AN INTRODUCTION TO DIMENSION THEORY AND FRACTAL GEOMETRY: FRACTAL DIMENSIONS AND MEASURES

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1. Historical context and motivation

Poincaré's topological reinterpretation of Euclid's initial concept of dimension:

> "When we say that space has the dimensions three, what do we mean? If to divide a continuum C it suffices to consider as cuts a certain number of distinguishable points, we say that this continuum is of dimension one. If, on the contrary, to divide a continuum it suffices to use cuts which form one or several continua of dimension one, then we say that C is a continuum of dimension two. If cuts which form one or several continua of at most dimension two suffice, we say that C is a continuum of dimension three; and so on."

This idea of dimension can be rephrased (à la Brouwer) inductively in more modern language:

(1) We take a single point to have dimension 0.
(2) If a set A contains points for which the boundaries of arbitrarily small neighbourhoods all have dimension n, then A is said to have dimension $n + 1$.

These basically justify thinking of dimension as the number of parameters required to identify a point in a given space. This idea was turned on its head by Peano at the beginning of this century, when he constructed a continuous function of the unit interval with the unit square. This construction of a space-filling curve showed how the dimension of an object could be changed by a continuous transformation and thus contradicted the idea of dimension as "minimum number of parameters". [PJS]

At about the same time, Cantor showed the cardinality of the line and plane to be equal, prompting the construction of a bijection between them. This lead naturally to the question of whether a continuous bijection could be found between sets of dimension one and sets of dimension two. If so, the conclusion would be that dimension (at least in this sense) has no real topological meaning or value.

In 1911, Brouwer (building on the foundations of Lüroth) demonstrated a nonconstructive proof that $\mathbb{R}^n \cong \mathbb{R}^m \iff n = m$. "Nonconstructive" in the sense that it did not identify any characteristic of n-space that would allow it to be distinguished from m-space. However, two years later he did construct a topologically invariant integer-valued function. At about the same time, Henri Lebesgue's approach to this problem (via covering sets) elicited a way to distinguish between Euclidean spaces of different topological dimension. This idea was developed by Hausdorff, and led to the formulation of dimension as the proper "mode" in which to measure a set. By this I mean that the intuitive idea is as follows: before you can accurately measure the size of the set, you need to ascertain the appropriate manner of measurement. Consider the example of a line segment $(a, b) \subset \mathbb{R}$:

(1) In terms of cardinality, (a, b) has measure ∞. (Measuring w/r dim0)
(2) In terms of length, (a, b) has measure $b - a$. (Measuring w/r dim1)
(3) In terms of area, (a, b) has measure 0. (Measuring w/r dim2)

Hausdorff's idea was to find the value at which the measurement changes from infinite to zero. It was also part of his challenge to articulate the measure in such a way that this value is unique.

Why is the study of dimension important or useful? Dimension is at the heart of all fractal geometry, and provides a reasonable basis for an invariant between different fractal objects. There are also experimental techniques capable of calculating the dimension of a given object, and these methods have proven useful in several applied areas: rate of heat flow through the boundary of a domain, calculating metabolic rates where exchange functions are based on surface area, etc. (See [PJS, p.210] for a discussion of kidney, blood, and urinary systems.

2. Requirements for a Good Definition of *Dimension*

Before we begin defining Hausdorff and other dimensions, it is a good idea to clearly state our objectives. What should be the features of a good definition of dimension? Based on intuition, we would expect that the dimension of an object would be related to its measurement at a certain scale. For example, when an object is scaled by a factor of 2,

- for a line segment, its measure will increase by $2^1 = 2$
- for a rectangle, its measure will increase by $2^2 = 4$
- for a parallelipiped, its measure will increase by $2^3 = 8$

In each case, we extract the exponent and consider this to be the dimension. More precisely, $\dim F = \log \Delta\mu(F) / \log {}^1\!/_p$ where p is the precision (${}^1\!/_p$ is the scaling factor) and $\Delta\mu(F)$ is the change in the 'measure' of F when scaled by ${}^1\!/_p$.

Falconer suggests that most of following criteria also be met [Falc2], by anything called a *dimension*:

(1) **Smooth manifolds.** If F is any smooth, n-dimensional manifold, $\dim F = n$.
(2) **Open sets.** For an open subset $F \subset \mathbb{R}^n$, $\dim F = n$.
(3) **Countable sets.** $\dim F = 0$ if F is finite or countable.
(4) **Monotonicity.** $E \subset F \quad \Rightarrow \quad \dim E \leqslant \dim F$.
(5) **Stability.** $\dim (E \cup F) = \max (\dim E, \dim F)$.
(6) **Countable stability.** $\dim (\bigcup_{i=1}^{\infty} F_i) = \sup_i \{\dim F_i\}$.
(7) **Lipschitz Mappings.** If $f : E \to \mathbb{R}^m$ is Lipschitz, then $\dim f(E) \leqslant \dim (E)$.
(8) **Bi-Lipschitz Mappings.** If $f : E \to \mathbb{R}^m$ is bi-Lipschitz, then $\dim f(E) = \dim (E)$.
(9) **Geometric invariance.** $\dim f(F) = \dim F$, if f is a similarity, or affine transformation.

Recall that $f : E \to \mathbb{R}^m$ is *Lipschitz* iff $\exists c$ such that

$$|f(x) - f(y)| \leqslant c\,|x - y| \quad \forall x, y \in E;$$

and that f is *bi-Lipschitz* iff $\exists c_1, c_2$ such that

$$c_1\,|x - y| \leqslant |f(x) - f(y)| \leqslant c_2\,|x - y| \quad \forall x, y \in E;$$

and f is a *similarity* iff $\exists c$ such that

$$|f(x) - f(y)| = c\,|x - y| \quad \forall x, y \in E;$$

Thus (9) is a special case of (8), which is a special case of (7).

The first three properties on our list of "requirements" are formalizations of the historical ideas discussed previously and ensure that the classical definition is preserved. We pay particular attention to bi-Lipschitz functions, as they seem to be the prime candidate for what kind of functions preserve the dimension of a set.

Dimensionally concordant vs. dimensionally discordant. It is the hope that many of these properties hold true under different definitions of dimension. For such sets as this is true, we use the term dimensionally concordant. [Mand]

Relation to a measure. Although the discussion has mentioned "measure" a few times already, this is somewhat sloppy language. It is not necessarily the case that a definition of dimension will be based on a measure. Clearly there are advantages to using a measure-based definition, as this allows the analyst to exploit a large body of thoroughly-developed theory. However, it will be shown that some very useful ideas are decidedly *not* measure-based. To emphasize this distinction, **content** may occasionally be used as a more generic synonym for the volume/mass/measure of a set.

