Undergraduate Research Opportunity Programme in Science

Fractal Geometry with Applications

Arshay Nimish Sheth

Supervisor: Assistant Professor Graeme Wilkin

Department of Mathematics
National University of Singapore

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1. Introduction

The concept of fractals has been known for a long time and they have appeared frequently through history, especially in art forms. However, it is only quite recently that they have begun to gather attention from the mathematical community. The study of fractals from a mathematical point of view first began with Benoit Mandelbrot's seminal work entitled *The Fractal Geometry of Nature*, published in 1982. The word 'fractal' was also coined by Mandelbrot from the Latin word *fractus* which means broken.

As mentioned in Falconer's *Fractal Geometry, Mathematical Foundations and Applications* [5], there is no precise mathematical definition of a fractal. Instead, we think of fractals as possessing some common characteristics. The most important of them are:

- Fractals have fine structure, that is detail on arbitrary small scales
- Fractals usually possess some form of self similarity, that is smaller parts of the fractal are geometrically similar to the larger parts up to a scaling factor
- Fractals are usually too irregular to be described using usual Euclidean geometry.

Let us now give an example of a well known fractal called the Sierpinski Triangle. We construct the triangle as follows:

- Start with an equilateral triangle of length 1 and denote it by S_0.
- Join the midpoints of each side of the triangle to form a new triangle inside the bigger triangle
- Remove this newly formed triangle (leaving the boundary) and obtain a new set S_1
- Continue this process for each of the three smaller triangles obtained

Hence we get a sequence of sets S_k such that $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$. Define the Sierpinski Triangle to be $S = \cap_{k \in \mathbb{N}} S_k$. The following picture depicts the above process:

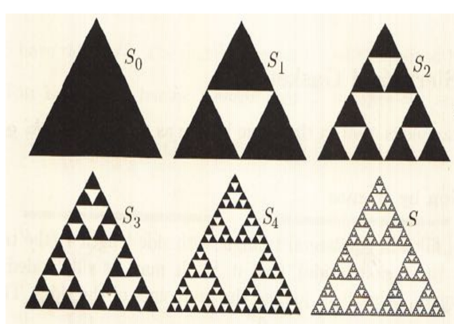

Figure 1. Iterations of the Sierpinski triangle along with the final fractal S. (Source: [4])

Let L_k and A_k denote the length and area respectively of S_k, the set formed in the kth stage in the construction. At the k th stage of the construction we have 3^k equilateral triangles each of length 2^{-k}. Hence the $L_k = 3^k \cdot 3 \cdot 2^{-k} = 3 \cdot (\frac{3}{2})^k$ and so $L_k \to \infty$ as $k \to \infty$. From this we can conclude that length may not be a suitable parameter to measure the size of the Sierpinski Triangle. Now similarly notice that using the formula for the area of an equilateral triangle, $A_k = 3^k \cdot (\frac{\sqrt{3}}{4} \cdot (2^{-k})^2) = \frac{\sqrt{3}}{4} \cdot (\frac{3}{4})^k$ and so $A_k \to 0$ as $k \to \infty$. From this we can conclude that area also may not be a suitable parameter to measure its size.

In summary, we have seen that the length of the triangle is infinite but its area is zero; neither length nor area provides a useful description of S [5]. So in some sense, the dimension of the triangle should be greater than 1 but at the same time it should be less than 2. The questions arise: What should be the dimension of the Sierpinski triangle? How should we assign the dimension? We will address this question in Section 6 of the paper, where we will see that the dimension of the Sierpinski triangle is $\frac{\log 3}{\log 2} \approx 1.58$.

This paper aims to study the geometry of such fractals through the lens of the notion of dimension. In the first part of the paper, we shall introduce some basic terminology and results which will be useful for our purposes later. Equipped with these, we will then define the notion of a Hausdorff measure and subsequently the notion of Hausdorff dimension. We will also study some useful properties of these notions. We will then introduce another different and computationally more useful dimension called the box-counting dimension. Then finally we will answer the question we have raised here; that is we will compute the fractal dimension of the Sierpinski triangle in a mathematically rigorous manner and also compute the dimension of some other well known-fractals.

Fractals have many applications both within mathematics and also in other disciplines such as engineering, geography and physics. In this paper we will discuss a well-known geometric problem called the Kakeya's needle problem, whose solution involves fractals. We will close by taking a look at the application of fractals in geomorphology; in particular we shall investigate the fractal properties of river networks and how understanding this helps us in studying and predicting floods.

2. Preliminaries

We need to develop a few of the very basic concepts of metric topology and measure theory since they are essential for understanding material in the later sections of this paper. We will begin with many definitions and examples. We will also state some properties related to these definitions that will be useful later.

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be points in the Euclidean n space $\mathbb{R}^n$. The distance between x and y is defined to be $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$

Definition 2.2. An open ball in $\mathbb{R}^n$ centered at x with radius r is defined to be $B(x; r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$. A closed ball in $\mathbb{R}^n$ centered at x with radius r is defined to be $\overline{B(x; r)} = \{y \in \mathbb{R}^n : d(x, y) \leq r\}$.

In $\mathbb{R}$ for example, open and closed balls are the usual open and closed intervals respectively.

Definition 2.3. Let $A \subseteq \mathbb{R}^n$. The **diameter of A** denoted by $|A|$ is defined to be $|A| = \sup\{d(x, y) : x, y \in A\}$, $d(x, y)$ being the distance function in the Euclidean $n-$space as defined in Definition 2.1.

For example, in $\mathbb{R}$, the sets $[0, 1]$ and $[0, 1)$ both have diameter 1. In $\mathbb{R}^2$, the diameter of a circle is indeed the usual diameter.

Definition 2.4. A point $x \in A \subseteq \mathbb{R}^n$ is called a **boundary point** if for any $\epsilon > 0, B(x; \epsilon)$ contains a point in A and a point not in A. Denote the set of boundary points in A by ∂A. The **closure of A** denoted by $\overline{A}$ is defined as $\overline{A} = A \cup \partial A$.

It is known that $\overline{A}$ is the smallest closed set containing A; see for instance *Measure and Integral: An Introduction to Real Analysis* (Wheeden and Zygmund) [18].

Definition 2.5. A collection $\mathcal{F}$ of subsets of X is called a σ **algebra on X** if

- $\emptyset, X \in \mathcal{F}$
- If $A \in \mathcal{F}, X \setminus A \in \mathcal{F}$
- If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup\limits_{n \in \mathbb{N}} A_n \in \mathcal{F}$

For example, the power set $\mathcal{P}(X)$ is a σ algebra on X. Note that since $A \cap B = X \setminus (X \setminus A \cup X \setminus B)$, a σ algebra is also closed under countable intersection and since $A \setminus B = A \cap (X \setminus B)$, a σ algebra is closed under countable complement as well. Intuitively a σ algebra is a subset of $\mathcal{P}(X)$ that is "rich" enough to be closed under set-theoretic operations.

Definition 2.6. Let X be a set and $\mathcal{F}$ be a σ algebra on X. A **measure** μ on $\mathcal{F}$ is a function $\mu : \mathcal{F} \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$
- **Countable additivity**: If $\{A_n\} \in \mathcal{F}$ is a sequence of disjoint sets, then $\mu(\bigcup\limits_{n \in \mathbb{N}} A_n) = \sum\limits_{n=1}^{\infty} \mu(A_n)$

A measure is intuitively a function which captures the notion of the "size" of a set. For each set $F \in \mathcal{F}$, $\mu(F)$ is real number reflecting the size of the set. It would be appropriate to take note that the σ algebra $\mathcal{F}$ in the above definition is never taken to be $\mathcal{P}(X)$. This is because there indeed exists subsets that we cannot measure; more precisely there are subsets for which the second condition in Definition 2.6 above fails. One famous example of such a set is the Vitali set; for more details of construction of such sets see Kubrusly's *Essentials of Measure Theory* [9]. For the rest of this paper we shall assume that the every set in our desired σ algebra is measurable. Two properties of measures are important for us in this paper. The first is that if $\{A_n\}$ is a collection of arbitrary sets, then $\mu(\bigcup\limits_{n \in \mathbb{N}} A_n) \leq \sum\limits_{n=1}^{\infty} \mu(A_n)$. This property is known as **countable subadditivity**. The second property states that if $E \subseteq F$, then $\mu(E) \leq \mu(F)$. This property is known as **monotonicity**. For proofs of these properties, also refer to [9].

Definition 2.7. Let $X = \mathbb{R}^n$ and let $A \subseteq X$ be set in the σ algebra $\mathcal{F}$ on X (the σ algebra that we use here is called the Lebesgue σ algebra, for details see [9]).Let $A = \{(x_1, \ldots, x_n\} \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$. Define $\text{vol}^n(A) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$. Define a function $\mathcal{L}^n : \mathcal{F} \to [0, \infty]$ by setting $\mathcal{L}^n(A) = \inf\{\sum_{i=1}^{\infty} \text{vol}^n(A_i) : A \subset \cup_{i=1}^{\infty} A_i\}$. This function $\mathcal{L}^n$ is called **the Lebesgue measure on** $\mathbb{R}^n$.

Although the above definition looks a bit daunting, Lebesgue measure is nothing but the generalization of the notions of length, area and volume. The other technical features of Lebesgue measure need not concern us here; for the purposes of this paper it is enough to view them as indicating the length, area or volume of a subset of $\mathbb{R}^n$. It is important to note that for regular sets, the Lebesgue measure and the length, area or volume of these sets indeed coincide. For example, $\mathcal{L}([2, 5]) = 3$ and $\mathcal{L}^2([0, 3] \times [0, 3]) = 9$.

Definition 2.8. If μ is a measure on X and $0 < \mu(X) < \infty$, then the measure μ is called a **mass distribution** on X.

For example, let $X = \mathbb{R}^2$. Define for any A in the σ algebra of measurable sets:

$$\mu(A) = \begin{cases} 1 & \text{if} \quad (0,0) \in A \\ 0 & \quad \text{otherwise} \end{cases}$$

Then by checking the two conditions in Definition 2.6, it is clear that μ is measure. It is also a mass distribution on $\mathbb{R}^2$ since indeed we have $0 < 1 = \mu(\mathbb{R}^2) < \infty$.

