Modeling Sea Ice *in a changing climate*

Kenneth M. Golden
Department of Mathematics
University of Utah

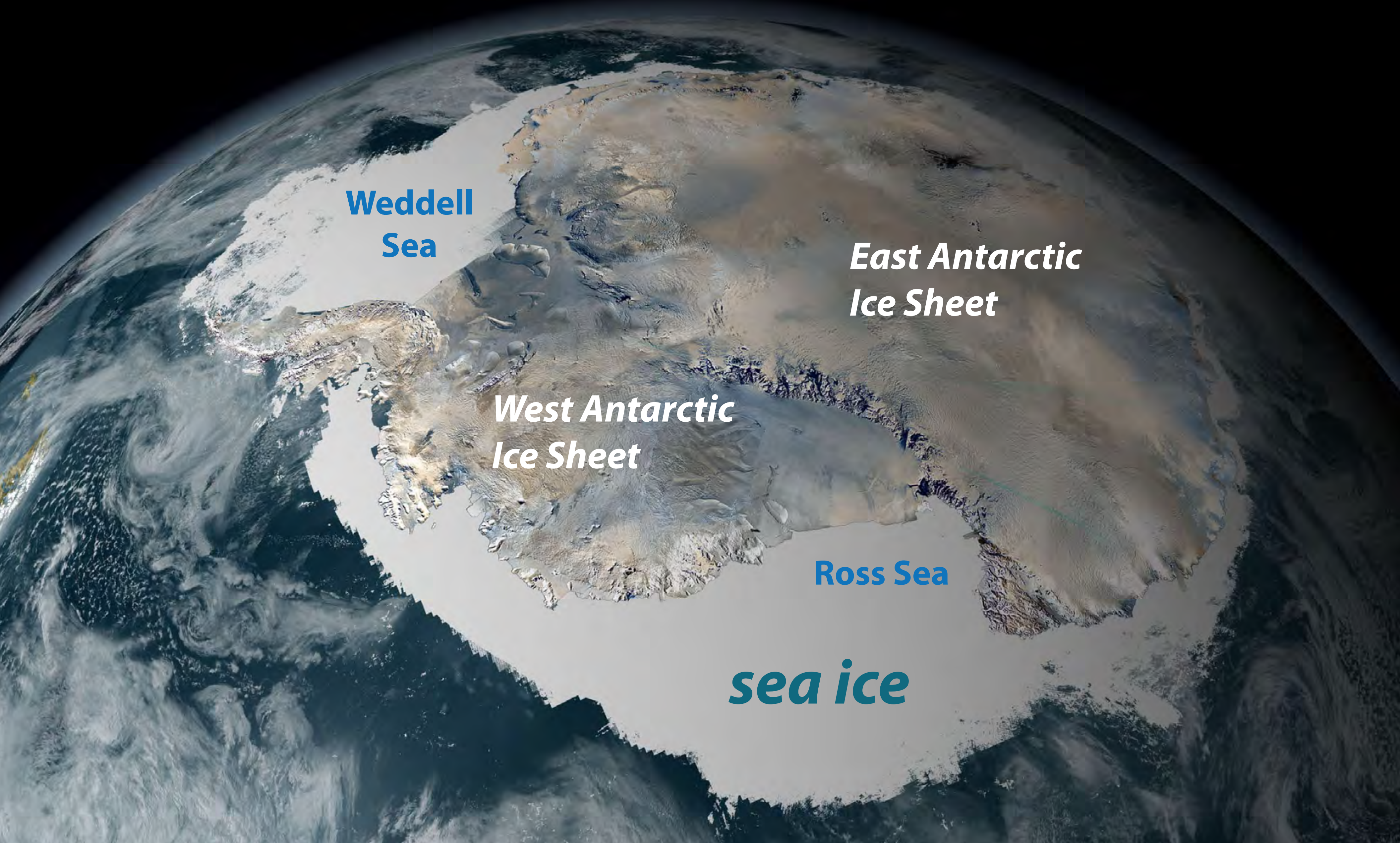
Alison Kohout
September 2012

Workshop on Applied Mathematics
Stanford University

In Memory of Joseph B. Keller
20 May 2017

ANTARCTICA

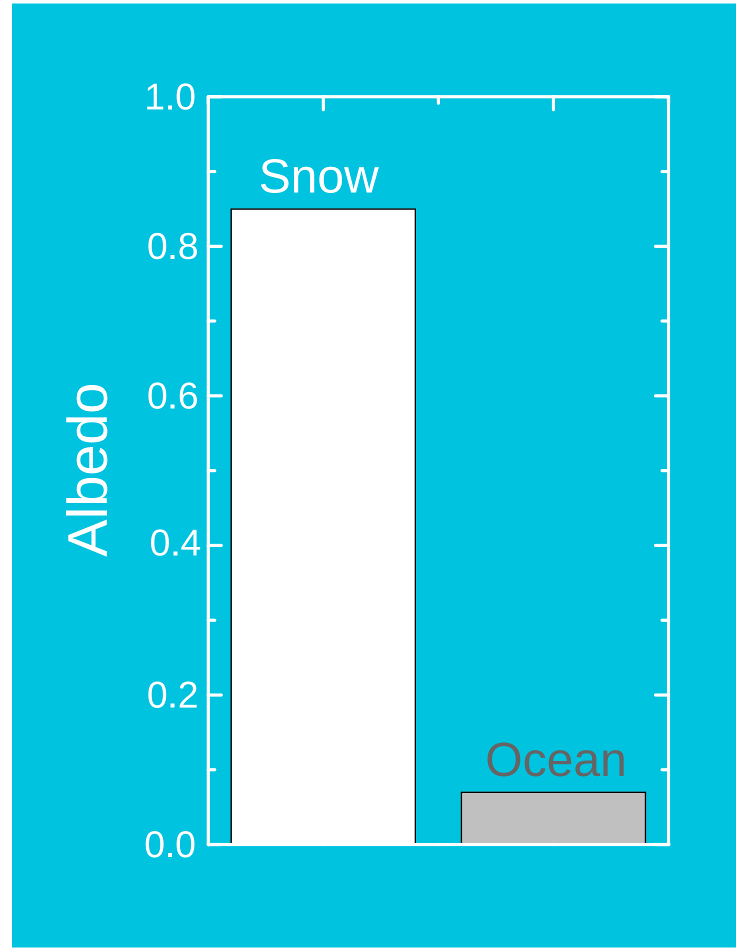
southern cryosphere




SEA ICE covers 7 - 10% of earth's ocean surface
• boundary between ocean and atmosphere
• mediates exchange of heat, gases, momentum
• global ocean circulation
• indicator and agent of climate change

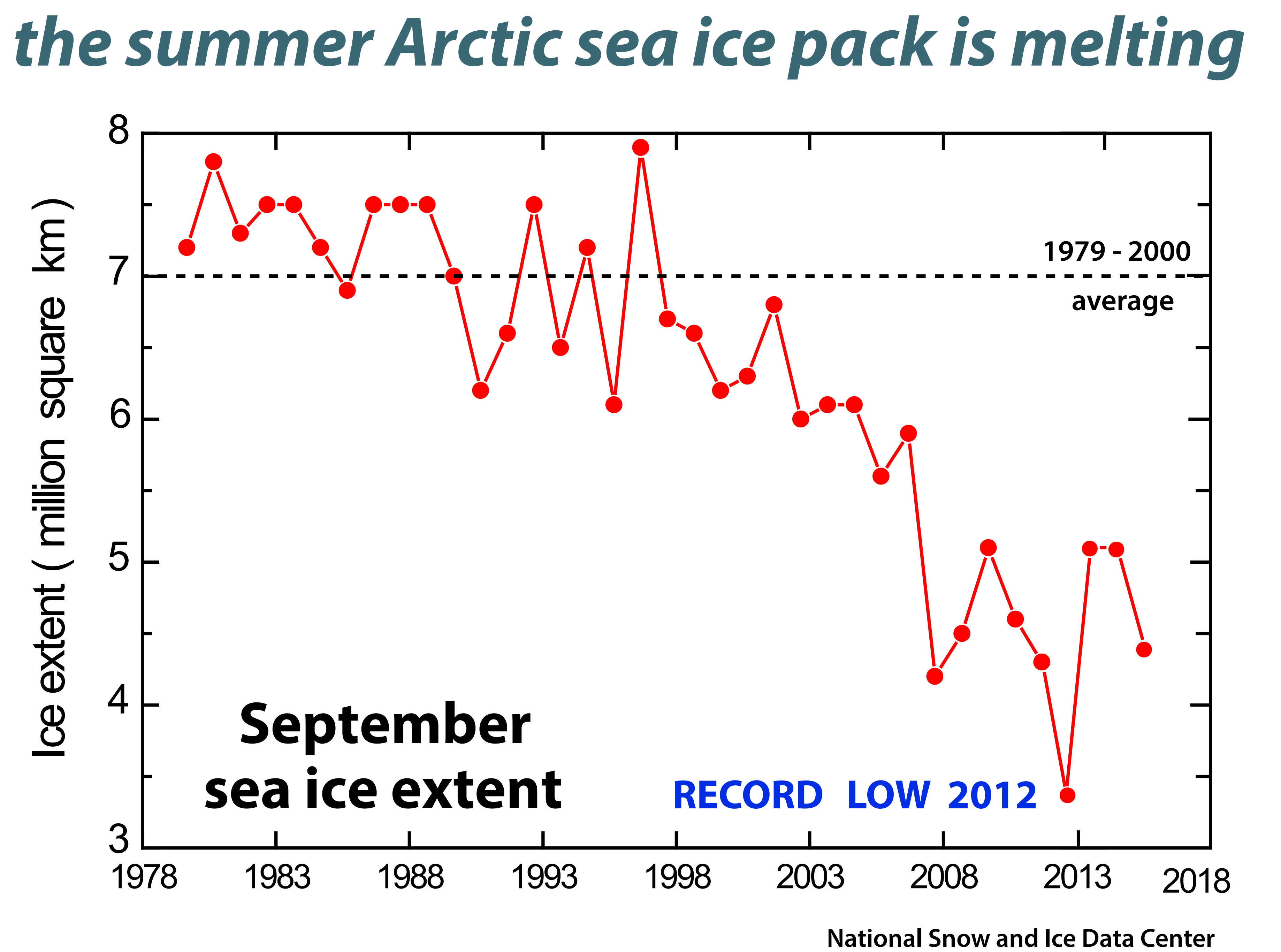
polar ice caps critical to global climate in reflecting incoming solar radiation

white snow and ice reflect



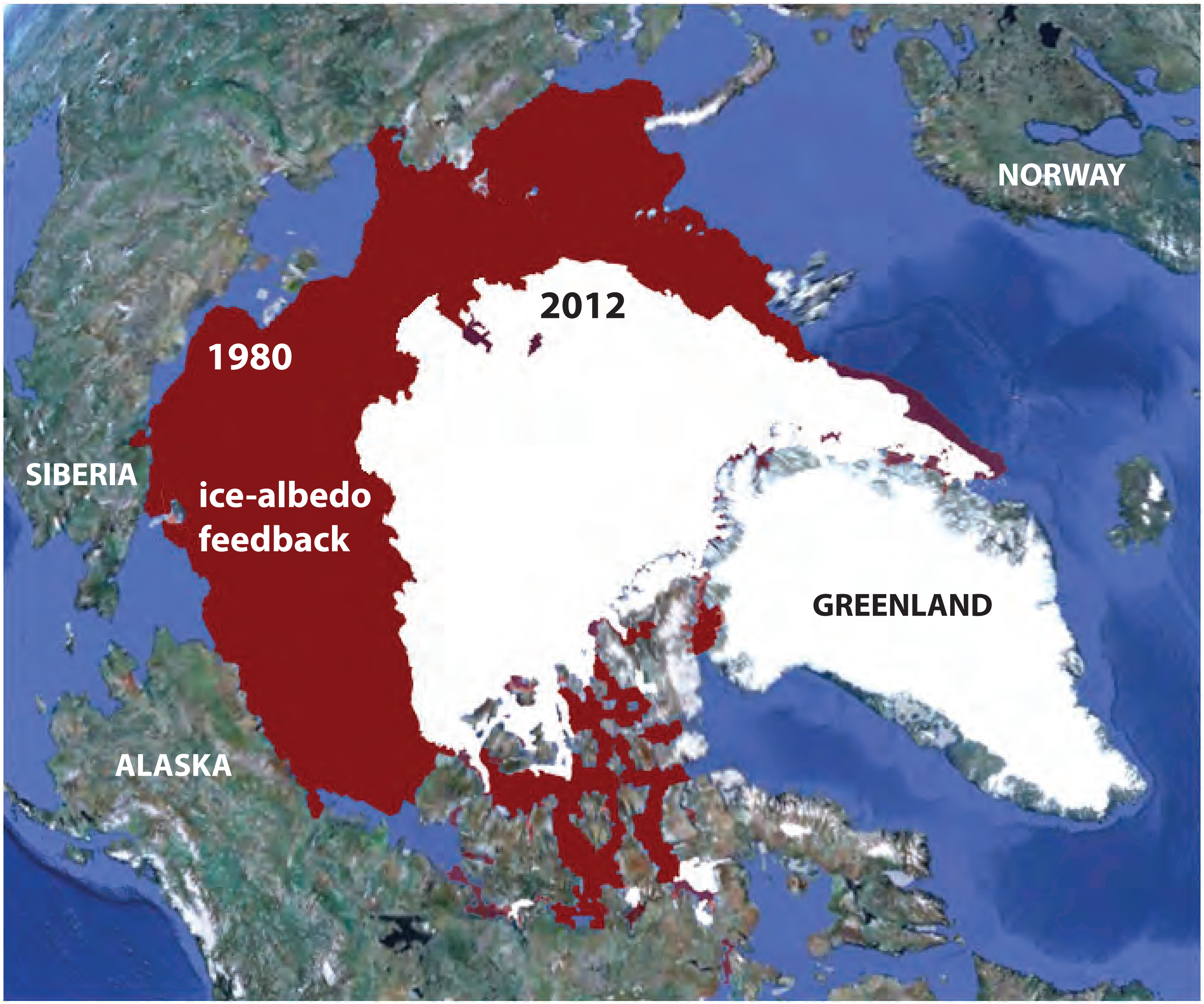
dark water and land absorb

$$\textbf{\textit{albedo}}\ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$


the summer Arctic sea ice pack is melting
8
7
6
5
4
3
Ice extent (million square km)
1979 - 2000
average
September
sea ice extent
RECORD LOW 2012
1978
1983
1988
1993
1998
2003
2008
2013
2018
National Snow and Ice Data Center

Change in Arctic Sea Ice Extent

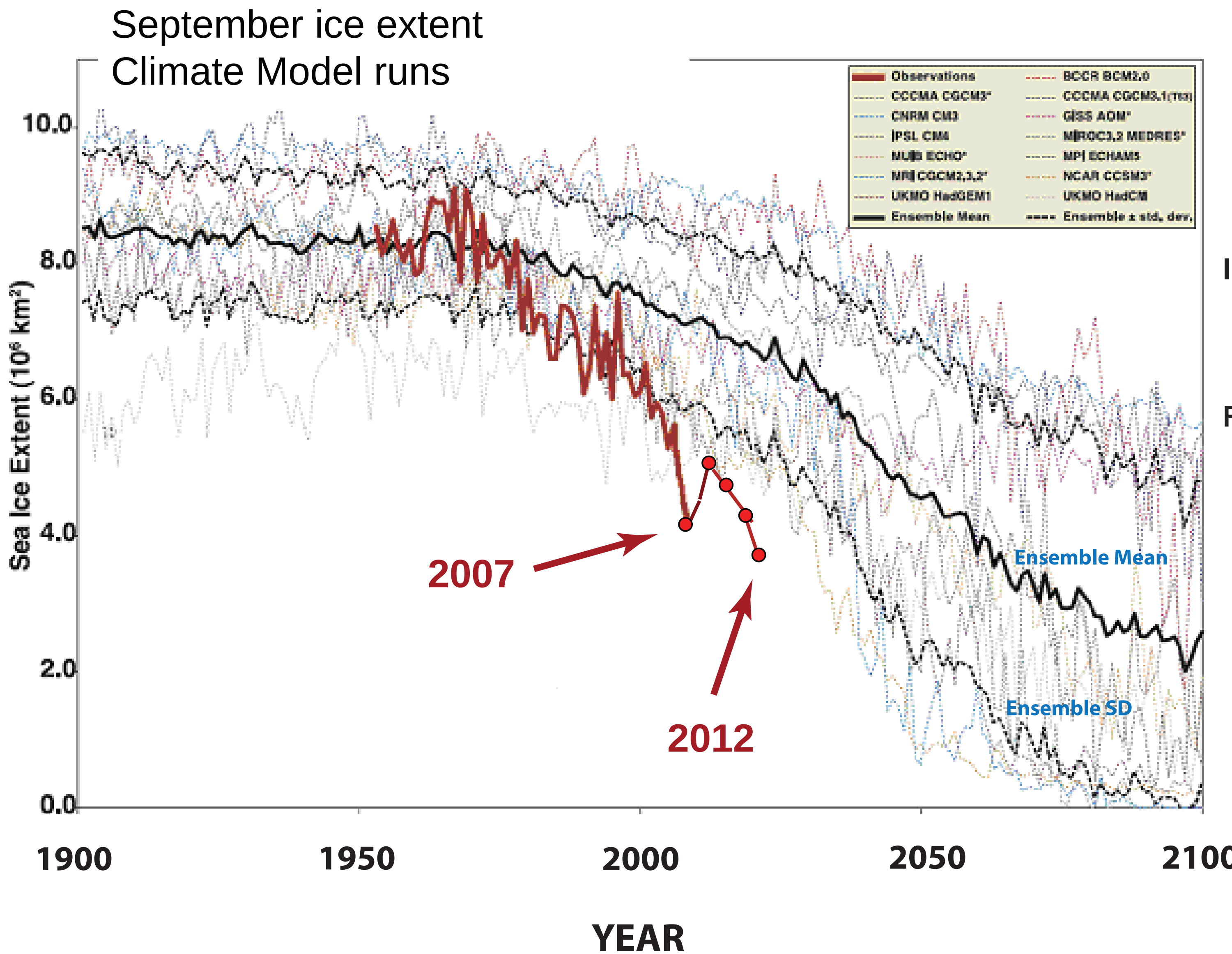
September 1980 -- 7.8 million square kilometers
September 2012 -- 3.4 million square kilometers



Perovich

Arctic sea ice decline - faster than predicted by climate models

Stroeve et al., GRL, 2007



IPCC AR4 Models

Intergovernmental Panel on Climate Change (IPCC)

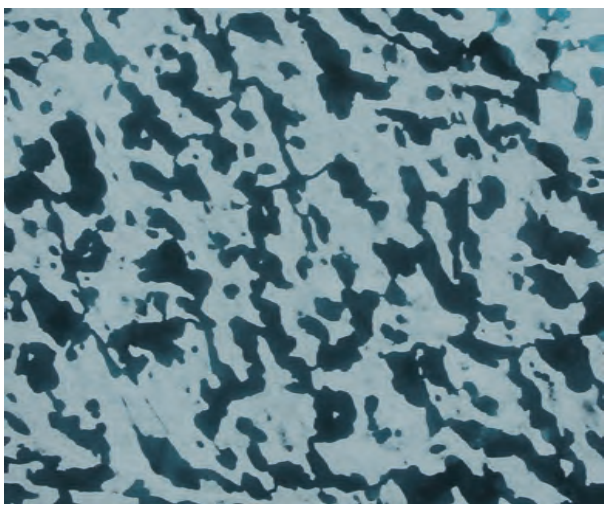
Fourth Assessment AR4, 2007

challenge

represent sea ice more rigorously in climate models

account for key processes

such as melt pond evolution



Impact of melt ponds on Arctic sea ice simulations from 1990 to 2007

Flocco, Schroeder, Feltham, Hunke, *JGR Oceans* 2012

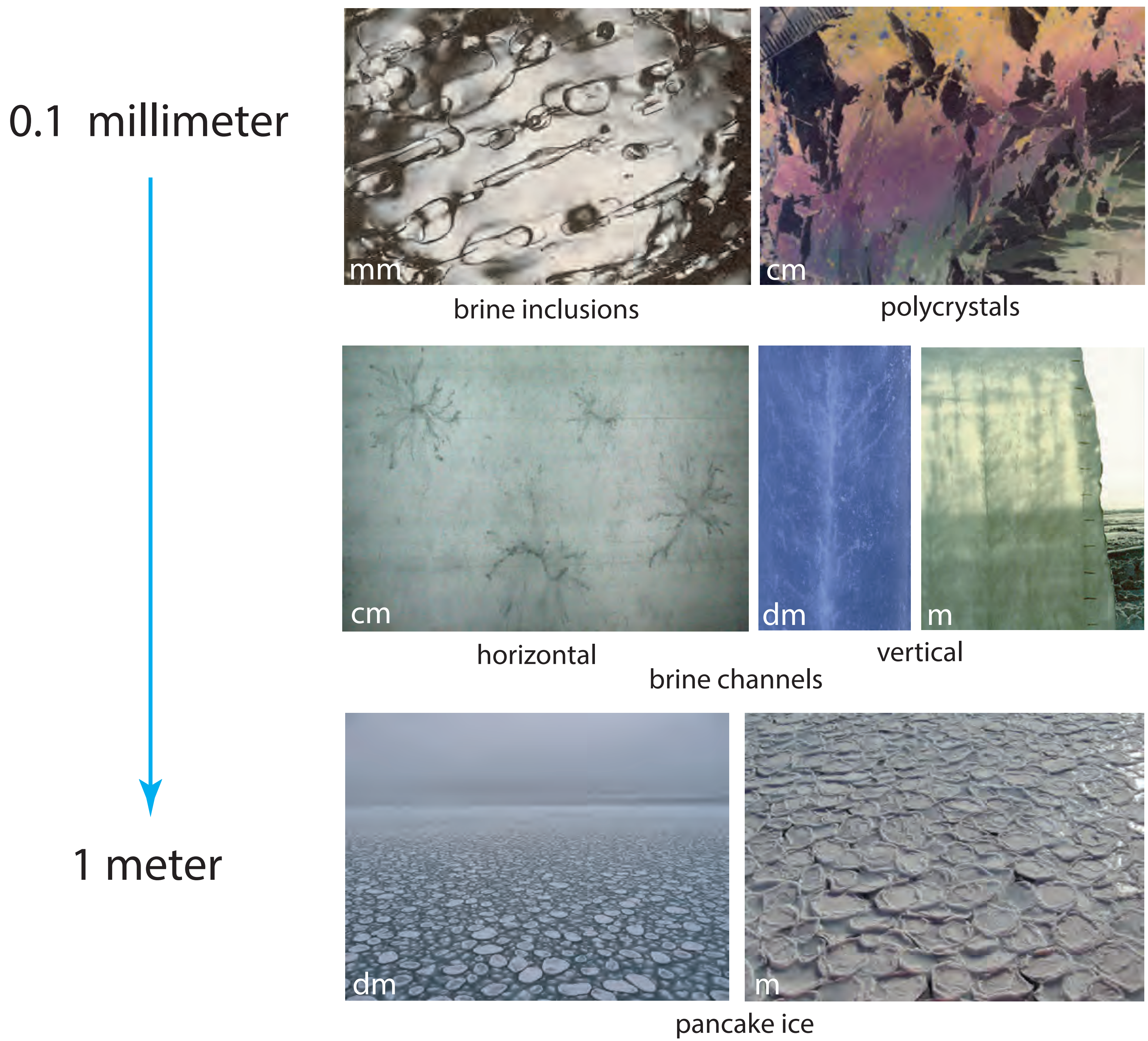
For simulations with ponds September ice volume is nearly 40% lower.

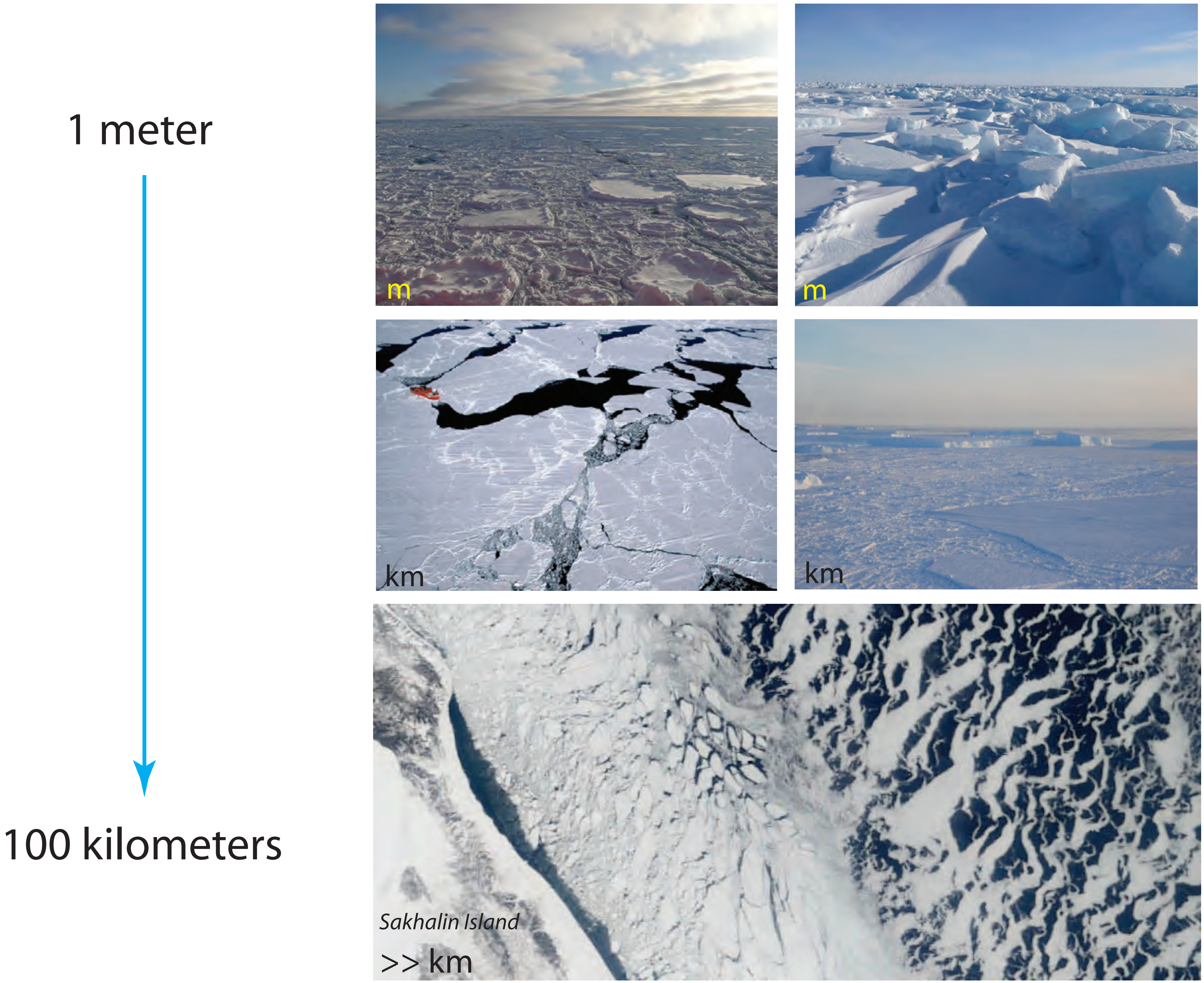
... and other sub-grid scale structures and processes

linkage of scales

sea ice is a multiscale composite

displaying structure over 10 orders of magnitude




1 meter
100 kilometers
m
m
km
km
Sakhalin Island
>> km

What is this talk about?

Using the mathematics of composite materials and statistical physics to study sea ice structures and processes ... to improve projections of climate change.

... and a brief tour through some of Joe's work!