3. Compass Dimension

Motivated by a famous paper entitled, "How Long is the Coastline of Britain?", a new concept of dimension was developed via power law. The crux of the problem was that the length of a coastline seems to expand exponentially as the measurement is refined. For example [PJS]:

p=Compass Setting	l=Coastline Length
500 km	2600 km
100 km	3800 km
54 km	5770 km
17 km	8640 km

One would hope (and naively expect) that the measured length would "calm down" at some point, and submit to reasonable approximation for precise enough measurement. This sample data indicate, however, that precisely the opposite occurs: the more detailed the measurements become, the faster the total length diverges. This is in sharp contrast the measurement of a smooth curve in the same fashion, for example, a circle of diameter 1000 km:

p=Compass Setting	# sides	l=Coastline Length
500.00 km	6	3000 km
259.82 km	12	3106 km
131.53 km	24	3133 km
65.40 km	48	3139 km
32.72 km	96	3141 km
16.36 km	192	3141 km

Due to the very large distance between compass settings at higher scales, and the small distance between compass settings at lower scales, it is more convenient to graph this data as a log/log plot. Also, this tack was suggested by our intuition in the discussion on requirements for a good definition. (See Figure: 1).

Doing a best fit for the data points of the coastline, we see a line with slope $D \approx 0.3$ emerge. If the equation of this line is $y = mx + b$, we can rewrite the relationship between l and p as [PJS]:

$$\log l \quad = \quad D \log {}^{1}\!/_{p} + b$$

or

$$\begin{aligned} l &= e^{D \log {}^{1}\!/_{p} + b} \\ &= e^{\log\left({}^{1}\!/_{p}\right)^{D}} e^{b} \\ &= p^{-D} e^{b} \end{aligned}$$

(1)

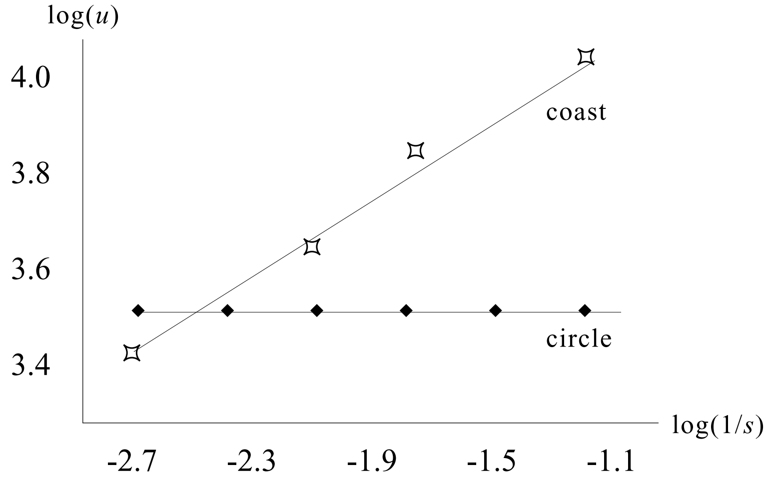

FIGURE 1. Log/log plot of the circle vs. the coastline.

Choosing this for our function and plugging in the original data, we obtain $D \approx 0.36$. Thus, our conclusion (stated as a power law) is:

$$l \sim p^{-0.36} \tag{2}$$

The number $D \approx 0.36$ is our candidate for dimension as noted in (2).

Now we apply our results to a well-known fractal: the von Koch curve. (See Figure: 2). Due to the manner in which the Koch curve is constructed, it is relatively

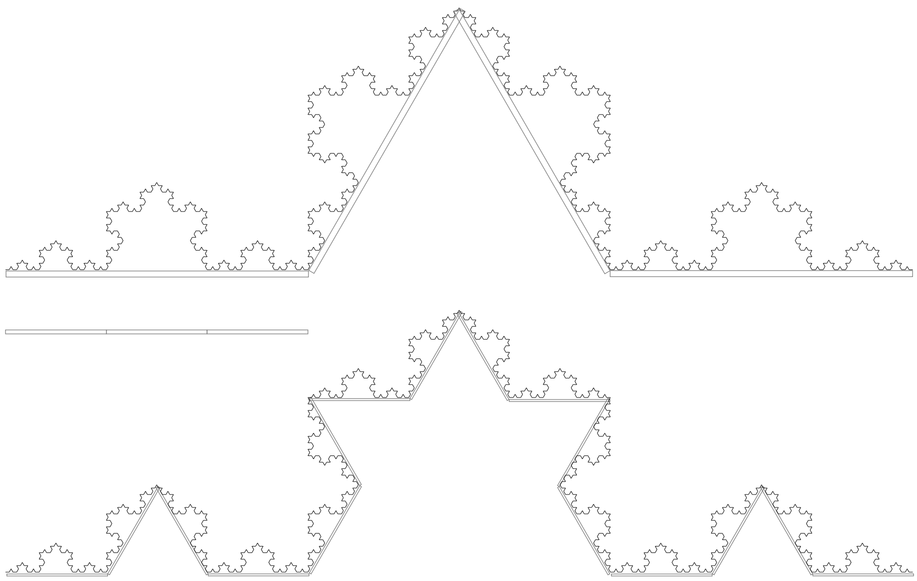

FIGURE 2. Measuring the Koch curve with different compass settings.

easy to "measure with compass" if we restrict our settings to those of the form 3^{-n}:

p=Compass Setting	l=Curve Length
1/3	4/3
1/9	16/9
$\vdots$	$\vdots$
$\frac{1}{3}^n$	$\left(\frac{4}{3}\right)^n$

Graphing these results, we see that the log-log plot (with log base 3 for convenience) is exactly linear. See Figure: 3.

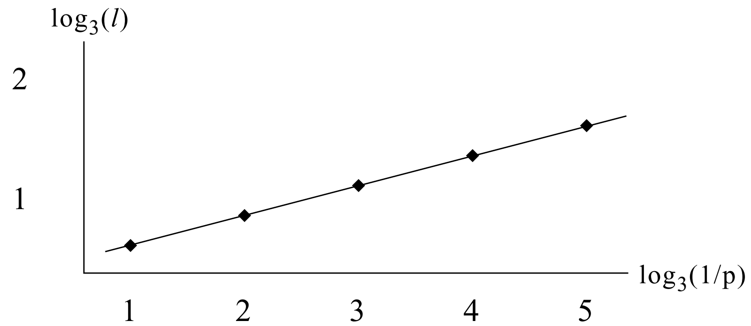

FIGURE 3. Log/log plot for the Koch curve.

So for compass setting $p = \left(\frac{1}{3}\right)^n$ and corresponding length $l = \left(\frac{4}{3}\right)^n$, we obtain $\log_3 {}^1\!/_p = n$ and $\log_3 l = n \log_3 {}^4\!/_3$. Solving for n and combining, we get $\log_3 l = (\log_3 {}^4\!/_3)(\log_3 {}^1\!/_p)$, or rewritten as a power law: $l \sim p^{-D}$ for $D = \log_3 \frac{4}{3} \approx 0.2619$. This number is lower than the D we found for the coastline, indicating that the coast is more convoluted (or detailed).

4. SELF-SIMILARITY DIMENSION

After compass-measuring the Koch curve, it is evident that the scaling properties of some objects can be measured in a slightly different, and more direct way. The Koch curve, like many fractals, is *self-similar*: the entire curve can be seen as a union of scaled copies of itself. In the case of the Koch curve K, K is the union of 4 copies of K, each scaled by a factor of 1/3. See Figure 4.

