

Mathematics and the

melting polar ice caps



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*Alison Kohout
September 2012*

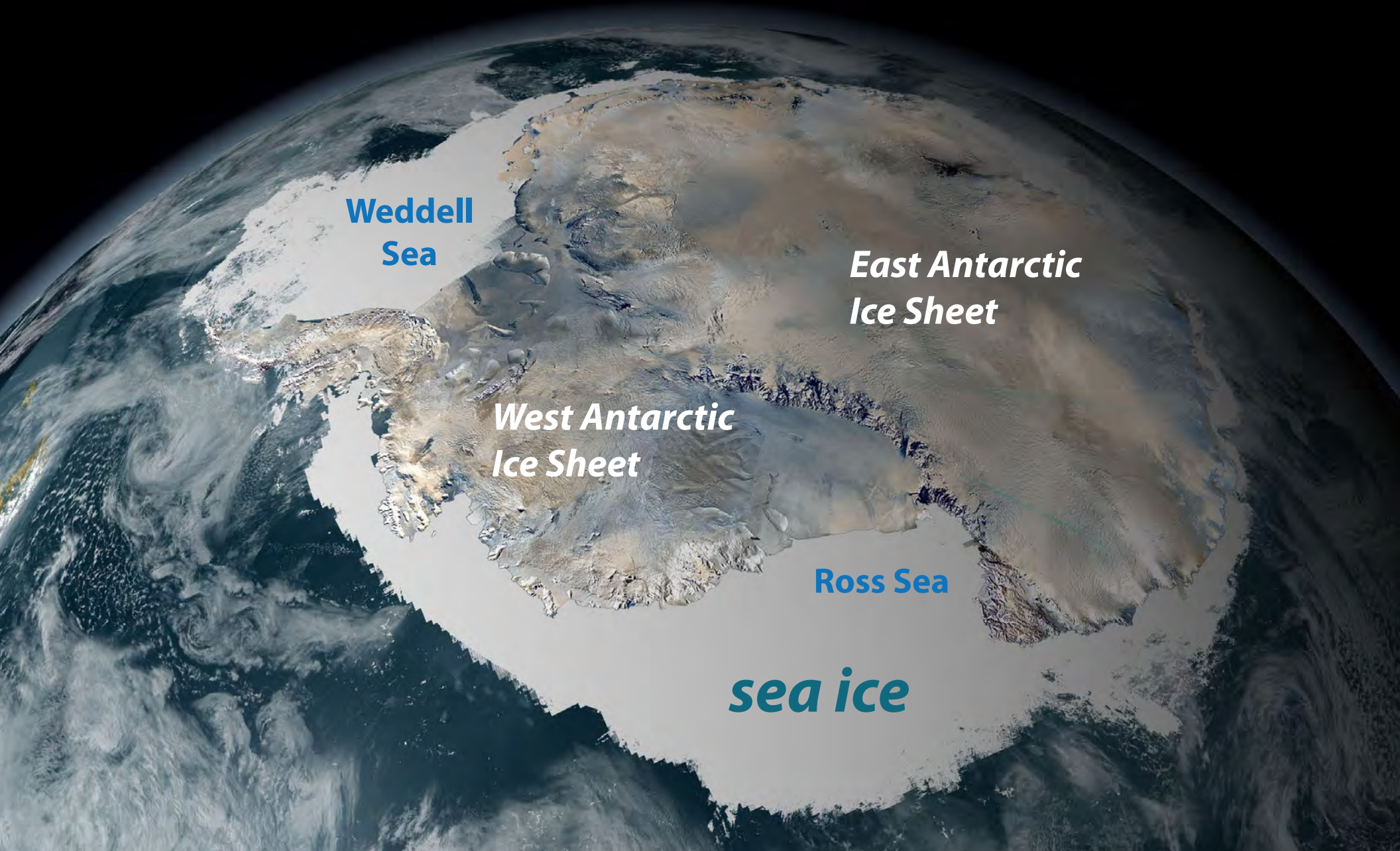
Mathematics of Planet Earth

***Platform for Mathematics
Utrecht, Netherlands***

21 December 2013

ANTARCTICA

southern cryosphere



**Weddell
Sea**

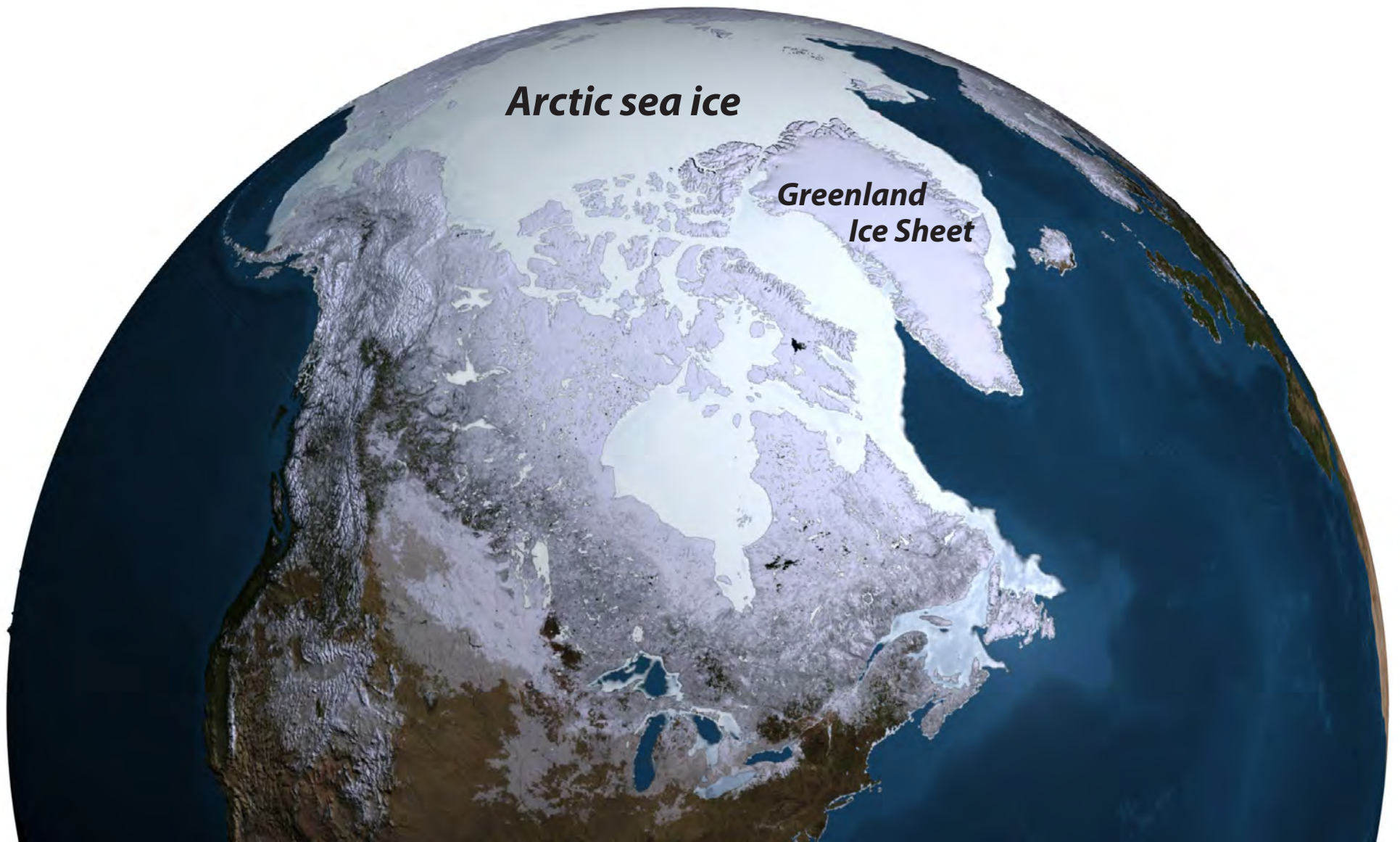
***East Antarctic
Ice Sheet***

***West Antarctic
Ice Sheet***

Ross Sea

sea ice

northern cryosphere

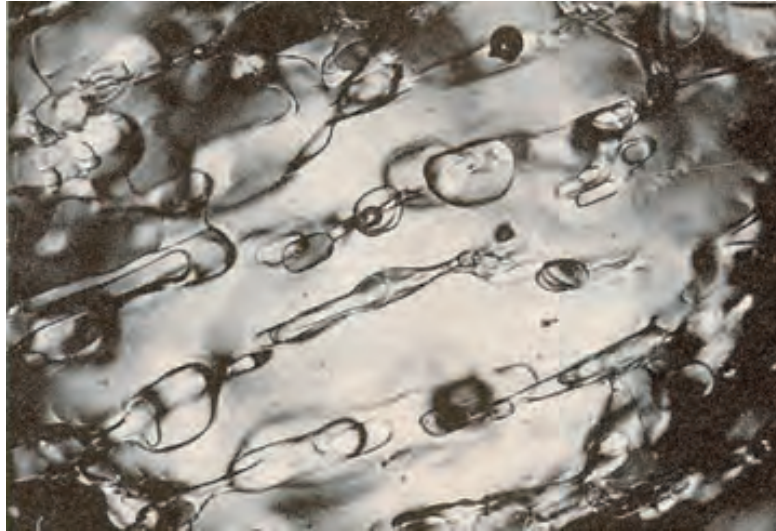


SEA ICE covers 7 - 10% of earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- indicator and agent of **climate change**



sea ice is a multiscale composite



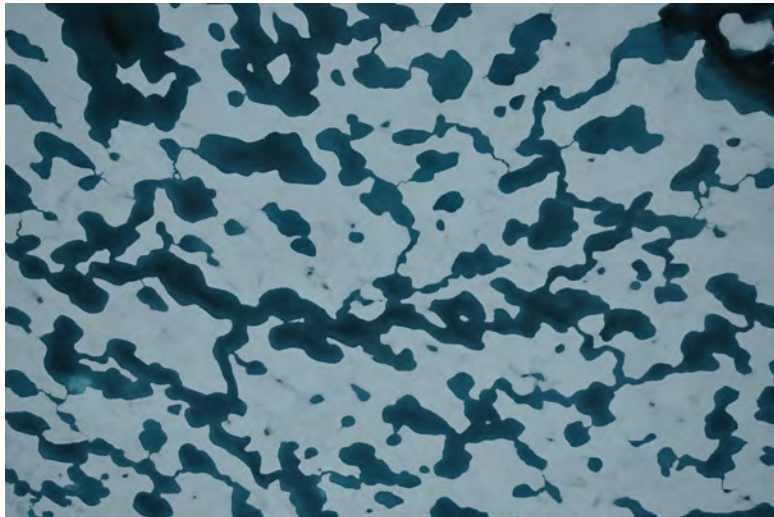
*brine
inclusions*

sub-millimeter



pancakes

centimeters



*melt
ponds*

meters



*ice
floes*

kilometers

polar ice caps critical to global climate in reflecting incoming solar radiation



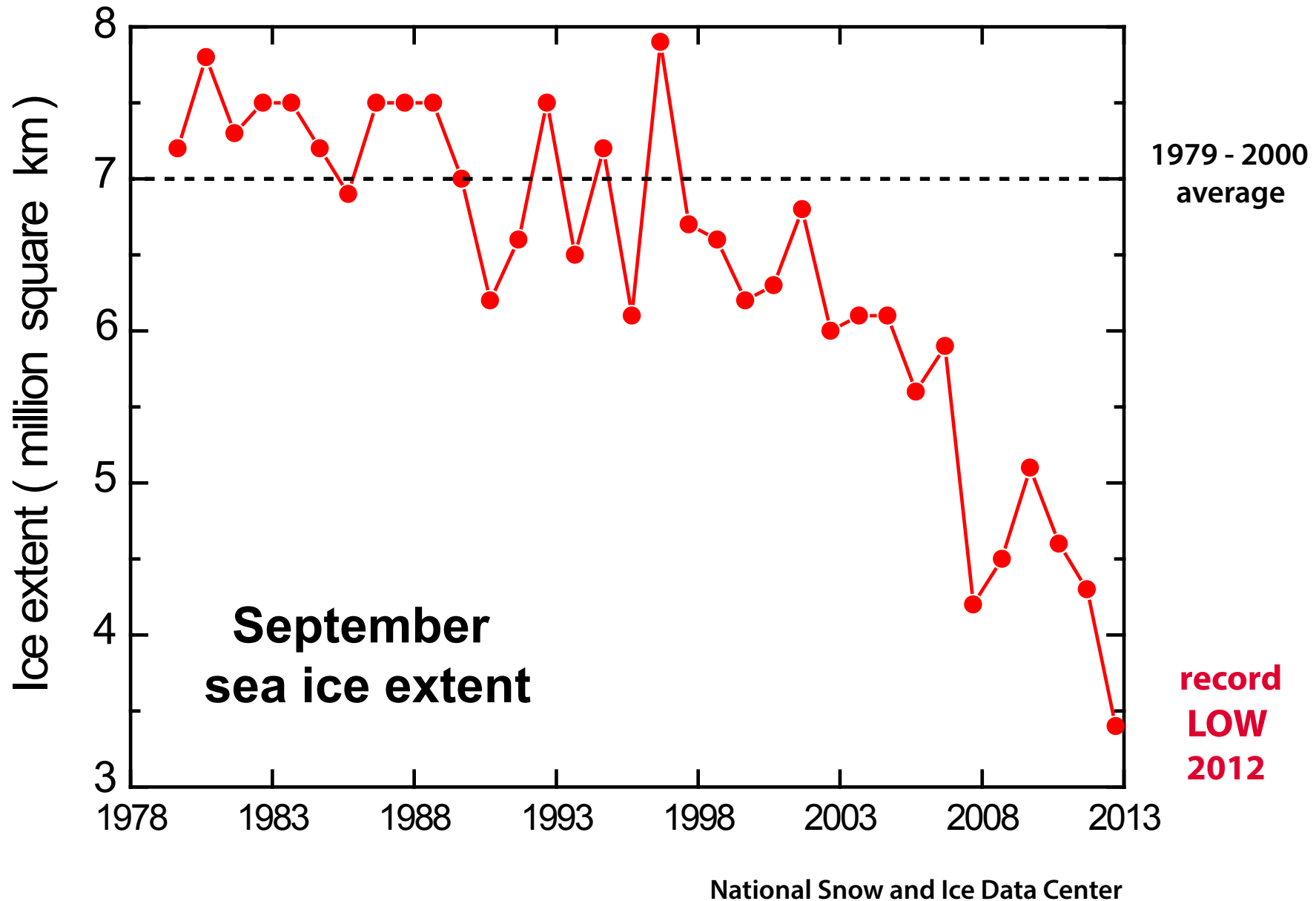
white snow and ice
reflect



dark water and land
absorb

$$\text{albedo } \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

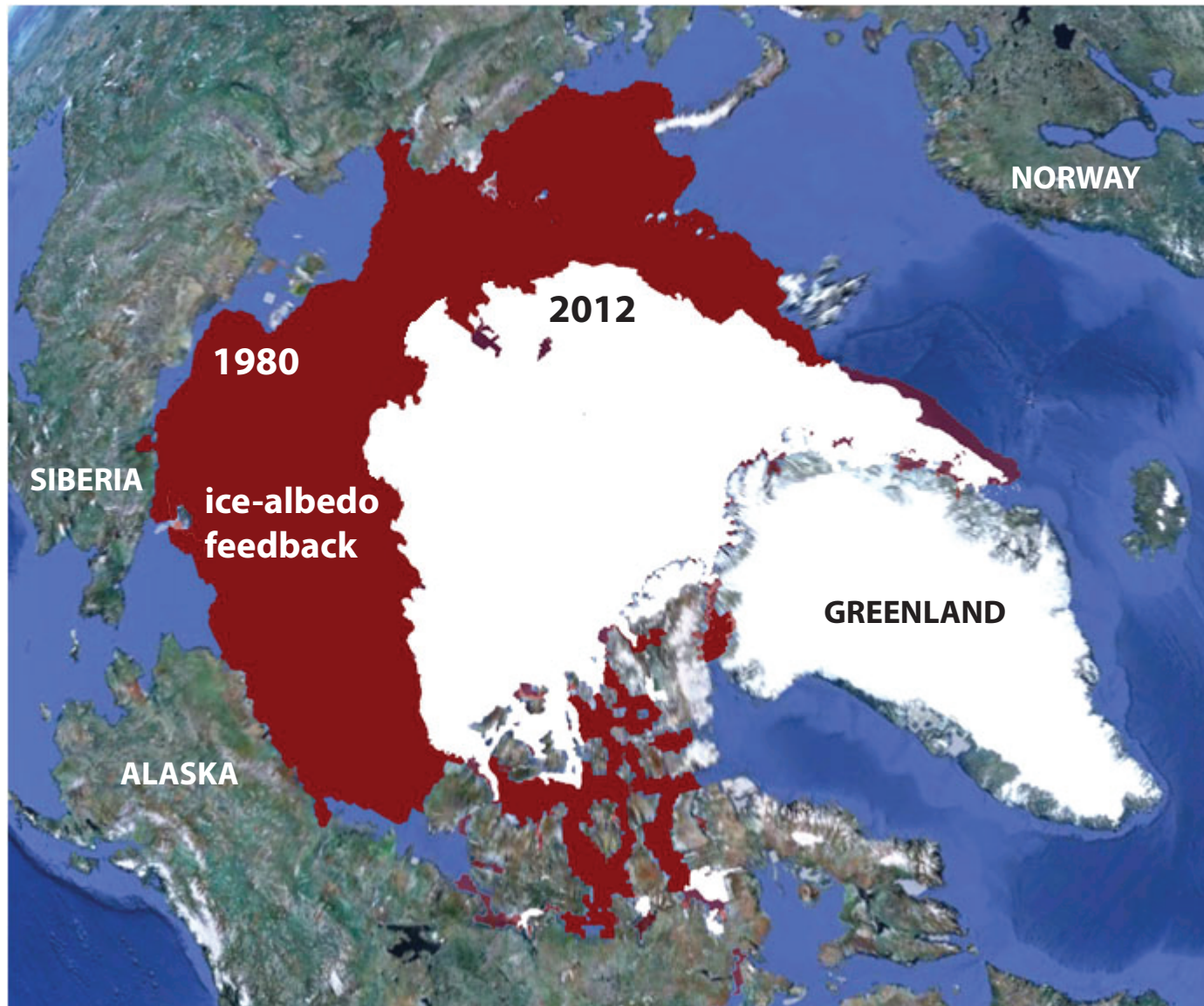
the summer Arctic sea ice pack is melting