1. Climate modeling and the polar ice packs

partial differential equations, stochastic differential equations

2. Sea ice microphysics and composite structure

homogenization, fluid flow, diffusion processes, percolation theory

3. EM monitoring of sea ice

homogenization, integral representations, random matrix theory

4. Advection diffusion; polycrystals; waves in the MIZ

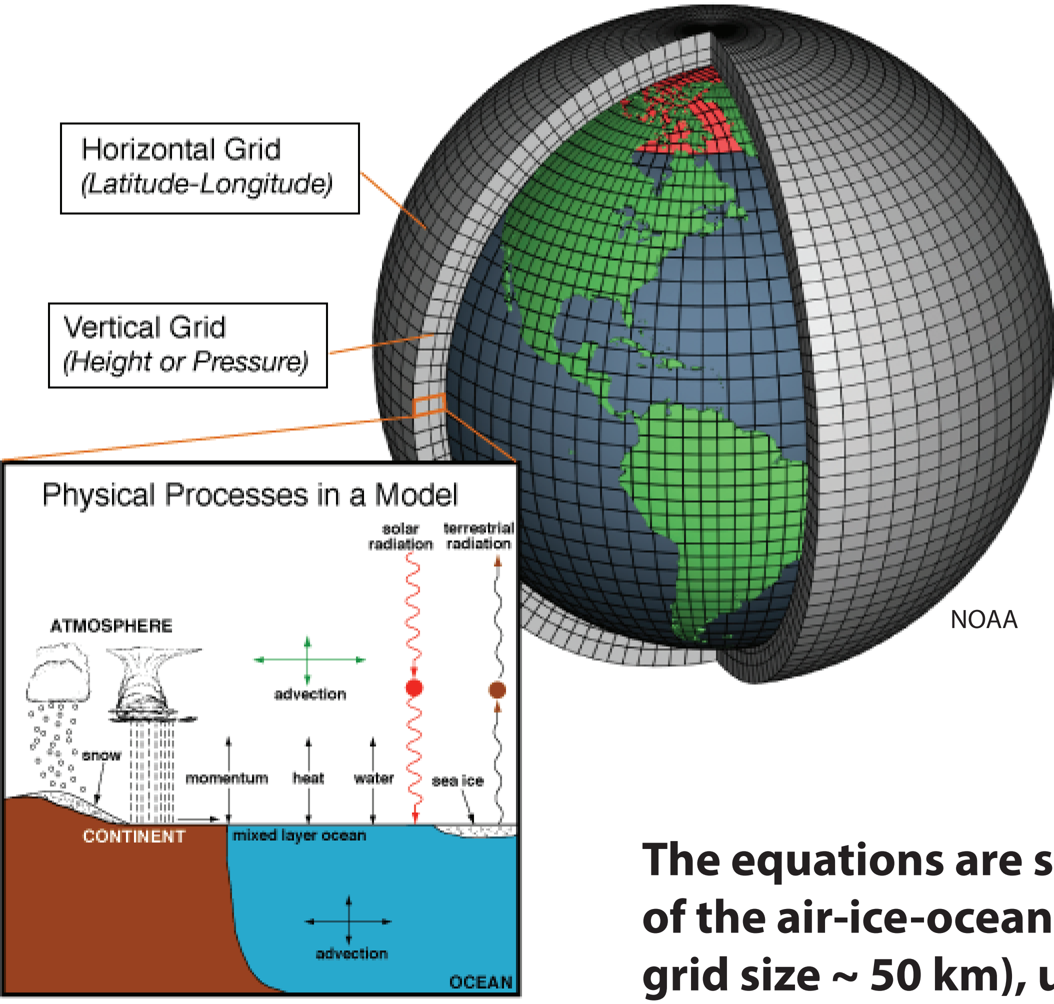
homogenization, integral representations, bounds

5. Evolution of Arctic melt ponds, fractal geometry

continuum percolation, network and Ising models

critical behavior ***cross - pollination***

Global Climate Models



NOAA

Climate models are systems of partial differential equations (PDE) derived from the basic laws of physics, chemistry, and fluid motion.

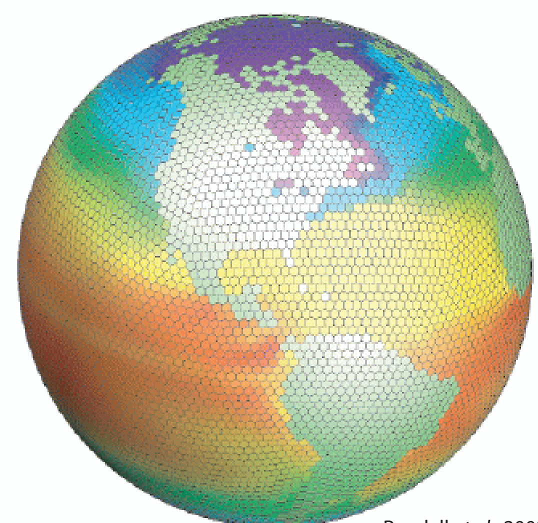
They describe the state of the ocean, ice, atmosphere, land, and their interactions.

The equations are solved on 3-dimensional grids of the air-ice-ocean-land system (with horizontal grid size ~ 50 km), using very powerful computers.

key challenge :

incorporating sub - grid scale processes

linkage of scales



Randall *et al.*, 2002

sea ice components of GCM's

What are the key ingredients -- or ***governing equations***
that need to be solved on grids using powerful computers?

1. Ice thickness distribution $g(x, y, h, t)$ evolution equation

(Thorndike *et al.* 1975)

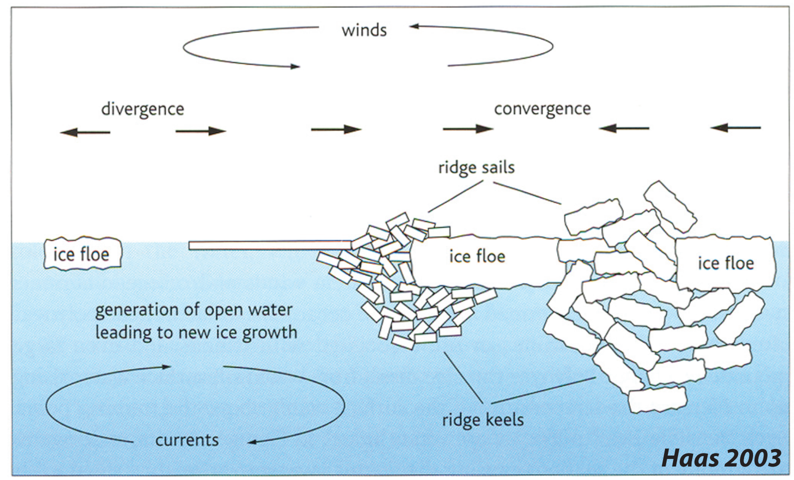
dynamics
+
thermodynamics

$$\frac{Dg}{Dt} = -g\nabla \cdot \mathbf{u} + \Psi(g) - \frac{\partial}{\partial h}(\tau g) + \mathcal{L}$$

nonlinear PDE with ice velocity field

ice growth ice melting

mechanical redistribution - ridging and opening



2. Conservation of momentum, stress *vs.* strain relation

(Hibler 1979)

$$m\frac{D\mathbf{u}}{Dt} = -mf\mathbf{k} \times \mathbf{u} + \boldsymbol{\tau}_a + \boldsymbol{\tau}_o - mg\nabla H + \mathbf{F}_{int}$$

F = ma **for sea ice** ***dynamics***

3. Heat equation of sea ice and snow

(Maykut and Untersteiner 1971)

$$\frac{\partial T}{\partial t} + \mathbf{u}_{br} \cdot \nabla T = \nabla \cdot k(T)\, \nabla T$$

thermodynamics

\+ ***balance of radiative and thermal fluxes on interfaces***

transform ice thickness distribution equation to Fokker-Planck type equation; Boltzmann framework

Toppaladoddi and Wettlaufer, *PRL*, 2015

thickness h is a diffusion process with probability density g(h,t)

"microscopic" mechanical processes that influence
ice thickness distribution— rafting, ridging, and open water formation
occur over very rapid time scales relative to geophysical-scale changes of *g(h)*

$$\Psi = \int_0^\infty [g(h+h')w(h+h',h') - g(h)w(h,h')]dh'$$

w = ***transition probability***

moments k_1 , k_2

Fokker-Planck

$$\frac{\partial g}{\partial t} = -\frac{\partial}{\partial h}\left[\left(\frac{\epsilon}{h} - k_1\right) g\right] + \frac{\partial^2}{\partial h^2}(k_2 g)$$

Langevin

$$\frac{dh}{dt} = \left(\frac{\epsilon}{h} - k_1\right) + \sqrt{2k_2}\,\xi(t)$$

$\xi(t)$ = ***Gaussian white noise***

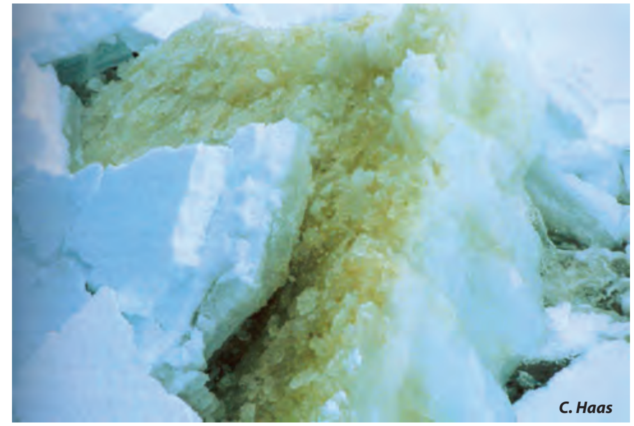
sea ice microphysics

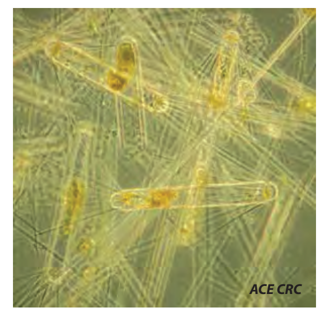
fluid transport

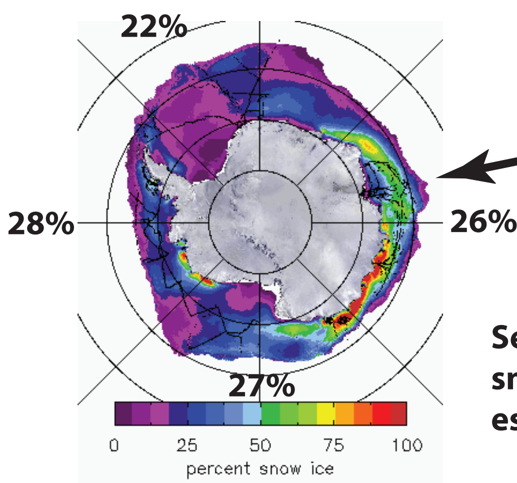
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities







T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

sea ice ecosystem

sea ice algae
support life in the polar oceans

fluid permeability *k* of a porous medium

porous concrete

how much water gets through the sample per unit time?

HOMOGENIZATION

mathematics for analyzing effective behavior of heterogeneous systems

Random media Joseph B. Keller

1964 F. Karal and J. B. Keller, *J. Math. Phys.*
Elastic, Electromagnetic, and Other Waves in a Random Medium

1979 J. B. Keller, Mathematics of Wave Propagation in Random Media

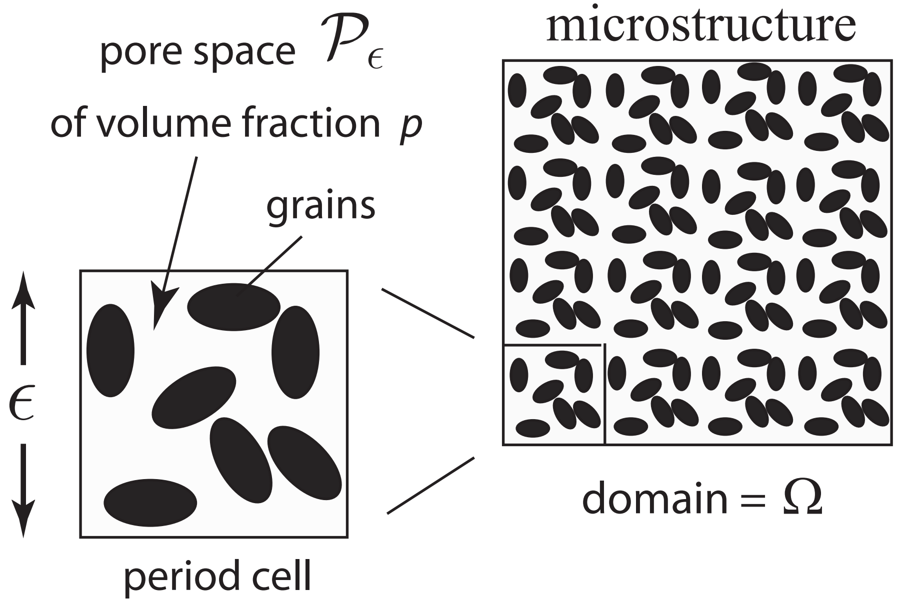
1980 J. B. Keller, *Nonlinear PDE in Engineering and Applied Science*
Darcy's law for flow in porous media and the two-space method

1991 R. Burridge and J. B. Keller, *J. Acoustical. Soc. Amer.*
Poroelasticity equations derived from microstructure

2001 J. B. Keller, *Transport in Porous Media*
Flow in random porous media

HOMOGENIZE as $\epsilon \to 0$

Stokes equations for fluid velocity $\mathbf{v}^\epsilon$, pressure p^ϵ, force $\mathbf{f}$:



$$\nabla p^\epsilon - \epsilon^2 \eta \Delta \mathbf{v}^\epsilon = \mathbf{f}, \qquad x \in \mathcal{P}_\epsilon$$

$$\nabla \cdot \mathbf{v}^\epsilon = 0, \qquad x \in \mathcal{P}_\epsilon$$

$$\mathbf{v}^\epsilon = 0, \qquad x \in \partial \mathcal{P}_\epsilon$$

η = fluid viscosity

via two-scale expansion

MACROSCOPIC EQUATIONS $\mathbf{v}^\epsilon \to \mathbf{v}$, $p^\epsilon \to p$ as $\epsilon \to 0$

Darcy's law $(\mathbf{f} = 0)$

$$\mathbf{v} = -\frac{1}{\eta}\,\mathbf{k}\,\nabla p, \qquad x \in \Omega$$

$$\nabla \cdot \mathbf{v} = 0, \qquad x \in \Omega$$

$\mathbf{k}(x)$ = effective fluid permeability tensor

[Keller '80, Tartar '80, Sanchez-Palencia '80, J. L. Lions '81, Allaire '89, '91,'97]

Darcy's Law for slow viscous flow in a porous medium

averaged fluid velocity

pressure gradient

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\,\nabla p$$

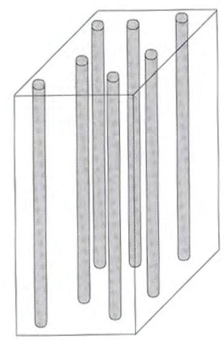
viscosity

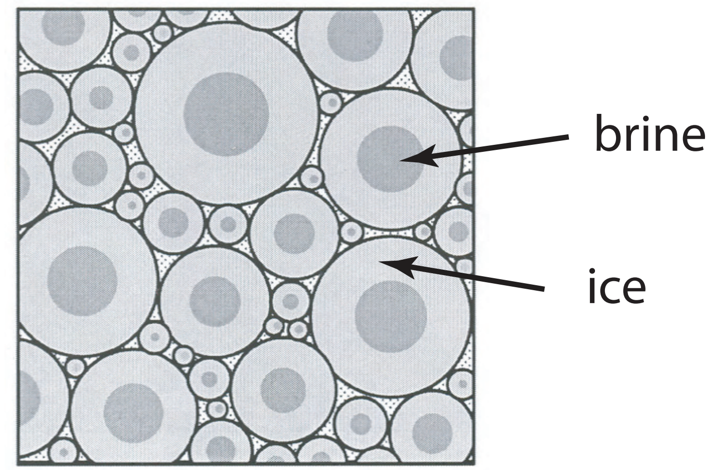
$\mathbf{k}$ = fluid permeability tensor

PIPE BOUNDS on vertical fluid permeability k

Golden, Heaton, Eicken, Lytle, Mech. Materials 2006
Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophys. Res. Lett. 2007

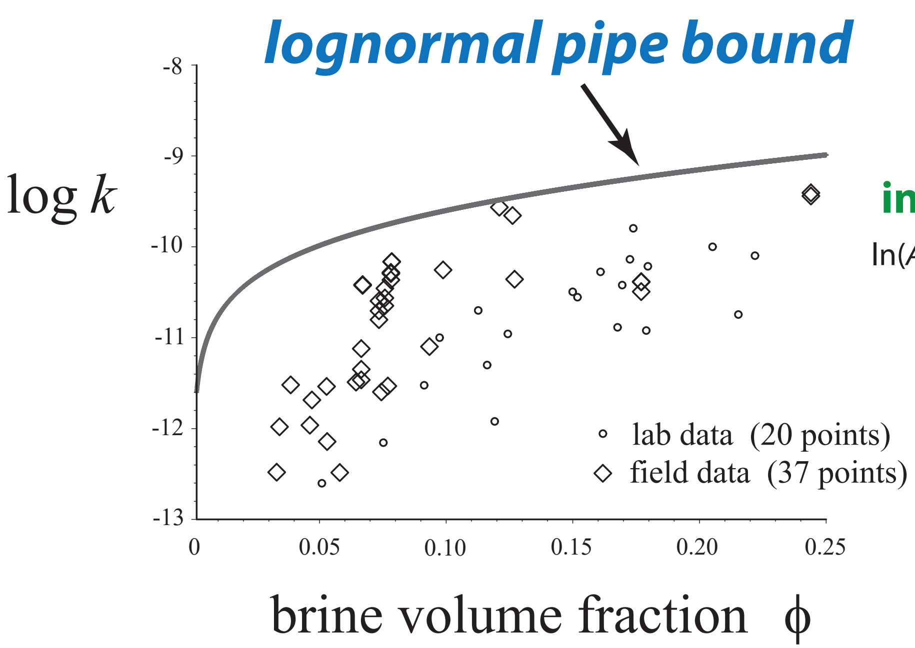
vertical pipes
with appropriate radii
maximize k





optimal coated cylinder geometry

fluid analog of arithmetic mean upper bound for effective conductivity of composites (Wiener 1912)



Golden et al., Geophys. Res. Lett. 2007

$$k \;\leq\; \frac{\phi\langle R^4\rangle}{8\langle R^2\rangle} \;=\; \frac{\phi}{8}\langle R^2\rangle e^{\sigma^2}$$

inclusion cross sectional areas A lognormally distributed

ln(A) normally distributed, mean μ (increases with T) variance σ^2 (Gow and Perovich 96)

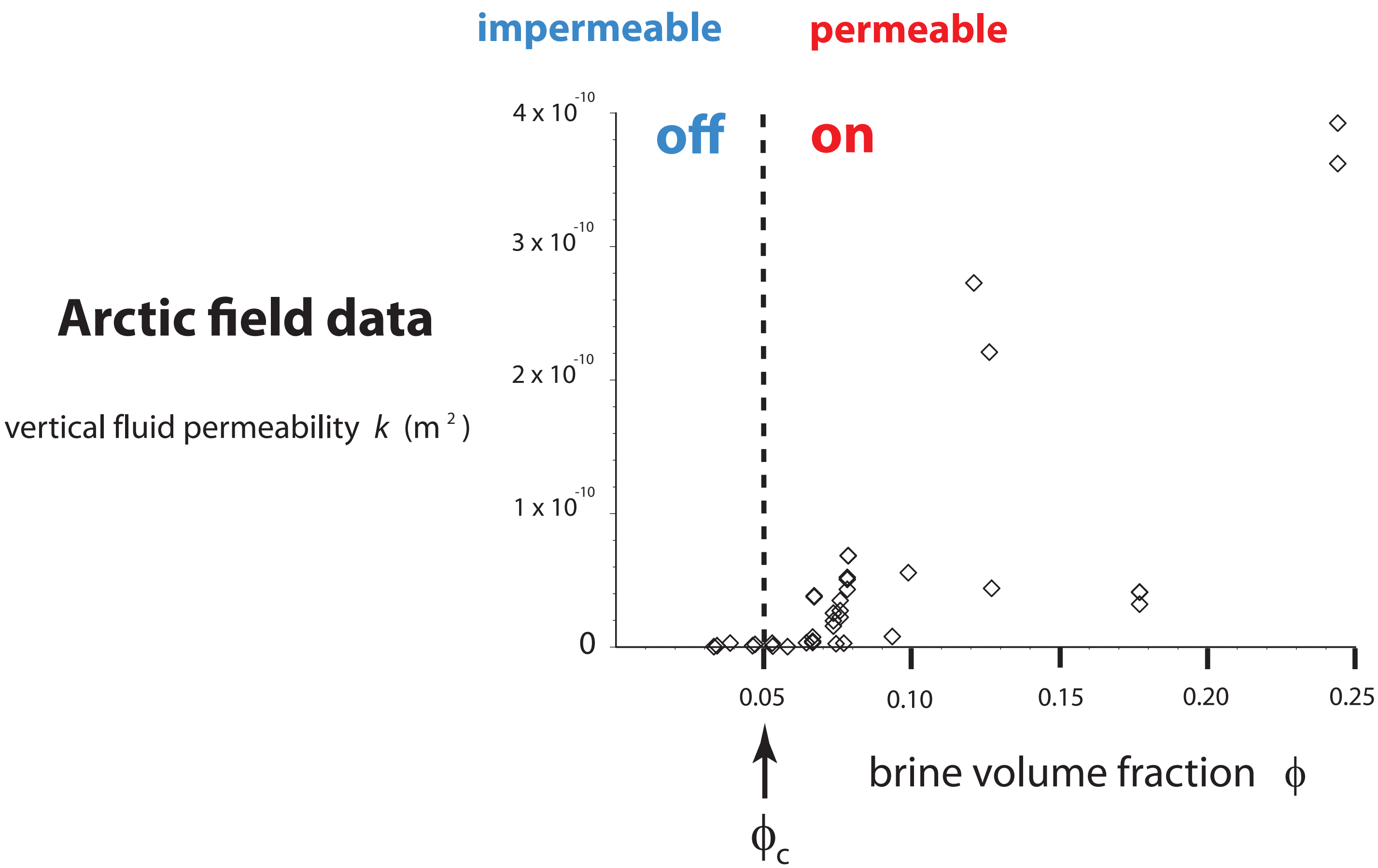
get bounds through variational analyis of ***trapping constant*** γ for diffusion process in pore space with absorbing BC

Torquato and Pham, *PRL* 2004

$$\mathbf{k} \;\leq\; \gamma^{-1}\mathbf{I}$$

for any ergodic porous medium (Torquato 2002, 2004)

Critical behavior of fluid transport in sea ice



on - off switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\%$ $\longleftrightarrow$ $T_c \approx -5^\circ C$, $S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

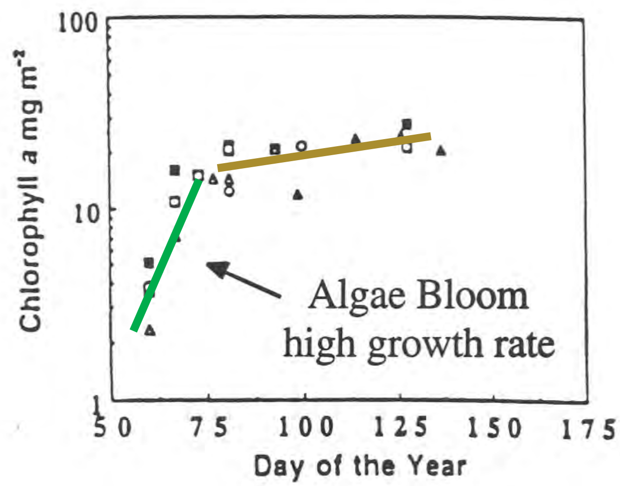

sea ice algal communities

D. Thomas 2004

nutrient replenishment
controlled by ice permeability

biological activity turns on
or off according to
rule of fives

critical behavior of microbial activity



Convection-fueled algae bloom
Ice Station Weddell

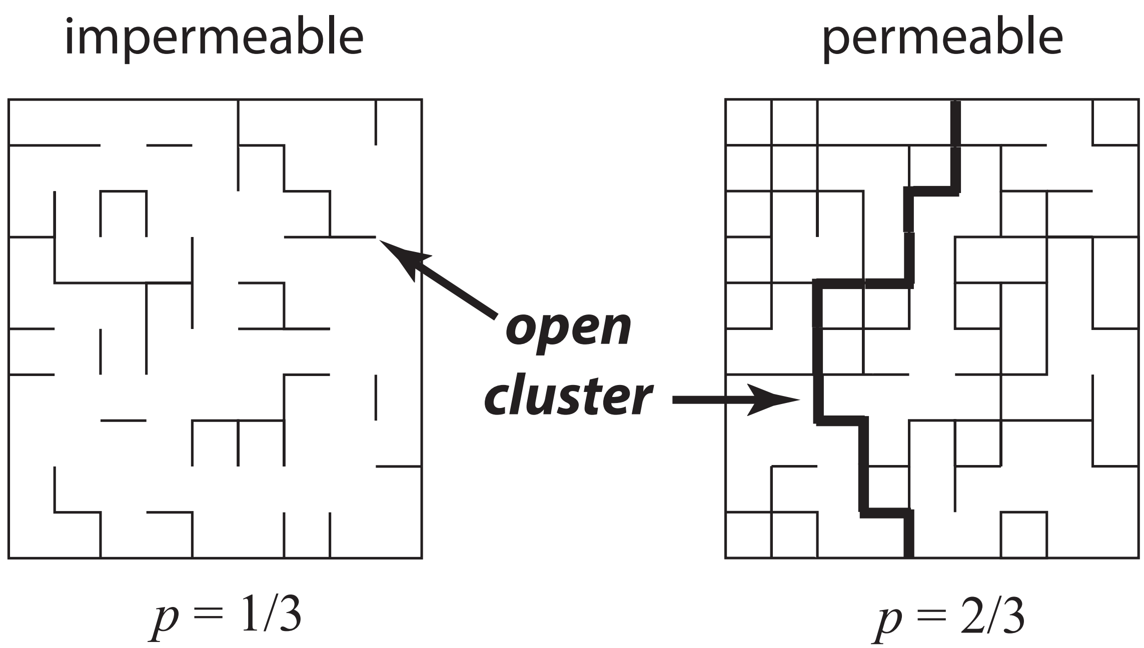
Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

Why is the rule of fives true?

