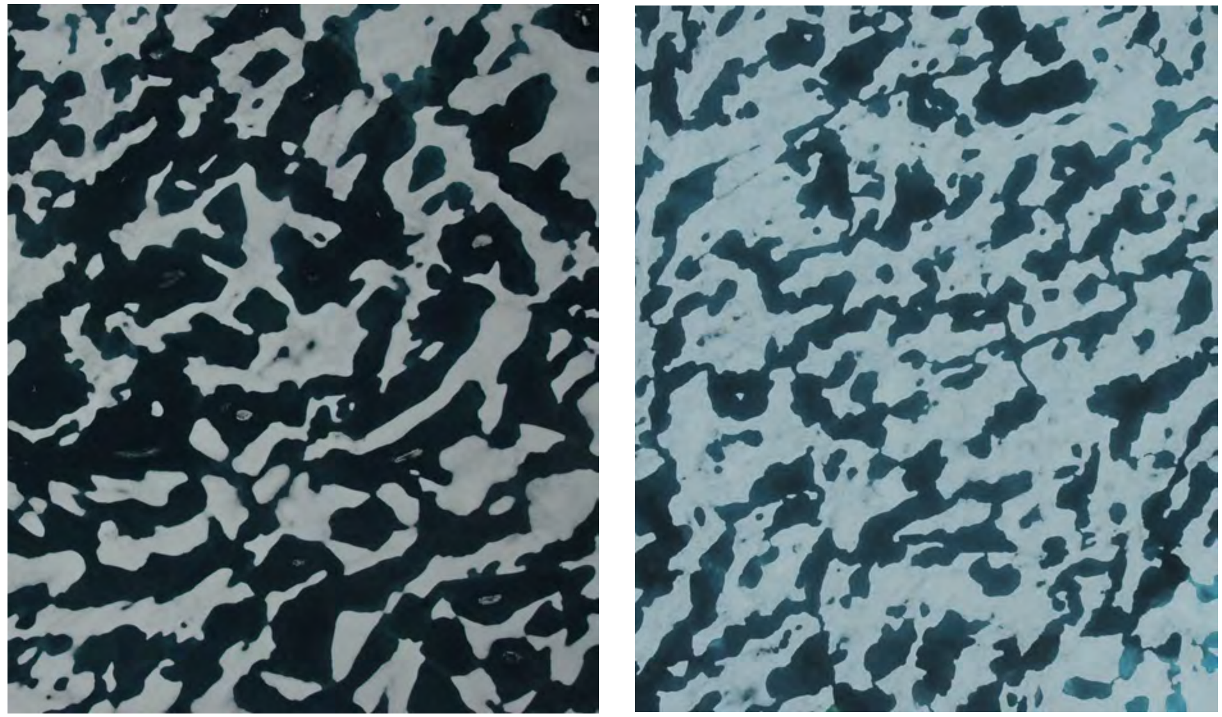
microns



(Davis, McKenzie, McPhedran, 1991)

Arctic melt ponds

kilometers



(Perovich, 2005)

