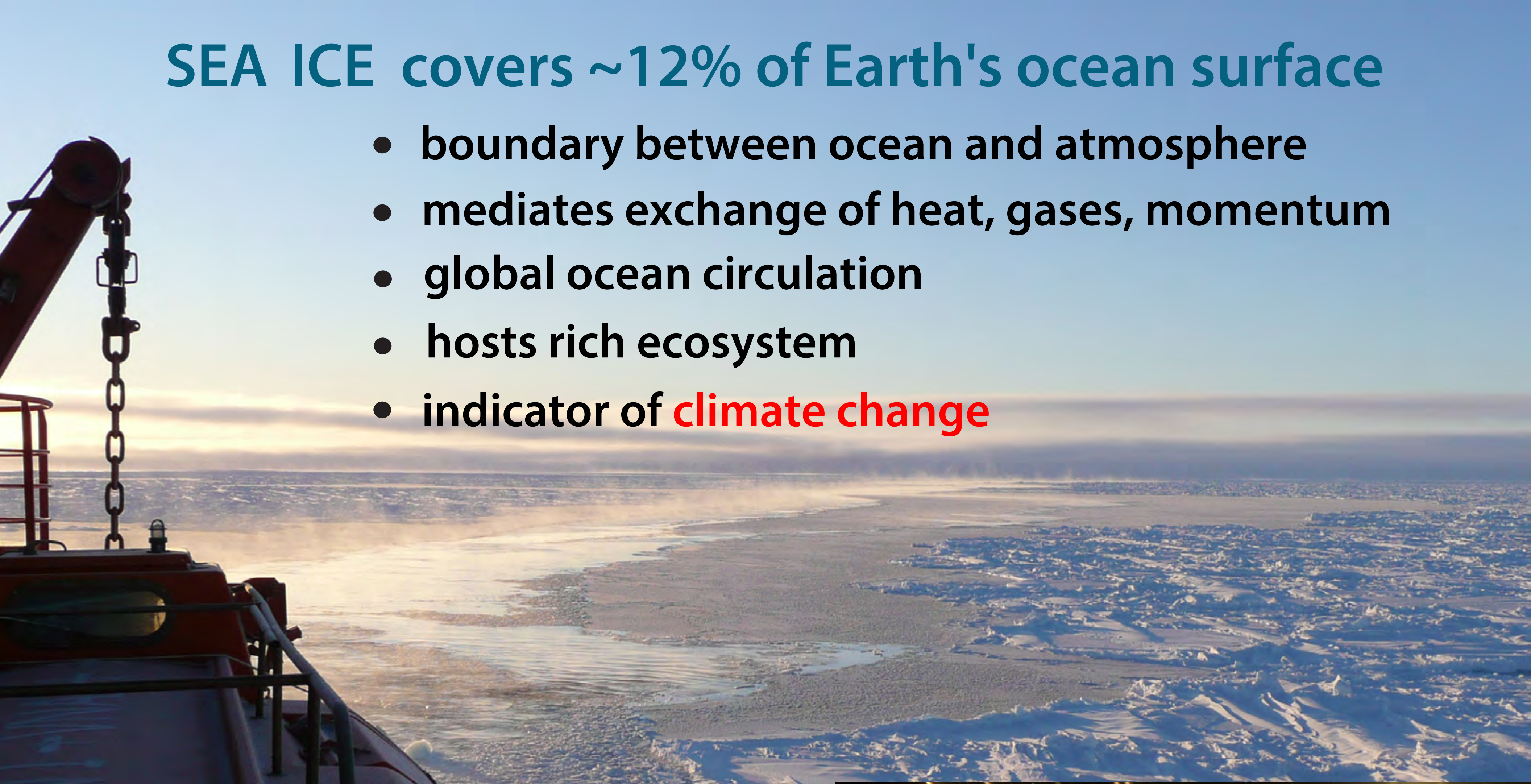
Waves in Sea Ice

Kenneth M. Golden **University of Utah**

ICIAM 2019 Valencia



SEA ICE covers ~12% of Earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- hosts rich ecosystem
- indicator of climate change

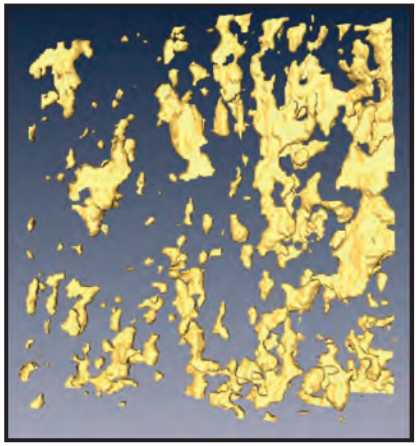
polar ice caps critical
to climate in reflecting
sunlight during summer

Sea Ice is a Multiscale Composite Material

sea ice microstructure

brine inclusions

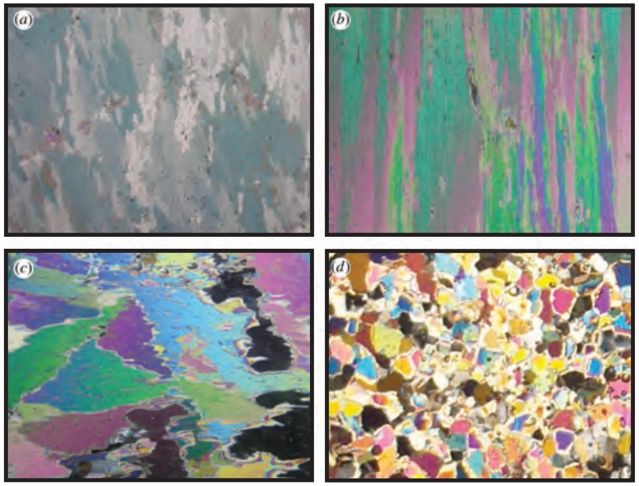
Weeks & Assur 1969



H. Eicken

Golden *et al.* *GRL* 2007

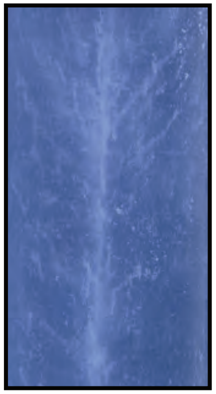
polycrystals



Gully *et al.* *Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

centimeters

---

sea ice mesostructure

sea ice macrostructure

Arctic melt ponds

K. Frey

Antarctic pressure ridges

K. Golden

sea ice floes

J. Weller

sea ice pack

NASA

meters

kilometers

What is this talk about? HOMOGENIZATION

Using methods of statistical physics and composite materials to **LINK SCALES** *in the sea ice system ... rigorously compute effective behavior and improve climate models.*

Tour wave phenomena in sea ice

1. ***Sea ice microphysics and fluid transport***

   homogenization and percolation theory

2. ***EM monitoring of sea ice, analytic continuation method***

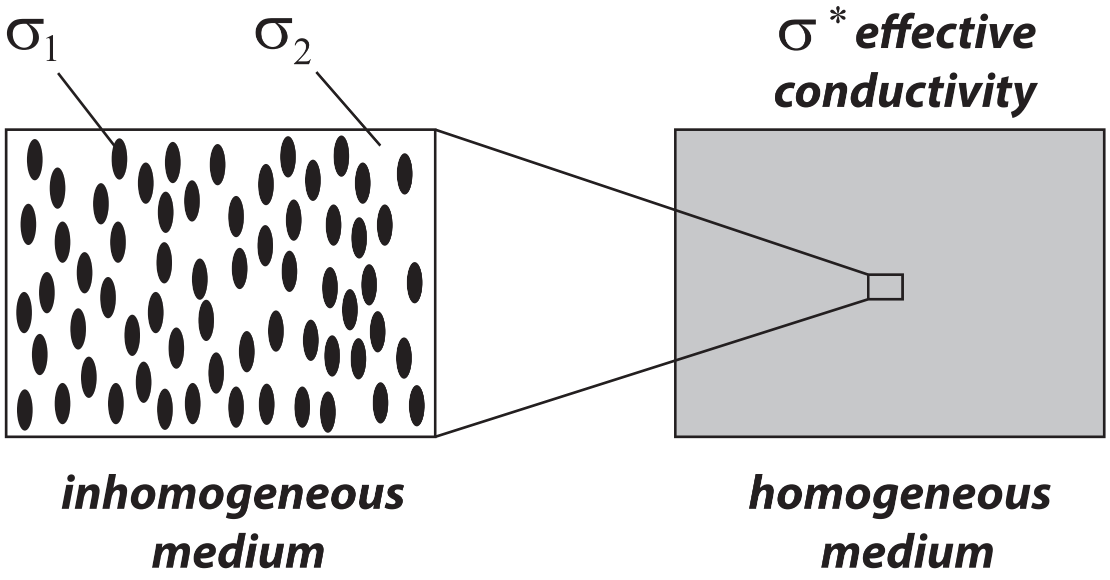
   random matrix theory and Anderson transitions

3. ***Extension of ACM to polycrystals, waves in sea ice***

   Stieltjes integral representations, spectral measures

4. ***Light in sea ice, melt ponds***

HOMOGENIZATION - Linking Scales in Composites



find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*
*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

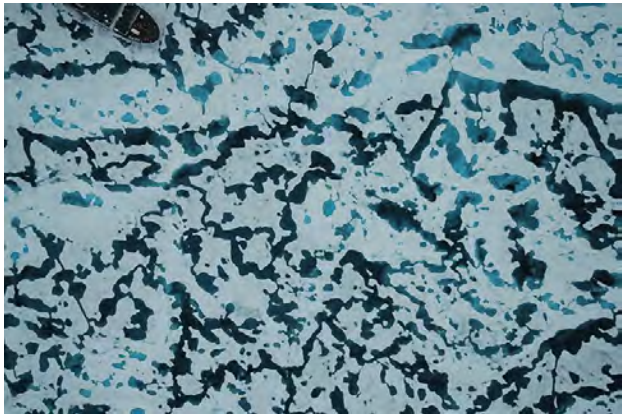
How do scales interact in the sea ice system?

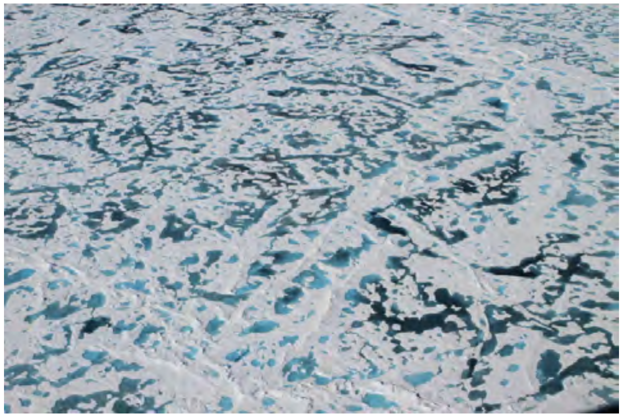
NASA

basin scale -
grid scale
albedo

Linking Scales

km
scale
melt
ponds





Perovich

km
scale
melt
ponds

Linking Scales

mm
scale
brine
inclusions

meter
scale
snow
topography

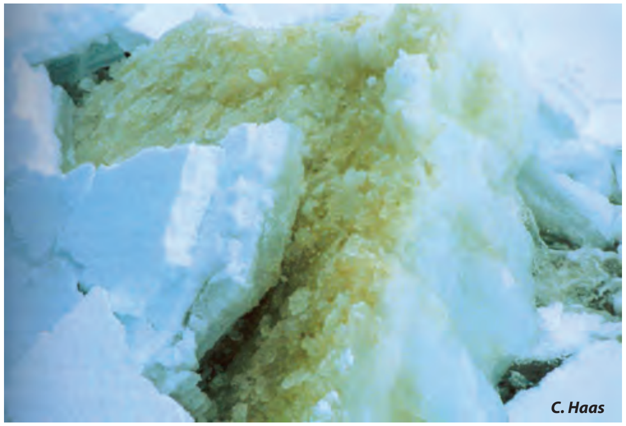
sea ice microphysics

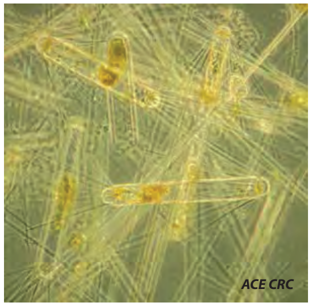
fluid transport

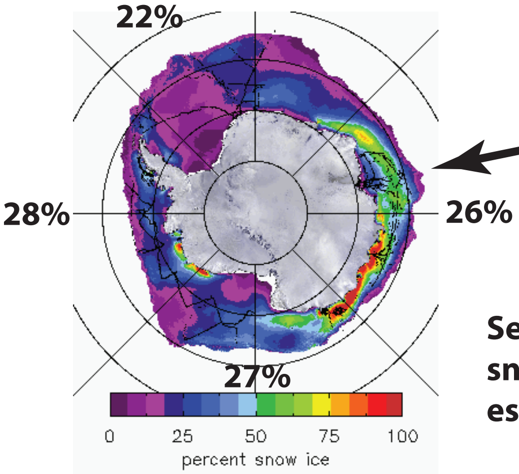
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities







