Name ______________________________

Student ID # ______________________________

Class (circle one) 9:40 10:45

Math 1210
Fall 2009
K. M. Golden

EXAM II
Friday, October 23, 2009

Problem	Points	Score
1.	30	
2.	25	
3.	20	
4.	15	
5.	10	
	TOTAL	

(30 points) 1. Calculate the following limits, and be sure to show all of your work. If a particular limit does not exist, state this clearly and tell why.

(a) $\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$

(b) $\lim_{x\to +\infty} \frac{7x^5 + \pi x^4 - 10^{20}x^3 - x + 1}{(x+1)^2(x+2)^3} = \lim_{x\to +\infty} \frac{7x^5}{x^5} = 7$

(c) $\lim_{x\to 0} \frac{1 - \cos x}{x}$

$= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x} = 0$

(d) $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \sin x, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x = \cos 0 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin x = \sin 0 = 0$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, the limit does not exist.

(e) $\lim_{x\to 0} \frac{\sin(\pi x)}{3x}$

$= \lim_{x\to 0} \frac{\sin(\pi x)}{3x} \cdot \frac{\pi x}{\pi x} = \lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} \cdot \frac{\pi x}{3x} = \lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} \cdot \lim_{x\to 0} \frac{\pi x}{3x}$

$= \frac{\pi}{3}$

(f) $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$

The limit does not exist. Because we have $\lim_{x\to 3} f(x) = 1$ where x is rational and $\lim_{t\to 3} f(t) = 0$ where t is irrational.

(25 points) 2. Find the following. Be sure to show all of your work.

(a) $\frac{d}{dx}(x \sin x) = \sin x + x \cos x$

Product Rule

(b) $\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{1 \cdot (1+x^2) - 2x \cdot x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

Qutient Rule

(c) $\frac{d^2x}{dt^2}$ where $x(t) = \sin(\sqrt{2}\, t)$

$\frac{dx}{dt} = \sqrt{2}\cos(\sqrt{2}t)$

$\frac{d^2x}{dt^2} = \sqrt{2}\left(\sqrt{2}\left(-\sin(\sqrt{2}t)\right)\right) = -2\sin(\sqrt{2}t)$

(d) $\frac{dm}{dv}$ where $m(v) = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

$\frac{dm}{dv} = -\frac{1}{2}m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \cdot \left(-\frac{2v}{c^2}\right) = \frac{m_0 v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}$

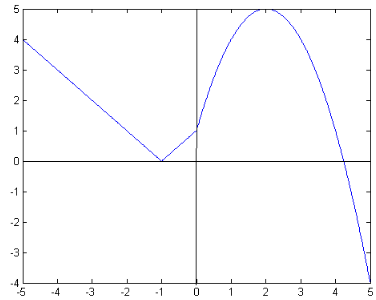
Chain Rule

(e) $\frac{dy}{dx}$ where $x^2 + y^2 = 1$

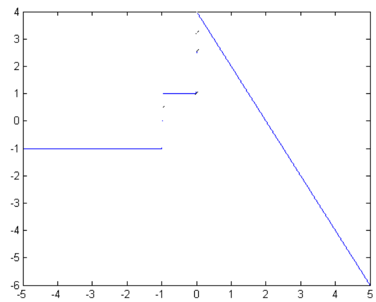
By implicit differentiation, we have $2x + 2y \cdot \frac{dy}{dx} = 0$. Therefore, we get $\frac{dy}{dx} = -\frac{x}{y}$.

(20 points) 3. Consider $f(x) = \begin{cases} |x+1|, & x \le 0 \\ 5-(x-2)^2, & x > 0 \end{cases}$, defined on all of $\mathbb{R}$.

(a) Sketch the graph of $f(x)$.



(b) Sketch the graph of the derivative of $f(x)$.



(c) Where is $f(x)$ continuous?

$f(x)$ is continuous everywhere.

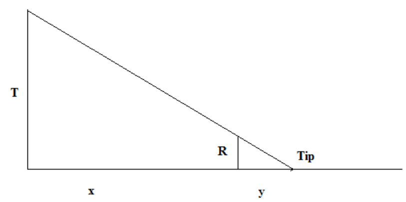
(d) Where is $f(x)$ differentiable?

$f(x)$ is differentiable everywhere except $x = -1$ and $x = 0$.

(e) Use the derivative to find the maximum value of $f(x)$ on the interval $[0, \infty)$. Where does the maximum occur? Justify your results.

On $[0, \infty)$, $f(x) = 5 - (x-2)^2$. The maximum is achieved at the places where the derivative is 0 or it is the boundary. We have $f'(x) = -2x+4$. It equals 0 if $x = 2$. Moreover we have $f(2) = 5$ and $f(0) = 1$. Thus the maximum is achieved at $x = 2$ and the maximum value is 5.

(15 points) 4. A primordial lightning bolt ignites a flame in the top of a 100 foot tall Cretaceous tree. A 20 foot tall *Tyrannosaurus rex* runs away from the flame at a speed of 10 f/s. How fast is the tip of his shadow moving when he is 150 feet away from the base of the tree? Be sure to show all your work.



The triangle made by the top of the tree, the base of the tree, and the tip of the shadow is similar to the triangle made by the top of the $Tyrannosaurus$, the base of the $Tyrannosaurus$, and the tip of the shadow. This implies:

$$\frac{T}{x+y} = \frac{R}{y}$$

Thus, we have $\dfrac{100}{x+y} = \dfrac{20}{y}$ and $y = \dfrac{1}{4}x$.

Since $\dfrac{dx}{dt} = 10f/s$, the speed of the tip of the shadow is

$$\frac{d(x+y)}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \frac{dx}{dt} + \frac{1}{4}\frac{dx}{dt} = \frac{5}{4} \cdot 10 = 12.5f/s$$

(10 points) 5. Use linear approximation (the differential) to estimate $\sqrt{4.25}$.

The question is not completely clear on the point at which we should construct the tangent line, although the obvious choice is $x = 4$. So, the solution uses $x = 4$. However, if you set everything up correctly but used a different point, we gave you full credit.

The derivative of the function $f(x) = \sqrt{x}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$, and so $f'(4) = \dfrac{1}{4}$. The tangent line to $f(x)$ at point $x = 4$ is $L(x) = \dfrac{1}{4}x + 1$, and so $f(4.25) \approx \dfrac{1}{4}(4.25) + 1 = \dfrac{33}{16}$.