From magnets to melt ponds

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When snow on top of Arctic sea ice begins to melt in late spring, small pools of water form on the surface. As the melt season progresses, these simply shaped meter scale pools grow and coalesce into beautiful kilometer scale labyrinths of cerulean blue with complex, self-similar boundaries. The fractal dimension of the boundaries transitions from 1 to about 2 as the area increases through a critical regime centered around 100 square meters [4]. While the white snowy sea ice surface reflects most of the incident sunlight, the darker melt ponds act like windows and allow significant light to penetrate into the ice and sea water underneath the ponds. Melt ponds are thus one of the key controls on how much solar energy is absorbed by the ice pack and upper ocean, and strongly influence ice melting rates as well as the ecology of the polar marine environment. They largely determine sea ice albedo, the ratio of reflected to incident sunlight, a key parameter in climate modeling.

Viewed from a helicopter, the labyrinthine patterns of dark and light seen on the surface of melting sea ice are reminiscent of the kinds of structures applied mathematicians sometimes see in studies of phase transitions and coarsening processes in materials science. They might

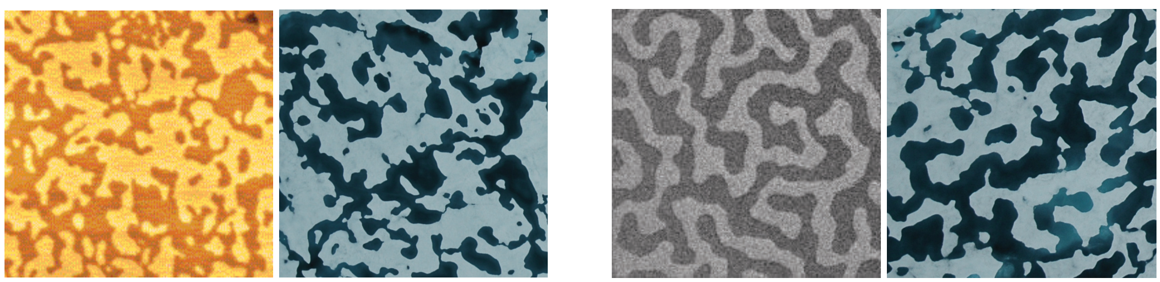

Figure 1: From left to right: image of magnetic domains in cobalt [9] about 20 microns, or 0.02 mm across; Arctic melt pond photo from a helicopter (Perovich), about 100 meters across; Magneto-optic Kerr effect (MOKE) microscope image of maze-like domain structures in thin films of the ferromagnetic alloy cobalt-iron-boron (CoFeB) [10], about 150 microns, or 0.15 mm across; similarly structured melt ponds (Perovich), about 70 meters across.

even remind an observer who has taken a course in statistical mechanics of the complex regions of aligned spins, or *magnetic domains*, that one can see in magnetic materials. In Figure 1 we show two examples of magnetic domains compared with similar patterns formed by melt ponds on Arctic sea ice. In these magnetic materials, energy is lowered when nearby spins align with each other, forming the domains. At higher temperatures thermal fluctuations dominate the tendency for the magnetic moments of the domains to align as well, with no net magnetization M of the material, unless an external magnetic field H is applied to induce alignment. However, at temperatures below the Curie point T_c, the

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tendency for overall alignment takes over, and the material remains magnetized ($M \neq 0$) even as the applied field H vanishes, often called *spontaneous* or *residual* magnetization.

The prototypical model of a magnetic material based on a lattice of interacting binary spins is the Ising model, which was proposed a century ago in 1920 by Wilhelm Lenz, who was Ising's Ph.D advisor. This model incorporates only the most basic physics of magnetic materials and the principle that natural systems tend toward minimum energy states. It has found unexpected applications in areas as diverse as neuroscience and spatial ecology.

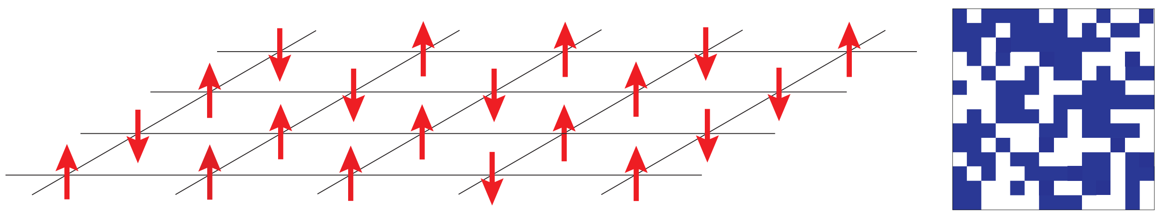

Figure 2: Left: Two dimensional Ising model, with spins at each lattice site either up or down. Right: Spin configuration, with spin up sites colored blue and spin down sites white.

