

# Herglotz Functions and Multiscale Homogenization for Sea Ice

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BIRS Workshop  
8 October 2019



# SEA ICE covers ~12% of Earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- hosts rich ecosystem
- indicator of **climate change**



polar ice caps critical  
to climate in reflecting  
sunlight during summer



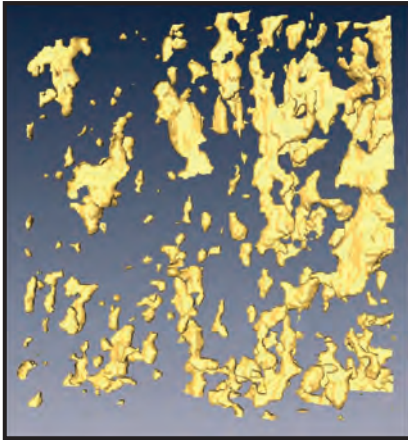
# Sea Ice is a Multiscale Composite Material

## *sea ice microstructure*

brine inclusions

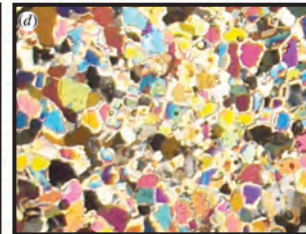
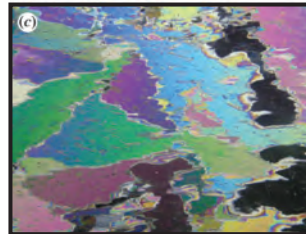
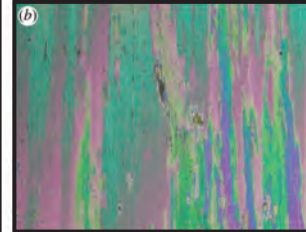
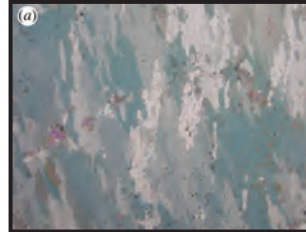


Weeks & Assur 1969



H. Eicken  
Golden et al. GRL 2007

polycrystals

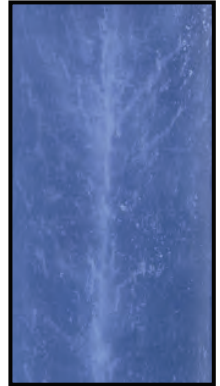


Gully et al. Proc. Roy. Soc. A 2015

brine channels



D. Cole



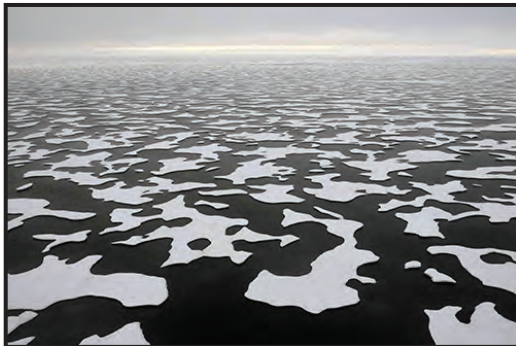
K. Golden

**millimeters**

**centimeters**

## *sea ice mesostructure*

Arctic melt ponds



K. Frey

Antarctic pressure ridges



K. Golden

**meters**

## *sea ice macrostructure*

sea ice floes



J. Weller

sea ice pack



NASA

**kilometers**



# What is this talk about?

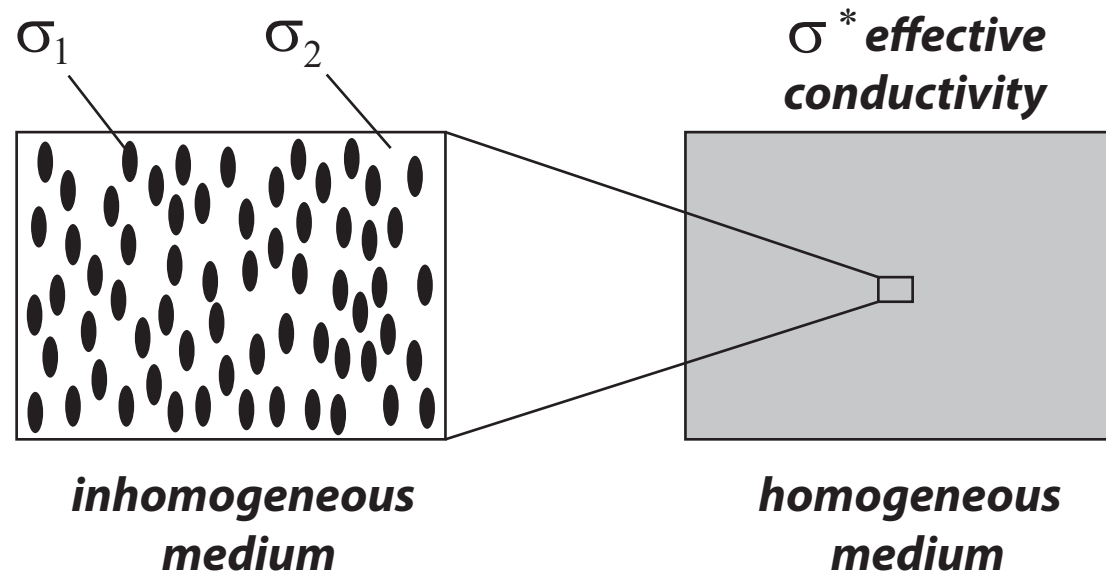
**A tour of Herglotz functions and how they arise in the study of sea ice and its role in the climate system.**

Use **statistical physics and homogenization for composites** to LINK SCALES in the sea ice system ... compute effective behavior on scales relevant to coarse-grained climate models, remote sensing, process studies, ...

1. Fluid flow through sea ice, percolation
2. Analytic continuation for two phase composites  
remote sensing, inversion, spectral measures  
**random matrix theory and Anderson transitions**
3. Stieltjes representations for advection diffusion, polycrystals, ocean waves in the sea ice pack
4. Melt ponds, the Ising model, and Stieltjes integrals



# HOMOGENIZATION - Linking Scales in Composites



**find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium**

*Maxwell 1873 : effective conductivity of a dilute suspension of spheres*

*Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid*

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*

*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties



# How do scales interact in the sea ice system?

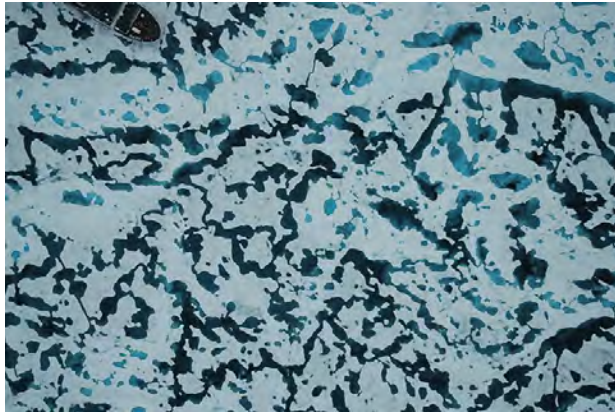


basin scale -  
grid scale  
albedo

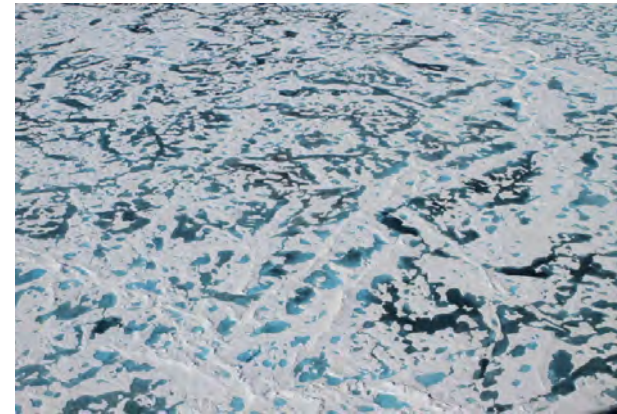
NASA

## Linking Scales

km  
scale  
melt  
ponds



km  
scale  
melt  
ponds

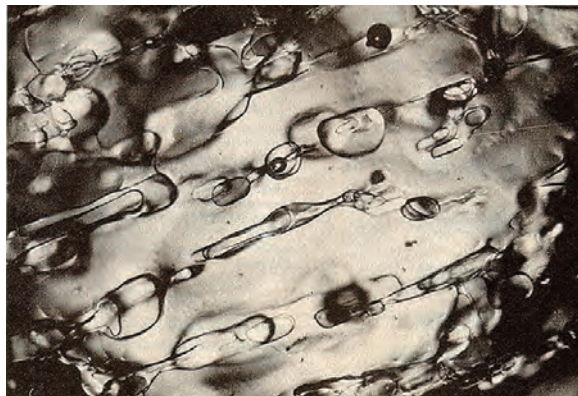


Perovich

## Linking

## Scales

mm  
scale  
brine  
inclusions



meter  
scale  
snow  
topography





***sea ice microphysics***

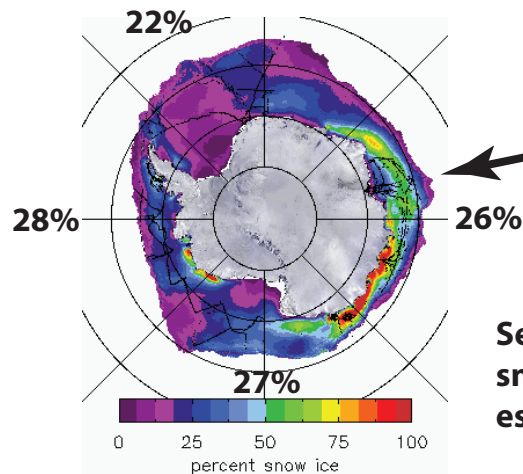
***fluid transport***

# fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

*evolution of Arctic melt ponds and sea ice albedo*



*nutrient flux for algal communities*



T. Maksym and T. Markus, 2008

*Antarctic surface flooding  
and snow-ice formation*

September  
snow-ice  
estimates

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO<sub>2</sub>*



# fluid permeability of a porous medium



how much water gets through the sample per unit time?

## *Darcy's Law*

for slow viscous flow in a porous medium

averaged  
fluid velocity

pressure  
gradient

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$

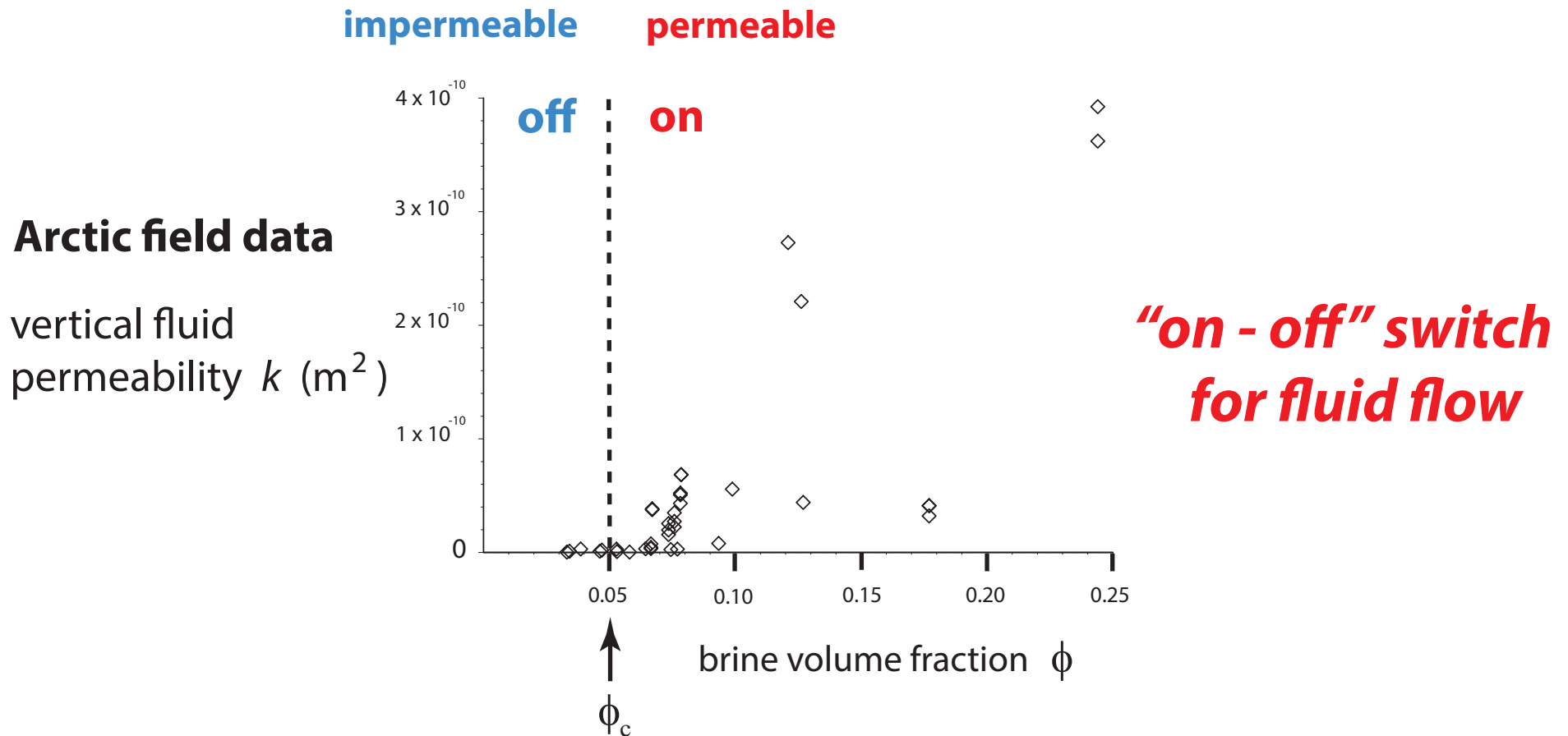
viscosity

$\mathbf{k}$  = fluid permeability tensor

## *HOMOGENIZATION*

*mathematics for analyzing effective behavior of heterogeneous systems*

# Critical behavior of fluid transport in sea ice



critical brine volume fraction  $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ \text{C}, S \approx 5 \text{ ppt}$