percolation theory

probabilistic theory of connectedness



bond → *open* with probability p
closed with probability $1-p$

percolation threshold

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

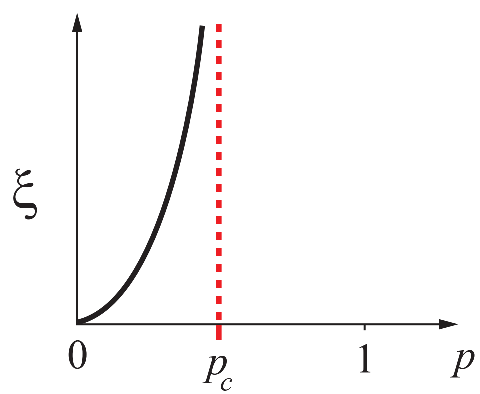
smallest *p* for which there is an infinite open cluster

order parameters in percolation theory

geometry

correlation length

characteristic scale
of connectedness

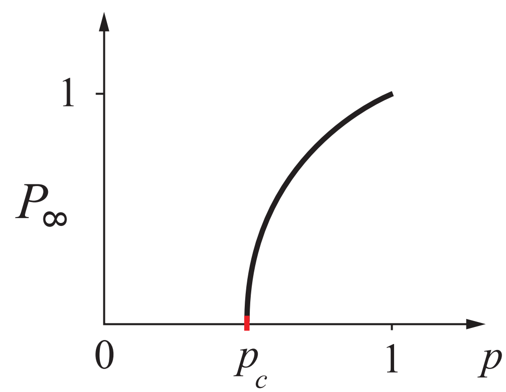


$$\xi(p) \sim |p - p_c|^{-\nu}$$

$$p \to p_c$$

infinite cluster density

probability the origin
belongs to infinte cluster

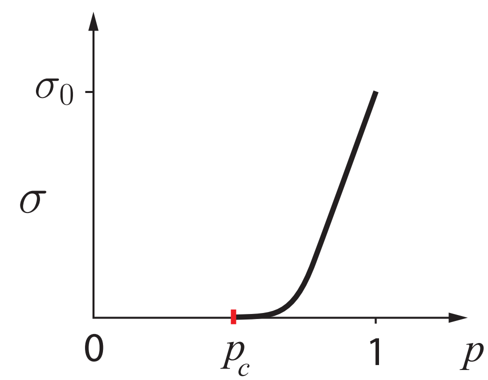


$$P_\infty(p) \sim (p - p_c)^\beta$$

$$p \to p_c^+$$

transport

effective conductivity or fluid permeability



$$\sigma(p) \sim \sigma_0 (p - p_c)^t$$

$$p \to p_c^+$$

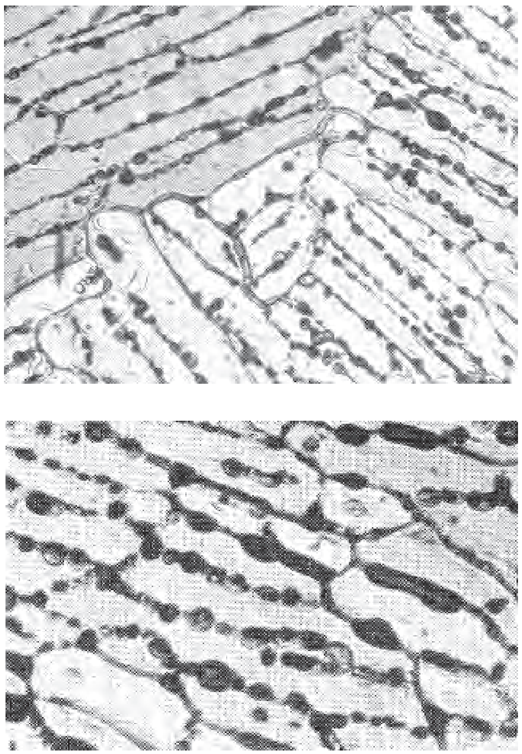
UNIVERSAL critical exponents for lattices -- depend only on dimension

1 ≤ *t* ≤ 2 (for idealized model), Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992

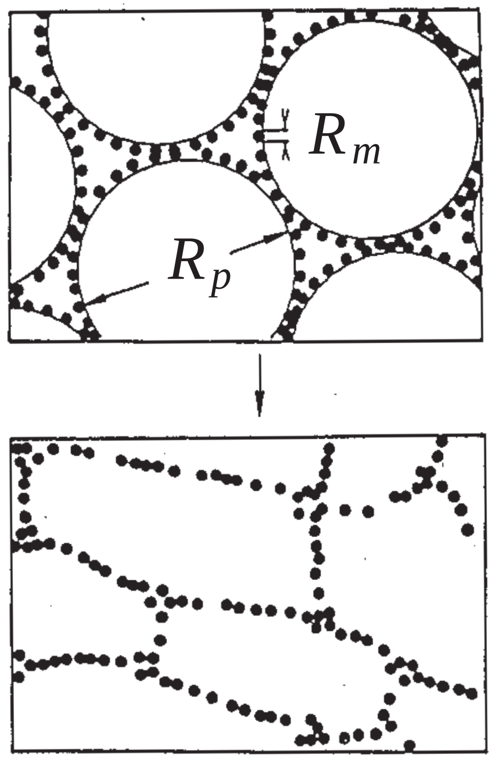
non-universal behavior in continuum

Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

$\phi_c \approx 5\,\%$ Golden, Ackley, Lytle, *Science*, 1998



sea ice



compressed powder

radar absorbing composite

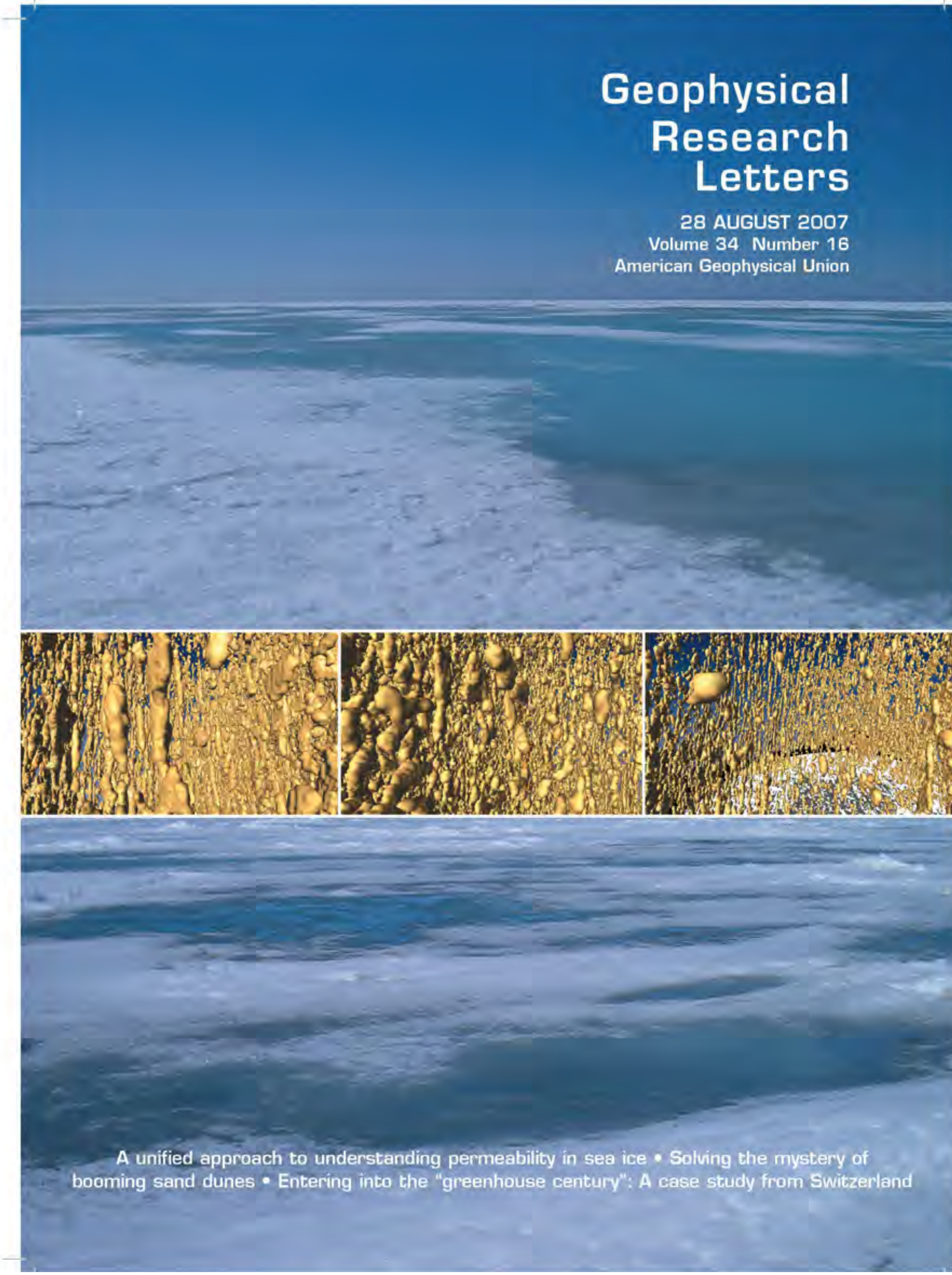
sea ice is radar absorbing

Thermal evolution of permeability and microstructure in sea ice — Golden, Eicken, Heaton, Miner, Pringle, Zhu

micro-scale

controls

macro-scale processes



rigorous bounds
percolation theory
hierarchical model
network model

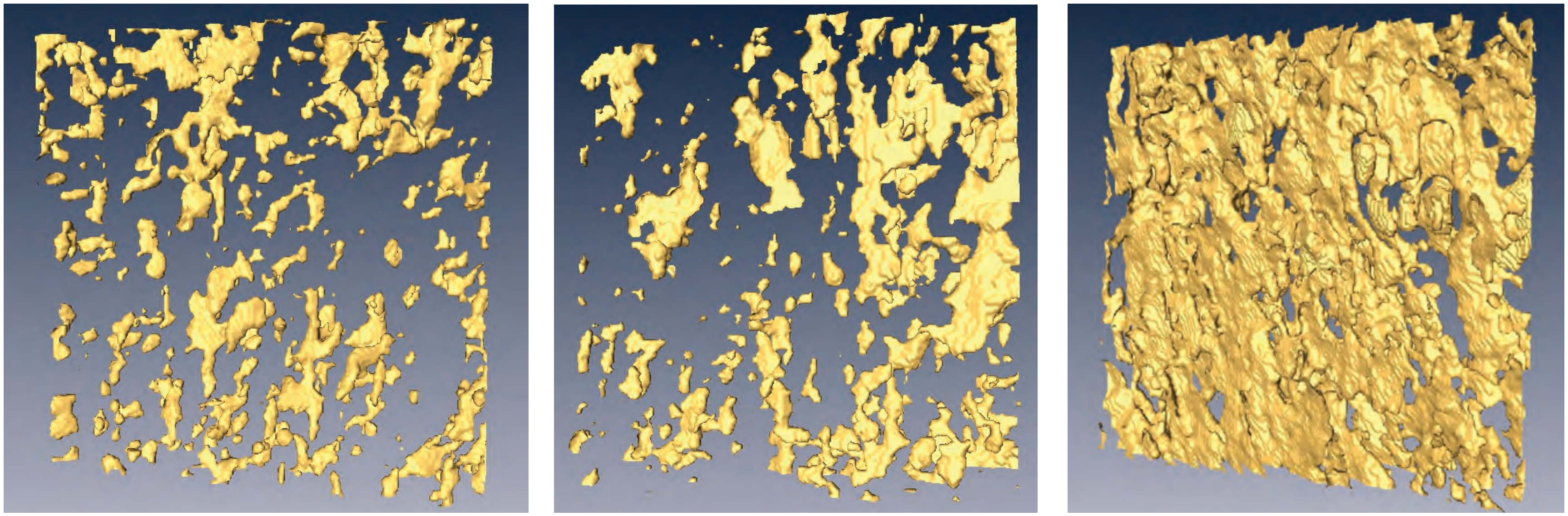
field data

X-ray tomography for brine inclusions

unprecedented look at thermal evolution of brine phase and its connectivity

brine connectivity (over cm scale)

8 x 8 x 2 mm



-15 °C, $\phi = 0.033$ -6 °C, $\phi = 0.075$ -3 °C, $\phi = 0.143$

X-ray tomography confirms percolation threshold

3-D images
pores and throats → 3-D graph
nodes and edges

analyze graph connectivity as function of temperature and sample size

- ***use finite size scaling techniques to confirm rule of fives***
- ***order parameter data from a natural material***

Pringle, Miner, Eicken, Golden, J. Geophys. Res. 2009

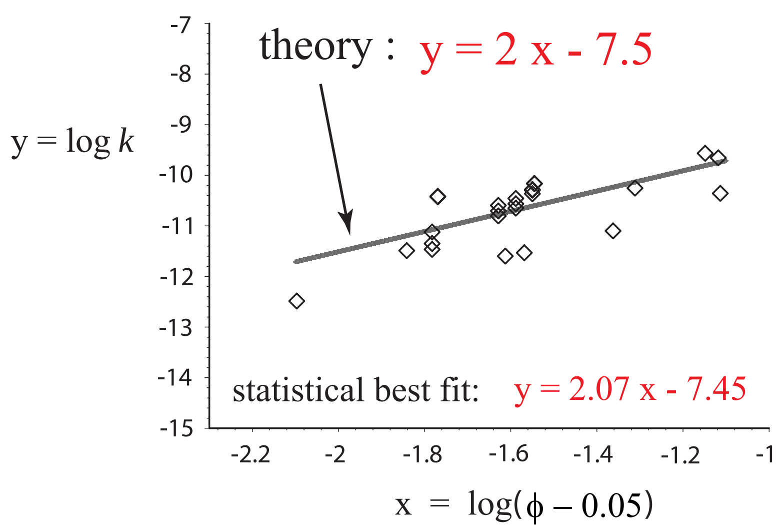
lattice and continuum percolation theories yield:

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

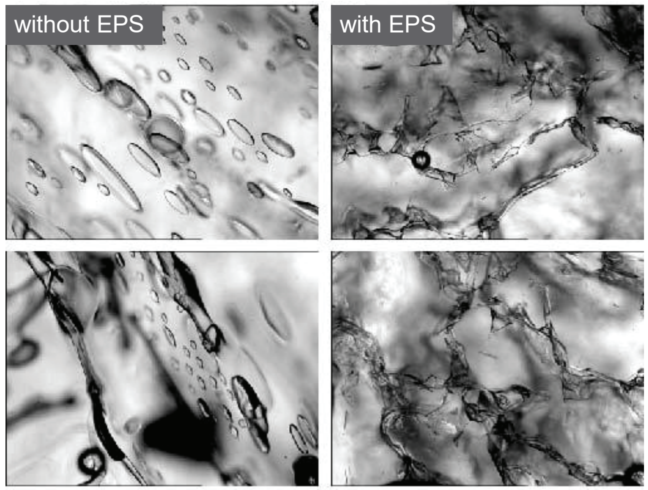
$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

- exponent is UNIVERSAL lattice value $t \approx 2.0$
- sedimentary rocks like sandstones also exhibit universality
- critical path analysis -- developed for electronic hopping conduction -- yields scaling factor k_0

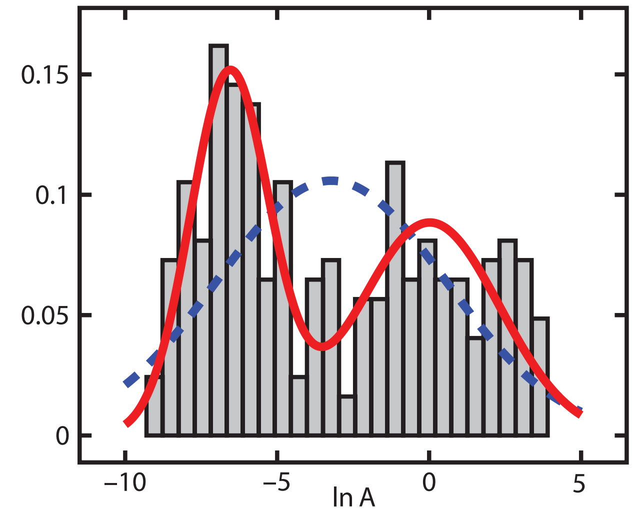


Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

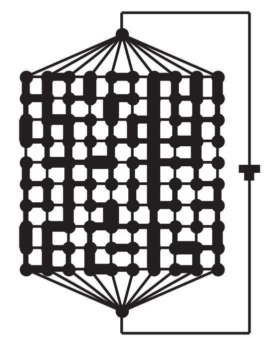
How does EPS affect fluid transport?

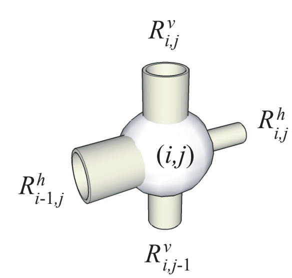


Krembs, Eicken, Deming, PNAS 2011



- **Bimodal** lognormal distribution for brine inclusions
- Develop random pipe network model with bimodal distribution; Use numerical methods that can handle larger variances in sizes.
- Results predict observed drop in fluid permeability *k*.
- **Rigorous bound on *k* for bimodal distribution of pore sizes**





Steffen, Epshteyn, Zhu, Bowler, Deming, Golden 2017

Zhu, Jabini, Golden, Eicken, Morris
Ann. Glac. 2006

How does the biology affect the physics?

Remote sensing of sea ice

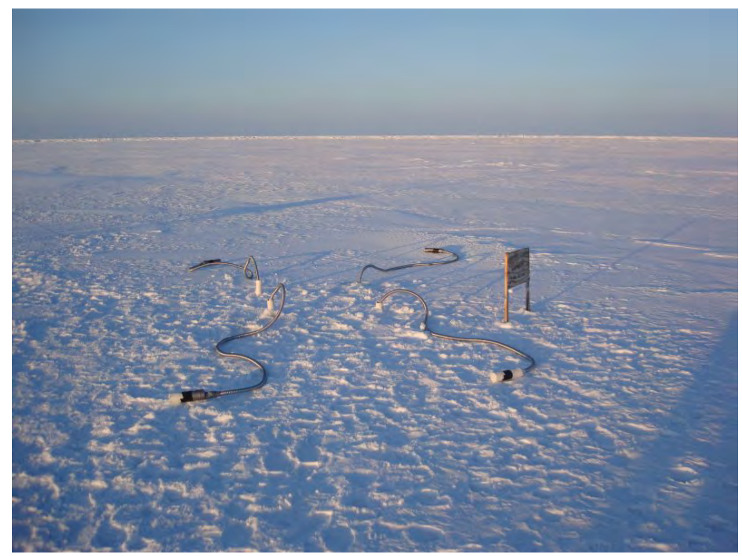
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

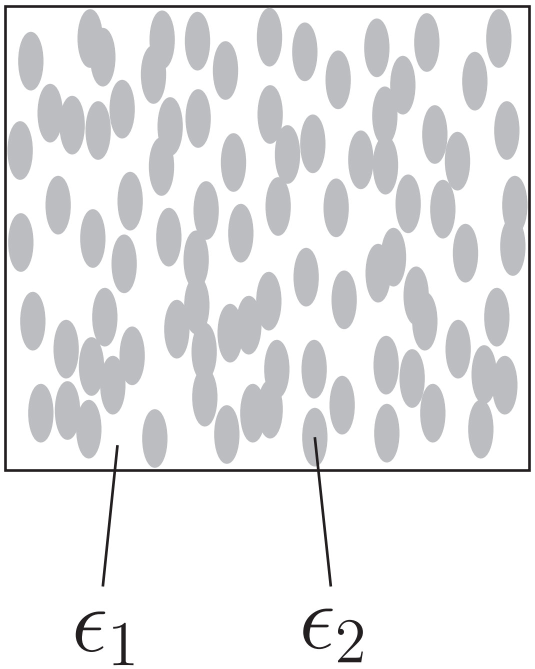
ε^*

effective complex permittivity (dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\Big\} \; \epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \; , \text{ composite geometry} \right)$$

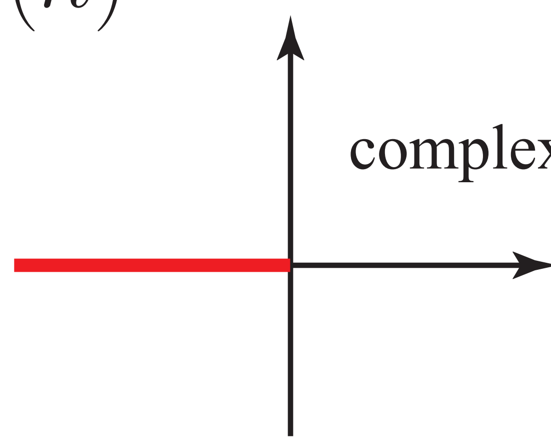
Analytic continuation method for bounding complex ϵ^*

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

$$m(h) = \frac{\epsilon^*}{\epsilon_2}\left(\frac{\epsilon_1}{\epsilon_2}\right) \qquad h = \frac{\epsilon_1}{\epsilon_2}$$

Exploit analytic properties of $m(h)$

- $m(h)$ analytic off negative real axis



- $m(h)$: UHP $\longrightarrow$ UHP

Theory of Effective Electromagnetic Behavior of Composites

analytic continuation method

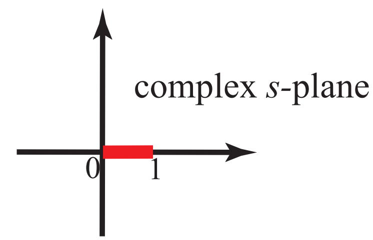
Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)
Theory of Composites, Milton (2002)

composite geometry
(spectral measure μ) ⟶ ε^*

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

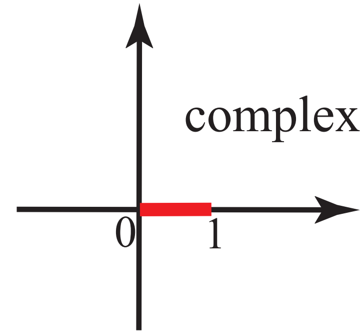
ε^* ⟶ ***composite geometry***
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

Stieltjes integral representation

separation of geometry from parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

μ
- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = (s + \Gamma\chi)^{-1} e_k$$

Golden and Papanicolaou, Comm. Math. Phys. 1983

using the Stieltjes integral representation to obtain bounds

"linear programming" Golden and Papanicolaou, *CMP* 1983

M_1 = **the set of positive Borel measures on [0,1], compact, convex**

$F_s(\mu): \; M_1 \longrightarrow \mathbb{C}$ **linear functional**

extremal values (bounds) are images of extreme points of M_1

Dirac point measures $\dfrac{\mu_0}{s - z^*}$

---

higher order bounds -- iterated fractional linear transformations

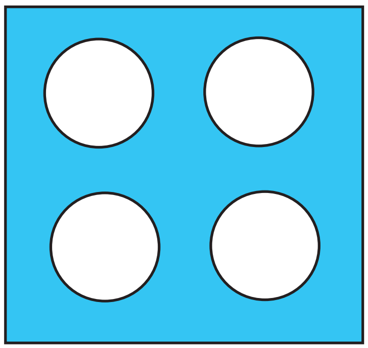
$$F_1(s) = \frac{1}{\mu_0} - \frac{1}{sF(s)}$$

←

Baker 1969
Milton 1981
Bergman 1982
Felderhof 1984
Golden 1986

Keller Interchange Theorem

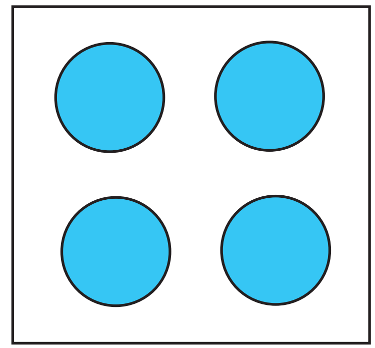
J. B. Keller, *J. Math. Phys.* 1964



two dimensional

two phase composites

$\sigma_1 \quad \sigma_2$



$$\sigma^*_{xx}(\sigma_1, \sigma_2)\,\sigma^*_{yy}(\sigma_2, \sigma_1) = \sigma_1\sigma_2$$

isotropic media

$$\sigma^*(\sigma_1, \sigma_2)\,\sigma^*(\sigma_2, \sigma_1) = \sigma_1\sigma_2$$

$d = 3$ inequality
Shulgasser *JMP* 1976

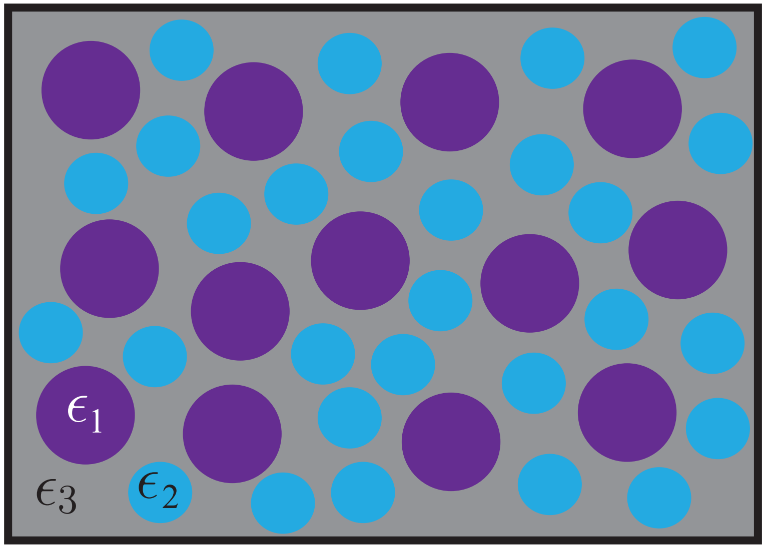
$d = 2$ square bond lattice 1, h
conductivity of random checkerboard

$$\sigma^*(p_c = 1/2) = \sqrt{h}\,, \quad h \longrightarrow 0$$

effective transport in two dimensional quasiperiodic media

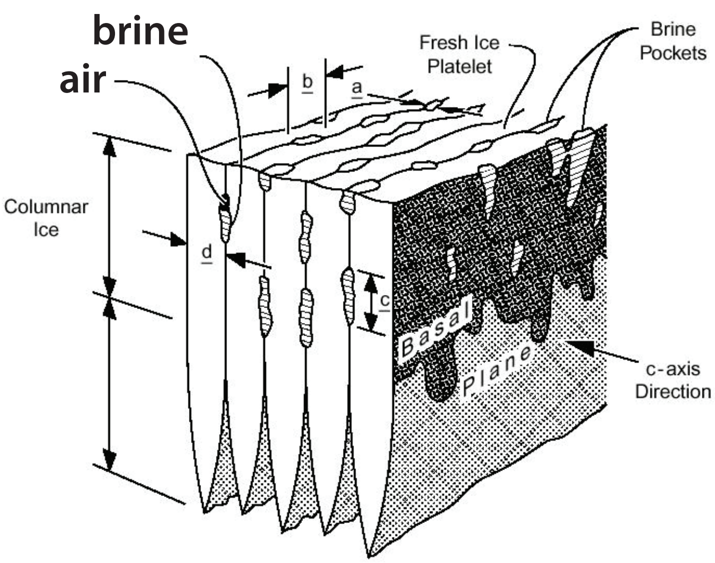
Golden, Goldestein, Lebowitz, *J. Stat. Phys.* 1990

Multiphase Media



polydisc representation formula for Herglotz function in several complex variables; bounds on effective parameters

Golden and Papanicolaou, *J. Stat. Phys.* 1985
Golden, *J. Mech. Phys. Solids* 1986



$$m(h_1, h_2) = \epsilon^*/\epsilon_3 \qquad h_i = \epsilon_i/\epsilon_3 \qquad i = 1, 2$$

$$F(s_1, s_2) = 1 - m(h_1, h_2)$$

$$s_i = 1/(1 - h_i) \qquad \zeta_j = \frac{s_j - i}{s_j + i}$$

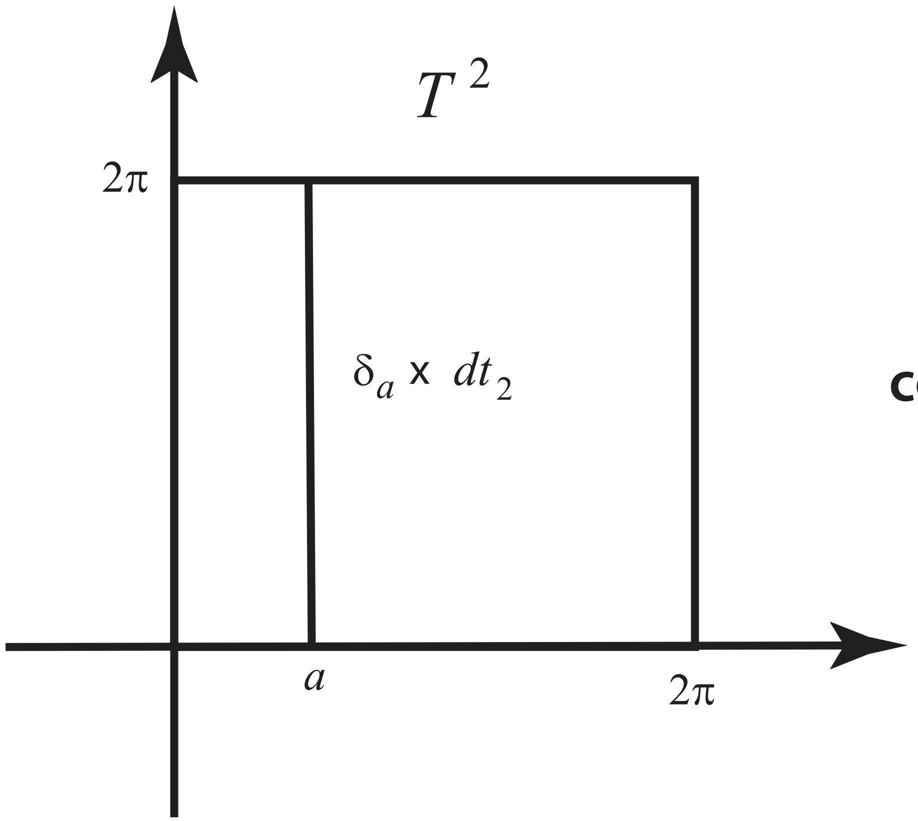
$$f(\zeta_1, \zeta_2) = iF(s_1, s_2) \qquad f(\zeta_1, \zeta_2) : \; D^2 \to \{Ref > 0\}$$

$$f(\zeta_1, \zeta_2) = iImf(0,0) + \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} (H_1 H_2 + H_1 + H_2 - 1)\mu(dt_1, dt_2)$$

$$H_1 = (e^{it_1} + \zeta_1)/(e^{it_1} - \zeta_1) \qquad H_2 = (e^{it_2} + \zeta_2)/(e^{it_2} - \zeta_2)$$

μ is a positive Borel measure on the torus T^2

satisfying a Fourier condition (excludes point measures)



conjectured extremals yield bounds

set of extremal measures = ??

