Homogenization, Anderson Transitions, and Waves in Sea Ice

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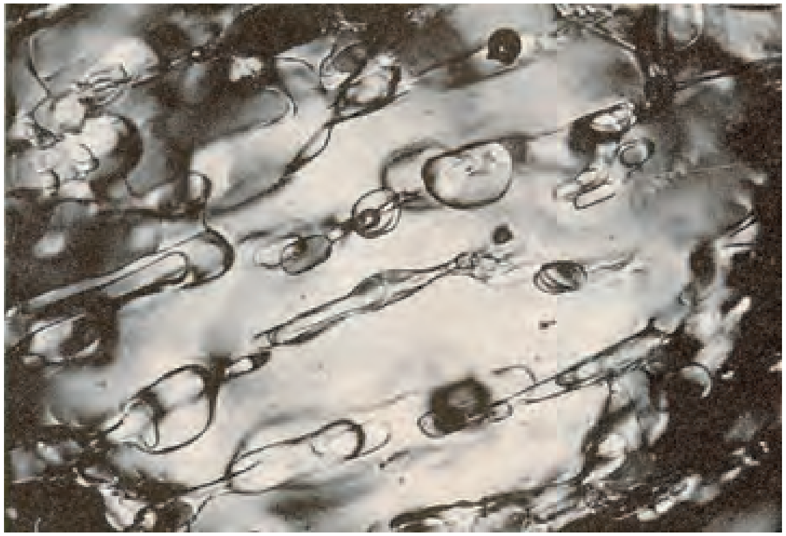
sea ice is a multiscale composite

structured on many length scales - from tenths of mm's to tens of km's

brine inclusions

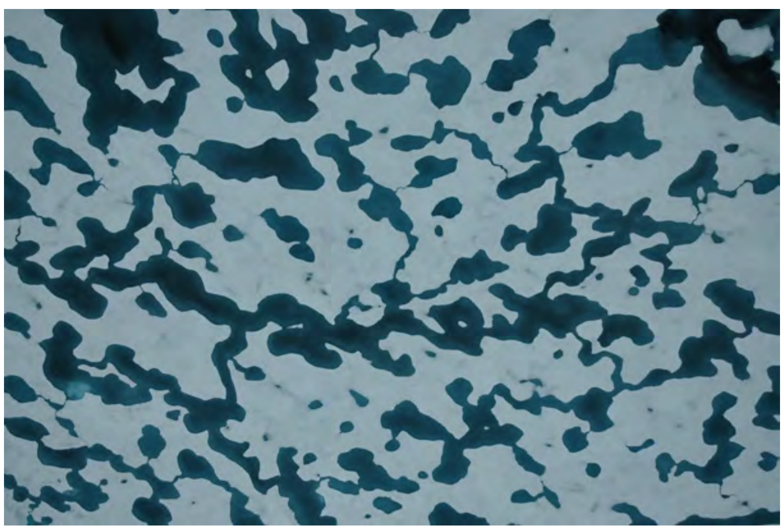


millimeters

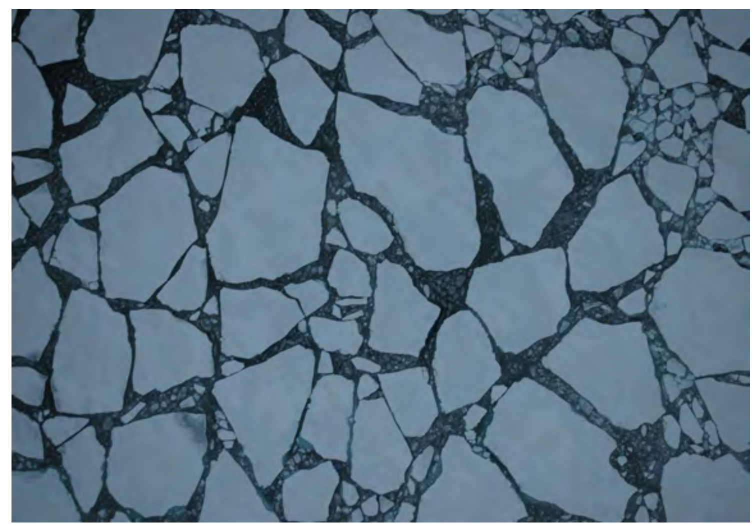
pancakes

centimeters

melt ponds



meters



ice floes

kilometers

What is this talk about? HOMOGENIZATION

Using methods of statistical physics and composite materials to **LINK SCALES** ***in the sea ice system ... rigorously compute effective behavior and improve climate models.***

Find unexpected Anderson transition in composites along the way!

1. ***Sea ice microphysics and fluid transport***

   homogenization and percolation theory

2. ***EM monitoring of sea ice, analytic continuation method***

   random matrix theory and Anderson transitions

3. ***Extension of ACM to advection diffusion, waves in sea ice***

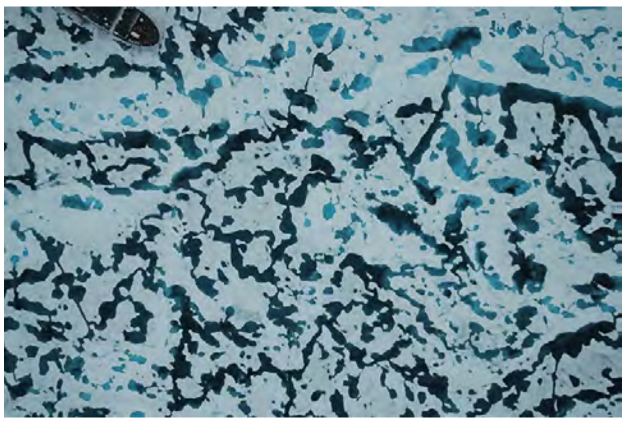
   Stieltjes integral representations, spectral measures

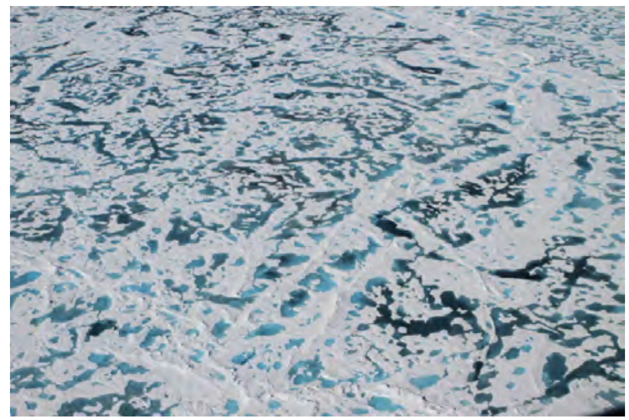
How do scales interact in the sea ice system?

basin scale -
grid scale
albedo

Linking Scales

km
scale
melt
ponds

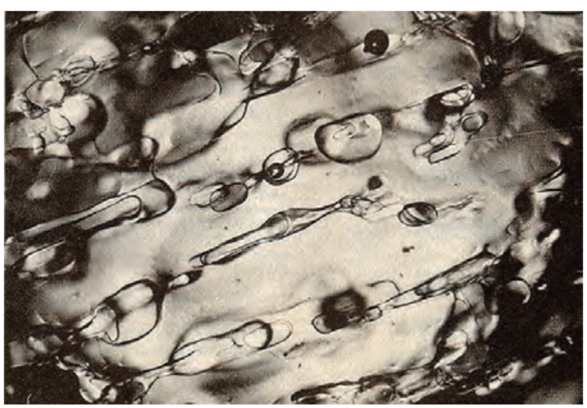




km
scale
melt
ponds

Linking Scales

mm
scale
brine
inclusions



meter
scale
snow
topography

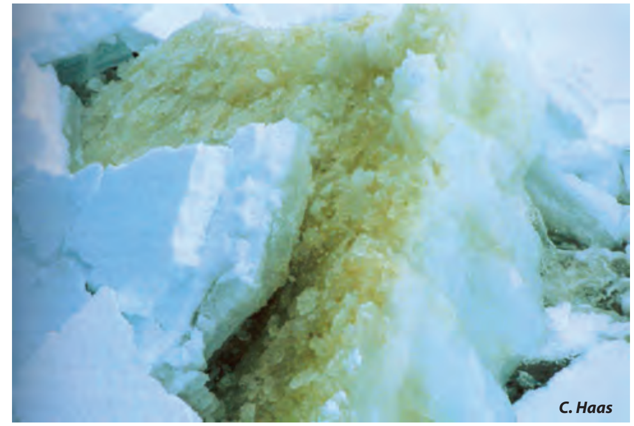
sea ice microphysics

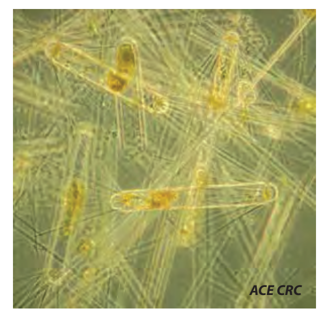
fluid transport

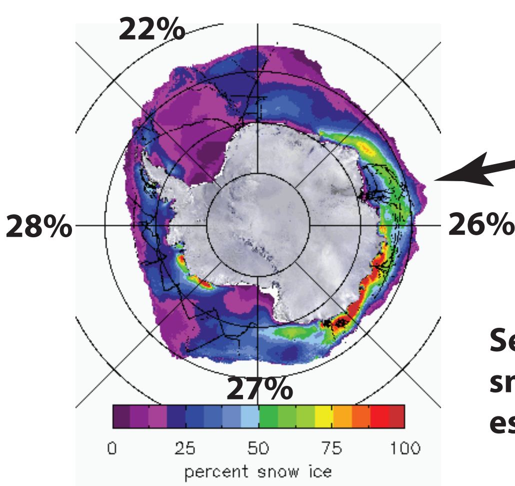
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities







T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

- evolution of salinity profiles

- ocean-ice-air exchanges of heat, CO_2

Darcy's Law for slow viscous flow in a porous medium

averaged
fluid velocity

pressure
gradient

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$

viscosity

$\mathbf{k}$ = fluid permeability tensor

example of *homogenization*

mathematics for analyzing effective behavior of heterogeneous systems

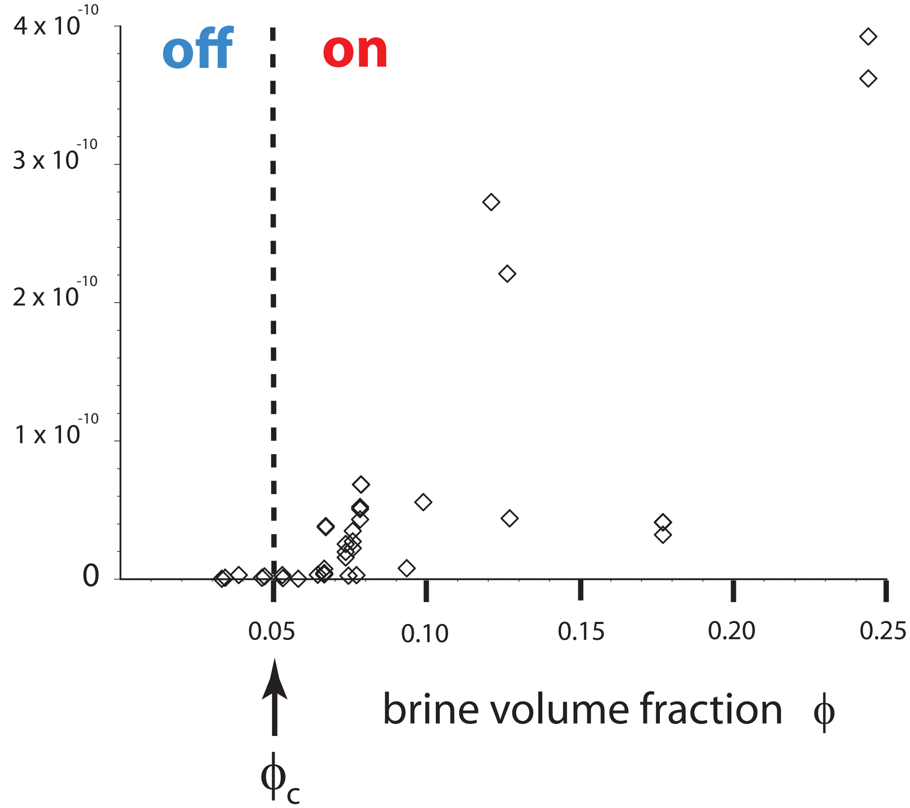
e.g. transport properties of composites - electrical conductivity, thermal conductivity, etc.