Definition 2.9. Let A be a set. A collection of sets $\{U_i\}$ is said to **cover** A if $A \subseteq \bigcup_{i=1}^{\infty} U_i$. Furthermore if $0 \leq |U_i| \leq \delta$ for each i then we say that $\{U_i\}$ is a δ **cover** of A.

For example, the sequence of sets $[\frac{1}{n}, 1]$ cover $[\frac{1}{10}, 1]$.

Definition 2.10. A set $A \subseteq \mathbb{R}^n$ is said to be **compact** if every covering of A by open sets has a finite subcovering.

Two remarks are important to make here. Firstly, it can be shown that a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Secondly, if $\{x_n\}_{n=1}^{\infty}$ is a sequence in a compact set A, then the sequence has a subsequence which converges to a point in A. For proofs of these properties, see Rudin's *Principles of Mathematical Analysis* [14].

3. Hausdorff Measure and its properties

3.1. Definition of Hausdorff measure.

Definition 3.1. Let $\mathcal{H}^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s \mid \{U_i\}$ is a δ cover of F $\}$, where s and δ are positive real numbers. Then define the Hausdorff measure $\mathcal{H}^s(F)$ to be $\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$.

If $\delta_1 < \delta_2$, then not all δ_2 covers are δ_1 covers. Hence as $\delta \to 0$, the number of possible δ covers decreases. More precisely, if $\{\delta_i\}$ is a sequence decreasing to zero then $\{\mathcal{H}^s_{\delta_i}(F)\}$ is a monotone increasing function and so $\{\mathcal{H}^s_{\delta_i}(F)\}$ converges (possibly to $+\infty$). Hence the above limit exists and may equal ∞ [12].
The intuitive meaning of the Hausdorff measure is the idea of covering the target set as accurately as possible.

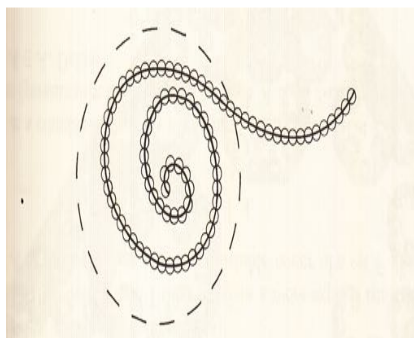

Figure 2. Idea of covering a set: the length of the spiral in the picture is well-estimated by the sum of the diameters of the tiny balls but grossly underestimated by the diameter of the huge ball. (Source: [5])

3.2. Properties of Hausdorff measure. It can be shown be shown by checking Definition 2.6 that the Hausdorff measure is indeed a measure. Since this is not especially relevant for the purposes of the paper, we refer the reader to [12] for a proof.

In this section, we will discuss some properties of Hausdorff measure [5] which will help us to understand the concept better.

The most important property of the Hausdorff measure is the scaling property which will be also used to compute the dimension of fractals.

(1) **Scaling Property:** If $F \subseteq \mathbb{R}^n$ and $\lambda > 0$, then $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a δ cover of F. Then $\{\lambda U_i\}_{i=1}^{\infty}$ is a $\lambda\delta$ cover of λF. Hence

$$\begin{aligned}\mathcal{H}^s_{\lambda\delta}(\lambda F) &\leq \sum_{i=1}^{\infty} |\lambda U_i|^s \qquad \text{(By the property of infimum)}\\ &= \lambda^s \sum_{i=1}^{\infty} |U_i|^s \qquad \text{(By factoring out } \lambda\text{, see Lemma 5.3 for a proof)}\end{aligned}$$

Since these steps hold for any δ cover, they also hold in particular for the δ cover for which the infimum of the sums occurs in Definition 3.1. Hence we get $\mathcal{H}^s_{\lambda\delta}(\lambda F) \leq \lambda^s \mathcal{H}^s_\delta(F)$. Letting $\delta \to 0$, gives $\mathcal{H}^s(\lambda F) \leq \lambda^s \mathcal{H}^s(F)$. To obtain the opposite inequality, replace F by λF and λ by $\frac{1}{\lambda}$ in the inequality we just obtained: we get $\mathcal{H}^s(\frac{1}{\lambda}(\lambda F)) \leq (\frac{1}{\lambda})^s \mathcal{H}^s(\lambda F)$ which is $\lambda^s \mathcal{H}^s(F) \leq \mathcal{H}^s(\lambda F)$. ☐

(2) **Meaning of Hausdorff measure for different values of s:**. When $s = 0$, $\mathcal{H}^0(F)$ is the counting measure: if a set F is finite and has n elements then $\mathcal{H}^0(F) = n$ and if F is infinite then $\mathcal{H}^0(F) = \infty$.

Proof. First pick any $a \in F$ and consider the singleton set {a}. Clearly at least one set would be required to cover $\{a\}$. Hence $\mathcal{H}^0_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^0 \mid \{U_i\} \text{ is a } \delta \text{ cover of } F\ \} = 1$ and so $\mathcal{H}^s(\{a\}) = 1$. Hence by the countable additivity property of measures, $\mathcal{H}^0(F) = n$ if F is finite and has n elements and $\mathcal{H}^0(F) = \infty$ otherwise. ☐

In general, it is true that $\mathcal{H}^s(F) = c_s \mathcal{L}^s(F)$ where $\mathcal{L}^s(F)$ is the n-dimensional Lebesgue measure and c_s is a scaling constant. The proof of this result is quite technical and the reader is encouraged to consult Edgar's *Measure, Topology, and Fractal Geometry* [4] for a proof. However, it is important to note from this formula that since for $s = 1, 2, 3$ the Lebesgue measure equals the length, area and volume respectively, the Hausdorff measure also equals the length, area and volume up to a scaling constant.

(3) **Countable sets:** If F is a countable set, $\mathcal{H}^s(F) = 0$ for any $s > 0$.

Proof. Since F is countable, we can enumerate the elements of F as $x_1, x_2, \ldots$. Pick any $\delta > 0$. Let $U_i = B(x_i; \frac{\delta}{2^i})$ for $i \geq 1$. This just means that we enclose each $x_i \in F$ by a ball of radius $\frac{\delta}{2^i}$ with centre x_i. Hence the diameter of each set U_i equals $\frac{\delta}{2^{i-1}}$ and so $\{U_i\}$ is indeed a δ cover of F. Hence

$$\begin{aligned}\mathcal{H}^s_\delta(F) &\le \delta^s + (\frac{\delta}{2})^s + (\frac{\delta}{4})^s + \cdots && \text{(By property of infimum)}\\ &= \delta^s \cdot (1 + \frac{1}{2^s} + \frac{1}{4^s} + \cdots)\\ &= \delta^s \cdot \frac{1}{1-\frac{1}{2^s}} \to 0 \text{ as } \delta \to 0\end{aligned}$$

So we get $\mathcal{H}^s(F) \le 0$ and since measures take on non-negative values, $\mathcal{H}^s(F) = 0$. □

(4) **Mass distribution principle:** If μ is a mass distribution on F and suppose that for some $s > 0, \delta > 0$ we have $\mu(U) \le c \cdot |U|^s$ for some constant c and all sets U with $|U| \le \delta$. Then $\frac{\mu(F)}{c} \le \mathcal{H}^s(F)$.

Proof. Note that if $\{U_i\}_{i=1}^\infty$ is a sequence of δ covers of F we have

$$\begin{aligned}0 &< \mu(F) && \text{(Since } \mu \text{ is a mass distribution)}\\ &\le \mu(\bigcup_{i=1}^\infty U_i) && \text{(By monotonicity of measures)}\\ &\le \sum_{i=1}^\infty \mu(U_i) && \text{(By the countable subadditivity property of measures)}\\ &\le c\sum_{i=1}^\infty |U_i|^s && \text{(By assumption)}\end{aligned}$$

Again we observe that since these steps hold for any δ cover, they also hold for the δ cover for which the infimum of the sums occur in Definition 3.1. Using this observation, we have $\mu(F) \le c \cdot \mathcal{H}^s_\delta(F)$ and finally letting $\delta \to 0$ yields $\frac{\mu(F)}{c} \le \mathcal{H}^s(F)$. □

A natural question to ask at this point is: why is a mass distribution necessary, why not any ordinary measure? This is because since we use a mass distribution, $\mu(F)$ is non-zero and finite and so we indeed get a positive finite bound on the Hausdorff measure from below; this may not be the case with arbitrary measures. In Section 5, we shall see that this property of bounding the Hausdorff measures from below will be very useful.

3.3. **Hausdorff Dimension.** The following theorem will serve as a motivation for the introduction of the notion of Hausdorff Dimension.

Theorem 3.1. If $\mathcal{H}^p(A) < \infty$ then $\mathcal{H}^q(A) = 0$ for all $q > p$. Also if $\mathcal{H}^p(A) > 0$ then $\mathcal{H}^q(A) = \infty$ for all $q < p$. Hence if $\mathcal{H}^p(A)$ is positive and finite, then $\mathcal{H}^q(A) = 0$ for all $q > p$ and $\mathcal{H}^q(A) = \infty$ for all $q < p$.

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a δ cover of A and suppose $q > p$.
Then

$$\begin{aligned}\sum_{i=1}^{\infty} |U_i|^q &= \sum_{i=1}^{\infty} |U_i|^{q-p+p} \\ &= \sum_{i=1}^{\infty} |U_i|^{q-p} |U_i|^p \\ &\leq \delta^{q-p} \sum_{i=1}^{\infty} |U_i|^p.\end{aligned}$$

Since this holds for any δ cover of A,

$$\mathcal{H}_\delta^q(A) \leq \delta^{q-p} \mathcal{H}_\delta^p(A)$$

Taking $\delta \to 0$ gives, $\mathcal{H}^q(A) \leq 0$ which implies that $\mathcal{H}^q(A) = 0$ since measures take on non-negative values.