Given a self-similar object, we can generalize this relationship as $n = p^{-s}$ where p is the reduction factor, and n is the number of pieces. The basis for this relation is easily drawn from a comparison with more familiar, non-fractal self-similar objects like line, square, and cube. Extending this relation to other self-similar sets allows us to calculate the self-similarity dimension by the formula:

$$\dim_{sim}(F) = \frac{\log n}{\log {}^1\!/_p} \tag{3}$$

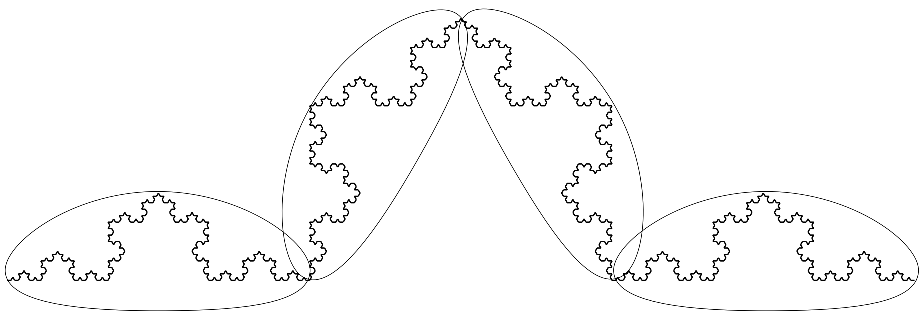

FIGURE 4. Self-similarity of the Koch curve K. $K = \bigcup_{j=1}^{4} f_j(K)$, where each of the four maps f_j is a contraction similitude. That is, f_j is the composition of a contraction (by a factor of 1/3) and an isometry.

For the Koch curve, this formula yields $\dim_{sim}(K) = \frac{\log 4}{\log 3} = \log_3 4 \approx 1.2619$, a number which is strikingly similar to the compass dimension of K, $\dim_{com}(K) \approx 0.2619$.

From compass dimension we have

$$\log l = \dim_{com}(F) \cdot \log {}^1\!/_p \tag{4}$$

and from self-similarity we have

$$\log n = \dim_{sim}(F) \log {}^1\!/_p \tag{5}$$

The connection between length and number of pieces is given by

$$l = n \cdot p$$

from which we get

$$\log l = \log n + \log p$$

Substituting (4) and (5) into this expression, we get

$$\dim_{com}(F) \cdot \log {}^1\!/_p = \dim_{sim}(F) \log {}^1\!/_p - \log {}^1\!/_p$$

Which simplifies to

$$\dim_{com}(F) = \cdot \dim_{sim}(F) - 1$$

So $\dim_{sim} = 1 + \dim_{com}$, just as suggested by our results for the von Koch curve.

We can make two conclusions from this result, at least one of which is surprising:

(1) Compass dimension and self-similarity dimension are essentially the same.
(2) We can compute the self-similarity dimension of irregular shapes (e.g., coastlines) by means of compass measurements.

Result #2 justifies the description of highly irregular objects as being self-similar; it even offers a mathematical basis for rigor in such a notion.

5. Box-Counting Dimension

Now we return to the idea of measurement at scale δ: if $M_\delta(F) \sim c\delta^{-s}$, then we think of F as having dimension s and having s-dimensional content c. From $M_\delta(F) \sim c\delta^{-s}$, we can take logarithms to get

$$\log M_\delta(F) \simeq \log c - s\log\delta$$

and isolate s as

$$s = \lim_{\delta\to 0} \frac{\log M_\delta(F)}{-\log\delta}. \tag{6}$$

This is the idea behind the box-counting or box dimension. Now the trick is to come up with a good definition of $M_\delta(F)$ that can be used on unwieldy sets F. Box-counting dimension derives its name from the following measurement technique:

(1) Consider a mesh of boxes in $\mathbb{R}^n$, of side length δ.
(2) Define $M_\delta(F)$ to be the number of boxes in the mesh that intersect F, or (equivalently) define $M_\delta(F)$ as the number of boxes in the mesh required to cover F.

The interpretation of this measure is an indication of how irregular or spread out the set is when examined at scale δ. [Falc1] However, it should be pointed out that s, as defined by a limit in (6), may not exist! Since $\underline{\lim}$ and $\overline{\lim}$ do always exist, we define the *upper* and *lower box-counting dimensions* as

$$\underline{\dim}_B F = \underline{\lim}_{\delta\to 0} \frac{\log M_\delta(F)}{-\log\delta} \tag{7}$$

$$\overline{\dim}_B F = \overline{\lim}_{\delta\to 0} \frac{\log M_\delta(F)}{-\log\delta} \tag{8}$$

so that s is well-defined when the two are equal.

An equivalent definition is formulated as follows: let $N_\delta(F)$ be defined as the least number of sets of diameter at most δ that are required to cover F. Here we define the diameter of a set $U \subset \mathbb{R}^n$ as $|U| = \sup\left\{|x-y| \,\vdots\, x, y \in U\right\}$.

$N_\delta(F)$ can be seen as equivalent to $M_\delta(F)$ as follows:

The cubes $[m_1\delta, (m_1+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta]$ which intersect F form a cover of $M_\delta(F)$ sets of diameter $\delta\sqrt{n}$. Thus, it is intuitively clear that

$$\lim_{\delta\sqrt{n}\to 0} \frac{\log N_{\delta\sqrt{n}}(F)}{-\log\delta\sqrt{n}} = \lim_{\delta\to 0} \frac{\log M_\delta(F)}{-\log\delta}.$$

More formally, we note that

$$N_{\delta\sqrt{n}}(F) \leqslant M_\delta(F).$$

But then since we can take $\delta\sqrt{n} < 1$ for $\delta \to 0$, we get

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n})} \leqslant \frac{\log M_\delta(F)}{-\log\sqrt{n} - \log\delta}$$

and then

$$\underline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \leqslant \underline{\dim}_B F \tag{9}$$

and

$$\overline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \leqslant \overline{\dim}_B F. \tag{10}$$

Then, since any set of diameter at most δ is contained in 3^n cubes of a δ-mesh, we get

$$M_\delta(F) \leqslant 3^n N_\delta(F),$$

from which the opposite inequalities follow in a similar manner.

This approach leads to more equivalent definitions. We can take $M_\delta(F)$ to be the smallest number of arbitrary cubes needed to cover F (i.e., they need not be aligned in a mesh). Similarly, since any set of diameter at most δ is contained in a ball of radius δ, we can take $M_\delta(F)$ to be the smallest number of balls of radius δ needed to cover F. We can even take $M_\delta(F)$ to be the largest number of disjoint balls of radius δ with centers in F. See Figure 5.

Now that we have some tools to work with, let's examine some of the implications of this definition.