Critical behavior of fluid transport in sea ice

impermeable permeable

Arctic field data

vertical fluid permeability k (m^2)



on - off switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\%$ $\longleftrightarrow$ $T_c \approx -5^\circ C$, $S \approx 5$ ppt

RULE OF FIVES

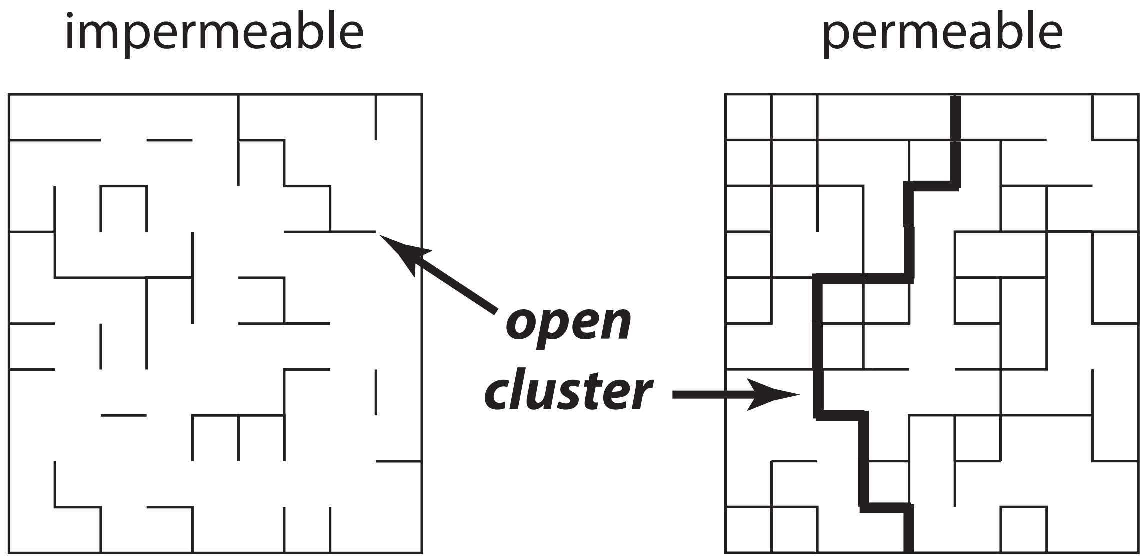
Golden, Ackley, Lytle *Science* 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

percolation theory

probabilistic theory of connectedness



$p = 1/3$ $p = 2/3$

bond → ***open*** *with probability p*
closed *with probability $1\text{-}p$*

percolation threshold

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

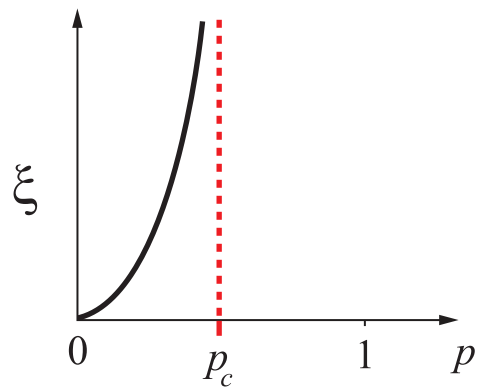
smallest *p* for which there is an infinite open cluster

order parameters in percolation theory

geometry

correlation length

characteristic scale
of connectedness

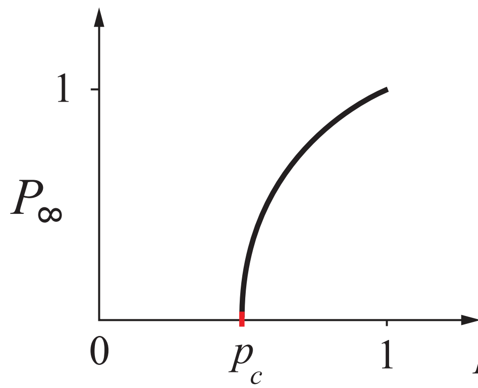


$$\xi\,(p) \sim |\,p - p_c|^{-\nu}$$

$$p \to p_c$$

infinite cluster density

probability the origin
belongs to infinte cluster

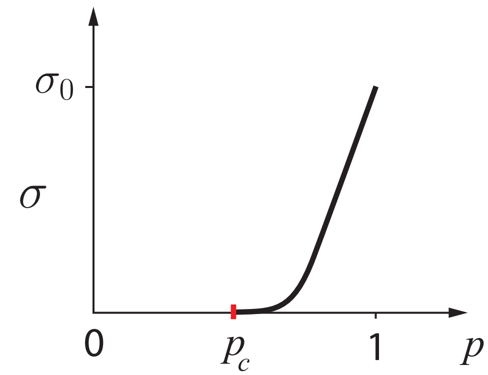


$$P_\infty(p) \sim (\,p - p_c)^{\beta}$$

$$p \to p_c^{+}$$

transport

effective conductivity or fluid permeability



$$\sigma\,(p) \sim \sigma_0\,(p - p_c)^{t}$$

$$p \to p_c^{+}$$

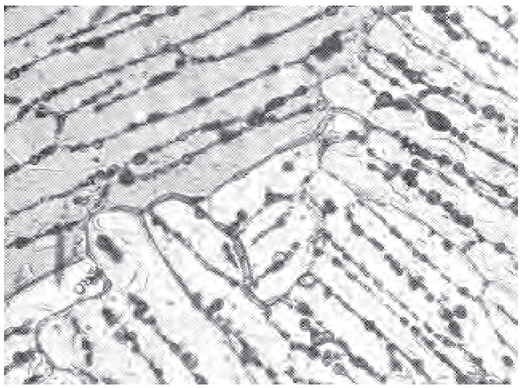
UNIVERSAL critical exponents for lattices -- depend only on dimension

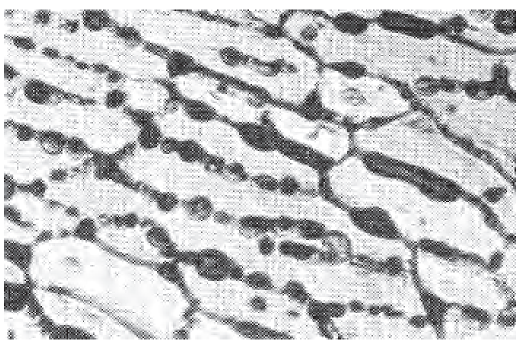
$1 \leq t \leq 2$ (for idealized model), Golden, *Phys. Rev. Lett.* 1990 **;** *Comm. Math. Phys.* 1992

non-universal behavior in continuum

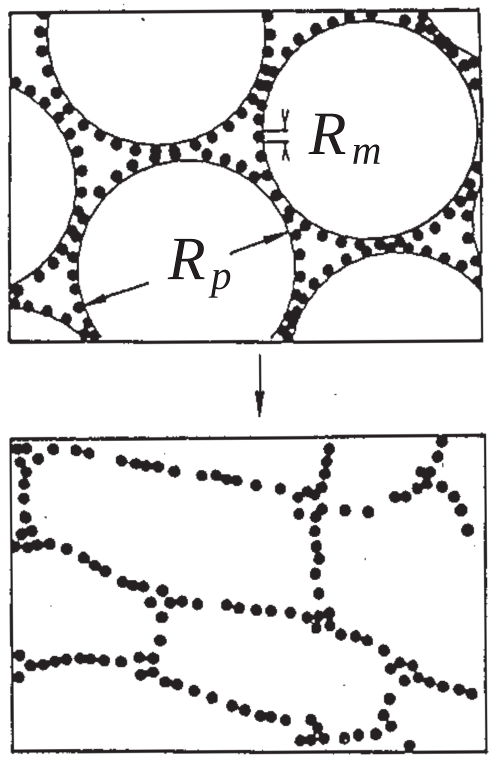
Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