The other case when $p > q$ is similar:

$$\begin{aligned}\sum_{i=1}^{\infty} |U_i|^p &= \sum_{i=1}^{\infty} |U_i|^{p-q+q} \\ &= \sum_{i=1}^{\infty} |U_i|^{p-q} |U_i|^q \\ &\leq \delta^{p-q} \sum_{i=1}^{\infty} |U_i|^q.\end{aligned}$$

Since this holds for any δ cover of A,

$$\mathcal{H}_\delta^p(A) \leq \delta^{p-q} \mathcal{H}_\delta^q(A)$$

.

This is equivalent to $\frac{\mathcal{H}_\delta^p(A)}{\delta^{p-q}} \leq \mathcal{H}_\delta^q(A)$ and so taking $\delta \to 0$ gives $\infty \leq \mathcal{H}^q(A)$ which means $\mathcal{H}^q(A) = \infty$.

$\square$

The above theorem can also be pictorially as a graph. (See Figure 3).

The theorem also naturally gives rise to this definition:

Definition 3.2. The Hausdorff dimension $\dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$

Corollary 3.1.1. If $\mathcal{H}^s(F) = d$, where $0 < d < \infty$ then $\dim_H(F) = s$

Proof. By the previous theorem, $\mathcal{H}^q(F) = \infty$ for all $q < s$ and $\mathcal{H}^q(F) = 0$ for all $q > s$. Hence $\dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\} = s$. $\square$

Let us state a few properties of the Hausdorff dimension.

(1) If F is a countable set, $\dim_H(F) = 0$.

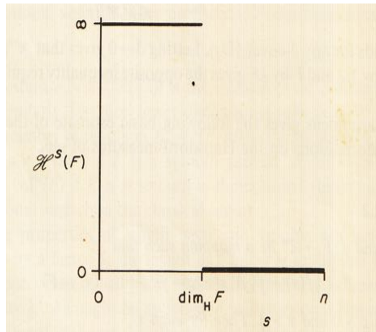

FIGURE 3. Graph of $\mathcal{H}^s(F)$ against s. $\dim_H(F)$ is the value of s for which the jump from ∞ to 0 occurs. (Source: [5])

Proof. We proved before that if F is a countable set then for any $s > 0$, $\mathcal{H}^s(F) = 0$. By definition, $\dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\}$, and so we have $\dim_H(F) = 0$. □

(2) Monotonicity: If $E \subseteq F$, then $\dim_H(E) \leq \dim_H(F)$

Proof. We prove by taking cases.
Case 1: $\dim_H(F) = \infty$. In this case, the inequality is trivial.
Case 2: $\dim_H(F) = k < \infty$. Then by Theorem 3.1, for all $c > k$, $\mathcal{H}^s(F) = 0$ and so by hypothesis $\mathcal{H}^s(E) \leq 0$ also and since measures taken on non-negative values, $\mathcal{H}^s(E) = 0$. Then by Definition 3.2, $\dim_H(E) \leq c$ and since this holds for all $c > k = \dim_H(F), \dim_H(E) \leq k$.

□

(3) $\dim_H(F)$ is countably stable. That is, if $\{F_n\}$ is a countable sequence of sets, then $\dim_H \bigcup\limits_{n\in\mathbb{N}} F_n = \sup_{n\in\mathbb{N}}\{\dim_H F_n\}$

Proof. For all $j \in \mathbb{N}$, $\dim_H(F_j) \leq \dim_H \bigcup\limits_{n\in\mathbb{N}} F_n$ by the monotonicity property and so $\sup_{n\in\mathbb{N}}\{\dim_H F_n\} \leq \dim_H \bigcup\limits_{n\in\mathbb{N}} F_n$. We prove the other inequality by taking cases.
Case 1: Suppose there exists $s \in \mathbb{R}$ with $s > 0$ such that for all $j \in \mathbb{N}$, $\dim_H(F_j) < s$. Then by theorem 3.1, for all $j \in \mathbb{N}$ we have $\mathcal{H}^s(F_j) = 0$. By the countable subadditivity property of measures $\mathcal{H}^s(\bigcup\limits_{n\in\mathbb{N}} F_n) \leq \sum\limits_{n=1}^{\infty} \mathcal{H}^s(F_j) = 0$ and so, $\mathcal{H}^s(\bigcup\limits_{n\in\mathbb{N}} F_n) = 0$. Hence $\dim_H(\bigcup\limits_{n\in\mathbb{N}} F_n) \leq s$ by definition of Hausdorff dimension. This is true for all $s > \dim_H(F_j)$ for any $j \in \mathbb{N}$. Hence $\dim_H \bigcup\limits_{n\in\mathbb{N}} F_n \leq \sup_{n\in\mathbb{N}}\{\dim_H F_n\}$

Case 2: Suppose all $s \in \mathbb{R}$ there exists $j \in \mathbb{N}$ such that $\dim_H(F_j) \geq s$. Then $\dim_H(F_j) = \infty$ in which case the inequality is trivial. $\square$

4. Box counting dimension

In this section, we will encounter another dimension for computing the dimension of fractals. This method is known as the box counting dimension and is advantageous because it is much easier to compute than the Hausdorff dimension. However, as we shall see later, it also has some disadvantages as compared to Hausdorff dimension. First let us proceed to motivate the definition of the box counting dimension [3].

Consider a line segment. If it is scaled down by a factor of $\frac{1}{2}$, the combining two of these scaled down copies will give back the original line segment. Instead if we consider a square and scale it down by a factor of $\frac{1}{2}$ then we would need to combine four of these copies to get back the original square. Finally, if we perform the same operation on a cube, we would need 8 of the scaled copies to recover the original cube. If we let N denote the number of copies we must glue back together, d denote the dimension of the set and δ denote the scaling factor where $0 < \delta < 1$, then it seems from our discussion that $N = (\frac{1}{\delta})^d$. Taking log on both sides yields $d = \frac{\log N}{\log \frac{1}{\delta}}$. The following definition formalizes our argument below with the idea of letting δ tend very close to zero to get a precise answer for dimension.

Definition 4.1. Let $N_\delta(F)$ be the smallest number of sets of diameter atmost δ required to cover a set F. Then the box counting dimension is defined to be $\dim_B F = \lim\limits_{\delta\to 0^+} \frac{\log N_\delta(F)}{\log \frac{1}{\delta}}$

Remark. In some cases, the above defined limit may not exist. In that case, we define the upper dimension box counting dimension and lower box counting dimension respectively to be $\overline{\dim}_B F = \limsup_{\delta\to 0^+} \frac{\log N_\delta(F)}{\log \frac{1}{\delta}}$ and $\underline{\dim}_B F = \liminf_{\delta\to 0^+} \frac{\log N_\delta(F)}{\log \frac{1}{\delta}}$. We always have $\underline{\dim}_B F \leq \overline{\dim}_B F$.

Remark. There are many equivalent definitions of the Box counting dimension, see for example [5]. One particular variant that will be useful to us is: instead of taking $N_\delta(F)$ to be the smallest number of sets of diameter atmost δ that cover F, we could let $N_\delta(F)$ to be the smallest number of closed balls of radius atmost δ that cover F.

Let us compute the box-counting dimension of the unit interval $[0,1]$

Example 4.1. $\dim_B([0,1]) = 1$

Proof. Pick any δ with $0 < \delta < 1$. Let $U_i = [i\delta, (i+1)\delta]$, where $i \in \mathbb{N}$ and $0 \leq i \leq \lfloor \frac{1}{\delta} \rfloor + 1$. Then $\{U_i\}_{i=1}^{i=\lfloor \frac{1}{\delta} \rfloor + 1}$ is a δ cover of $[0,1]$. Since $N_\delta([0,1])$ is the smallest number of sets of diameter atmost δ that cover $[0,1]$, $N_\delta([0,1]) \leq \lfloor \frac{1}{\delta} \rfloor + 1 \leq \frac{1}{\delta} + 1 \leq \frac{2}{\delta}$. Also since we are looking for δ covers, there must be at least $\frac{1}{\delta}$ covers. That is, $\frac{1}{\delta} \leq N_\delta$. In summary we get $\frac{1}{\delta} \leq N_\delta([0,1]) \leq \frac{2}{\delta}$. Taking log on both sides yields $\log(\frac{1}{\delta}) \leq \log(N_\delta([0,1])) \leq \log 2 + \log(\frac{1}{\delta})$. Since $\delta < 1, \log(\frac{1}{\delta}) > 0$ and so multiplying each side by

$\log(\frac{1}{\delta})$ we have $\frac{\log(\frac{1}{\delta})}{\log(\frac{1}{\delta})} \leq \frac{\log N_\delta([0,1])}{\log(\frac{1}{\delta})} \leq \frac{\log 2+\log(\frac{1}{\delta})}{\log(\frac{1}{\delta})}$. Now $\lim_{\delta\to 0^+} \frac{\log(\frac{1}{\delta})}{\log(\frac{1}{\delta})} = 1$ and $\lim_{\delta\to 0^+} \frac{\log 2+\log(\frac{1}{\delta})}{\log(\frac{1}{\delta})} = 1$ by L'Hospital's rule and so by the squeeze theorem for limits $\lim_{\delta\to 0^+} \frac{\log N_\delta([0,1])}{\log(\frac{1}{\delta})} = 1$. □

Let us now present an example where the box- counting dimension and the Hausdorff dimension do not agree. We know that the Hausdorff dimension of any countable set is zero; in particular, the Hausdorff dimension of the set of rational numbers in $[0,1]$ is zero. However, the next example shows that box- counting dimension of this set actually turns out to be 1.

Example 4.2. The box counting dimension of the set of rational numbers in $[0,1]$ is 1.