## RULE OF FIVES

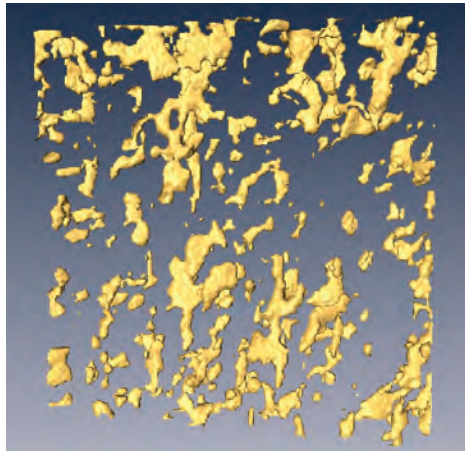
**Golden, Ackley, Lytle Science 1998**

**Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007**

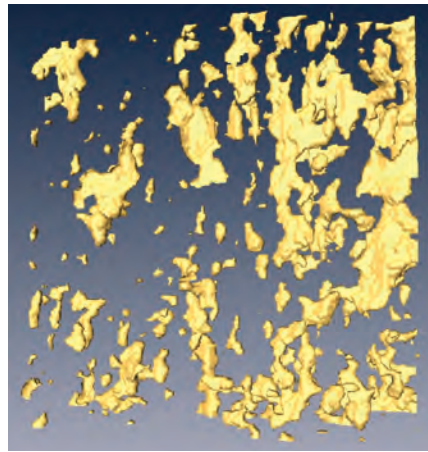
**Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009**



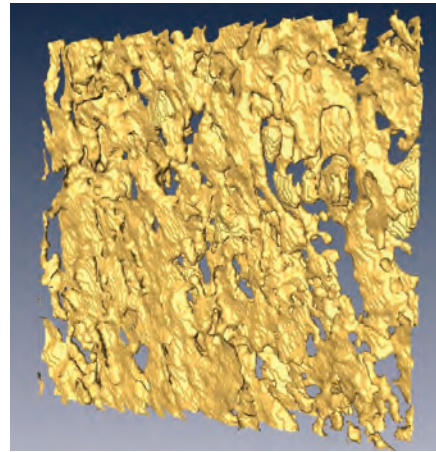
brine volume fraction and **connectivity** increase with temperature



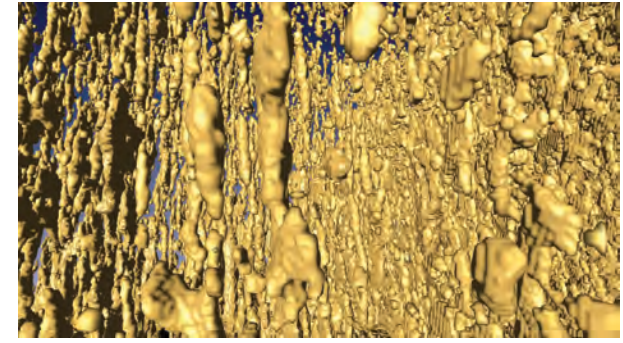
$T = -15\text{ }^{\circ}\text{C}$ ,  $\phi = 0.033$



$T = -6\text{ }^{\circ}\text{C}$ ,  $\phi = 0.075$



$T = -3\text{ }^{\circ}\text{C}$ ,  $\phi = 0.143$



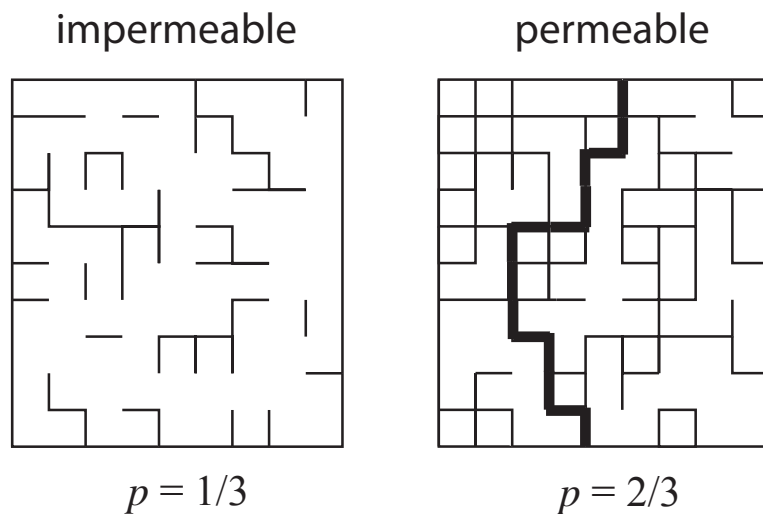
$T = -4\text{ }^{\circ}\text{C}$ ,  $\phi = 0.113$

**X-ray tomography for brine phase in sea ice**

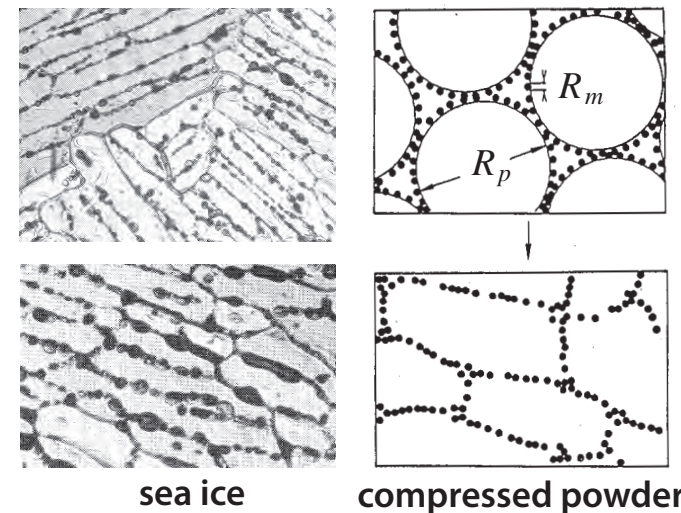
Golden, Eicken, *et al.*, *Geophysical Research Letters* 2007

**PERCOLATION THRESHOLD**  $\phi_c \approx 5\%$

Golden, Ackley, Lytle, *Science* 1998



**lattice percolation**

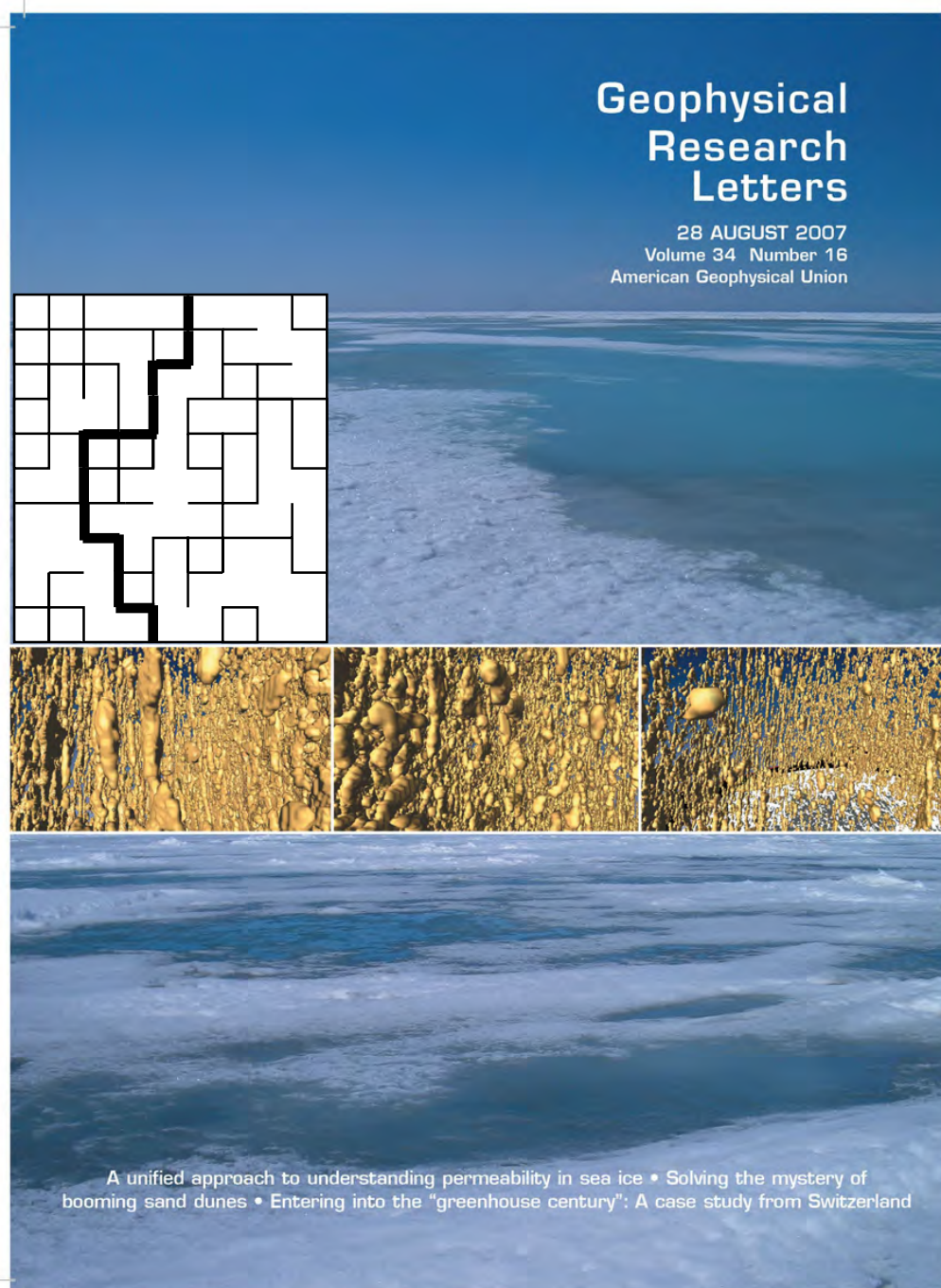


**continuum percolation**

Kusy, Turner  
*Nature* 1971

# ***Thermal evolution of permeability and microstructure in sea ice***

***Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007***



micro-scale  
controls  
macro-scale  
processes

***percolation theory***

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical  
exponent  
*t*

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

***hierarchical model  
network model  
rigorous bounds***

agree closely  
with field data

***X-ray tomography for  
brine inclusions***

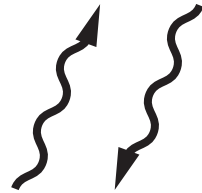
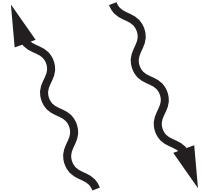
***unprecedented look  
at thermal evolution  
of brine phase and  
its connectivity***