$\phi_c \approx 5\,\%$ Golden, Ackley, Lytle, *Science*, 1998

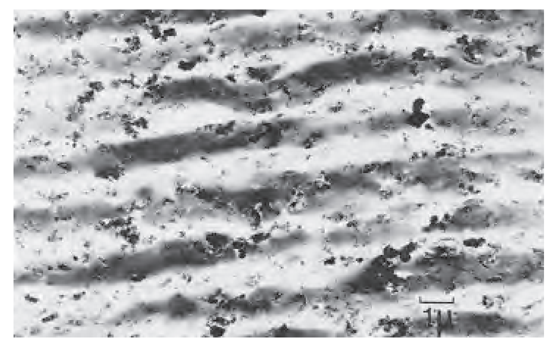




sea ice



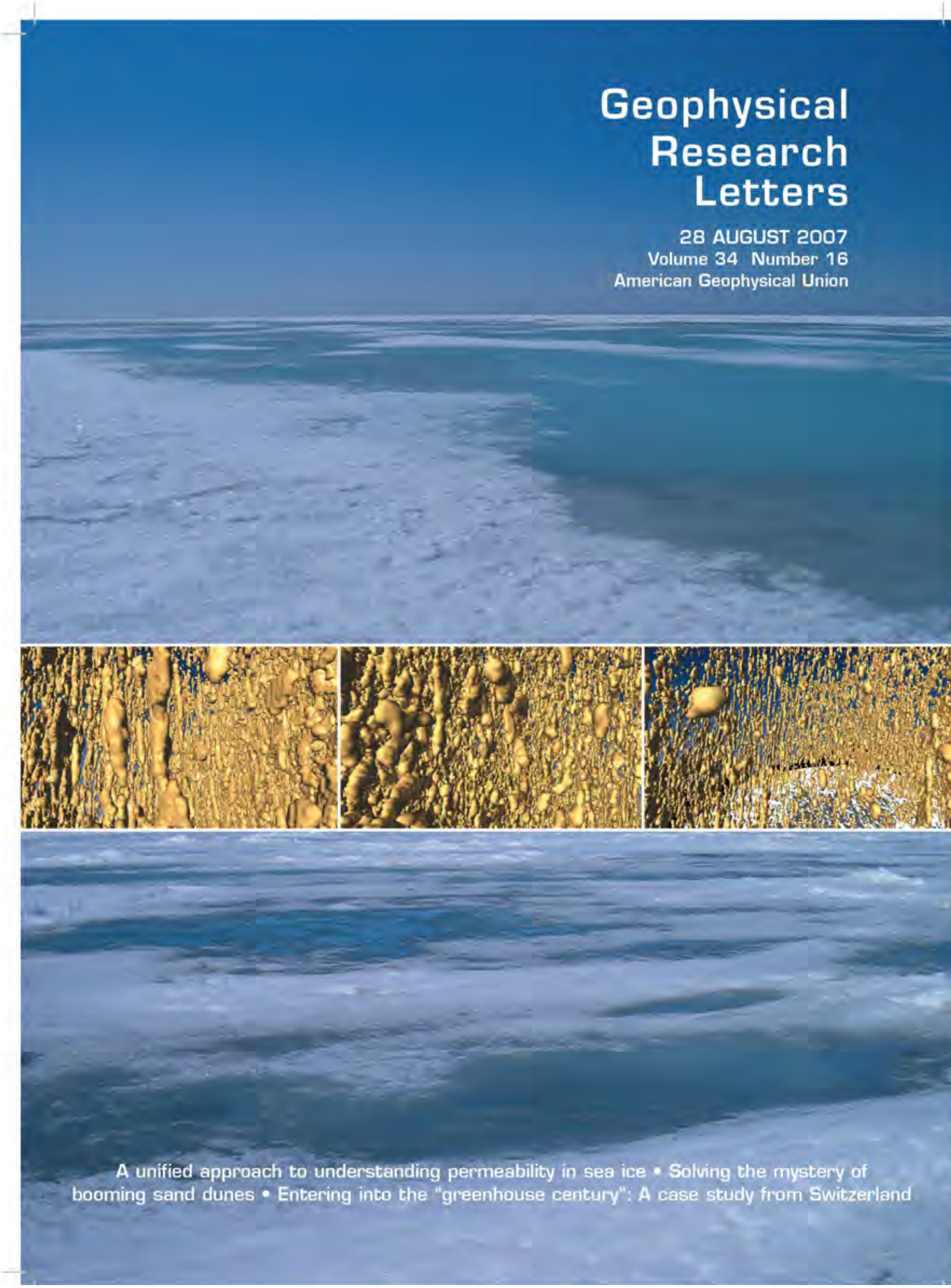
compressed powder



radar absorbing composite

sea ice is radar absorbing

Thermal evolution of permeability and microstructure in sea ice *Golden, Eicken, Heaton, Miner, Pringle, Zhu*



micro-scale

controls

macro-scale processes

rigorous bounds
percolation theory
hierarchical model
network model

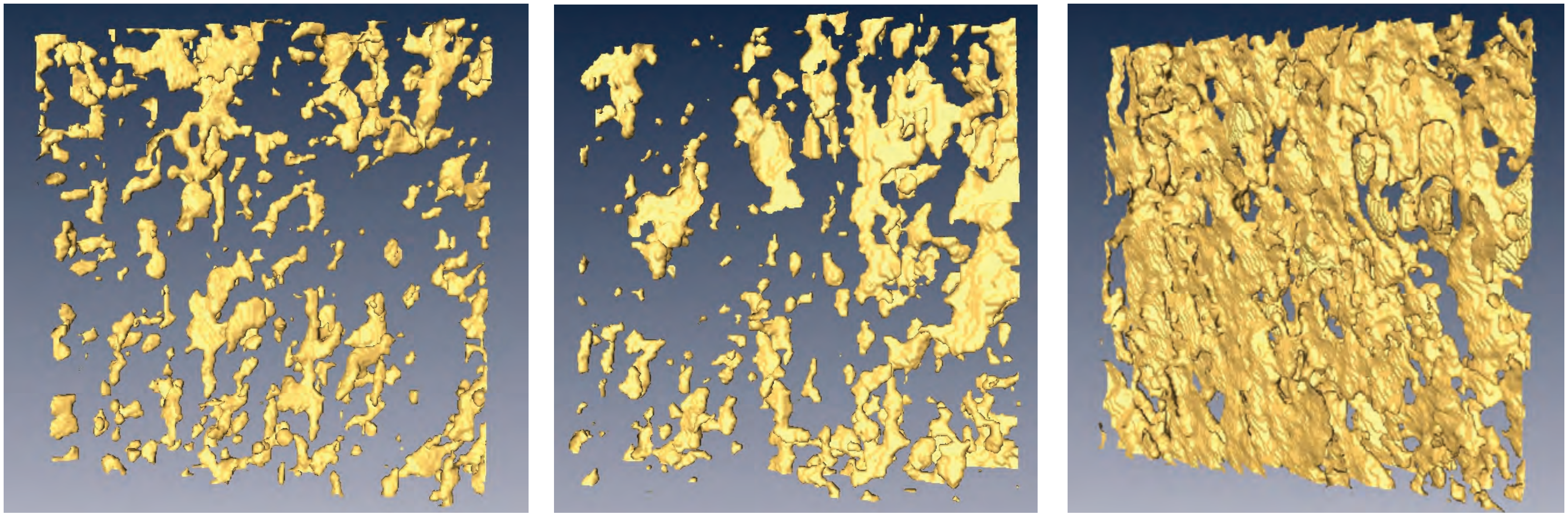
field data

X-ray tomography for brine inclusions

unprecedented look at thermal evolution of brine phase and its connectivity

brine connectivity (over cm scale)

8 x 8 x 2 mm



-15 °C, $\phi = 0.033$ -6 °C, $\phi = 0.075$ -3 °C, $\phi = 0.143$

X-ray tomography confirms percolation threshold

3-D images
pores and throats → 3-D graph
nodes and edges

analyze graph connectivity as function of temperature and sample size

- ***use finite size scaling techniques to confirm rule of fives***
- ***order parameter data from a natural material***

Pringle, Miner, Eicken, Golden, J. Geophys. Res. 2009

Remote sensing of sea ice

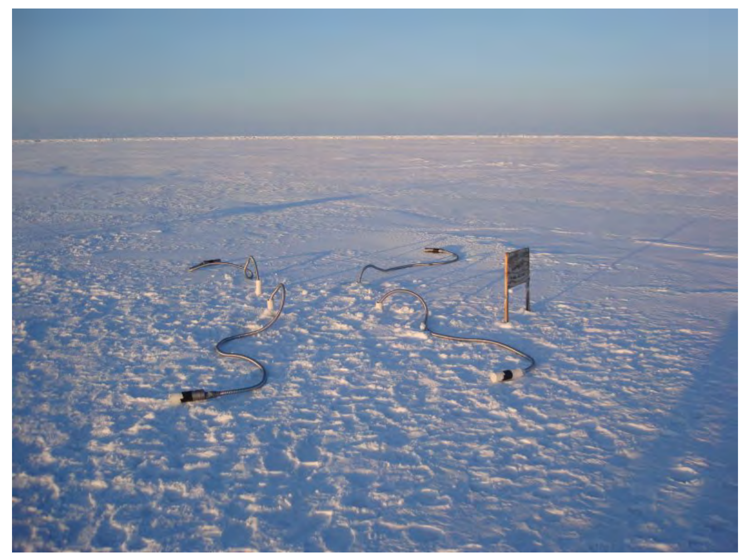
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

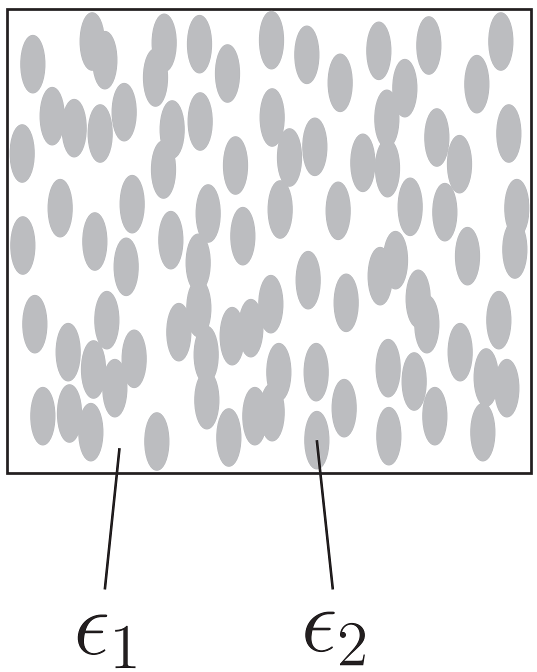
$$\varepsilon^*$$

effective complex permittivity (dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



ϵ^*

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry} \right)$$

Herglotz function

Theory of Effective Electromagnetic Behavior of Composites

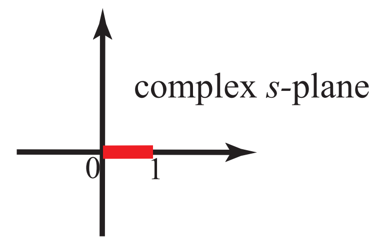
analytic continuation method

Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)
Theory of Composites, Milton (2002)

composite geometry
(spectral measure μ) $\longrightarrow$ ε^*

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z} \qquad s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* $\longrightarrow$ ***composite geometry***
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

Stieltjes integral representation

separates geometry from parameters

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

geometry

material parameters

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = (s + \Gamma\chi)^{-1} e_k$$

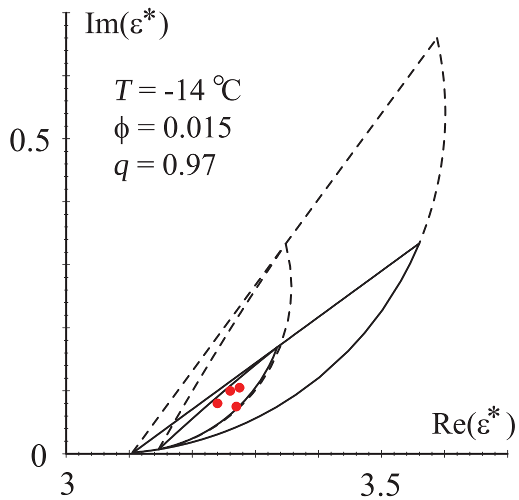
$\Gamma\chi$: microscale $\longrightarrow$ macroscale