Proof. First let us show the following general claim: for any set V, $\dim_B \overline{V} = \dim_B V$. By the previous remark, it is sufficient if we consider our covering sets to be closed balls. If $\overline{B_1}, \overline{B_2}, \ldots, \overline{B_k}$ are any collection of closed balls that cover V, they also cover $\overline{V}$, since $\overline{V}$ is the smallest closed set that contains V (see Section 2) and so $\dim_B \overline{V} = \dim_B V$ from the definition of box counting dimension.
Let Q_1 denote the rational numbers in $[0,1]$. Then $\overline{Q_1} = [0,1]$ by the density of rational and irrational numbers in $\mathbb{R}$. Hence by the above claim, $\dim_B(Q_1) = \dim_B([0,1])$ and so $\dim_B(Q_1) = 1$. □

Notice from above also that the countable stability does not hold for the Box-counting dimension as it does for the Hausdorff dimension. This is because the Box-counting dimension of each individual rational point x in $[0,1]$ is 0 (since $N_\delta(\{x\}) = 1$ for a singleton set) but the countable union of these singleton sets (namely the set of rational numbers in $[0,1]$) has box counting dimension 1. Hence it is not true in general that countable stability holds for box counting dimension, as it does for the Hausdorff dimension.

At this stage, it is quite natural to wonder about the relationship between the Hausdorff dimension and the Box counting dimension. The following theorem [5] answers our question:

Theorem 4.1. $\dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F)$

Proof. Pick s so that $1 < \mathcal{H}^s(F)$. Then from Figure 3 and Theorem 3.1 it is clear that $\dim_H(F) \leq s$. Suppose that F can be covered by $N_\delta(F)$ sets of diameter δ. Then by definition 3.1, $\mathcal{H}^s_\delta(F) \leq N_\delta(F)\delta^s$. Now also by definition 3.1, $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$. Hence for any $\epsilon > 0$, there exists a δ neighbourhood such that, $-\epsilon < \mathcal{H}^s_\delta(F) - \mathcal{H}^s(F) < \epsilon$. This means in particular that $\mathcal{H}^s(F) < \mathcal{H}^s_\delta(F)+\epsilon$ and since ϵ is arbitrary $\mathcal{H}^s(F) \leq \mathcal{H}^s_\delta(F)$ in the δ neighbourhood. So combining this inequality with $1 < \mathcal{H}^s(F)$ and with $\mathcal{H}^s_\delta(F) \leq N_\delta(F)\delta^s$ we get $1 < N_\delta(F)\delta^s$ and hence $0 < \log(N_\delta(F)) + s\log\delta$.

This implies that $s < \frac{\log(N_\delta(F))}{\log(\frac{1}{\delta})}$ and so $s \leq \underline{\dim}_B(F)$. Hence $\dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F)$. $\square$

5. The Open Set Condition

By using the tools we have developed so far, we will be able solve our initial goal of computing the dimension of the Sierpinski Triangle. However the proof below is still not completely rigorous as we shall explain after we present the proof. Indeed the goal of this section is to develop the tools that will rigorously enable us to compute the dimension of fractals.

Theorem 5.1. The Hausdorff dimension of the Sierpinski Triangle is $\frac{\log 3}{\log 2}$.

Heuristic proof. Let S denote the Sierpinski Triangle. From Figure 1, it is clear that the Sierpinski Triangle contains three copies of itself, each scaled by a factor of $\frac{1}{2}$. Let R_1 R_2 and R_3 denote these three copies of the Sierpinski Triangle. Also from Figure 1 note that R_1,R_2 and R_3 are disjoint sets and that $S = R_1 \cup R_2 \cup R_3$. By using the countable additivity property of measures, we get $\mathcal{H}^s(S) = \mathcal{H}^s(R_1) + \mathcal{H}^s(R_2) + \mathcal{H}^s(R_3)$. Now using the scaling property of Hausdorff measures we proved in Section 3 with scaling factor $\lambda = \frac{1}{2}$ we get $\mathcal{H}^s(S) = (\frac{1}{2})^s \cdot \mathcal{H}^s(S) + (\frac{1}{2})^s \cdot \mathcal{H}^s(S) + (\frac{1}{2})^s \cdot \mathcal{H}^s(S)$. Dividing on both sides by $\mathcal{H}^s(S)$ we get $1 = (\frac{1}{2})^s + (\frac{1}{2})^s + (\frac{1}{2})^s$ and finally solving for s yields $s = \frac{\log 3}{\log 2}$.

$\square$

.

Notice that in the above proof we make the *assumption* that there exists a critical value s such that $\mathcal{H}^s(S)$ is positive and finite. It is this assumption that lets us divide $\mathcal{H}^s(S)$ on both sides. In other words, we do not consider the case when for all $s \geq 0$, $\mathcal{H}^s(S)$ is either zero or ∞ in which case $\dim_H(S) = 0$ or ∞ respectively. We will now prove a theorem which, if fulfilled, will make the preceding heuristic argument rigorous. The proof of this theorem is adapted from [5] and [11] and is quite long and intricate. We will break down the proof into many intermediate definitions and lemmas in order to simplify the proof.

Definition 5.1. Let $D \subseteq \mathbb{R}^n$. A map from $S : D \to D$ is called a **contraction** on D if all $x, y \in D, d(S(x), S(y)) \leq c \cdot d(x, y)$ where $0 < c < 1$. If $d(S(x), S(y)) = c \cdot d(x, y)$, then S is called a **similarity**.

Definition 5.2. Let $S_1, \ldots, S_m$ be contractions. Then F is called an invariant set if $F = \bigcup_{i=1}^{m} S_i(F)$.

For example, if C is the Cantor Set and we let $S_1(x) = \frac{x}{3}$ and $S_2(x) = \frac{x}{3} + \frac{2}{3}$, then $C = S_1(C) \cup S_2(C)$.

Definition 5.3. We say that similarities S_i where $1 \leq i \leq m$ satisfy the **open set condition** if there exists a non-empty, open and bounded set V such that $\cup_{i=1}^{m} S_i(V) \subset V$ where the union is disjoint.

We can now state our goal theorem mentioned earlier that would help in making our heuristic proof rigorous.

Theorem 5.2. Suppose the open set condition holds for similarities S_i on $\mathbb{R}^n$ with similarity ratios c_i where $1 \leq i \leq m$. Suppose also that a set F is invariant under these similarities: $F = \cup_{i=1}^m S_i(F)$. Then $\dim_H(F) = s$ where s satisfies $\sum_{i=1}^m c_i^s = 1$. Moreover, we have that $0 < \dim_H(F) < \infty$.

All through this section, we will assume all the hypotheses mentioned in Theorem 5.2 .We will begin the proof of the theorem by introducing the following notation.

Notation. Let $A_{i_1,i_2,\ldots,i_k} = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}(A)$.

Notation. Let J_k denote the set of all k-term sequences $(i_1, i_2, \ldots, i_k)$, where $1 \leq i_j \leq m$ and $1 \leq j \leq k$. Note that J_k can be thought as an indexing set and also note that repetitions are allowed in any sequence in J_k.

Lemma 5.3. If S is a similarity function with ratio c and $A \subseteq \mathbb{R}^n$ then $|S(A)| = c|A|$

Proof.

$$\begin{aligned}
|S(A)| &= \sup\{d(S(x), S(y)) : x, y \in A\} \\
&= \sup\{c \cdot d(x, y) : x, y \in A\} \\
&= c \cdot \sup\{d(x, y) : x, y \in A\} \\
&= c \cdot |A|
\end{aligned}$$

□

Lemma 5.4. If S_1 and S_2 are similarity functions with ratios c_1 and c_2 then their composition $S_1 \circ S_2$ is also a similarity function with ratio $c_1 \cdot c_2$.

Proof.

$$\begin{aligned}
d(S_1 \circ S_2(x), S_1 \circ S_2(y)) &= d(S_1(S_2(x)), S_1(S_2(y))) \\
&= c_1 \cdot d(S_2(x), S_2(y)) && \text{(Since } S_1 \text{ is a similarity with ratio } c_1\text{)} \\
&= c_1 \cdot (c_2 \cdot d(x, y)) && \text{(Since } S_2 \text{ is a similarity with ratio } c_2\text{)} \\
&= (c_1 \cdot c_2)(d(x, y))
\end{aligned}$$

□

Lemma 5.5. For any function f and any sets A, B, $f(A \cup B) = f(A) \cup f(B)$.

Proof. Pick $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$. This means that $x \in A$ or $x \in B$; without loss of generality assume that $x \in A$. Then $y = f(x) \in f(A)$ and so $f(A \cup B) \subseteq f(A) \cup f(B)$. Now pick any $y \in f(A) \cup f(B)$. Without loss of generality assume that $y \in f(A)$. Then $\exists x \in A$ such that $f(x) = y$. Hence $x \in A \cup B$ also with $f(x) = y$. Hence $y \in f(A \cup B)$ and so $f(A) \cup f(B) \subseteq f(A \cup B)$. □

This next theorem can be regarded as the first part of the proof of Theorem 5.2. It gives an upper bound for the Hausdorff measure and hence shows that the measure is finite.

Theorem 5.6. If F is an invariant set satisfying $F = \bigcup_{i=1}^{m} S_i(F)$, then $\mathcal{H}^s(F) \leq |F|^s$.