Change in Arctic Sea Ice Extent

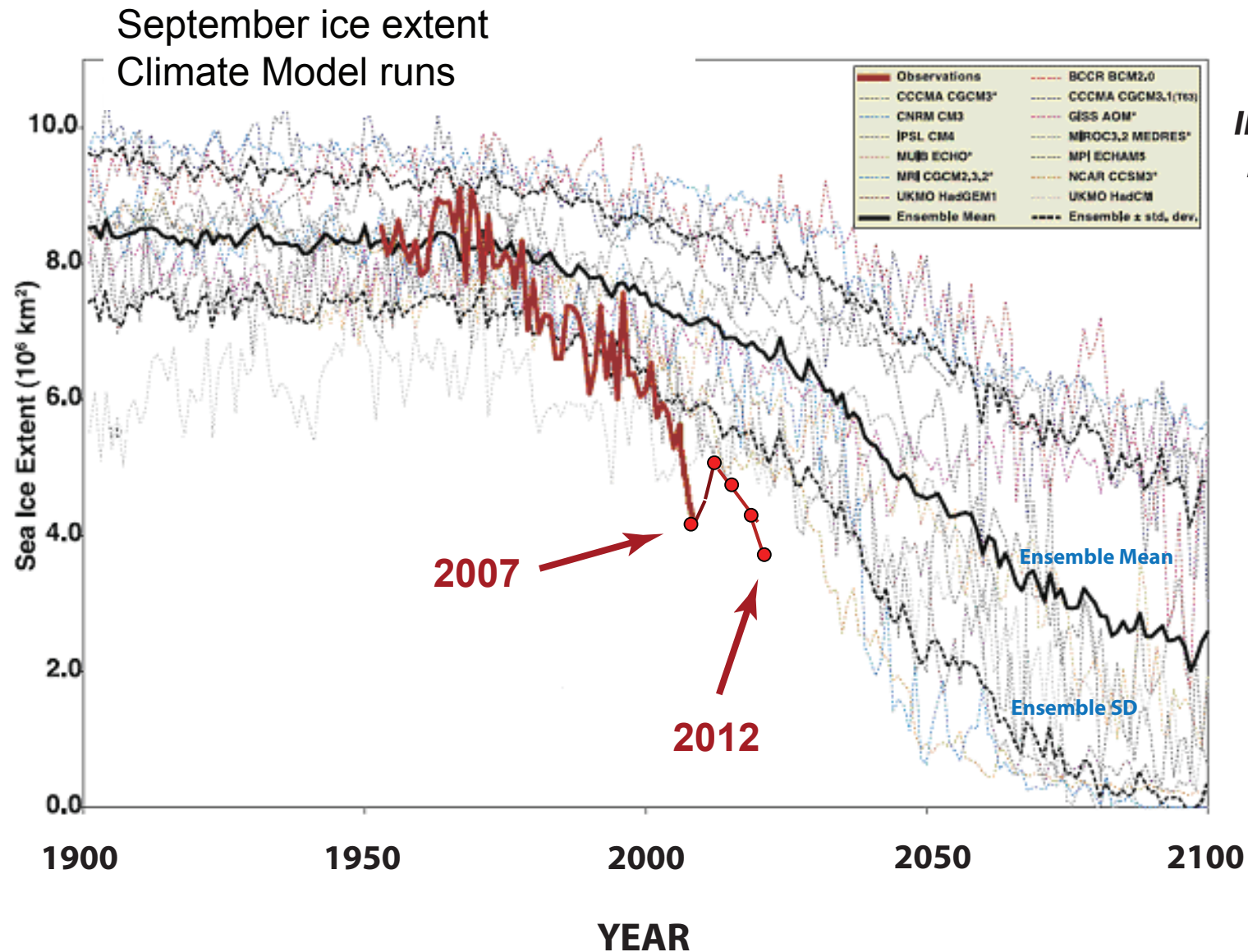
September 1980 -- **7.8** million square kilometers

September 2012 -- **3.4** million square kilometers



Arctic sea ice decline - faster than predicted by climate models

Stroeve et al., GRL, 2007



IPCC AR4
Models

What is this talk about?

Using the mathematics of composite materials and statistical physics to study sea ice structures and processes ... to improve projections of climate change.

1. Fluid flow through sea ice - percolation

homogenization for composite materials

2. Electromagnetic monitoring of sea ice

homogenization for larger scale structures

3. Arctic and Antarctic experiments

4. Fractal geometry of Arctic melt ponds

critical behavior

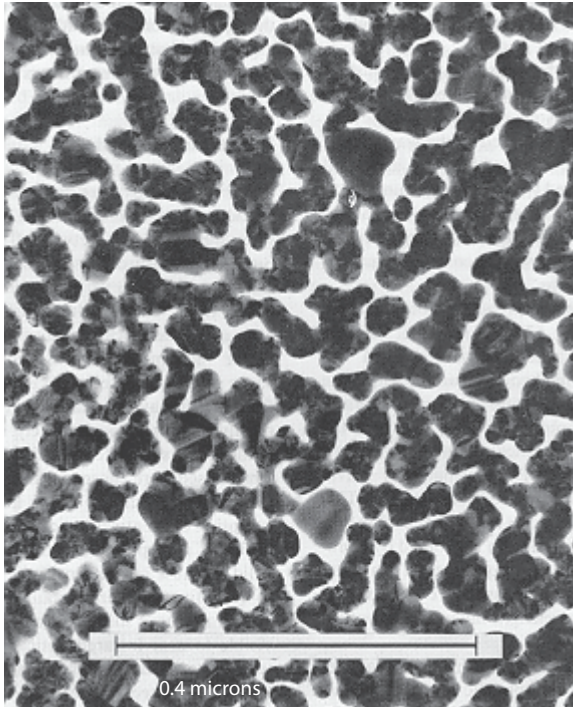
linkage of scales

cross-pollination

Develop rigorous representations of sea ice in climate models.

thin silver film

microns



(Davis, McKenzie, McPhedran, 1991)

Arctic melt ponds

kilometers



(Perovich, 2005)



optical properties

composite geometry -- area fraction of phases, connectedness, necks

Global Climate Models

Climate models are systems of partial differential equations (PDE) derived from the basic laws of physics, chemistry, and fluid motion.

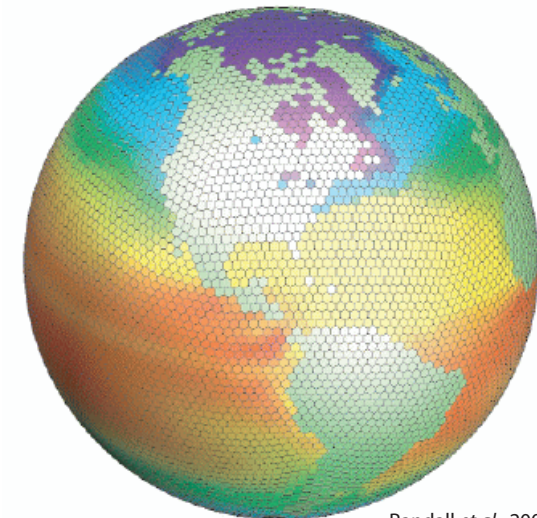
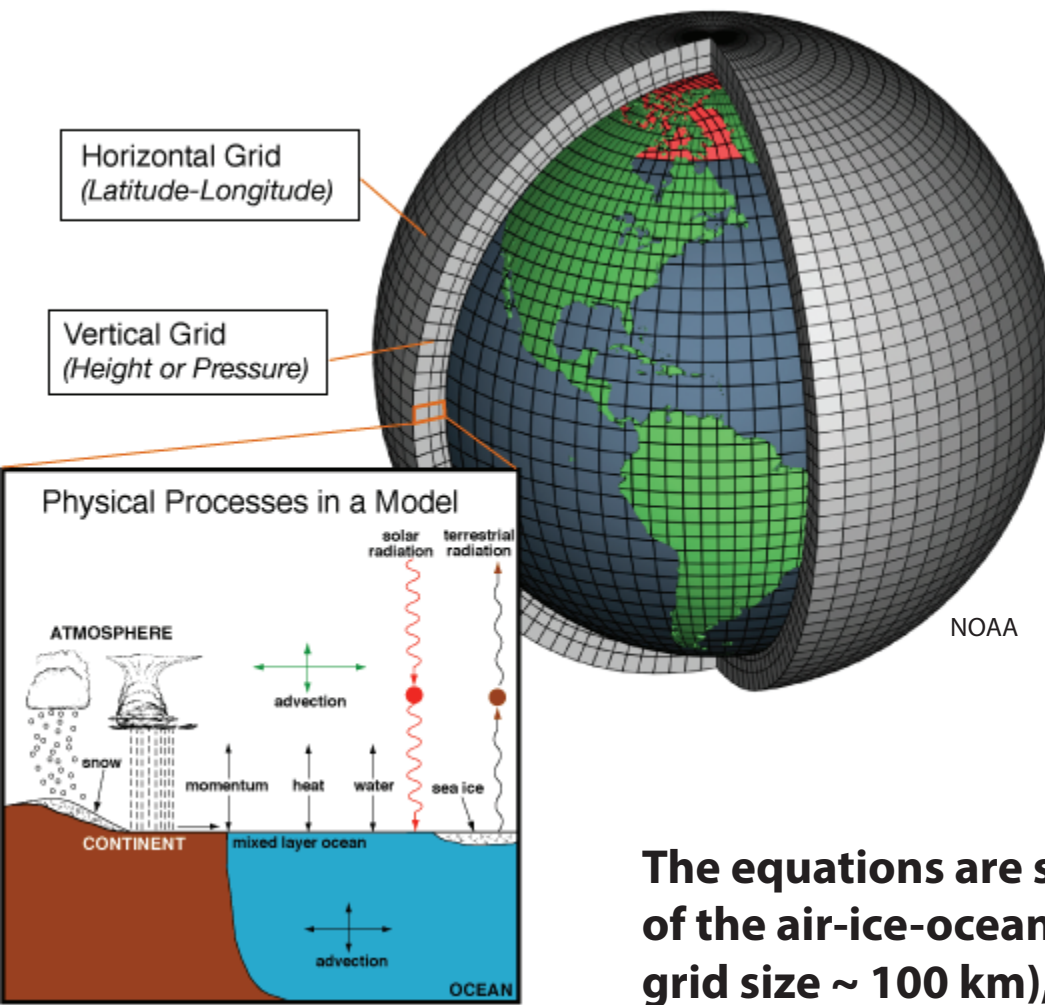
They describe the state of the ocean, ice, atmosphere, land, and their interactions.

The equations are solved on 3-dimensional grids of the air-ice-ocean-land system (with horizontal grid size ~ 100 km), using very powerful computers.

key challenge :

incorporating sub - grid scale processes

linkage of scales



sea ice component of a Global Climate Model

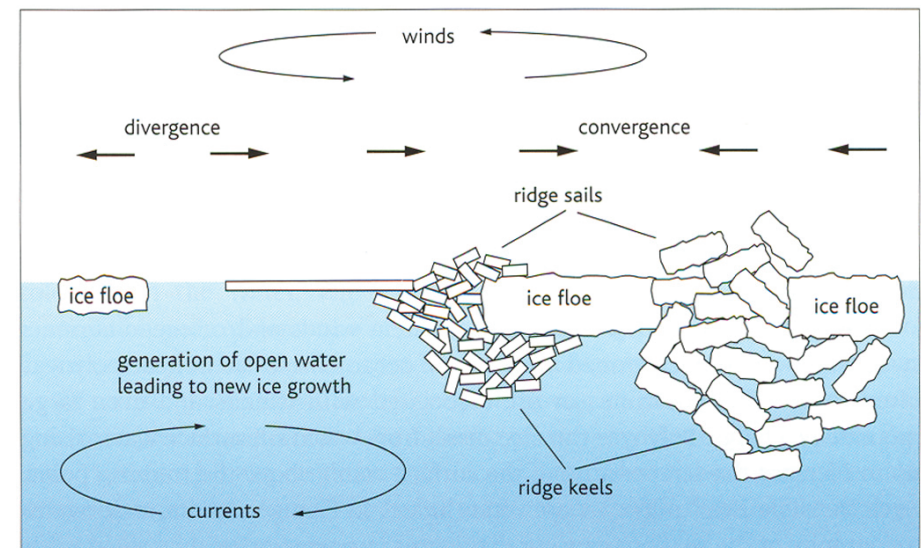
What are the key ingredients -- or **governing equations** that need to be solved on grids using powerful computers?

1. Ice thickness distribution evolution equation

(Thorndike et al. 1975)

dynamics
+
thermodynamics

*PDE incorporating ice velocity field
ice growth and melting
mechanical redistribution
ridging and opening*



2. Conservation of momentum, stress vs. strain relation (Hibler 1979)

$F = ma$ for sea ice

dynamics

3. Heat equation of sea ice and snow

coupling ocean and atmosphere

thermodynamics

(Maykut and Untersteiner 1971)



**Who cares if
Arctic sea ice
disappears?**



Ralph (Malik) Ahkivgak, c. 20 Oct 1988

© Bill Hess – Running Dog Publications; <http://wasillaalaskaby300.squarespace.com/>

Drew Barrymore

John Krasinski



INSPIRED BY THE INCREDIBLE TRUE STORY
that united the world

BIG MIRACLE

seaice.alaska.edu/gi

- The Arctic holds 25% of the world's undiscovered oil & gas reserves
- Sea ice is both a hazard and a supporting feature for hydrocarbon exploration & production

oil companies care about Arctic sea ice loss



Source: UNEP/GRID-Arendal

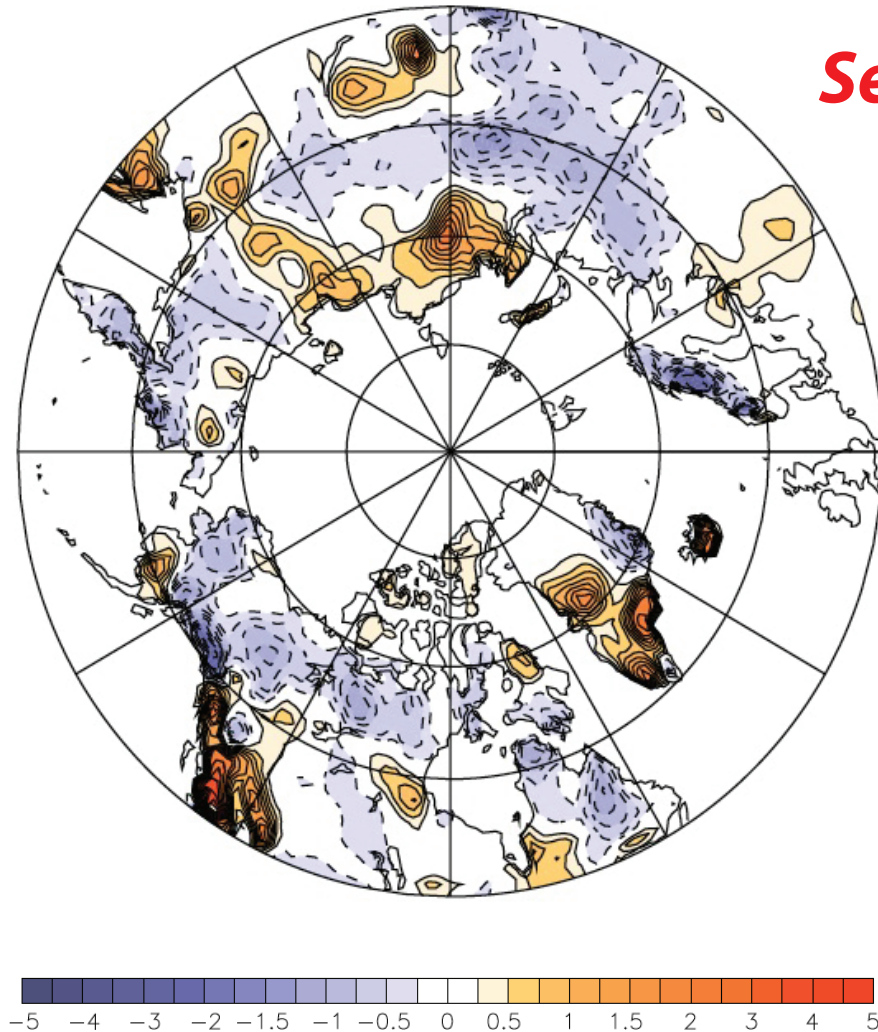
Sea-ice loss: impacts beyond the Arctic

changes in precipitation and temperature patterns, storm tracks, ...

- One climate model projects reduced precipitation in American West (Sewall & Sloan, 2005)

Utah - greatest snow on Earth?