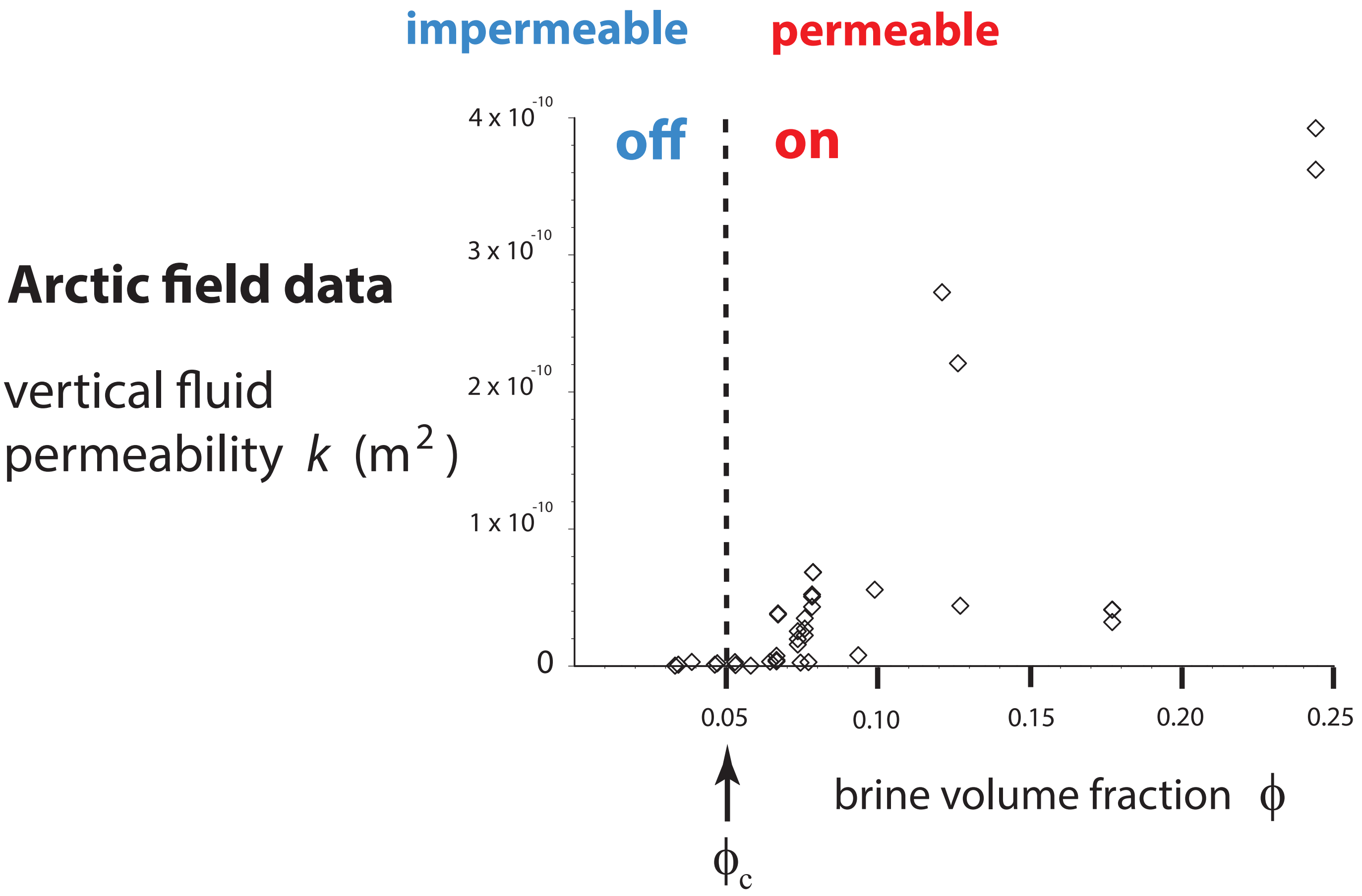
T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

Critical behavior of fluid transport in sea ice



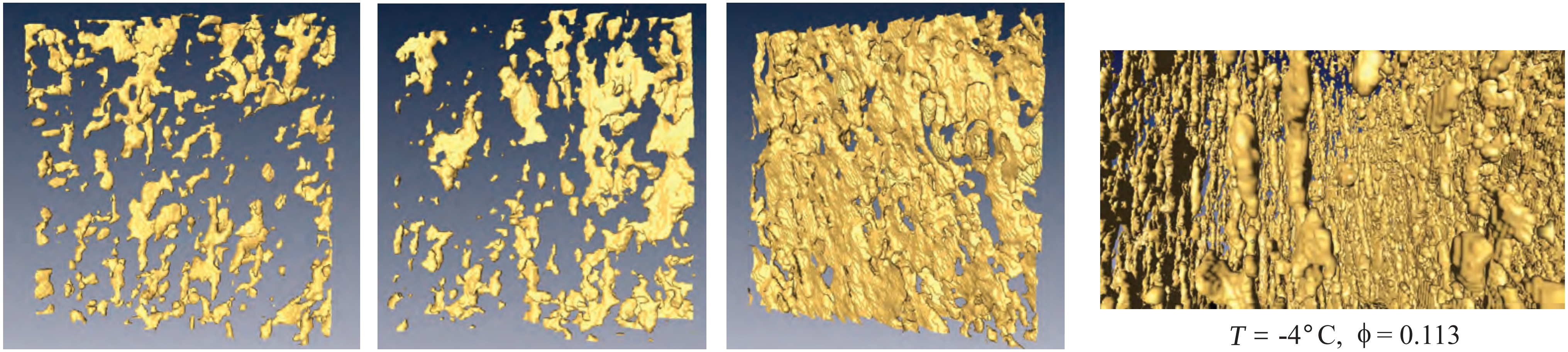
"on - off" switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\%$ ⟷ $T_c \approx -5^\circ\mathrm{C}, \; S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

brine volume fraction and ***connectivity*** increase with temperature



T = -15 °C, $\phi = 0.033$ T = -6 °C, $\phi = 0.075$ T = -3 °C, $\phi = 0.143$ T = -4 °C, $\phi = 0.113$

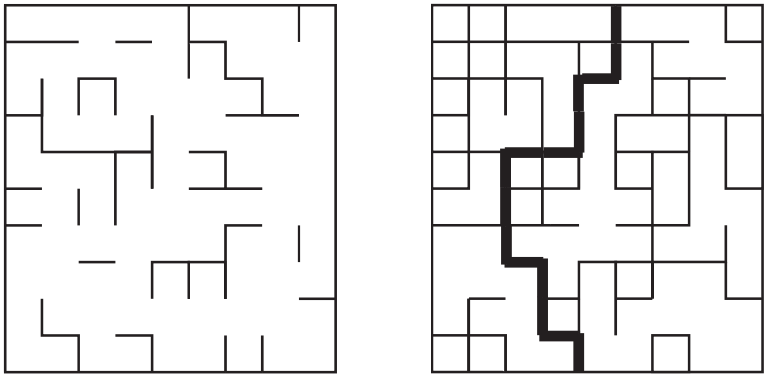
X-ray tomography for brine phase in sea ice

Golden, Eicken, *et al.*, *Geophysical Research Letters* 2007

PERCOLATION THRESHOLD $\phi_c \approx 5\ \%$

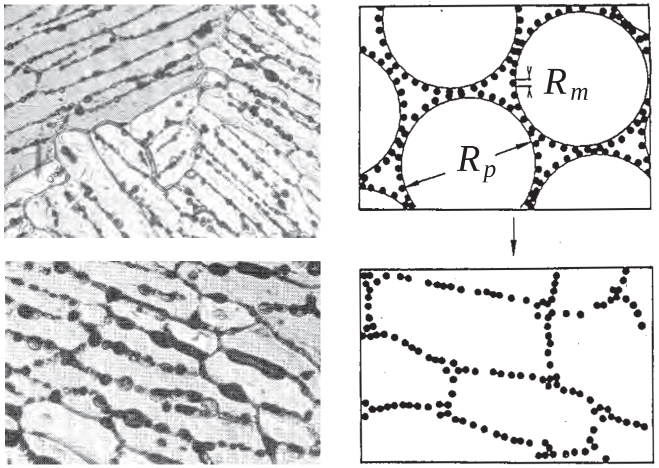
Golden, Ackley, Lytle, *Science* 1998

impermeable permeable



$p = 1/3$ $p = 2/3$

lattice percolation



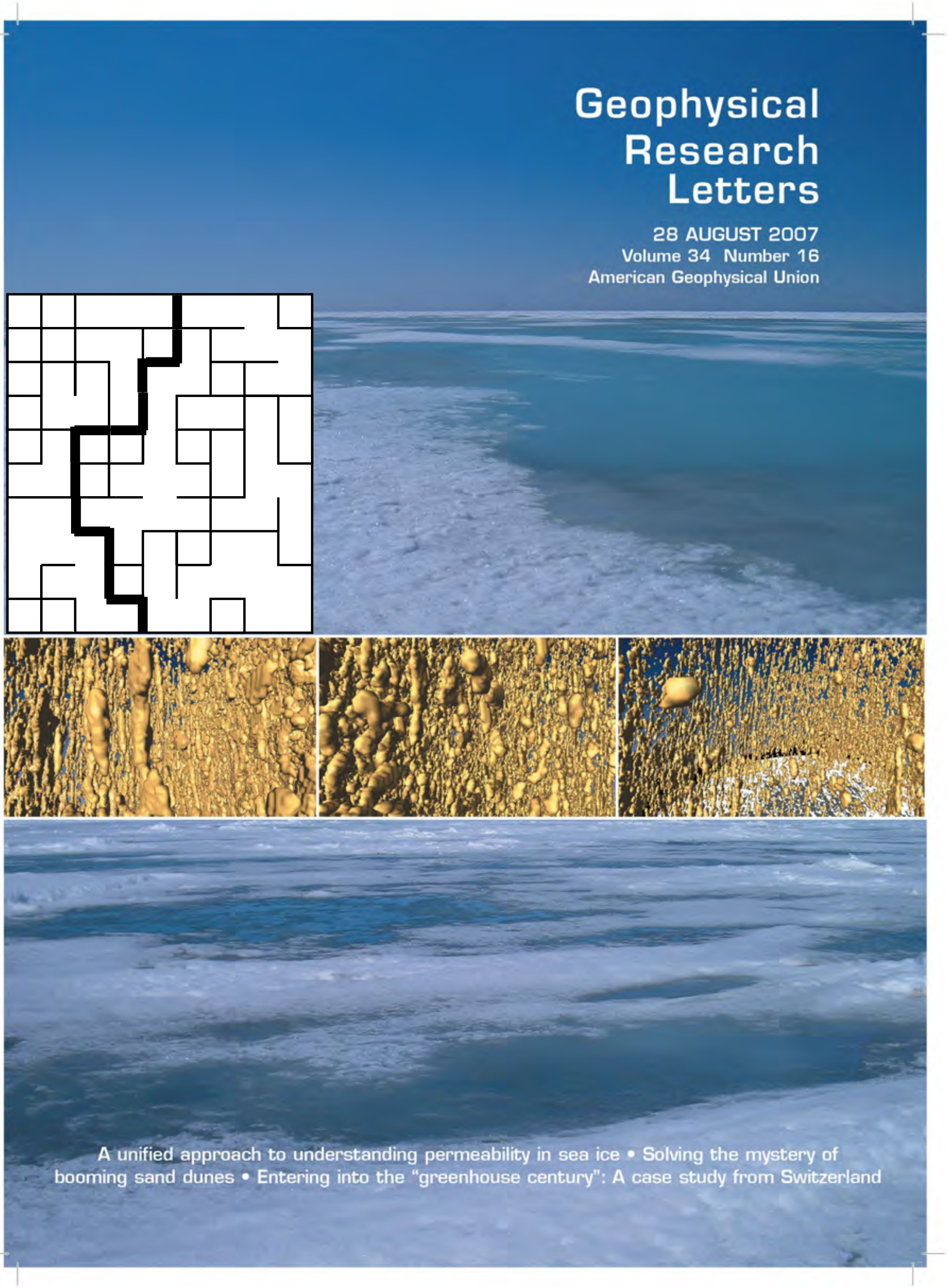
Kusy, Turner *Nature* 1971

sea ice compressed powder

continuum percolation

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007



Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

micro-scale

controls

macro-scale

processes

percolation theory

$$k(\phi) = k_0 (\phi - 0.05)^2 \longleftarrow \text{critical exponent } t$$

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

hierarchical model
network model
rigorous bounds

agree closely
with field data

X-ray tomography for brine inclusions

unprecedented look
at thermal evolution
of brine phase and
its connectivity

confirms rule of fives

Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

Remote sensing of sea ice

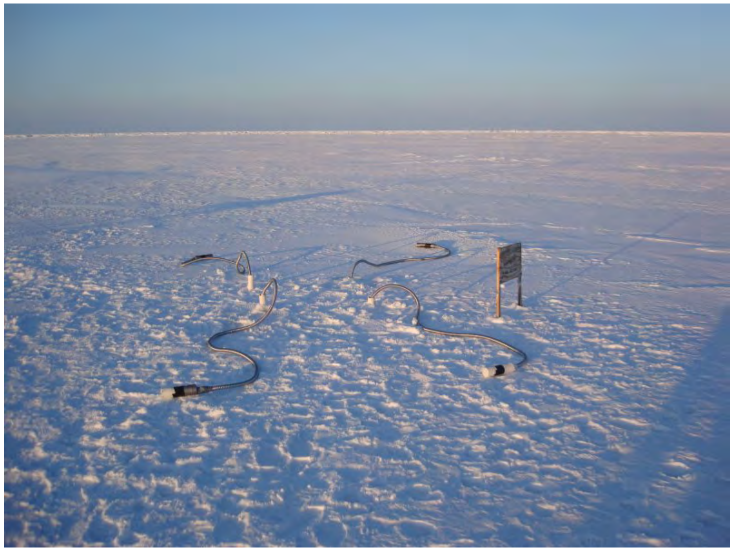
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

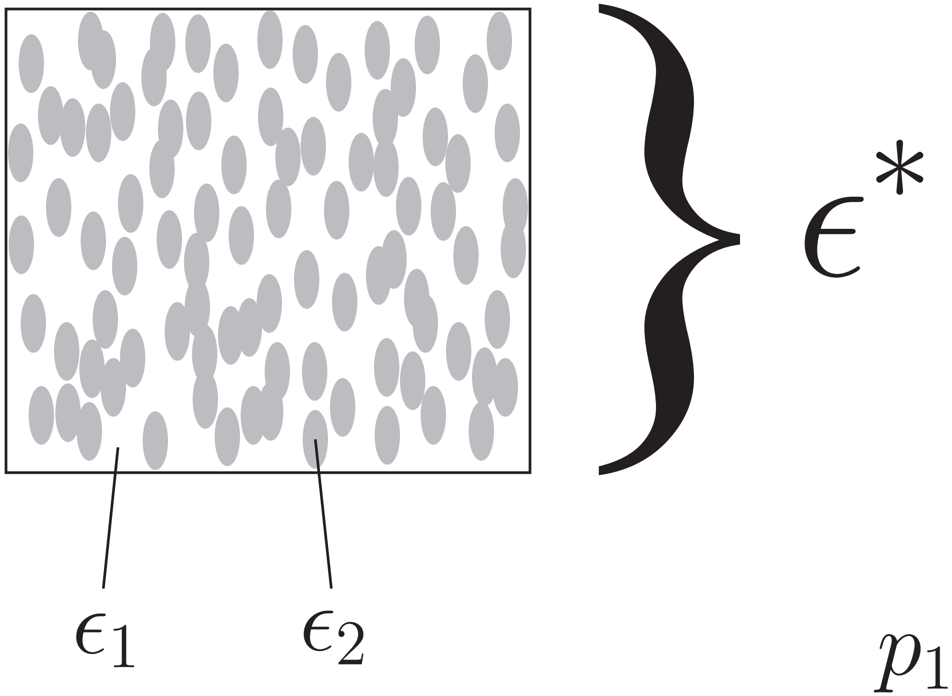
$$\varepsilon^*$$

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry} \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