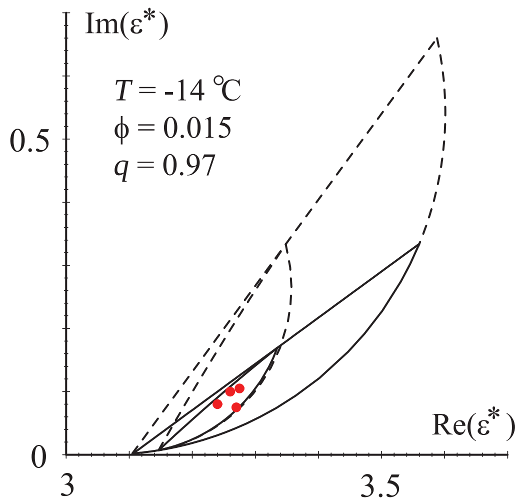
Herglotz-Nevanlinna functions and their applications
8-12 May 2017, Institut Mittag-Leffler, Stockholm

field equation recursion method
Milton, *Comm. Math. Phys. (I and II)* 1987

effective elasticity tensor
Ou, *Complex Vars. Elliptic. Eqs. 2011*

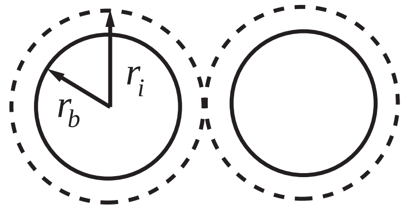
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle



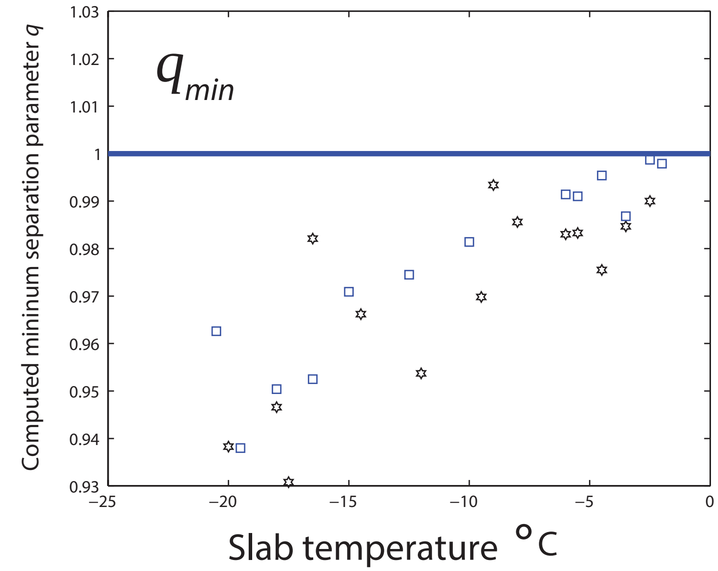
$q = r_b / r_i$

$0 < q < 1$

Golden 1995, 1997

Bruno 1991

inverse bounds



inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

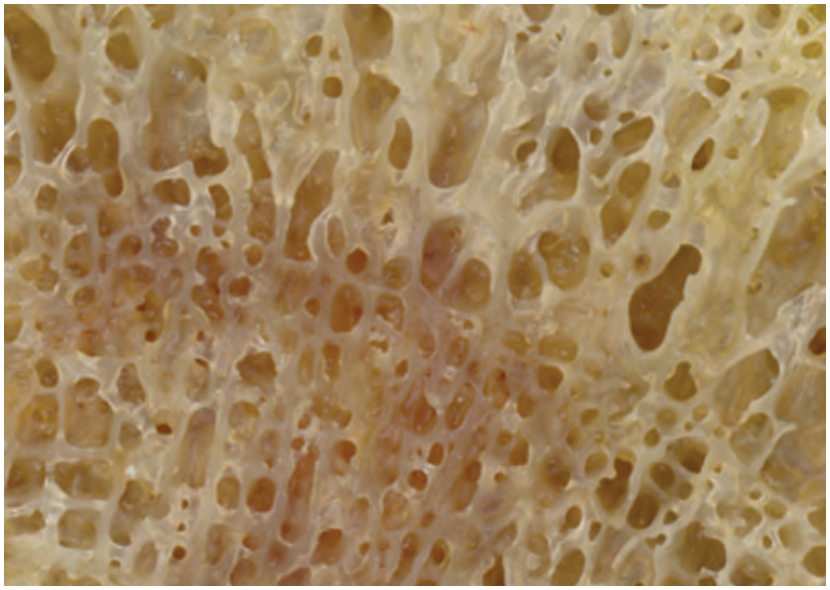
rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

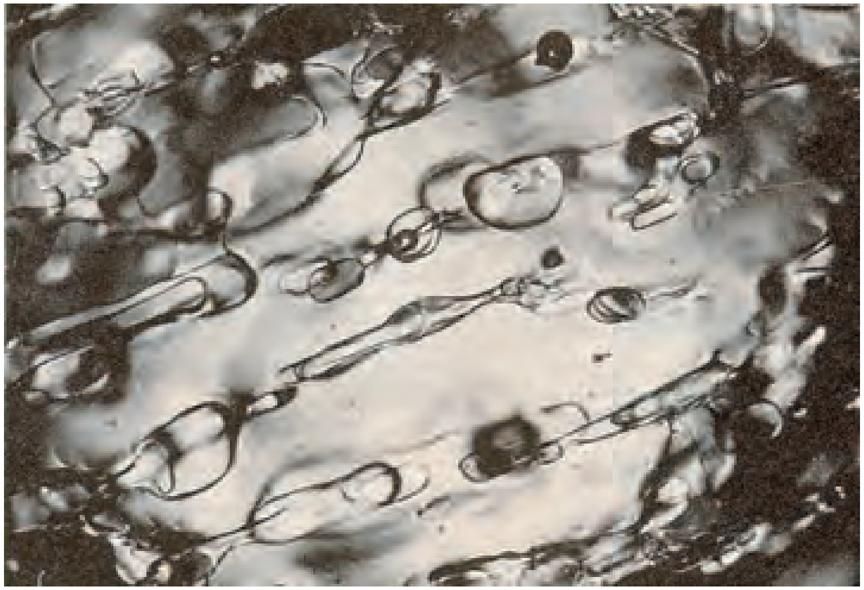
Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

the math doesn't care if it's sea ice or bone!

HUMAN BONE



SEA ICE



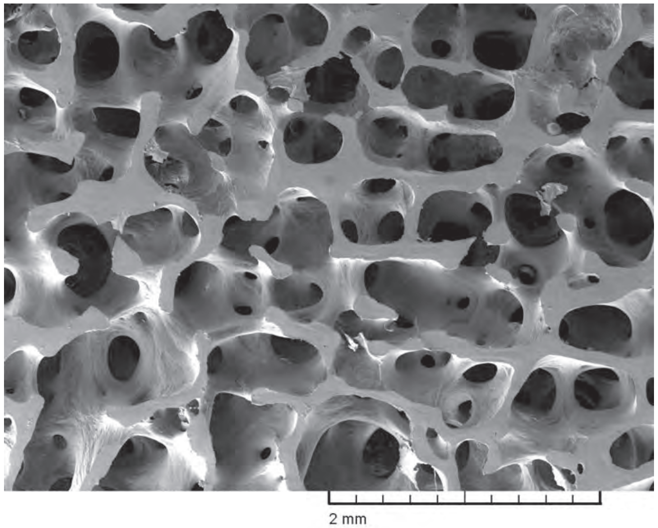
apply spectral measure analysis of brine connectivity and spectral inversion to electromagnetic monitoring osteoporosis

Golden, Murphy, Cherkaev, J. Biomechanics 2011

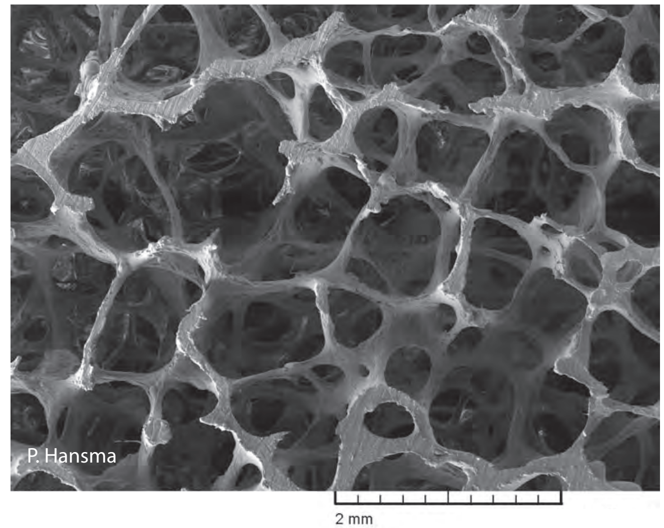
spectral characterization of porous microstructures in bone

Golden, Murphy, Cherkaev, J. Biomechanics 2011

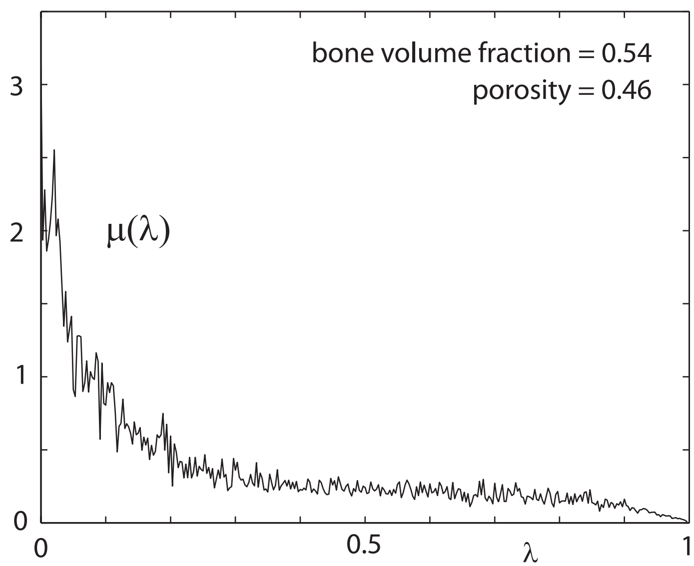
(a) young healthy trabecular bone



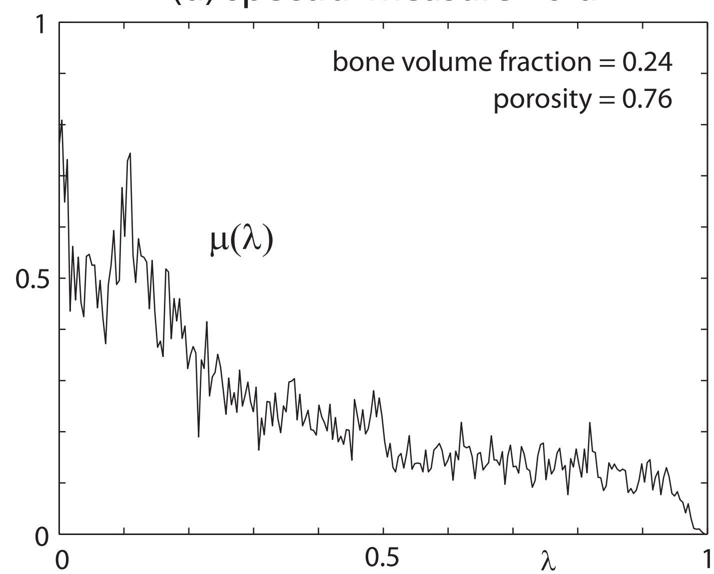
(b) old osteoporotic trabecular bone



(c) spectral measure - young



(d) spectral measure - old



+

reconstruction of spectral measures from complex permittivity data

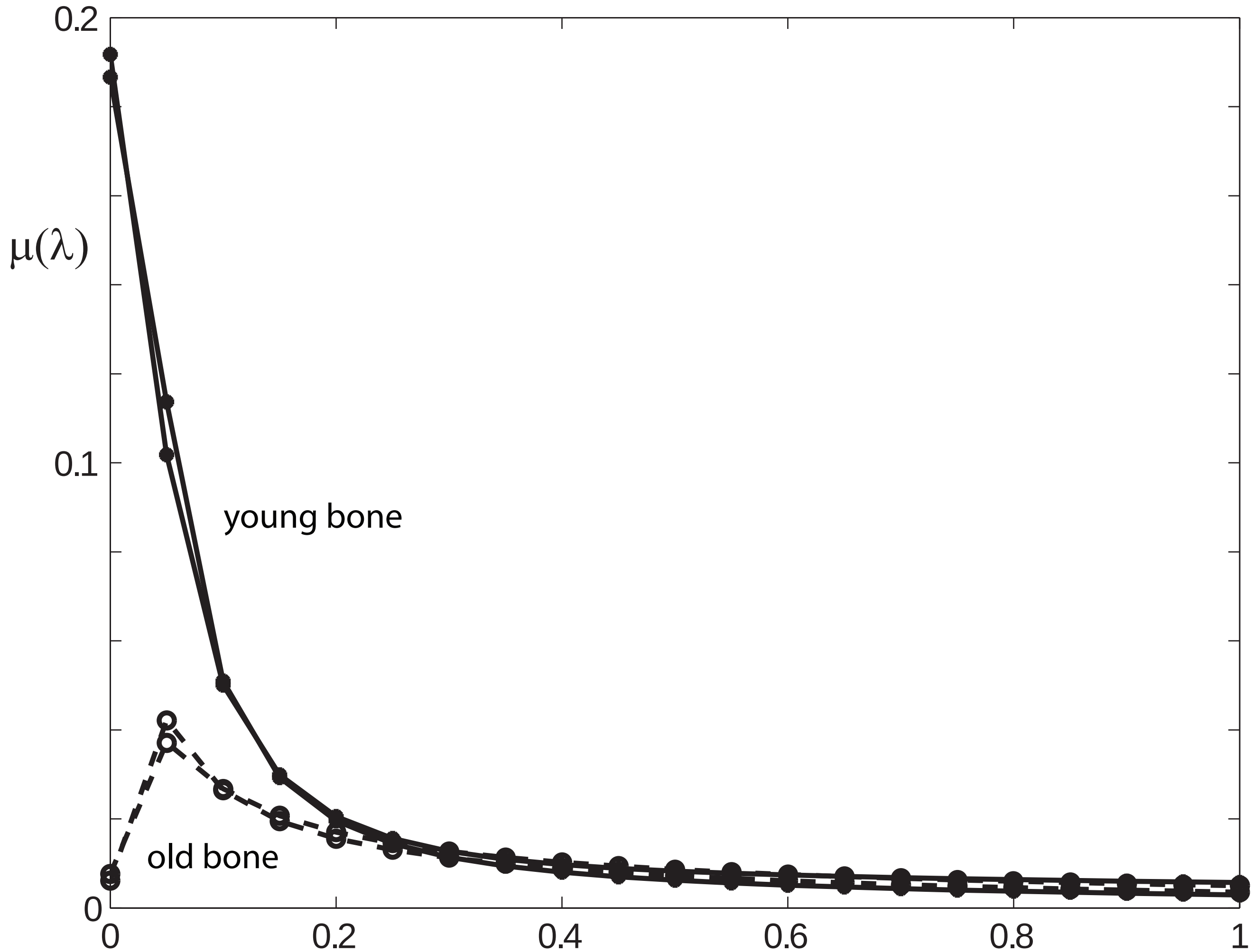
using regularized inversion scheme

↓

EM monitoring of osteoporosis

loss of bone connectivity

reconstruction of spectral measures from simulated complex permittivity data



regularized inversion scheme

Golden, Murphy, Cherkaev, J. Biomech. 2011

direct calculation of spectral measure

1. Discretization of composite microstructure gives lattice of 1's and 0's (random resistor network).

2. The fundamental operator $\chi\Gamma\chi$ becomes a random matrix depending only on the composite geometry.

3. Compute the eigenvalues λ_i and eigenvectors of $\chi\Gamma\chi$ with inner product weights α_i

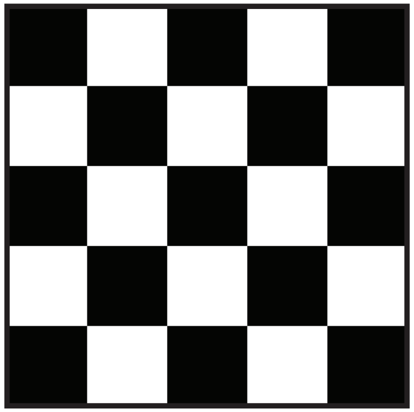
$$\mu(\lambda) = \sum_i \alpha_i \, \delta(\lambda - \lambda_i)$$

Dirac point measure (Dirac delta)

earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Continuum composite

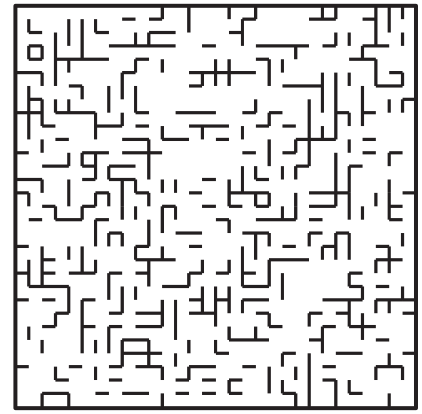


Spectral measures of

$$\chi_1 \Gamma \chi_1$$

Murphy, Hohenegger, Cherkaev, Golden
Comm. Math. Sci. 2015

Discrete composite



Integro-differential projection operator

$$\Gamma = \vec{\nabla}(\Delta^{-1})\vec{\nabla}\cdot$$

Point-wise indicator function

$$\chi_1$$

Resolvent representation of electric field

$$\chi_1 \vec{E} = sE_0 (sI - \chi_1 \Gamma \chi_1)^{-1} \chi_1 \vec{e}_k$$

Integral representation

$$F(s) = \int_0^1 \frac{d\mu(\lambda)}{s - \lambda}$$

Projection matrix

$$\Gamma = \nabla (\nabla^{\mathrm{T}} \nabla)^{-1} \nabla^{\mathrm{T}}$$

Diagonal projection matrix

$$\chi_1$$

Series representation of electric field

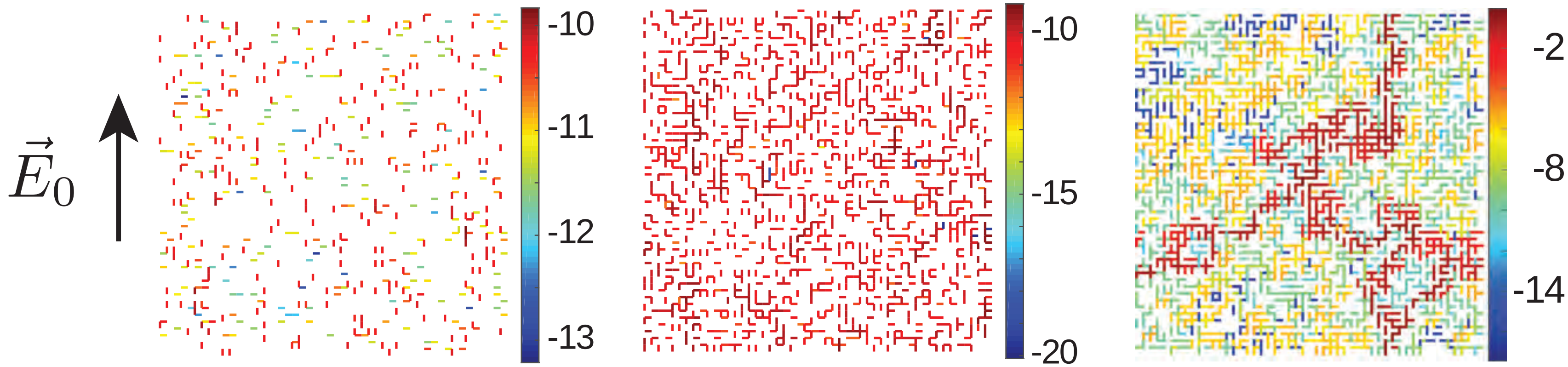
$$\chi_1 \vec{E} = sE_0 \sum_j \frac{\vec{v}_j \cdot \chi_1 \vec{e}_k}{s - \lambda_j} \vec{v}_j$$

Series representation

$$F(s) = \sum_j \frac{(\vec{v}_j \cdot \chi_1 \vec{e}_k)^2}{s - \lambda_j}$$

Spectral statistics for 2D random resistor network

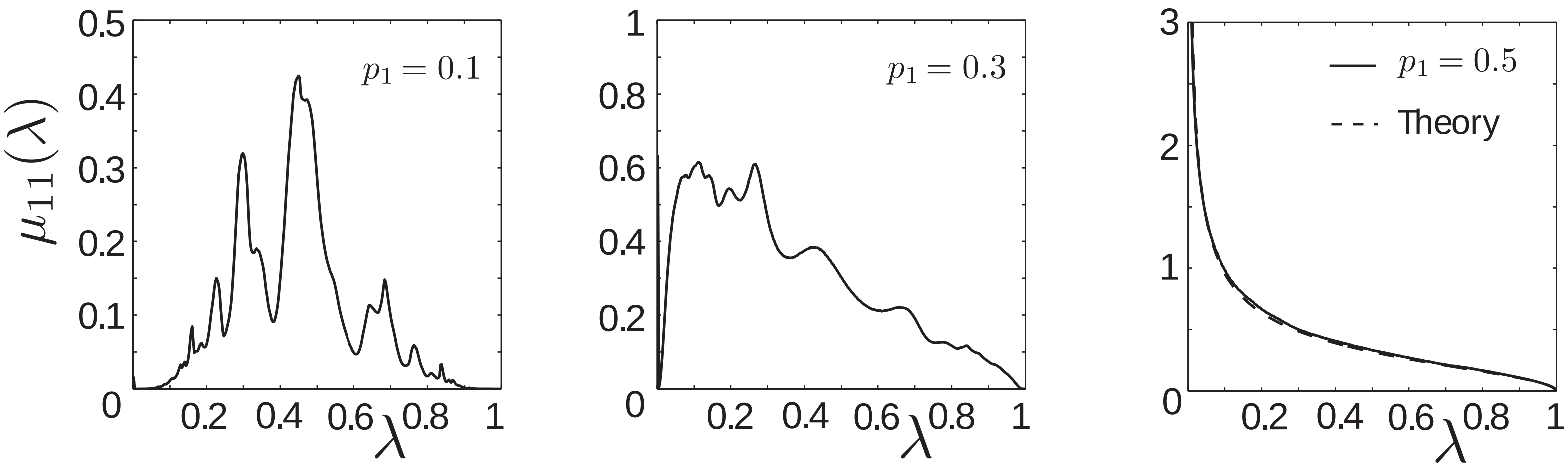
Log Electric Field



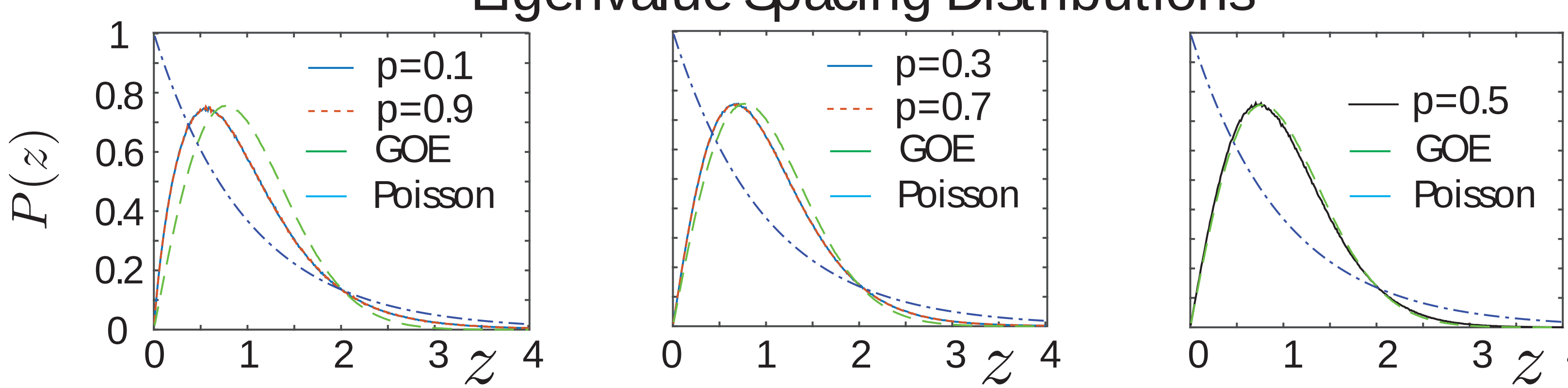
Murphy and Golden, J. Math. Phys., 2012

Murphy et al. Comm. Math. Sci., 2015

Spectral Measures



Eigenvalue Spacing Distributions



Murphy, Cherkaev, Golden, 2016

Eigenvalue Statistics of Random Matrix Theory

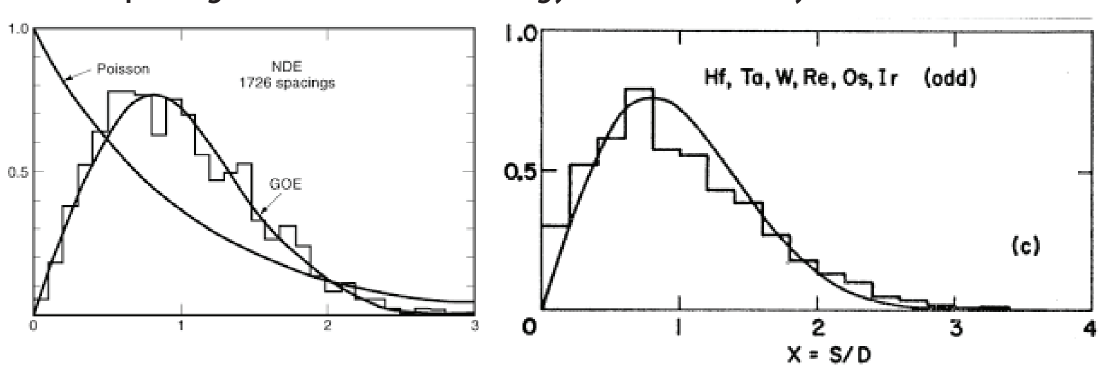
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^T)/2$ ***Gaussian orthogonal ensemble (GOE)***