- Analysis of 2007 ice minimum suggests above normal snow deposition in NW North America (Orsolini et al., 2011)
- Colder weather in SE Asia, possibly in Eastern US (Hondo et al., 2009)



Orsolini et al., 2011

sea ice microphysics

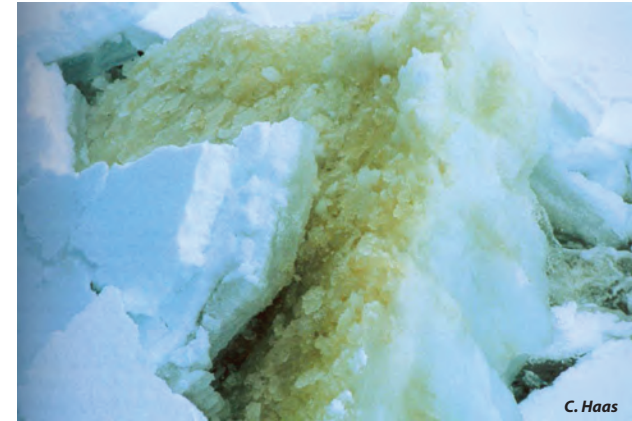
fluid transport

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo



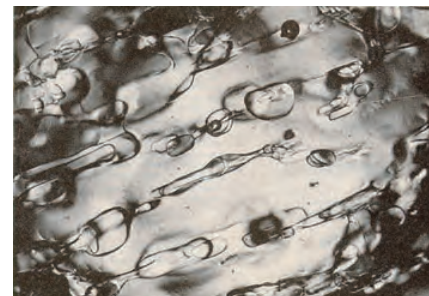
nutrient flux for algal communities



- *drainage of brine and melt water*
- *ocean-ice-air exchanges of heat, CO₂*
- *Antarctic surface flooding and snow-ice formation*
- *evolution of salinity profiles*



linkage of scales



Darcy's Law for slow viscous flow in a porous medium

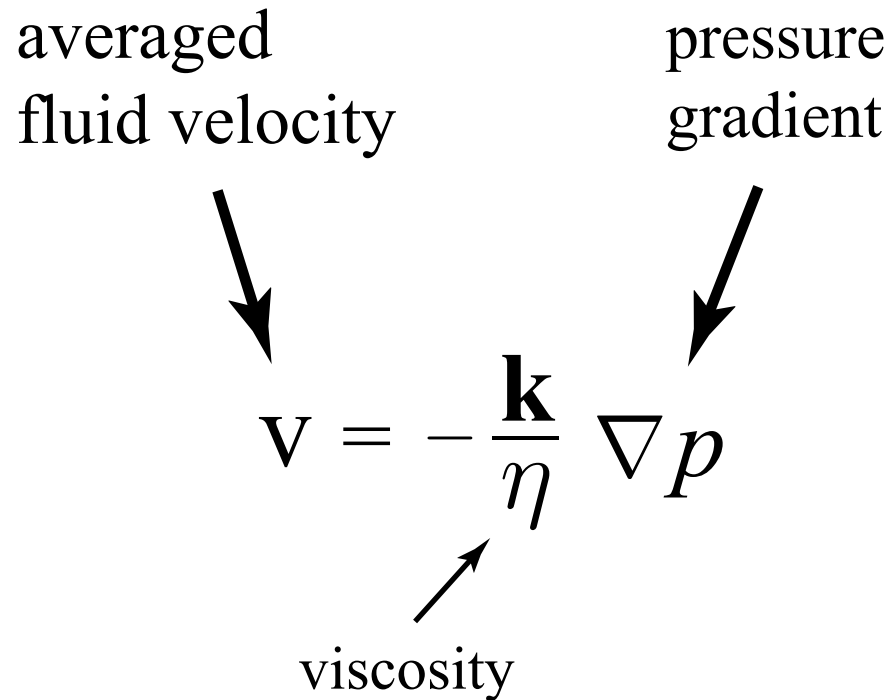


Diagram illustrating Darcy's Law for slow viscous flow in a porous medium. The equation is shown as $\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$. Arrows point from the following labels to the corresponding terms in the equation:

- averaged fluid velocity points to \mathbf{v}
- pressure gradient points to ∇p
- viscosity points to η

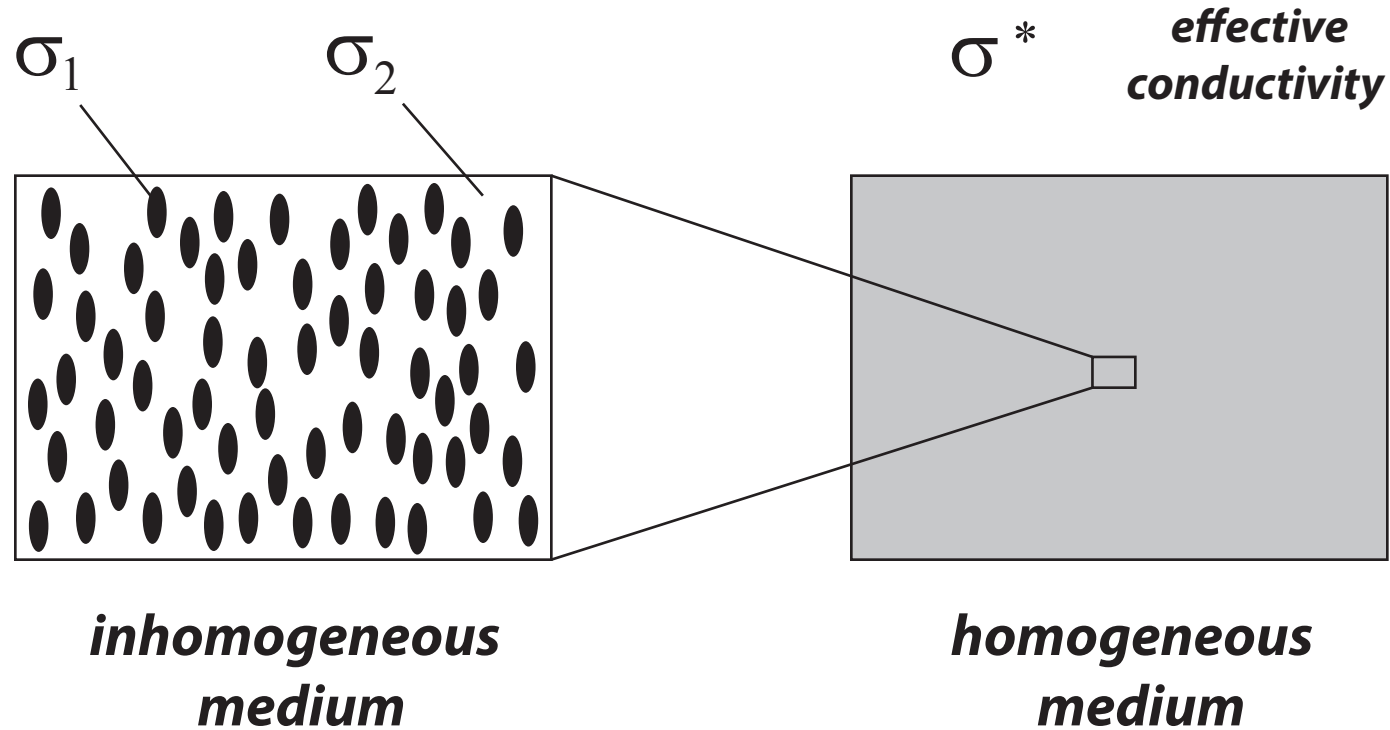
\mathbf{k} = fluid permeability tensor

example of *homogenization*

mathematics for analyzing effective behavior of heterogeneous systems

e.g. transport properties of composites - electrical conductivity, thermal conductivity, etc.

HOMOGENIZATION



**find the homogeneous medium which
behaves macroscopically the same as
the inhomogeneous medium**

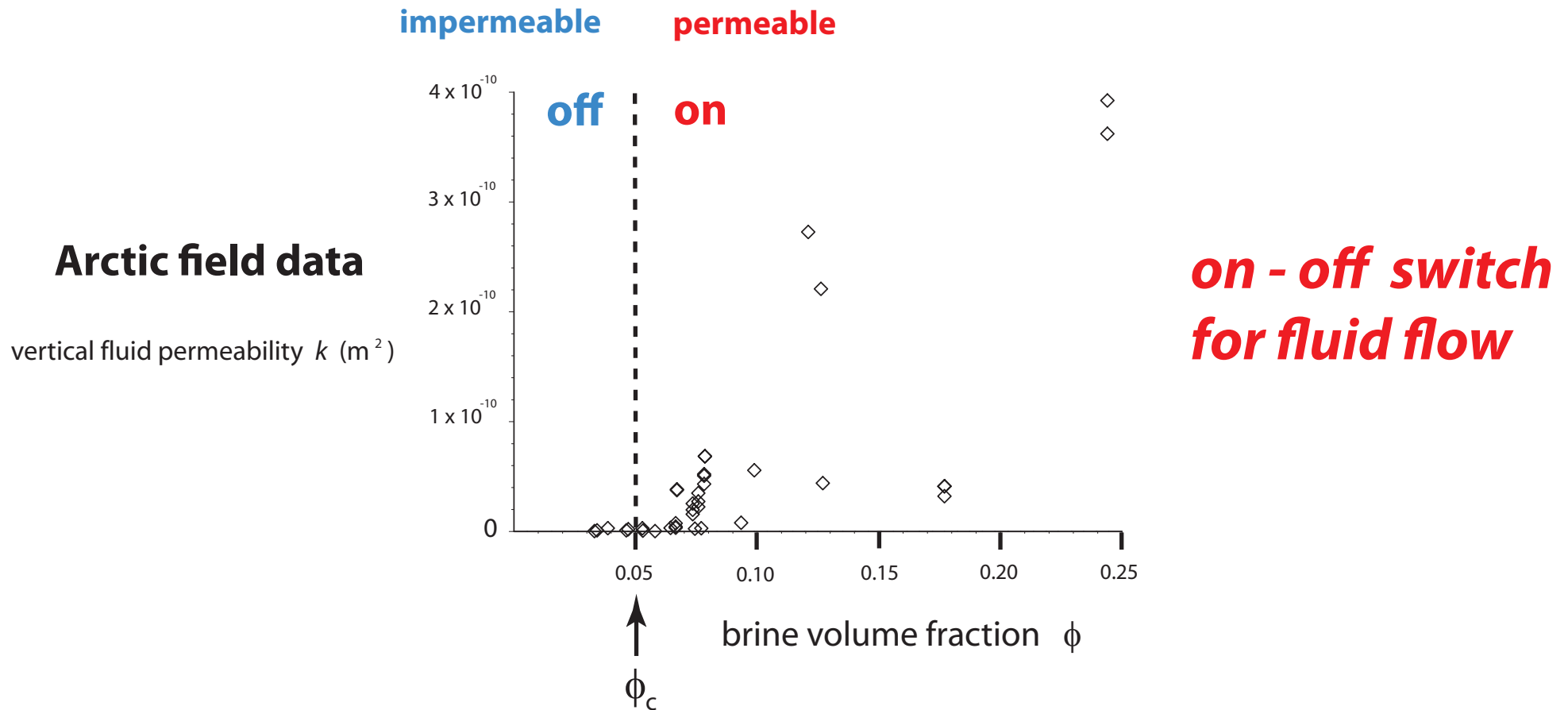
Maxwell 1873 : effective conductivity of a dilute suspension of spheres

Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*

*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

Critical behavior of fluid transport in sea ice



critical brine volume fraction $\phi_c \approx 5\%$ \longleftrightarrow $T_c \approx -5^\circ \text{C}$, $S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009



sea ice algal communities

D. Thomas 2004

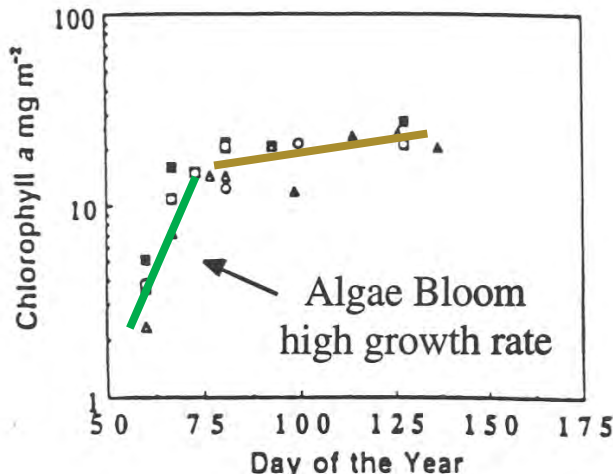
nutrient replenishment
controlled by ice permeability

biological activity turns on
or off according to
rule of fives

Golden, Ackley, Lytle *Science* 1998

Fritsen, Lytle, Ackley, Sullivan *Science* 1994

critical behavior of microbial activity

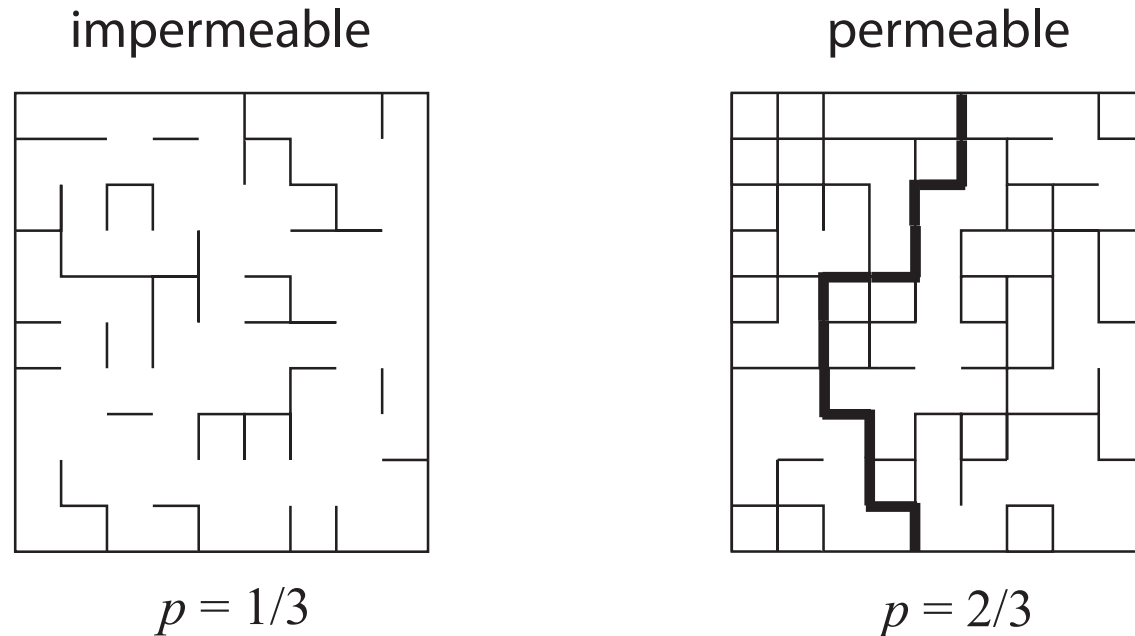


Convection-fueled algae bloom
Ice Station Weddell

Why is the rule of fives true?

percolation theory

mathematical theory of connectedness



bond \longrightarrow *open* with probability p
closed with probability $1-p$

percolation threshold

$$p_c = 1/2 \quad \text{for } d = 2$$

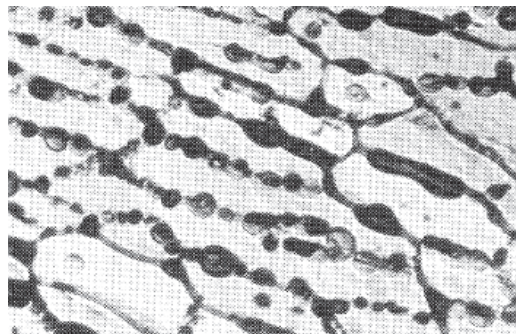
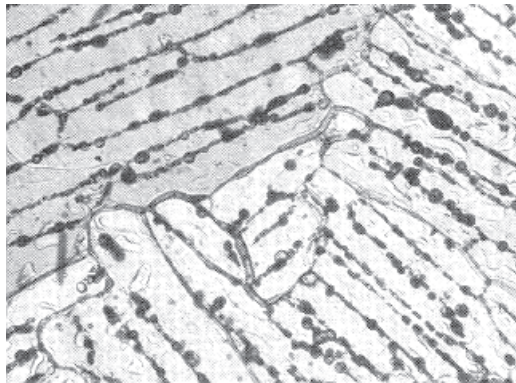
first appearance of infinite cluster