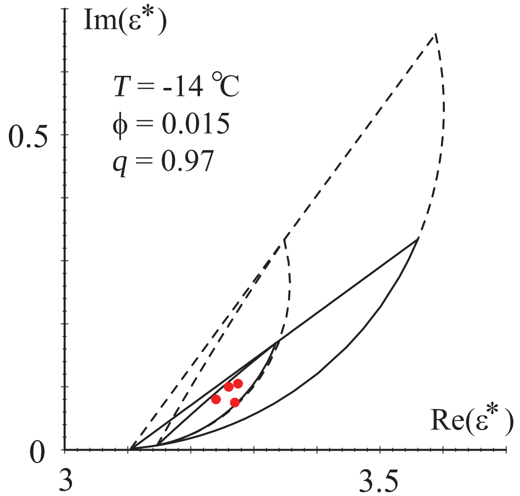
$\Gamma\chi$: **microscale** $\longrightarrow$ **macroscale**

$\Gamma\chi$ ***links scales***

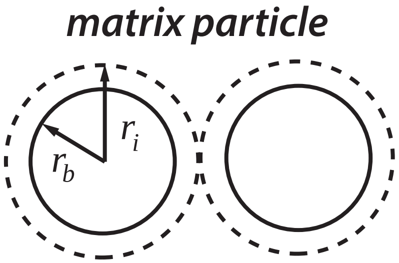
Golden and Papanicolaou, Comm. Math. Phys. 1983

forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data



$$q = r_b / r_i$$

$$0 < q < 1$$

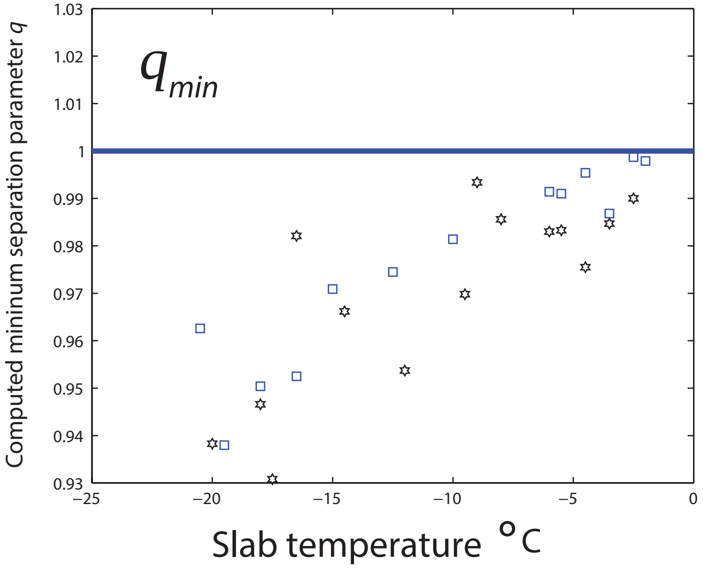
Golden 1995, 1997

Bruno 1991

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inverse bounds



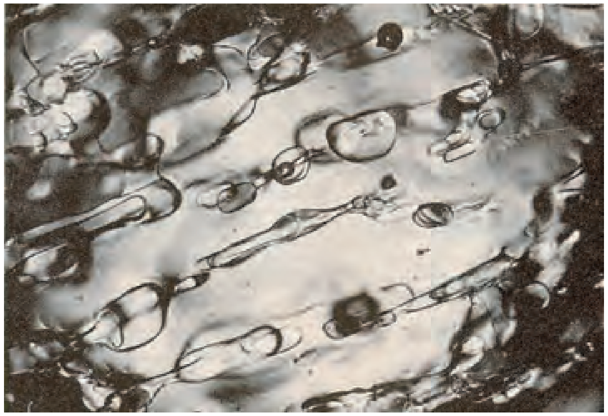
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

rigorous inverse bound
on spectral gap

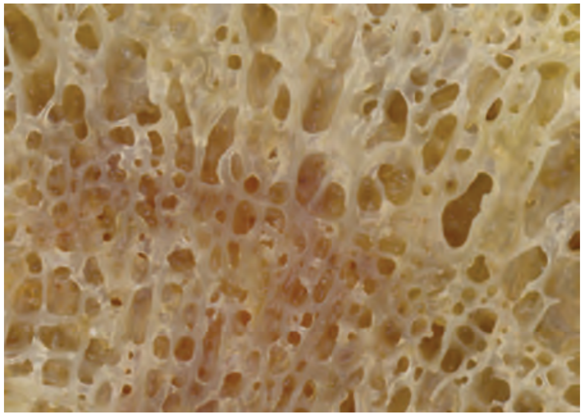
construct algebraic curves which bound
admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

SEA ICE

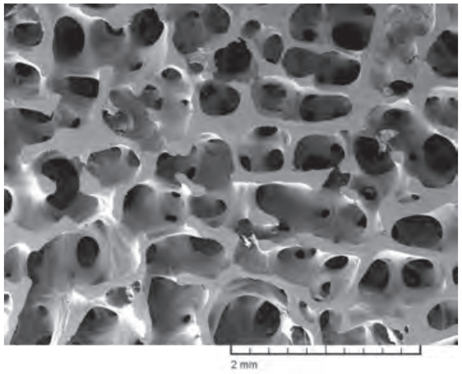


HUMAN BONE

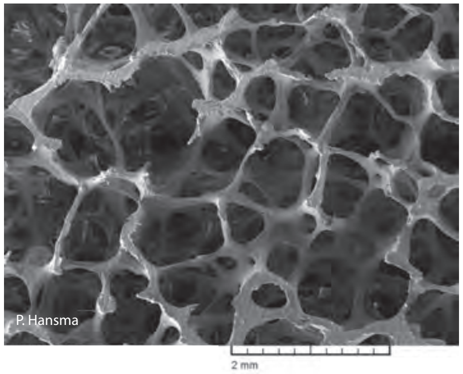


spectral characterization of porous microstructures in human bone

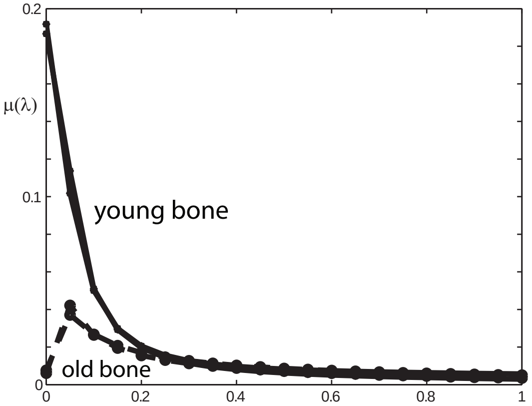
young healthy trabecular bone



old osteoporotic trabecular bone



reconstruct spectral measures from complex permittivity data



use regularized inversion scheme

apply spectral measure analysis of brine connectivity and spectral inversion to electromagnetic monitoring of osteoporosis

Golden, Murphy, Cherkaev, J. Biomechanics 2011

the math doesn't care if it's sea ice or bone!

direct calculation of spectral measures

Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

once we have the spectral measure μ it can be used in Stieltjes integrals for other transport coefficients:

electrical and thermal conductivity, complex permittivity, magnetic permeability, diffusion, fluid flow properties

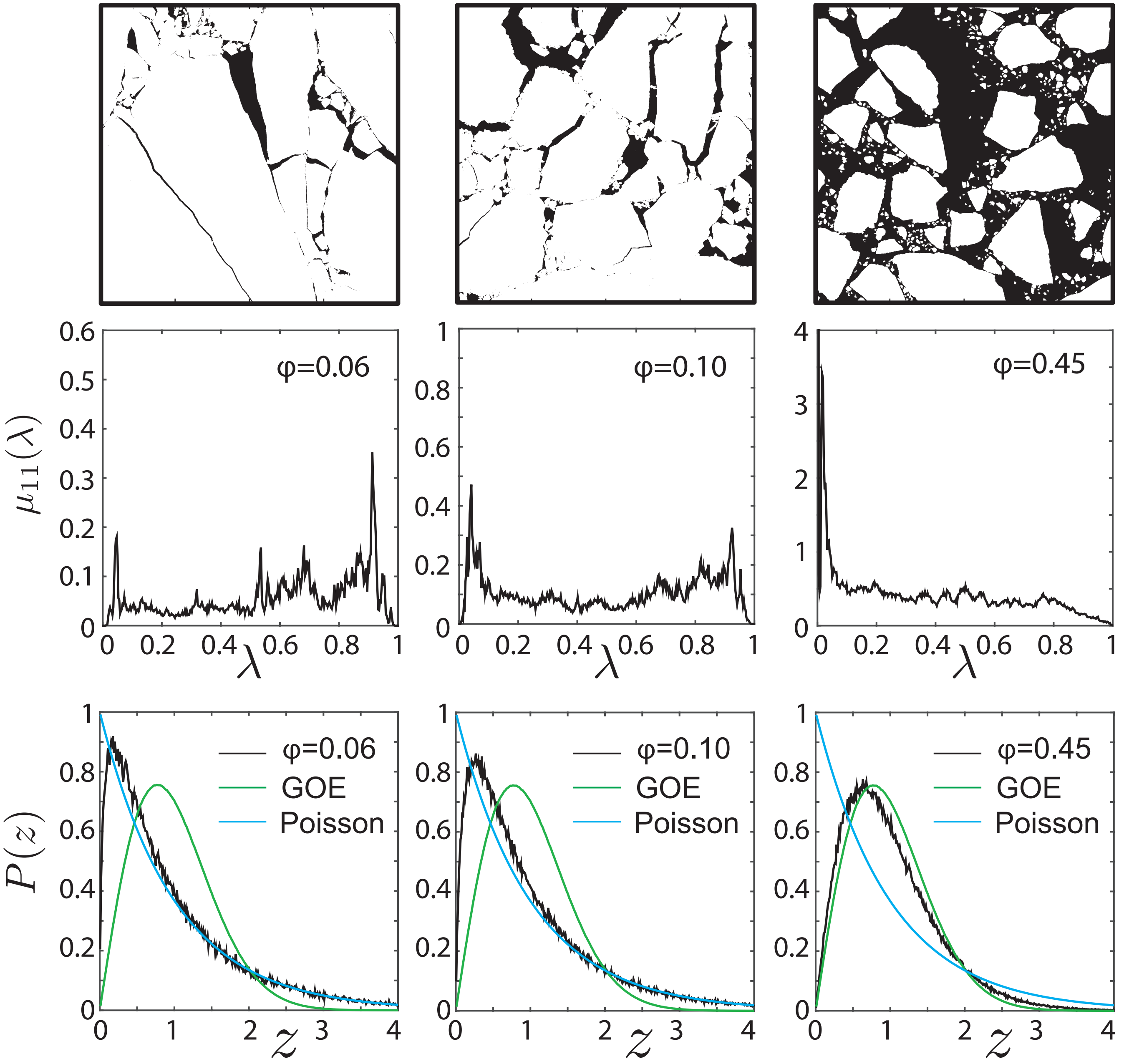
earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral computations for sea ice floe configurations

spectral measures

eigenvalue spacing distributions

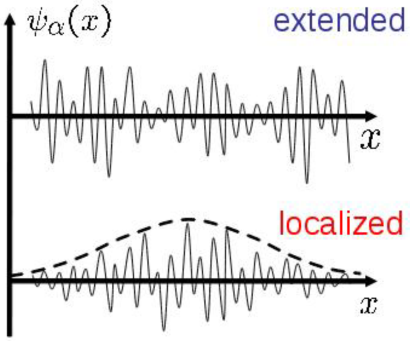


UNIVERSAL Wigner-Dyson distribution

uncorrelated → **level repulsion**

ANDERSON TRANSITION

Murphy, Cherkaev, Golden
Phys. Rev. Lett. 2017



metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

we find a surprising analog

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION TRANSITION → **transition to universal eigenvalue statistics (GOE) extended states, mobility edges**

-- but without wave interference or scattering effects ! --

Eigenvalue Statistics of Random Matrix Theory

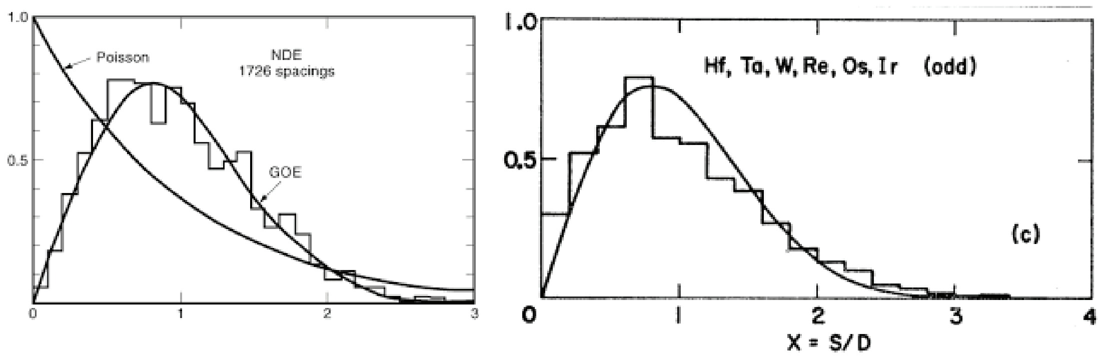
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^T)/2 \quad$ ***Gaussian orthogonal ensemble (GOE)***

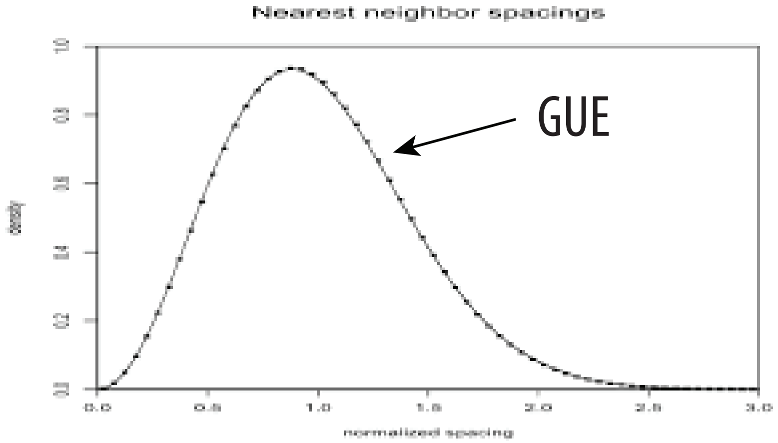
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\quad \mathbf{A} = (\mathbf{N}+\mathbf{N}^\dagger)/2 \quad$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.

Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



RMT used to characterize disorder-driven transitions in mesoscopic conductors, neural networks, random graph theory, etc.

Universal eigenvalue statistics arise in a broad range of "unrelated" problems!