Proposition 1. *If we let $\overline{F}$ denote the closure of F in $\mathbb{R}^n$, then*

$$\underline{\dim}_B\overline{F} = \underline{\dim}_B F$$

and

$$\overline{\dim}_B\overline{F} = \overline{\dim}_B F.$$

Proof. [Falc1] Let $B_1, B_2, \ldots, B_k$ be a finite collection of closed balls, each with radius δ. If the closed set $\bigcup_{i=1}^k B_i$ contains F, it also contains $\overline{F}$. Hence, $N_\delta(F) = N_\delta\left(\overline{F}\right)$, where $N_\delta(F)$ is interpreted as the least number of closed balls of radius δ that cover F. Hence,

$$\underline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} = \underline{\lim}_{\delta\to 0}\frac{\log N_\delta\left(\overline{F}\right)}{-\log\delta}$$

and the result follows immediately. ∎

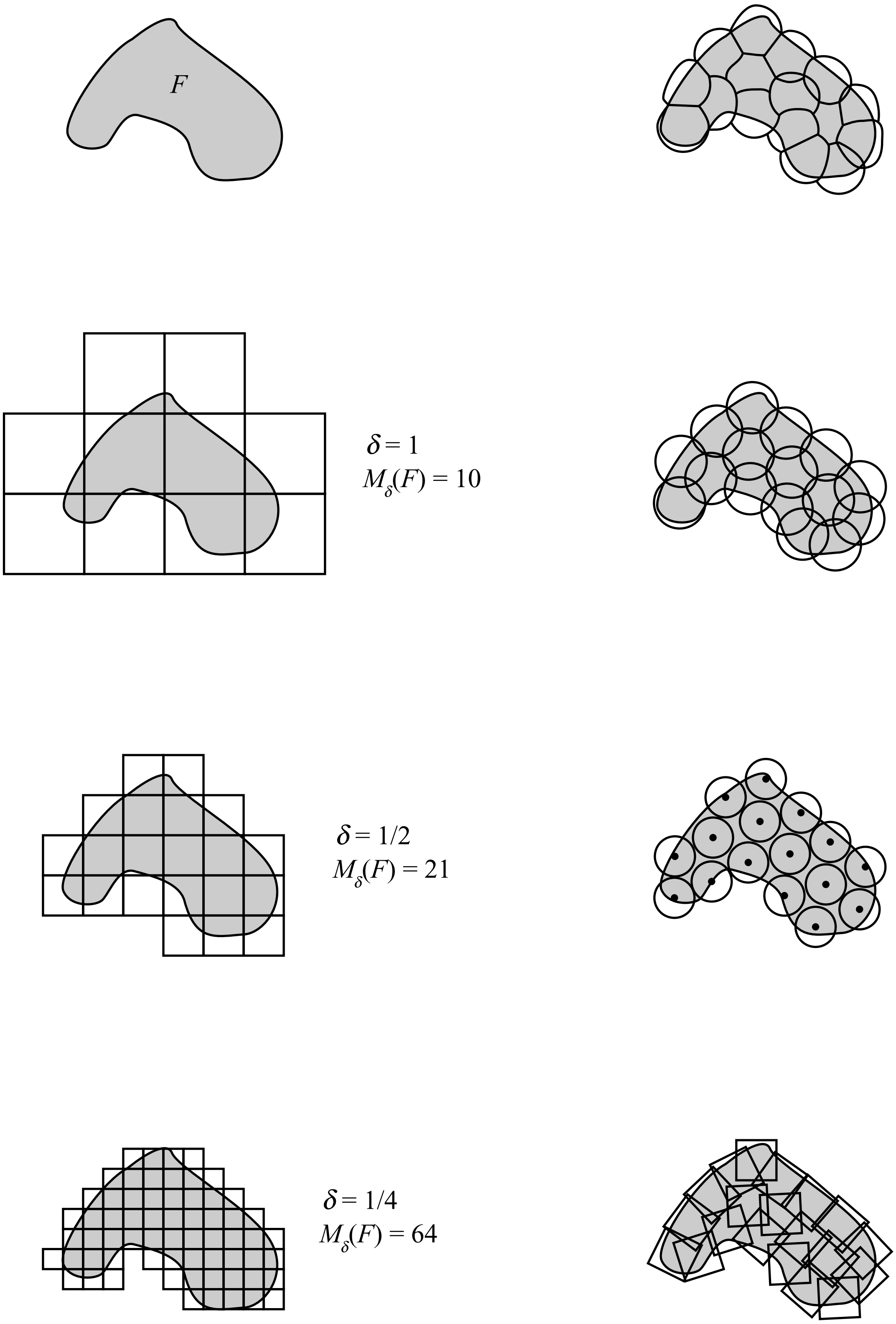

FIGURE 5. Equivalent definitions of $M_\delta(F)$.

Example 2. $F = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ has $\dim_B F = \frac{1}{2}$.

Proof. [Falc1] For $|U| = \delta < \frac{1}{2}$, let k be the integer such that $\frac{1}{k(k+1)} \leqslant \delta < \frac{1}{(k-1)k}$. On one hand, U can cover at most one of the points $\left\{1, \frac{1}{2}, \ldots, \frac{1}{k}\right\}$, so at least k sets of diameter δ are required to cover F, indicating that

$$\frac{\log N_\delta (F)}{-\log \delta} \geqslant \frac{\log k}{\log k\,(k+1)}.$$

Then letting $\delta \to 0$, we get $\underline{\dim}_B F \geqslant \frac{1}{2}$.

On the other hand, $(k+1)$ intervals of length δ cover $\left[0, \frac{1}{k}\right]$, leaving $k-1$ points of F which can be covered by another $k-1$ intervals. Thus,

$$\frac{\log N_\delta (F)}{-\log \delta} \leqslant \frac{\log (2k)}{\log k\,(k-1)},$$

which gives $\underline{\dim}_B F \leqslant \frac{1}{2}$. Then the result follows from

$$\frac{1}{2} \leqslant \underline{\dim}_B F \leqslant \overline{\dim}_B F \leqslant \frac{1}{2}.$$

□

This Proposition both serves to illustrate a serious shortcoming of box dimension. While intuition (or maybe experience) might make $\underline{\dim}_B \overline{F} = \underline{\dim}_B F$ seem like an attractive and straightforward result, it has the unattractive consequence of neatly illustrating that box dimension is not countably stable. For example, the rationals in $[0,1]$ are a countable union of singletons, each with $\dim_B (\{x\}) = 0$. However, as shown by the proposition,

$$\overline{\mathbb{Q} \cap [0,1]} = [0,1] \quad \Rightarrow \quad \dim_B (\mathbb{Q} \cap [0,1]) = \dim_B ([0,1]) = 1,$$

indicating in general that countable dense subsets don't behave well under this definition.

Similarly, the Example indicates another instance where box dimension is shown to be not countably stable. It is included here because it somehow indicates a more severe failing of box dimension: the Example only has one non-isolated point, and still fails to have dimension 0!

Box Dimension Summary

Advantages of working with box dimension:

- **Computationally robust.:** This technique lends itself readily to experimental work and analysis.
- **Flexible.:** A variety of equivalent approaches may be used, allowing the analyst to choose whichever formulation is easiest to work with on a given application. (For $M_\delta (F)$ and $\delta \to 0$.)

Widely applicable.: Box-counting may be applied to non-self-similar sets, and sets that are not easily "compass-able".