“tipping point” for connectivity

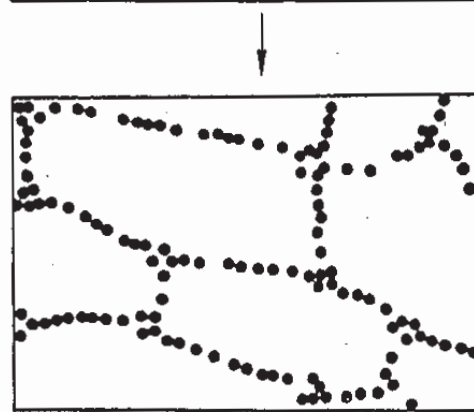
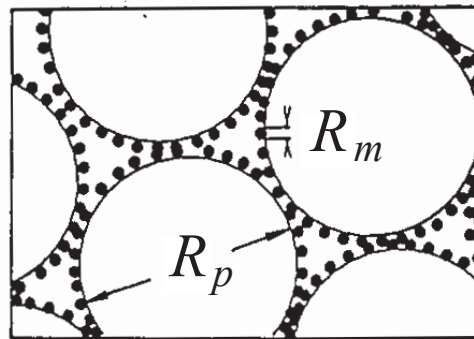
Continuum percolation model for **stealthy** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on **ice production** and **algal growth**

$$\phi_c \approx 5 \%$$

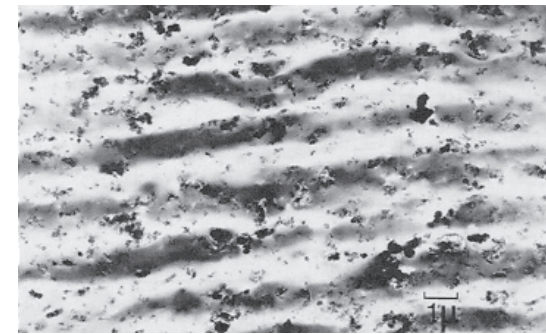
Golden, Ackley, Lytle, *Science*, 1998



sea ice



compressed
powder



radar absorbing
composite

sea ice is radar absorbing



***rigorous bounds
percolation theory
hierarchical model
network model***

field data

X-ray tomography for
brine inclusions

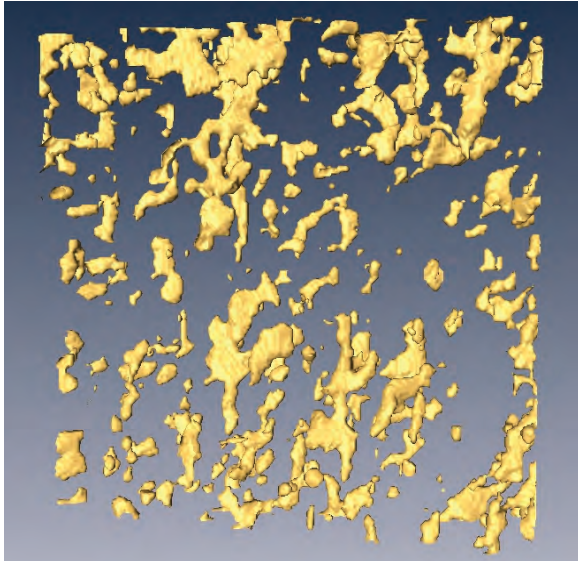
***unprecedented look
at thermal evolution
of brine phase and
its connectivity***

micro-scale
controls
macro-scale
processes

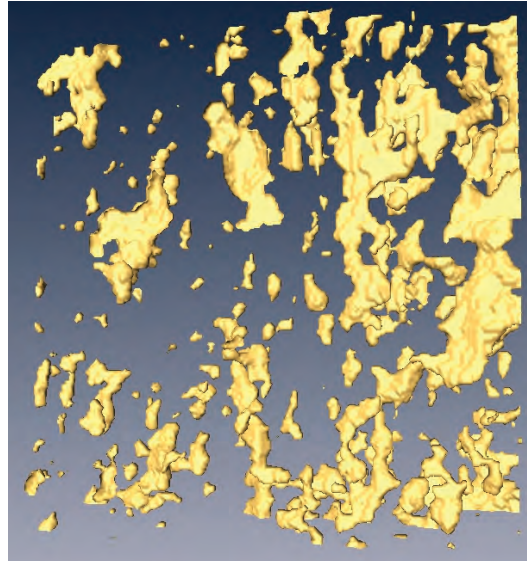
A unified approach to understanding permeability in sea ice • Solving the mystery of
booming sand dunes • Entering into the “greenhouse century”: A case study from Switzerland

brine connectivity (over cm scale)

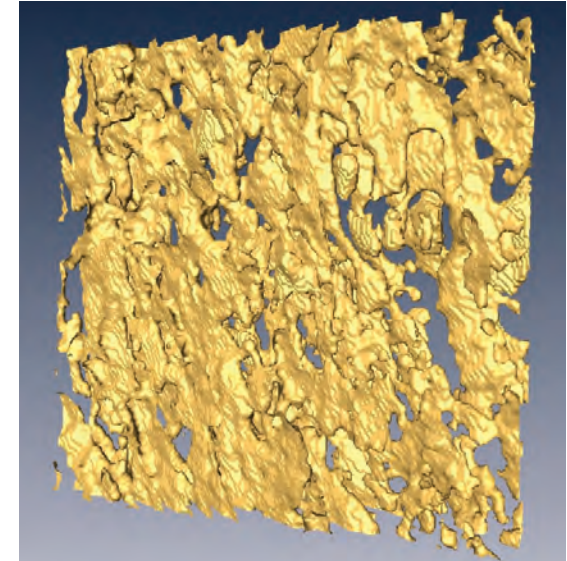
8 x 8 x 2 mm



-15 °C, $\phi = 0.033$



-6 °C, $\phi = 0.075$



-3 °C, $\phi = 0.143$

X-ray tomography confirms percolation threshold

3-D images
pores and throats



3-D graph
nodes and edges

analyze graph connectivity as function of temperature and sample size

- ***use finite size scaling techniques to confirm rule of fives***
- ***order parameter data from a natural material***

lattice and continuum percolation theories yield:

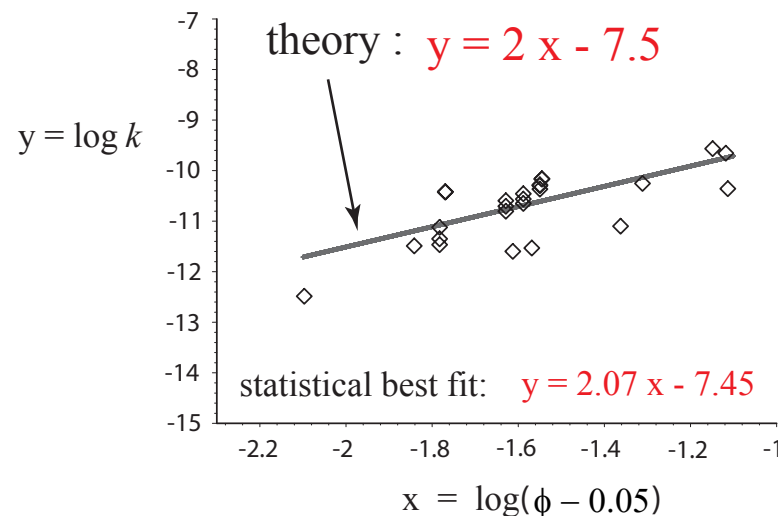
$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical
exponent

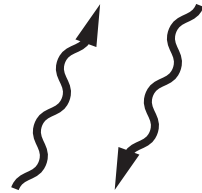
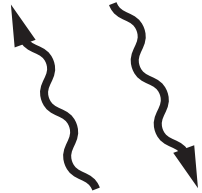
$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

t

- exponent is **UNIVERSAL** lattice value $t \approx 2.0$
- **sedimentary rocks** like sandstones also exhibit universality
- **critical path analysis** -- developed for electronic hopping conduction -- yields scaling factor k_0



Remote sensing of sea ice



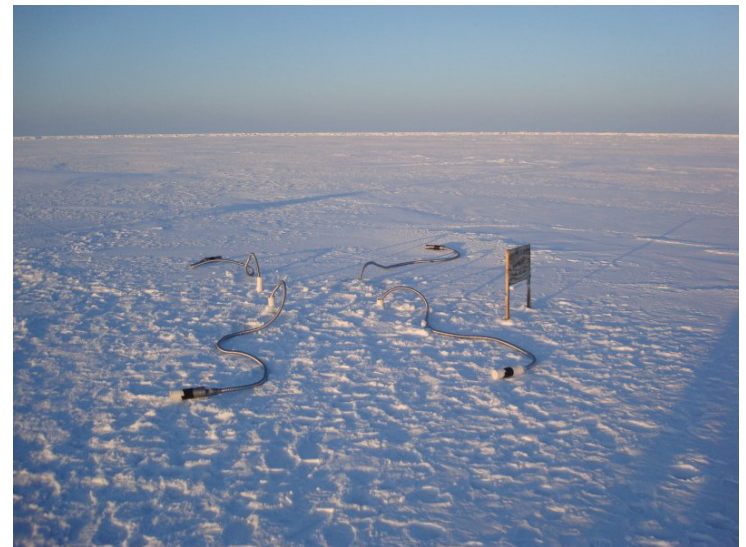
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice
properties from
electromagnetic
(EM) data

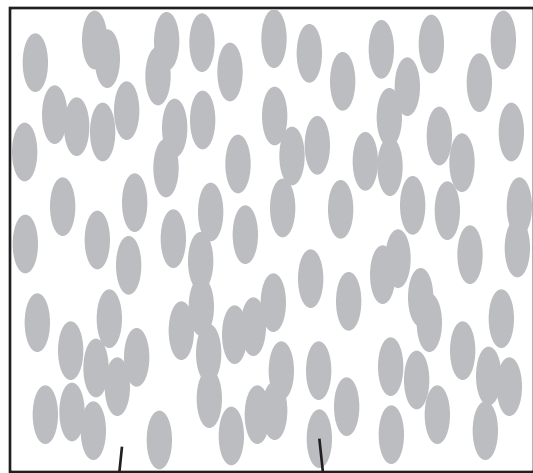
$$\epsilon^*$$

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



ϵ_1

ϵ_2



ϵ^*

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1, p_2 = volume fractions of
the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2}, \text{ composite geometry} \right)$$

ocean swells propagating through a vast field of pancake ice

HOMOGENIZATION: long wave sees an effective medium, not individual floes



Theory of Effective Electromagnetic Behavior of Composites

analytic continuation method

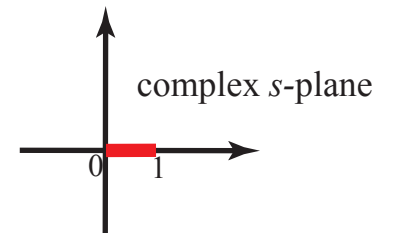
Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

composite geometry
(spectral measure μ) \longrightarrow ϵ^*

integral representations, rigorous bounds, approximations, etc.

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



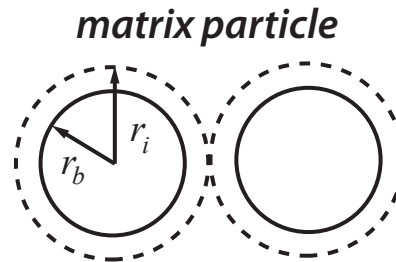
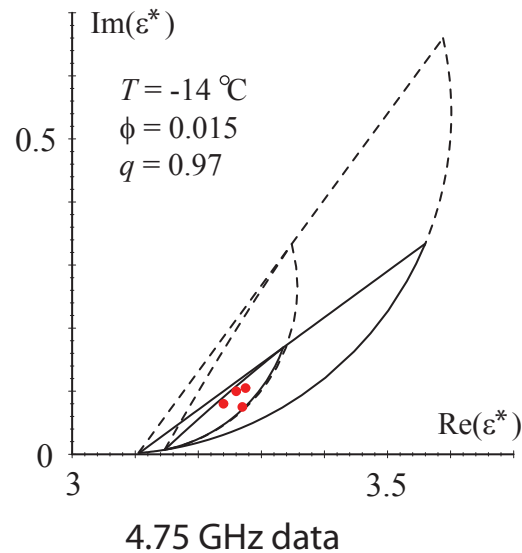
Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
(McPhedran, McKenzie, and Milton, 1982)

ϵ^* \longrightarrow **composite geometry**
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

forward and inverse bounds for sea ice

forward bounds

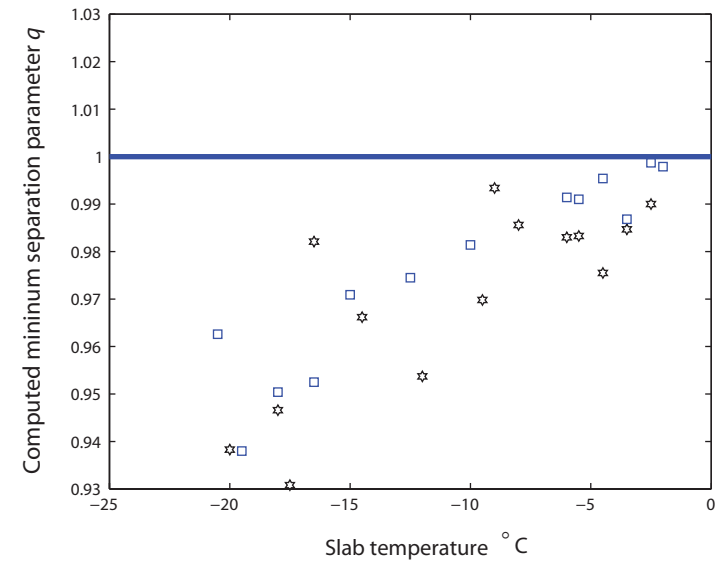


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



inverse bounds and recovery of brine porosity

**Gully, Backstrom, Eicken, Golden
Physica B, 2007**

polycrystalline bounds two-scale homogenization

Gully, Lin, Cherkaev, Golden, 2013

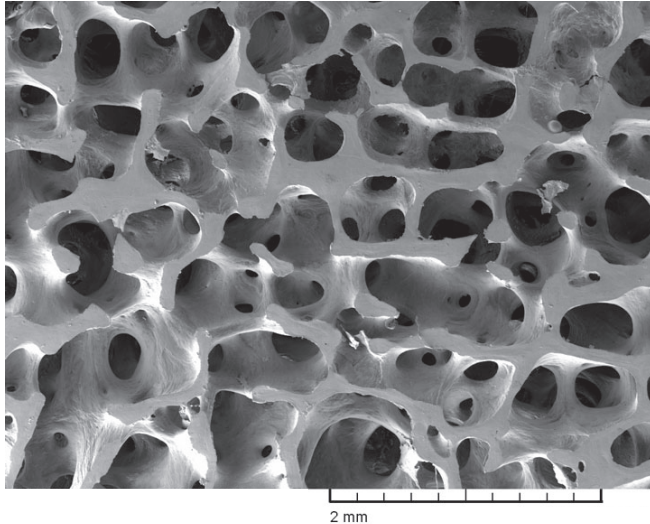
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ϵ^*

**Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012**

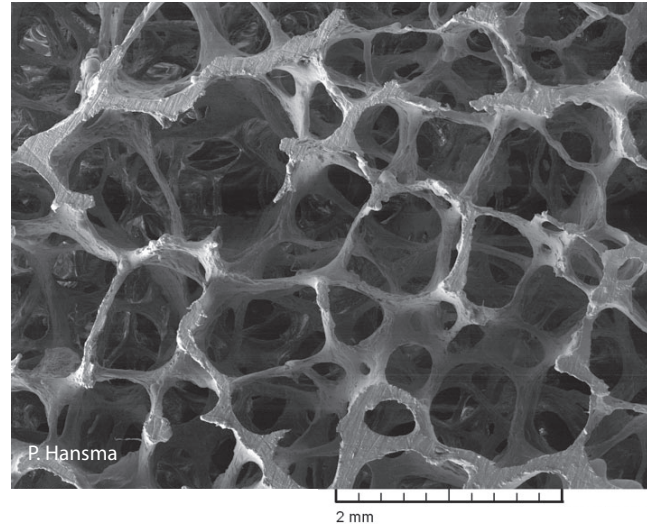
spectral characterization of porous microstructures in bone