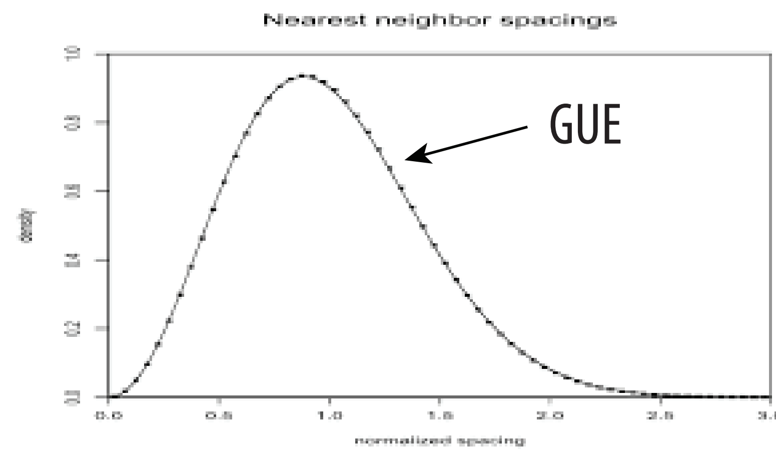
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^\dagger)/2$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics

Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



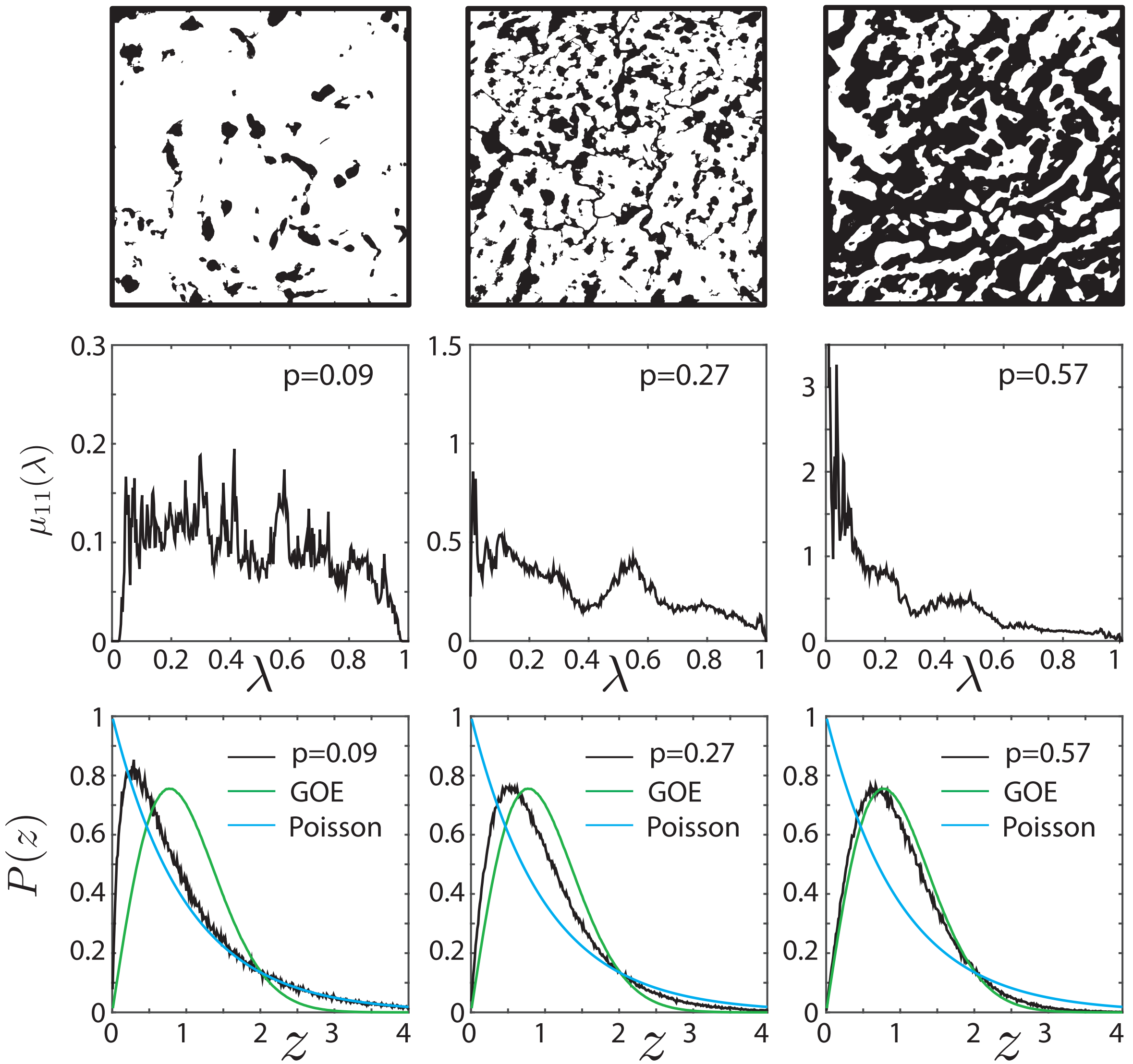
RMT used to characterize disorder-driven transitions in mesoscopic conductors, neural networks, random graph theory, etc.

Phase transitions ~ transitions in universal eigenvalue statistics.

Spectral computations for Arctic melt ponds

spectral measures

eigenvalue spacing distributions

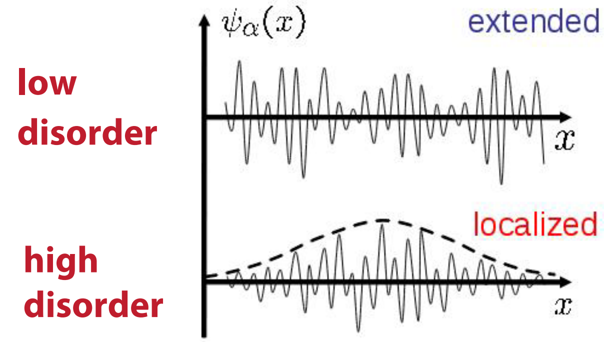


uncorrelated → **level repulsion**

TRANSITION

Ben Murphy
Elena Cherkaev
Ken Golden
2017

eigenvalue statistics for transport tend toward the UNIVERSAL Wigner-Dyson distribution as the "conducting" phase percolates



metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

we find a surprising analog

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden   Phys. Rev. Lett.   2017

PERCOLATION TRANSITION → **transition to universal eigenvalue statistics (GOE) extended states, mobility edges**

-- but without wave interference or scattering effects ! --

eigenvector localization and mobility edges

Inverse Participation Ratio: $I(\vec{v}_n) = \sum_{i=1}^{N} |(\vec{v}_n)_i|^4$

Completely Localized: $I(\vec{e}_n) = 1$

Completely Extended: $I\left(\frac{1}{\sqrt{N}}\vec{1}\right) = \frac{1}{N}$

Anderson Model

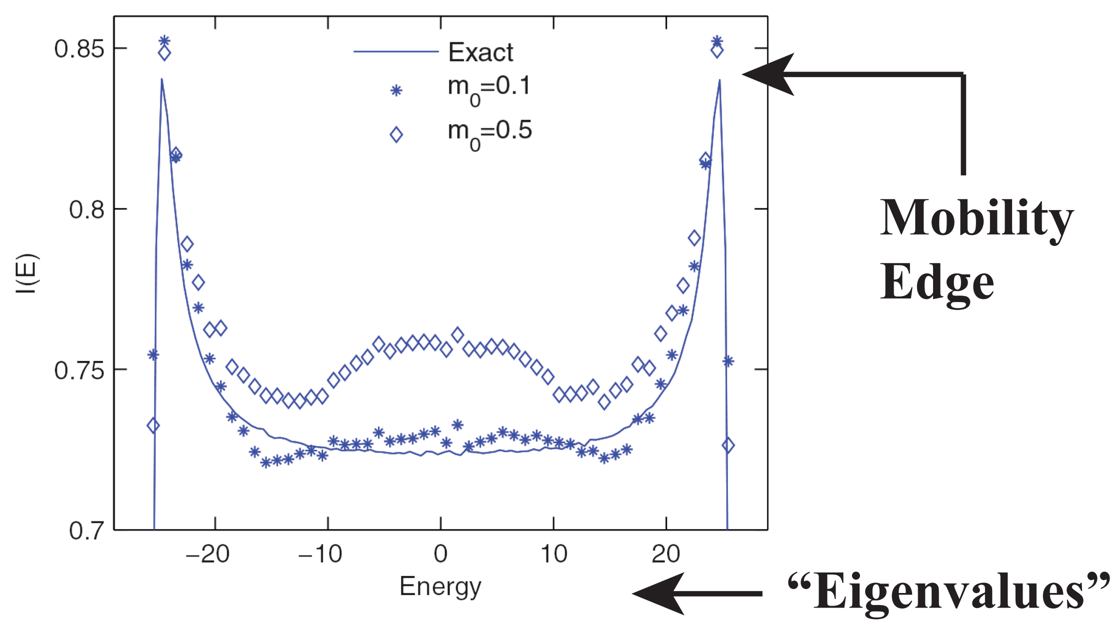
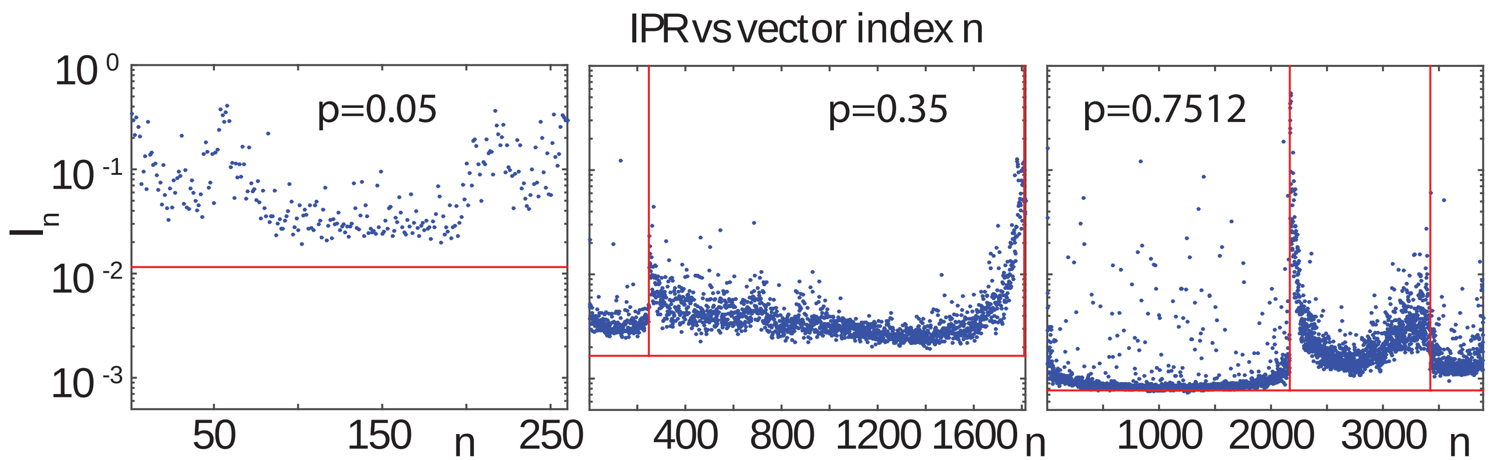
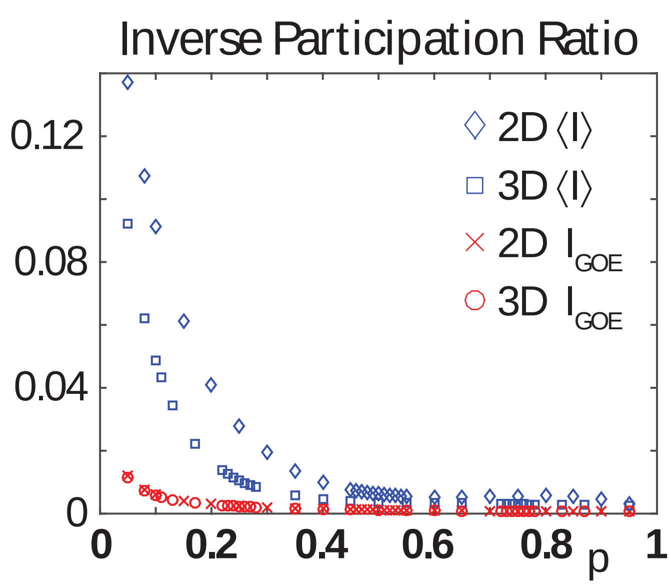


FIG. 4. (Color online) IPR for Anderson model in two dimensions with $x = 6.25$ ($w = 50$) from exact diagonalization (solid line) and from LDRG with different values of the cutoff m_0. LDRG data are averaged over 100 runs of systems with 100×100 sites.

PHYSICAL REVIEW B 90, 060205(R) (2014)

Localization properties of eigenvectors in random resistor networks

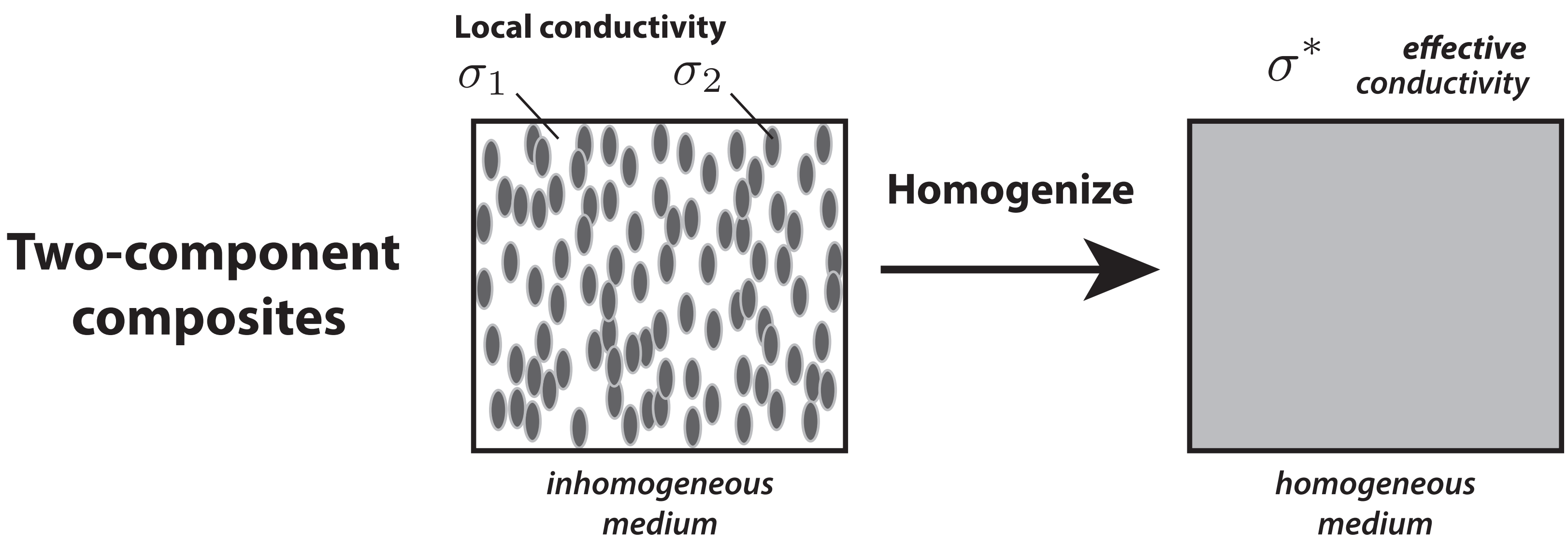




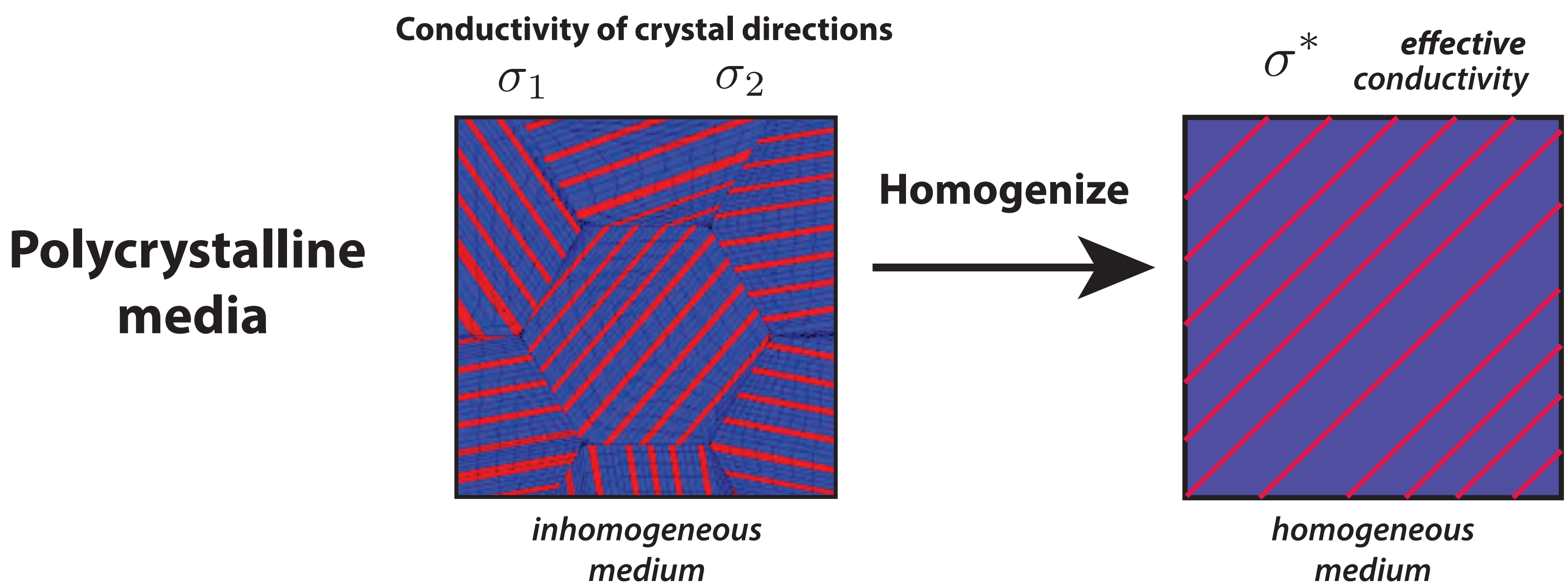
$$I_n = \sum_i (\vec{v}_n)_i^4$$

Murphy, Cherkaev, Golden, 2017

Homogenization for composite materials



Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium



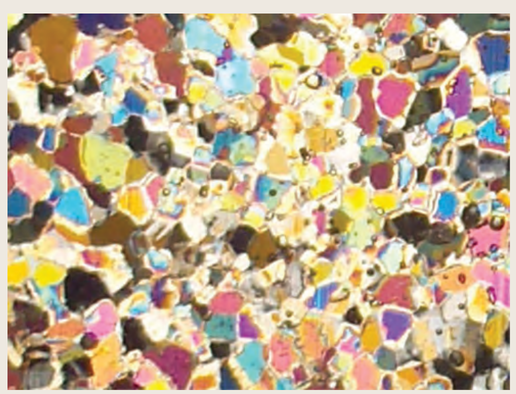
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...

- **Forward and inverse bounds**
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS OF SCIENTIFIC PUBLISHING

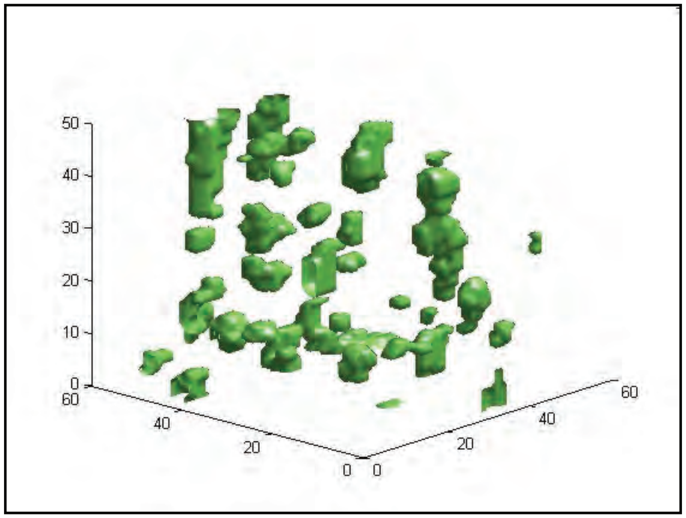
An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

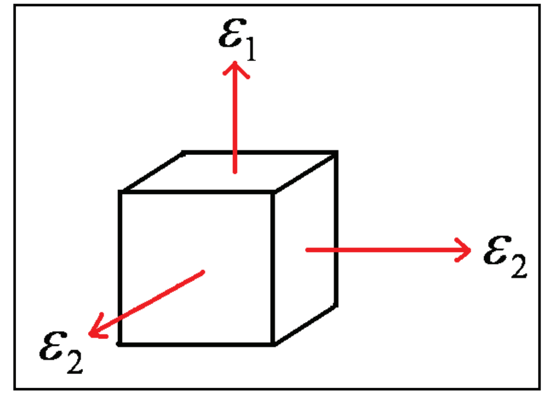
A computer model to determine how a human should walk so as to expend the least energy

THE ROYAL SOCIETY PUBLISHING

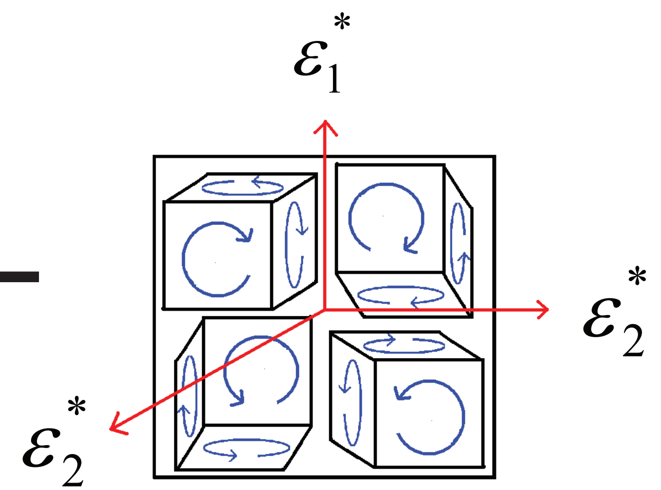
two scale homogenization for polycrystalline sea ice



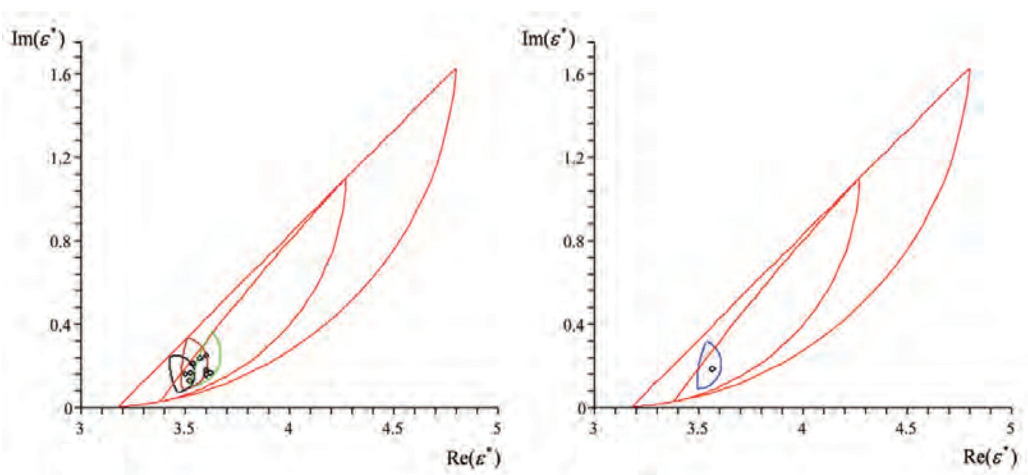
numerical homogenization for single crystal



analytic continuation for polycrystals



bounds

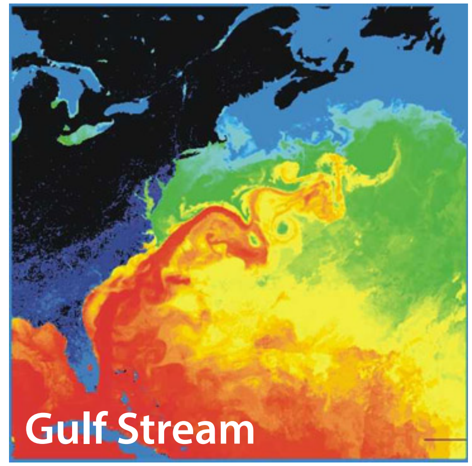


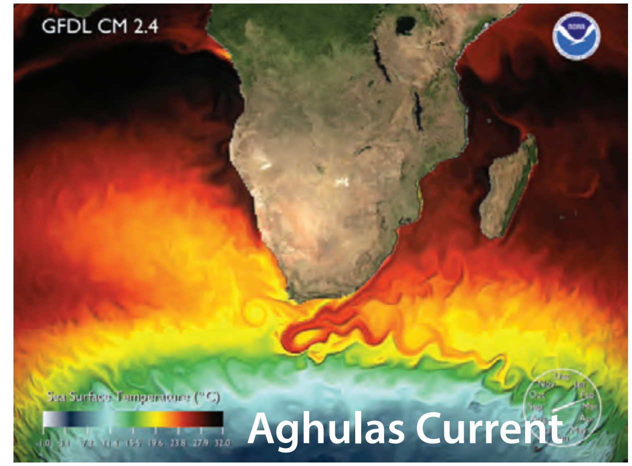
Gully, Lin, Cherkaev, Golden, Proc. Roy. Soc. A (and cover) 2015

advection enhanced diffusion

effective diffusivity

tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere
salt and heat transport in ocean
heat transport in sea ice with convection





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

↓ ***homogenize***

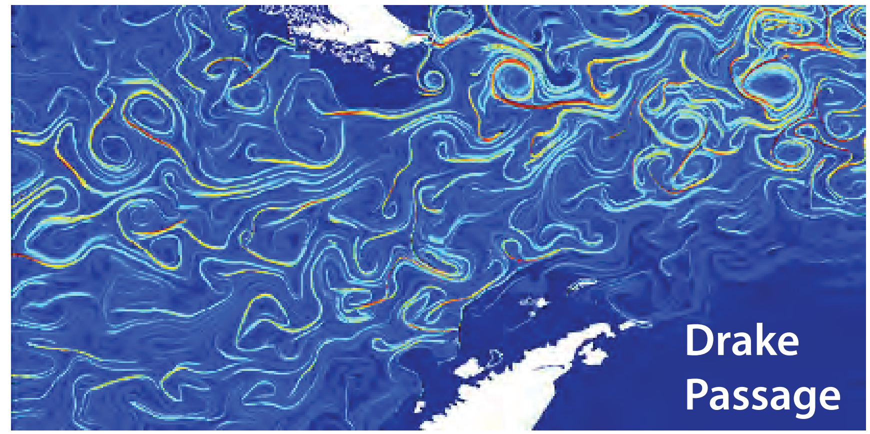
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