***confirms rule of fives***

***Pringle, Miner, Eicken, Golden  
J. Geophys. Res. 2009***



# Remote sensing of sea ice



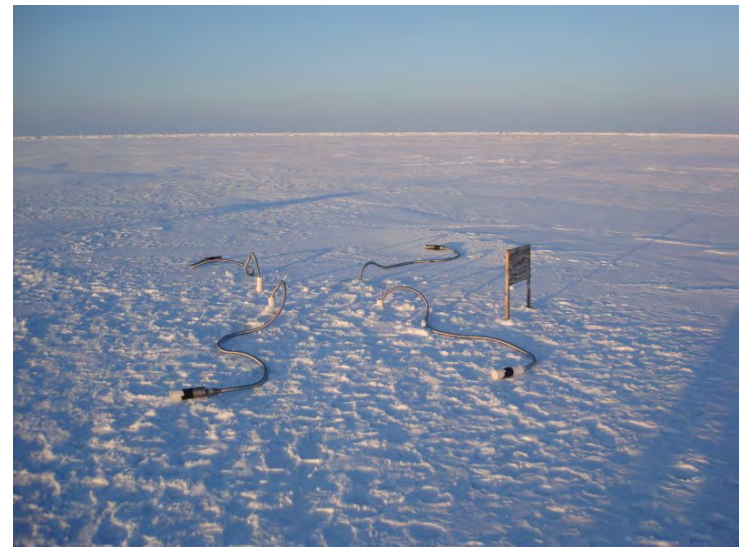
*sea ice thickness*  
*ice concentration*

## **INVERSE PROBLEM**

Recover sea ice  
properties from  
electromagnetic  
(EM) data

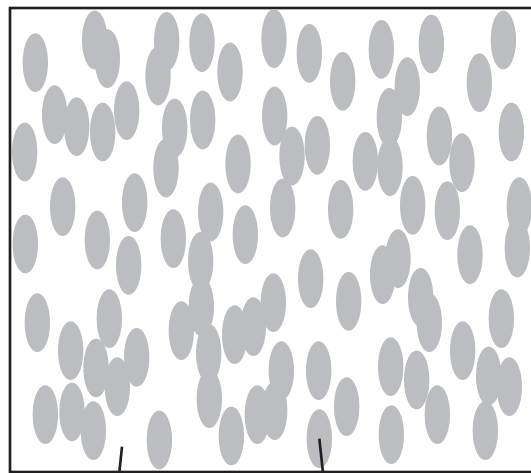
$$\epsilon^*$$

effective complex permittivity  
(dielectric constant, conductivity)



*brine volume fraction*  
*brine inclusion connectivity*

# Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$\epsilon_1$

$\epsilon_2$



$\epsilon^*$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

$p_1, p_2$  = volume fractions of  
the components

$$\epsilon^* = \epsilon^* \left( \frac{\epsilon_1}{\epsilon_2}, \text{ composite geometry} \right)$$

**What are the effective propagation characteristics  
of an EM wave (radar, microwaves) in the medium?**



# Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

## Stieltjes integral representation for homogenized parameter

*separates geometry from parameters*

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

← geometry

← material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

$\mu$

- spectral measure of self adjoint operator  $\Gamma\chi$
- mass =  $p_1$
- higher moments depend on  $n$ -point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla.$$

$\chi$  = characteristic function of the brine phase

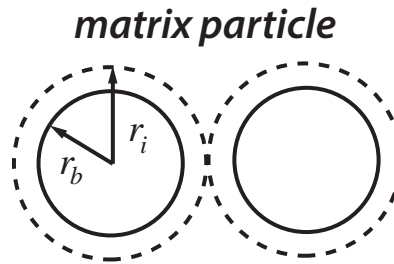
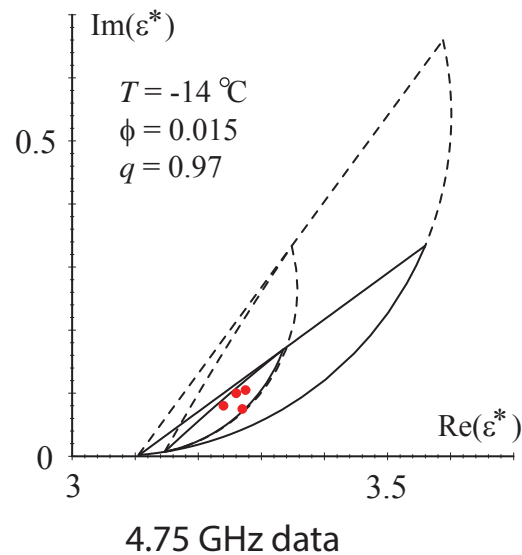
$$E = s (s + \Gamma\chi)^{-1} e_k$$

$\Gamma\chi$  : microscale  $\rightarrow$  macroscale

$\Gamma\chi$  *links scales*

# forward and inverse bounds on the complex permittivity of sea ice

## forward bounds



$$q = r_b / r_i$$

$$0 < q < 1$$

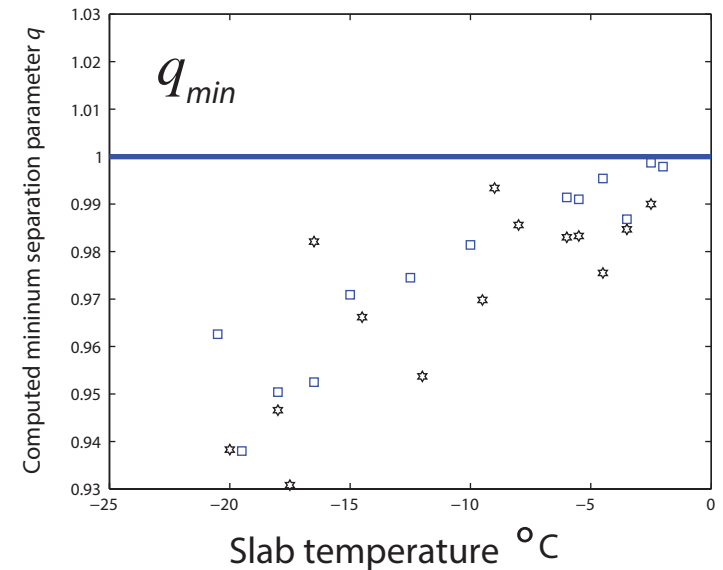
**Golden 1995, 1997**

**Bruno 1991**

## inverse bounds and recovery of brine porosity

**Gully, Backstrom, Eicken, Golden  
Physica B, 2007**

## inverse bounds



## inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity $\epsilon^*$

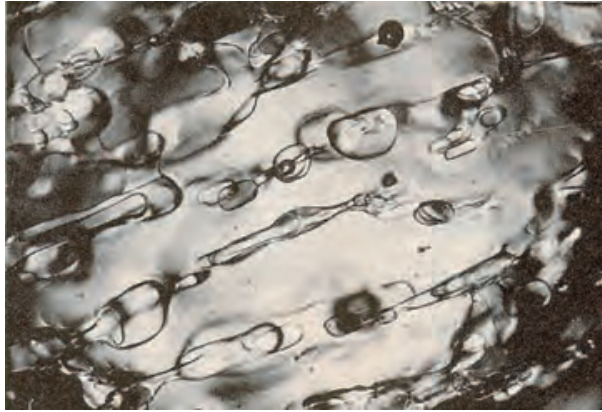
### rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in  $(p, q)$ -space

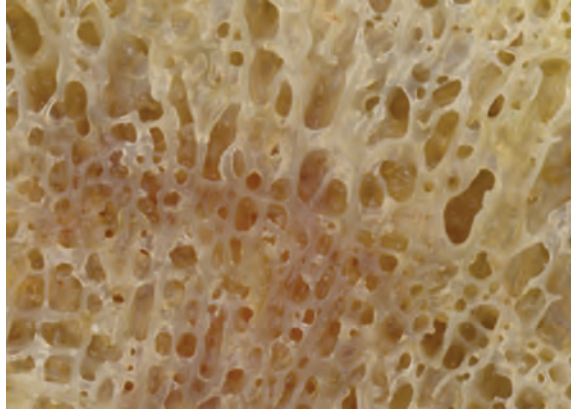
**Orum, Cherkaev, Golden  
Proc. Roy. Soc. A, 2012**



## SEA ICE

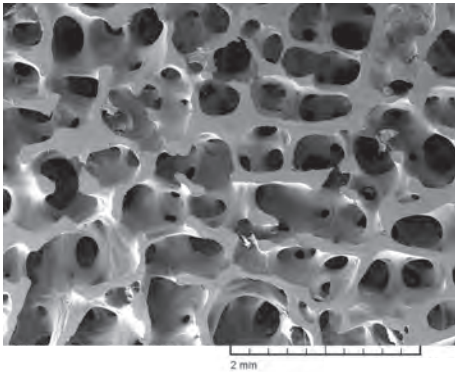


## HUMAN BONE

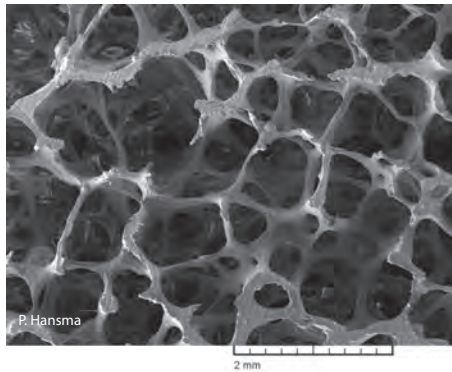


*spectral characterization  
of porous microstructures  
in human bone*

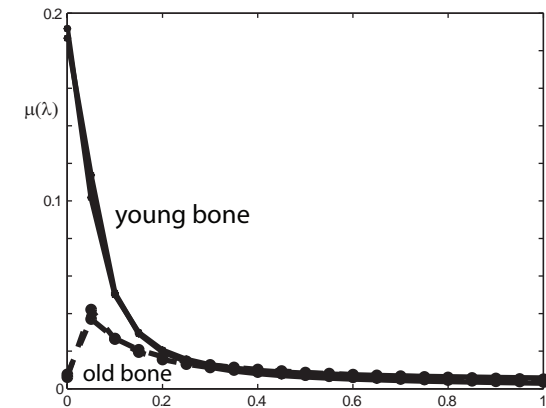
young healthy trabecular bone



old osteoporotic trabecular bone



reconstruct spectral measures  
from complex permittivity data



use regularized inversion scheme

*apply spectral measure analysis of brine connectivity and  
spectral inversion to electromagnetic monitoring of osteoporosis*

Golden, Murphy, Cherkaev, J. Biomechanics 2011

*the math doesn't care if it's sea ice or bone!*

# direct calculation of spectral measures

Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

**once we have the spectral measure  $\mu$  it can be used in  
Stieltjes integrals for other transport coefficients:**

***electrical and thermal conductivity, complex permittivity,  
magnetic permeability, diffusion, fluid flow properties***