Disadvantages of working with box dimension:

May not always exist.: If the upper and lower box-counting dimensions are not equal, $\dim_B F$ is not well defined.
Instability.: On our list of requirements for a definition of dimension, upper box-counting dimension may not be countably stable (req#6) and lower box-counting dimension may not even be finitely stable (req#5)!

6. Minkowski Dimension

The Minkowski dimension of a set $F \subset \mathbb{R}^n$ is defined via the δ-neighbourhood of F:

$$F_\delta = \left\{ x \in \mathbb{R}^n \,\vdots\, |x - y| < \delta \text{ for some } y \in F \right\} \tag{11}$$

i.e., the set of points within δ of F. Note that F_δ is always an open set of $\mathbb{R}^n$ and hence has dimension n. Now consider the rate at which the n-dimensional volume of F_δ decreases, as δ decreases. Some familiar examples in $\mathbb{R}^3$:

F	$\dim F$	F_δ	$\mathrm{vol}_{13}\left(F_\delta\right)$
single point	0	ball	$\frac{4}{3}\pi\delta^3$
line segment	1	"sausage"	$\sim \pi\delta^2 l$
rectangle of area A	2	"mouse pad"	$\sim 2\delta A$

In each case, $\mathrm{vol}_{13}\left(F_\delta\right) \sim c\delta^{3-s}$ where s is the dimension of F. The coefficient c of δ^{n-s} is known as the s-dimensional Minkowski content of F, and is defined when the values of

$$^*M^s\left(F\right) = \overline{\lim_{\delta\to 0}} \frac{\mathrm{vol}_{1n}\left(F_\delta\right)}{\delta^{n-s}} \tag{12}$$

and

$$_*M^s\left(F\right) = \underline{\lim_{\delta\to 0}} \frac{\mathrm{vol}_{1n}\left(F_\delta\right)}{\delta^{n-s}} \tag{13}$$

are equal. (12) and (13) are known as upper and lower s-dimensional Minkowski content, respectively. Now for F embeddable in $\mathbb{R}^n$, we define the Minkowski dimension of F as:

$$\dim_M(F) = \inf\left\{ s \,\vdots\, {}^*M^s\left(F\right) < \infty \right\} \quad = \quad \sup\left\{ s \,\vdots\, {}^*M^s\left(F\right) = \infty \right\} \tag{14}$$

Returning to the relation $\mathrm{vol}_{1n}(F_\delta) \sim c\delta^{n-s}$, we derive

$$\log \mathrm{vol}_n(F_\delta) \sim (n-s)\log c\delta$$
$$s \sim n - \frac{\log \mathrm{vol}_{1n}(F_\delta)}{\log c\delta}$$

(15)

and relate this to the box dimension of F as follows:

Proposition 3. *For $F \subset \mathbb{R}^n$,*

$$\underline{\dim}_B F = n - \overline{\lim_{\delta\to 0}} \frac{\log \mathrm{vol}_{1n}(F_\delta)}{\log \delta}$$

and

$$\overline{\dim}_B F = n - \underline{\lim_{\delta\to 0}} \frac{\log \mathrm{vol}_{1n}(F_\delta)}{\log \delta}$$

Proof. [Falc1] If F can be covered by $N_\delta(F)$ balls of radius δ, then F_δ can be covered by the concentric balls of radius 2δ. If we denote the volume of the unit ball in $\mathbb{R}^n$ by c_n, this gives

$$\begin{aligned}
\mathrm{vol}_{1n}(F_\delta) &\leqslant N_\delta(F)c_n(2\delta)^n \\
\frac{\log \mathrm{vol}_n(F_\delta)}{-\log\delta} &\leqslant \frac{\log 2^n c_n + n\log\delta + \log N_\delta(F)}{-\log\delta} \\
\underline{\lim_{\delta\to 0}} \frac{\log \mathrm{vol}_n(F_\delta)}{-\log\delta} &\leqslant -n + \underline{\dim}_B F
\end{aligned}$$

(16)

Now we use an alternate but equivalent (as shown previously) formulation of box dimension to show the opposite inequality. If there are $N_\delta(F)$ disjoint balls of radius δ with centres in F, then

$$N_\delta(F)c_n\delta^n \leqslant \mathrm{vol}_{1n}(F_\delta).$$

Taking logs as above clearly leads to the opposite of (16), and together they yield the first equality of the proposition. The equality for upper box dimension follows by nearly identical inequalities. □

Thus, we have shown that the Minkowski dimension is essentially just another formulation of box dimension.

7. Hausdorff Measure

Now we develop a concept of dimension due to Hausdorff. As it is based on a measure, it provides a more sophisticated concept of dimension. We define Hausdorff measure for subsets of $\mathbb{R}$, and then extend this to arbitrary subsets $F \subset \mathbb{R}^n$.

For a nonnegative real s, $\delta > 0$, and $F \subset \mathbb{R}$, we define

$$H_\delta^s(F) = \inf\left\{\sum\nolimits_{i=1}^{\infty} |U_i|^s \vdots\, 0 < |U_i| < \delta,\ \forall i\right\}, \tag{17}$$

where the infimum is taken over all sequences of Borel sets $\{U_i\}_{i=1}^{\infty}$ such that $F \subset \bigcup_{i=1}^{\infty} U_i$. In other words, the infimum is taken over all δ-covers of F.

This allows us to define the *s-dimensional Hausdorff outer measure* by

$$H^s(F) \quad = \quad \lim_{\delta\to 0^+} H_\delta^s(F) \quad = \quad \sup_{\delta>0} H_\delta^s(F) \quad \in \quad [0,\infty]. \tag{18}$$

Finally, *s-dimensional Hausdorff measure* is the restriction of this outer measure to the σ-algebra of H^s-measurable sets.

Justifications.

$H^s(F)$ **is well-defined:**
If we allow δ to vary, note that for $\delta_1 < \delta_2$, any δ_1-cover is also a δ_2-cover. Thus, decreasing δ restricts the range of permissible covers. Conversely, increasing δ allows more covers to be considered, possibly allowing the infimum to drop. Hence, $H_{\delta_1}^s(F) \geqslant H_{\delta_2}^s(F)$, so $H_\delta^s(F)$ is a nonincreasing function of δ on $(0,\infty)$.
Further, for any monotonic sequence $\{\delta_i\}_{i=1}^{\infty}$ tending to 0, $\left\{H_{\delta_i}^s(F)\right\}_{i=1}^{\infty}$ is a nondecreasing sequence bounded above by ∞ and below by 0. Since every bounded monotonic sequence converges, $\left\{H_{\delta_i}^s(F)\right\}$ converges (possibly to ∞).