Consider a finite box $\Lambda \subset \mathbb{Z}^2$ containing N sites. At each site there is a spin variable s_i which can take the values $+1$ or -1, as shown in Figure 2. To illustrate how our melt pond Ising model works, we briefly formulate the problem of finding the magnetization $M(T, H)$, or *order parameter*, of an Ising ferromagnet at temperature T in a field H. The Hamiltonian $\mathscr{H}$ with ferromagnetic interaction $J \geq 0$ between nearest neighbor pairs is given by

$$\mathscr{H}_\omega = -\, H \sum_i s_i - J \sum_{<i,j>} s_i s_j \,,$$

for any configuration $\omega \in \Omega = \{-1, 1\}^N$ of the spin variables. The canonical partition function Z_N, from which the observables of the system can be calculated, is given by

$$Z_N(T, H) = \sum_{\omega \in \Omega} \exp(-\beta \mathscr{H}_\omega) = \exp(-\beta N f_N) \,,$$

where $\beta = 1/kT$, k is Boltzmann's constant, $\exp(-\beta\mathscr{H}_\omega)$ is the Gibbs factor, and f_N is the free energy per site, $f_N(T, H) = (-1/\beta N) \log Z_N(T, H)$.

The magnetization $M(T, H) = \lim_{N\to\infty} \frac{1}{N} \sum_i s_i$ averaged over $\omega \in \Omega$ with Gibbs' weights, can then be expressed in terms of the free energy $f(T, H) = \lim_{N\to\infty} f_N(T, H)$,

$$M(T, H) = -\frac{\partial f}{\partial H} \,.$$

Emblematic of the rich behavior exhibited by this model is the existence of a critical temperature T_c, the Curie point, where $M(T) = \lim_{H\to 0} M(T, H) > 0$ for $T < T_c$ and $M(T) = 0$ for $T \geq T_c$, with universal power law asymptotics for $M(T) \to 0$ as $T \to T_c^-$, independent of the type of lattice or other local details of the model.

A widely used method to numerically construct equilibrium states of the Ising ferromagnet is the Metropolis algorithm. A spin is chosen at random, and whether or not it is flipped depends on if it would lower or raise energy. With ΔE the change in magnetostatic energy from a potential flip, as measured by $\mathscr{H}_\omega$, then the spin is flipped if $\Delta E \leq 0$. If $\Delta E > 0$, the spin is flipped with probability given by the Gibbs' factor for ΔE. By sweeping through the whole lattice and then iterating the process many times, a local minimum of the energy of the system is attained.

In our recent work [5] published in *New Journal of Physics*, we have adapted the classical Ising model to study and explain the observed geometry of melt pond configurations and capture the fundamental physical mechanism of pattern formation for melt ponds on Arctic sea ice. While there have been instructive, important numerical models of melt pond evolution developed previously [2, 3], they have tended to be somewhat detailed, accounting for much of the physics of ponded sea ice, and not focused on how melt water is distributed over the sea ice surface. What distinguishes our new model is its simplicity. It accounts for only the most basic physics of the system, in the same way the classical nearest neighbor Ising model incorporates only the essential features of magnetic materials, yet still accurately predicts complex behavior. In fact, the only measured parameter specified in the model is the 1 meter lattice spacing, as determined by snow topography data. The simulated ponds are metastable equilibria of our melt pond Ising model, obtained using a type of Metropolis algorithm. They have geometrical characteristics that agree very closely with observed scaling of pond sizes [6] and the transition in pond fractal dimension [4]. In this connection we also mention the advances made in [1] and [8], where continuum percolation models which reproduce these geometrical features have been developed. In the random surface model [1], a melt pond boundary is the intersection of a surface representing the snow topography with a horizontal plane representing the water level. As the plane rises the ponds grow and coalesce. Snow topography data are used to generate random Fourier surfaces with resulting realistic ponds. In the void model [8], disks of varying size which represent the snow surface are placed randomly on the plane, and the voids between them represent the ponds, with data on pond sizes and correlations incorporated into the model.

With our Ising model approach, we aim to introduce into cryosphere modeling a predictive capability based on ideas of statistical mechanics and energy minimization, and just the essential physics of the system. The model consists of a two dimensional lattice of N square patches or pixels of melt water ($s_i = +1$) or ice ($s_i = -1$), corresponding to the spin up or spin down states in the Ising ferromagnet. Configurations $\omega \in \Omega = \{-1, 1\}^N$ of the spin field s_i represent the distribution of melt water on the sea ice surface. Each spin or patch interacts only with its nearest neighbors, and is influenced by a forcing field. However, for our melt pond Ising model the role of the applied field is played by the surface topography of sea ice, which can vary from site to site and influence whether a patch is water or ice, spin up or spin down. Our melt pond model is actually then a type of *random field* Ising model, and the Hamiltonian can be written as

$$\mathscr{H}_\omega = -\sum_i (H - h_i)s_i - J \sum_{<i,j>} s_i s_j \,,$$

where the h_i are the surface heights, taken to be independent Gaussian variables with mean 0, and H is a reference height, which is taken to be 0 in the simplest form of the model. The basic idea is to reorganize the spin field s_i to lower the free energy, based on local update rules that describe how the "spin" at each site is influenced by its four nearest neighbors and the "forcing" field h_i. The order parameter in the melt pond Ising model is the pond area fraction $F = (M+1)/2$, which is directly related to sea ice albedo, where M is the magnetization of the corresponding Ising ferromagnet. A temperature T can be defined which characterizes the strength of thermal fluctuations, but we set $T = 0$ assuming for simplicity that environmental noise does not significantly influence melt pond geometry.