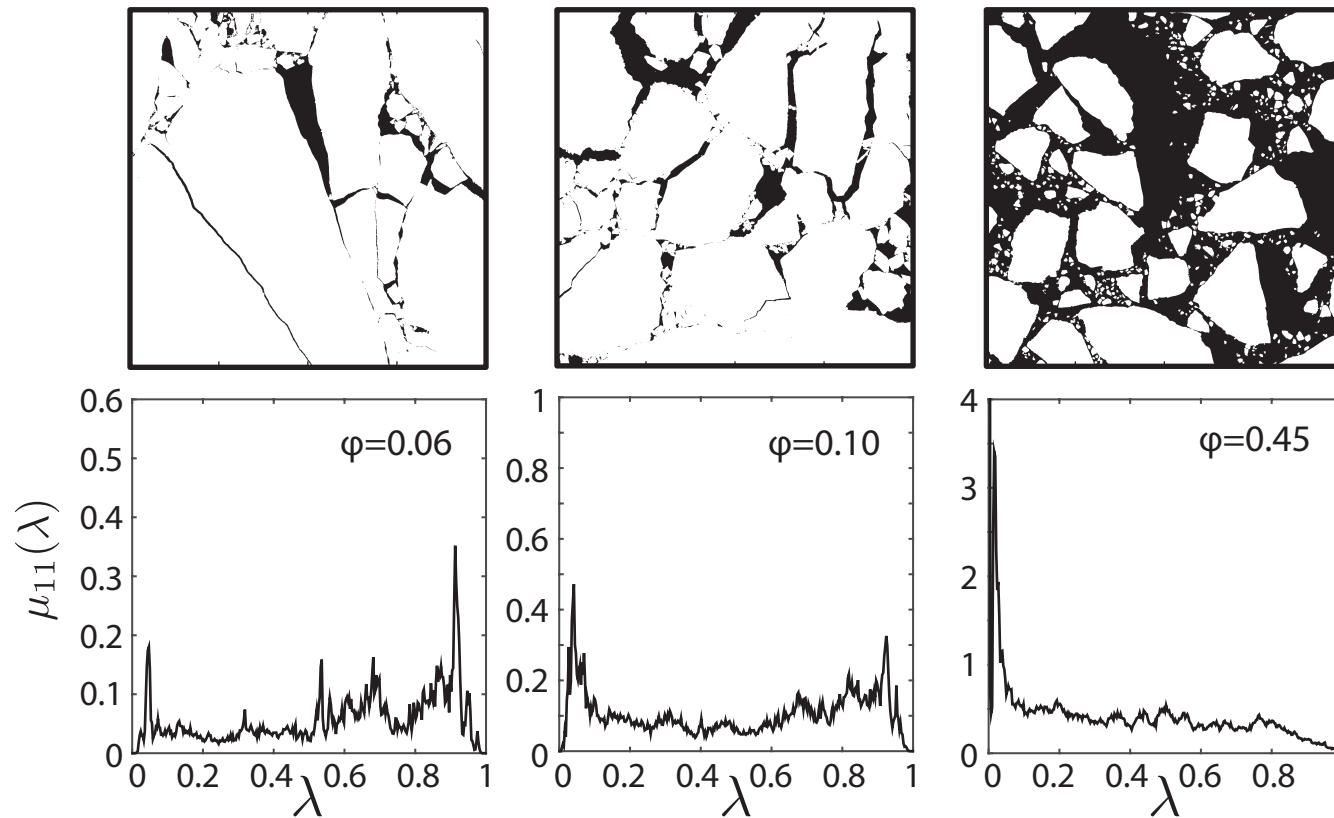
earlier studies of spectral measures

Day and Thorpe 1996

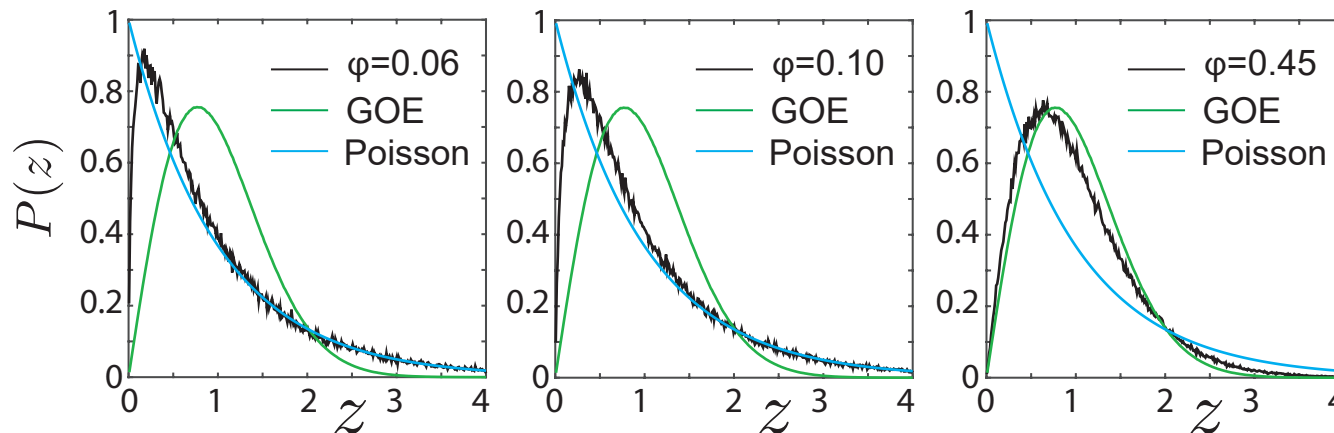
Helsing, McPhedran, Milton 2011

# Spectral computations for sea ice floe configurations

spectral  
measures



eigenvalue  
spacing  
distributions



uncorrelated



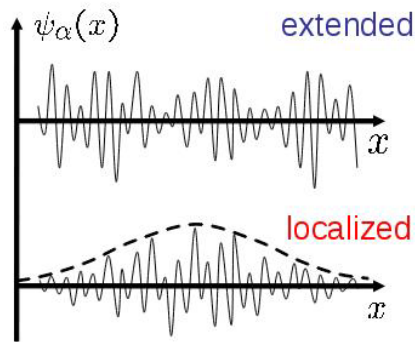
level repulsion

**ANDERSON TRANSITION**

**UNIVERSAL  
Wigner-Dyson  
distribution**

Murphy, Cherkhev, Golden  
*Phys. Rev. Lett.* 2017





## metal / insulator transition **localization**

*Anderson 1958*  
*Mott 1949*  
*Shklovshii et al 1993*  
*Evangelou 1992*

**Anderson transition in wave physics:  
quantum, optics, acoustics, water waves, ...**

**we find a surprising analog**

***Anderson transition for classical transport in composites***

*Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017*

**PERCOLATION  
TRANSITION**



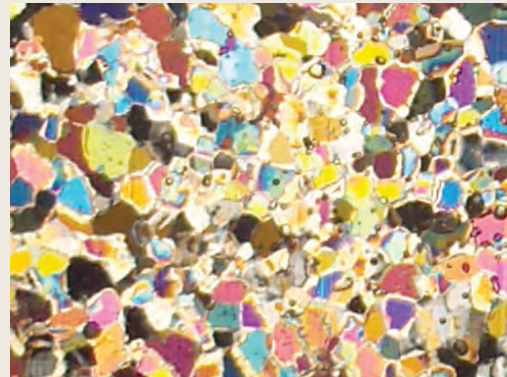
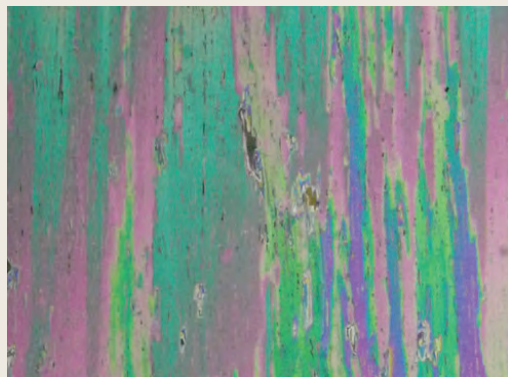
**transition to universal  
eigenvalue statistics (GOE)  
extended states, mobility edges**

**-- but without wave interference or scattering effects ! --**

# Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,  
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**  
Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**  
*orientation statistics*
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



## PROCEEDINGS A

350 YEARS  
OF SCIENTIFIC  
PUBLISHING

An invited review  
commemorating 350 years  
of scientific publishing at the  
Royal Society

A method to distinguish  
between different types  
of sea ice using remote  
sensing techniques

A computer model to  
determine how a human  
should walk so as to expend  
the least energy



THE  
ROYAL  
SOCIETY  
PUBLISHING

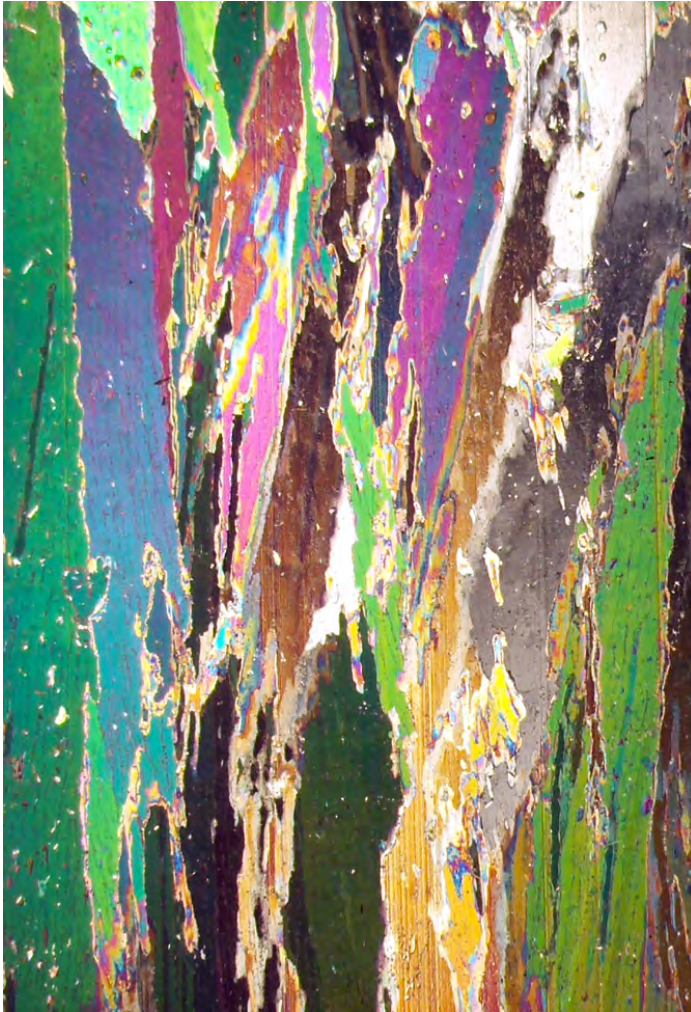


# ***higher threshold for fluid flow in Antarctic granular sea ice***

columnar

granular

**5%**



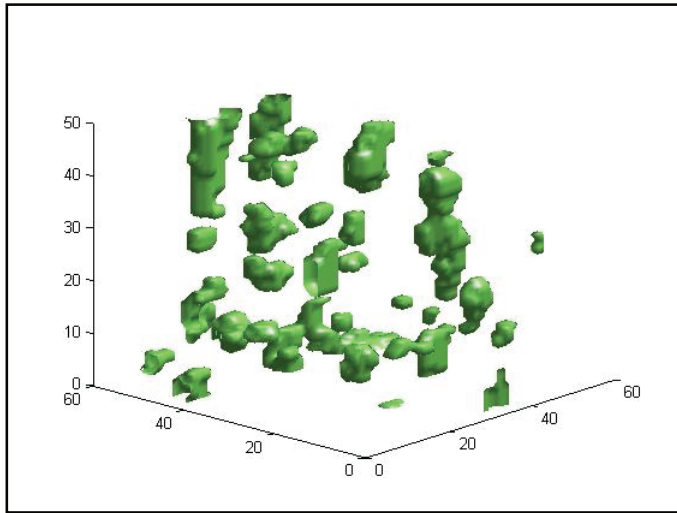
**10%**



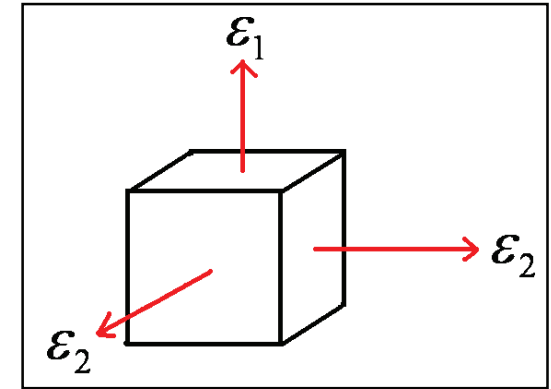
***Golden, Sampson, Gully, Lubbers, Tison 2019***



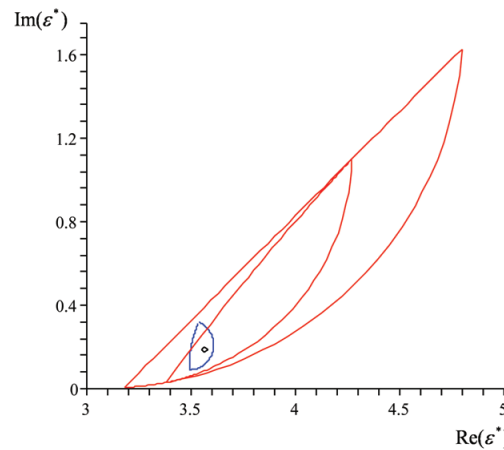
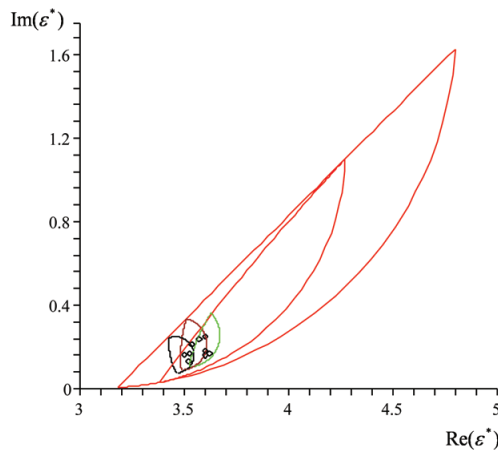
# two scale homogenization for polycrystalline sea ice