optical properties

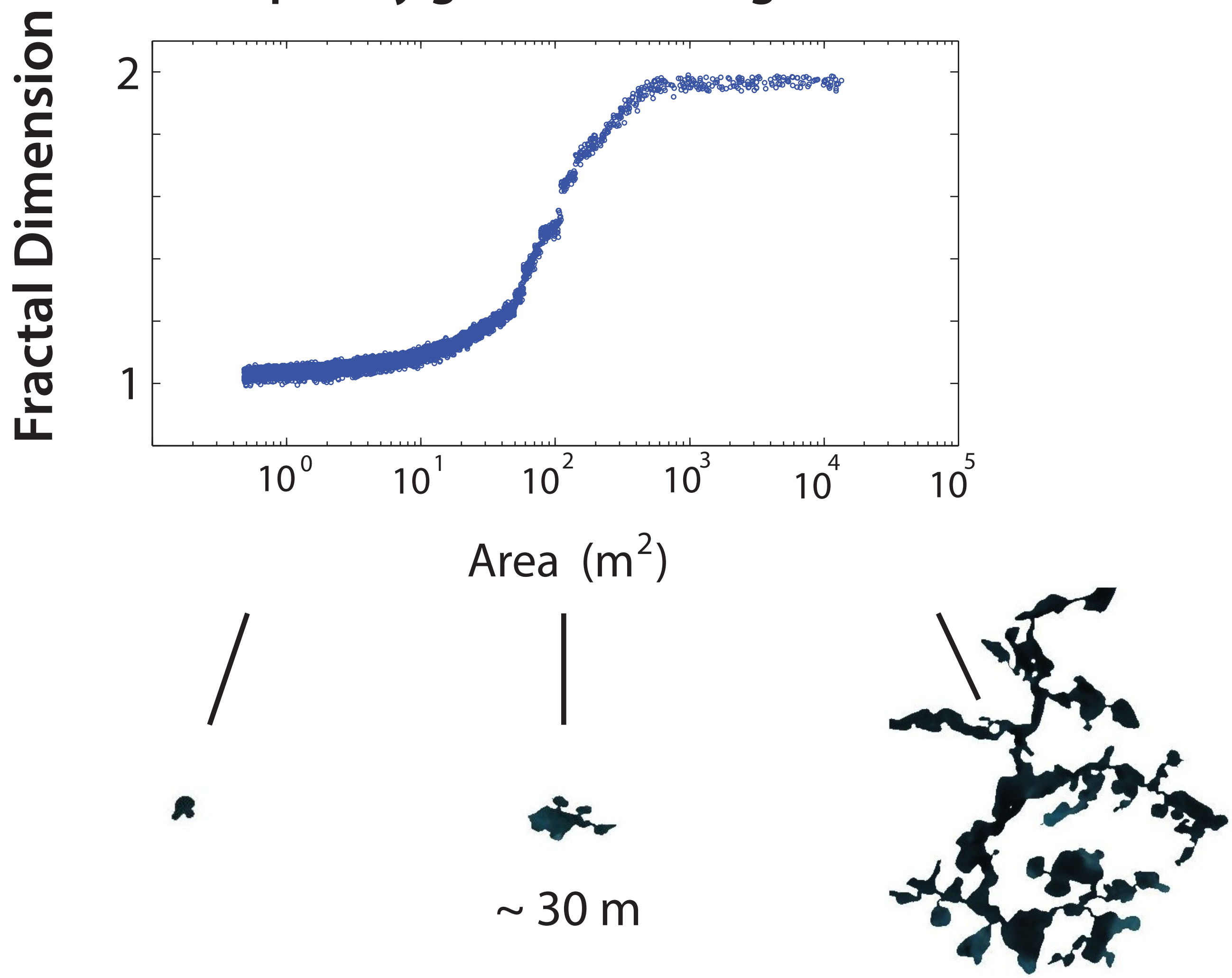
composite geometry -- area fraction of phases, connectedness, necks

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

complexity grows with length scale

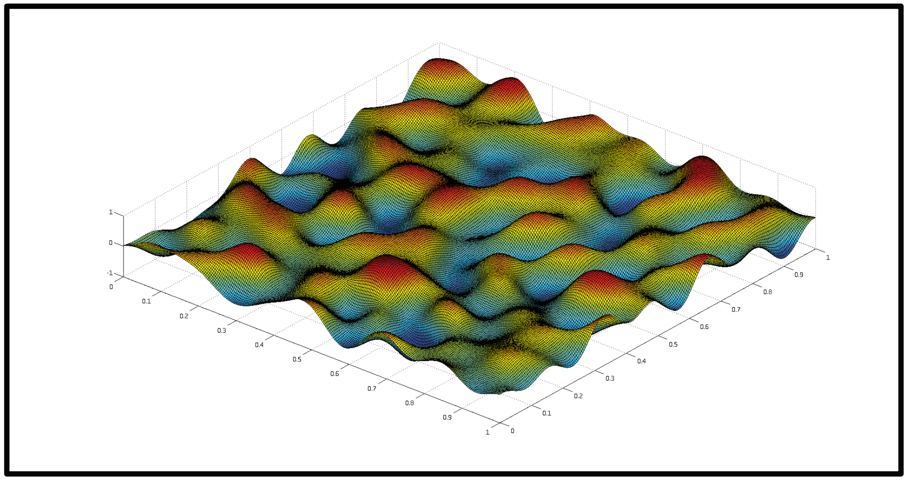


simple pond ***transitional pond*** ***complex pond***

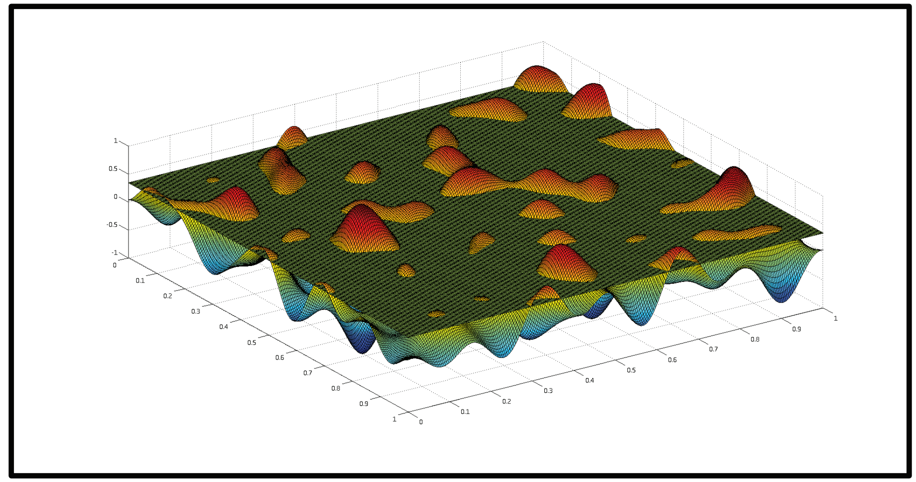
Continuum percolation model for melt pond evolution