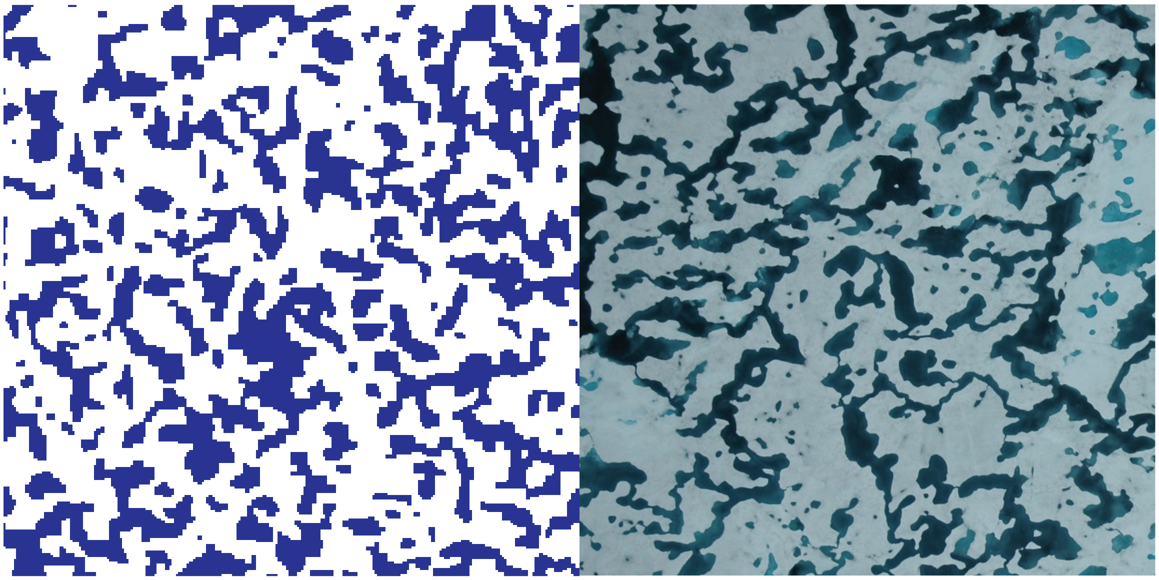

Figure 3: Ising model simulation on the left; melt pond photo on the right (Perovich).

The system is initialized with random configurations determined by independent flips of a weighted coin with probability $p < 0.5$ of spin "up" or melt water pixels at each of the sites of the two dimensional square lattice. The update rules are simple: pick a random site i and update s_i as follows. If there is a majority among its four nearest neighbors, then we assume that heat diffusion drives s_i to agree with this majority. Otherwise, we assume the tendency for water to fill troughs, as determined by the local value of the random field h_i. This update step is iterated until the field s_i becomes steady. The up spin or melt water clusters in the final configurations of the spin field s_i are then found to exhibit geometric characteristics that agree surprisingly well with observations of Arctic melt ponds, as indicated in Figure 3. The reorganization process corresponds to minimization of a free energy via Glauber spin flip dynamics, or the Metropolis algorithm at zero temperature. The final configuration is a metastable state, or local minimum of the Hamiltonian $\mathscr{H}_\omega$. As neighboring sites exchange heat, to minimize energy spins tend to align, that is, causing a coarsening away from the purely random initial state. The emergence of this order from disorder (initially random states) is a central theme in statistical physics, and an attractive feature of our approach.

The ability to efficiently generate realistic pond spatial patterns may enable advances in how melt ponds are handled in global climate models (GCMs). Typical GCM grid spacing is tens to hundreds of kilometers so melt ponds are subgrid-scale, meaning too small to resolve on the model grid. Instead, *parameterizations* are used to specify a pond fraction which then impacts fluxes and other variables resolved on the large-scale grid. To specify the pond fraction, modern parameterizations track a thermodynamically-driven melt water volume and distribute it over the sea ice thickness classes present in a grid cell beginning with the thinnest class assumed to have the lowest ice height [3]. This yields a pond fraction F and hence a first order approximation to sea ice albedo, $\alpha_{sea\ ice} = F\alpha_{water} + (1-F)\alpha_{snow}$, but does not address how the pond area is organized spatially. The simple model presented here provides a framework for prescribing a subgrid-scale spatial organization whose realistic fractal dimension or area-perimeter relation could have important influences on pond evolution via changes in, for example, the efficiency of lateral melt dynamics [7].

At this stage, total agreement between this simple model and the real world is too much to ask for. The Ising model is unable to resolve features smaller than the lattice constant. The metastable state also inherits certain unrealistic features from the purely random initial condition. Nonetheless, by using more sophisticated rules, the model may be able to reproduce actual melt pond evolution. We anticipate that emerging techniques such as machine learning will be able to deduce such evolution rules from observational data.

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