$\Gamma\chi$ *links scales*

Golden and Papanicolaou, Comm. Math. Phys. 1983

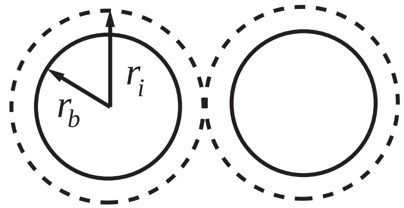
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle



$$q = r_b / r_i$$

$$0 < q < 1$$

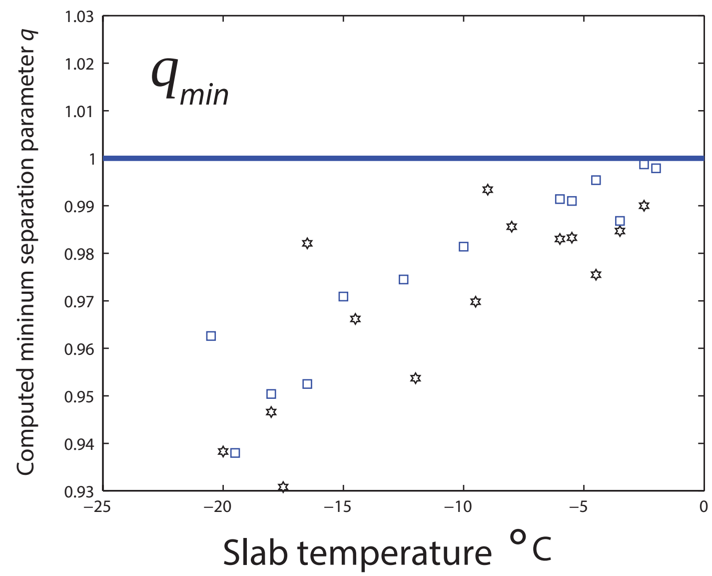
Golden 1995, 1997

Bruno 1991

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inverse bounds



inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

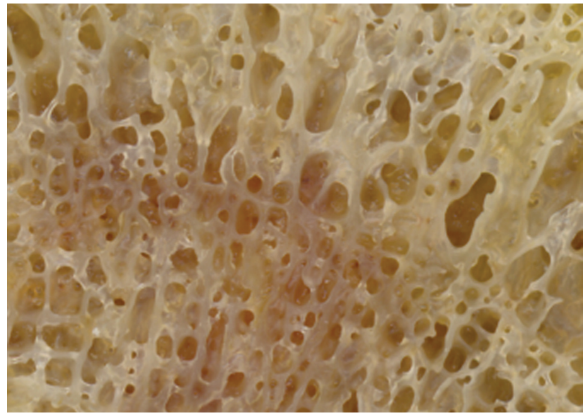
rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

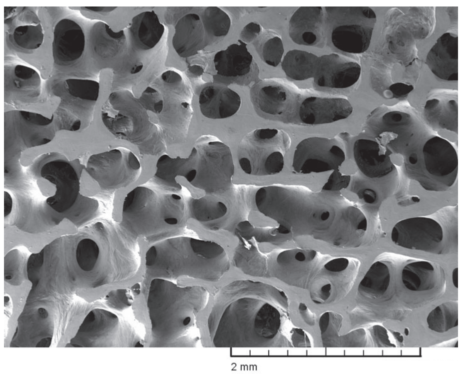
SEA ICE

HUMAN BONE

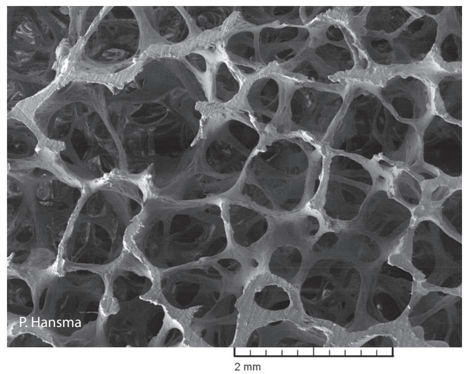


spectral characterization of porous microstructures in human bone

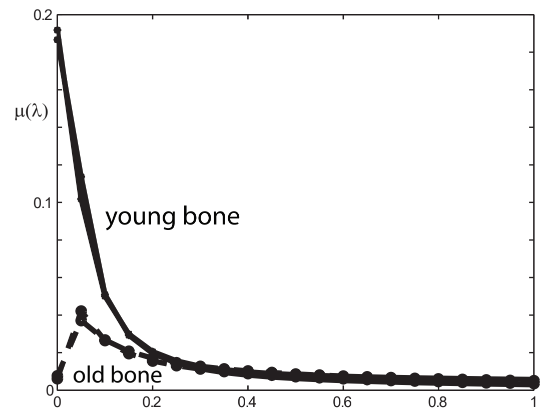
young healthy trabecular bone



old osteoporotic trabecular bone



reconstruct spectral measures from complex permittivity data



use regularized inversion scheme

apply spectral measure analysis of brine connectivity and spectral inversion to electromagnetic monitoring of osteoporosis

Golden, Murphy, Cherkaev, J. Biomechanics 2011

the math doesn't care if it's sea ice or bone!

direct calculation of spectral measure

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).

2. The fundamental operator $\chi\Gamma\chi$ becomes a random matrix depending only on the composite geometry.

3. Compute the eigenvalues λ_i and eigenvectors of $\chi\Gamma\chi$ with inner product weights α_i

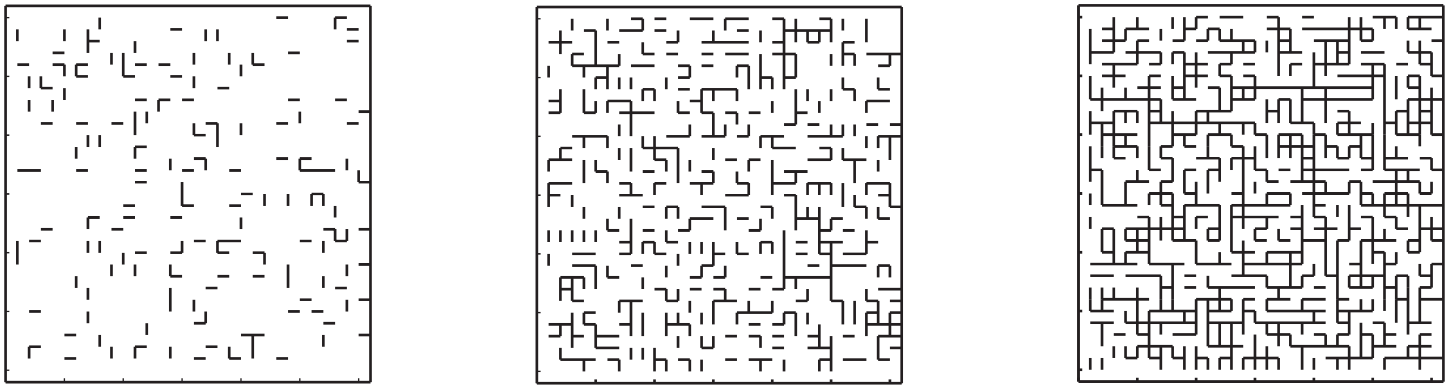
$$\mu(\lambda) = \sum_i \alpha_i \, \delta(\lambda - \lambda_i)$$

Dirac point measure (Dirac delta)

earlier studies of spectral measures

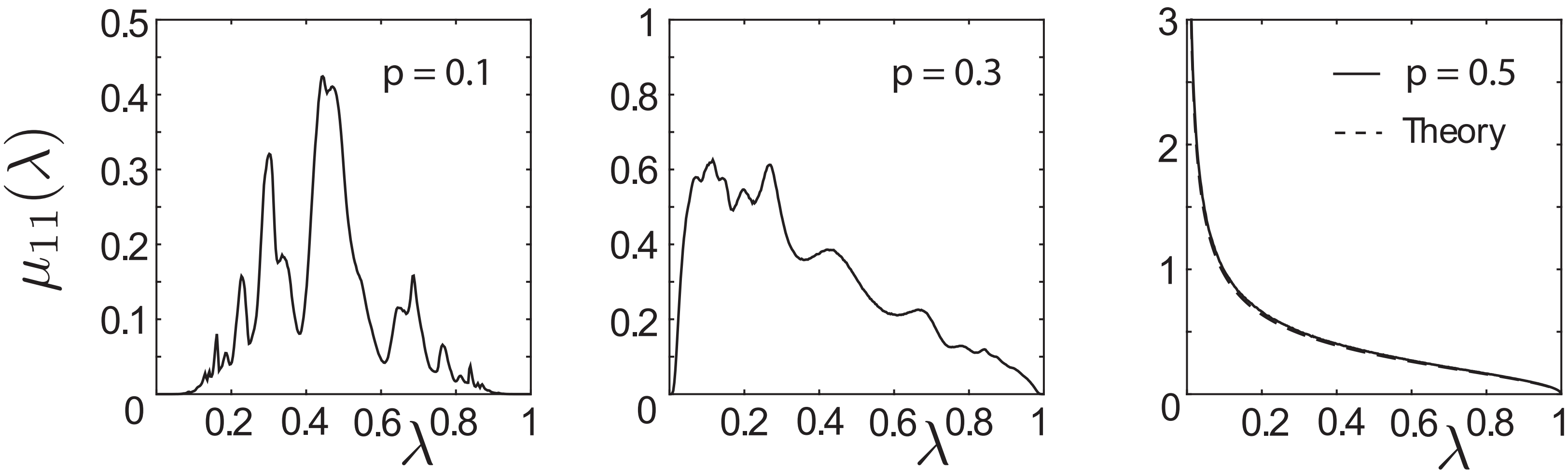
Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral statistics for 2D random resistor network