Proof.

$$
\begin{aligned}
F &= \bigcup_{i_1=1}^{m} S_{i_1}(F) \\
&= \bigcup_{i_1=1}^{m} S_{i_1}(\bigcup_{i_2=1}^{m} S_{i_2}(\cdots \bigcup_{i_k=1}^{m} S_{i_k}(F))) && \text{(By repeated application of the above equality)} \\
&= \bigcup_{i_1=1}^{m} \bigcup_{i_2=1}^{m} \cdots \bigcup_{i_k=1}^{m} S_{i_1} \circ S_{i_2} \cdots S_{i_k}(F) && \text{(By repeated application of Lemma 5.5)} \\
&= \bigcup_{J_k} S_{i_1} \circ S_{i_2} \cdots S_{i_k}(F) && \text{(By definition of } J_k \text{ as an indexing set)} \\
&= \bigcup_{J_k} F_{i_1,i_2,\ldots,i_k} && \text{(By the notation adopted)}
\end{aligned}
$$

Hence we see that $\bigcup_{J_k} F_{i_1,i_2,\ldots,i_k}$ is a cover of F. Recall that to calculate $\mathcal{H}^s_\delta(F)$, we need to compute the diameter of each set in the cover raised to the power of s and sum these resulting values. To this end notice that

$$
\begin{aligned}
\sum_{J_k} |F_{i_1,i_2,\ldots,i_k}|^s &= \sum_{J_k} |S_{i_1} \circ S_{i_2} \circ \cdots S_{i_k}(F)|^s \\
&= \sum_{J_k} |D(F)|^s && \text{(where by Lemma 5.4 } D = S_{i_1} \circ \cdots \circ S_{i_k} \text{ with ratio } c_1 c_2 \cdots c_k \text{)} \\
&= \sum_{J_k} ((c_1 \cdot c_2 \cdots c_k)|F|)^s && \text{(By Lemma 5.3)} \\
&= \sum_{J_k} (c_1 \cdot c_2 \cdots c_k)^s |F|^s \\
&= (\sum_{i_1=1}^{m} c_{i_1}^s) \cdots (\sum_{i_k=1}^{m} c_{i_k}^s)|F|^s \\
&= |F|^s && \text{(By assumption in Theorem 5.2, each of these sums equal 1)}
\end{aligned}
$$

Pick any $\delta > 0$. We have to show that this cover is a δ cover. We will do this by choosing k large enough. Let $d = \max\{c_i : 1 \leq i \leq m\}$. Since for every i, $0 < c_i < 1$, $0 < d < 1$ also. Hence $\lim_{k\to\infty} d^k = 0$ and so we can choose k large enough so that $d^k < \frac{\delta}{|F|}$. Now note that

$$\begin{aligned} |F_{i_1,i_2,\ldots i_k}| &= c_{i_1}c_{i_2}\cdots c_{i_k}|F| && \text{(By the above argument)} \\ &\leq d^k|F| \\ &< \frac{\delta}{|F|}|F| \\ &= \delta \end{aligned}$$

Hence by choosing k large enough, $\bigcup\limits_{J_k} F_{i_1,i_2,\ldots,i_k}$ is a δ cover of F. Finally by the definition of Hausdorff measure, $\mathcal{H}^s_\delta(F) \leq |F|^s$ and then by letting $\delta \to 0$ we have $\mathcal{H}^s(F) \leq |F|^s$ $\square$

The next part of the proof will consist of bounding the Hausdorff measure from below. First, we will need the following lemma.

Lemma 5.7. Let $\{W_i\}$ be a collection of disjoint open sets in $\mathbb{R}^n$ such that each W_i contains a ball of radius $a_1 r$ and is contained within a ball of radius $a_2 r$. Let $\{\overline{W_i}\}$ denote the corresponding set of closures. Consider any ball B of radius r and let m denote the number of sets in $\{\overline{W_i}\}$ that intersect B. Then $m \leq (\frac{1+2a_2}{a_1})^n$.

Proof. Suppose $\overline{W}_i$ intersects B. Then $\overline{W}_i$ is contained in a ball concentric with B of radius $\leq r + 2|W_i| \leq r + 2a_2 r = (1+2a_2)r$. Since m sets of $\{\overline{W}_i\}$ intersect B and since each of these contain a ball of radius $a_1 r$ by hypothesis, we can sum the volumes on both sides to obtain $m(a_1 r)^n \leq (1+2a_2)^n r^n$ giving the desired equality for m. $\square$

Notation. Let $I = \{(i_1, i_2, \cdots) : 1 \leq i_j \leq m\}$ be the set consisting of all infinite sequences where each term in the infinite sequence is between 1 and m. For example if we take F to be the Cantor Set, $(1,2,1,1,2,\cdots) \in I$.
Notation. Let $I_{i_1,i_2,\cdots,i_k} = \{(i_1, i_2, \cdots, i_k, q_{k+1}, \cdots) : 1 \leq q_j \leq m\}$ be the set consisting of all infinite sequences whose first k terms are $i_1, i_2, ..., i_k$. Again if we take F to be the Cantor Set, $(1,2,2,1,1,1,1,\cdots) \in I_{1,2,2}$ and $(1,2,2,2,1,2,1\cdots) \in I_{1,2,2}$ also.
Since we want to bound the Hausdorff measure from below, we want to make use of the mass distribution principle. The idea behind doing this is to define a mass distribution on I and then "transfer" it to F. Hence first define $\mu(I_{i_1,i_2,\cdots,i_k}) = (c_{i_1} \cdot c_{i_2} \cdots c_{i_k})^s$ for subsets $I_{i_1,i_2,\cdots,i_k}$ of I.

Lemma 5.8. μ as defined above is a mass distribution on I

Proof. Note that $I = I_1 \cup I_2 \cup \cdots \cup I_m$ where the union is disjoint. By the countable additivity property of measures,

$$\begin{aligned}\mu(I) &= \mu(I_1) + \mu(I_2) + \cdots + \mu(I_m) \\ &= c_1^s + c_2^s + \cdots + c_m^s && \text{(By definition of the measure } \mu) \\ &= 1 && \text{(By assumption in Theorem 5.2)}\end{aligned}$$

Hence indeed we have $0 < \mu(I) = 1 < \infty$ and so μ is a mass distribution on I. □

Now we "transfer" the mass distribution on F by defining $\mu'(A) = \mu\{(i_1, i_2, \cdots) : x_{i_1,i_2,\ldots} \in A\}$ for subsets A of F. It can be shown by a similar argument, that μ' is a mass distribution with $\mu'(F) = 1$.

Now consider any ball B of radius r where $r < 1$. The ball B will play the role of the set U and the number 2 will play the role of δ in the Mass-distribution principle mentioned in Section 3.

Lemma 5.9. For each sequence $(i_1, i_2, \ldots) \in I$, there is a term i_k such that $(\min c_i)r \le c_{i_1}c_{i_2}\cdots c_{i_k} \le r$.

Proof. Since for each i, $c_i < 1$, there exists a $k \in \mathbb{N}$ such that $c_{i_1}c_{i_2}\cdots c_{i_k} \le r < c_{i_1}c_{i_2}\cdots c_{i_{k-1}}$. Now considering this second inequality, we have $r \min c_i \le c_{i_1}c_{i_2}\cdots c_{i_{k-1}}c_{i_k}$. Combining these results yield $(\min c_i)r \le c_{i_1}c_{i_2}\cdots c_{i_k} \le r$. □

Curtail each sequence $(i_1, i_2, \ldots,) \in I$ to a finite sequence $(i_1, i_2, \ldots, i_k)$ such that $(\min c_i)r \le c_{i_1}c_{i_2}\cdots c_{i_k} \le r$. By the previous lemma, such an i_k must exist. Let Q denote the set of all such finite sequences curtailed in this way.

Let V be the set satisfying the open set condition. Suppose that V contains a ball of radius a_1 and is contained within a ball of radius a_2. Then by a calculation in Theorem 5.6, $|V_{i_1,\ldots i_k}| = c_{i_1}\cdots c_{i_k}|V|$ for any finite sequence $(i_1, \ldots, i_k) \in Q$. Hence $|V_{i_1,\ldots i_k}| \le c_{i_1}\cdots c_{i_k}a_2$ and by Lemma 5.8, $|V_{i_1,\ldots i_k}| \le a_2 r$ which implies that $V_{i_1,\ldots,i_k}$ is contained in a ball of radius $a_2 r$. Similarly $|V_{i_1,\ldots i_k}| = c_{i_1}\cdots c_{i_k}|V| \ge c_{i_1}\cdots c_{i_k}a_1 \ge (\min c_i)ra_1$ and so $V_{i_1,\ldots,i_k}$ contains a ball of radius $(\min c_i)a_1 r$.

Note that $V_1 = S_1(V), \ldots, V_m = S_m(V)$ are disjoint by hypothesis of the open set condition. Hence $V_{i_1,\ldots,i_k,1}, \ldots, V_{i_1,\ldots i_k,m}$ are also disjoint. Thus all the sets $\{V_{i_1,\ldots,i_k} : (i_1, \ldots, i_k) \in Q\}$ are also disjoint. We have arrived at a disjoint collection of open sets and are now in a position to apply Lemma 5.7. Let Q_1 denote the sequences $(i_1, \ldots, i_k) \in Q$ such that B intersects $\overline{V}_{i_1,\ldots,i_k}$. Let q denote the cardinality of the set Q_1. By Lemma 5.7, $q \le (1 + 2a_2)^n a_1^{-n}(\min c_i)^n$ and so q is a positive finite number. We will need one final lemma before we can bound the Hausdorff measure from below.

Lemma 5.10. $F \subseteq \bigcup_Q F_{i_1,\ldots,i_k} \subseteq \bigcup_Q \overline{V}_{i_1,\ldots,i_k}$.

Proof. By hypothesis F is an invariant set and so $F = S_1(F) \circ \cdots \circ S_m(F)$. Pick any $x \in F$. Then $x \in S_1(F) \circ \cdots \circ S_m(F)$ and so $x \in F_{1,2,\ldots m}$ by the notation we introduced earlier. Hence $x \in \bigcup\limits_{Q} F_{i_1,\ldots,i_k}$ which means that $F \subseteq \bigcup\limits_{Q} F_{i_1,\ldots,i_k}$. Now it can be shown, using methods that are outside the scope of this paper, that $F \subseteq \overline{V}$. For a proof of this, see [5]. Using this fact it is then clear that for any sequence finite sequence $(i_1, \ldots, i_k)$ in Q $F_{i_1,\ldots,i_k} \subseteq \overline{V}_{i_1,\ldots,i_k}$ (using our earlier notation and the fact that for any function f and for any sets A and B, $A \subseteq B$ implies $f(A) \subseteq f(B)$). Hence $\bigcup\limits_{Q} F_{i_1,\ldots,i_k} \subseteq \bigcup\limits_{Q} \overline{V}_{i_1,\ldots,i_k}$. □

Theorem 5.11. If F is an invariant set and the open set condition holds, $\mathcal{H}^s(F)$ is bounded below by a positive constant.