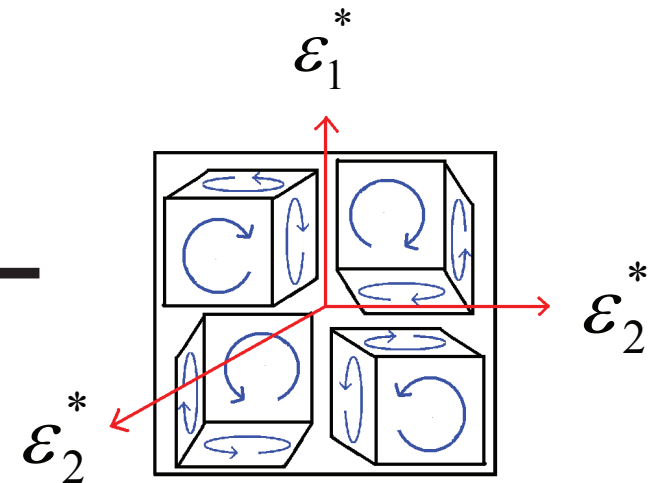
numerical homogenization  
for single crystal



analytic continuation  
for polycrystals



bounds



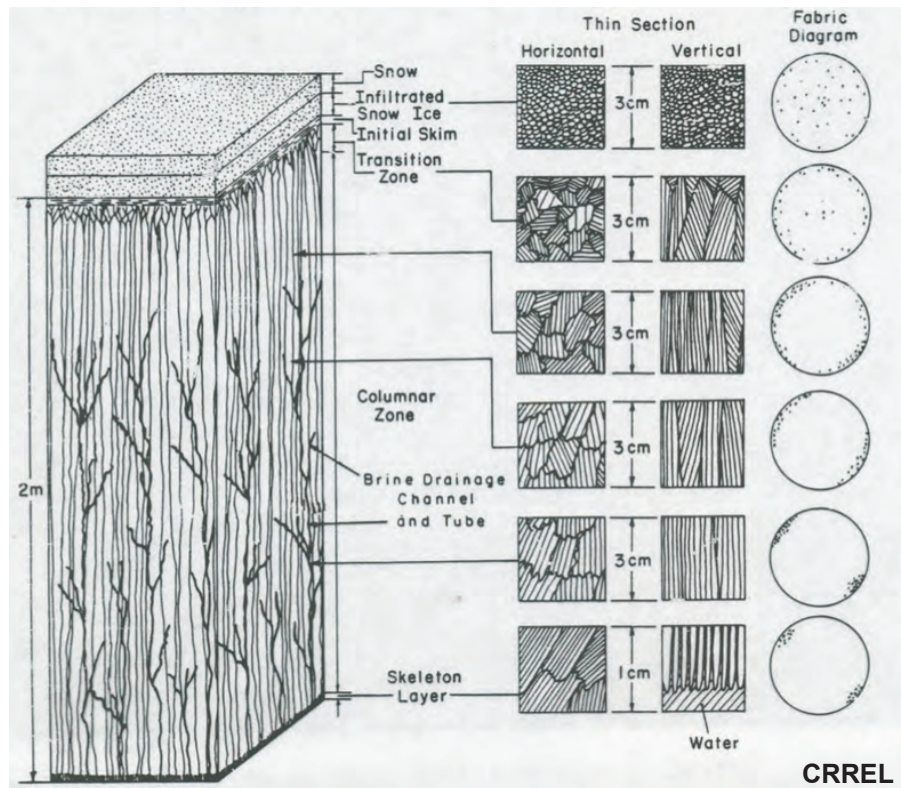
# Rigorous bounds on the complex permittivity tensor of sea ice with polycrystalline anisotropy within the horizontal plane

McKenzie McLean, Elena Cherkaev, Ken Golden 2019

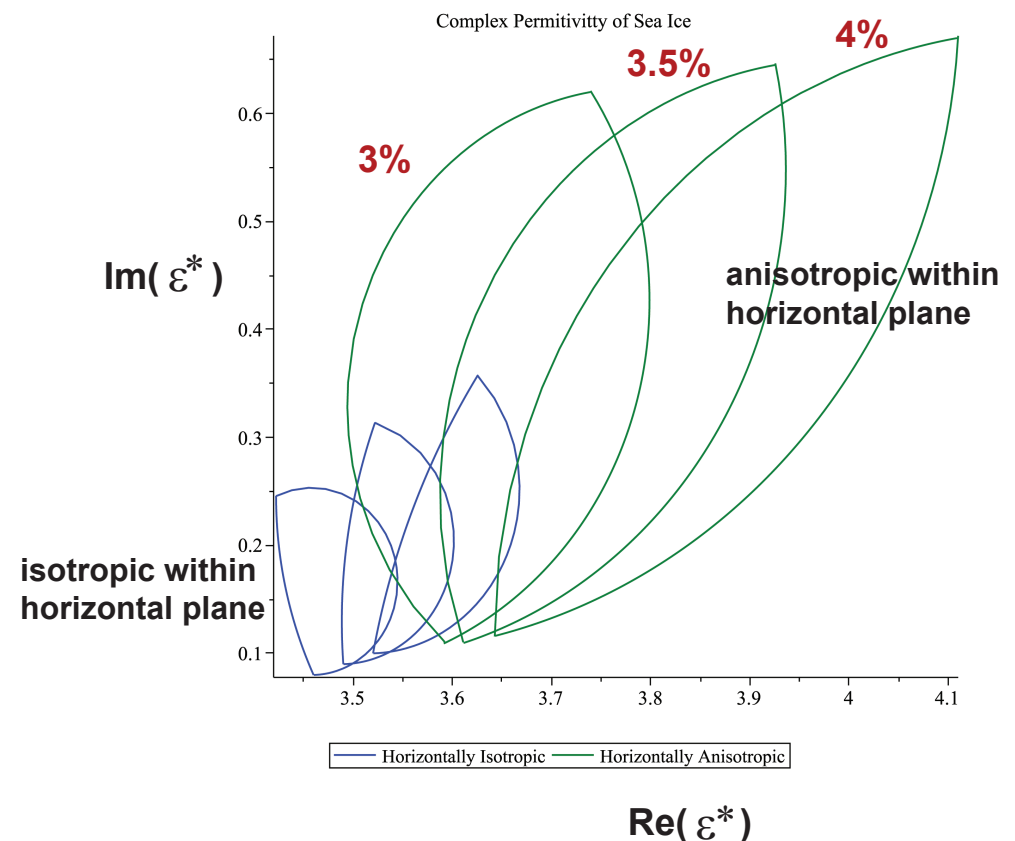
motivated by **Weeks and Gow, *JGR* 1979: c-axis alignment in Arctic fast ice off Barrow**

**Golden and Ackley, *JGR* 1981: radar propagation model in aligned sea ice**

input: orientation statistics



output: bounds



# advection enhanced diffusion

## effective diffusivity

nutrient and salt transport in sea ice  
heat transport in sea ice with convection  
sea ice floes in winds and ocean currents  
tracers, buoys diffusing in ocean eddies  
diffusion of pollutants in atmosphere

advection diffusion equation with a velocity field  $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$



homogenize

$$\frac{\partial \bar{T}}{\partial t} = \kappa^* \Delta \bar{T}$$

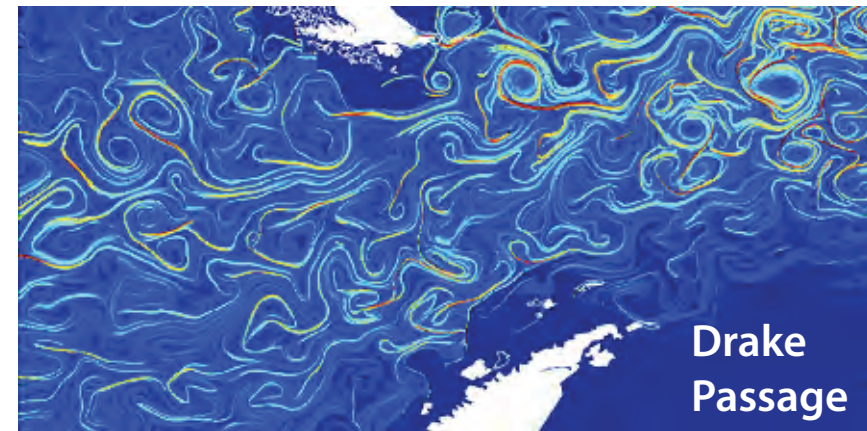
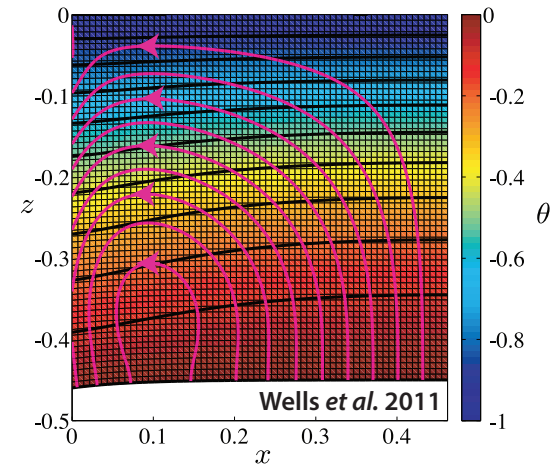
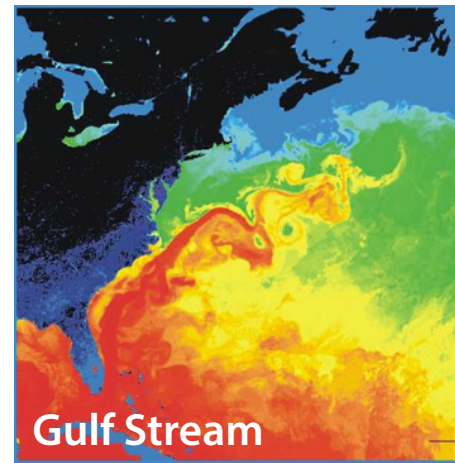
$\kappa^*$  effective diffusivity

**Stieltjes integral for  $\kappa^*$  with spectral measure**

*Avellaneda and Majda, PRL 89, CMP 91*

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019





# Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019

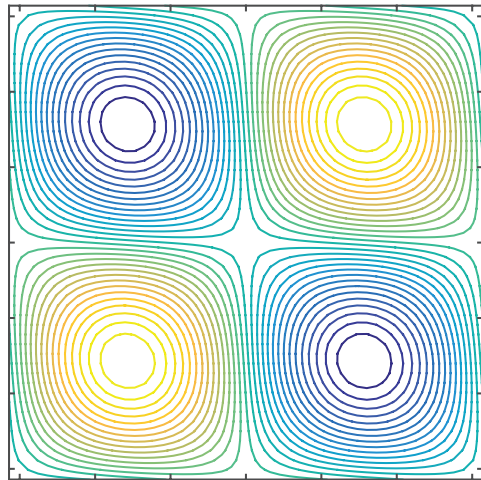
$$\kappa^* = \kappa \left( 1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2} \right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

- $\mu$  is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator  $i\Gamma H\Gamma$
- $H$  = stream matrix ,  $\kappa$  = local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla$  ,  $\Delta$  is the Laplace operator
- $i\Gamma H\Gamma$  is bounded for time independent flows
- $F(\kappa)$  is analytic off the spectral interval in the  $\kappa$ -plane