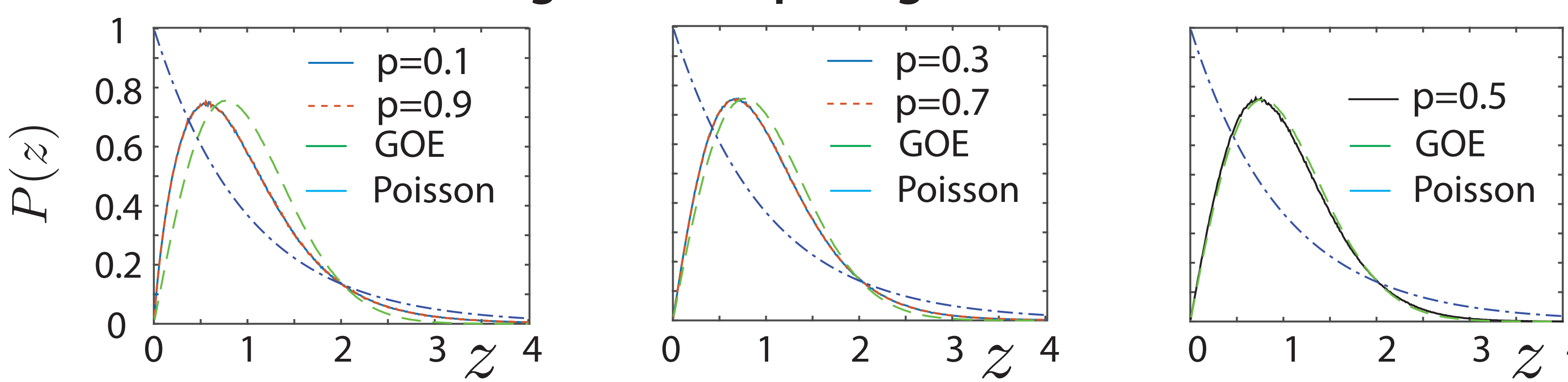
Spectral Measures

Murphy and Golden, *J. Math. Phys.*, 2012
Murphy et al. *Comm. Math. Sci.*, 2015



$p_c = 0.5$

Eigenvalue Spacing Distributions



Murphy, Cherkaev, Golden, *PRL*, 2017

Eigenvalue Statistics of Random Matrix Theory

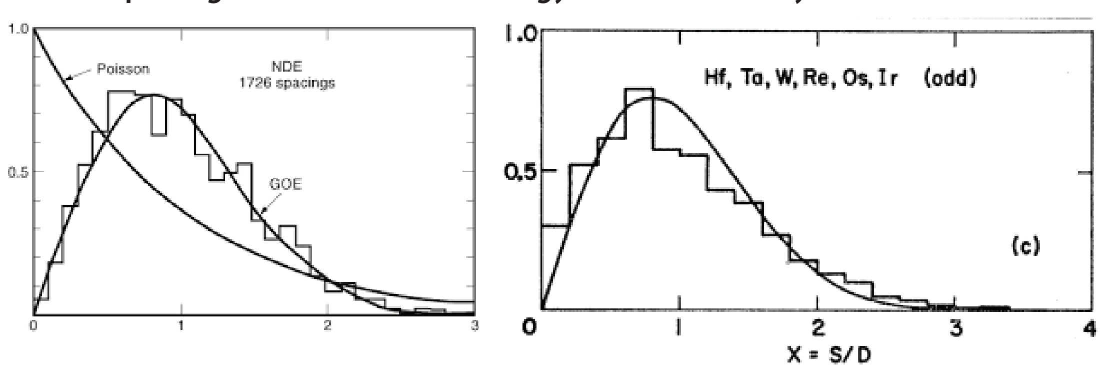
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^{T})/2$ ***Gaussian orthogonal ensemble (GOE)***

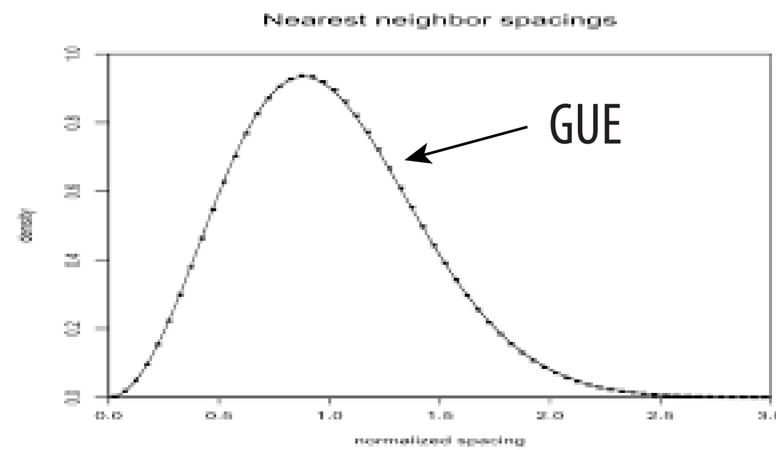
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^{\dagger})/2$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics

Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



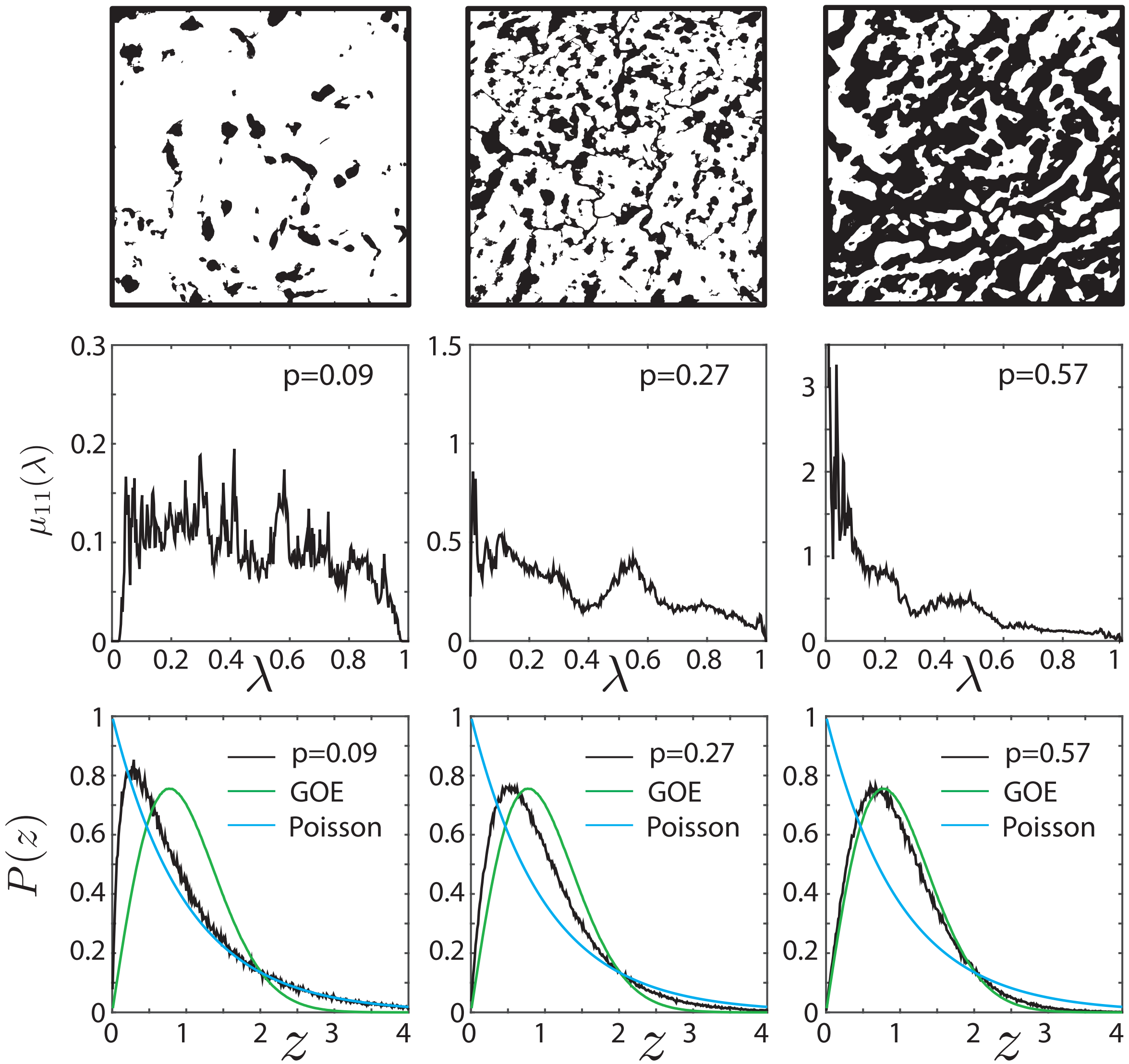
RMT used to characterize disorder-driven transitions in mesoscopic conductors, neural networks, random graph theory, etc.

Phase transitions ~ transitions in universal eigenvalue statistics.

Spectral computations for Arctic melt ponds

spectral measures

eigenvalue spacing distributions

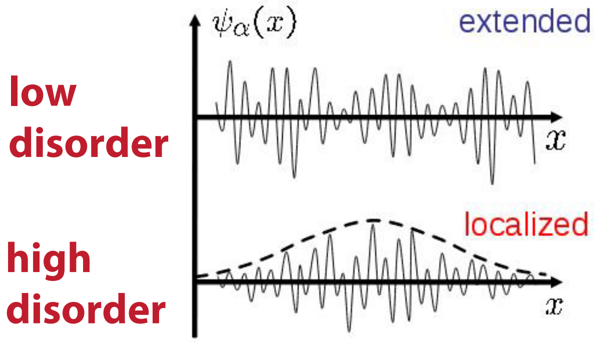


uncorrelated → **level repulsion**

TRANSITION

Ben Murphy
Elena Cherkaev
Ken Golden
2017

eigenvalue statistics for transport tend toward the UNIVERSAL Wigner-Dyson distribution as the "conducting" phase percolates



metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

we find a surprising analog

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION TRANSITION → **transition to universal eigenvalue statistics (GOE) extended states, mobility edges**