H^s **is an outer measure:**
We begin by showing that for each fixed $\delta > 0$, $H_\delta^s(F)$ is an outer measure. $H_\delta^s(\varnothing) = 0$ and monotonicity follow immediately from the properties of covers. To show σ-subadditivity, pick an $\varepsilon > 0$ and find an open cover $\{B_{n_i}\}_{i=1}^{\infty}$ for each component F_n such that

$$\sum\nolimits_{i=1}^{\infty} |B_{n_i}|^s \ \leqslant\ H_\delta^s(F_n) + \frac{\varepsilon}{2^n} \tag{19}$$

Now $F_n \subset \bigcup_i B_{n_i} \quad \Rightarrow \quad \bigcup_n F_n \subset \bigcup_n \bigcup_i B_{n_i}$, so

$$\begin{aligned}
H_\delta^s\left(\bigcup F_n\right) &\leqslant H_\delta^s\left(\bigcup_n \bigcup_i B_{n_i}\right) && \text{by monotonicity} \\
&\leqslant \sum_n \sum_i |B_{n_i}|^s && H_\delta^s \text{ is an inf over such sums} \\
&\leqslant \sum_n \left(H_\delta^s(F_n) + \frac{\varepsilon}{2^n}\right) && \text{by (19)} \\
&= \sum_n H_\delta^s(F_n) + \sum_n \frac{\varepsilon}{2^n} \\
&= \sum_n H_\delta^s(F_n) + \varepsilon
\end{aligned}$$

Since this is true for arbitrary ε, we let $\varepsilon \to 0$ and get $H_\delta^s\left(\bigcup F_n\right) \leqslant \sum_n H_\delta^s(F_n)$.

Now that we've established H_δ^s as an outer measure, H^s can easily be shown to be an outer measure: note that the supremum definition of H^s in (18) gives us

$$H^s(F_n) = \sup_{\delta>0} H_\delta^s(F_n) \quad \Rightarrow \quad H^s(F_n) \geqslant H_\delta^s(F_n). \tag{20}$$

Then as shown previously for any fixed δ,

$$\begin{aligned}
H_\delta^s\left(\bigcup F_n\right) &\leqslant \sum_{n=1}^\infty H_\delta^s(F_n) \\
&\leqslant \sum_{n=1}^\infty H^s(F_n)
\end{aligned}$$

where the second inequality follows by (20). Letting $\delta \to 0$, we obtain

$$H^s\left(\bigcup F_n\right) \leqslant \sum_{n=1}^\infty H^s(F_n).$$

Extension of H^s to $F \subset \mathbb{R}^n$.

The definition of *s-dimensional Hausdorff measure* remains essentially the same for subsets of higher dimensions; the difference is just that the covering sets U_i are now taken to be subsets of $\mathbb{R}^n$. The only real work involved in the extension is in the proofs of the various properties. For example, the volume estimates of the covers will need to be refined in the justification for H^s being an outer measure.

Properties of H^s.

1. **Scaling**. For $\lambda > 0$, $H^s(\lambda F) = \lambda^s H^s(F)$

If $\{U_i\}$ is a δ-cover of F, then $\{\lambda U_i\}$ is a $\lambda\delta$-cover of λF. Hence

$$\begin{aligned}
H^s_{\lambda\delta}(\lambda F) &\leqslant \sum |\lambda U_i|^s \\
&= \lambda^s \sum |U_i|^s && \text{(factoring)} \\
&\leqslant \lambda^s H^s_\delta(F) && \text{(since this is true for any } \delta\text{-cover)} \\
\Rightarrow H^s(\lambda F) &\leqslant \lambda^s H(F) && \text{(letting } \delta \to 0)
\end{aligned}$$

The reverse inequality is obtained by replacing λ with $\frac{1}{\lambda}$ and F with λF. ■

2. **Hölder Transforms**. For $F \subset \mathbb{R}^n$, let $f : F \to \mathbb{R}^m$ be such that $|f(x) - f(y)| \leqslant c|x-y|^\alpha$. Then $H^{s/\alpha}(f(F)) \leqslant c^{s/\alpha} H^s(F)\ \forall s$.
   This follows by an argument similar to the above.
   Note that for $\alpha = 1$, this shows that Hausdorff measure satisfies the Lipschitz criterion (7). Further, in the case when $c = 1$ (i.e., f is an isometry), then this shows that $H^s(f(F)) = H^s(F)$.

3. **Lebesgue agreement**. Using the previous property, it may be shown that for any integer n, $H^n = c_n \mu_n$, where μ_n is Lebesgue measure in $\mathbb{R}^n$ and $c_n = 2^n\,(n/2)!/\pi^{n/2}$ is a renormalization constant. H^0 is counting measure, H^1 is length, H^2 is area, etc.

8. Hausdorff Dimension

Now that we have an idea of what s-dimensional Hausdorff measure is and how it works, let's consider what happens when s is allowed to vary. Since we will eventually consider $\delta \to 0$, suppose $\delta < 1$ and consider the definition

$$H^s_\delta(F) = \inf\left\{\sum_{i=1}^\infty |U_i|^s \,\vdots\, 0 < |U_i| < \delta,\ \forall i\right\}.$$

$|U_i| < \delta < 1$ implies that $H^s_\delta(F)$ is nonincreasing as a function of s, and therefore that $H^s(F)$ is also. Suppose that $s < t$, so that $s - t < 0$, and that $\{U_i\}$ is a δ-cover of F. Then we get

$$\begin{aligned}
\sum |U_i|^t &\leqslant \delta^{t-s} \sum |U_i|^s \\
H^t_\delta(F) &\leqslant \delta^{t-s} H^s_\delta(F) && \text{(taking infima)}
\end{aligned}$$

Now if we let $\delta \to 0$, then $H^s(F) < \infty \quad \Rightarrow \quad H^t(F) = 0$. What is the interpretation of this result? There is a critical value of s at which $H^s(F)$ changes from ∞ to 0. For $F \subset \mathbb{R}^n$, we define this unique number to be the Hausdorff dimension of F and denote it $\dim_H F$. See Figure 6.

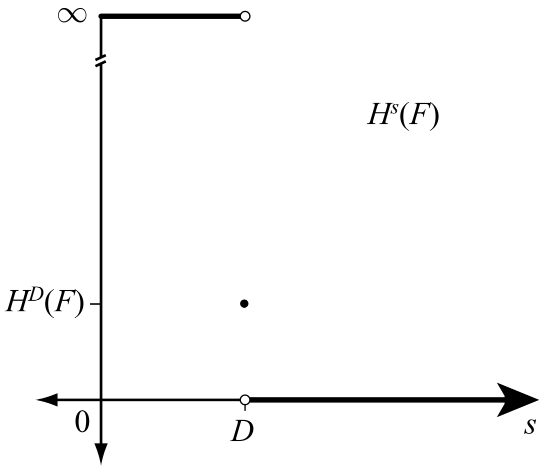

FIGURE 6. The graph of $H^s(F)$ as a function of s.

$$\dim_H F = \inf\left\{s \vdots H^s(F) = 0\right\} = \sup\left\{s \vdots H^s(F) = \infty\right\} \tag{21}$$

Note that $\dim_H F \in [0, \infty]$ and that if $\dim_H(F) = s_0$, then $H^{s_0}(F) \in [0, \infty]$. In other words, when measured in its appropriate dimension, the Hausdorff measure of an object may take any nonnegative value, including 0 and ∞.

Properties of $\dim_H$. Hausdorff dimension satisfies all the definition requirements suggested by Falconer. These relations can be seen readily from the definition of Hausdorff measure, and the results from the previous section.