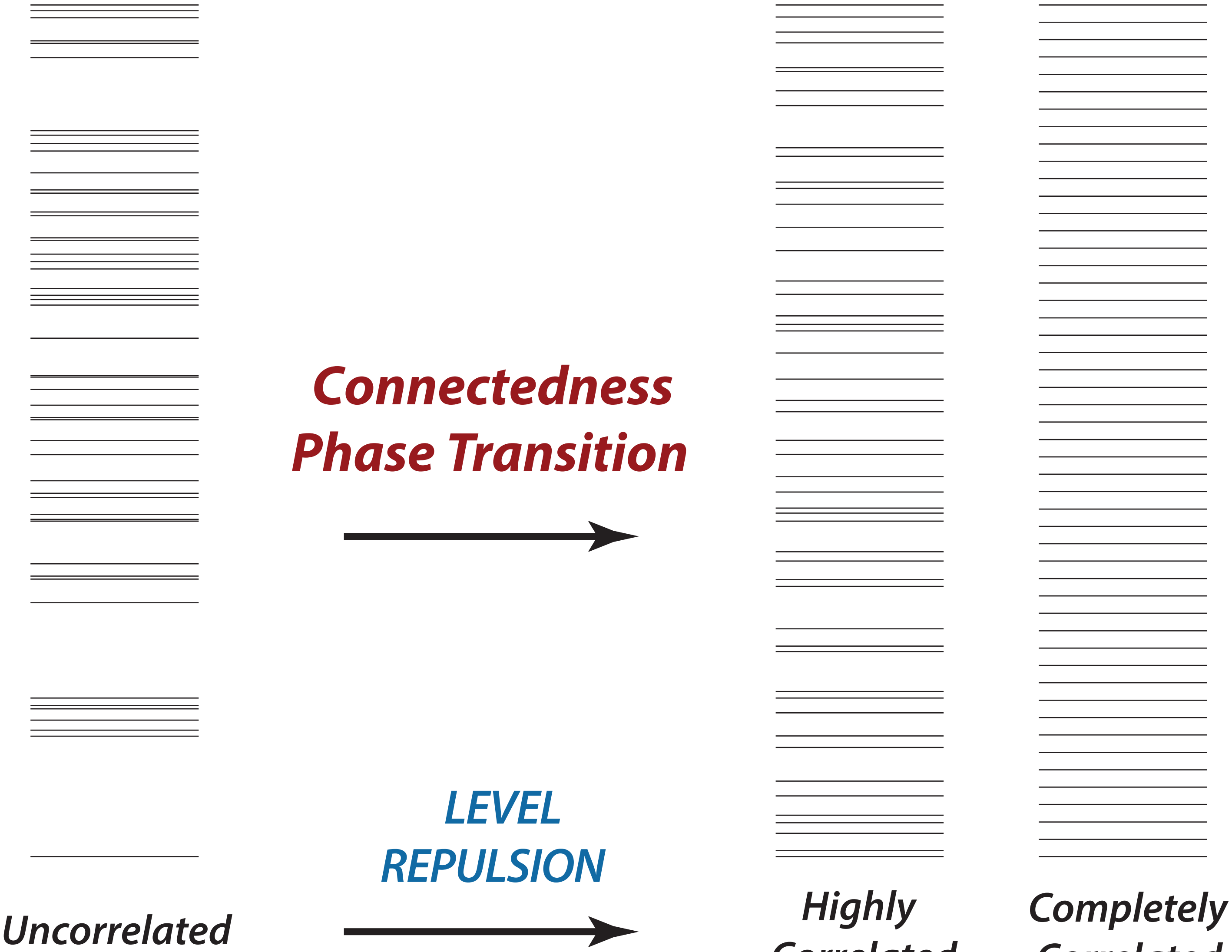
Transition in Eigenvalue Correlations

$P(z) = \exp(-z)$

Eigenvalue Spacing Distribution

$P(z) \approx \frac{\pi z}{2} \exp(-\pi z^2/4)$ **Wigner surmise**

Eigenvalue Spacing Distribution



Poisson Spectra

GOE Spectra

Picket Fence

Connectedness Phase Transition

LEVEL REPULSION

Uncorrelated

Highly Correlated

Completely Correlated

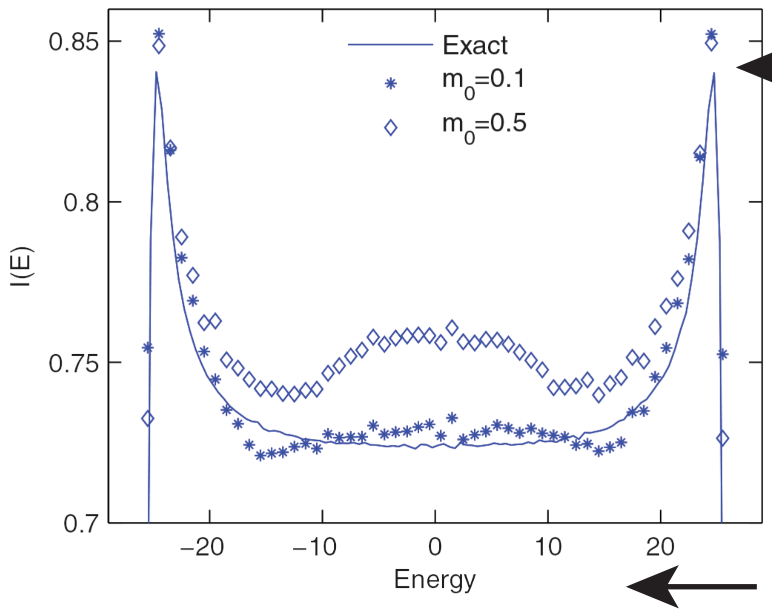
eigenvector localization and mobility edges

Inverse Participation Ratio: $I(\vec{v}_n) = \sum_{i=1}^{N} |(\vec{v}_n)_i|^4$

Completely Localized: $I(\vec{e}_n) = 1$

Completely Extended: $I\left(\frac{1}{\sqrt{N}}\vec{1}\right) = \frac{1}{N}$

Anderson Model



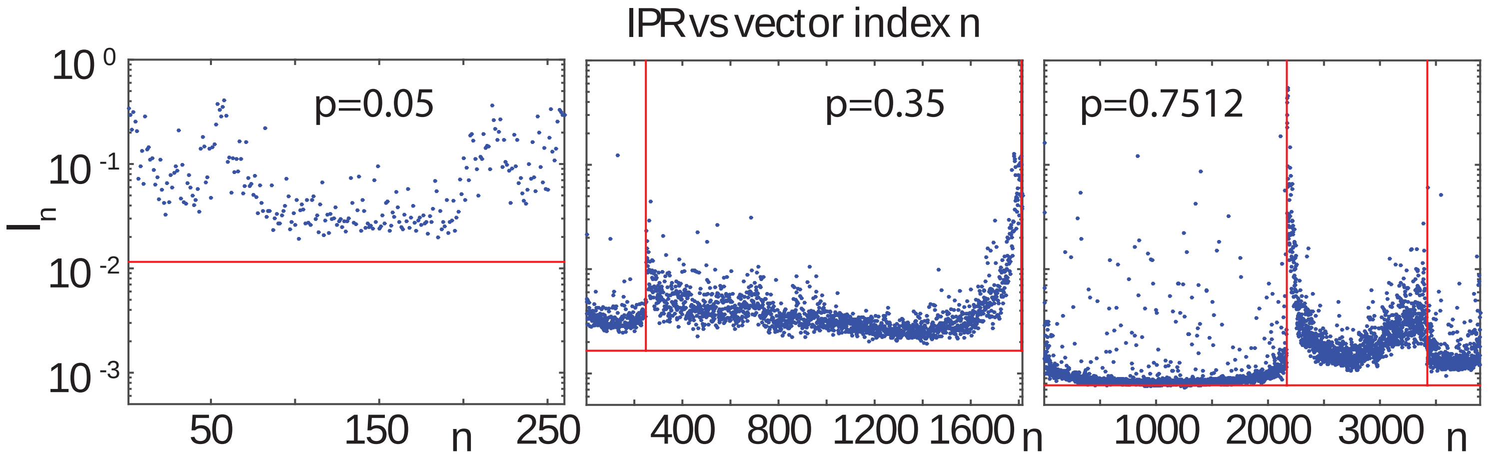
Mobility Edge

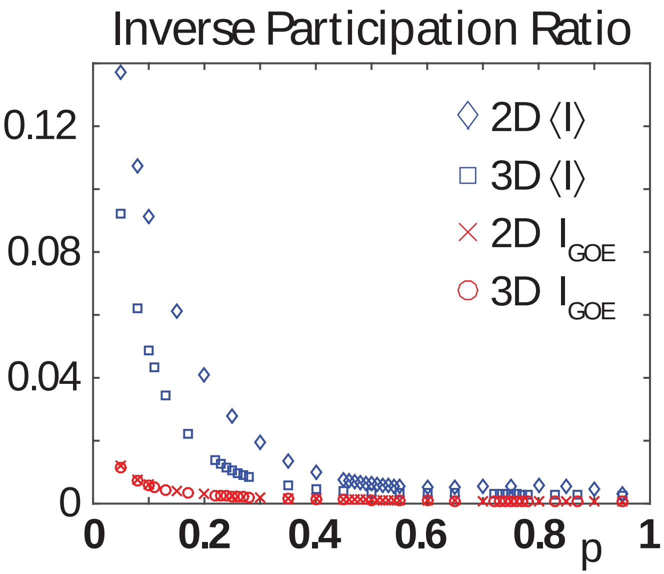
“Eigenvalues”

FIG. 4. (Color online) IPR for Anderson model in two dimensions with $x = 6.25$ ($w = 50$) from exact diagonalization (solid line) and from LDRG with different values of the cutoff m_0. LDRG data are averaged over 100 runs of systems with 100×100 sites.

PHYSICAL REVIEW B 90, 060205(R) (2014)

Localization properties of eigenvectors in random resistor networks





$$I_n = \sum_i (\vec{v}_n)_i^4$$

Murphy, Cherkaev, Golden, 2017

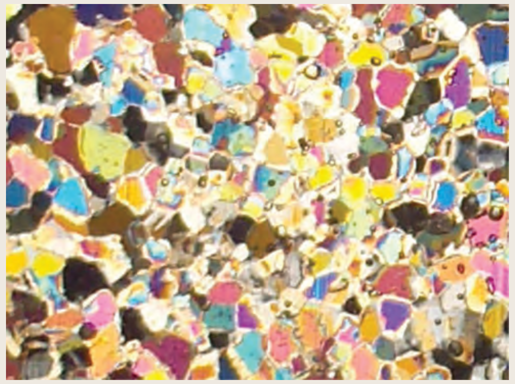
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**

  orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A   8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

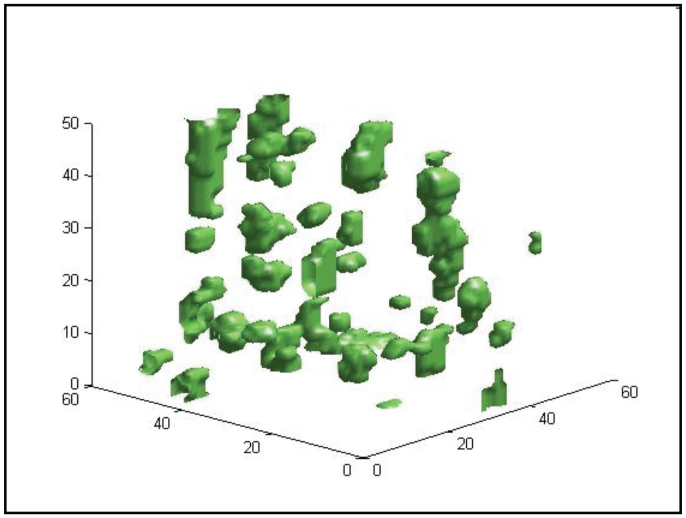
A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

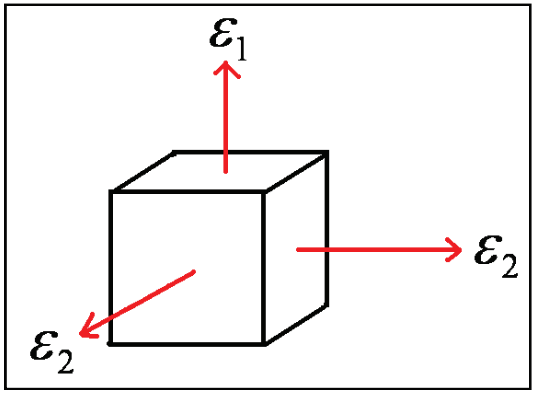
THE
ROYAL
SOCIETY
PUBLISHING

21/01/15 1:47 PM

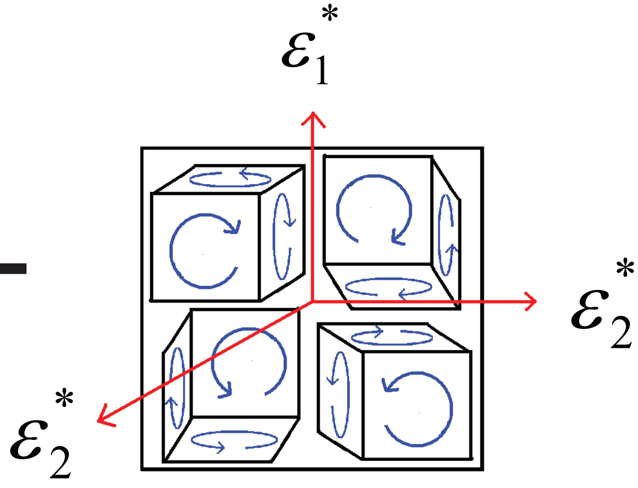
two scale homogenization for polycrystalline sea ice