-- but without wave interference or scattering effects ! --

eigenvector localization and mobility edges

Inverse Participation Ratio: $I(\vec{v}_n) = \sum_{i=1}^{N} |(\vec{v}_n)_i|^4$

Completely Localized: $I(\vec{e}_n) = 1$

Completely Extended: $I\left(\frac{1}{\sqrt{N}}\vec{1}\right) = \frac{1}{N}$

Anderson Model

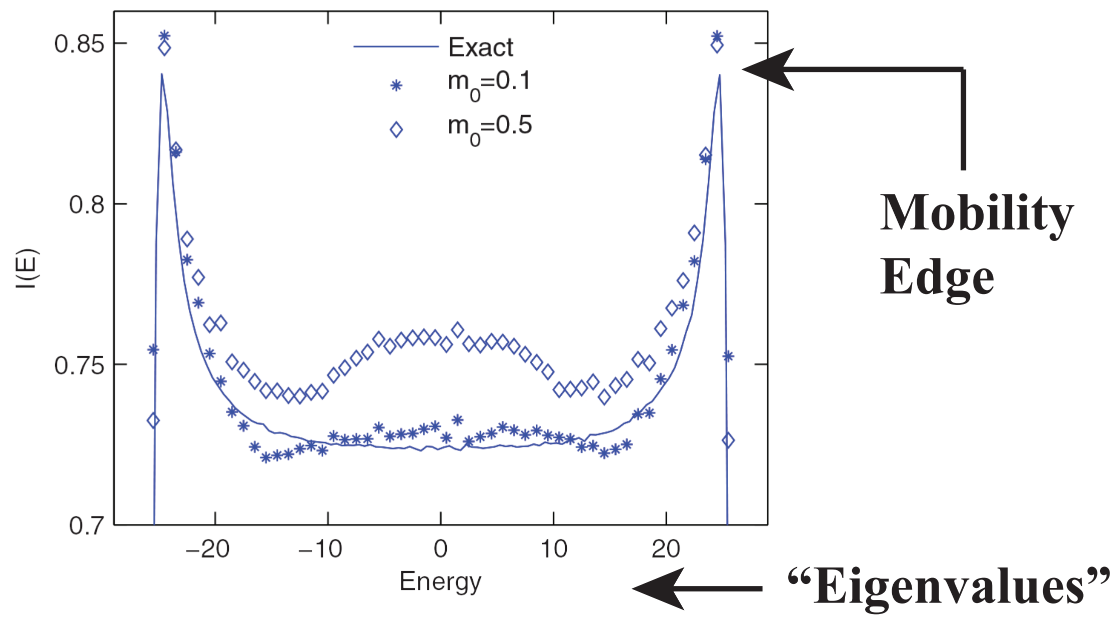
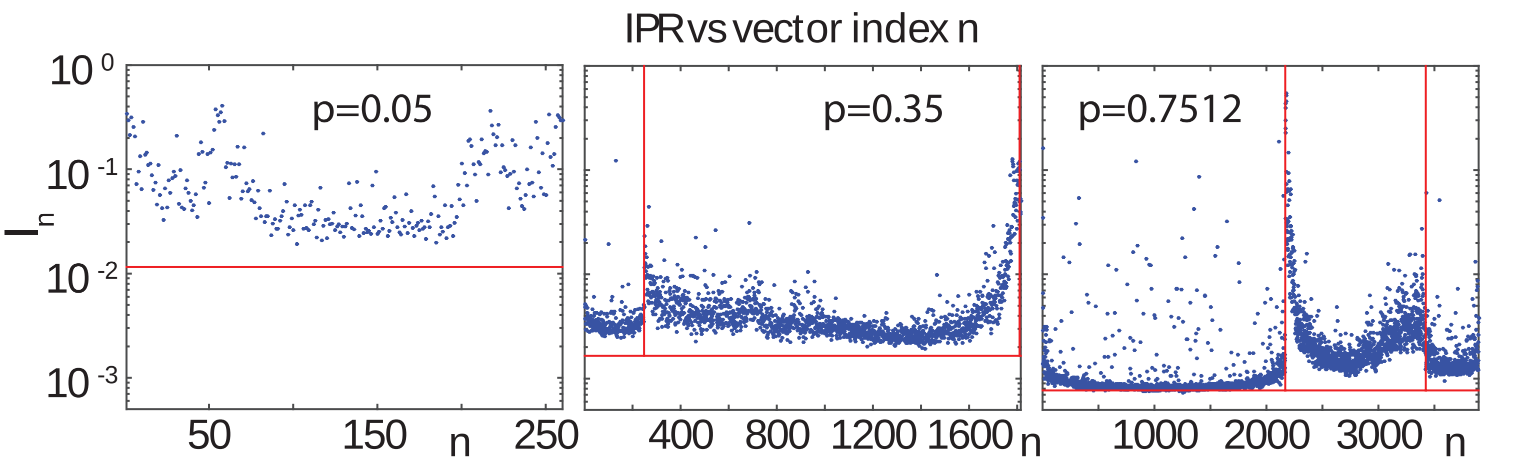
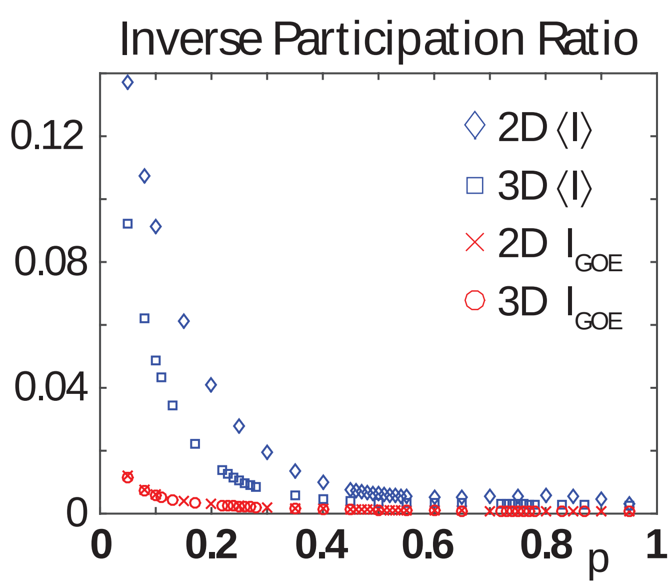


FIG. 4. (Color online) IPR for Anderson model in two dimensions with $x = 6.25$ ($w = 50$) from exact diagonalization (solid line) and from LDRG with different values of the cutoff m_0. LDRG data are averaged over 100 runs of systems with 100×100 sites.

PHYSICAL REVIEW B 90, 060205(R) (2014)

Localization properties of eigenvectors in random resistor networks





$$I_n = \sum_i (\vec{v}_n)_i^4$$

Murphy, Cherkaev, Golden, 2017

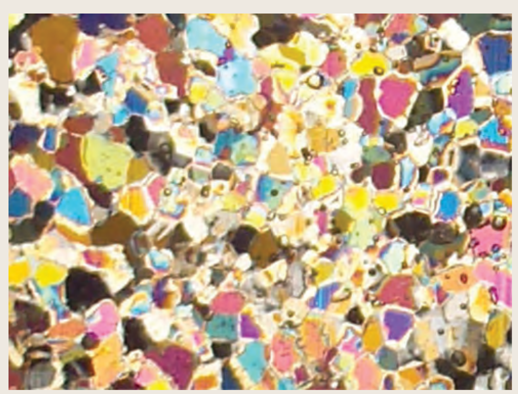
Proc. Roy. Soc. A 8 Feb 2015

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



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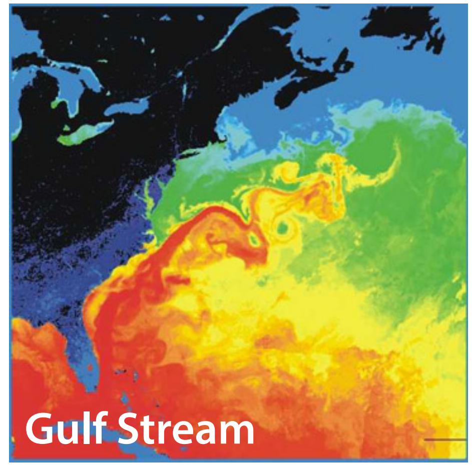
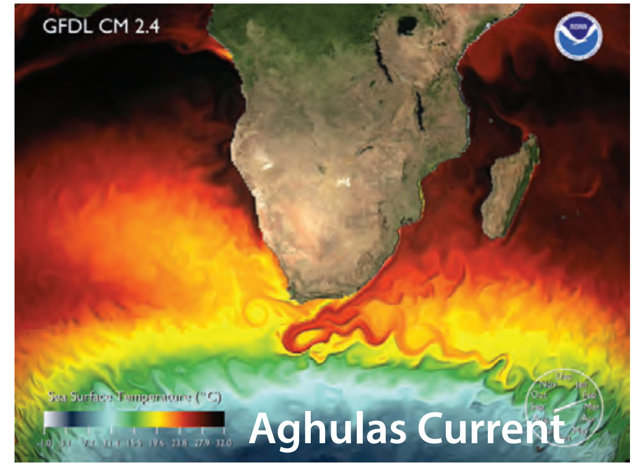
advection enhanced diffusion

effective diffusivity

sea ice floes diffusing in ocean currents
diffusion of pollutants in atmosphere
salt and heat transport in ocean
heat transport in sea ice with convection





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\downarrow \textit{homogenize}$$

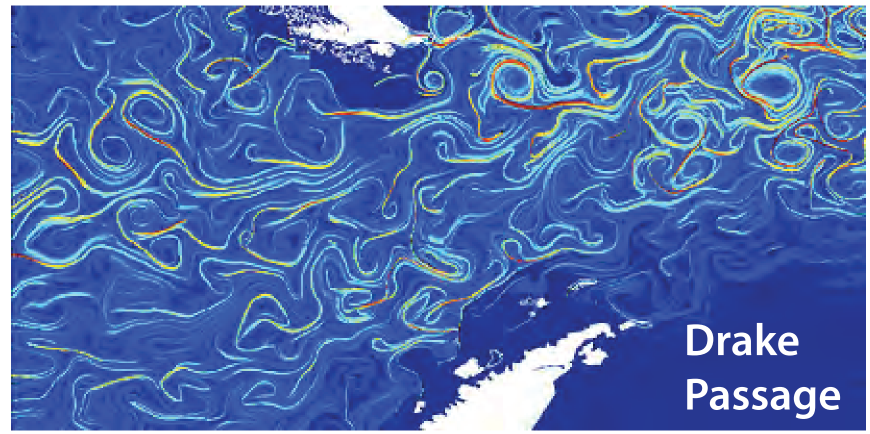
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

Stieltjes integral for κ^* with spectral measure

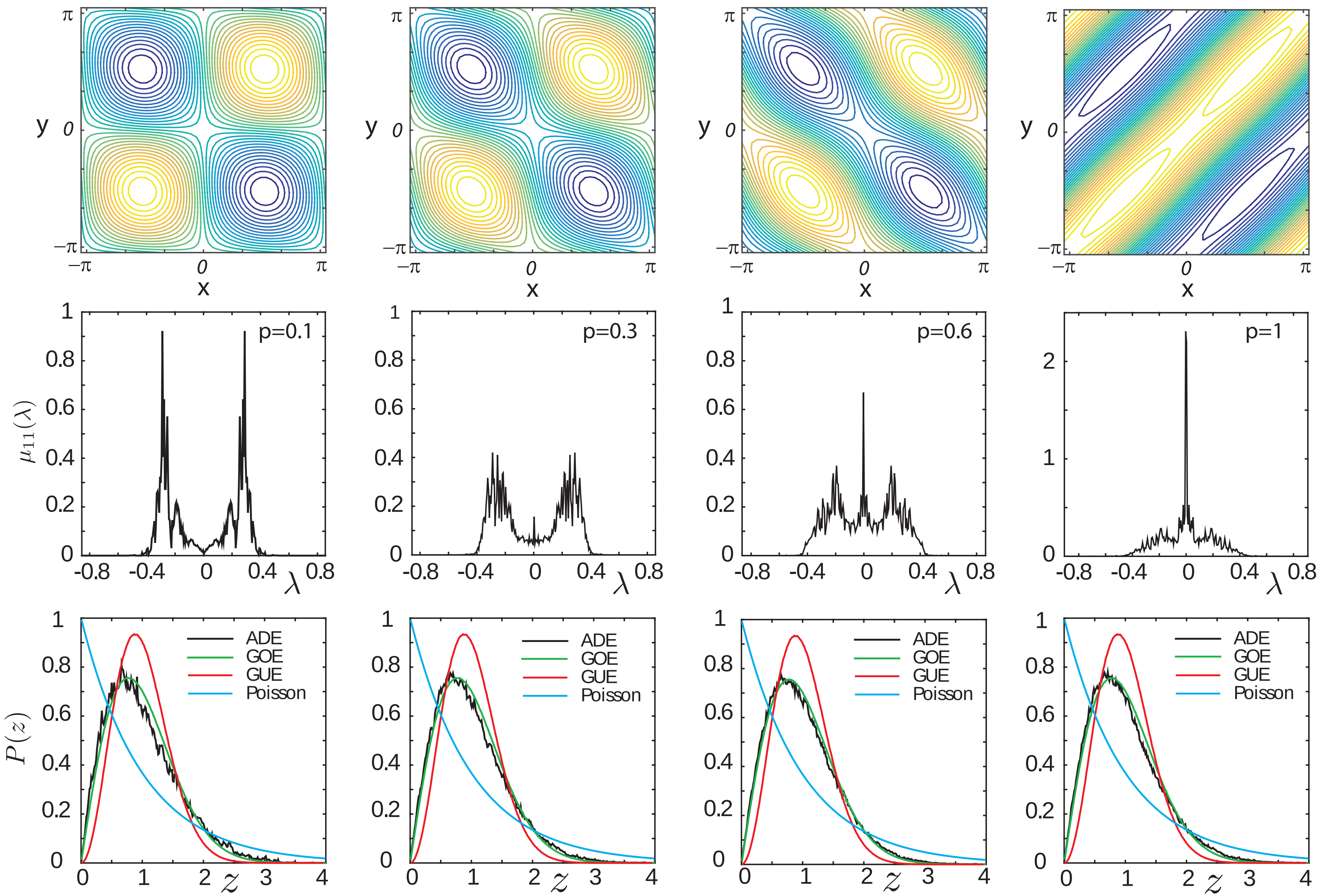
Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, 2018