separation of material properties and flow field  
spectral measure calculations

# Rigorous bounds on convection enhanced thermal conductivity of sea ice

Kraitzman, Hardenbrook, Dinh, Murphy, Zhu, Cherkaev, Golden 2019

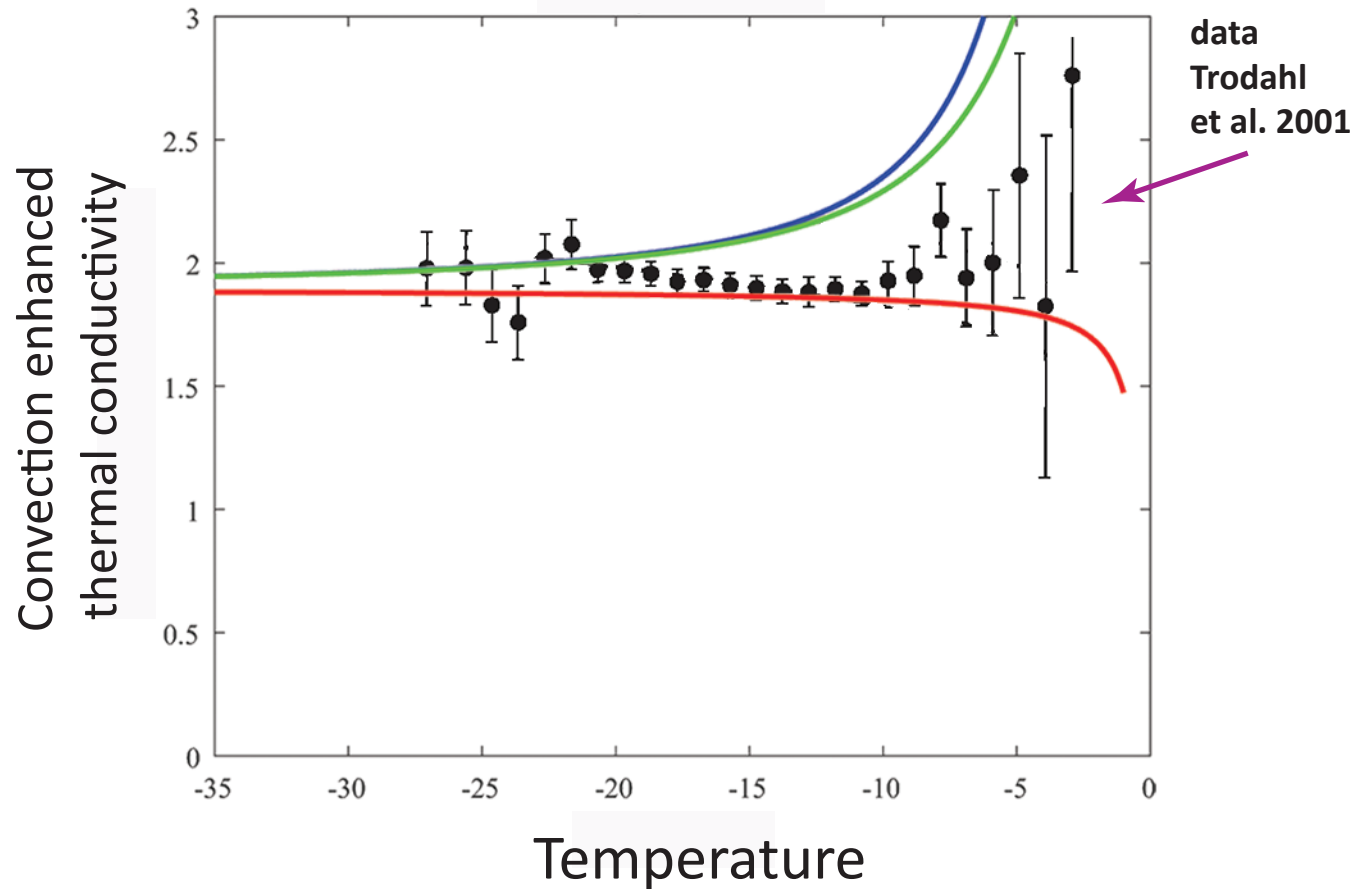


cat's eye flow model for  
brine convection cells

similar bounds  
for shear flows

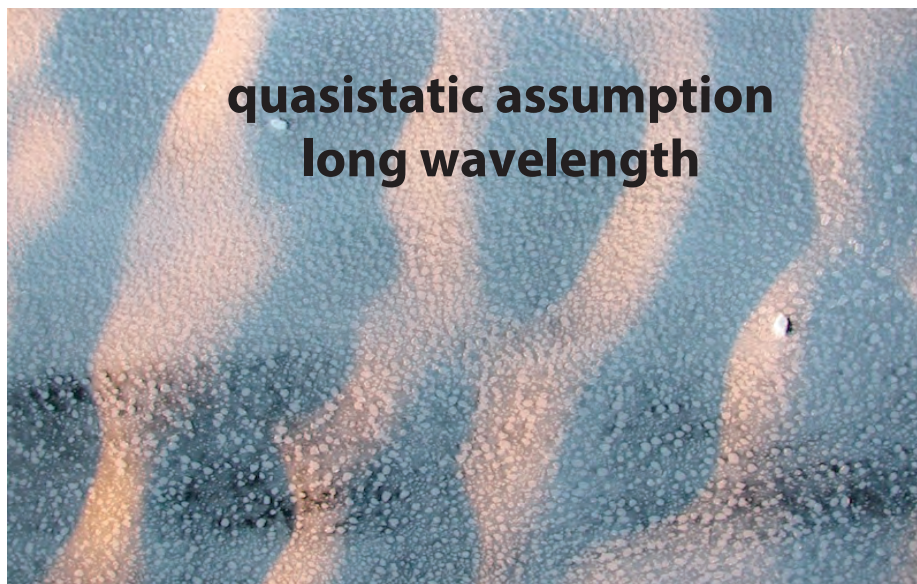
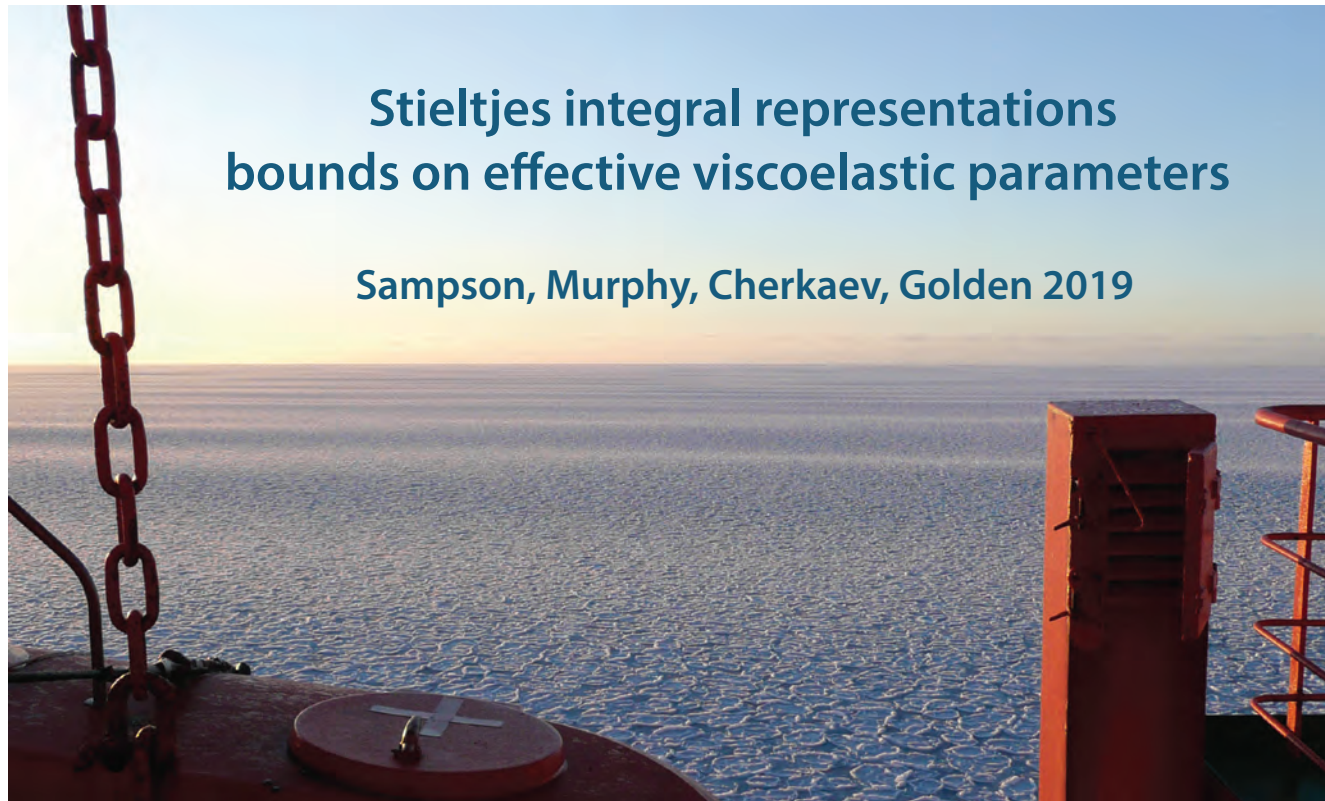
**rigorous bounds assuming information  
on flow field INSIDE inclusions**

Kraitzman, Cherkaev, Golden  
*SIAM J. Appl. Math* (in revision), 2019



rigorous Padé bounds from Stieltjes integral +  
analytical calculations of moments of measure

# wave propagation in the marginal ice zone





# bounds on the effective complex viscoelasticity

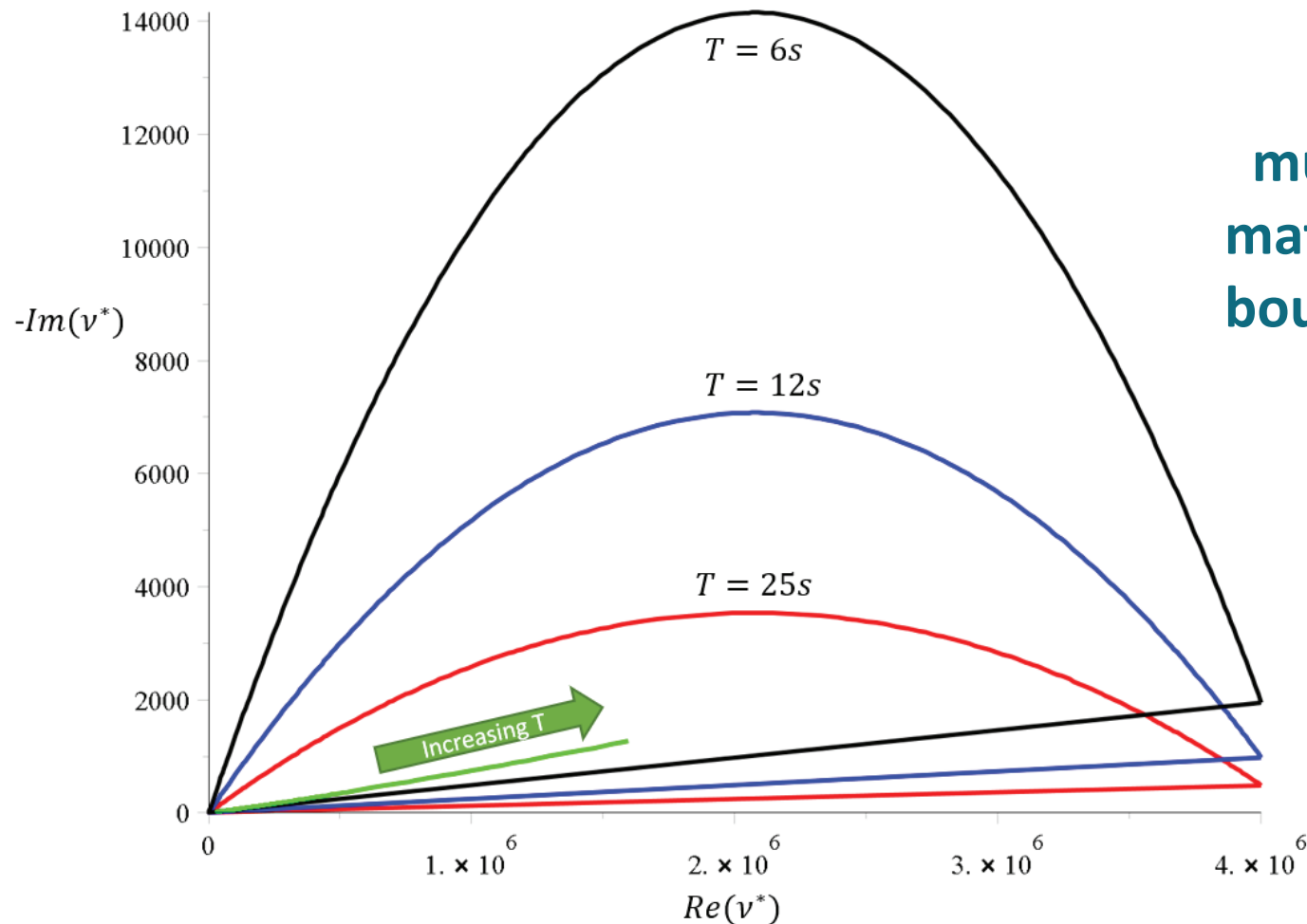
complex elementary bounds  
(fixed area fraction of floes)

$$V_1 = 10^7 + i 4875$$

pancake ice

$$V_2 = 5 + i 0.0975$$

slush / frazil



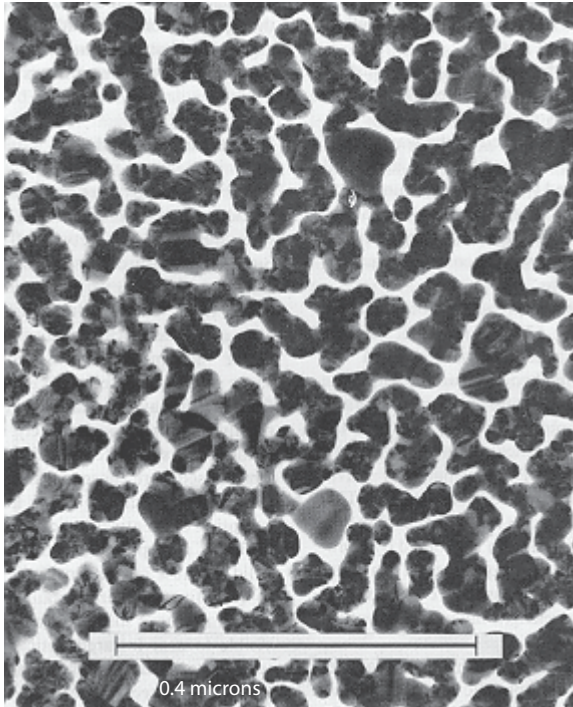
+  
much tighter  
matrix particle  
bounds + data

Sampson, Murphy, Cherkaev, Golden 2019

# Interaction of light with sea ice

## thin silver film

microns



(Davis, McKenzie, McPhedran, 1991)