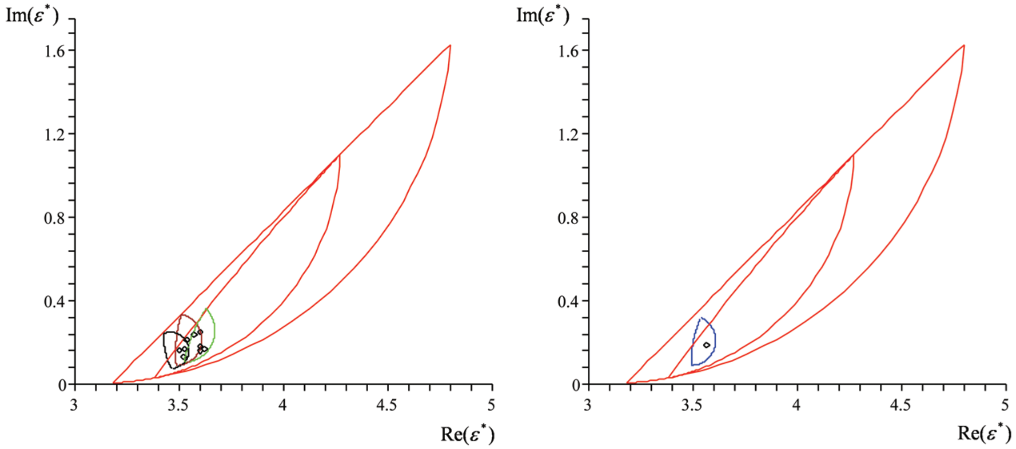
numerical homogenization for single crystal



analytic continuation for polycrystals



bounds

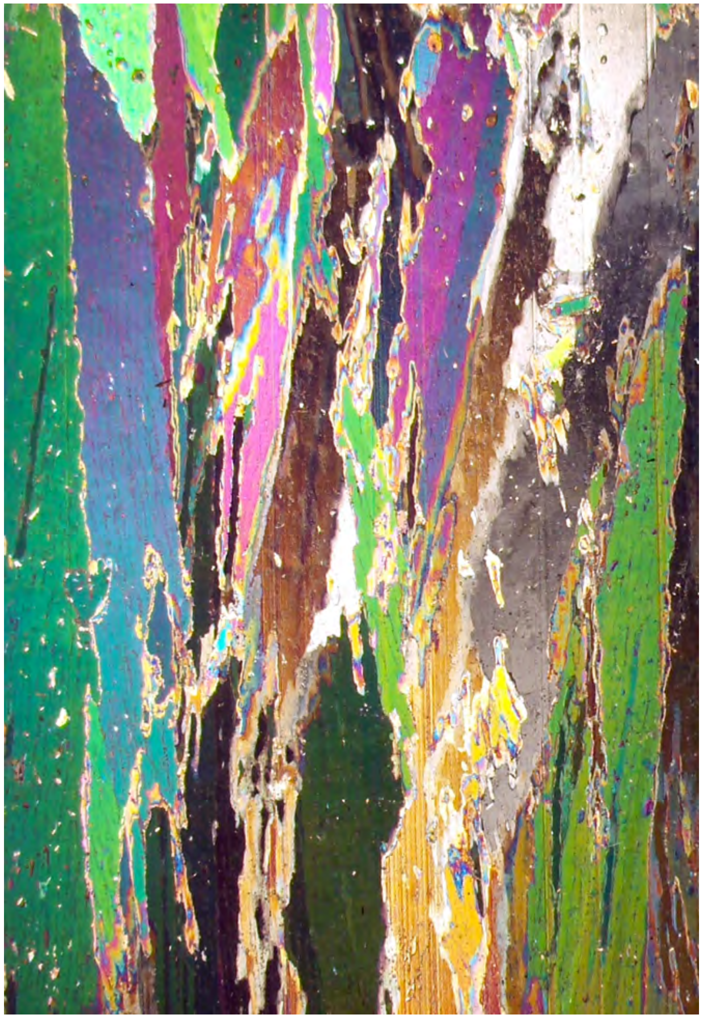


Gully, Lin, Cherkaev, Golden, Proc. Roy. Soc. A (and cover) 2015

higher threshold for fluid flow in Antarctic granular sea ice

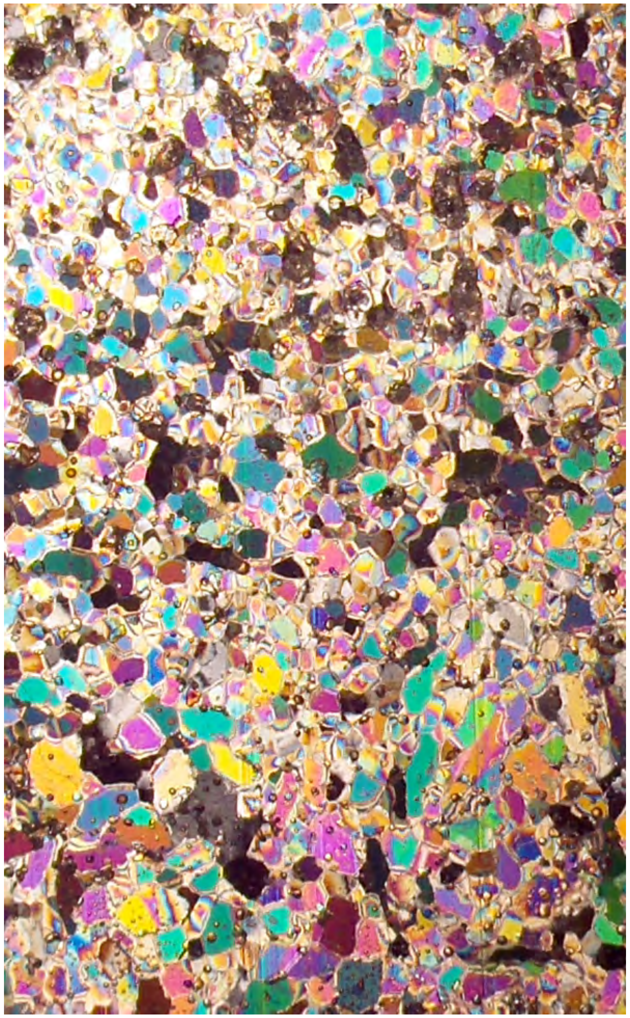
columnar

5%



granular

10%



Golden, Sampson, Gully, Lubbers, Tison 2019

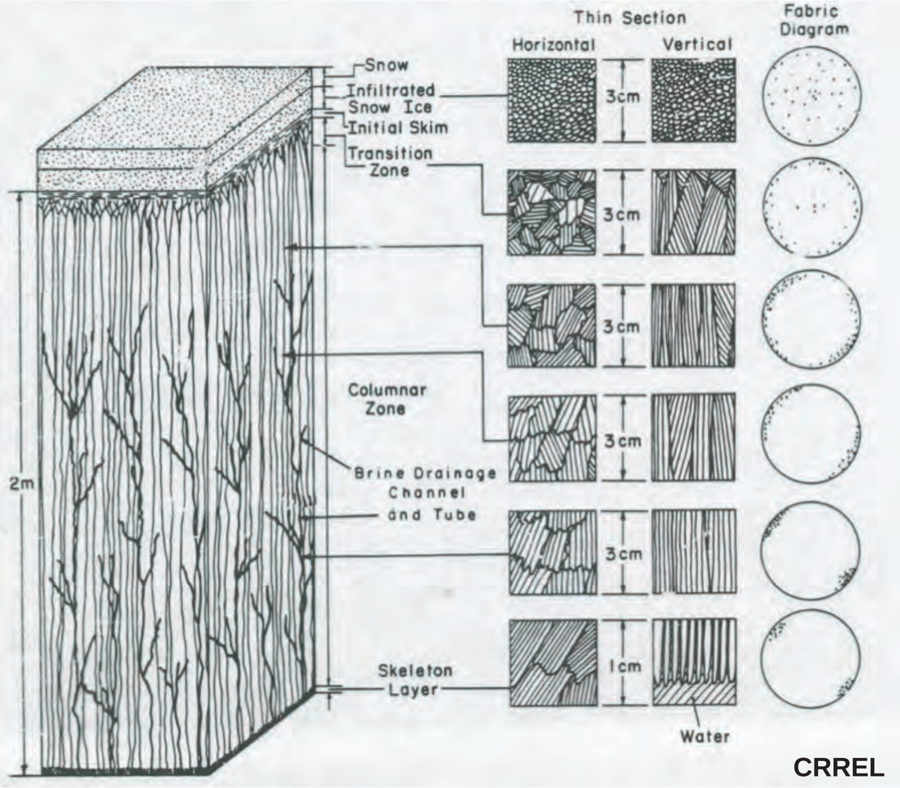
Rigorous bounds on the complex permittivity tensor of sea ice with polycrystalline anisotropy within the horizontal plane

McKenzie McLean, Elena Cherkaev, Ken Golden 2019

motivated by **Weeks and Gow, *JGR* 1979: c-axis alignment in Arctic fast ice off Barrow**

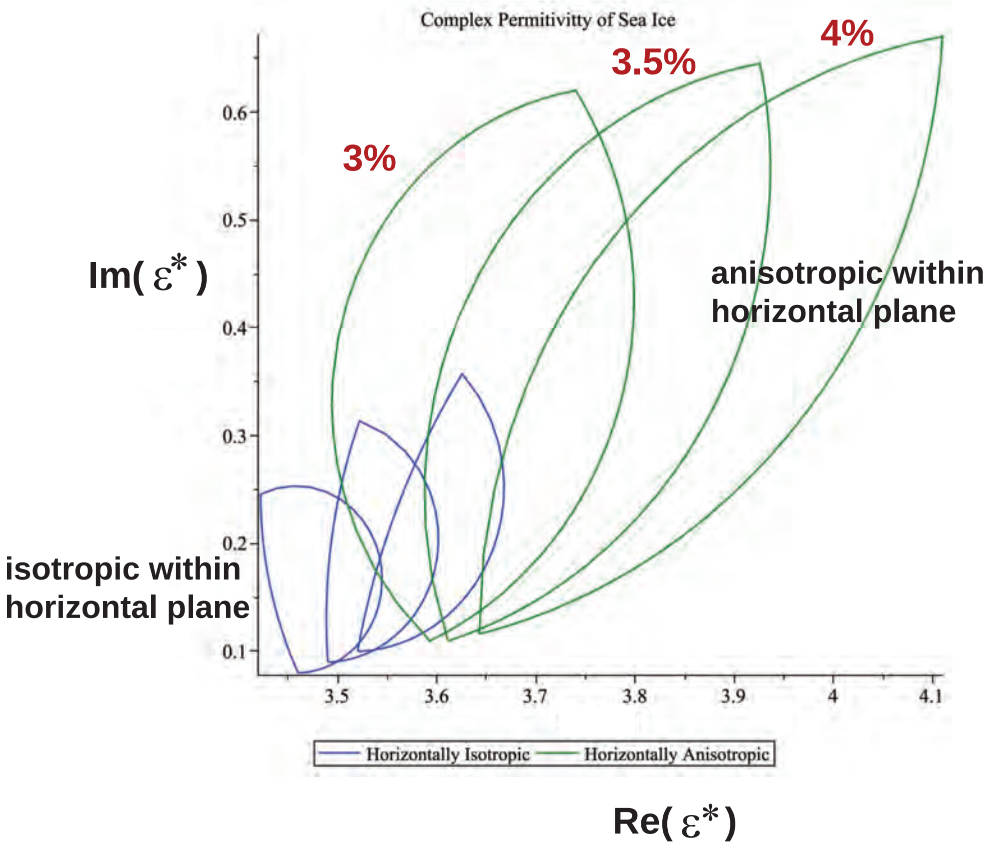
Golden and Ackley, *JGR* 1981: radar propagation model in aligned sea ice

input: orientation statistics



CRREL

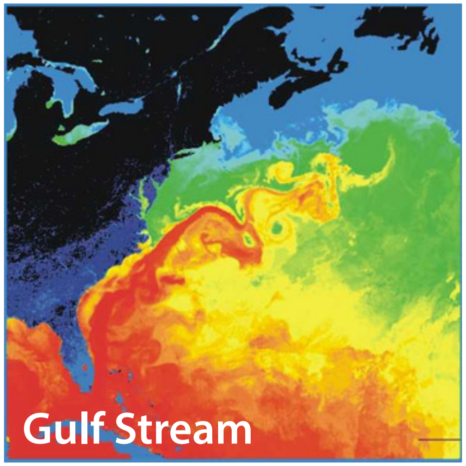
output: bounds

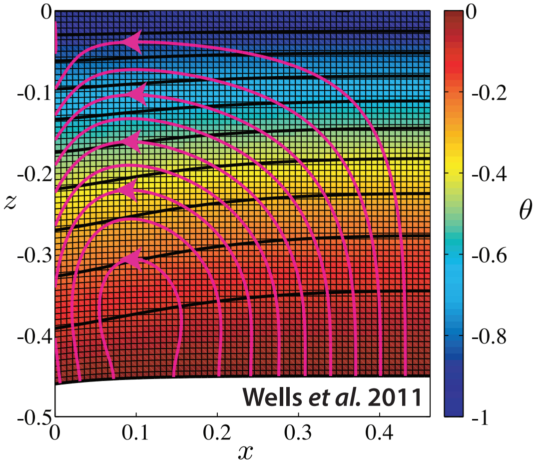


advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere


Gulf Stream



Wells *et al.* 2011

advection diffusion equation with a velocity field $\vec{u}$

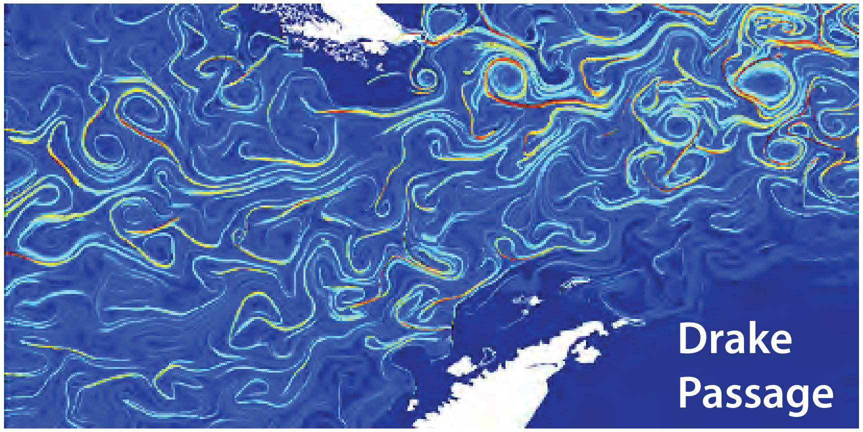
$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\Big\downarrow$$

homogenize

$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$


Drake Passage

κ^* *effective diffusivity*

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91


sea ice vortices

Masters, 1989

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019

Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019

$$\kappa^* = \kappa\left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}\right), \quad F(\kappa) \ = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