Golden, Murphy, Cherkaev, J. Biomechanics 2011

(a) young healthy trabecular bone



(b) old osteoporotic trabecular bone



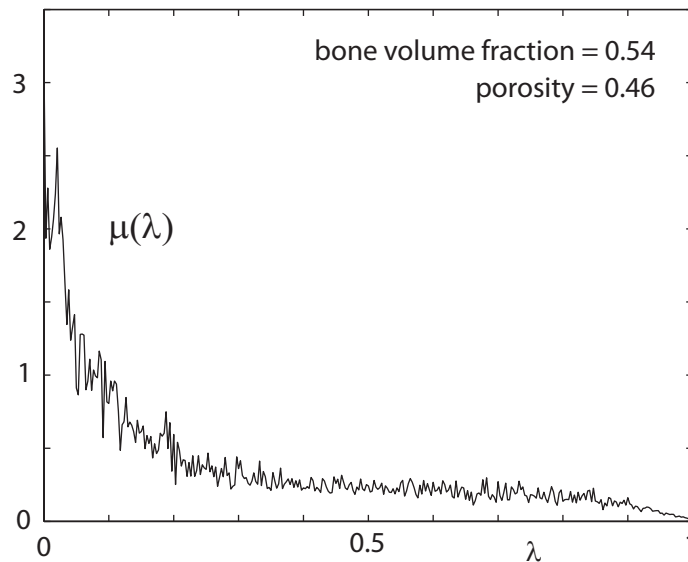
+

reconstruction of spectral
measures from complex
permittivity data

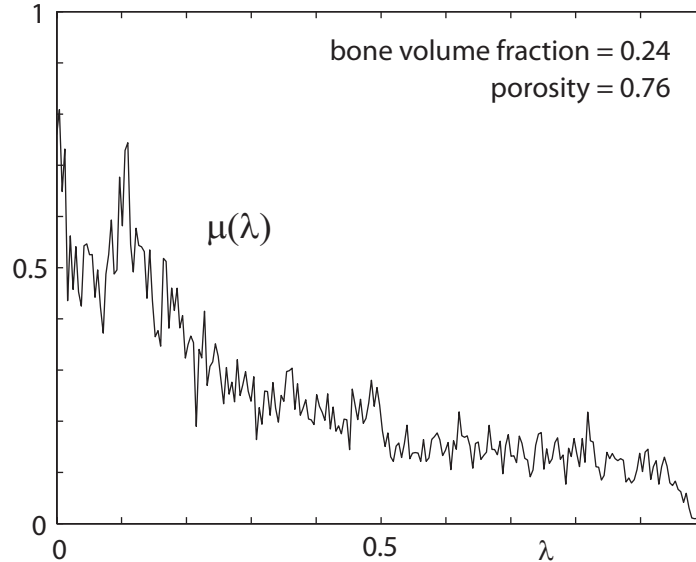
*using regularized
inversion scheme*



(c) spectral measure - young



(d) spectral measure - old



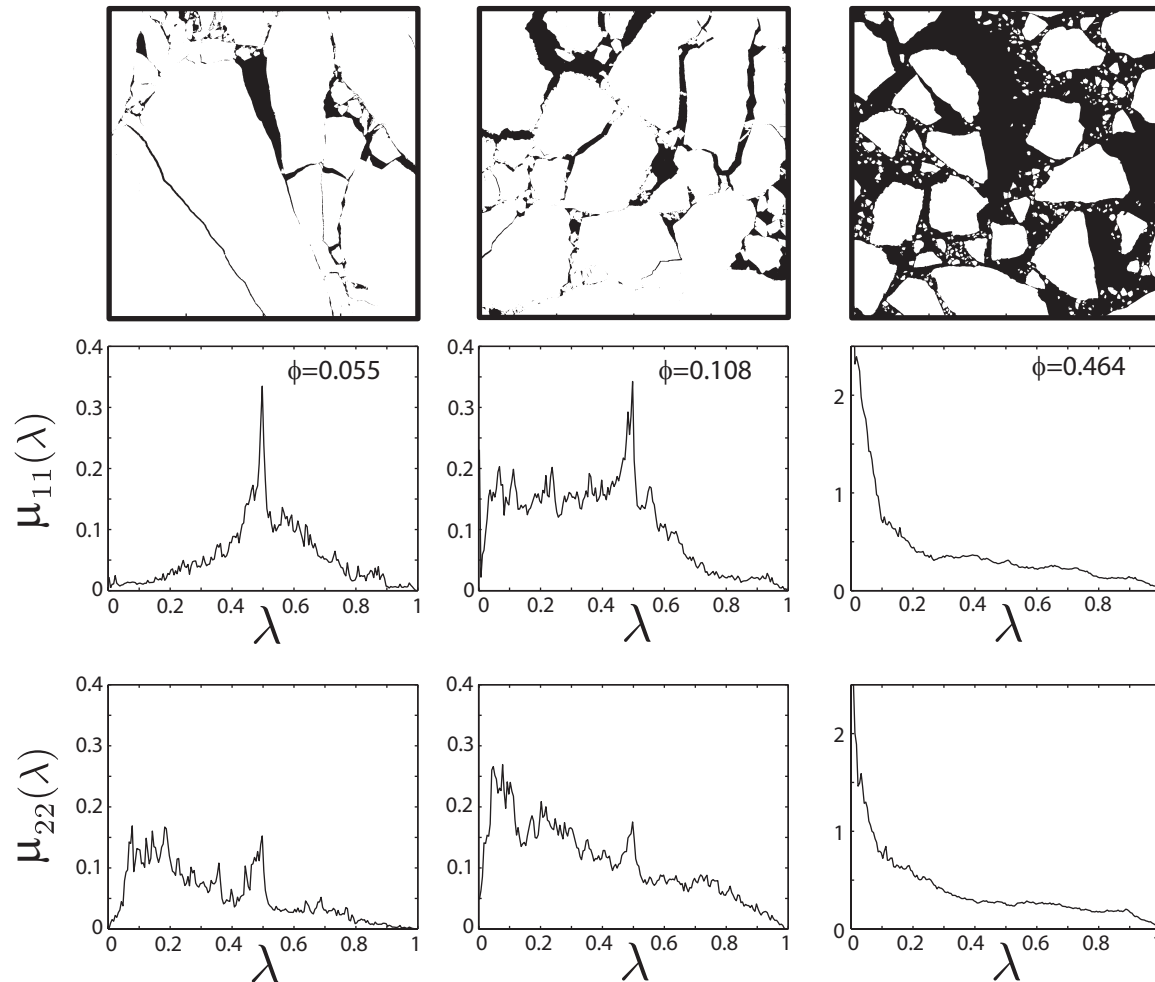
***EM monitoring
of osteoporosis***

***loss of bone
connectivity***

the math doesn't care if it's sea ice or bone!

spectral measures provide a path toward rigorously incorporating
“composite microstructure” into calculations of effective behavior on larger scales

spectral measures for the Arctic sea ice pack



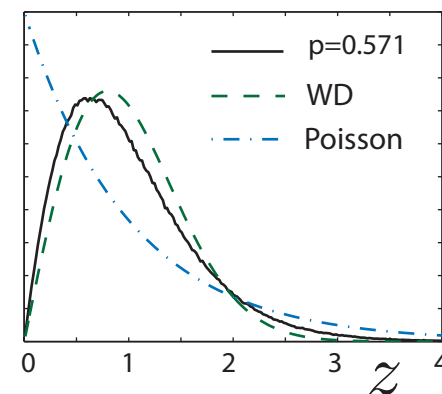
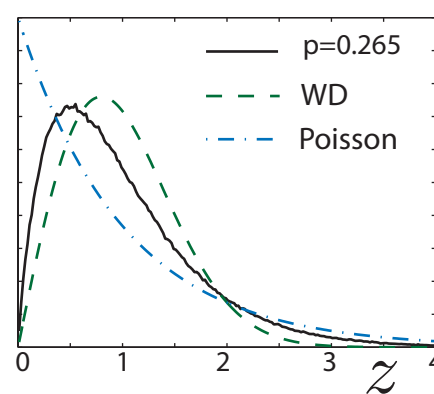
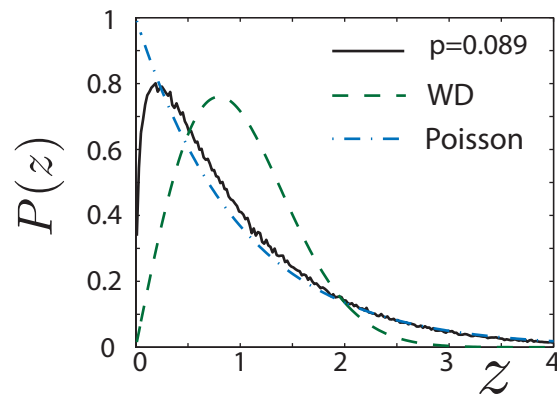
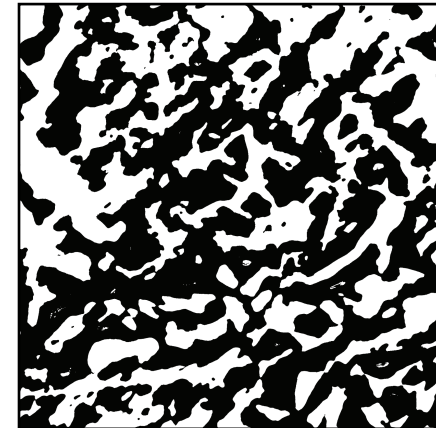
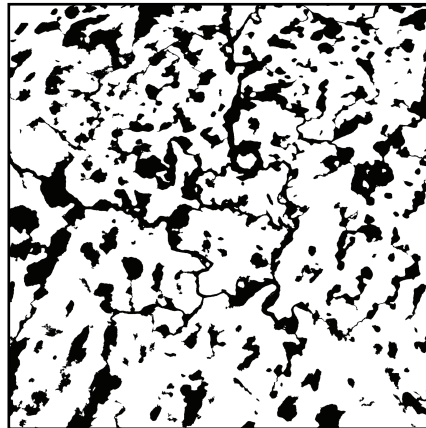
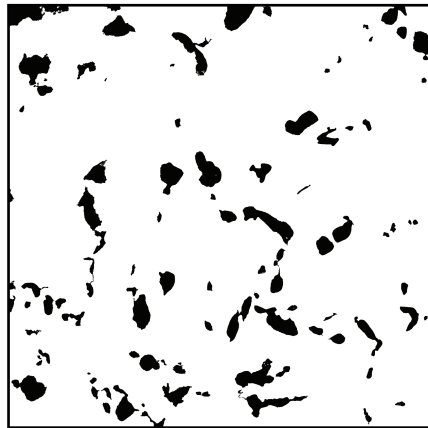
area under curve = ϕ = open water fraction

spectral gap closes as ocean phase becomes connected

random matrix characterization of connectedness transition -- discretization of $\chi\Gamma\chi$

Unfolded Eigenvalue Spacing Distribution

ARCTIC MELT PONDS



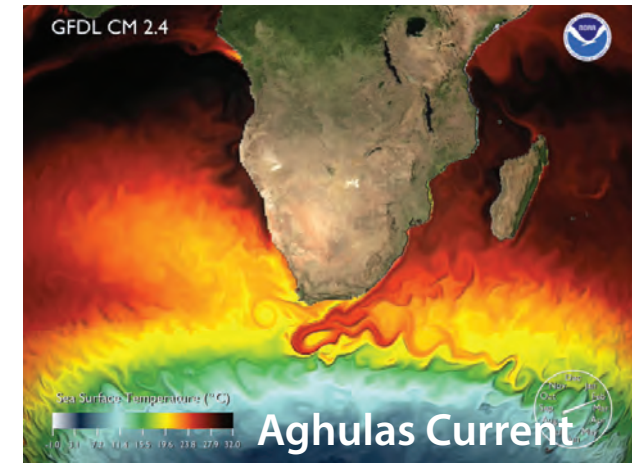
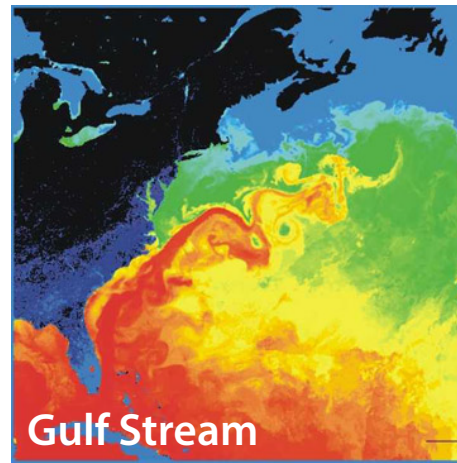
*eigenvalue statistics for transport tend toward the **UNIVERSAL Wigner-Dyson distribution** as the “conducting” phase becomes connected over large scales*

uncorrelated \longrightarrow “level repulsion”

advection enhanced diffusion

effective diffusivity

tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere
salt and heat transport in ocean



advection diffusion equation with a velocity field \vec{u}

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

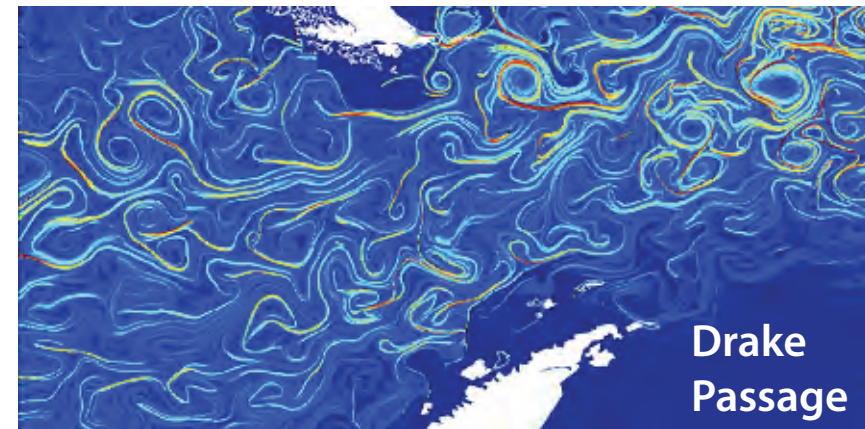
homogenize

$$\frac{\partial \bar{T}}{\partial t} = \kappa^* \Delta \bar{T}$$

κ^* effective diffusivity

Stieltjes integral for κ^* with spectral measure

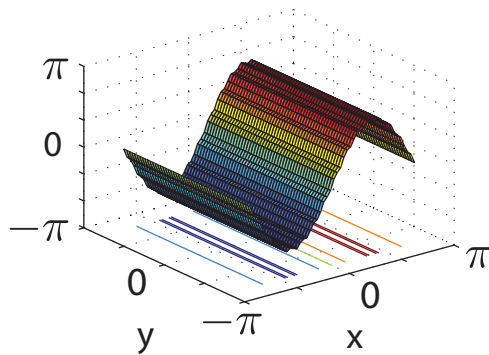
Avellaneda and Majda, PRL 89, CMP 91



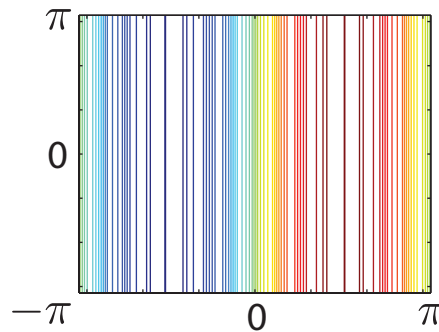
effective diffusivities for sample flows

Shear Flow: $H(x, y) = \sin x + (0.5 + \eta) \sin(15x)/15$, $\eta \sim \text{Uniform}(-0.1, 0.1)$

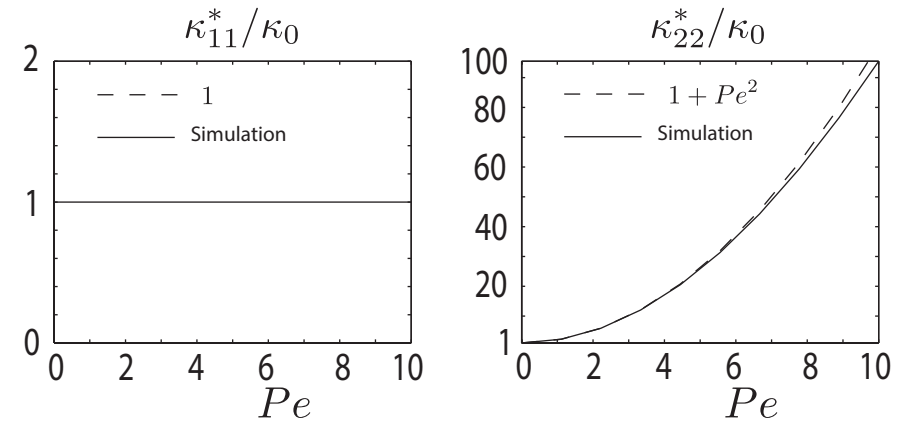
stream function



streamlines

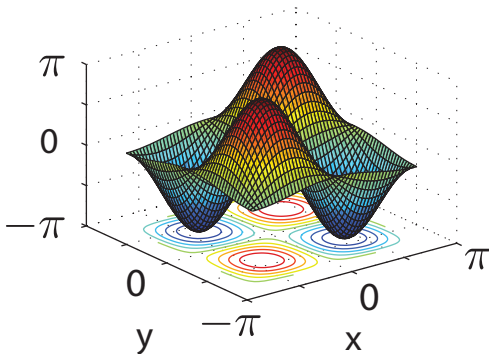


effective diffusivities

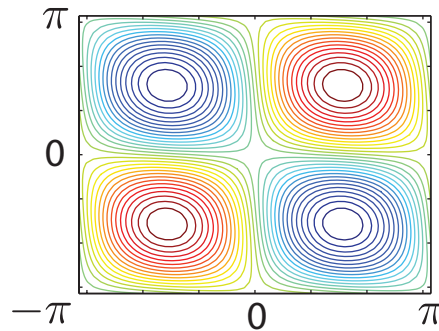


Modified Cat's Eye Flow: $H(x, y) = \sin x \sin y + \eta \cos x \cos y$, $\eta \sim \text{Uniform}(-0.1, 0.1)$

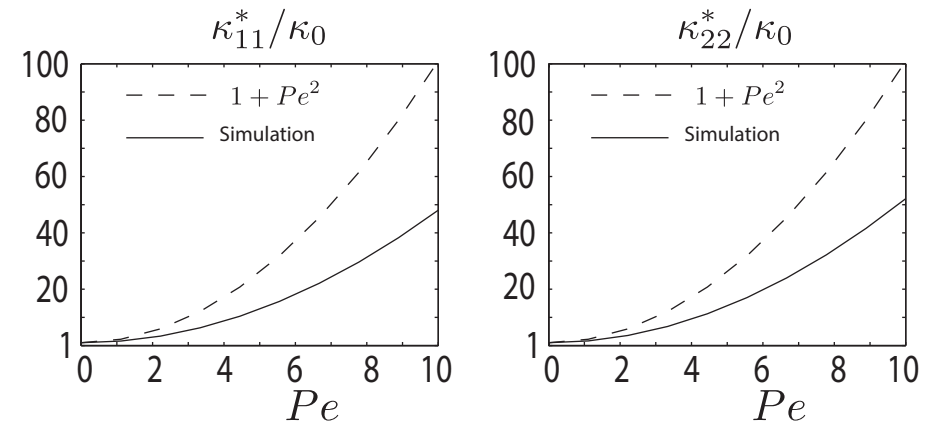
stream function



streamlines



effective diffusivities



Arctic and Antarctic field experiments

*develop electromagnetic methods
of monitoring fluid transport and
microstructural transitions*

extensive measurements of fluid and
electrical transport properties of sea ice:

2007 Antarctic SIPEX

2010 Antarctic McMurdo Sound

2011 Arctic Barrow AK

2012 Arctic Barrow AK

2012 Antarctic SIPEX II

2013 Arctic Barrow AK



Notices

of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and
the Mathematics of
Transport in Sea Ice