Spectral measures and eigenvalue spacings for cat's eye flow

$$H(x,y) = \sin(x)\sin(y) + A\cos(x)\cos(y), \qquad A \sim U(-p,p)$$

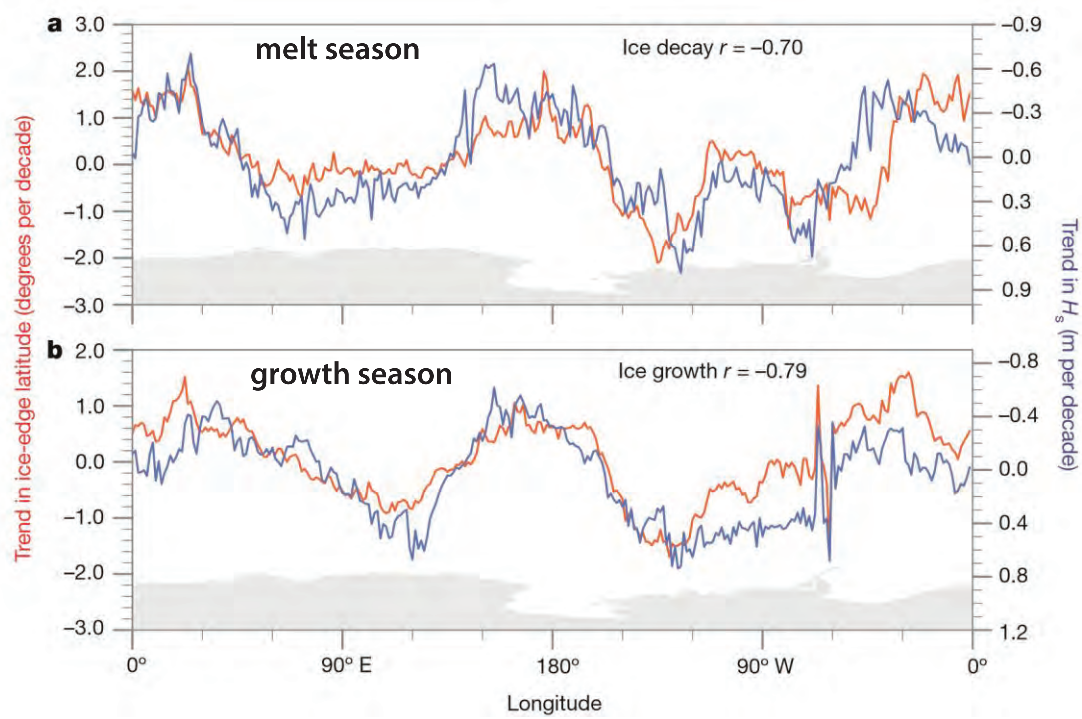


Murphy, Cherkaev, Zhu, Xin, Golden

Storm-induced sea-ice breakup and the implications for ice extent

Kohout et al., *Nature* 2014

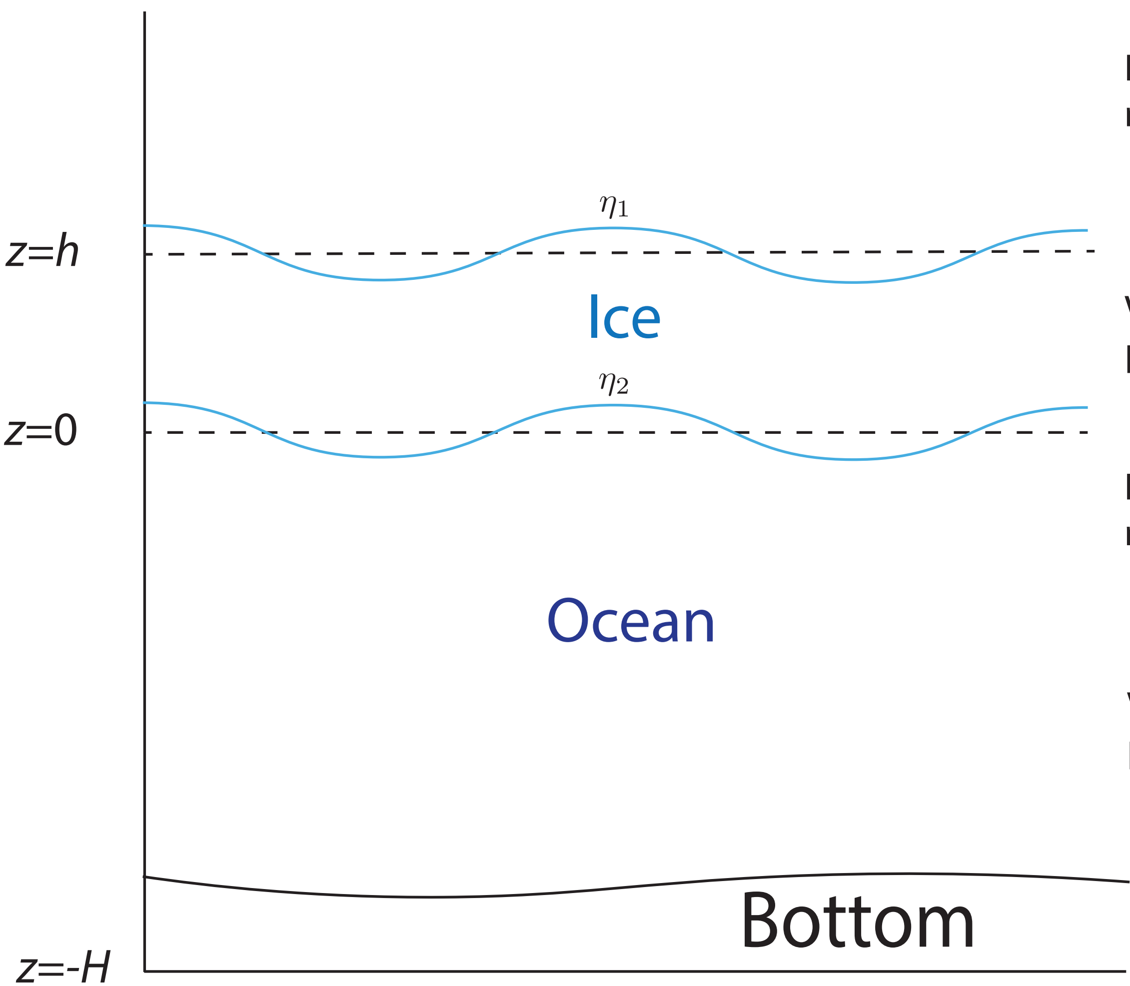
- during three large-wave events, significant wave heights did not decay exponentially, enabling large waves to persist deep into the pack ice.
- large waves break sea ice much farther from the ice edge than would be predicted by the commonly assumed exponential decay



ice extent compared with significant wave height

Waves have strong influence on both the floe size distribution and ice extent.

Two Layer Models and Effective Parameters



Viscous fluid layer (Keller 1998)
Effective Viscosity $\boxed{\nu}$

Equations of motion:
$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 U + g$$

Viscoelastic fluid layer (Wang-Shen 2010)
Effective Complex Viscosity $\boxed{\nu_e = \nu + iG/\rho\omega}$

Equations of motion
$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu_e \nabla^2 U + g$$

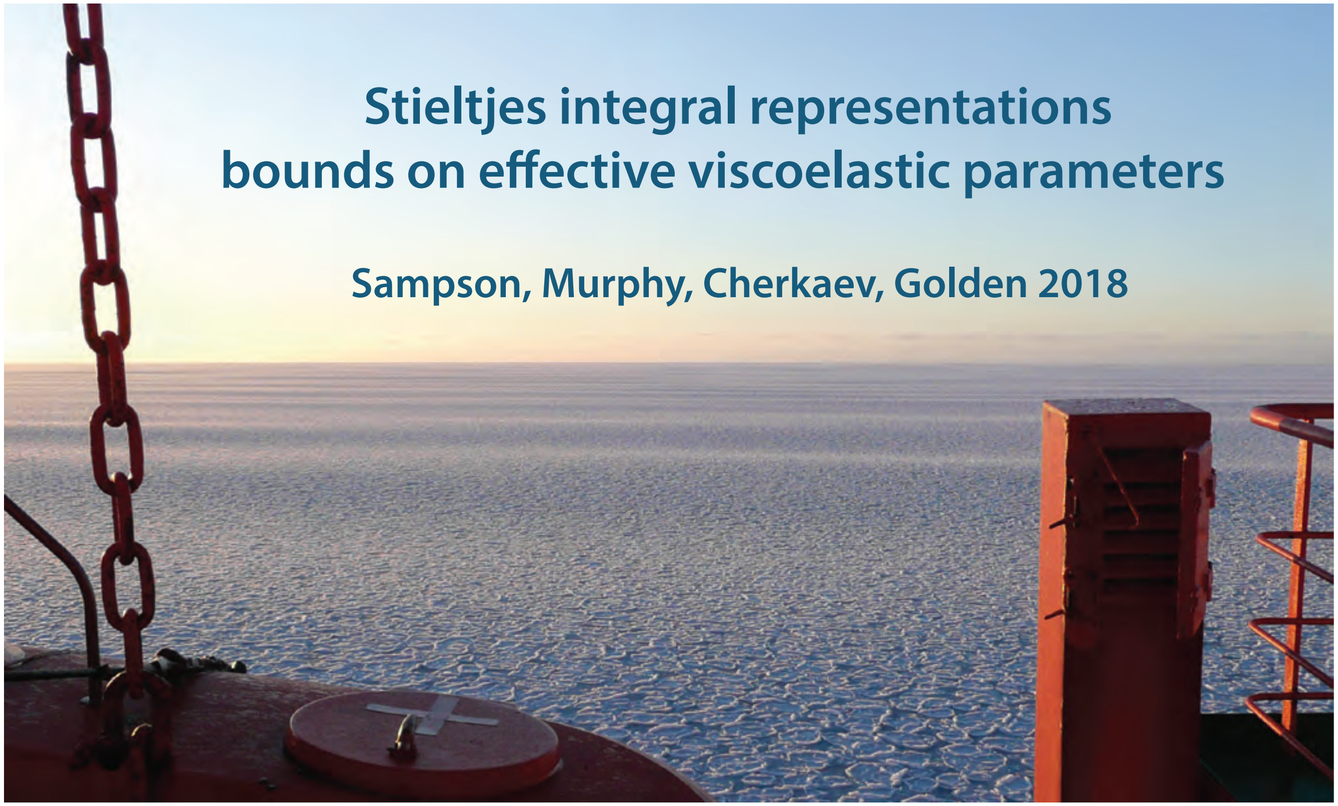
Viscoelastic thin beam (Mosig *et al.* 2015)
Effective Complex Shear Modulus $\boxed{G_v = G - i\omega\rho\nu}$