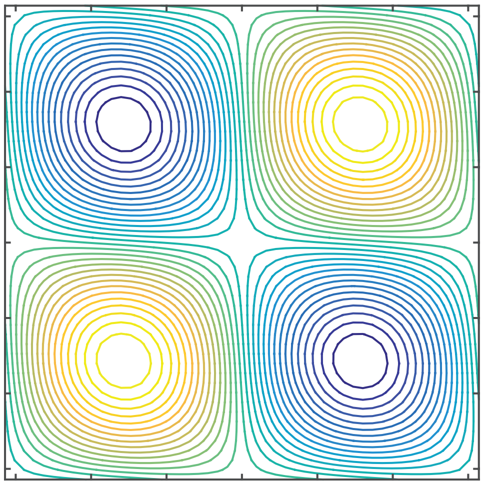
- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H\Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H\Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

separation of material properties and flow field

spectral measure calculations

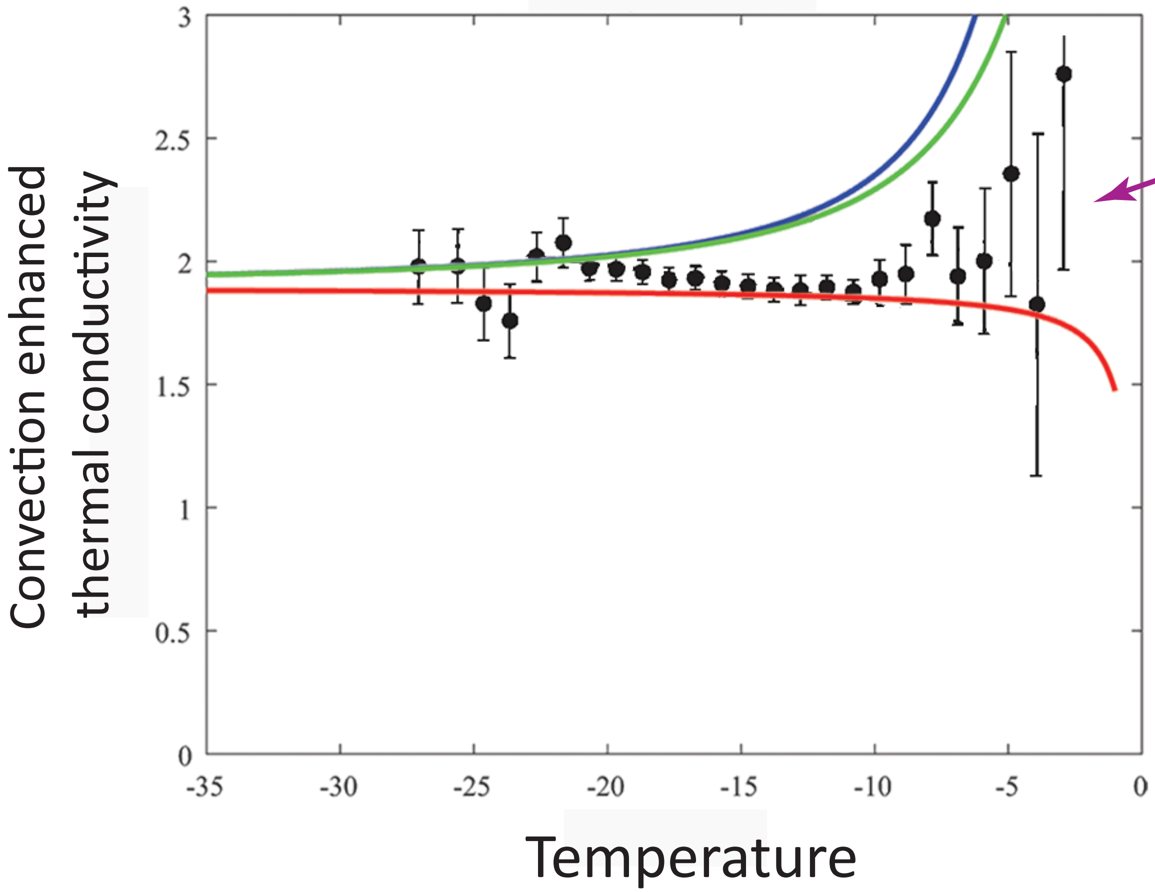
Rigorous bounds on convection enhanced thermal conductivity of sea ice

Kraitzman, Hardenbrook, Murphy, Zhu, Cherkaev, Strong, Golden 2019



cat's eye flow model for
brine convection cells

similar bounds
for shear flows



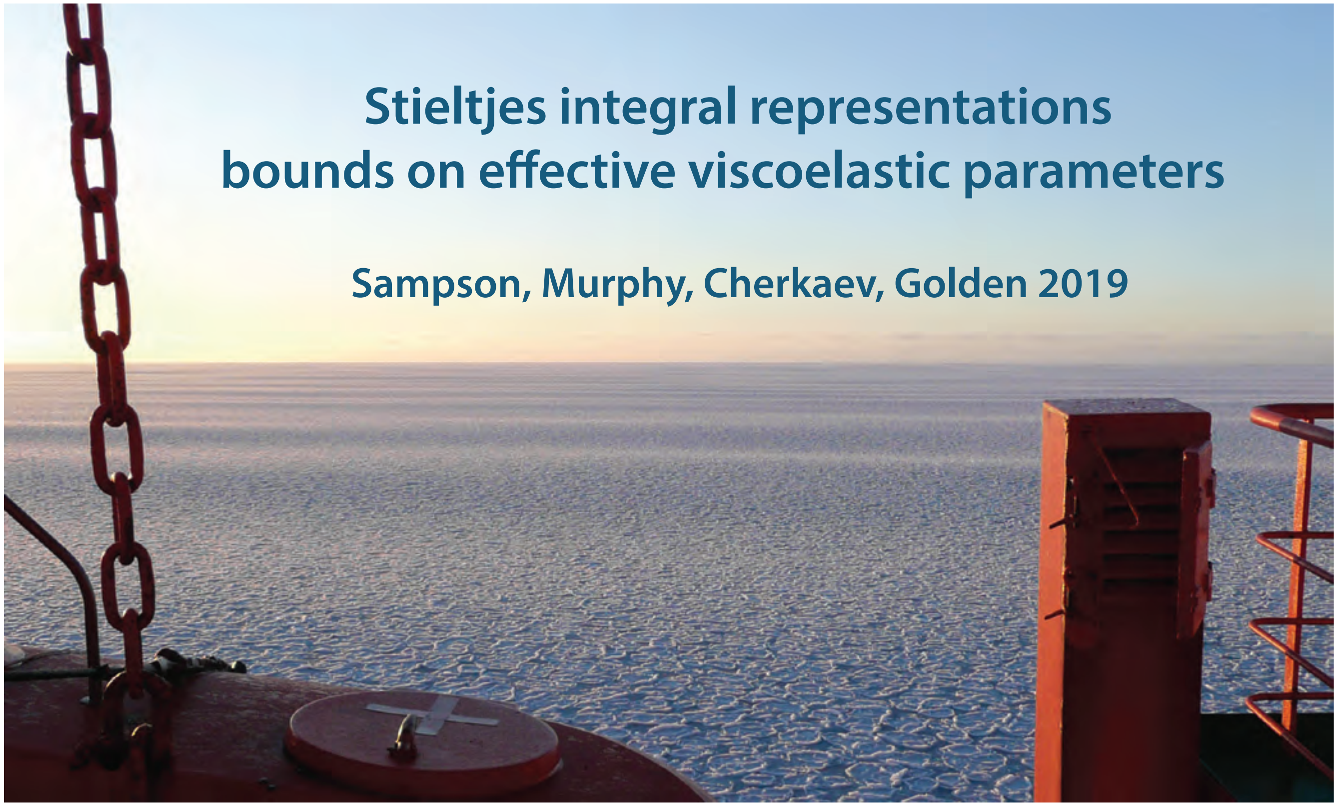
**data
Trodahl
et al. 2001**

rigorous Padé bounds from Stieltjes integral +
analytical calculations of moments of measure

**rigorous bounds assuming information
on flow field INSIDE inclusions**

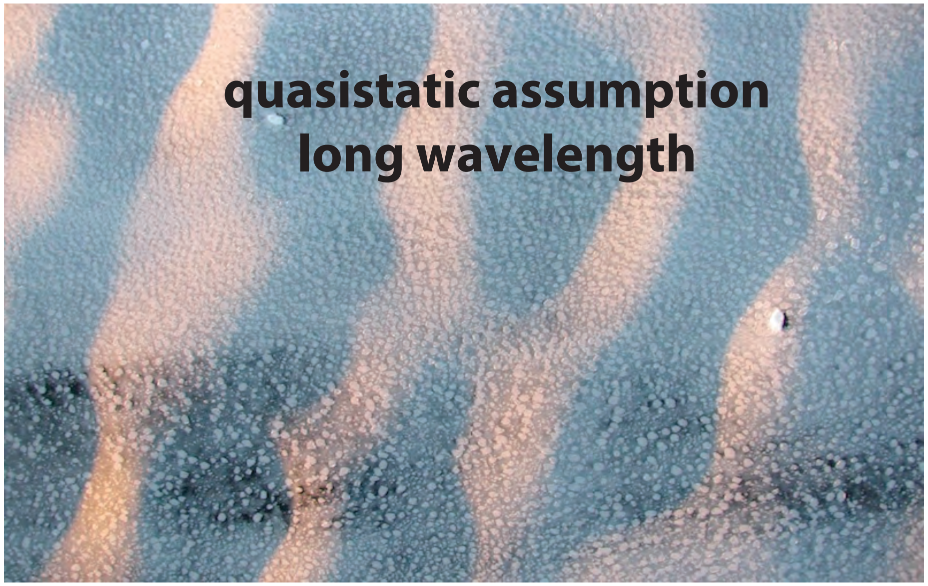
Kraitzman, Cherkaev, Golden
SIAM J. Appl. Math (in revision), 2019

wave propagation in the marginal ice zone



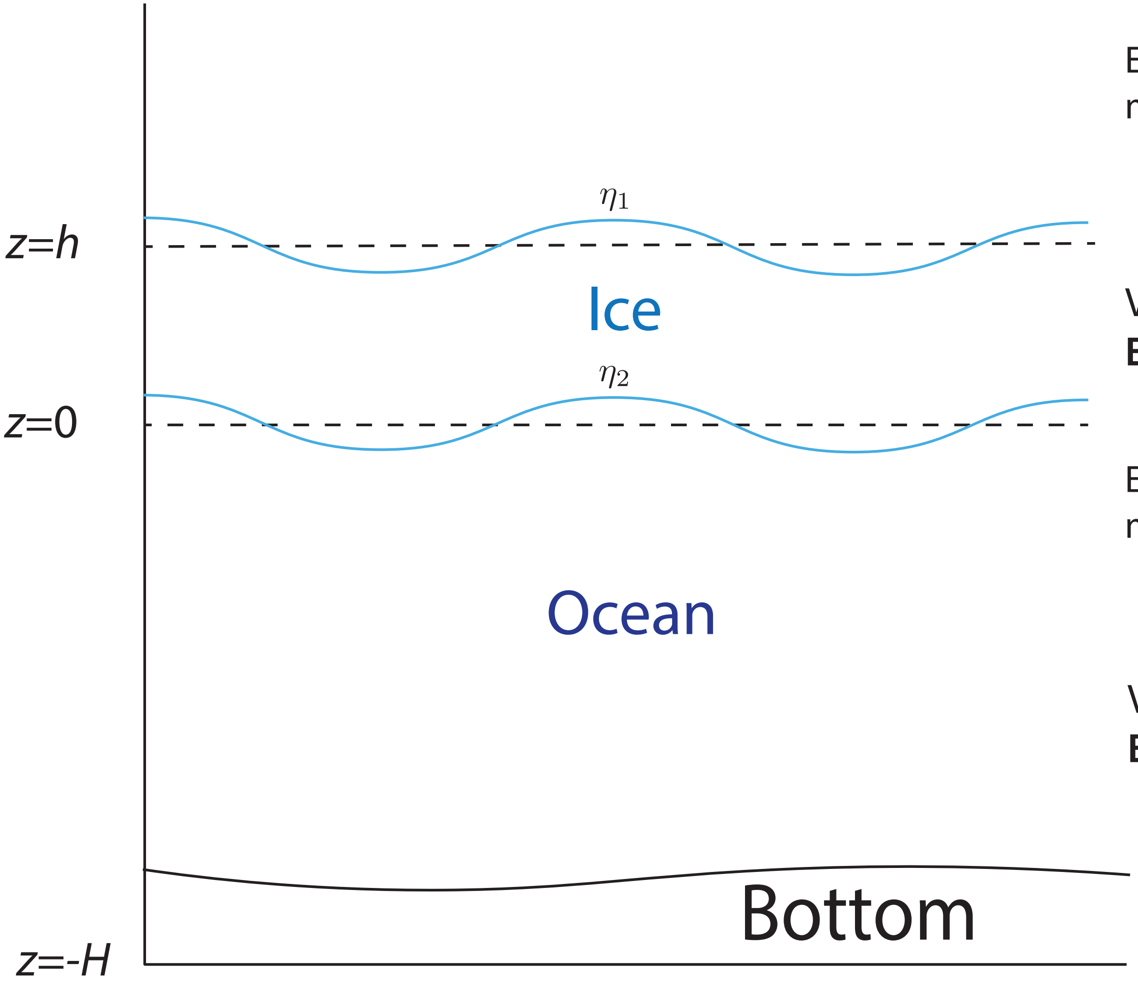
Stieltjes integral representations
bounds on effective viscoelastic parameters

Sampson, Murphy, Cherkaev, Golden 2019



quasistatic assumption
long wavelength

Two Layer Models and Effective Rheological Parameters



G shear modulus

P pressure

ω angular frequency

U velocity field

ν viscosity

λ Poission ratio

ρ density

g gravity

Viscous fluid layer (Keller 1998)

Effective Viscosity $\boxed{\nu}$

Equations of motion:

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 U + g$$

Viscoelastic fluid layer (Wang-Shen 2010)

Effective Complex Viscosity $\boxed{\nu_e = \nu + iG/\rho\omega}$

Equations of motion

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu_e \nabla^2 U + g$$

Viscoelastic thin beam (Mosig *et al.* 2015)

Effective Complex Shear Modulus $\boxed{G_v = G - i\omega\rho\nu}$

Stieltjes integral representation for effective complex viscoelastic parameter; bounds

Sampson, Murphy, Cherkaev, Golden 2019

Homogenization for two phase viscoelastic composite

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**

microscale $\sigma = C_{ijkl}\epsilon_{kl} = C : \epsilon$

$$C = 2(\chi_1\nu_1 + \chi_2\nu_2)\Lambda_s$$

macroscale $\langle\sigma\rangle = C^* : \langle\epsilon\rangle$

$$\langle\epsilon\rangle = \epsilon^0$$

quasistatic assumption $\nabla \cdot \sigma = 0$

Strain Field

$$\epsilon = \frac{1}{2}[\nabla u + (\nabla u)^T] = \nabla^s u \quad \nabla \cdot u = 0$$