Proof. Note that

$$
\begin{aligned}
\mu'(B) &= \mu'(F \cap B) \\
&= \mu\{(i_1, i_2, \ldots) : x_{i_1,i_2,\ldots} \in F \cap B\} \\
&\leq \mu\{(i_1, i_2, \ldots) : x_{i_1,i_2,\ldots} \in (\bigcup_{Q} \overline{V}_{i_1,\ldots,i_k}) \cap B\} && (\text{ since } F \subseteq \bigcup_{Q} \overline{V}_{i_1,\ldots,i_k} \text{ by Lemma 5.10}) \\
&= \mu\{(i_1, i_2, \ldots) : x_{i_1,i_2,\ldots} \in \bigcup_{Q_1} \overline{V}_{i_1,\ldots,i_k}.\} && (\text{By definition of } Q_1) \\
&\leq \mu\{\bigcup_{Q_1} I_{i_1,\ldots,i_k}\} && (\text{Since } x_{i_1,i_2,\ldots} \in \bigcup_{Q_1} \overline{V}_{i_1,\ldots,i_k} \text{ implies there exists} \\
& && \text{an integer } k \text{ such that } (i_1, \ldots, i_k) \in Q_1) \\
&\leq \sum_{Q_1} \mu(I_{i_1,\ldots,i_k}) && (\text{By countable subadditivity}) \\
&= \sum_{Q_1} (c_{i_1} c_{i_2} \cdots c_{i_k})^s && (\text{By definition of } \mu) \\
&\leq \sum_{Q_1} r^s && (\text{By Lemma 5.8}) \\
&= r^s q && (\text{Since the cardinality of } Q_1 \text{ is } q) \\
&\leq |B|^s q.
\end{aligned}
$$

By the mass distribution principle, $\mathcal{H}^s(F) \geq \frac{\mu'(F)}{q}$. But $\frac{\mu'(F)}{q} = \frac{1}{q}$ and so $\mathcal{H}^s(F) \geq \frac{1}{q} > 0$. □

We are now ready to prove Theorem 5.2 which we state again for convenience.

Theorem 5.2. Suppose the open set condition holds for similarities S_i on $\mathbb{R}^n$ with similarity ratios c_i. Suppose also that a set F is invariant under these similarities: $F = \cup_{i=1}^{m} S_i(F)$. Then $\dim_H(F) = s$ where s satisfies $\sum_{i=1}^{m} c_i^s = 1$. Moreover, we have that $0 < \mathcal{H}^s(F) < \infty$.

Proof. Theorem 5.6 bounds the Hausdorff measure from above and Theorem 5.11 bounds it from below. Hence $0 < \mathcal{H}^s(F) < \infty$ as required. Now from Corollary 3.1.1, $\dim_H(F) = s$. $\square$

6. Computing the dimension of fractals

We are now finally equipped with the tools that will enable us to rigorously calculate the dimension of the Sierpinski Triangle.

Theorem 6.1. The Hausdorff dimension of the Sierpinski triangle is $\frac{\log 3}{\log 2}$.

Proof. As before, let S denote the Sierpinski Triangle. We can view the Sierpinski Triangle on $\mathbb{R}^2$ having vertices at $(0,0),(1,0)$ and $(\frac{1}{2},\frac{\sqrt{3}}{2})$. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be a map with $f((x,y))=(\frac{x}{2},\frac{y}{2})$ for all $(x,y)\in\mathbb{R}^2$. It can easily be checked that f is a similarity with ratio $\frac{1}{2}$. Similarly define ratio $\frac{1}{2}$ similarity functions $g:\mathbb{R}^2\to\mathbb{R}^2$ by setting $g((x,y))=(\frac{x}{2}+\frac{1}{2},\frac{y}{2})$ and $h:\mathbb{R}^2\to\mathbb{R}^2$ by setting $h((x,y))=(\frac{x}{2}+\frac{1}{4},\frac{y}{2}+\frac{\sqrt{3}}{4})$. Intuitively each similarity function just maps the the Sierpinski Triangle S to a smaller copy of itself in one of the three triangles in S_1 (see Figure 1) and so $S=f(S)\cup g(S)\cup H(S)$. Let $V=S_0-\partial S_0$. That is, V is just the interior of the original equilateral triangle we started with. Hence V is a non-empty bounded open set and from construction of the similarity functions, V satisfies the Open Set Condition. Then by Theorem 5.2, $\dim_H(S)=s$ where s satisfies $\sum_{i=1}^{3}(\frac{1}{2})^s=1$. Solving for s yields, $s=\frac{\log 3}{\log 2}$. □

We can also compute the dimension of the famous Cantor Set.

Theorem 6.2. The Hausdorff dimension of the Cantor Set is $\frac{\log 2}{\log 3}$

Proof. Let $f:\mathbb{R}\to\mathbb{R}$ be a similarity function by setting $f(x)=\frac{x}{3}$. Let $g:\mathbb{R}\to\mathbb{R}$ be another similarity function by setting $g(x)=\frac{x}{3}+\frac{2}{3}$. Both f and g are similarities with ratio $\frac{1}{3}$. If we let C to be the Cantor Set, then $C=f(C)\cup g(C)$ which shows that C is invariant under these similarities. Let $V=(0,1)$ be an open, non-empty and bounded set. Then $f(V)=(0,\frac{1}{3})$ and $g(V)=(\frac{2}{3},1)$ and so $f(V)\cup g(V)\subseteq V$. Hence the Open Set Condition Holds. By Theorem 5.2, $\dim_H=s$ where s satisfies $(\frac{1}{3})^s+(\frac{1}{3})^s=1$. Solving for s yields $s=\frac{\log 2}{\log 3}$. □

Finally, let us compute the dimension of another famous fractal called the von-Koch curve. First let us highlight the construction of this fractal.

- Start with a unit line segment, say [0,1] for concreteness.
- Draw equilateral triangle of side length $\frac{1}{3}$ with vertices at $x=\frac{1}{3}$ and $x=\frac{2}{3}$ but delete the base
- Continue this procedure for each line segment obtained above; but in each successive iteration decrease the scaling factor by $\frac{1}{3}$

The following figure depicts the above process.

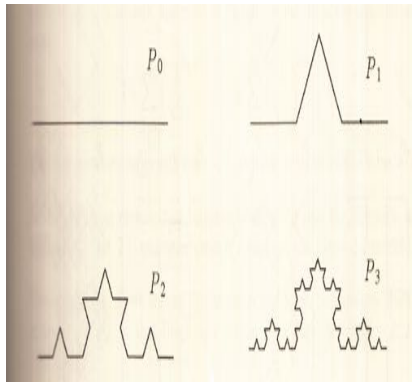

FIGURE 4. Iterations of the von-Koch curve. (Source: [4])

Theorem 6.3. The Hausdorff dimension of the von-Koch curve is $\frac{\log 4}{\log 3}$.

Proof. [11] Again the idea of the proof is to check that the conditions of Theorem 5.2 hold; but this time we will not explicitly compute the similarity functions. Just note that the von-Koch curve is invariant under the four similarities each of ratio $\frac{1}{3}$ that map the unit interval $[0,1]$ to each of the four intervals in P_1 (See Figure 4). Then the Open Set Condition holds by taking V to be the open isosceles triangle with base $[0,1]$ and height $\frac{1}{2\sqrt{3}}$. Intuitively: V is the "tightest" isosceles triangle that contains the curve. The base of the triangle is as given in P_0 and the height is as given as the height of the triangle in P_1. Let P denote the von-Koch curve. Then by Theorem 5.2, $\dim_H(P) = s$ where s satisfies $\sum_{i=1}^{4}(\frac{1}{3})^s = 1$. Solving for s yields $s = \frac{\log 4}{\log 3}$.

$\square$

7. Kakeya needle problem

In this section, we will discuss the Kakeya needle problem that was first posed by Japanese mathematician Kakeya in 1917. The solution to the problem, given by the Russian mathematician Besicovitch, involves fractals; we shall try to sketch the solution provided. This problem is indeed one of the many areas of mathematics where fractals have found applications. The original Kakeya needle problem has given rise to a new problem called the Kakeya conjecture which still remains unsolved in its full generality. We shall also briefly mention the conjecture and survey some important results that have been obtained with respect to this conjecture.

Before we delve into understanding the Kakeya problem, it will be meaningful to look at another problem known as the Besicovitch problem in Riemann Integration. This is because the solution to the Besicovitch problem, with some modifications, will also solve the Kakeya needle problem. The study of Besicovitch's problem is also a good example to show how different areas of mathematics are inter-related since in this case a problem in analysis was used to solve a problem in geometry.

7.1. **Besicovitch problem.**

In 1917, the Russian mathematician Besicovitch was working on a problem in Riemann integration. The problem was as follows:

Suppose $f(x, y)$ is a function Riemann integrable on $\mathbb{R}^2$. Does there exist a perpendicular system of co-ordinate axes (u, v) such that for any fixed v, $\int f(u, v)du$ exists as a Riemann integral ?

To answer the question, we will need the following lemmas.

Lemma 7.1. The function f defined on $[a, b]$ as follows is not Riemann integrable:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Proof. Take any partition $\mathcal{P} = \{a = x_0 < x_1 < x_2 < ... < x_n = b\}$ of $[a, b]$. We exploit the property that the set of rational numbers as well as irrational numbers are dense on the real line; that is between any two real numbers a and b there exists a rational number c and an irrational number d such that $a < c < b$ and $a < d < b$. Hence $\sup_{x\in[x_{i-1},x_i]} f(x) = 1$ and $\inf_{x\in[x_{i-1},x_i]} f(x) = 0$. Hence the upper sum

$$\begin{aligned}
\mathcal{U}(f;\mathcal{P}) &= \sum_{i=0}^{n}(\sup_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1}) && \text{(By definition)} \\
&= \sum_{i=0}^{n} 1 \cdot (x_i - x_{i-1}) && \text{(By the above observation)} \\
&= b - a && \text{(By summing the telescoping series)}
\end{aligned}$$

Similarly, $\mathcal{L}(f;\mathcal{P}) = \sum_{i=0}^{n}(\inf_{x\in[x_{i-1},x_i]} f(x))(x_i - x_{i-1}) = \sum_{i=0}^{n} 0 \cdot (x_i - x_{i-1}) = 0$. Hence we have shown that for any partition $\mathcal{P}, \mathcal{U}(f;\mathcal{P}) \neq \mathcal{L}(f;\mathcal{P})$. Hence f is not Riemann integrable.