An extension of the Hölder Transform property 2 from the previous section is as follows:

Proposition 4. *Let $F \subset \mathbb{R}^n$. Then for $f : F \to \mathbb{R}^n$ s.t.*

$$|f(x) - f(y)| \leqslant c\,|x - y|^\alpha \quad \forall x, y \in F,$$

we have $\dim_H f(F) \leqslant \left(\frac{1}{\alpha}\right) \dim_H F$.

Proof. [Falc1] If $s > \dim_H F$, then from 2, we get $H^{s/\alpha}(f(F)) \leqslant c^{s/\alpha} H^s(F) = 0$. But then $\dim_H f(F) \leqslant \frac{s}{\alpha}$ for all $s > \dim_H F$. □

Equivalent definitions of $\dim_H$. Just as there are different but equivalent formulations of box dimension, there are alternate formulations of Hausdorff dimension. Instead of using sets of diameter at most δ, we can take coverings by n-dimensional spheres. We also get the save values for $H^s(F)$ and $\dim_H F$ if we use just open sets or just closed sets to cover F.

In the case when F is compact, we can restrict coverings to those which consist of only open sets, and then take a finite subcover. Hence, we get the same values for $H^s(F)$ and $\dim_H F$ if we only use finite covers.

We can also use a δ-mesh similar to the one discussed previously (in Box-Counting Dimension) to provide a more computationally convenient version of Hausdorff measure (and dimension). Instead of the cubes $[m_1\delta, (m_1+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta]$, we now use the half-open cubes $[m_1\delta, (m_1+1)\delta) \times \cdots \times [m_n\delta, (m_n+1)\delta)$. Consider

$$N_\delta^s(F) = \inf\left\{\sum |U_i|^s \colon \{U_i\} \text{ is a cover of } F \text{ by } \delta\text{-boxes}\right\},$$

and let

$$N^s(F) = \lim_{\delta\to 0} N_\delta^s(F).$$

Note that for any two δ-boxes in the mesh, they are either disjoint, or one is contained in the other. Thus, any cover by δ-boxes can be reduced to a cover by *disjoint* δ-boxes.

These equivalences follow by similar arguments to those used to show the equivalent formulations of box dimension, earlier.

Proposition 5.

$$\dim_H F \leqslant \dim_B F.$$

Proof. [Falc1] Since we always have $\underline{\dim}_B F \leqslant \overline{\dim}_B F$, it suffices to show $\dim_H F \leqslant \underline{\dim}_B F$. Let F be covered by $M_\delta(F)$ sets of diameter δ. Then, by the definition 17, we get $H_\delta^s(F) \leqslant M_\delta(F)\,\delta^s$ (compare the quantities at the bottom of this page), and hence that $H^s(F) \leqslant \underline{\lim}\, M_\delta(F)\,\delta^s$ (letting $\delta \to 0$).

Consider those s for which $1 < H^s(F) = \lim_{\delta\to 0} H_\delta^s(F)$. Then

$$\begin{aligned}
&1 < H^s(F) \leqslant \underline{\lim}\, M_\delta(F)\,\delta^s \\
&1 < M_\delta(F)\,\delta^s && \text{for sufficiently small } \delta \\
&0 < \log M_\delta(F)\,\delta^s && \text{taking logs} \\
&0 < \log M_\delta(F) + s\log\delta \\
&s < \frac{\log M_\delta(F)}{-\log\delta} \\
&s \leqslant \underline{\lim}_{\delta\to 0} \frac{\log M_\delta(F)}{-\log\delta}
\end{aligned}$$

so that $\dim_H F \leqslant \underline{\dim}_B F \;\forall F \subset \mathbb{R}^n$. □

This proposition is very useful because it allows us to use the readily computable $\dim_B F$ for an upper estimate on $\dim_H F$. Essentially, box dimension is easier to calculate because the covering sets are all taken to be of equal size, while Hausdorff incorporates the "weight" of each covering set. To see this, we can write

$$M_\delta(F)\,\delta^s = \inf\left\{\sum \delta^s \vdots \ \{U_i\} \text{ is a } \delta - \text{cover of } F\right\}$$
$$H_\delta^s(F) = \inf\left\{\sum |U_i|^s \vdots \ \{U_i\} \text{ is a } \delta - \text{cover of } F\right\}$$

9. Applications

The purpose of this section is to provide a couple of examples of how to calculate the dimension of a set.

Let C be the familiar middle-thirds Cantor set.

$\dim_{sim} C = \frac{\log 2}{\log 3}$: .

C is clearly seen to be $n = 2$ copies of itself, each scaled by a factor of $p = \frac{1}{3}$. Thus

$$\dim_{sim} C = \frac{\log n}{\log 1/p} = \frac{\log 2}{\log 3} = D.$$

$\dim_B C = \frac{\log 2}{\log 3}$: .

[Falc1] Cover C by 2^k intervals of length 3^{-k}. Then $3^{-k} < \delta \leqslant 3^{-k+1}$ implies that $N_\delta(F) \leqslant 2^k$. Now the definition (8) gives

$$\overline{\dim}_B C = \varlimsup_{\delta\to 0} \frac{\log N_\delta(C)}{-\log\delta} \leqslant \varlimsup_{\delta\to 0} \frac{\log 2^k}{\log 3^{k-1}} = \frac{\log 2}{\log 3}$$

For the other inequality, note that for $3^{-k} < \delta \leqslant 3^{-k+1}$, any interval of length δ can intersect at most one of the basic intervals of length 3^{-k} used in the construction of C. Since there are 2^k such intervals, it must be that 2^k intervals of length δ are required to cover C, whence $N_\delta(C) \geqslant 2^k$ implies

$$\underline{\dim}_B C = \varliminf_{\delta\to 0} \frac{\log N_\delta(C)}{-\log\delta} \geqslant \varliminf_{\delta\to 0} \frac{\log 2^k}{\log 3^{k-1}} = \frac{\log 2}{\log 3}$$

$\dim_H C = \frac{\log 2}{\log 3}$: .

C may be split into two disjoint compact subsets $C_0 = C \cap \left[0, \frac{1}{3}\right]$ and $C_2 = C \cap \left[\frac{2}{3}, 1\right]$. From this, we can derive

$$\begin{aligned}
H^s(C) &= H^s(C_0) + H^s(C_2) && \text{by the additivity of } H^s \\
&= H^s\left(\frac{1}{3}C\right) + H^s\left(\frac{1}{3}C\right) && C_0 \cong C_2 \cong pC \text{ for } p = \frac{1}{3} \\
&= 2 \cdot \left(\frac{1}{3}\right)^s H^s(C) && \text{by scaling property (\textbf{??})} \\
1 &= \frac{2}{3^s} && \text{divide by } H^s(C) \\
2 &= 3^s \\
s &= \log_3 2 = \frac{\log 2}{\log 3}
\end{aligned}$$

For $D = \frac{\log 2}{\log 3}$, $H^D(C) = 1$: .