Stieltjes integral for κ^* ***with spectral measure***

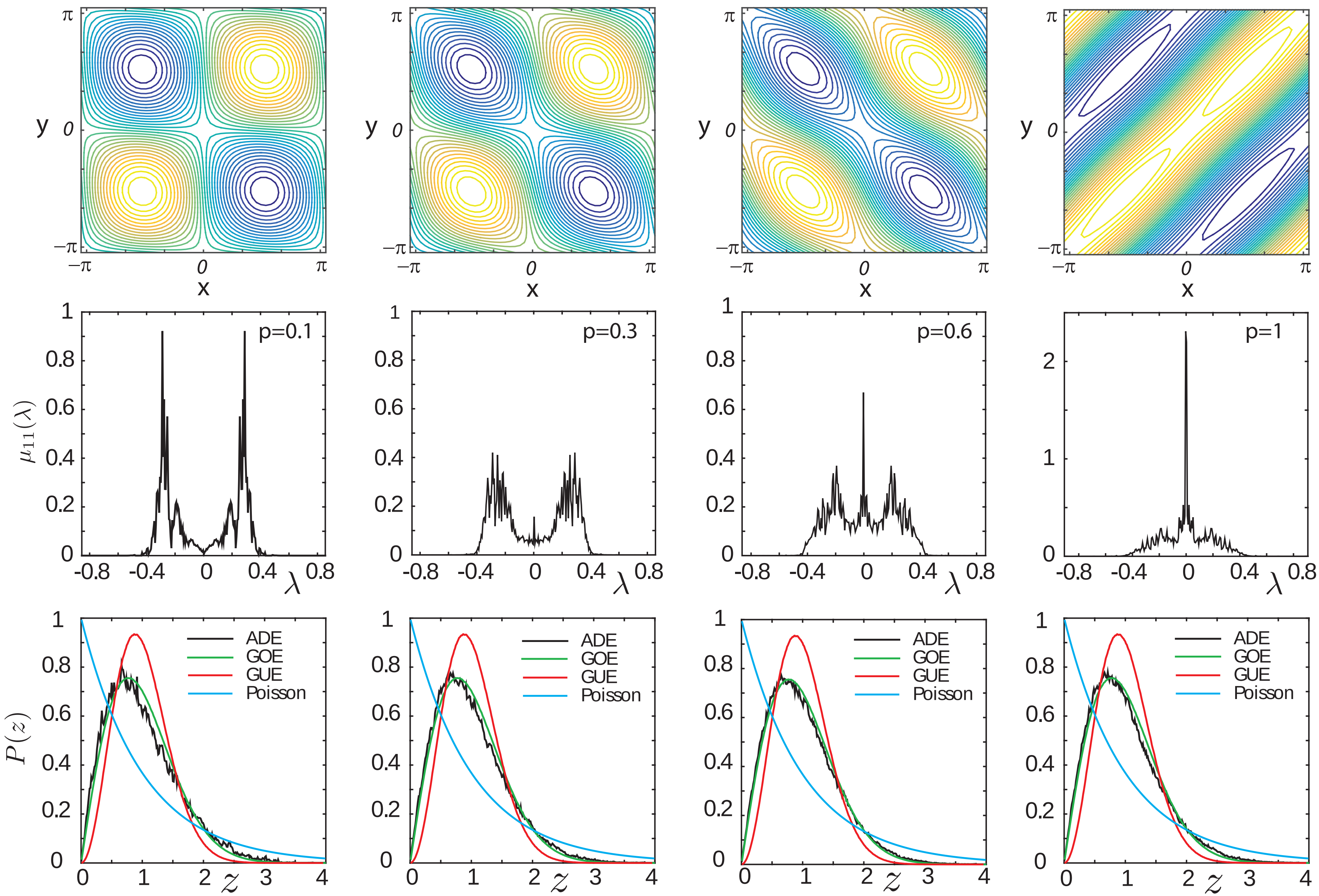
Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, 2017



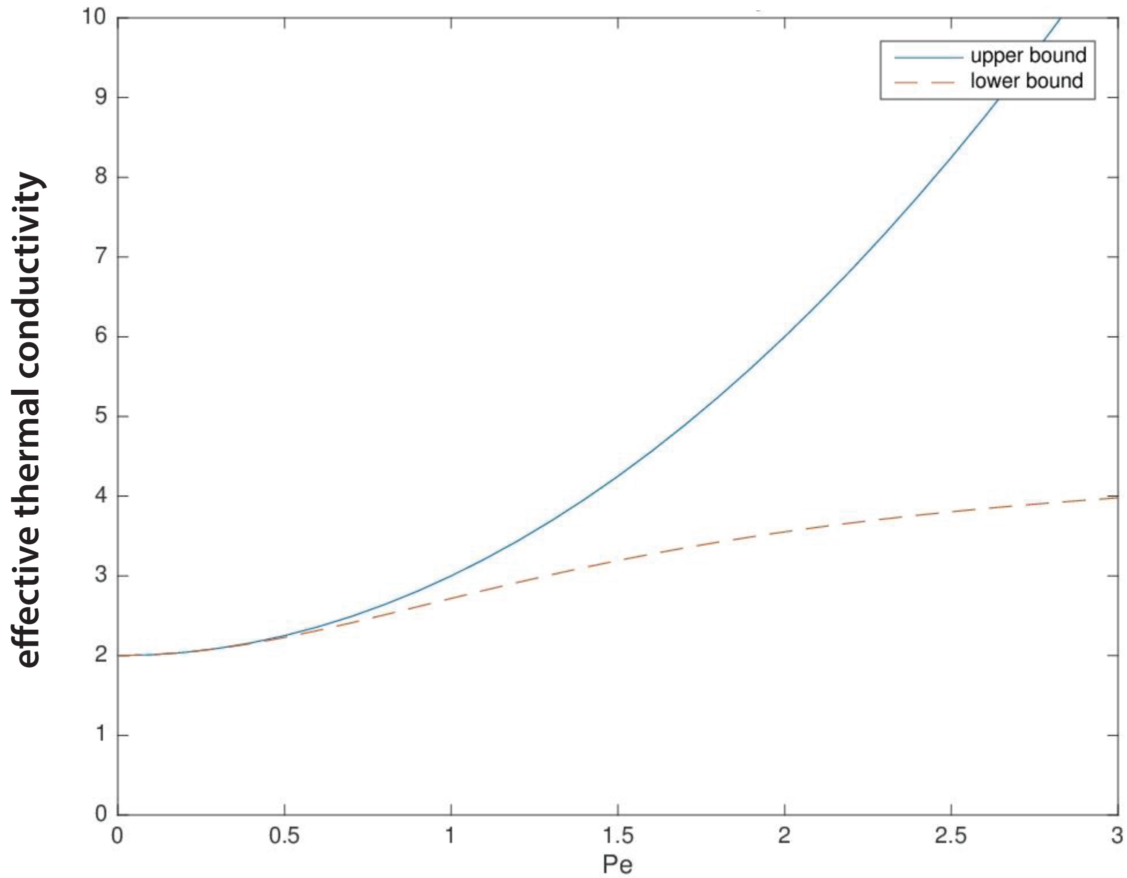
Spectral measures and eigenvalue spacings for cat's eye flow

$H(x,y) = \sin(x)\sin(y) + A\cos(x)\cos(y), \quad A \sim U(-p,p)$



Murphy, Cherkaev, Xin, Golden, 2017

bounds on the effective thermal conductivity of sea ice with convection (BC flow model)

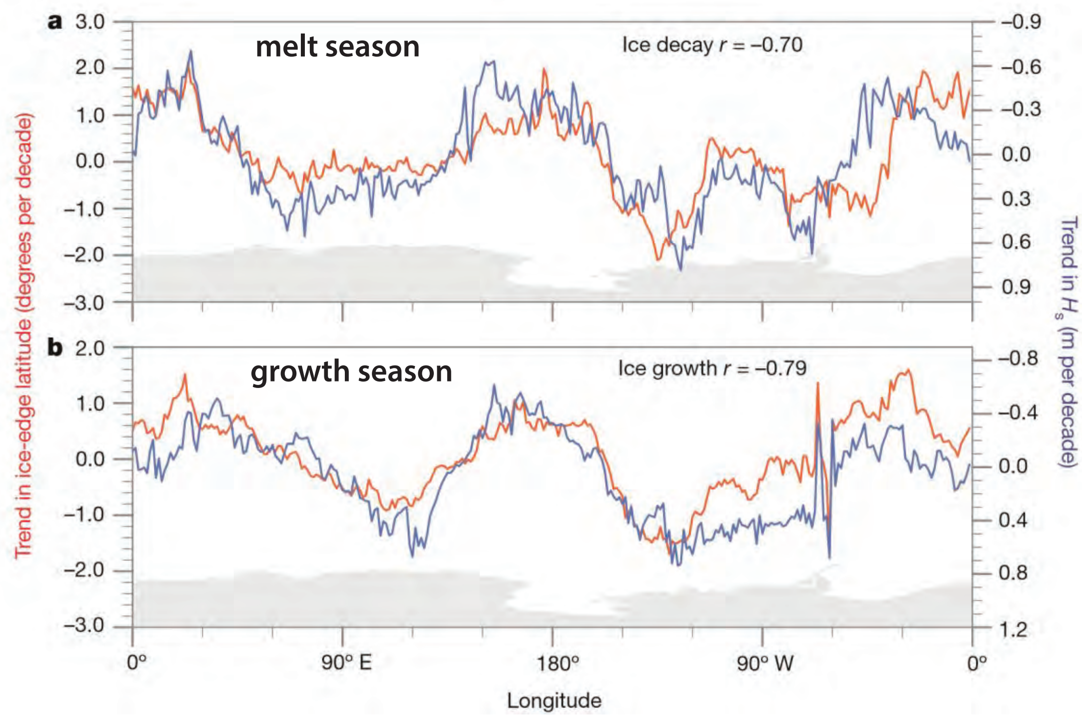


Hardenbrook, Kraitzman, Zhu, Murphy, Cherkaev, Golden

Storm-induced sea-ice breakup and the implications for ice extent

Kohout et al., *Nature* 2014

- during three large-wave events, significant wave heights did not decay exponentially, enabling large waves to persist deep into the pack ice.
- large waves break sea ice much farther from the ice edge than would be predicted by the commonly assumed exponential decay



ice extent compared with significant wave height

Waves have strong influence on both the floe size distribution and ice extent.

Waves on ice-covered seas

Joseph B. Keller

1950 M. L. Weitz and J. B. Keller, *Comm. Pure Appl. Math.*
Reflection of waves from floating ice in water of finite depth

1953 M. L. Weitz and J. B. Keller, *Comm. Pure Appl. Math.*
Reflection and transmission coefficients for water waves entering or leaving an ice field

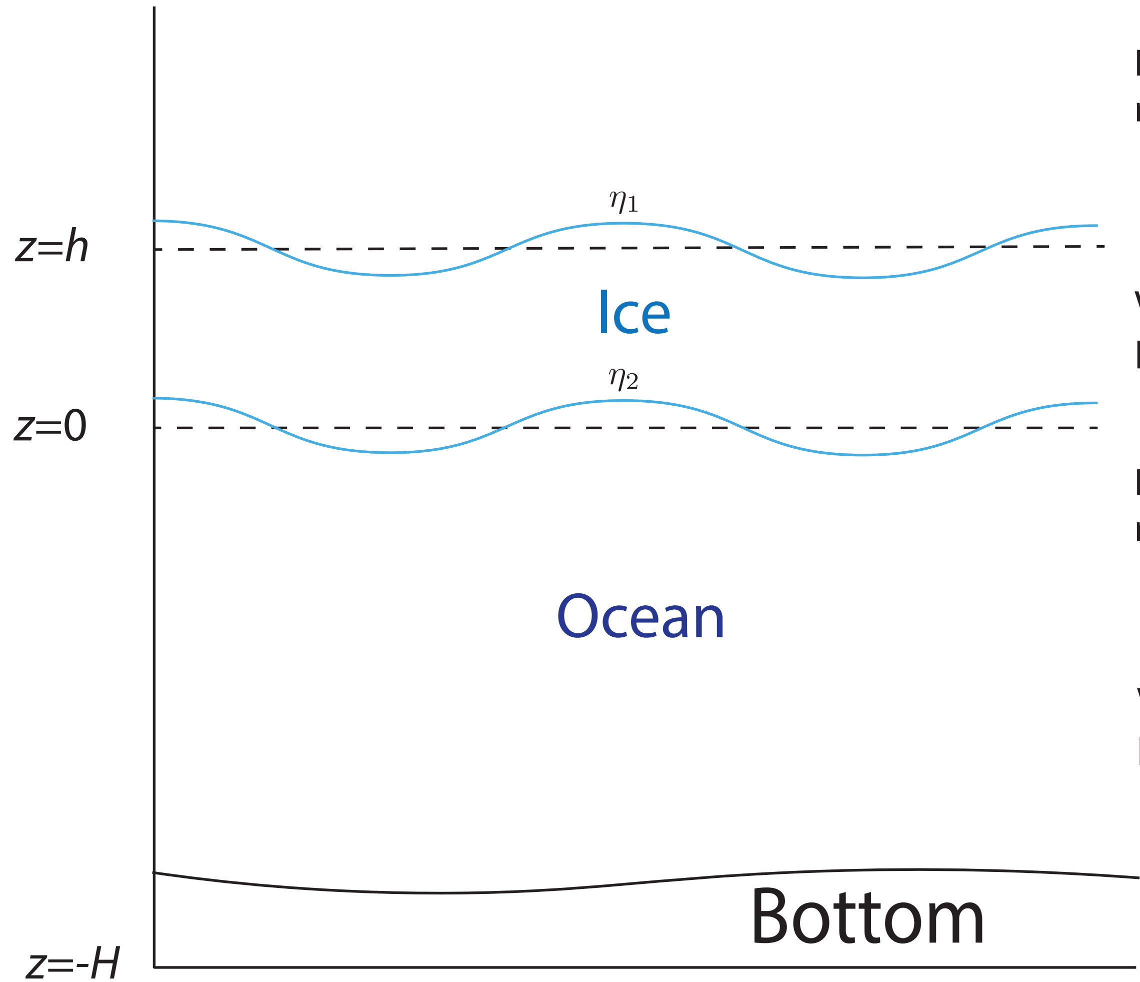
1953 E. Goldstein and J. B. Keller, *EOS Trans. AGU*
Water wave reflection due to surface tension and floating ice

1959 F. Karal and J. B. Keller, *J. Acoustical. Soc. Amer.*
Elastic wave propagation in homogeneous and inhomogeneous media

1984 K. C. Nunan and J. B. Keller, *J. Fluid Mech.*
Effective viscosity of a periodic suspension

1998 J. B. Keller, *J. Geophys. Res.* (*Oceans*)
Gravity waves on ice-covered water

Two Layer Models and Effective Parameters



G shear modulus | P pressure | ω angular frequency | U velocity field

ν viscosity | λ Poission ratio | ρ density | g gravity

Viscous fluid layer (Keller 1998)

Effective Viscosity ν

Equations of motion:

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 U + g$$

Viscoelastic fluid layer (Wang-Shen 2010)

Effective Complex Viscosity $\nu_e = \nu + iG/\rho\omega$

Equations of motion

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu_e \nabla^2 U + g$$

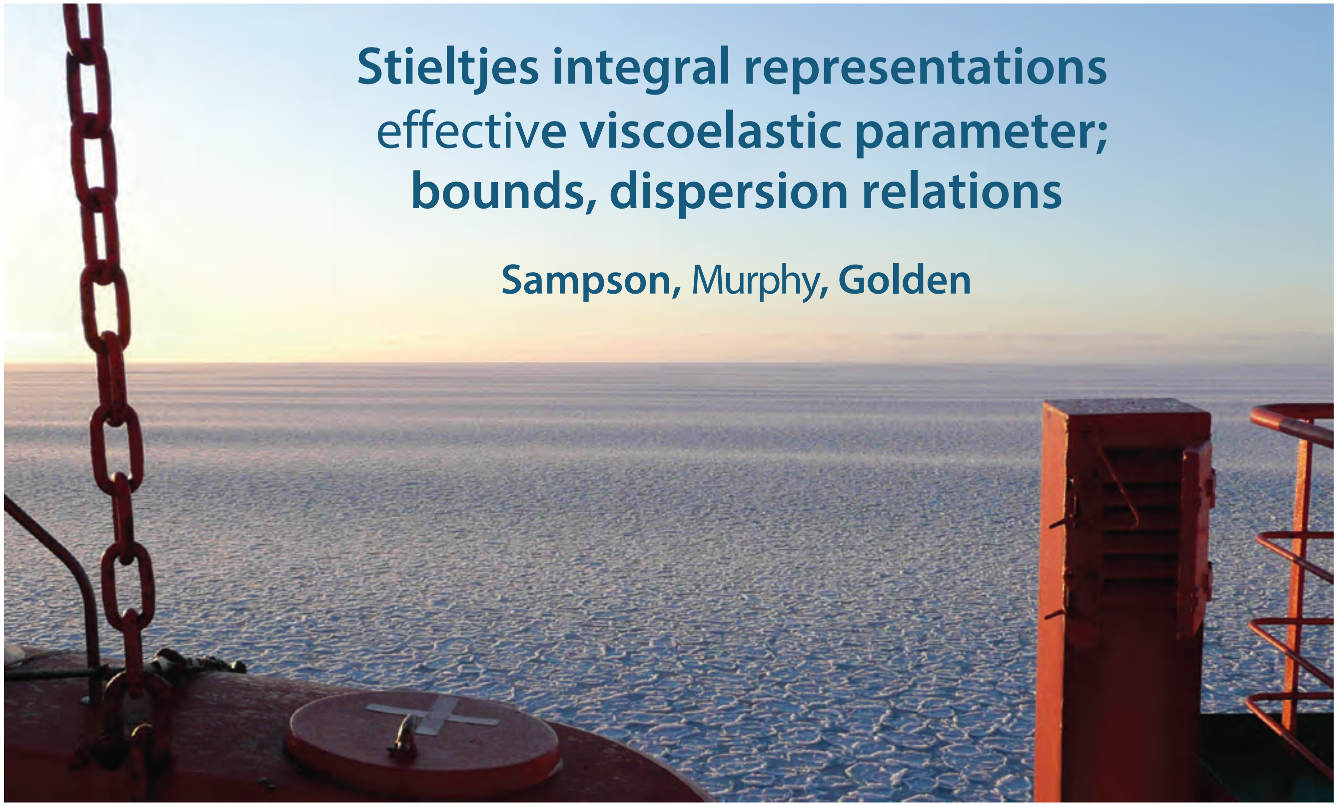
Viscoelastic thin beam (Mosig *et al.* 2015)

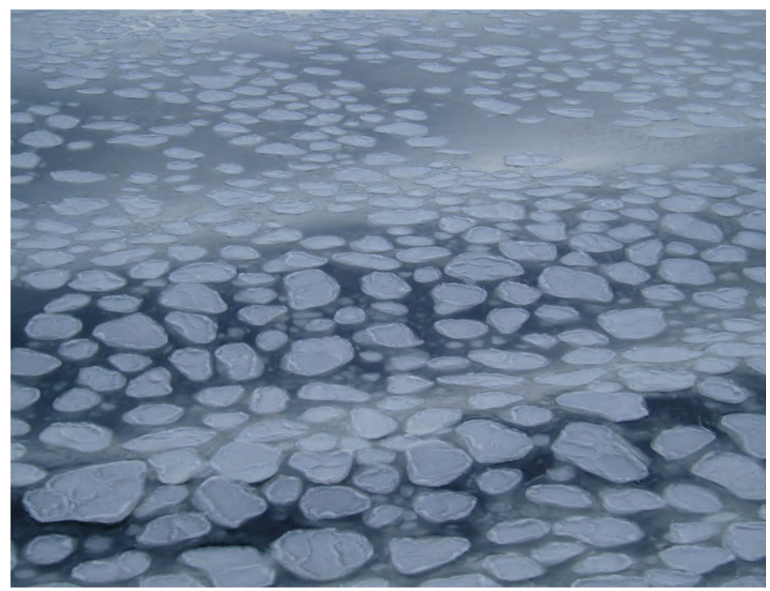
Effective Complex Shear Modulus $G_v = G - i\omega\rho\nu$

Stieltjes integral representation for effective complex viscoelastic parameter; bounds

Sampson, Murphy, Golden 2017

wave propagation in the marginal ice zone





Homogenization Problem for Quasistatic Waves

$$\nabla \cdot \sigma = 0 \qquad \sigma = C_{ijkl}\colon \epsilon_{kl} \qquad \langle \sigma \rangle = C^*_{ijkl} \langle \epsilon_{kl} \rangle$$

local $$C_{ijkl} = (\nu_1 \chi + (1-\chi)\nu_2)\lambda_s$$

$$\nabla \cdot \big((\nu_1 \chi + (1-\chi)\nu_2)\lambda_s \colon \epsilon \big) = 0$$

$$s = \frac{1}{1 - \frac{\nu_1}{\nu_2}}$$

Strain Field

$$\epsilon = \frac{1}{2}[\nabla u + (\nabla u)^T] = \nabla^s u$$

$$\epsilon = \epsilon_0 + \epsilon_f \text{ where } \epsilon_f = \nabla^s \phi$$

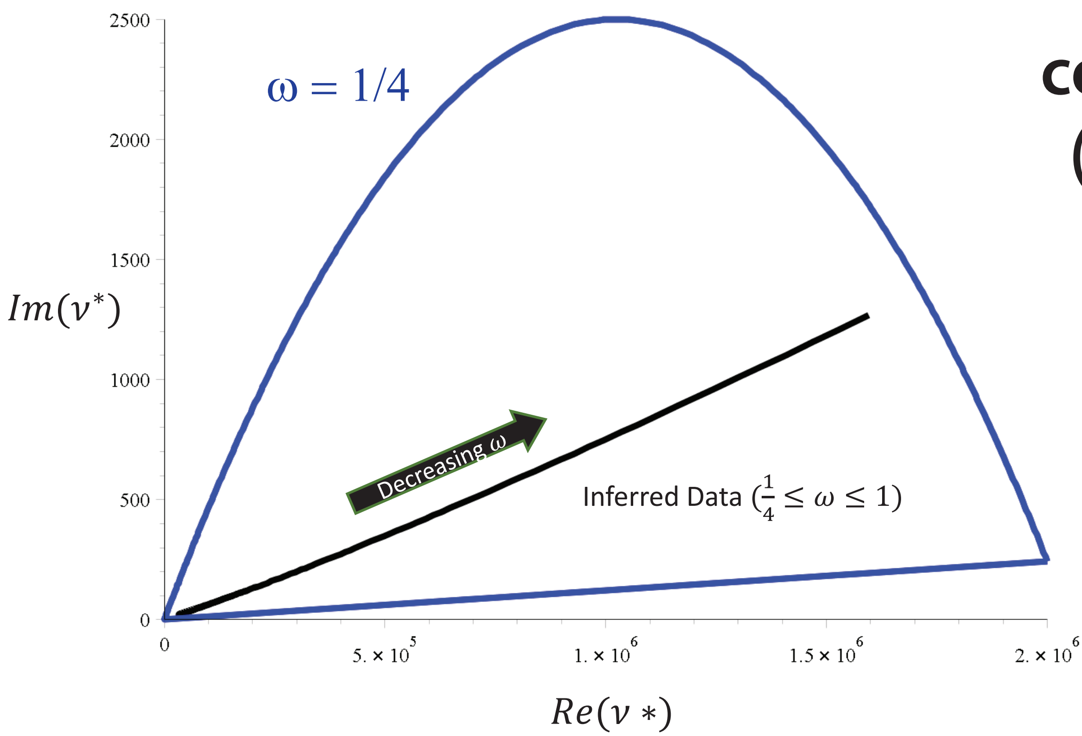
Elasticity Tensor

$$C^*_{ijkl} = \nu^* \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl} \right) = \nu^* \lambda_s$$

RESOLVENT $$\epsilon = \left(1 - \frac{1}{s}\Gamma\chi \right)^{-1} \epsilon_0 \qquad \Gamma = \nabla^s (\nabla \cdot \nabla^s)^{-1} \nabla \cdot$$ ϵ_0 **avg strain**

$$F(s) = 1 - \frac{\nu^*}{\nu_2} \qquad F(s) = ||\epsilon_0||^{-2} \int_\Sigma \frac{d\mu(\lambda)}{s - \lambda}$$

bounds on the effective complex viscoelasticity

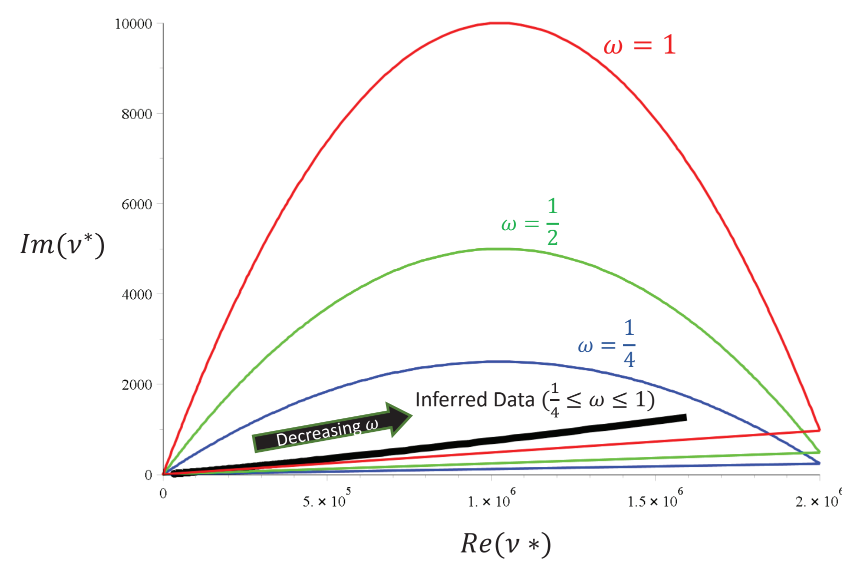


complex elementary bounds (fixed area fraction of floes)

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**

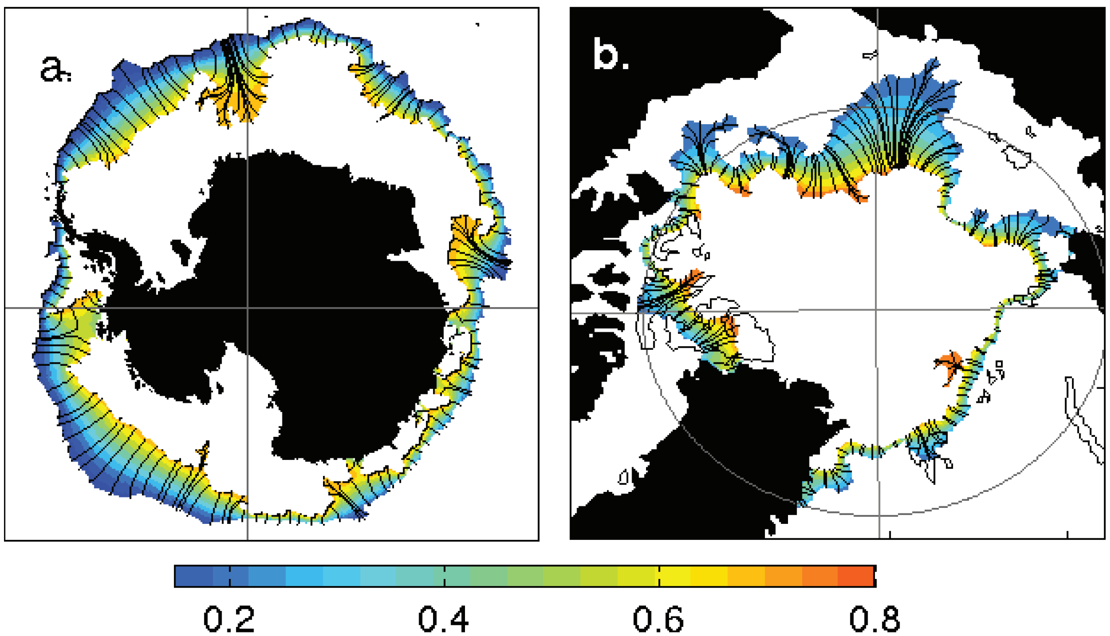
Sampson, Murphy, Golden 2017



Marginal Ice Zone

MIZ

- biologically active region
- intense ocean-sea ice-atmosphere interactions
- region of significant wave-ice interactions



transitional region between
dense interior pack ($c > 80\%$)
sparse outer fringes ($c < 15\%$)

MIZ WIDTH

fundamental length scale of
ecological and climate dynamics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

How to objectively measure the "width" of this complex, non-convex region?