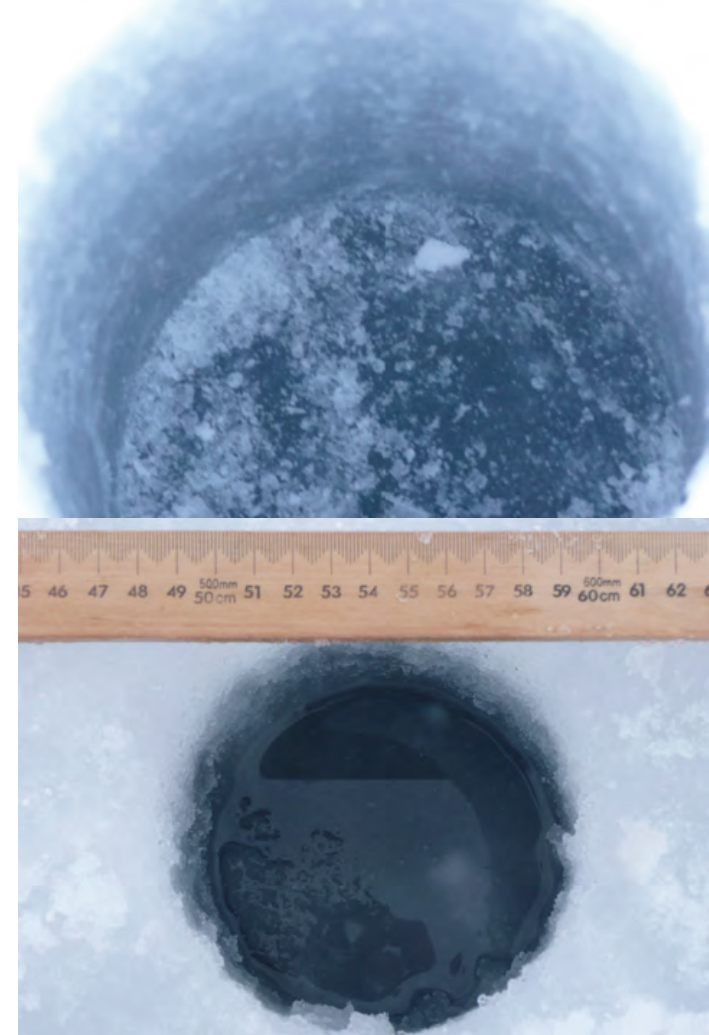
page 562

Mathematics and the
Internet: A Source of
Enormous Confusion
and Great Potential

page 586

photo by Jan Lieser

Real analysis in polar coordinates (see page 613)



**measuring
fluid permeability
of Antarctic sea ice**

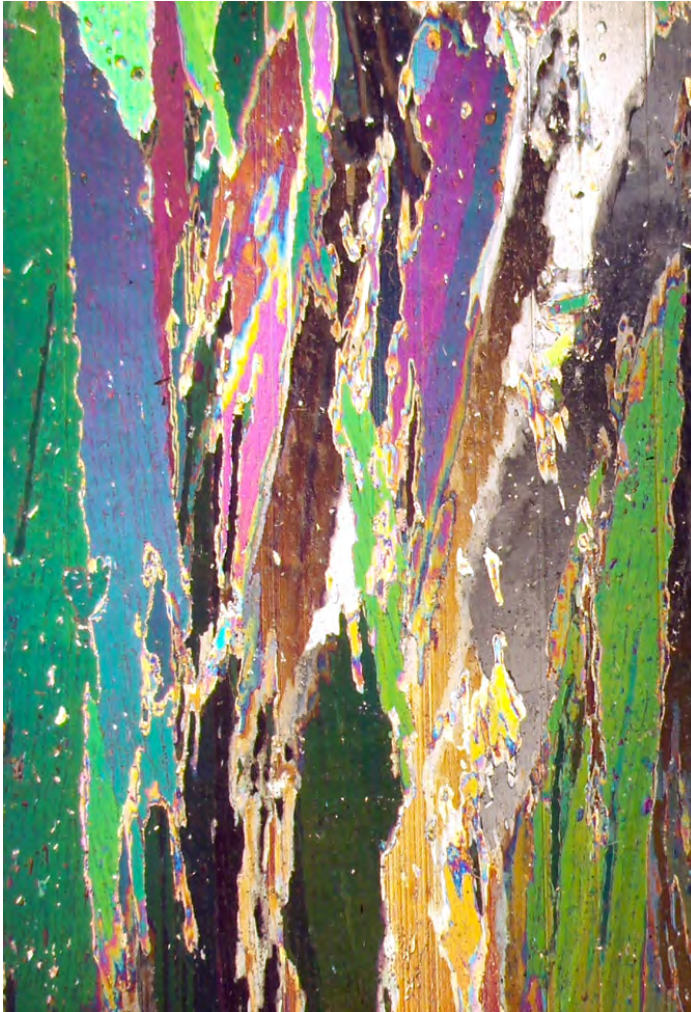
SIPEX 2007

higher threshold for fluid flow in Antarctic granular sea ice

columnar

granular

5%



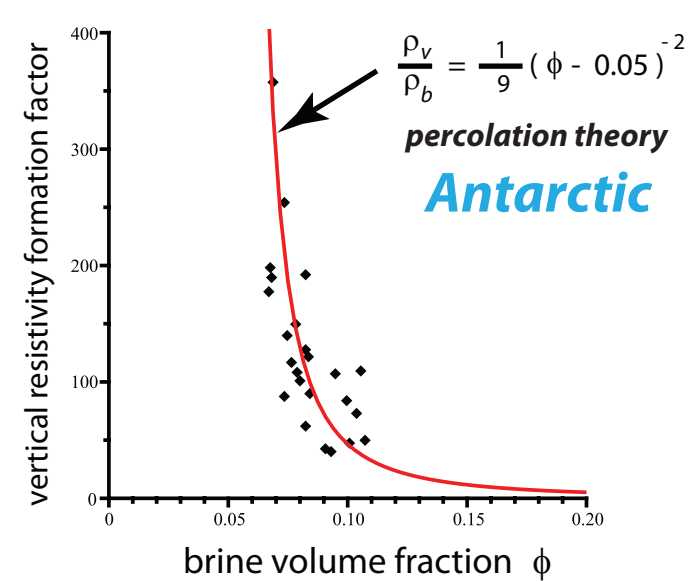
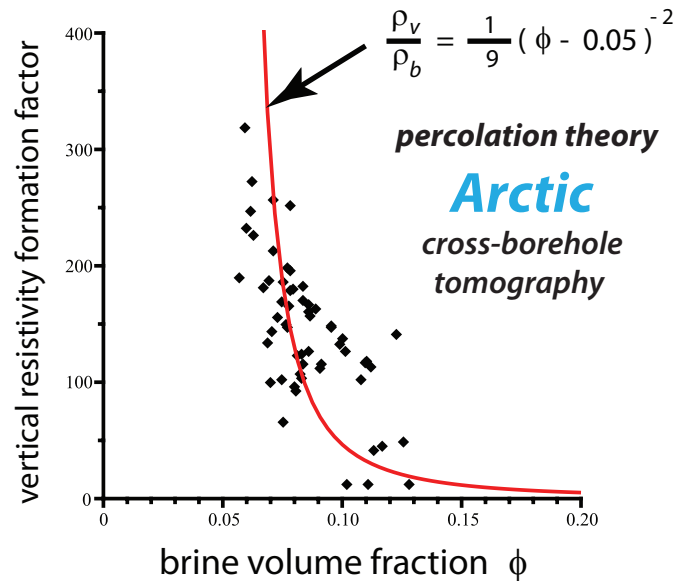
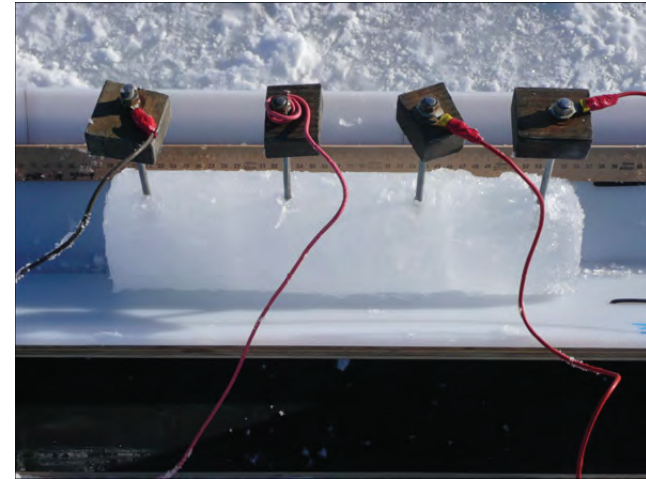
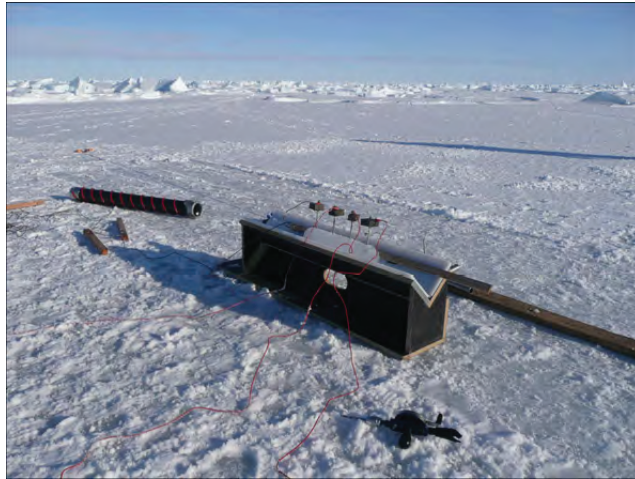
10%



Golden, Gully, Lubbers, Sampson, Tison 2013

critical behavior of electrical transport in sea ice

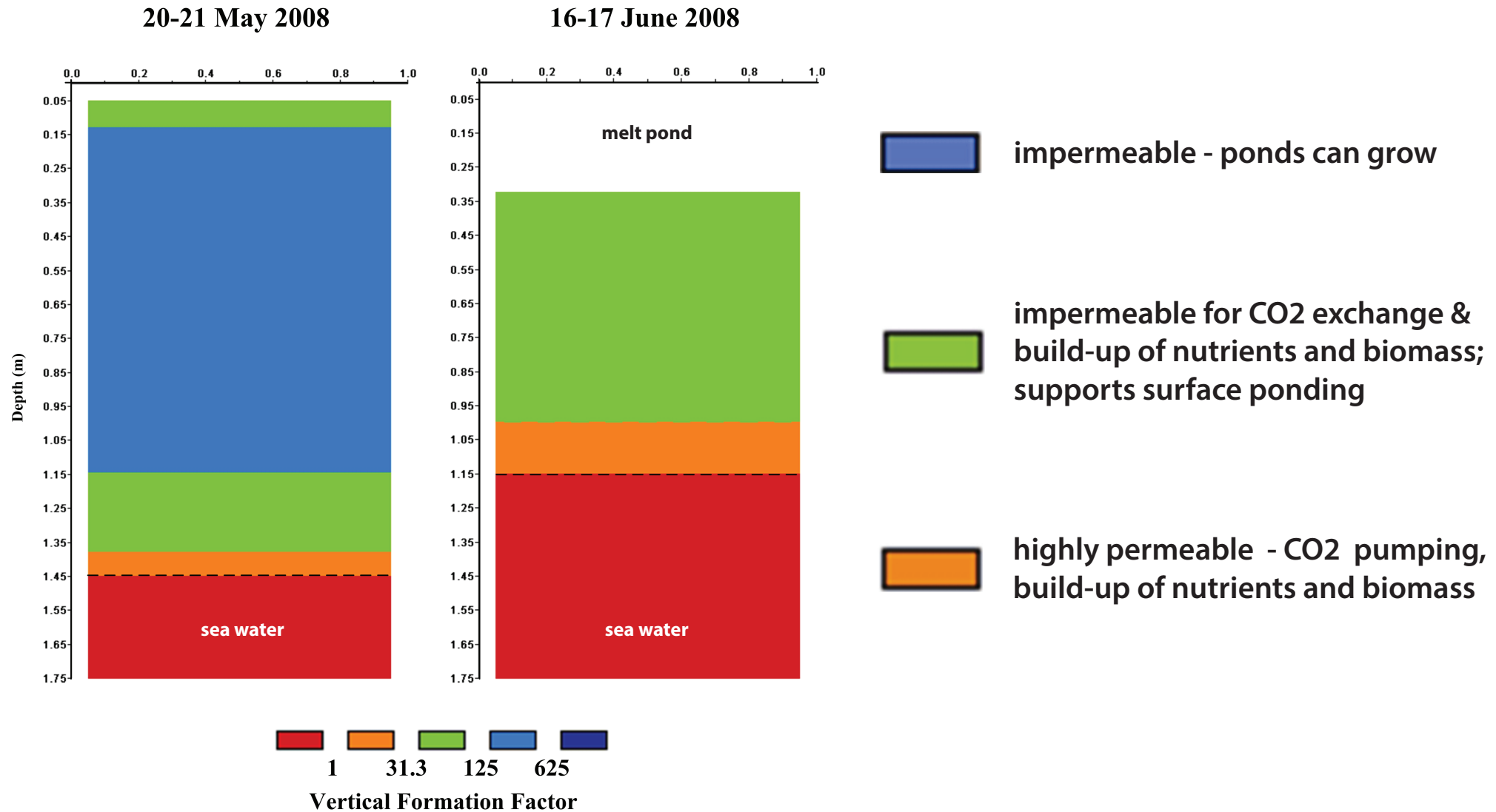
electrical signature of the on-off switch for fluid flow



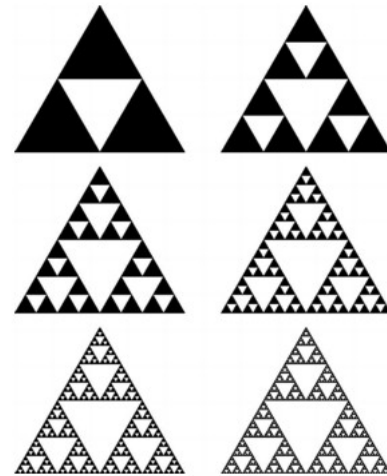
cross-borehole tomography - electrical classification of sea ice layers

Cross-borehole tomographic reconstructions of sea ice resistivity

before and after melt pond formation



fractals and multiscale structure



melt ponds on the surface of Arctic sea ice



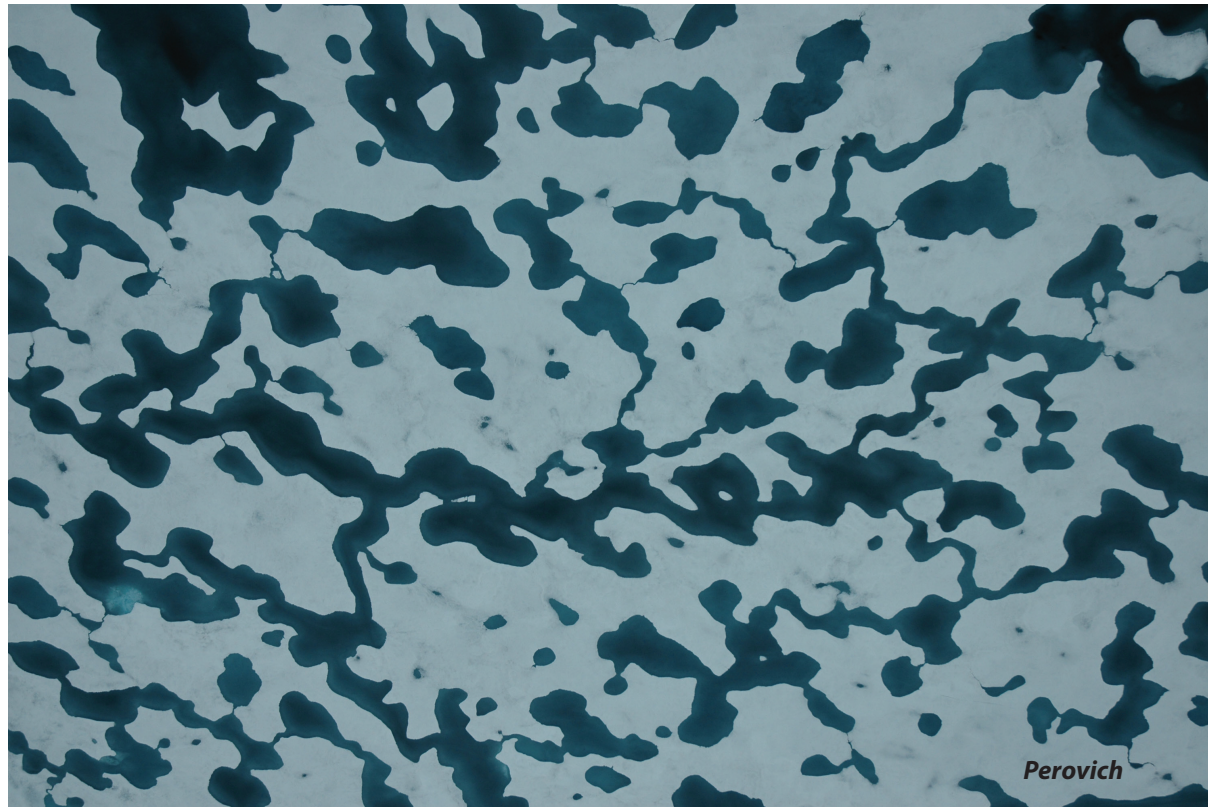
melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham,
Taylor, Worster 2006
Flocco, Feltham 2007

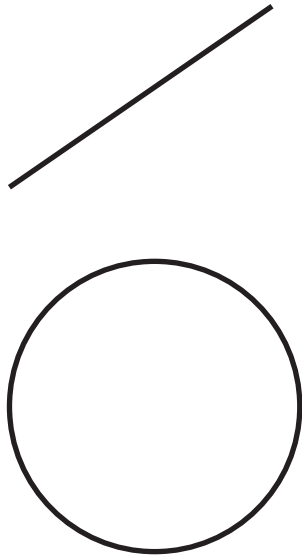
Skyllingstad, Paulson,
Perovich 2009
Flocco, Feltham,
Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