□

The proof of this next lemma is outside the scope of this paper. Interested readers can refer to Halmos' *Measure theory* [7].

Lemma 7.2. Suppose $f : S \to \mathbb{R}^n$ is a function where S is a bounded set and suppose also that the points of discontinuities of f lie on a set of measure zero. Then f is Riemann integrable over S.

In addition to the lemmas we will also need the following concept of a Besicovitch set to prove the main theorem.

Definition 7.1. A set of Lebesgue measure zero which contains a unit line segment in every direction is called a **Besicovitch set**.

At this stage, we shall assume that Besicovitch sets exist; we will prove this in section 7.2.

Theorem 7.3 (Besicovitch)**.** There exists a function $f(x, y)$ which is Riemann integrable on $\mathbb{R}^2$ but does not satisfy the conditions mentioned above

Proof. Let E be a Besicovitch set. Define a function f from $\mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} 1 & (x,y) \in E \text{ and } (x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}) \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 7.2, f is indeed Riemann integrable, since by definition of a Besicovitch set, E is of measure zero.
Now pick any co-ordinate system (u, v). Since E contains a line segment in every direction, it also contains a line segment parallel to the u axis. Let $v = c$ be this line segment for some $c \in \mathbb{R}$. Then the integral $\int f(u, c)du$ cannot be evaluated by Lemma 7.1. □

7.2. **Kakeya's needle problem.** First, let us begin with an intuitive description of Kakeya's needle problem. Consider a unit line segment in $\mathbb{R}^2$. What are the regions in $\mathbb{R}^2$ in which the unit needle can be rotated continuously by 360° within the region? An obvious example would be the circle centered at the origin with radius $\frac{1}{2}$. Another example where we can perform such an operation would be an equilateral triangle whose height is unity. Notice that the area of the circle is $\pi \cdot (\frac{1}{2})^2 \approx 0.785$ and the area of the equilateral triangle is $\frac{1}{\sqrt{3}} \approx 0.577$. The question arises: How small can such a region get?

Kakeya's needle problem: What is the set of smallest area inside which a unit line segment can be moved continuously by 360° ?

The answer is to this question is that surprisingly the set can be of arbitrarily small area. We will show this by constructing a well-known geometric figure called a 'Perron Tree.' The details of this construction are adapted from Falconer's *The Geometry of Fractal Sets* [6].

We outline the steps of the construction of the Perron Tree:

- Step 1 Let T_1 and T_2 be adjacent triangles with base length b on a line L'. Pick $\alpha \in \mathbb{R}$ such that $\frac{1}{2} < \alpha < 1$. Slide T_2 a distance of $2(1-\alpha)b$ along L to get a resulting figure S

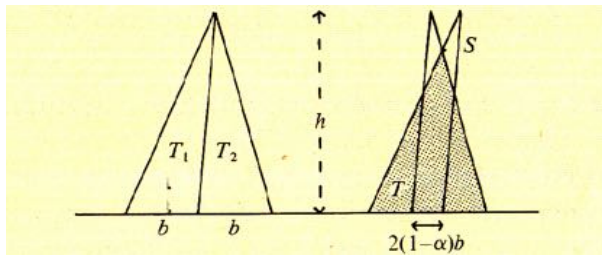

FIGURE 5. Step 1 of the construction. (Source: [6])

- Step 2 Pick a triangle T with a base on a line L. Divide the base of T into 2^k equal segments and join each point of division to the opposite vertex to form 2^k elementary triangles $T_1, \ldots, T_{2^k}$.
- Step 3 For each i, where $1 \le i \le 2^{k-1}$, move T_{2i} along T_{2i-1} in a manner similar to that in Step 1 to get a figure S_i^1.
- Step 4 Now for each S_{2i}^1, where now $1 \le i \le 2^{k-2}$, translate it relative to S_{2i-1}^1 to get S_i^2 in a manner similar to that of Step 1.
- Step 5 The idea now is to just repeat this construction till we get a single figure. That is, at the $(r+1)$ the stage of the construction, obtain S_i^{r+1} by moving S_{2i}^r relative to S_{2i-1}^r where the range of i has now decreased to $1 \le i \le 2^{k-r}$.

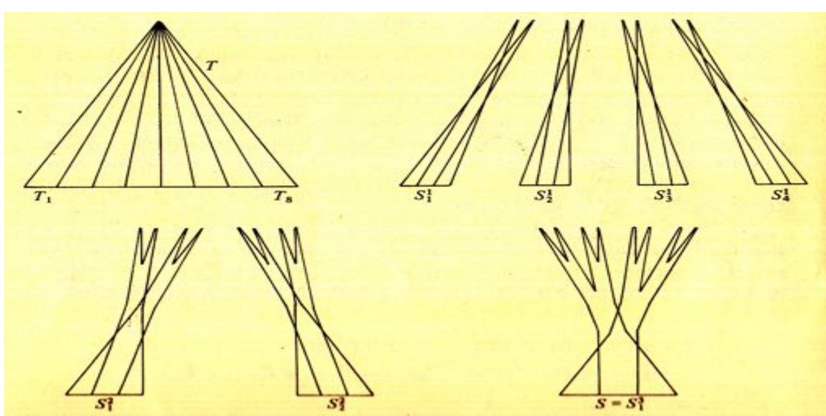

FIGURE 6. Stages of the construction of the Perron tree when $k = 3$. (Source: [6])

The following lemma will be important in answering the Kakeya Needle Problem. For the proof of the lemma, refer to [6].

Lemma 7.4. By choosing k large enough and performing the steps outlined above, it is possible to arrive at a resulting figure S whose area is as small as possible. Furthermore, if V is an open set containing the original triangle T in Step 2, this can be achieved with $S \subset V$.

Theorem 7.5 (Existence of Besicovitch sets)**.** There exists a set F which contains a unit line segment in every direction with $\mathcal{L}^2(F) = 0$

Proof. The idea of the proof is to use the triangles construction we had done previously. Hence we only construct a set which contains a unit line segment in a 60 degrees sector; the desired set can be obtained simply by taking the union of three copies of this set.
Let S_1 be an equilateral triangle of unit height and let V_1 be an open set with $S_1 \subset V_1$ and $\mathcal{L}^2(\overline{V_1}) \le 2\mathcal{L}^2(S_1)$. Using the steps outlined previously construct a new figure S_2 with $\mathcal{L}^2(S_2) \le \frac{1}{2^2}$ (This is possible by Lemma 7.4). Now find an open set V_2 with $S_2 \subset V_2 \subset V_1$ and $\mathcal{L}^2(\overline{V_2}) \le 2\mathcal{L}^2(S_2)$. We can repeat the same procedure on S_2.
In summary we get

- A sequence of figures $\{S_k\}$ with $\mathcal{L}^2(S_k) \le \frac{1}{2^k}$
- A sequence of open sets $\{V_k\}$ with $V_k \subset V_{k-1} \subset \cdots \subset V_2 \subset V_1$ and $\mathcal{L}^2(V_k) \le 2\mathcal{L}^2(S_k) \le \frac{1}{2^{k-1}}$

Now let $F = \bigcap_{k=1}^{\infty} V_k$ and we will show that F is our desired set. From above we have $\mathcal{L}^2(V_k) \le \frac{1}{2^{k-1}}$ and so $\mathcal{L}^2(V_k) \to 0$ as $k \to \infty$. Hence $\mathcal{L}^2(F) = 0$. By construction each S_k contains a unit segment in any direction making an angle of 60 degrees or more with L. Since $S_k \subset V_k \subseteq \overline{V_k}$, the same holds for $\overline{V_k}$ as well. Pick any direction θ with $0 \le \theta \le 60°$. Let $M_k \subset \overline{V_k}$ be a line segment in the direction θ. Now note by remarks made in Section 2, since each $\overline{V_k}$ is bounded by construction, V_k is compact. Hence there must exist at least a subsequence of $\{M_k\}$, which converges, say to M and hence M is also a unit line segment in direction θ. Finally, since $\{\overline{V_k}\}$ is a decreasing sequence of sets $M_k \subset \overline{V_j}$ for all $k \ge j$ and since $\overline{V_j}$ is closed, $M \subset V_j$ for each j. Hence $M \subset F$. $\square$

Lemma 7.6. Let L_1 and L_2 be lines in $\mathbb{R}^2$. Then given any $\epsilon > 0$, there exists a set E containing both L_1 and L_2 such that $\mathcal{L}^2(E) < \epsilon$. Moreover, a unit line segment can be moved continuously from L_1 to L_2 without leaving E.

Proof. Pick points x_1 and x_2 as shown in Figure 7. Let M be the line joining x_1 and x_2. Let E be the set consisting of L_1, L_2, x_1, x_2, M and the unit sectors centered at x_1 and x_2. If we take x_1 and x_2 to be sufficiently apart, then the area of the sectors can be made as small as we like; in particular the area of each sector can be made less then $\frac{\epsilon}{2}$. Hence since the only two-dimensional components of E are the two sectors, $\mathcal{L}^2(E) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Finally notice that a unit line segment can be moved from from L_1 and L_2 as follows: rotate the line segment on L_1 by an angle so that it lies on M, move it along M and rotate it again by the same angle so that it lies on L_2. Also since the set E was chosen to contain L_1, L_2 and M, the above procedure moves the line segment without leaving E. □

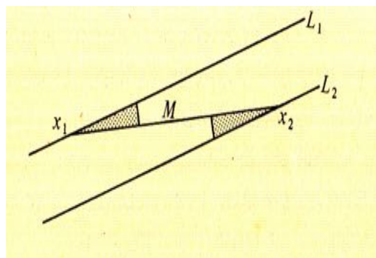

FIGURE 7. This picture shows how we can move the needle continuously from L_1 to L_2. (Source: [6])

Theorem 7.7 (Solution to Kakeya's needle problem)**.** Given any $\epsilon > 0$, there exists a set E with $\mathcal{L}^2(E) < \epsilon$ inside which a unit line segment can be moved continuously to lie in its original position but rotated through 180 °.