level sets of random surfaces

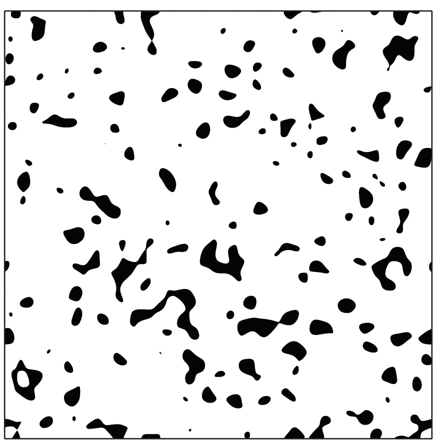
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018

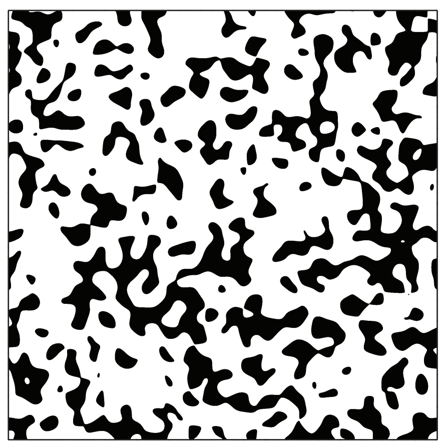


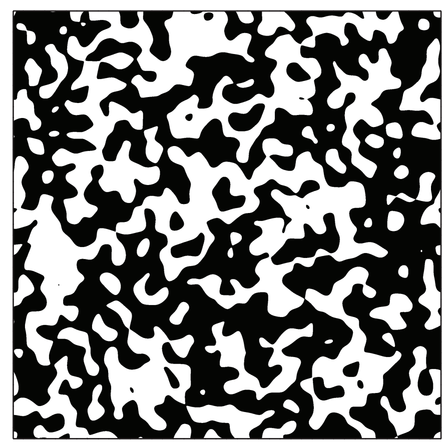
random Fourier series representation of surface topography



intersections of a plane with the surface define melt ponds





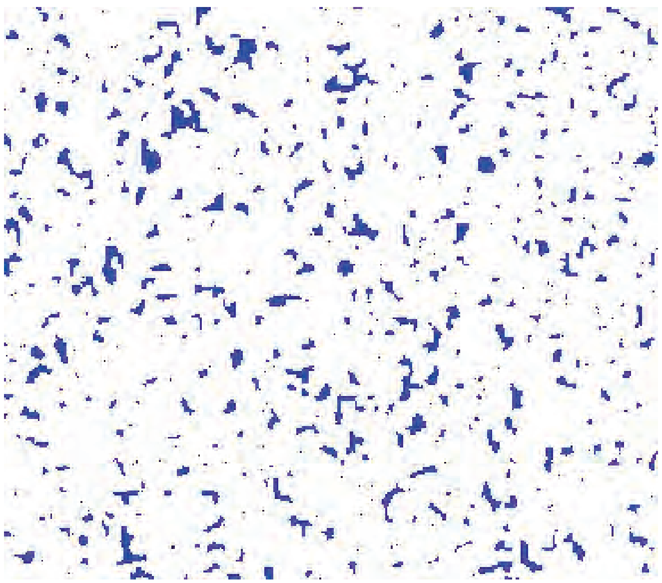


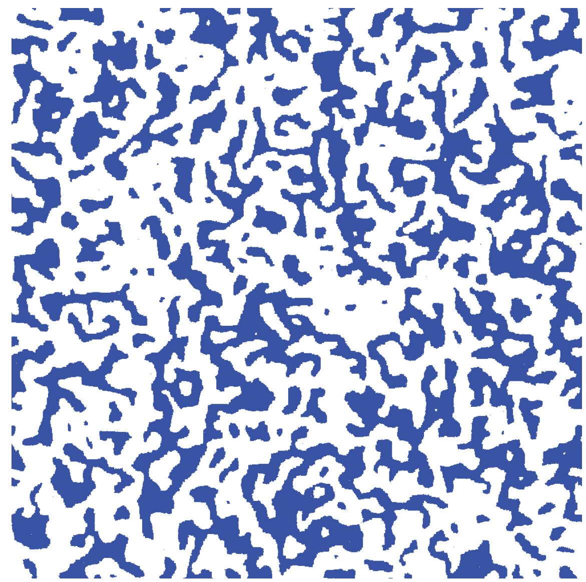
electronic transport in disordered media   *diffusion in turbulent plasmas*   *Isichenko, Rev. Mod. Phys., 1992*