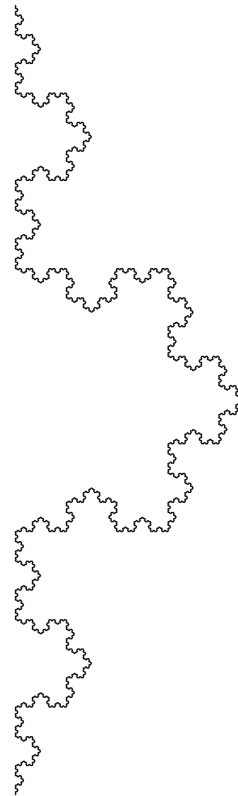
fractal curves in the plane

they wiggle so much that their dimension is >1



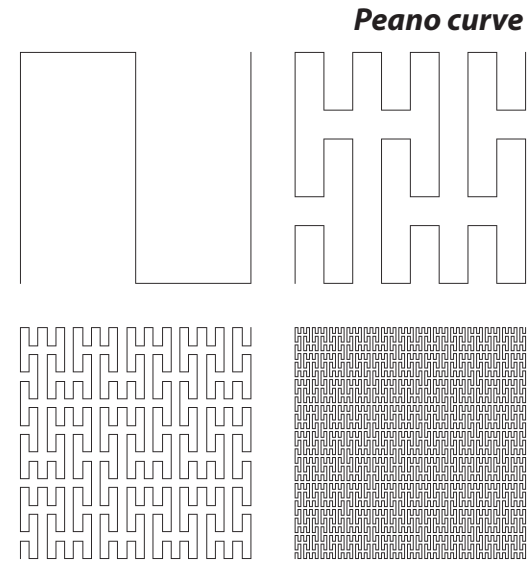
simple curves

$$D = 1$$



Koch snowflake

$$D = 1.26$$



Peano curve

Brownian motion

space filling curves

$$D = 2$$

clouds exhibit fractal behavior from 1 to 1000 km

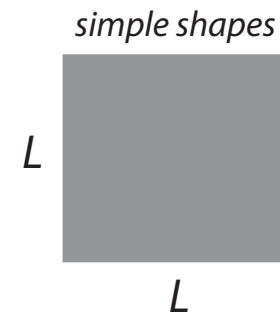
use **perimeter-area** data to find that
cloud and rain boundaries are fractals

$$D \approx 1.35$$

S. Lovejoy, Science, 1982

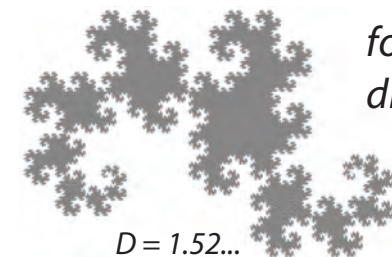


$$P \sim \sqrt{A}$$



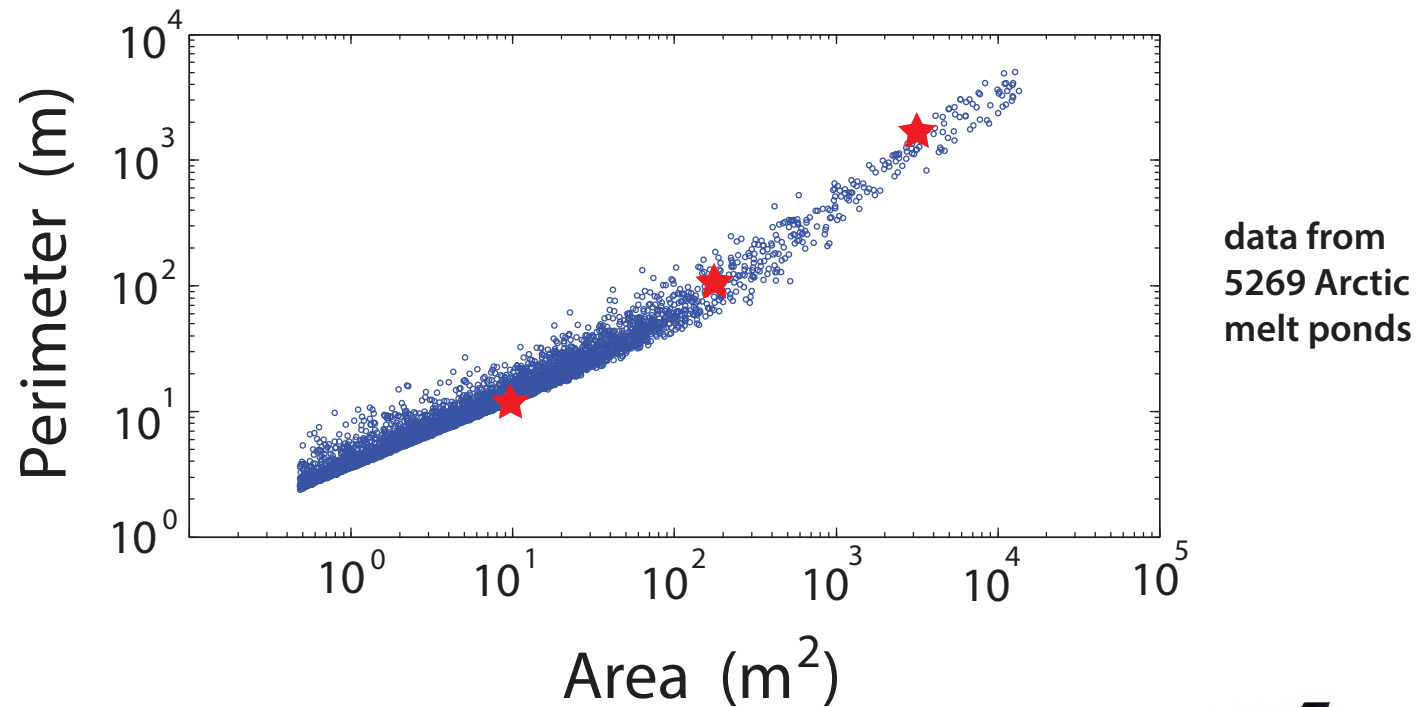
$$A = L^2$$
$$P = 4L = 4\sqrt{A}$$

$$P \sim \sqrt{A}^D$$

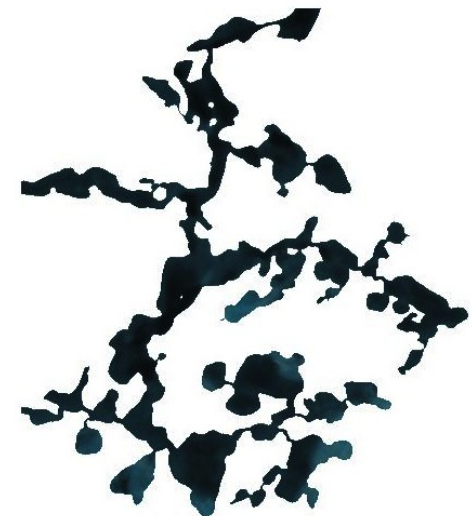


for fractals with
dimension D

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden



~ 30 m



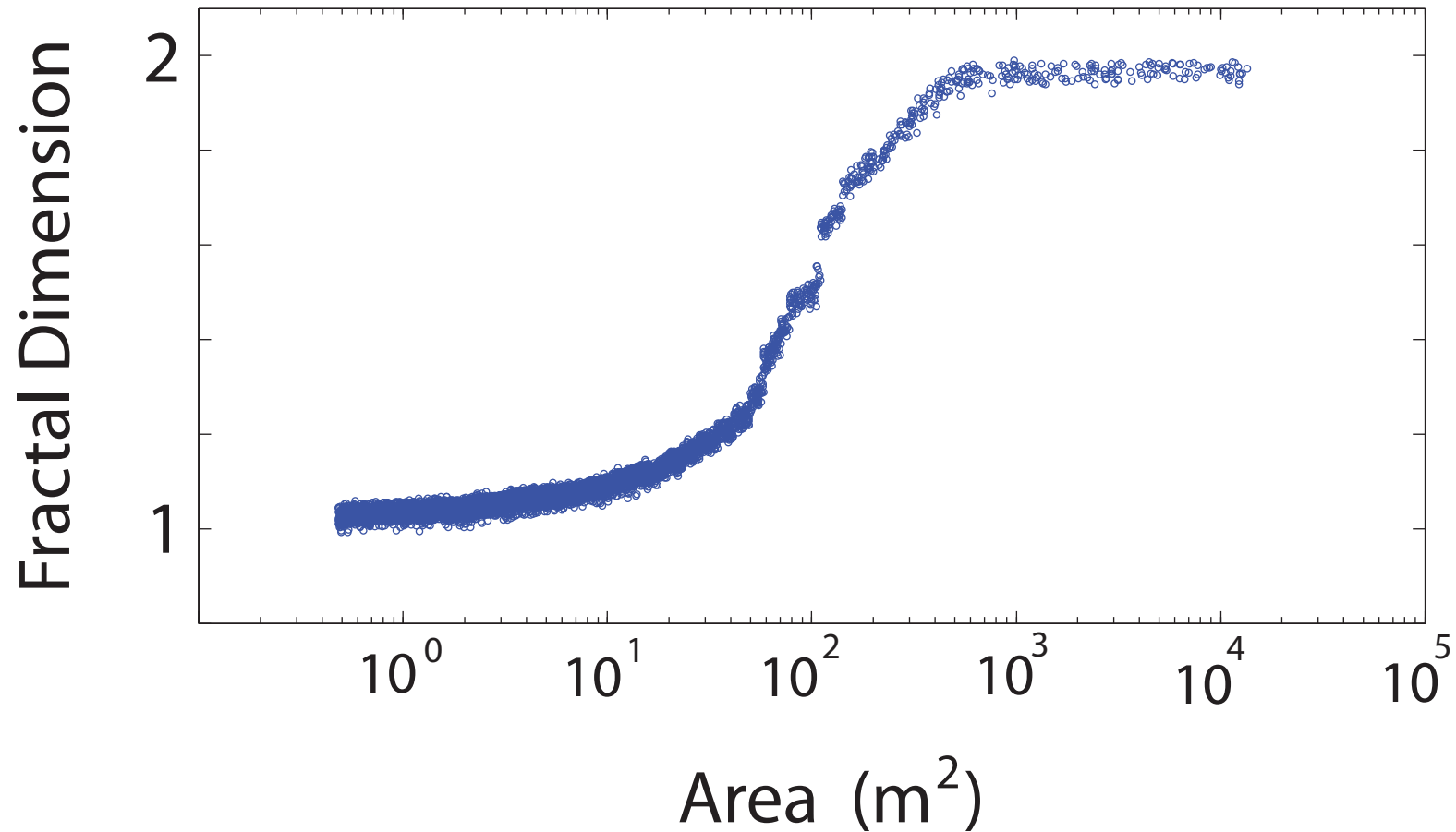
simple pond

transitional pond

complex pond

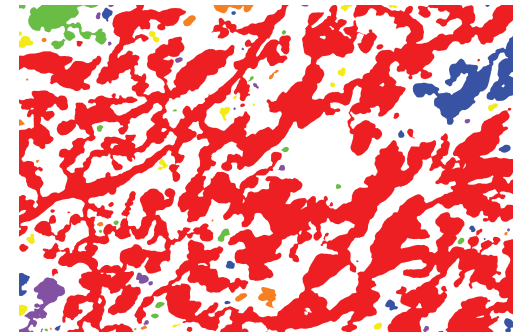
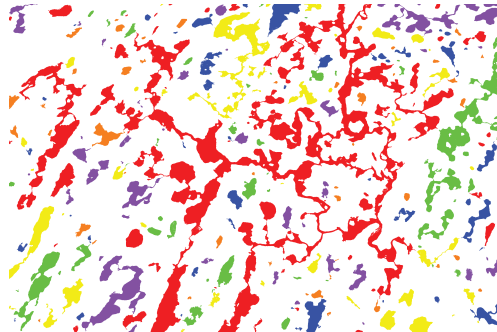
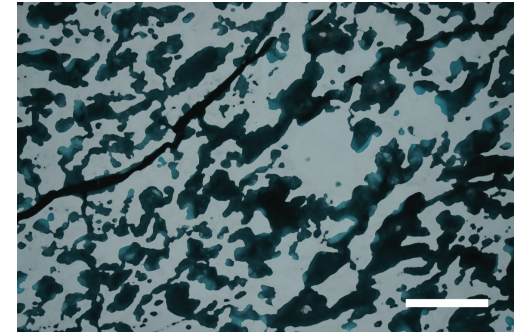
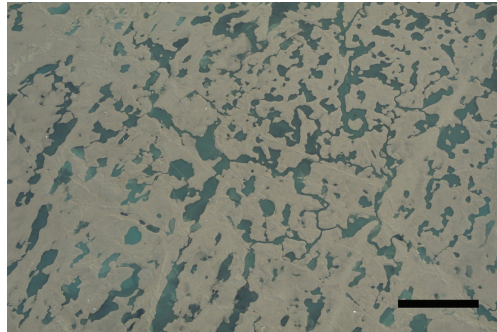
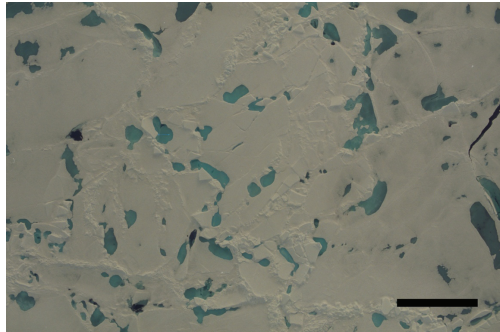
transition in the fractal dimension

complexity grows with length scale



compute “derivative” of area - perimeter data

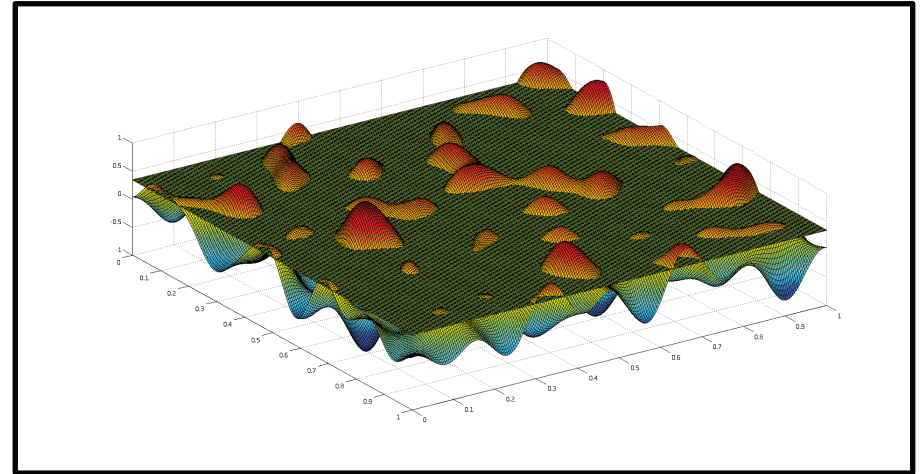
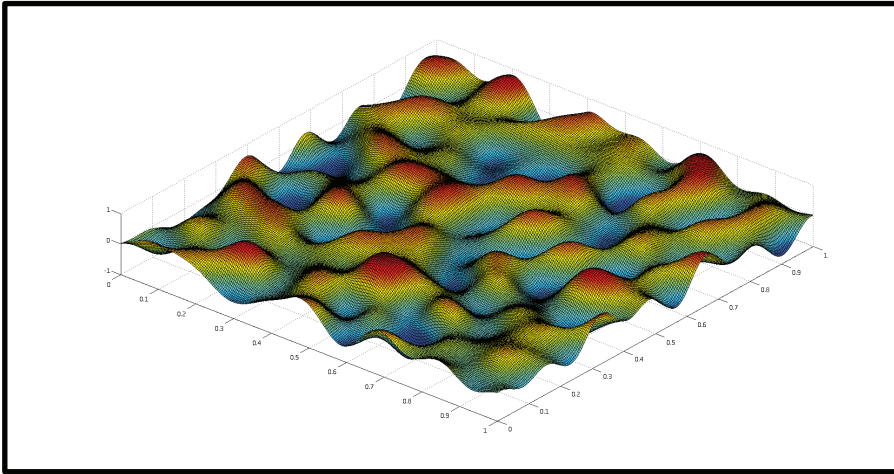
***small simple ponds coalesce to form
large connected structures with complex boundaries***



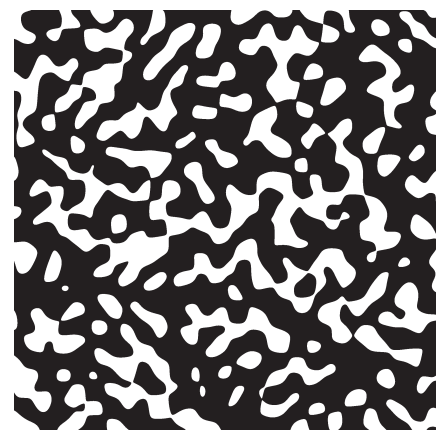
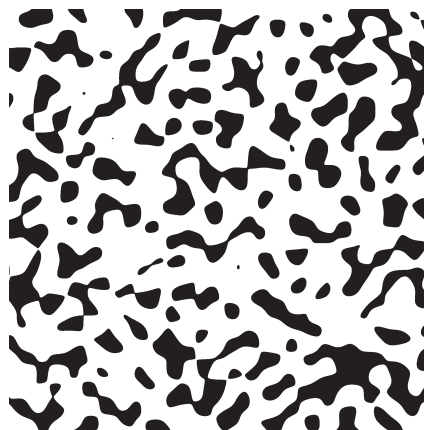
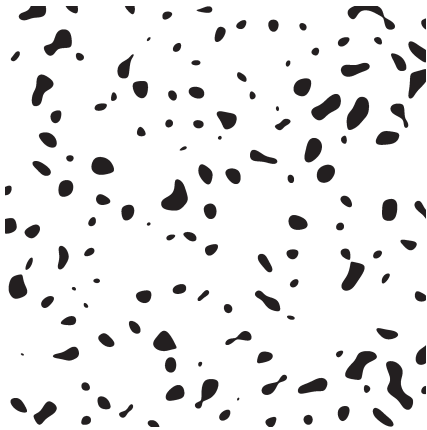
melt pond percolation