## Arctic melt ponds

kilometers



(Perovich, 2005)



***optical properties***

***composite geometry -- area fraction of phases, connectedness, necks***

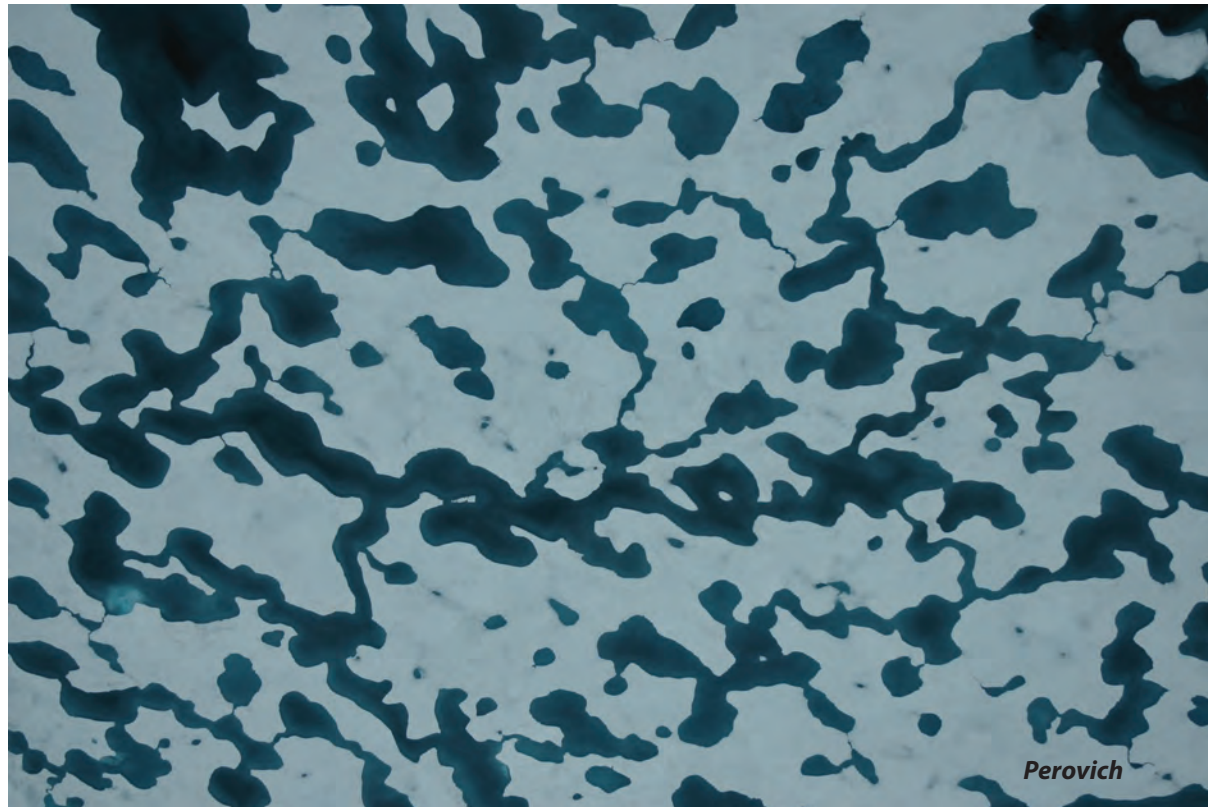
# *melt pond formation and albedo evolution:*

- *major drivers in polar climate*
- *key challenge for global climate models*

**numerical models of melt pond evolution, including topography, drainage (permeability), etc.**

Lüthje, Feltham,  
Taylor, Worster 2006  
Flocco, Feltham 2007

Skyllingstad, Paulson,  
Perovich 2009  
Flocco, Feltham,  
Hunke 2012



**Are there universal features of the evolution similar to phase transitions in statistical physics?**

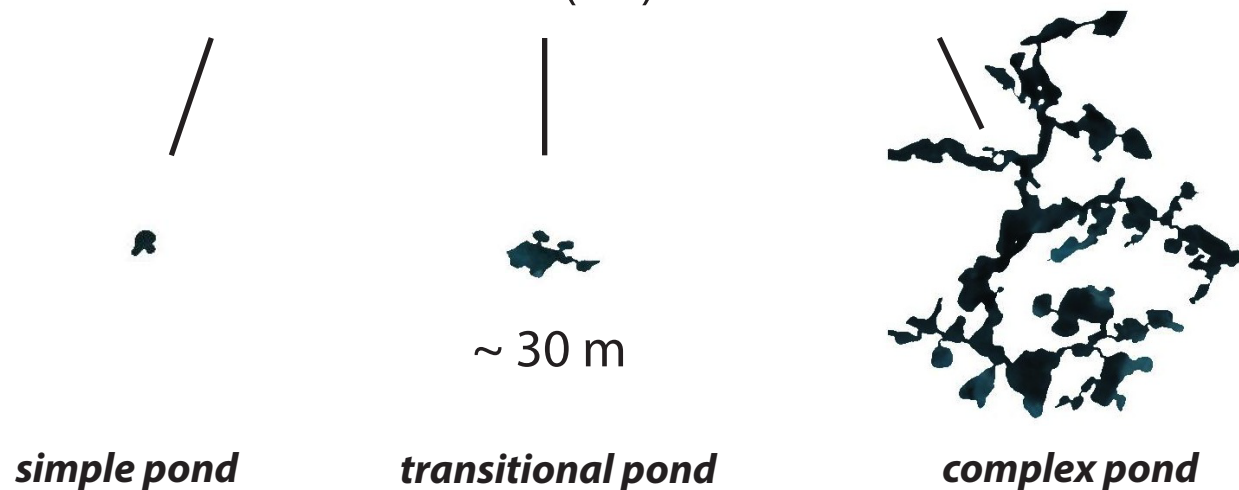
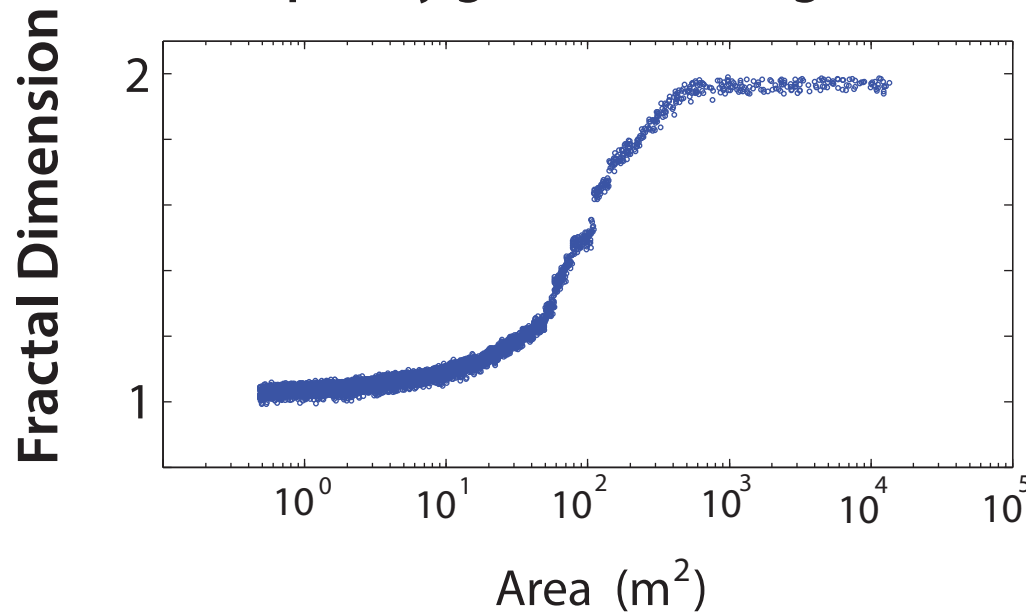


# *Transition in the fractal geometry of Arctic melt ponds*

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

*The Cryosphere, 2012*

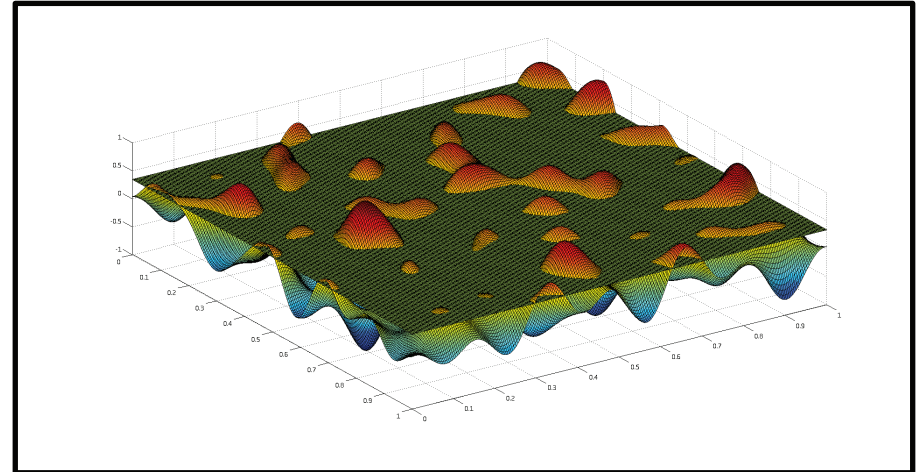
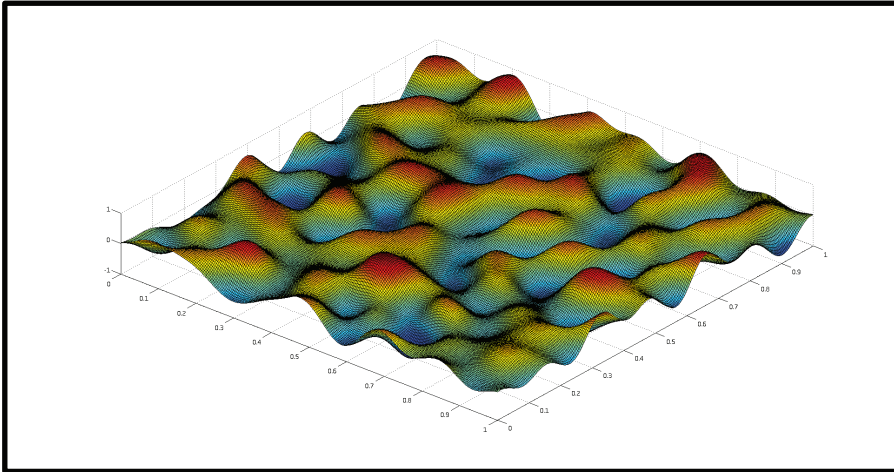
complexity grows with length scale



# Continuum percolation model for melt pond evolution

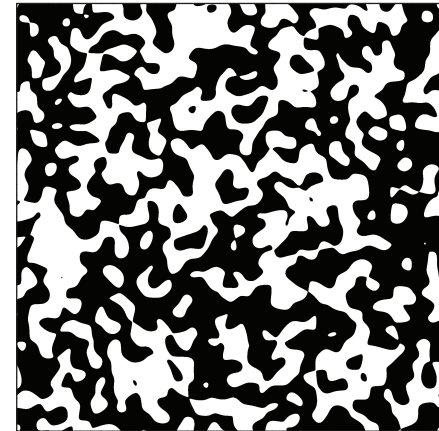
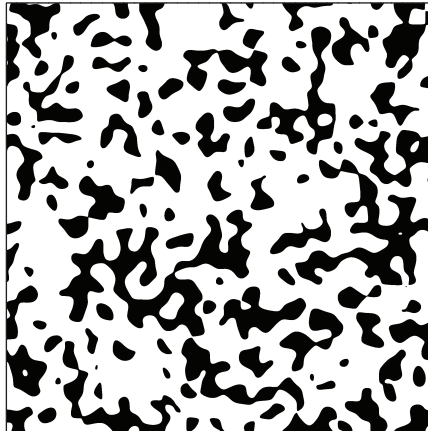
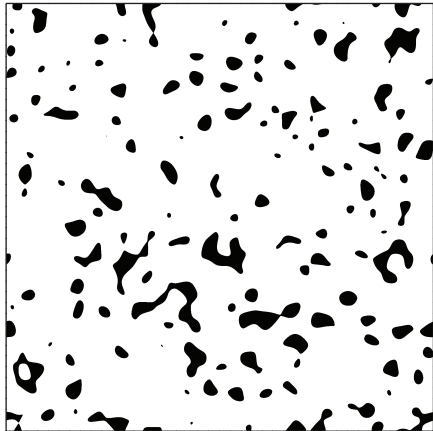
## *level sets of random surfaces*

*Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018*



random Fourier series representation of surface topography

intersections of a plane with the surface define melt ponds

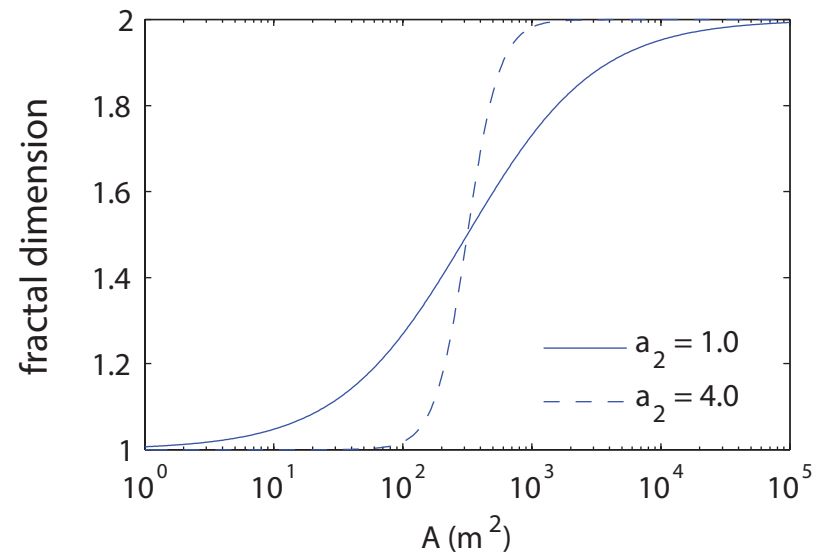
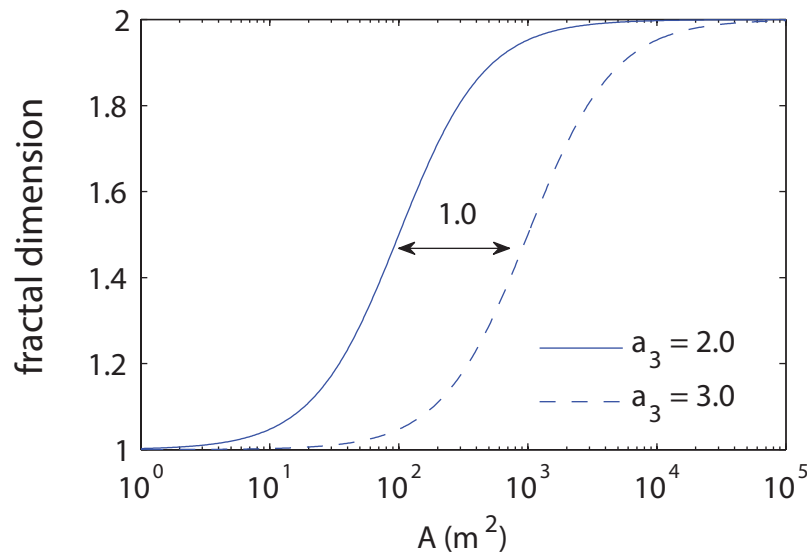


*electronic transport in disordered media*

*diffusion in turbulent plasmas*

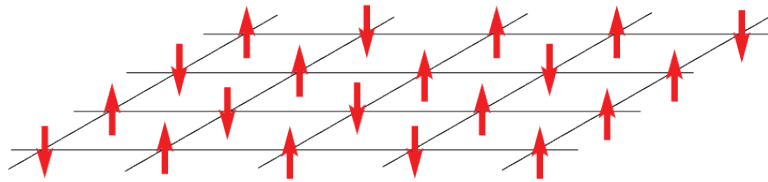
*Isichenko, Rev. Mod. Phys., 1992*

# fractal dimension curves depend on statistical parameters defining random surface





# Ising Model for a Ferromagnet



$$s_i = \begin{cases} +1 & \text{spin up} & \text{blue} \\ -1 & \text{spin down} & \text{white} \end{cases}$$

applied  
magnetic  
field



$H$

$$\mathcal{H} = -H \sum_i s_i - J \sum_{\langle i,j \rangle} s_i s_j$$

**nearest neighbor Ising Hamiltonian**

ferromagnetic interaction  $J \geq 0$

**magnetization**

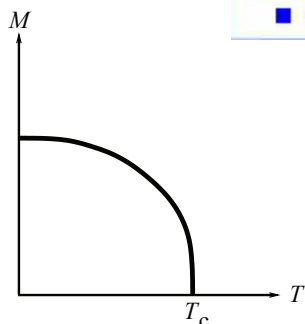
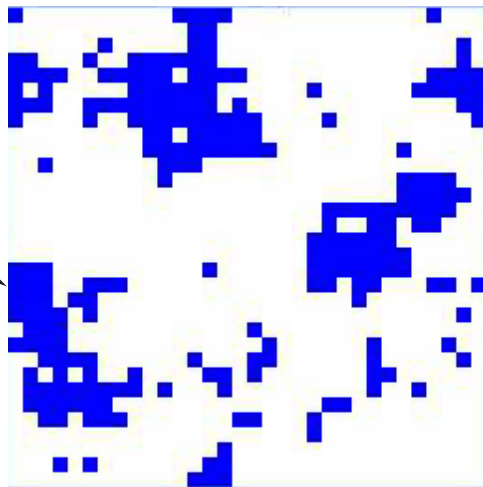
$$M(T, H) = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$

homogenized parameter  
like effective conductivity

**Stieltjes integral representation for  $M$**

**Baker, PRL 1968**

**islands or  
ponds of  
like spins**

**Curie point  
critical temperature**

# Ising model for ferromagnets $\longrightarrow$ Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

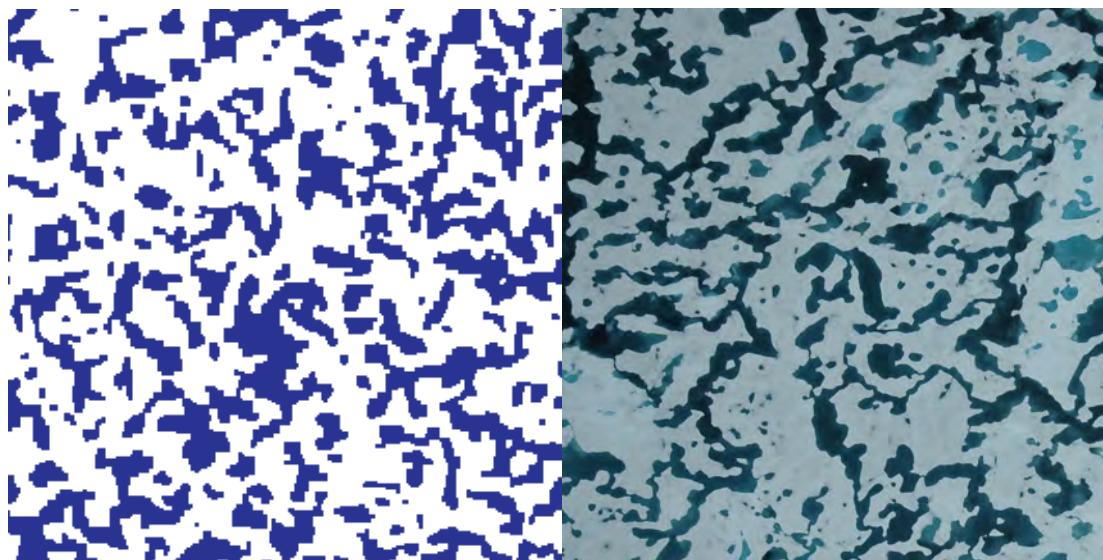
$$\mathcal{H} = - \sum_i^N H_i s_i - J \sum_{\langle i,j \rangle}^N s_i s_j \quad s_i = \begin{cases} \uparrow & +1 \text{ water (spin up)} \\ \downarrow & -1 \text{ ice (spin down)} \end{cases}$$

random magnetic field  
represents snow topography

magnetization  $M$       pond coverage  $\frac{(M+1)}{2}$   
 $\sim$  *albedo*      only nearest neighbor patches interact

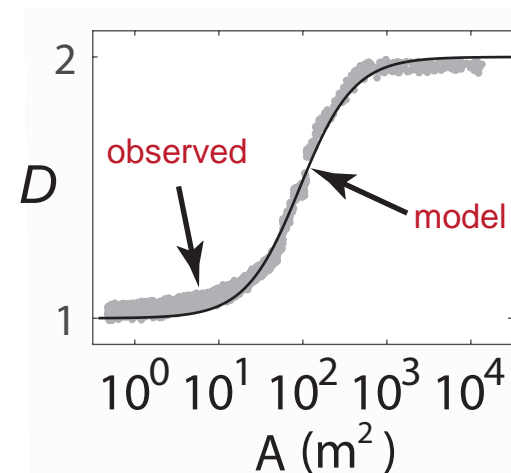
Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system “flows” toward metastable equilibria.

## *Order from Disorder*



Ising  
model

melt pond  
photo (Perovich)



pond size  
distribution exponent

observed -1.5

(Perovich, et al. 2002)

model -1.58

**ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data**

# Stieltjes integral representation for magnetization (~ albedo)

and scaling relations for critical exponents

Baker, *Phys. Rev. Lett.* 1968

$$M(\tau) = \tau + \tau(1 - \tau^2)G(\tau^2) \quad \tau = \tanh(\beta H)$$

$$G(\tau^2) = \int_0^\infty \frac{d\psi(y)}{1 + \tau^2 y} \quad \text{Herglotz} \quad (\text{Lee-Yang 1952})$$

---

parallel Herglotz structure for transport in composites  
analogous critical behavior and scaling relations hold near  $p_c$

Golden, *J. Math. Phys.* 1995 (C. Newman)  
*Phys Rev. Lett.* 1997

$$F(s) = 1 - m(h) = \int_0^1 \frac{d\mu(w)}{s - w}$$

$$m(h) = \frac{\sigma^*}{\sigma_2} \quad h = \frac{\sigma_1}{\sigma_2} \rightarrow 0 \quad \sigma^*(p, h) \quad \begin{array}{l} \text{effective conductivity} \\ \text{of two phase composite} \\ \text{*lattice or continuum*} \end{array}$$

$$m(h) = 1 + (h - 1)g(h) \quad g(h) = \int_0^\infty \frac{d\phi(y)}{1 + hy} \quad \text{Herglotz} \quad w = \frac{y}{y + 1}$$



# Ising model

## partition function

$$Z_N(z) = a_N \prod_{n=1}^N (z - z_n), \quad |z_n| = 1$$

## free energy

$$f(T, H) = \frac{-1}{\beta} \int_{|t|=1} \log(z - t) d\nu(t)$$

## order parameter

$$M(T) = -\frac{\partial f}{\partial H}$$

$$\frac{\partial^2 M}{\partial H^2} \leq 0$$

## *G.H.S. inequality*

Griffiths, Hurst, Sherman *JMP* 1970

# transport in composites

$$\mathcal{Z}_N(s) = \prod_{n=1}^N (s - s_n), \quad s_n \in [0, 1]$$

$$\Phi(p, s) = \int_0^1 \log(s - t) d\mu(t)$$

$$F(p, s) = \frac{\partial \Phi}{\partial s}$$

$$\frac{\partial^2 m}{\partial h^2} \leq 0$$

Golden, *JMP* 1995; *PRL* 1997

# Conclusions

1. Homogenization problems arise naturally in the multiscale sea ice system.
2. Theories of **statistical physics and composite materials** help us **link scales** and provide rigorous methods for finding effective behavior, advancing how sea ice is represented in climate models.
3. **Herglotz functions and Stieltjes integrals** provide powerful methods of homogenization for quasistatic EM waves, advection diffusion, polycrystalline media, and quasistatic surface waves in the ice pack.
4. Our research will help to **improve projections of climate change** and the fate of the Earth sea ice packs.

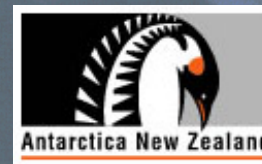
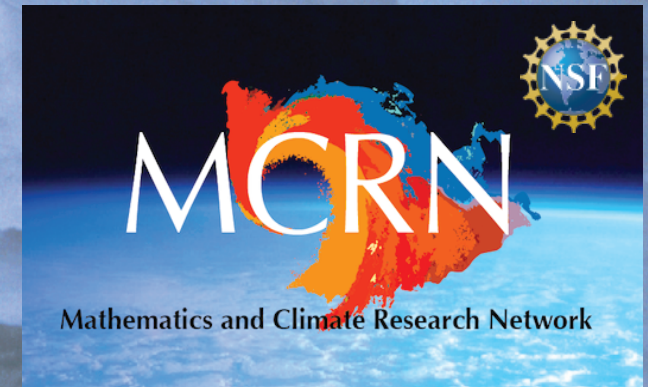
# THANK YOU

## Office of Naval Research

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## National Science Foundation

Division of Mathematical Sciences  
Division of Polar Programs



***Buchanan Bay, Antarctica    Mertz Glacier Polynya Experiment    July 1999***