Ising model for ferromagnets → Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - H \sum_{i}^{N} s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

magnetization $M = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$ **pond coverage** $\frac{(M+1)}{2}$





"melt ponds" are clusters of magnetic spins that align with the applied field

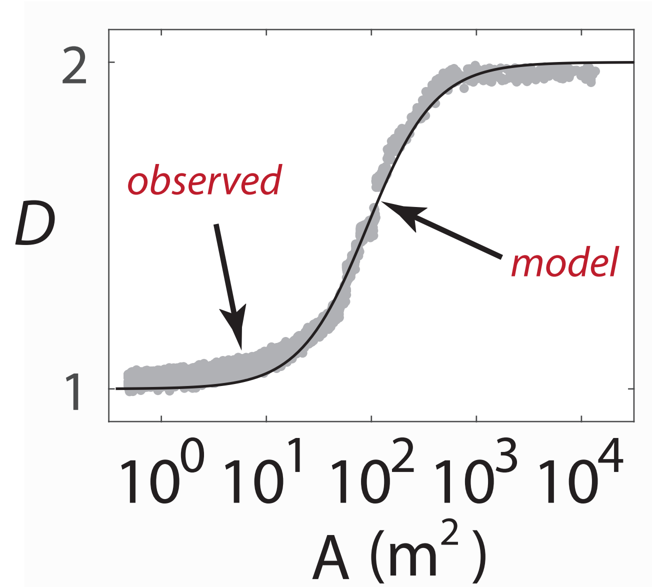
predictions of fractal transition, pond size exponent **Ma, Sudakov, Strong, Golden 2018**

Ising model results

Minimize Ising Hamiltonian energy

Random magnetic field represents snow topography; interaction term represents horizontal heat transfer.

Melt ponds —
metastable islands of like spins in our random field Ising model.

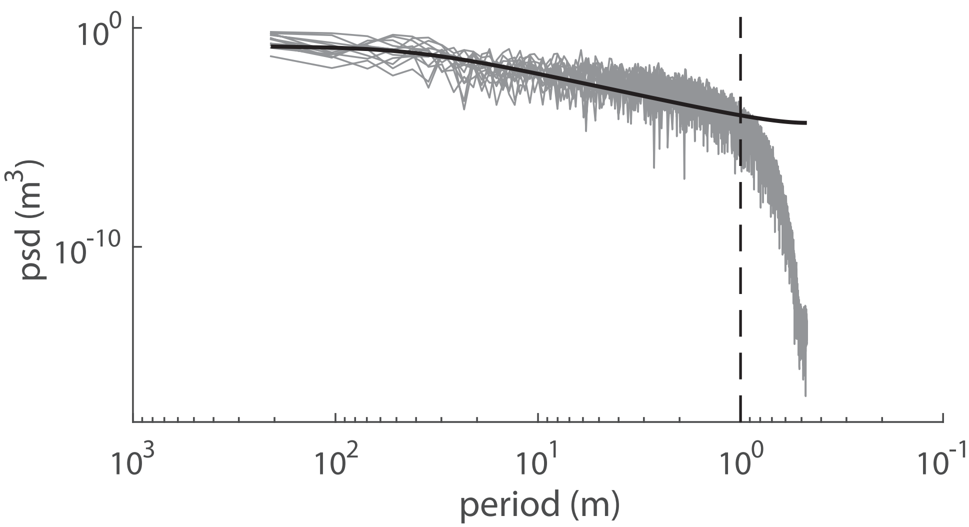


pond size distribution exponent

observed -1.5
(Perovich, et al 2002)

model -1.58

order out of disorder



The lattice constant *a* must be small relative to the 10-20 m length scales prominent in sea ice and snow topography. We set *a*=1 m as the length scale above which important spatially correlated fluctuations occur in the power spectrum of snow topography.

Conclusions

1. Summer Arctic sea ice is melting rapidly, and melt ponds and other processes must be accounted for in order to predict melting rates.

2. Fluid flow through sea ice mediates melt pond evolution and many processes important to climate change and polar ecosystems.

3. Statistical physics and homogenization help *link scales*, provide rigorous methods for finding effective behavior, and advance how sea ice is represented in climate models.

4. Random matrix theory and an unexpected Anderson transition arises in our studies of percolation in sea ice structures.

5. Our research will help to improve projections of climate change and the fate of the Earth sea ice packs.

THANK YOU

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

Office of Naval Research

Arctic and Global Prediction Program

Applied and Computational Analysis Program

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999