Proof. We will construct a set in which it is possible to move a unit line segment by 60° and then we will take three copies of such a set, hence proving the result.

Pick any $\epsilon > 0$. Let T be an equilateral triangle of unit height on a line L. By Lemma 7.4, we can choose a large enough k and set $m = 2^k$ so that if we divide T into m many equal triangles $T_1, T_2, \ldots, T_m$ and perform the construction of the Perron Tree to get a resulting figure S, $\mathcal{L}^2(S) < \frac{\epsilon}{6}$. Note that for each i, one side of T_i is parallel to the opposite side of T_{i+1}. Therefore by the previous lemma, we may for each i add a set of of measure of $\frac{\epsilon}{6m}$ to S to allow a unit line segment to be moved. Hence the set S has a measure of atmost $\frac{\epsilon}{6} + \frac{\epsilon(m-1)}{6m} < \frac{1}{2\epsilon}$ inside which a unit line segment can be rotated. □

7.3. **Kakeya conjecture.** We will end this section with an interesting description of an open conjecture that the Kakeya Needle problem has given birth to. This conjecture is called the Kakeya conjecture.

Definition 7.2. A **Kakeya set** is $\mathbb{R}^n$ is a set which contains a unit line segment in every direction.

It is important to note the difference between Kakeya sets and Besicovitch sets. Besicovitch sets are Kakeya sets with the additional property that they have Lebesgue measure zero; indeed Kakeya sets are generalizations of Besicovitch sets. Now note that sets of Lebesgue measure zero need not have Hausdorff dimension zero. For instance, the Cantor Set has Lebesgue measure zero but Hausdorff dimension $\frac{\log 2}{\log 3}$. The Kakeya needle problem asked about the smallest area of a Kakeya set and we showed in Theorem 7.7 that a Kakeya set can arbitrarily small Lebesgue measure. It is natural to ask: What is the minimum dimension of a Kakeya set? Indeed this is the Kakeya conjecture and it has two versions.

Weak Kakeya Conjecture: A Kakeya set in $\mathbb{R}^n$ has box-counting dimension n

Strong Kakeya Conjecture: A Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n

Recall that in Theorem 4.1 we proved that $\dim_H(F) \leq \dim_B(F)$. In other words, it is much harder to get a bound on the Hausdorff dimension (since if a set F has box counting dimension at least k, it does not imply that the Hausdorff dimension of F should also be at least k) . Hence the the usage of the word "strong" for the conjecture that concerns the Hausdorff dimension.

The following results have so far been obtained on the Strong Kakeya conjecture:

- Roy Davies, 1971 [2]: All Kakeya sets in $\mathbb{R}^2$ have Hausdorff dimension 2.

The question is still open for $n \geq 3$. However, the following partial progress has been made:

- Thomas Wolff, 1995 [19]: All Kakeya sets in $\mathbb{R}^d$ have Hausdorff dimension atleast $\frac{d-2}{2} + 2$.
- Nets Katz and Terence Tao, 2000 [8]: All Kakeya sets in $\mathbb{R}^d$ have Hausdorff dimension atleast $\frac{2-\sqrt{2}}{d-4} + 3$.

8. Applications in Geomorphology

Fractals play an important role in the field of geomorphology since many of the natural features around us exhibit fractal properties. Mandelbrot's pertinent remark- "Clouds are not spheres, mountains are not cones, coastlines are not circles and bark is not smooth"- shows the inadequacy of Euclidean geometry in describing natural phenomena. In this section, we will briefly touch upon the application of fractals in river networks.

It is well-known that appearance of floods brings about a substantial damage both in terms of loss of lives and in terms of destruction of infrastructure. Any model that helps us to understand floods better can be of great advantage. River networks due their branching nature usually show fractal properties and these properties can be used to predict the appearance of floods. We shall describe two methods for estimating the fractal dimension of river networks.

For the first method [16] , we shall have to establish some basic terminology of river networks. We first need to order the various streams in a river network and we shall do so by using a scheme of ordering called the Strahler ordering method.

- Streams that start from the source (such as mountains) are labelled 1
- If two streams of the same order, say i, combine to form a new stream, the resulting stream will be labelled $i + 1$
- If a stream combines of order i combines with a stream of order j and $i < j$, then the resulting stream will be of order j.

The following diagram depicts ordering streams of a typical river network using the Strahler method.

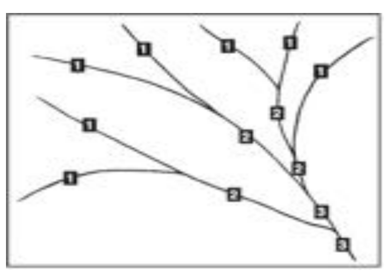

Figure 8. Ordering streams using the Strahler Method. (Source: [1])

We also need to make a few definitions:

Definition 8.1. The bifurcation ratio of a river network is defined as $R_B = \frac{N_i}{N_{i+1}}$ where N_i is the number of streams of order i.

Definition 8.2. The length ratio of a river network is defined as $R_L = \frac{L_{i+1}}{L_i}$ where L_i is the average length of streams of order i.

The bifurcation ratio can be obtained from a plot of $\log N_i$ versus i. Similarly, the length ratio can be obtained from a plot of $\log L_i$ versus i.

The following gives an expression for the fractal dimension of river networks

Theorem 8.1. The fractal dimension of a river network $D = \frac{\log R_B}{log R_L}$

The second method [20] to estimate the fractal dimension of river networks uses the box-counting method; the idea being derived from the box-counting dimension defined in Section 4.

- Place a box with spacing δ on the river network (where the network is assumed to be on a map)
- Count the minimum number of boxes N_δ that cover the network
- Repeat the above two steps for different spacings δ
- Obtain a table of values for $\log N_\delta$ and $\log \delta$ in each case
- Plot a graph of $\log N_\delta$ versus $\log \delta$ and compute the slope s. Then fractal dimension $D = -s$

The following picture demonstrates the process:

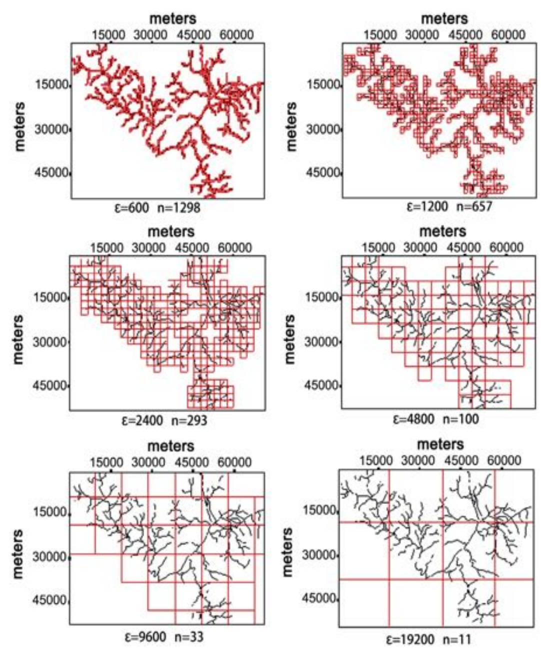

Figure 9. Example of the box-counting method for river networks (Source: [20])

Indeed the method outlined above is just an approximation to the definition of Box-counting dimension defined in Section 4.

It has been found that fractal dimension of a river network is inversely correlated with the probability of flooding [20]. In other words, the higher the fractal dimension the lower are the chances for flooding. Intuitively, this seems reasonable because the fractal dimension roughly indicates the complexity of the river network. So a higher fractal dimension indicates a more complex river network; this in turn suggests that there are more streams and tributaries in the network. Hence the water gets distributed in these many different streams and there is a less chance of accumulation and so a less chance of floods. Hence in conclusion, fractal dimension can be used as an important parameter in determining the probability of floods.

9. Conclusion and Further Extensions

This paper has provided a brief glimpse into the field of fractal geometry. We began with fundamental notions of a Hausdorff Measure and Hausdorff Dimension in Section 3 and of the Box-counting dimension in Section 4. We understood them better through their properties and with some examples and then we computed the dimensions of well-known fractals. Fractals are of paramount importance because of their applications; this paper would not have been complete if we did not include at least a few of the many myriad places in which fractals are being used. To this end, we answered the Kakeya Needle Problem by giving a fractal like construction and saw an application of fractal dimension of river networks in the study of floods.

One extension of the paper that would be worth considering would be to compute the dimension of some more complicated fractals, especially those that are not self-similar and also for those which the Open Set Condition proved in Section 5 does not apply. We could also have provided more exposition on the progress made on the Kakeya conjecture.

Another important extension of this paper would be related to the Section 8 material on geomorphology. It has been found that apart from drainage networks, the distribution of rainfall data [15], the distribution of discharge in a river [10] as well as the distribution of deaths and damages after floods [17] all follow a power law; that is their graphs show fractal properties. That is, many properties related to floods depict fractal behaviour. An interesting question to ask would be: What happens when we superimpose these fractals on each other? [17] More specifically, is the fractal behaviour of the resultant flood peaks and damage and deaths due to the fractal behaviour of the rainfall and drainage networks? If this is the case, human vulnerability is not just a function of human decisions and actions but there is an irreducible hydrological component that must be taken into account [17]. Considering this, are building embankments of no substantial use? More generally, is adaptation to floods more appropriate than control? These are all deep questions and further research is surely required to answer them and to come near in developing a comprehensive theory of flood disasters.

Fractal geometry is slowly emerging as a discipline of mainstream mathematics. Because of its capacity to model roughness which is inherent in the natural world, it has a great potential to solve many problems in the natural and earth sciences.

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