Resolvent

$$\epsilon = \left(1 - \frac{1}{s}\Gamma\chi_1\right)^{-1}\epsilon^0 \quad \Rightarrow \quad \boxed{\frac{\nu^*}{\nu_2} = \left(1 - \|\epsilon^0\|^{-2} F(s)\right)}$$

$$F(s) = \int_0^1 \frac{d\mu(\lambda)}{s - \lambda} \qquad s = \frac{1}{1 - \frac{\nu_1}{\nu_2}}$$

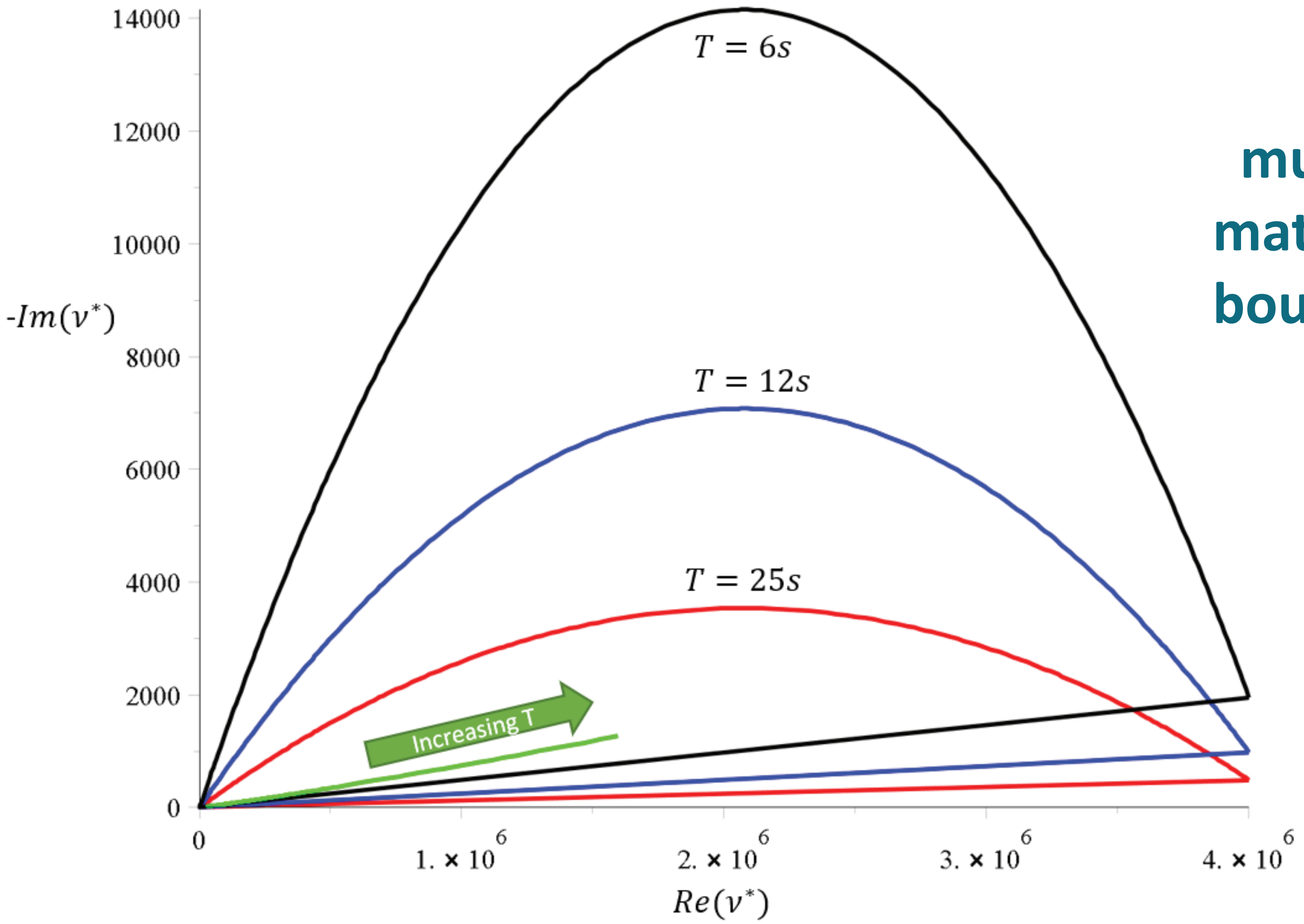
$$\Gamma = \nabla^s(\nabla \cdot \nabla^s)^{-1}\nabla \cdot$$

bounds on the effective complex viscoelasticity

complex elementary bounds (fixed area fraction of floes)

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**



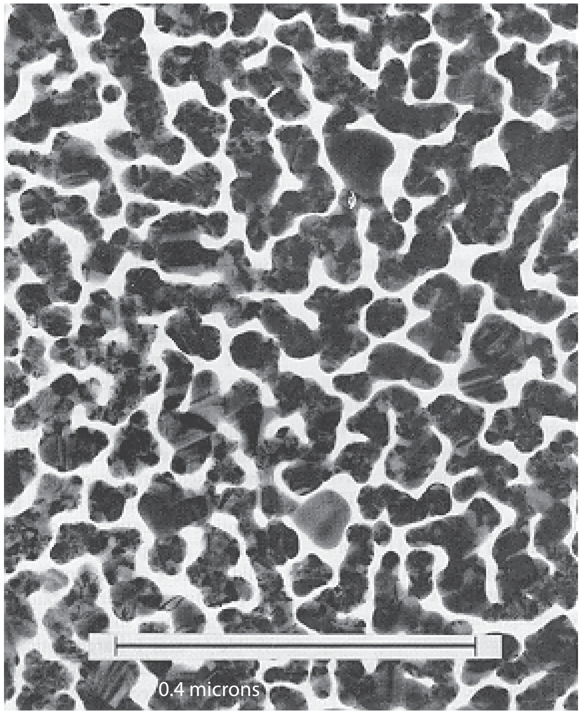
+ much tighter matrix particle bounds + data

Sampson, Murphy, Cherkaev, Golden 2019

Interaction of light with sea ice

thin silver film

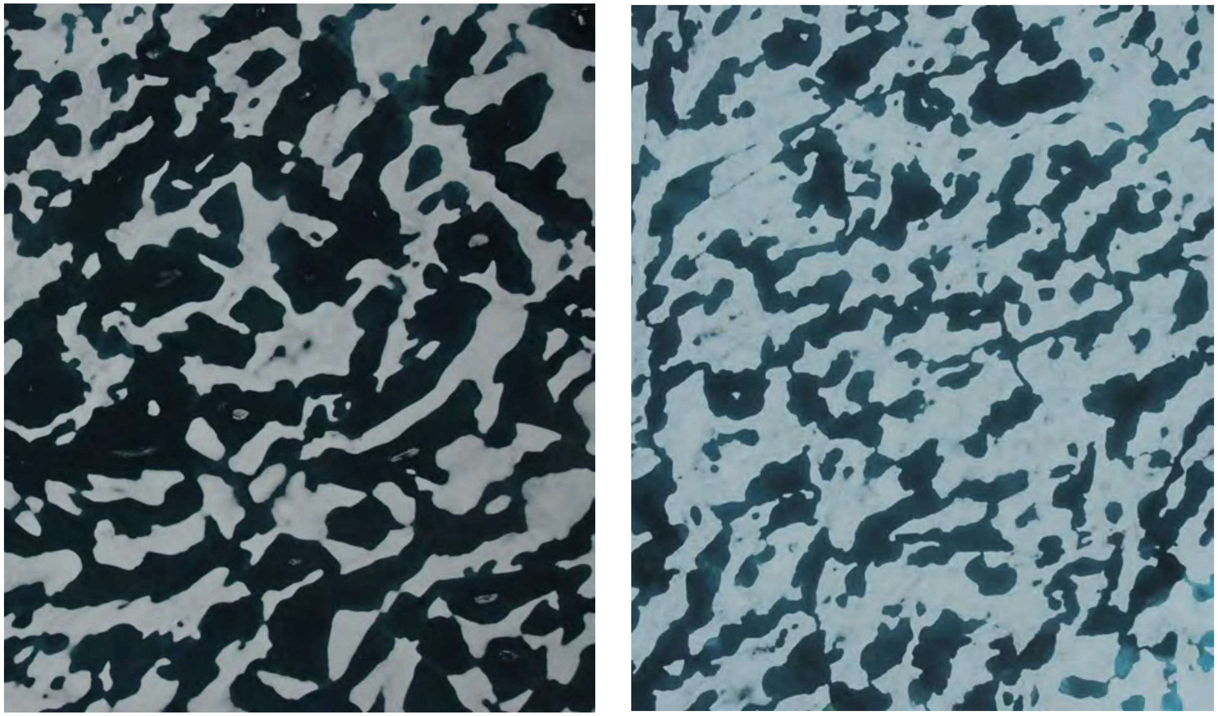
microns



(Davis, McKenzie, McPhedran, 1991)

Arctic melt ponds

kilometers



(Perovich, 2005)

optical properties

composite geometry -- area fraction of phases, connectedness, necks

melt pond formation and albedo evolution:

- ***major drivers in polar climate***
- ***key challenge for global climate models***

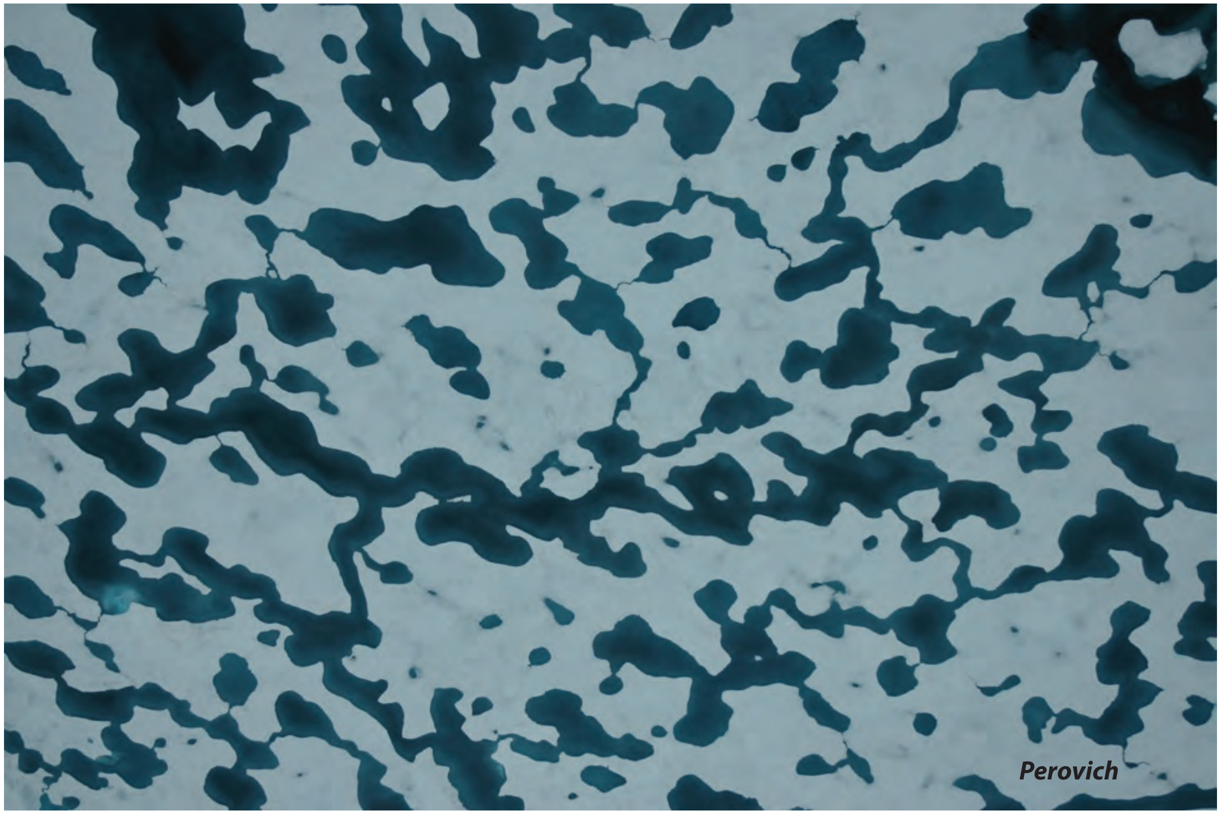
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012



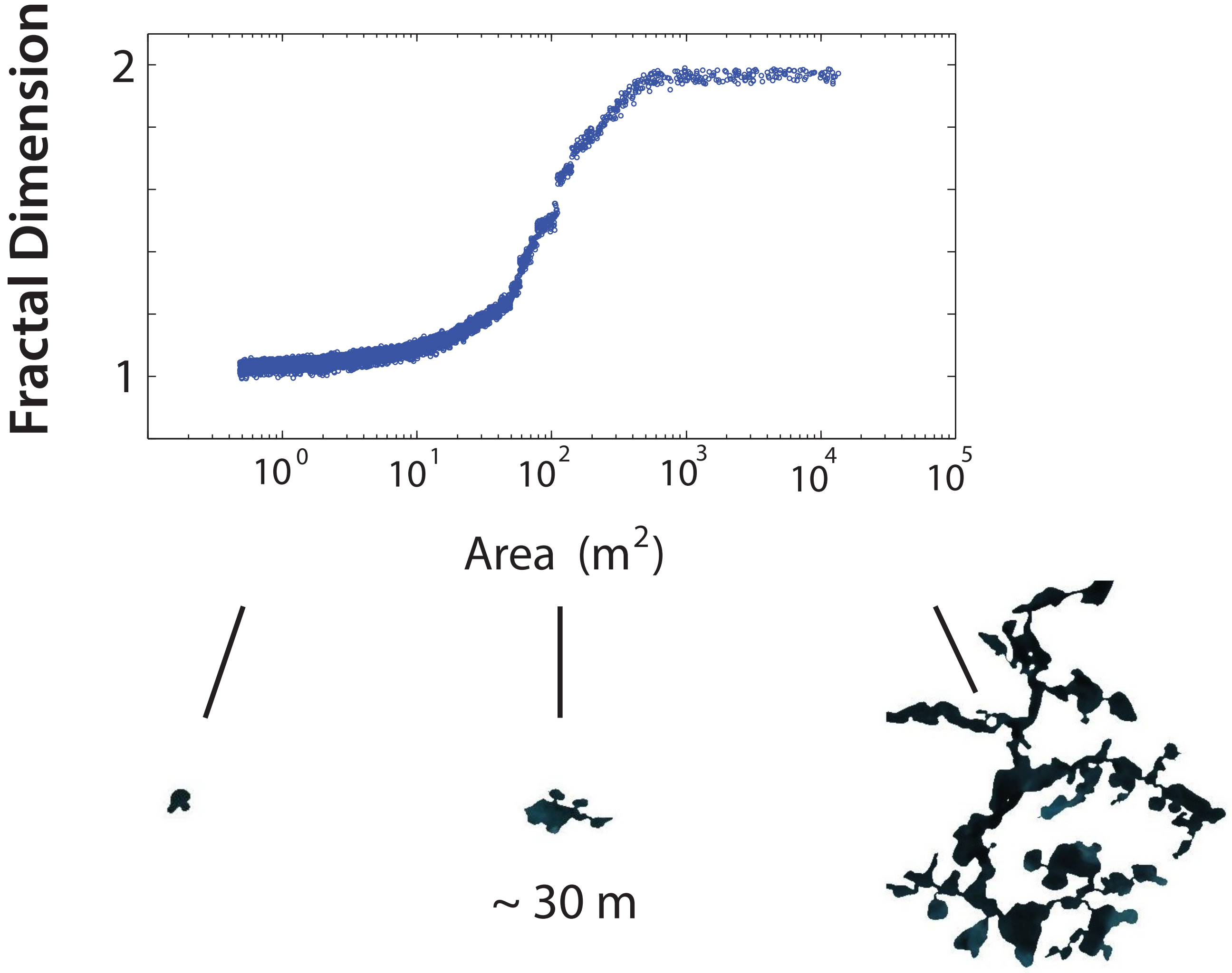
Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