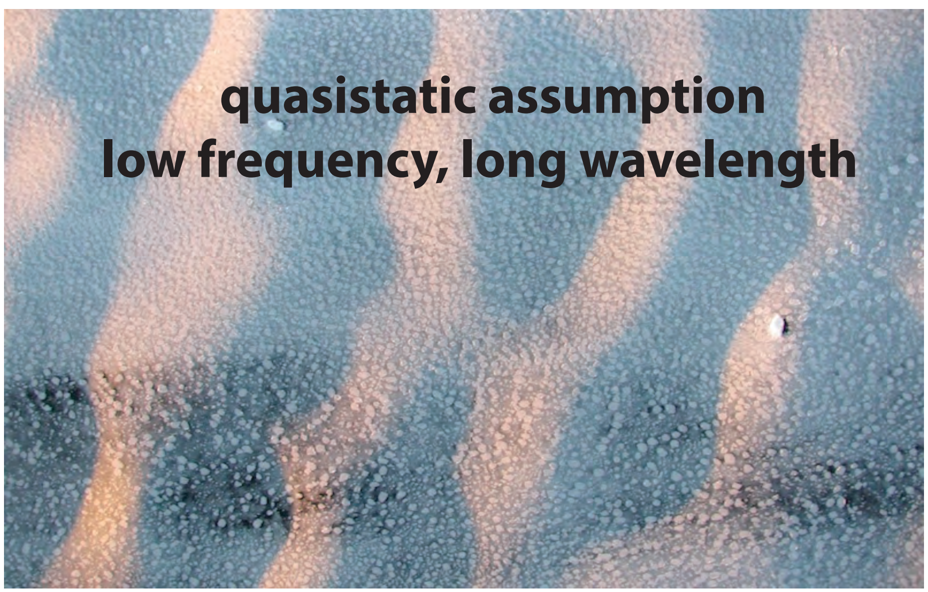
G shear modulus
P pressure
ω angular frequency
U velocity field
ν viscosity
λ Poission ratio
ρ density
g gravity

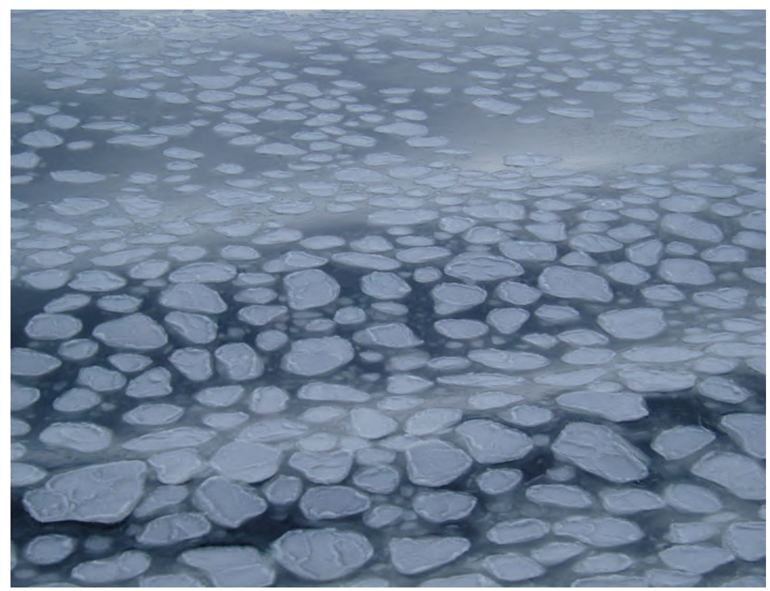
Stieltjes integral representation for effective complex viscoelastic parameter; bounds

Sampson, Murphy, Cherkaev, Golden 2018

wave propagation in the marginal ice zone







Stieltjes Integral Representation for Complex Viscoelasticity

homogenized $\langle \sigma_{ij} \rangle = C^*_{ijkl} \langle \epsilon_{kl} \rangle$

local $\nabla \cdot \sigma = 0$ $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$

$$C_{ijkl} = (\nu_1 \chi + (1-\chi)\nu_2)\lambda_s$$

$$\nabla \cdot \left((\nu_1 \chi + (1-\chi)\nu_2)\lambda_s : \epsilon \right) = 0$$

$$s = \frac{1}{1 - \frac{\nu_1}{\nu_2}}$$

Strain Field

$$\epsilon = \frac{1}{2}[\nabla u + (\nabla u)^T] = \nabla^s u$$

$$\epsilon = \epsilon_0 + \epsilon_f \text{ where } \epsilon_f = \nabla^s \phi$$

Elasticity Tensor

$$C^*_{ijkl} = \nu^* \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl} \right) = \nu^* \lambda_s$$

RESOLVENT $\epsilon = \left(1 - \frac{1}{s}\Gamma\chi \right)^{-1} \epsilon_0$ $\Gamma = \nabla^s (\nabla \cdot \nabla^s)^{-1} \nabla \cdot$ ϵ_0 **avg strain**

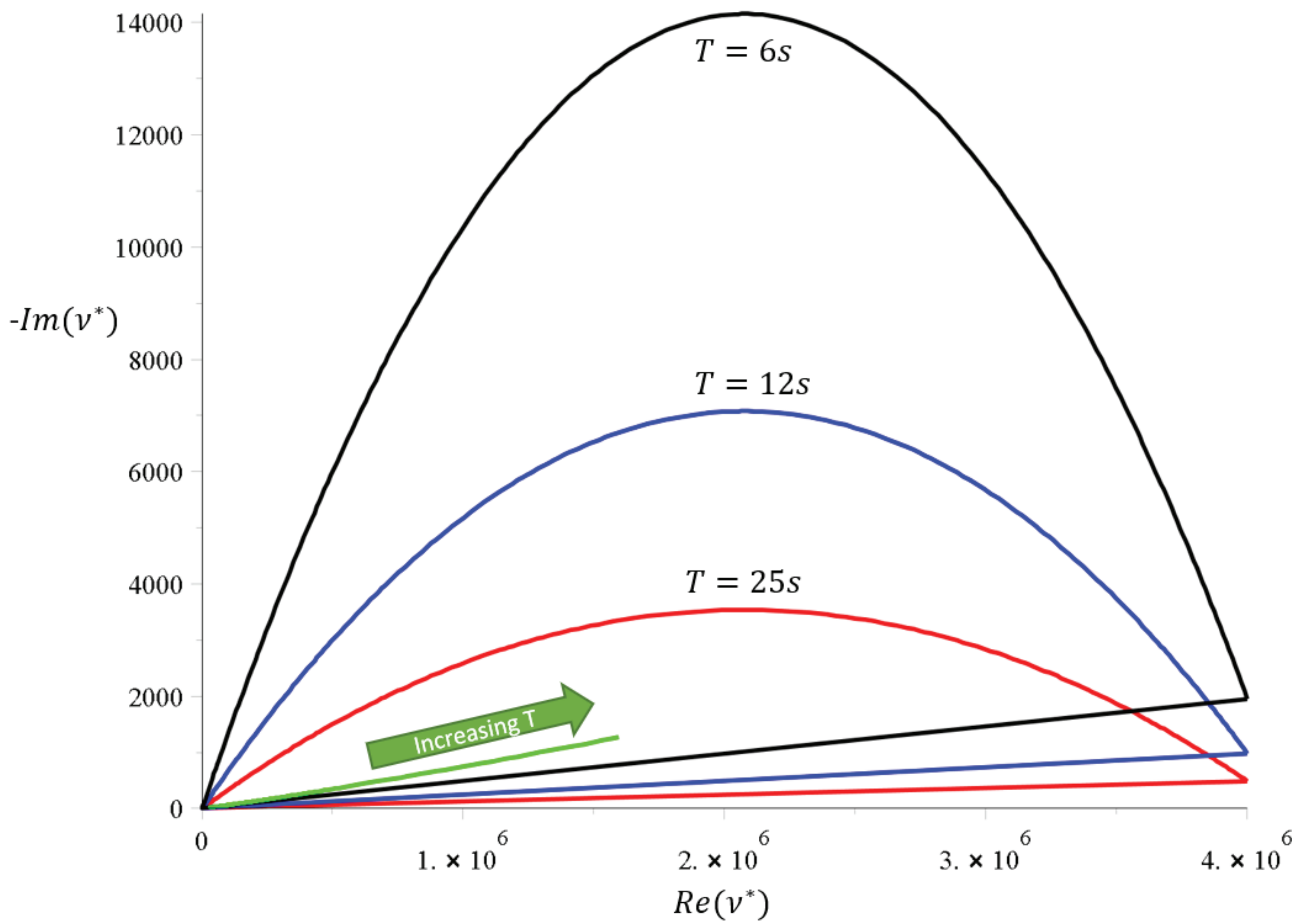
$$F(s) = 1 - \frac{\nu^*}{\nu_2} \qquad F(s) = ||\epsilon_0||^{-2} \int_\Sigma \frac{d\mu(\lambda)}{s - \lambda}$$

bounds on the effective complex viscoelasticity

complex elementary bounds
(fixed area fraction of floes)

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**



Sampson, Murphy, Cherkaev, Golden 2018

Conclusions

1. Summer Arctic sea ice is melting rapidly, and melt ponds and other processes must be accounted for in order to predict melting rates.

2. Fluid flow through sea ice mediates melt pond evolution and many processes important to climate change and polar ecosystems.

3. Statistical physics and homogenization help *link scales*, provide rigorous methods for finding effective behavior, and advance how sea ice is represented in climate models.

4. Random matrix theory and an unexpected Anderson transition arises in our studies of percolation in sea ice structures.

5. Our research will help to improve projections of climate change and the fate of the Earth sea ice packs.

THANK YOU

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