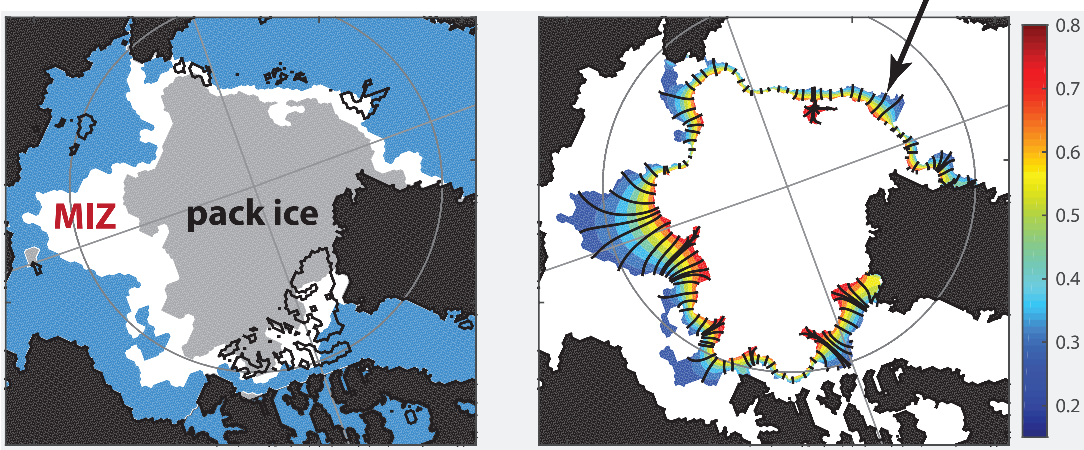
Objective method for measuring MIZ width motivated by medical imaging and diagnostics

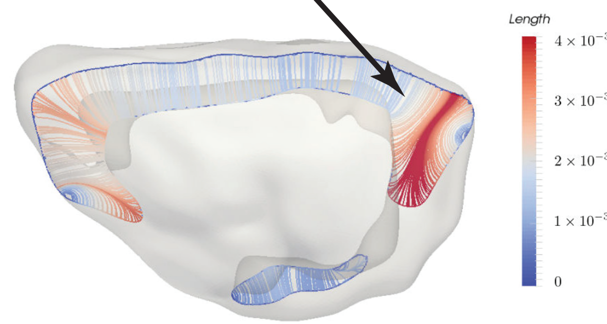
Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

39% widening
1979 - 2012

streamlines of a solution to Laplace's equation

"average" lengths of streamlines





Arctic Marginal Ice Zone

crossection of the cerebral cortex of a rodent brain

analysis of different MIZ WIDTH definitions

Strong, Foster, Cherkaev, Eisenman, Golden
J. Atmos. Oceanic Tech. 2017

Strong and Golden
Society for Industrial and Applied Mathematics News, April 2017

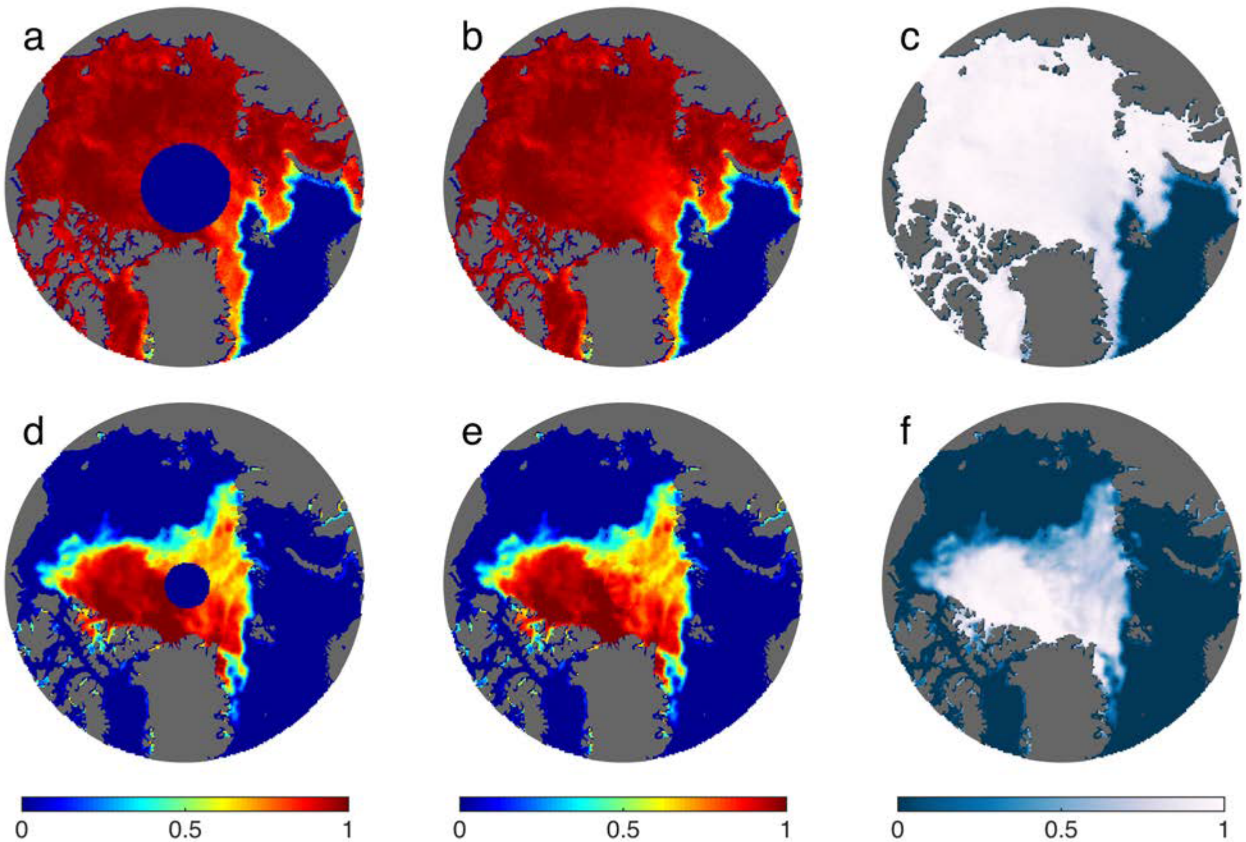
Filling the polar data gap

hole in satellite coverage of sea ice concentration field

previously assumed ice covered

Gap radius: 611 km
06 January 1985

Gap radius: 311 km
30 August 2007



fill with harmonic function satisfying satellite BC's plus stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

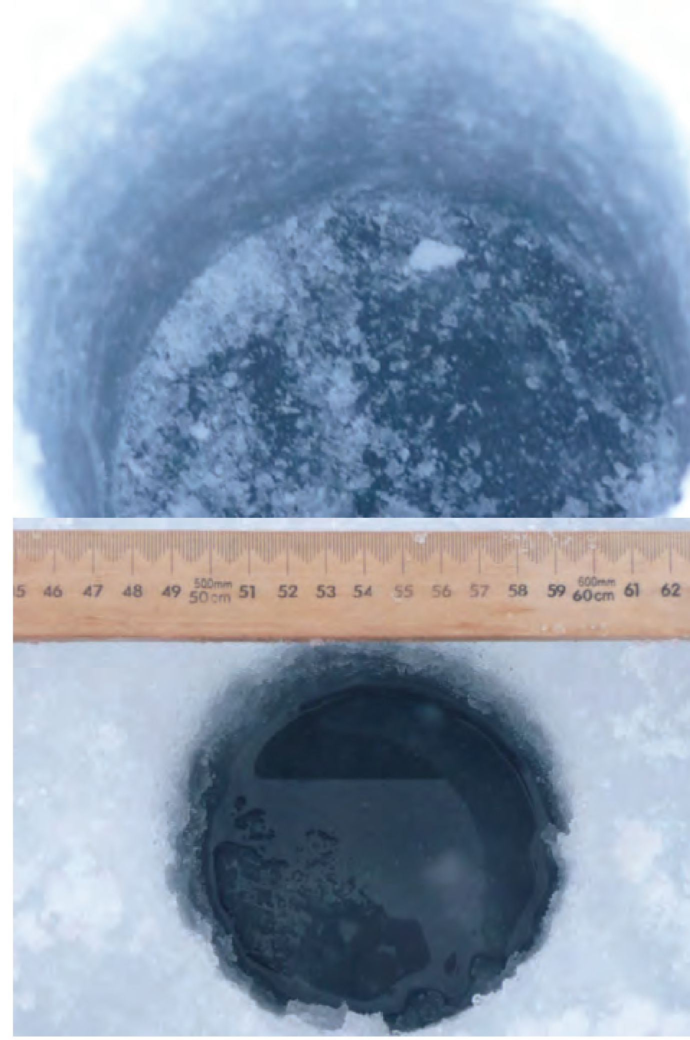
Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	*Antarctic*	*SIPEX*
2010	*Antarctic*	*McMurdo Sound*
2011	*Arctic*	*Barrow AK*
2012	*Arctic*	*Barrow AK*
2012	*Antarctic*	*SIPEX II*
2013	*Arctic*	*Barrow AK*
2014	*Arctic*	*Chukchi Sea*

photo by Jan Lieser



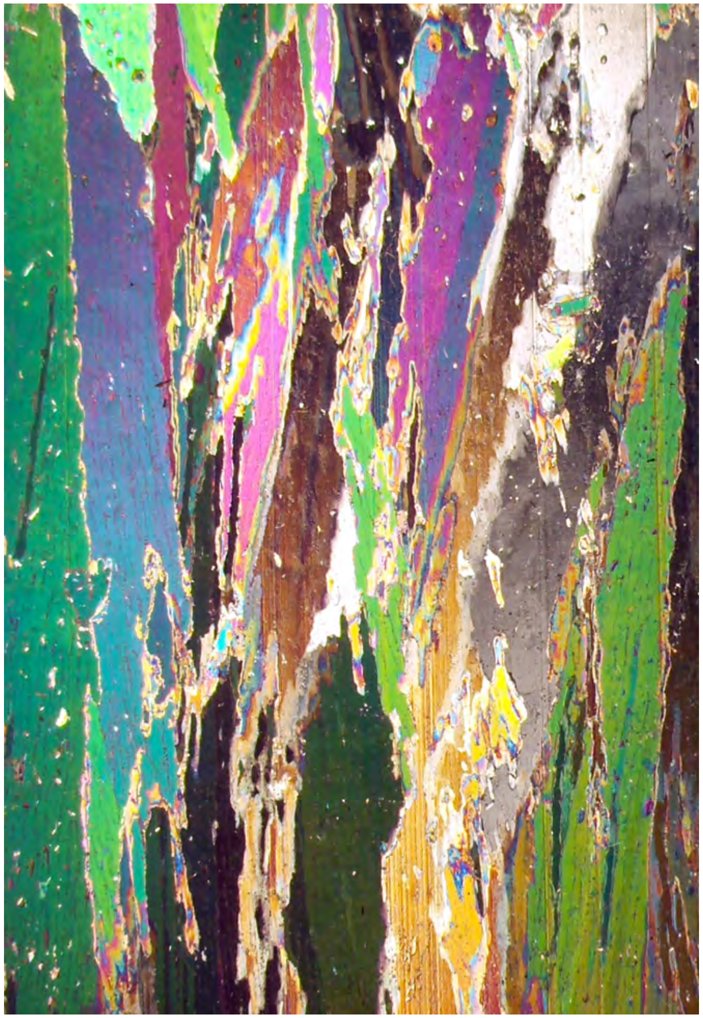
measuring fluid permeability of Antarctic sea ice

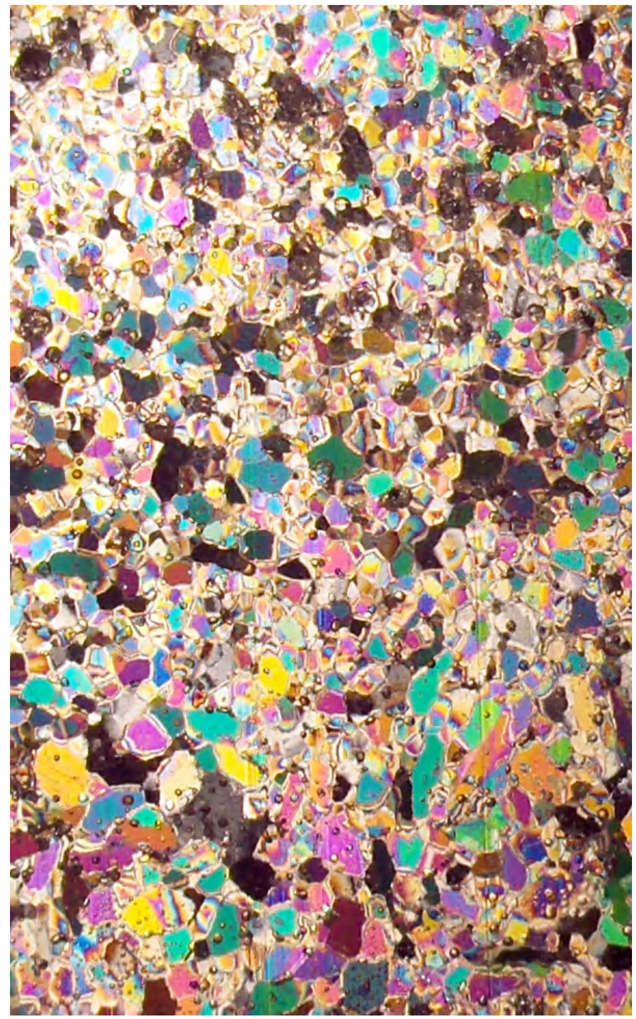
SIPEX 2007

higher threshold for fluid flow in Antarctic granular sea ice

columnar

granular



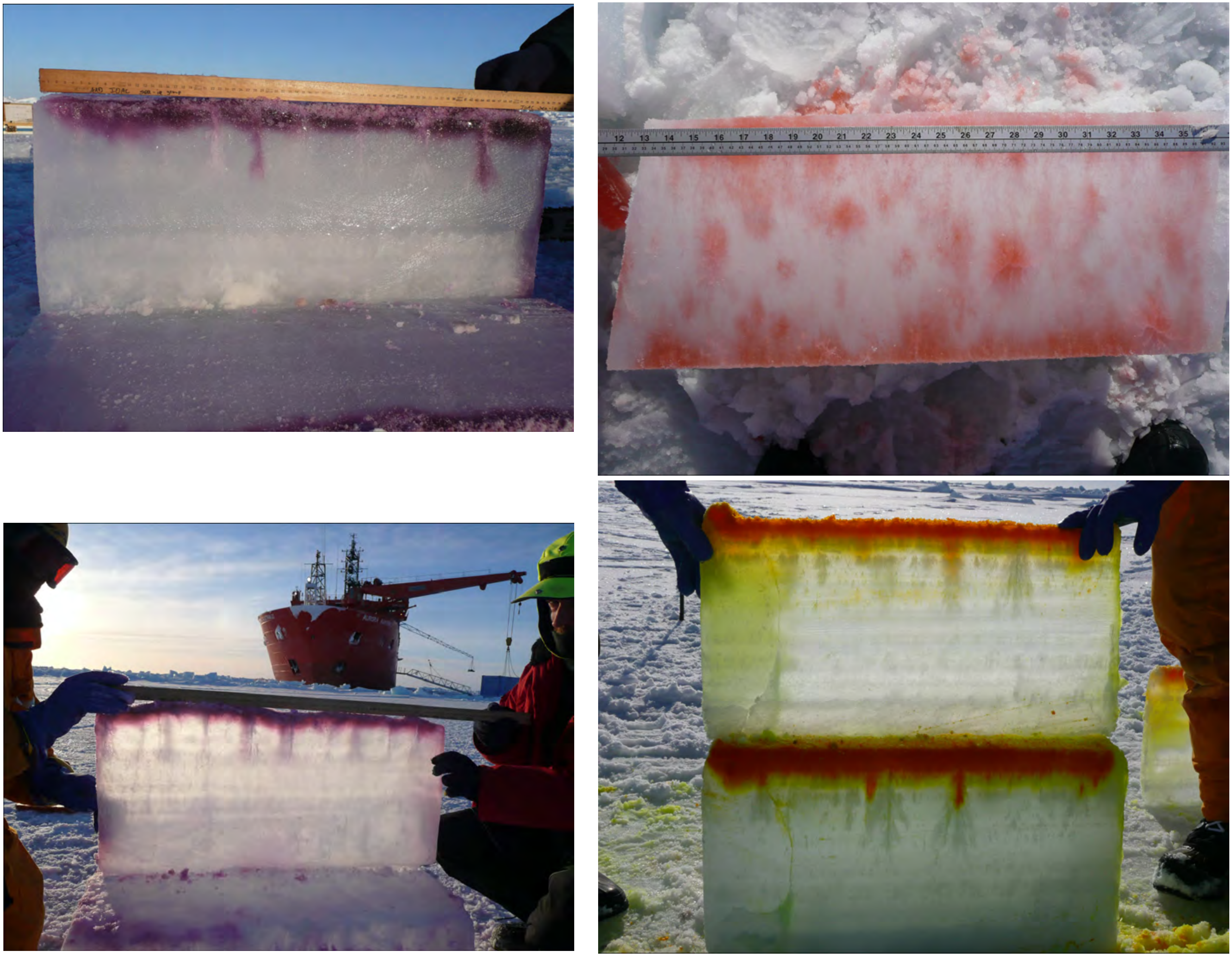


5%

10%

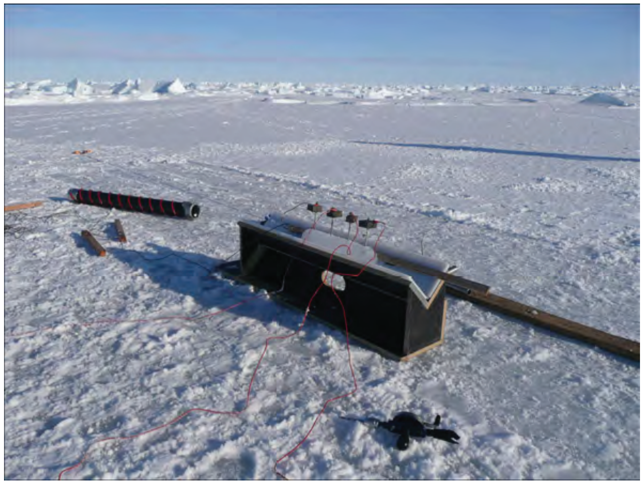
Golden, Sampson, Gully, Lubbers, Tison 2017

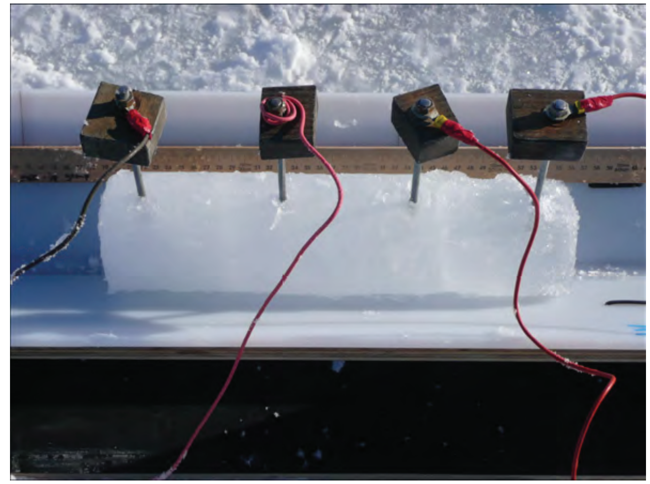
tracers flowing through inverted sea ice blocks

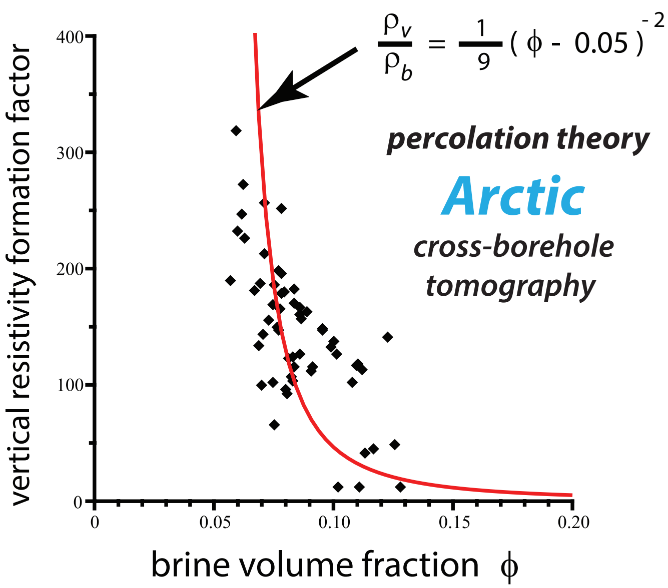


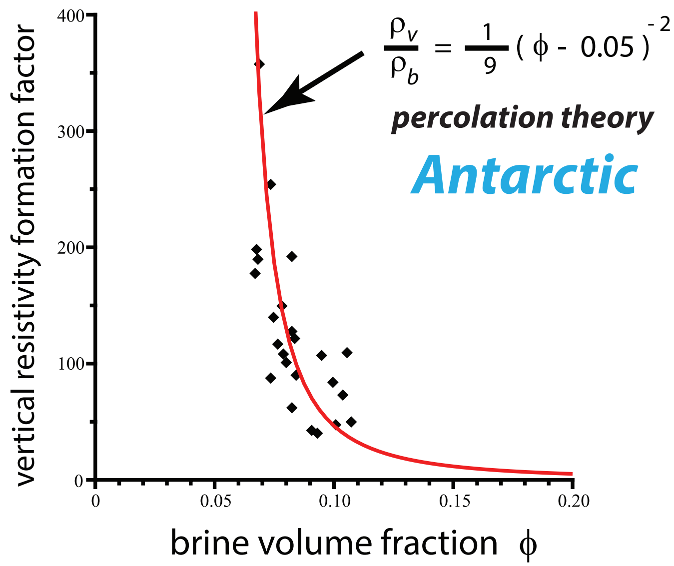
critical behavior of electrical transport in sea ice

electrical signature of the on-off switch for fluid flow





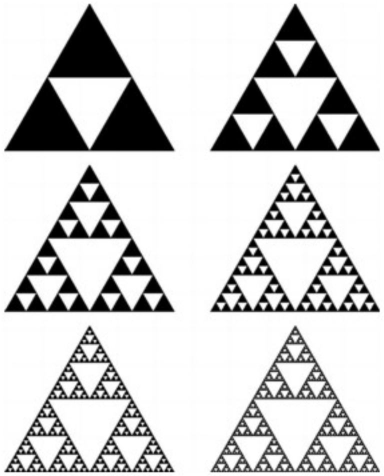




cross-borehole tomography - electrical classification of sea ice layers

Golden, Eicken, Gully, Ingham, Jones, Lin, Reid, Sampson, Worby 2017

fractals and multiscale structure



melt pond formation and albedo evolution:

- ***major drivers in polar climate***
- ***key challenge for global climate models***

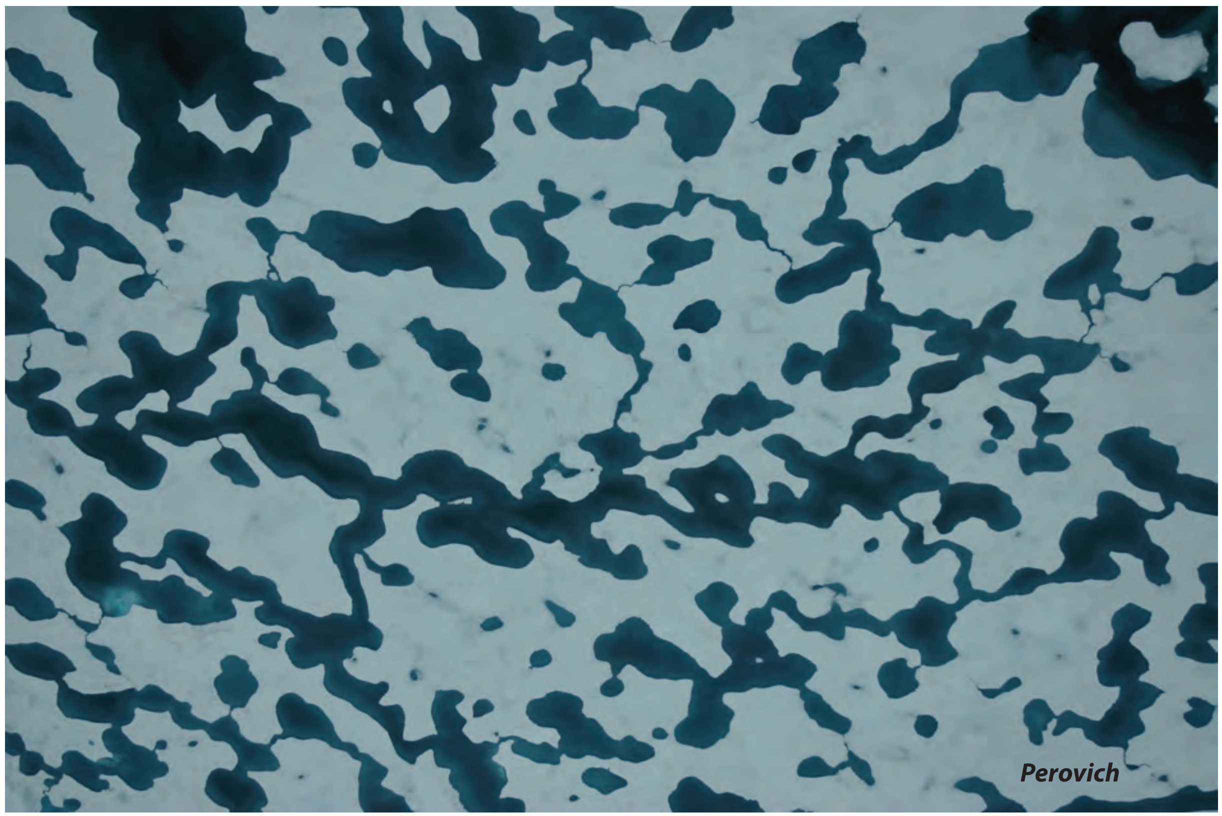
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012