complexity grows with length scale

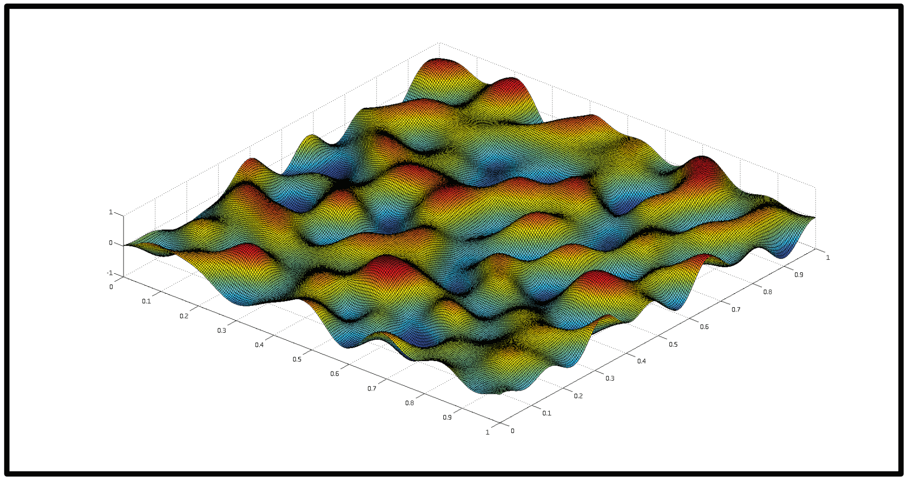


simple pond ***transitional pond*** ***complex pond***

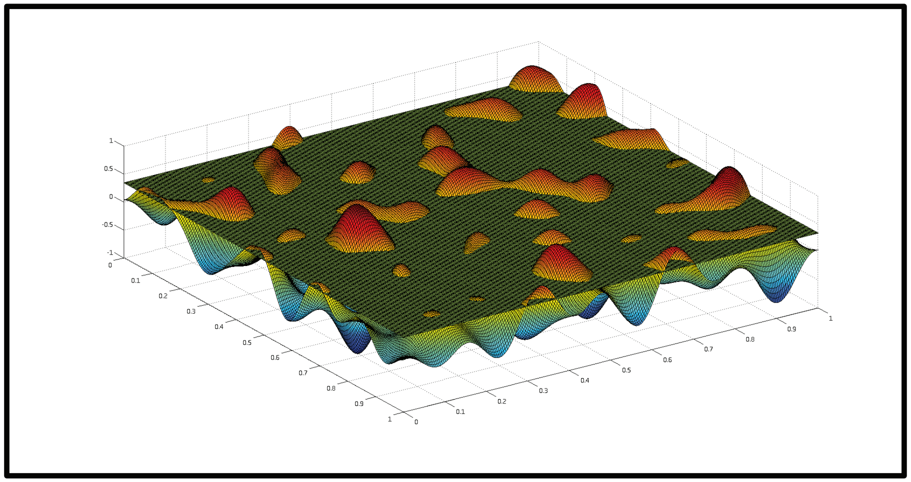
Continuum percolation model for melt pond evolution

level sets of random surfaces

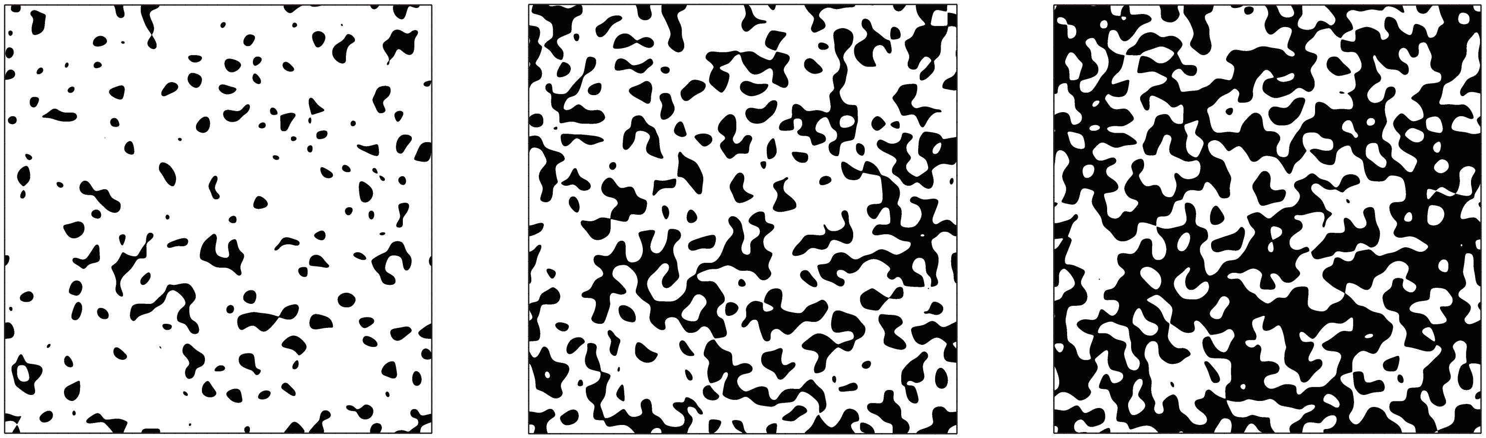
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography



intersections of a plane with the surface define melt ponds



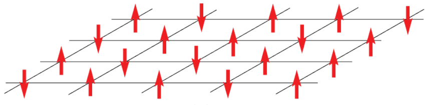
electronic transport in disordered media

diffusion in turbulent plasmas

Isichenko, Rev. Mod. Phys., 1992

Ising model for ferromagnets → Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.* 2019



2–D Ising Model

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - \sum_{i}^{N} H_i \, s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

random magnetic field represents snow topography

magnetization $M = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$

pond coverage $\frac{(M+1)}{2}$

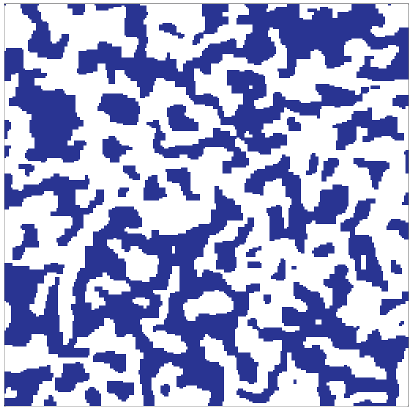
~ albedo

only nearest neighbor patches interact

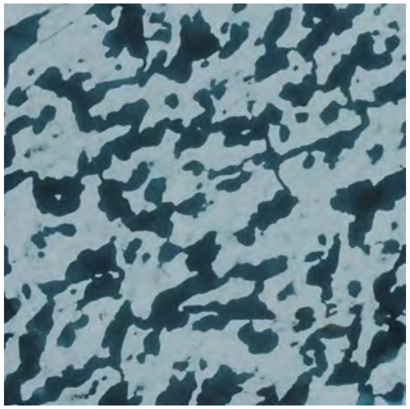
Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system “flows” toward metastable equilibria.

Melt ponds are metastable islands of like spins.

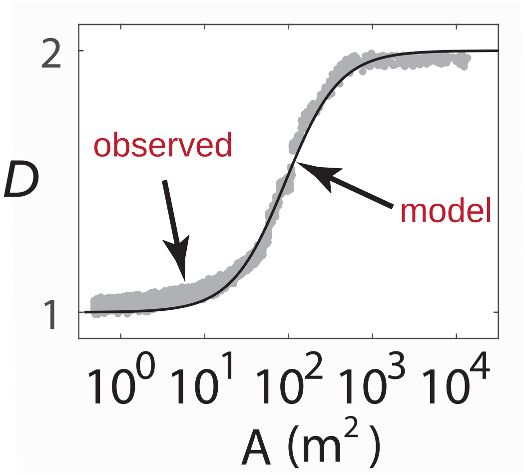
Order from Disorder



Ising model



melt pond photo **(Perovich)**



pond size distribution exponent

observed -1.5

(Perovich, *et al.* 2002)

model -1.58

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

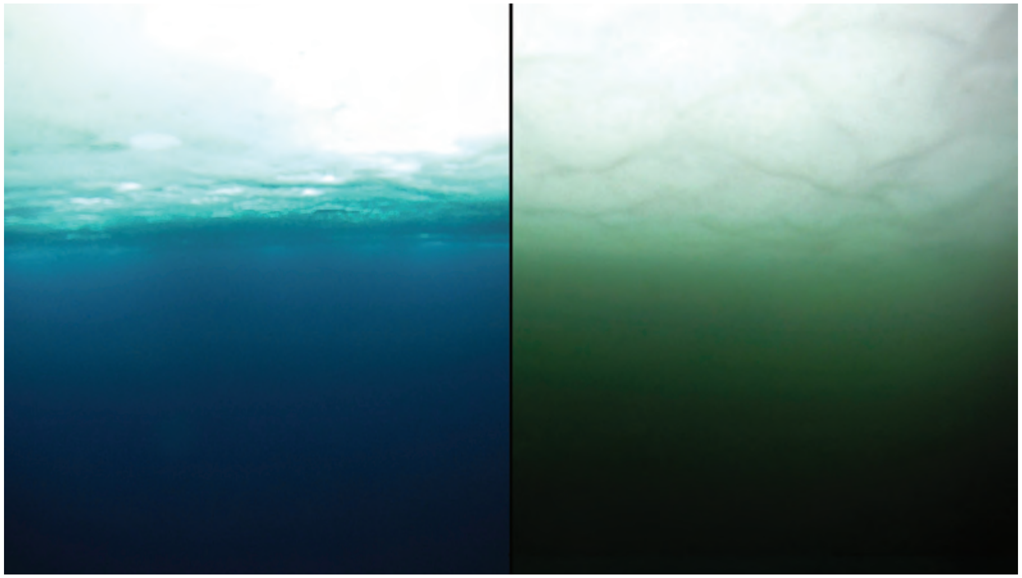
2011 massive under-ice algal bloom

Arrigo et al., *Science* 2012

melt ponds act as

WINDOWS

allowing light through sea ice



no bloom

bloom

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

Horvat, Rees Jones, Iams, Schroeder, Flocco, Feltham, ***Science Advances***, 2017

The distribution of solar energy under ponded sea ice

Horvat, Flocco, Rees Jones, Roach, Golden, 2019

(2015 AMS MRC)

The distribution of solar energy under ponded first-year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden, ***in revision,*** 2019

- Model for 3D light field under ponded sea ice.

- Distribution of solar energy at depth influenced by ***shape and connectivity*** of melt ponds, as well as area fraction.

- Aggregate properties of the sub-ice light field, such as a significant enhancement of available solar energy under the ice, are controlled by parameter closely related to pond fractal geometry.

- Model and analysis explain how melt pond geometry ***homogenizes*** under-ice light field, affecting habitability.

Pond geometry affects the ecology of the Arctic Ocean.

Conclusions

1. Wave phenomena arise naturally in the sea ice system.

2. Homogenization and statistical physics help *link scales* and provide rigorous methods for finding effective behavior, and advance how sea ice is represented in climate models.

3. Herglotz functions and Stieltjes integrals provide powerful methods of homogenization for wave phenomena in sea ice structures.

4. Our research will help to improve projections of climate change and the fate of the Earth sea ice packs.

THANK YOU

Office of Naval Research

Applied and Computational Analysis Program

Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

Department of the Navy ONR Science & Technology

MCRN Mathematics and Climate Research Network

Australian Government
Department of the Environment and Water Resources
Australian Antarctic Division

ace CRC
ANTARCTIC CLIMATE & ECOSYSTEMS
COOPERATIVE RESEARCH CENTRE

NSF

Antarctica New Zealand

UAF UNIVERSITY OF ALASKA FAIRBANKS

THE UNIVERSITY OF UTAH

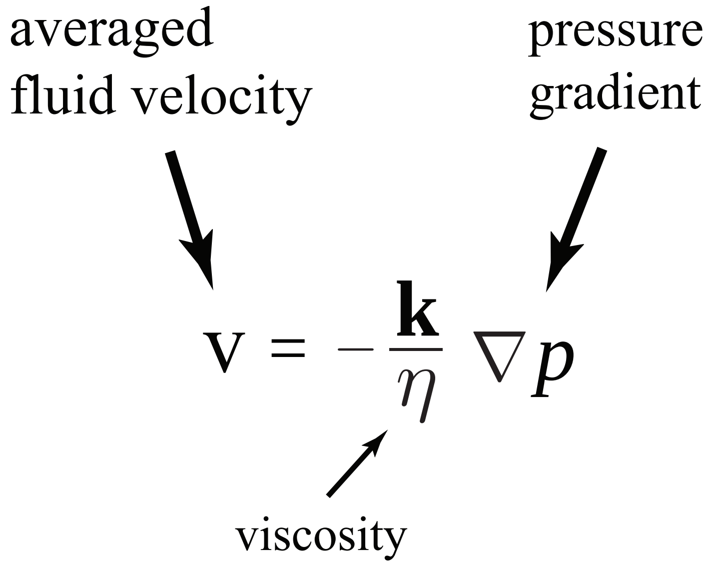
Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999

fluid permeability of a porous medium

how much water gets
through the sample
per unit time?

Darcy's Law

for slow viscous flow in a porous medium



averaged
fluid velocity

pressure
gradient

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\,\nabla p$$

viscosity

$\mathbf{k}$ = fluid permeability tensor

HOMOGENIZATION

mathematics for analyzing effective behavior of heterogeneous systems