Continuum percolation model for melt pond evolution

(Brady Bowen and Ken Golden, 2013)



intersections of a plane with the surface define melt ponds

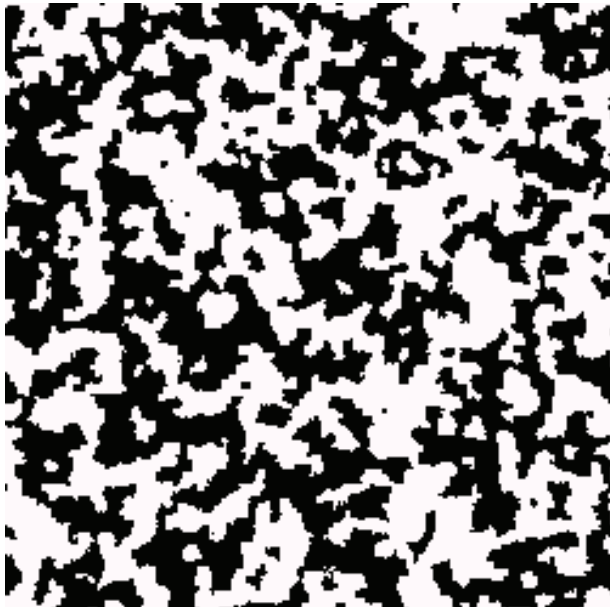
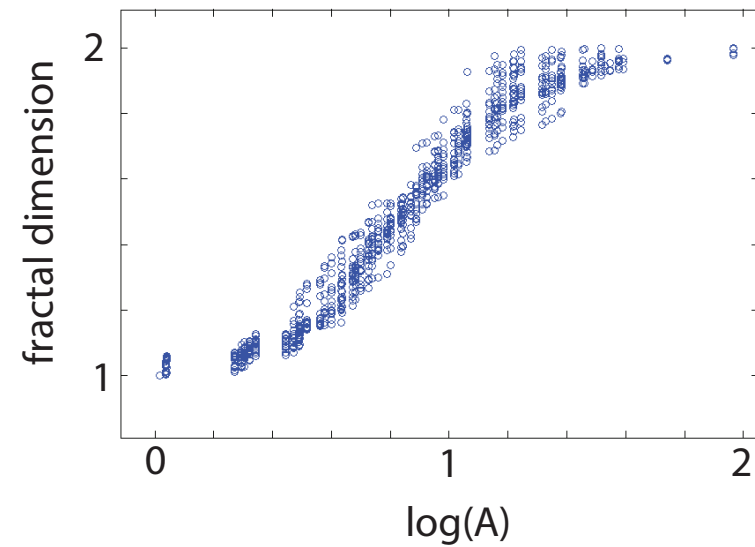
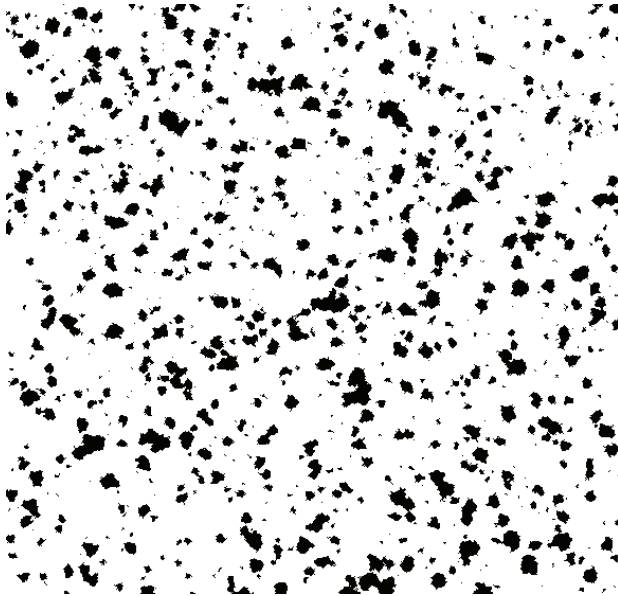


electronic transport in disordered media

diffusion in turbulent plasmas

(Isichenko, Rev. Mod. Phys., 1992)

simple stochastic growth model of melt pond evolution



voter
model

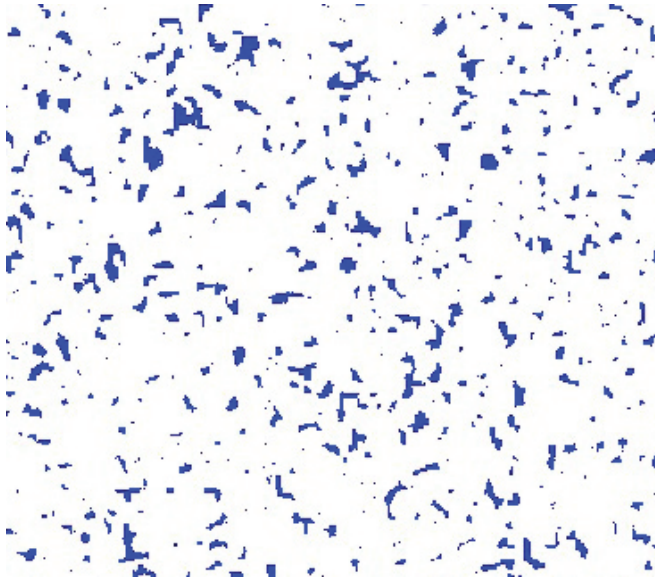
*a square is more likely to melt
if its neighbors have melted*

Ising model for ferromagnets

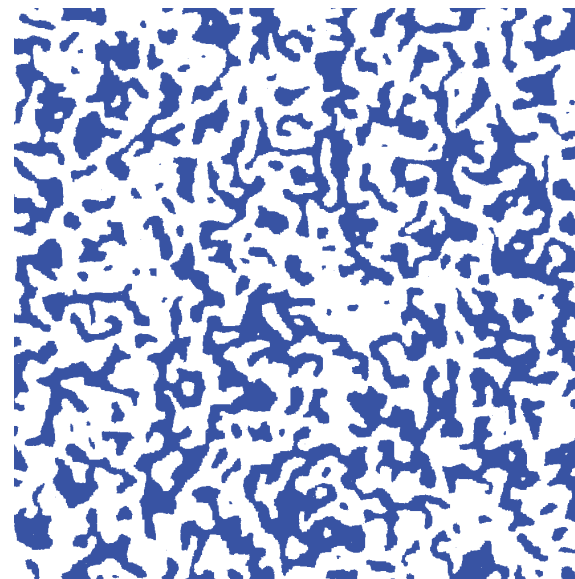


Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{\langle i,j \rangle}^N s_i s_j - H \sum_i^N s_i \quad s_i = \begin{cases} \uparrow & +1 & \text{ice} \\ \downarrow & -1 & \text{water} \end{cases} \quad M = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$



COLD



WARM

“melt ponds” are clusters of magnetic spins that align with the applied field

clusters exhibit transition in fractal dimension

Thekkedath, Alali, Strong, Golden
Sudakov, Ma, Golden

Conclusions

- 1. Summer Arctic sea ice is melting rapidly.**
- 2. Fluid flow through sea ice mediates many processes of importance to understanding climate change and the response of polar ecosystems.**
- 3. Mathematical models of composite materials and statistical physics help unravel the complexities of sea ice structure and processes, and provide a path toward rigorous representation of sea ice in climate models .**
- 4. Field experiments are essential to developing relevant mathematics.**
- 5. Our research will help to improve projections of climate change and the fate of the Earth sea ice packs.**

THANK YOU

National Science Foundation

Division of Mathematical Sciences

Arctic Natural Sciences

Office of Polar Programs

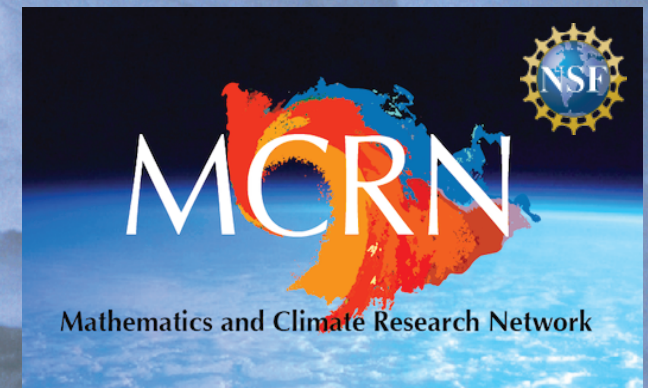
CMG Program

(Collaboration in Mathematical Geosciences)

Office of Naval Research

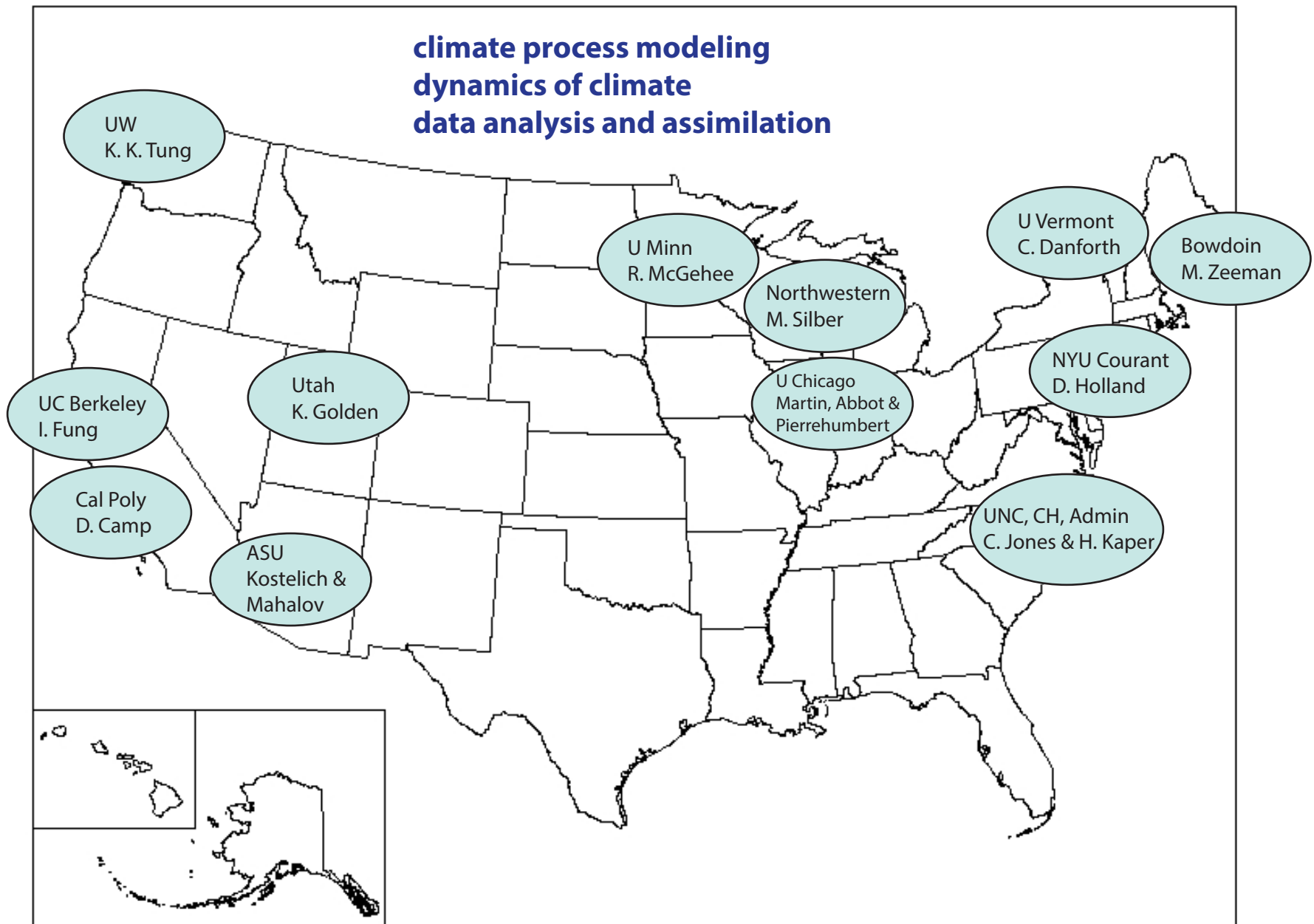
Applied Computational Analysis Program

Arctic and Global Prediction Program



Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999

Mathematics and Climate Research Network (MCRN)



NSF DMS 2010-2015, Lorenz postdocs, grad, undergrad, polar expeditions

Jones, Golden, Kaper, Zeeman

Fire endangers Hobart's ice ship

By DAVID CARRIGG

AN engine-room fire has left the Hobart-based Antarctic research ship *Aurora Australis* without power in dangerous sea ice off the Antarctic coast.

None of the 79 people on board was injured in the blaze, which broke out early yesterday morning while the ship was in deep water 185km off the coast.

The extent of the damage is not known.

Australian Antarctic Division director Rex Moncur said the fire was extinguished by flooding the engine room with an inert gas.

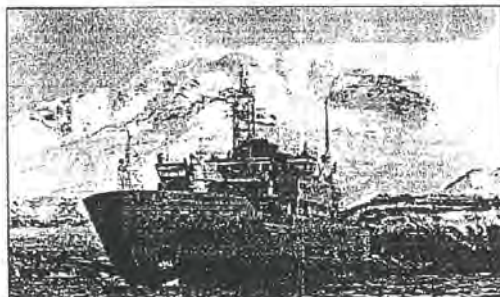
The gas had to be cleared before crew wearing breathing apparatus could enter and assess the situation.

He said it could be some time before the extent of damage was known.

The 25 crew and 54 expeditioners, mostly from Hobart, would wear thermal clothing and stay below decks to keep warm.

"There is always a risk of becoming ice-bound in these waters at this time of the year but at this stage we don't expect to launch a rescue mission from Hobart," Mr Moncur said.

The ship was in regular radio contact with the Antarctic Div-



A file photo of the *Aurora Australis* in Antarctica.

ision's Hobart office.

He expected the expeditioners and crew to abandon the pioneering winter voyage and return the ship to Hobart for repairs in about a week.

The Antarctic Division, which hires the ship from P&O Australia, would not be hiring another vessel for the expedition.

"It's a pretty specialist vessel so you couldn't get the sort of research capability that this ship has got readily available," Mr Moncur said.

"We hope the next voyage can still proceed on schedule, which is early September."

The *Aurora Australis* is owned by P&O Australia and chartered by the Antarctic Div-

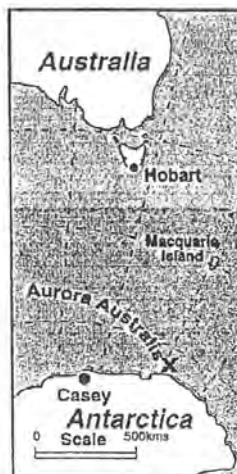
ision for about \$11 million a year.

P&O Australia managing director Richard Hein said yesterday the company was assessing the situation and a number of rescue options were being considered.

It was too early to say whether P&O would be liable for the cost of the aborted mission.

The vessel left Hobart last Wednesday for a seven-week voyage mainly to study a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

The ship was nearing the polynya when the fire broke out.



Oceanographers believe a closer study of the phenomenon will lead to a better understanding of climate change.

CSIRO Marine Research oceanographer Steve Rintoul said the dense bottom water, created only in a few places in Antarctica and to a lesser extent in the North Atlantic, was critical to the chemistry and biology of the world's oceans.

Fire strands Antarctic ship in sea ice

AN engine room fire has disabled the icebreaker *Aurora Australis* in sea ice, deep in Antarctic waters.

There were no injuries and the ship was not in danger after Tuesday night's fire.

Australian Antarctic Division director Mr Rex Moncur said. But Mr Moncur said he expected it would have to abandon its pioneering mid-winter voyage to the edge of the Ant-

arctic continent and return to Hobart for repairs.

The cause of the fire was not known but the engines have been turned off, with the ship 100 nautical miles from the Antarctic coast.

THE CANBERRA TIMES

Thursday 23 July 1998

Page 4

Antarctic voyage stopped by fire

HOBART: An engine room fire has disabled the Australian icebreaker *Aurora Australis* in sea ice, deep in Antarctic waters.

Australian Antarctic Division director Rex Moncur said there were no injuries and the ship was not in danger after Tuesday night's fire.

But Mr Moncur said he expected *Aurora Australis* would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent to return to Hobart for repairs.

The fire had been extinguished and the engines were turned off, leaving the ship in sea ice about 100 nautical miles from the Antarctic coast, he said. The weather was good.

Crew had to wear breathing apparatus to enter the engine room and it was likely to be 24 hours before the damage could be fully assessed.

The *Aurora*, with 54 expeditioners and 25 crew, left Hobart last Wednesday for a seven-week voyage which was to have focused on a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

Mr Moncur said, the cause of the fire was not yet known.

2:45 am July 22, 1998

"Please don't be alarmed but we have an uncontrolled fire in the engine room"

about 10 minutes later ...

"Please don't be alarmed but we're lowering the lifeboats"

Sydney Morning Herald
23 July, 1998

ICEBREAKER BURNS

A pioneering \$2-million Australian scientific voyage to the mid-winter Antarctic polynya is expected to be scrapped following an engine room fire on the *Aurora Australis* yesterday. The 54 people on board were forced on deck in the