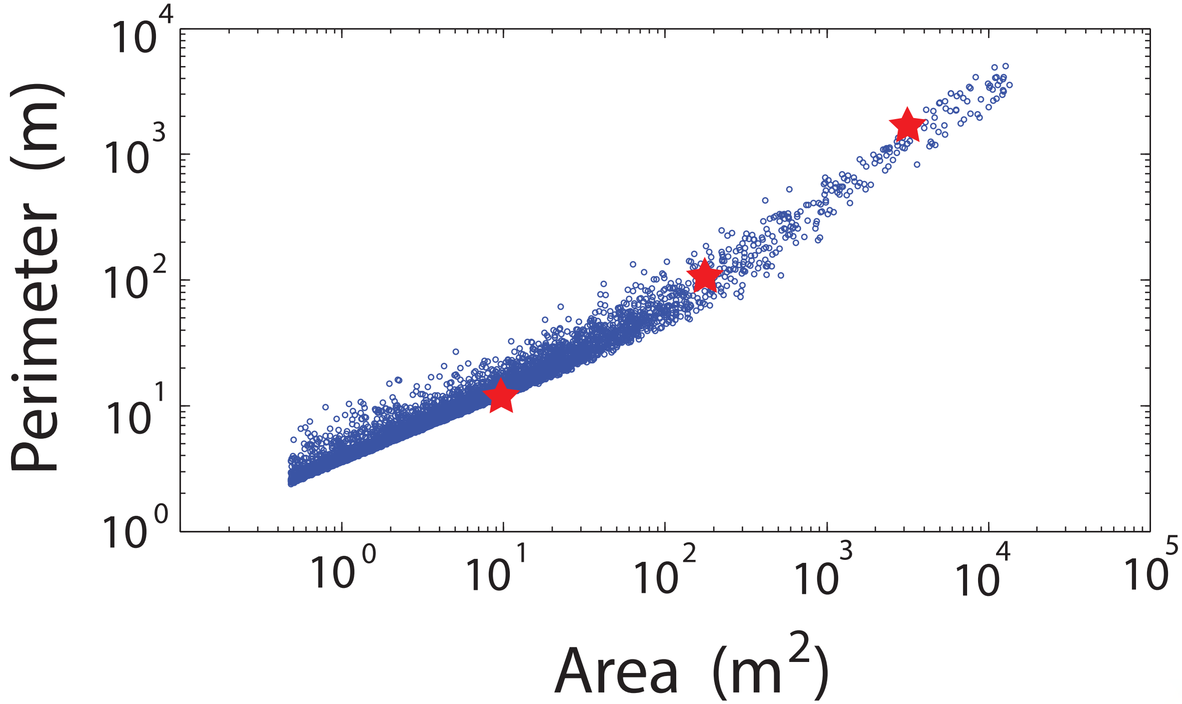


Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

The Cryosphere, 2012

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

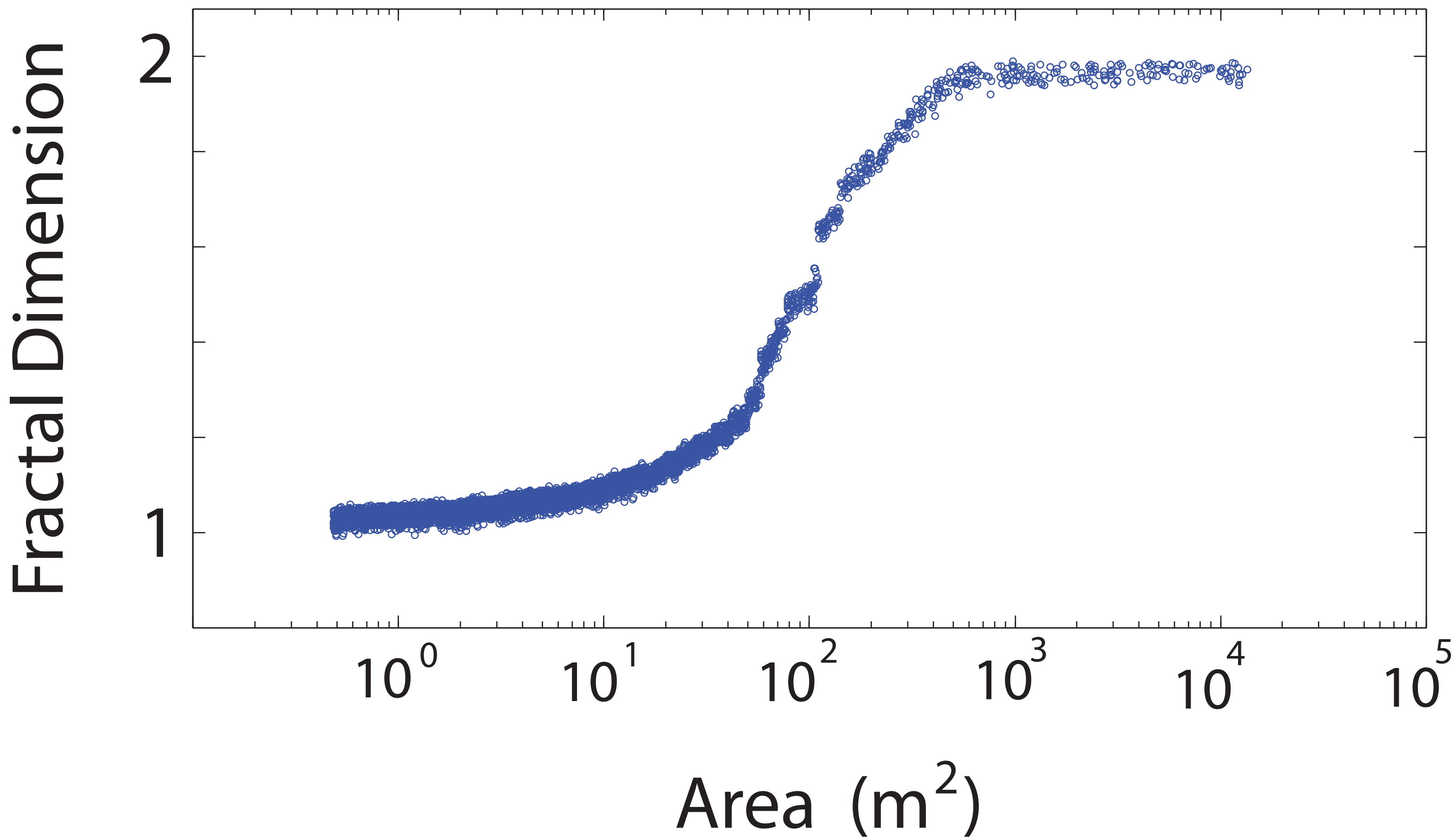


data from
5269 Arctic
melt ponds

simple pond ***transitional pond*** ***complex pond***

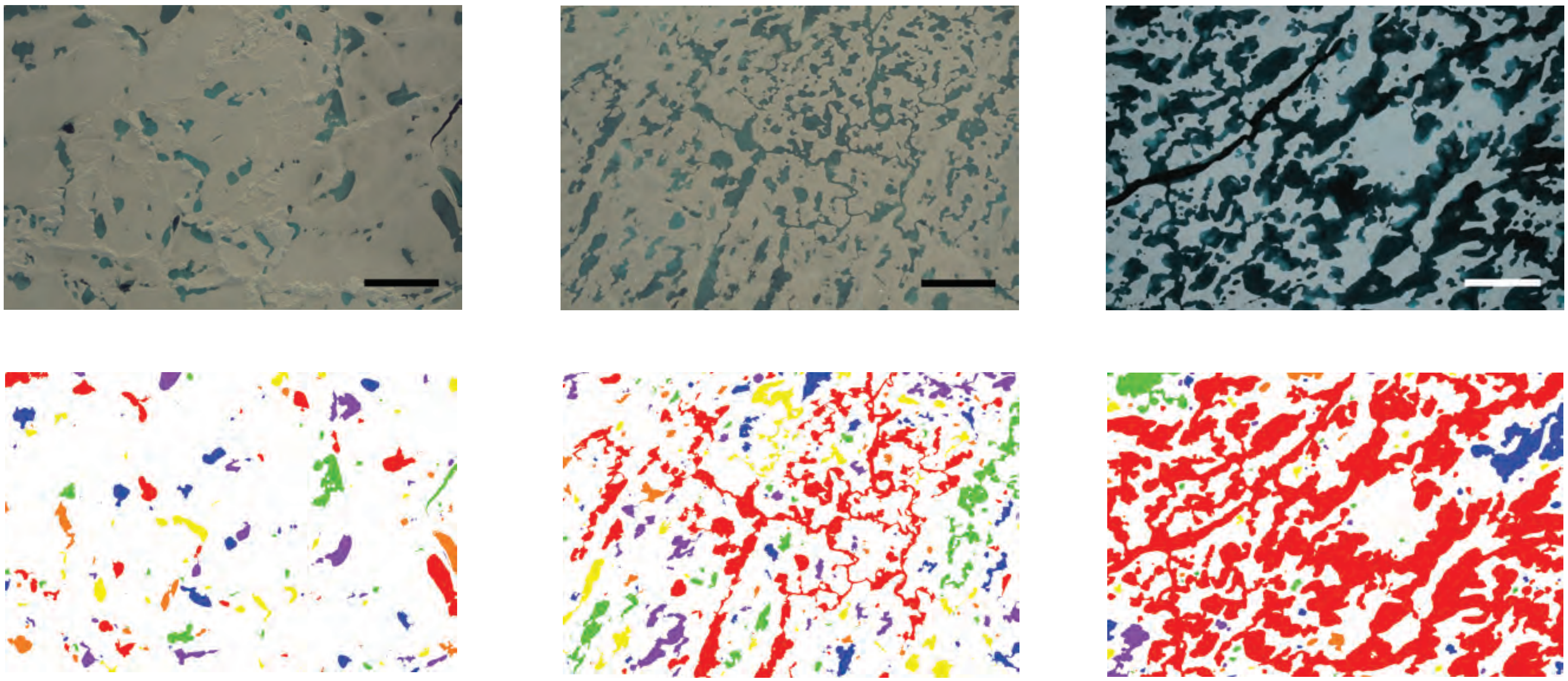
transition in the fractal dimension

complexity grows with length scale



compute "derivative" of area - perimeter data

small simple ponds coalesce to form large connected structures with complex boundaries



melt pond percolation

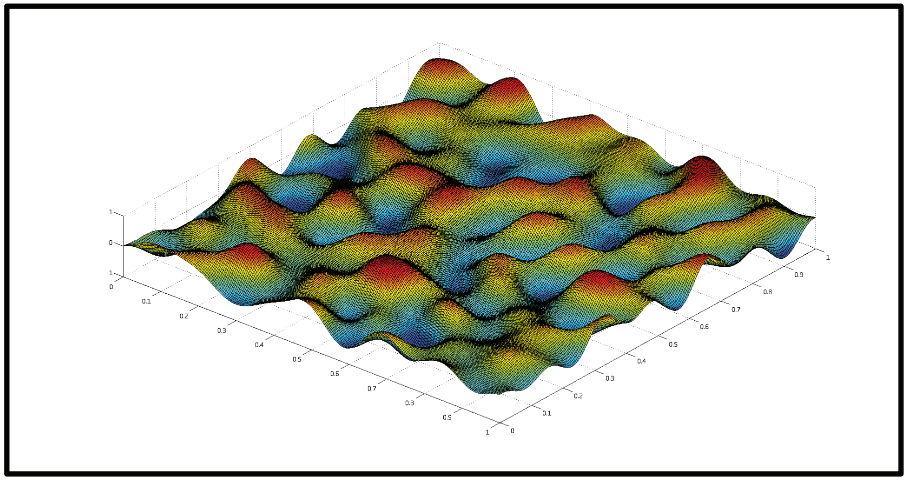
results on percolation threshold, correlation length, cluster behavior

Anthony Cheng (Hillcrest HS), Dylan Webb (Skyline HS), Court Strong, Ken Golden

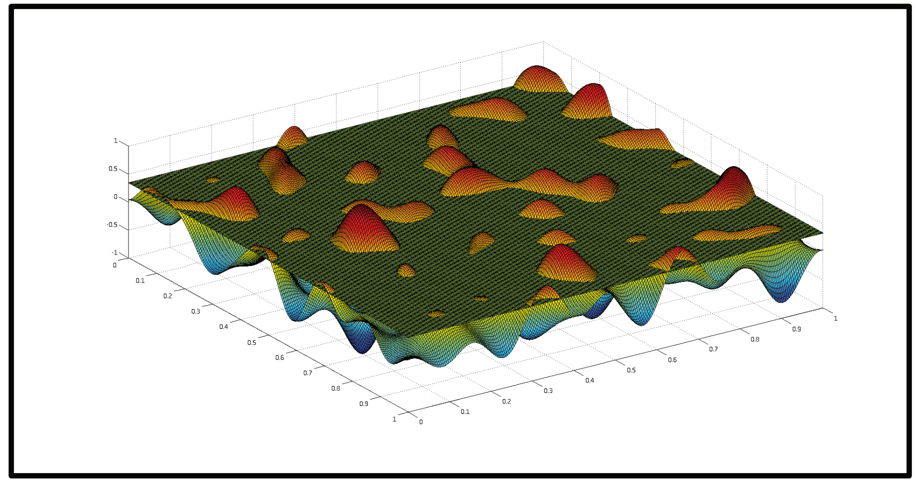
Continuum percolation model for melt pond evolution

level sets of random surfaces

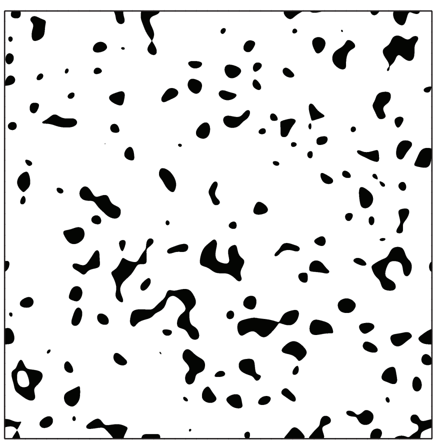
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2017

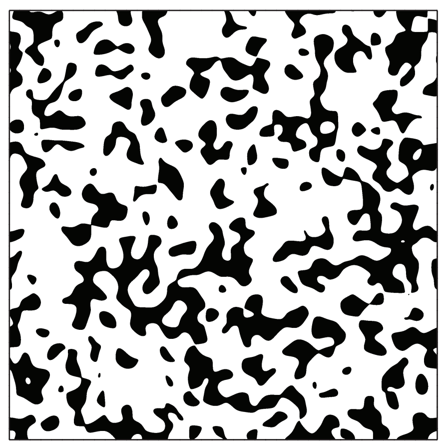


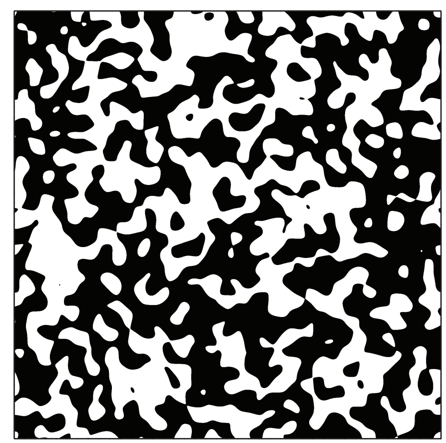
random Fourier series representation of surface topography



intersections of a plane with the surface define melt ponds



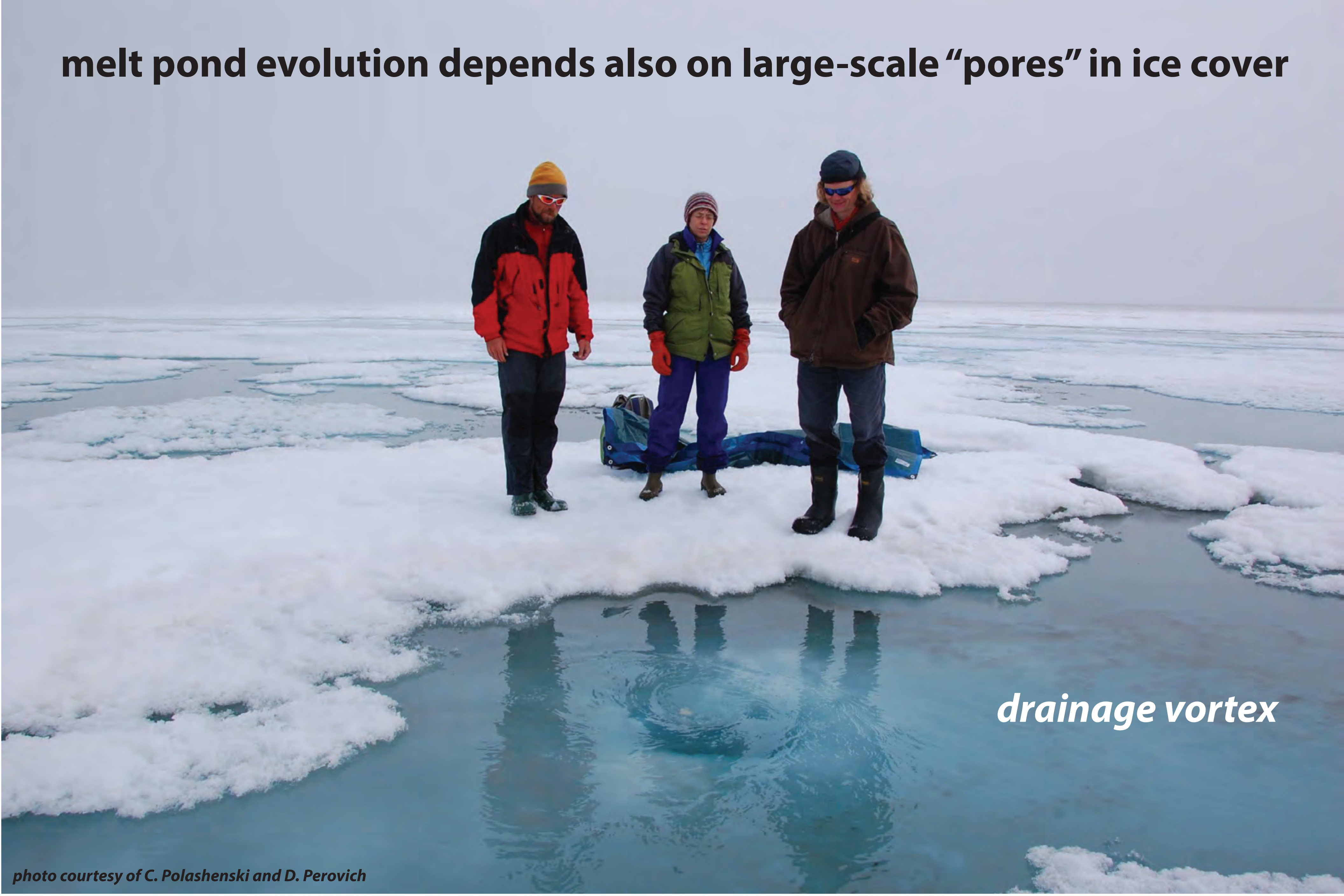




electronic transport in disordered media

diffusion in turbulent plasmas

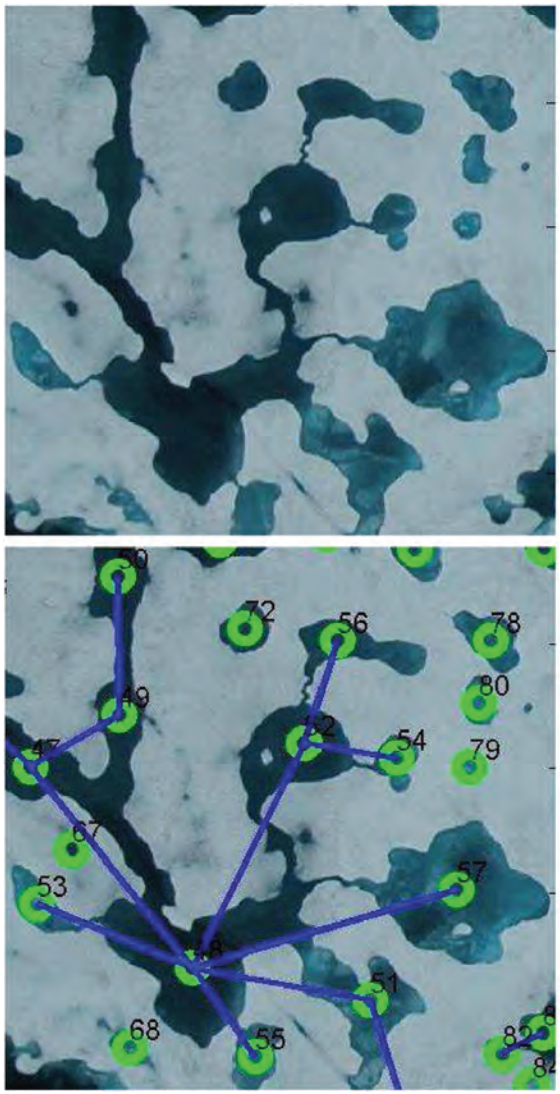
Isichenko, Rev. Mod. Phys., 1992



Melt pond connectivity enables vast expanses of melt water to drain down seal holes, thaw holes, and leads in the ice.

Network modeling of Arctic melt ponds

Barjatia, Tasdizen, Song, Sampson, Golden
Cold Regions Science and Tecnology, 2016



develop algorithms to map images of melt ponds onto

random resistor networks

graphs of nodes and edges with edge conductances

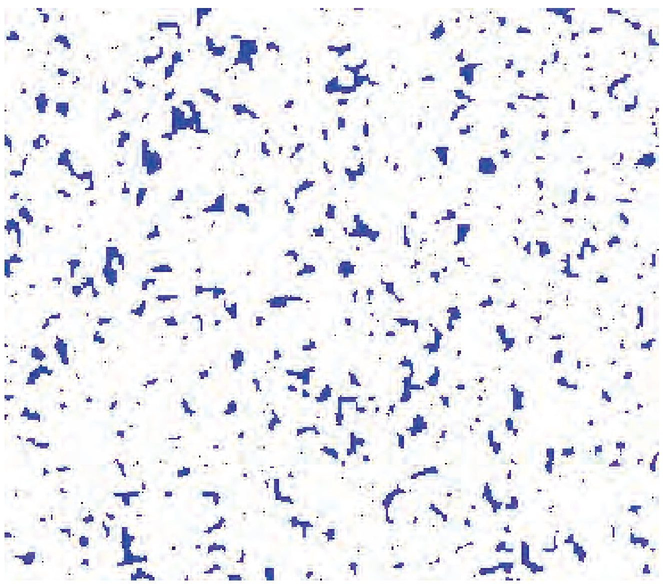
edge conductance ~ neck width

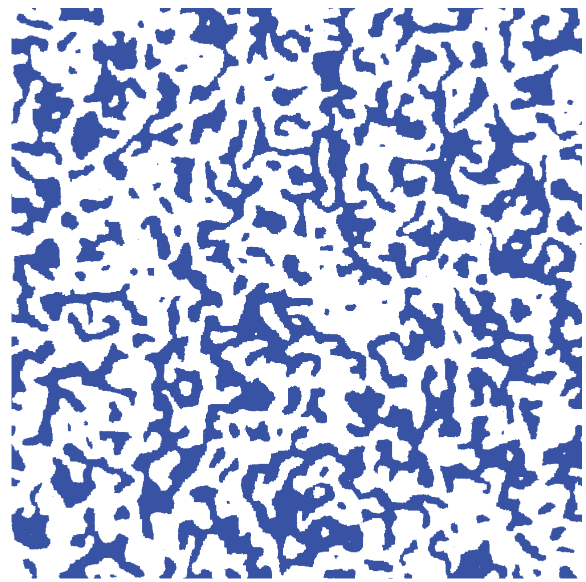
compute effective horizontal fluid conductivity

Ising model for ferromagnets → Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - H \sum_{i}^{N} s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

magnetization $M = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$ **pond coverage** $\frac{(M+1)}{2}$





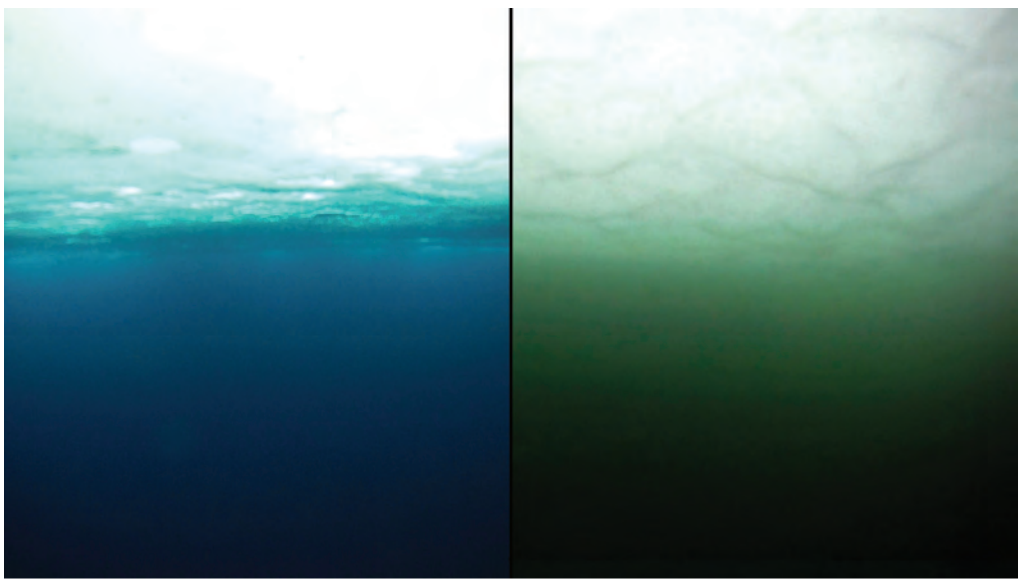
"melt ponds" are clusters of magnetic spins that align with the applied field

predictions of fractal transition, pond size exponent **Ma, Sudakov, Strong, Golden 2017**

2011 massive under-ice algal bloom

Arrigo et al., *Science* 2012

melt ponds act as ***WINDOWS*** allowing light through sea ice



no bloom   bloom

Have we crossed into a new ecological regime?

The Melt Pond Conundrum:

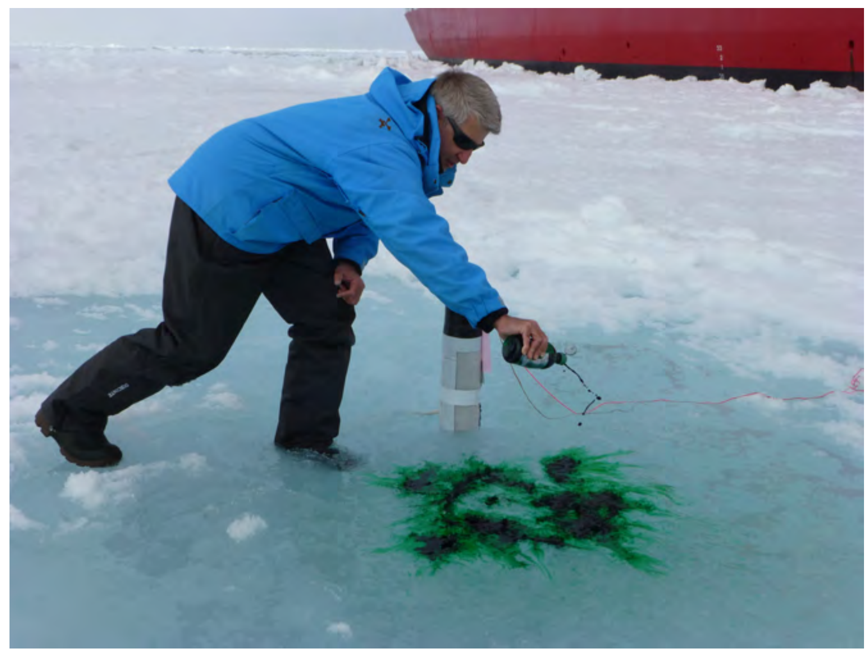
How can ponds form on top of sea ice that is highly permeable?

C. Polashenski, K. M. Golden, D. K. Perovich, E. Skyllingstad, A. Arnsten, C. Stwertka, N. Wright

Percolation Blockage: A Process that Enables Melt Pond Formation on First Year Arctic Sea Ice

J. Geophys. Res. Oceans 2017

2014 Study of Under Ice Blooms in the Chuckchi Ecosystem (SUBICE) aboard USCGC Healy



Conclusions

1. Summer Arctic sea ice is melting rapidly, and melt ponds and other processes must be accounted for in order to predict melting rates.

2. Fluid flow through sea ice mediates melt pond evolution and many processes important to climate change and polar ecosystems.

3. Statistical physics and homogenization help *link scales*, provide rigorous methods for finding effective behavior, and advance how sea ice is represented in climate models.

4. Critical behavior (in many forms) is inherent in the climate system.

5. Field experiments are essential to developing relevant mathematics.

6. Our research will help to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

Joe Keller had a broad impact on the math we use to model sea ice...

THANK YOU

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

Office of Naval Research

Arctic and Global Prediction Program

Applied and Computational Analysis Program

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999