Part 1. For any $\delta > 0$, pick the smallest n such that $\frac{1}{3^n} \leqslant \delta$. Choose $\varepsilon \leqslant \delta - \frac{1}{3^n}$ and cover C by 2^n intervals of the form $\left(a - \frac{\varepsilon}{2}, a + \left(\frac{1}{3^n} + \frac{\varepsilon}{2}\right)\right)$. The length of any such interval U is

$$|U| = \left|\left(a - \frac{\varepsilon}{2}, a + \left(\frac{1}{3^n} + \frac{\varepsilon}{2}\right)\right)\right| = \frac{1}{3^n} + \varepsilon \leqslant \delta. \tag{22}$$

But $H^D_\delta(C)$ is the infimum over all covers, and this is just one such, so

$$H^D_\delta(C) \leqslant \sum_{i=1}^{2^n} \left|\left(a - \frac{\varepsilon}{2}, a + \left(\frac{1}{3^n} + \frac{\varepsilon}{2}\right)\right)\right|^D$$

Now

$$\begin{aligned}
H^D_\delta(C) &\leqslant \inf_{\varepsilon > 0} \left\{ \sum_{i=1}^{2^n} \left|\left(a - \frac{\varepsilon}{2}, a + \left(\frac{1}{3^n} + \frac{\varepsilon}{2}\right)\right)\right|^D \right\} \\
&= \sum_{i=1}^{2^n} \left(\frac{1}{3^n}\right)^D && \text{by (22)} \\
&= \sum_{i=1}^{2^n} \left(\frac{1}{3^D}\right)^n \\
&= \sum_{i=1}^{2^n} \frac{1}{2^n} && 3^{\frac{\log 2}{\log 3}} = 2 \\
&= 1
\end{aligned}$$

Thus, $H^D_\delta(C) \leqslant 1\ \forall \delta$ implies that $H^D(C) \leqslant 1$.

Part 2. The opposite inequality is obtained as follows: for any $\delta > 0$, let $B = \{B_i\}_{i=1}^\infty$ be a δ-cover of C.

C is compact, so we can find a Lebesgue number $\eta > 0$ such that every $V \subset C$ with $|V| < \eta$ is contained entirely within *one* of the B_i's.

Pick the smallest m such that $\frac{1}{3^m} < \eta$, and choose $\varepsilon < \eta - \frac{1}{3^m}$.

Then we can find another cover of C by 2^m intervals $E_k = \left(a_k - \frac{\varepsilon}{2}, a_k + \frac{1}{3^m} + \frac{\varepsilon}{2}\right)$.

C is compact, so we can find a finite subcover $\{E_j\}_{j=1}^n$.

Now each E_j is entirely contained in a B_i, so let B_j be the set containing E_j, for each j.

Thus $\{B_j\}_{j=1}^n$ is a finite subcover of the original arbitrary cover, so we get

$$1 \leqslant \sum\nolimits_{j=1}^{2^m} |E_j|^D \leqslant \sum\nolimits_{j=1}^{2^m} |B_j|^D$$

by monotonicity. This indicates that 1 is a lower bound on the sum, for any cover B.

$H_\delta^D(C)$ is the greatest lower bound, so $H_\delta^D(C) \geqslant 1$.

As previously noted, $H_\delta^D(C)$ is a nonincreasing function of ε, so $H_\delta^D(C)$ can only increase or remain the same as $\varepsilon \to 0$. Thus, $1 \leqslant H_\delta^D(C) \leqslant H^D(C)$.

Proposition 6. *A set $F \subset \mathbb{R}^n$ with $\dim_H F < 1$ is totally disconnected.*

Proof. [Falc1] Let x and y be distinct points of F. Define a mapping $f : \mathbb{R}^n \to [0, \infty))$ by $f(z) = |z - x|$. Since f does not increase distances, i.e., $|f(z) - f(w)| \leqslant |z - w|$, we can use the Hölder scaling Proposition (4). With $\alpha = c = 1$, this gives us

$$\dim_H f(F) \leqslant \dim_H F < 1.$$

Thus $f(F)$ is a subset of $\mathbb{R}$ with $H^1(f(F)) = 0$ (i.e., $f(F)$ has length 0), and hence has a dense complement. Choosing r with $r \notin f(F)$ and $0 < r < f(y)$, it follows that

$$F = \left\{z \in F \,\vdots\, |z - x| < r\right\} \cup \left\{z \in F \,\vdots\, |z - x| > r\right\}.$$

Thus, F is contained in two disjoint open sets with x in one set and y in the other, so that x and y lie in different connected components of F. ☐

10. Further Dimensions

Modified Box Dimension. For $F \subset \mathbb{R}^n$, we decompose F into a countable number of pieces $F_1, F_2, ...$ in such a way that the largest piece has as small a dimension as possible. This leads to the *modified box-counting dimension*:

$$\underline{\dim}_{MB} F = \inf\left\{\sup_i \underline{\dim}_B F_i \,\vdots\, F \subset \bigcup\nolimits_{i=1}^{\infty} F\right\}$$
$$\overline{\dim}_{MB} F = \inf\left\{\sup_i \overline{\dim}_B F_i \,\vdots\, F \subset \bigcup\nolimits_{i=1}^{\infty} F\right\}$$

(23)

It is clear that $\underline{\dim}_{MB} F \leqslant \underline{\dim}_B F$ and $\overline{\dim}_{MB} F \leqslant \overline{\dim}_B F$, and for compact sets, the relation is even tighter.

Proposition 7. [Falc1] *Let $F \subset \mathbb{R}^n$ be compact, and suppose that $\overline{\dim}_B (F \cap V) = \overline{\dim}_B F$ for all open sets V that intersect F. Then $\overline{\dim}_B F = \overline{\dim}_{MB} F$. (And similarly for lower box dimensions)*

Packing Measure and Packing Dimension. As mentioned previously, the essential difference between Hausdorff measure and box-counting is that Hausdorff considers the size of each covering set, whereas box-counting only considers the number of them. Returning to the notion of $M_\delta (F)$ as the largest number of disjoint balls of radius δ with centers in F, we follow the footsteps of Hausdorff and define

$$P_\delta^s (F) = \sup \left\{ \sum |B_i|^s \right\} \tag{24}$$

where the supremum is taken over all collections $\{B_i\}$ of disjoint balls of radii at most δ with centers in F. Then we define

$$P_0^s (F) = \lim_{\delta \to 0} P_\delta^s (F) .$$

However, P_0^s is not a measure, as is seen by considering countable dense sets, and hence encounters the same difficulties as box dimension. To avoid this problem, we add an extra step to the definition of packing measure, and take

$$P^s (F) = \inf \left\{ \sum P_0^s (F_i) \ \vdots\ F \subset \bigcup F_i \right\} . \tag{25}$$

Now the *packing dimension* can be defined in the usual way as

$$\dim_P F = \sup \left\{ s \vdots P^s (F) = \infty \right\} = \inf \left\{ s \vdots P^s (F) = 0 \right\} . \tag{26}$$

Proposition 8. $\dim_P F = \overline{\dim}_{MB} F$. [Falc1]

In light of the previous two propositions, we have established the following relations:

$$\dim_H F \leqslant \underline{\dim}_{MB} F \leqslant \dim_P F = \overline{\dim}_{MB} F \leqslant \overline{\dim}_B F$$

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