

Sea Ice, Climate, and Multiscale Composites

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*Alison Kohout
September 2012*

SIAM Materials Conference

Philadelphia

June 9, 2013

SEA ICE covers 7 - 10% of earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- indicator and agent of **climate change**



polar ice caps critical to global climate in reflecting incoming solar radiation



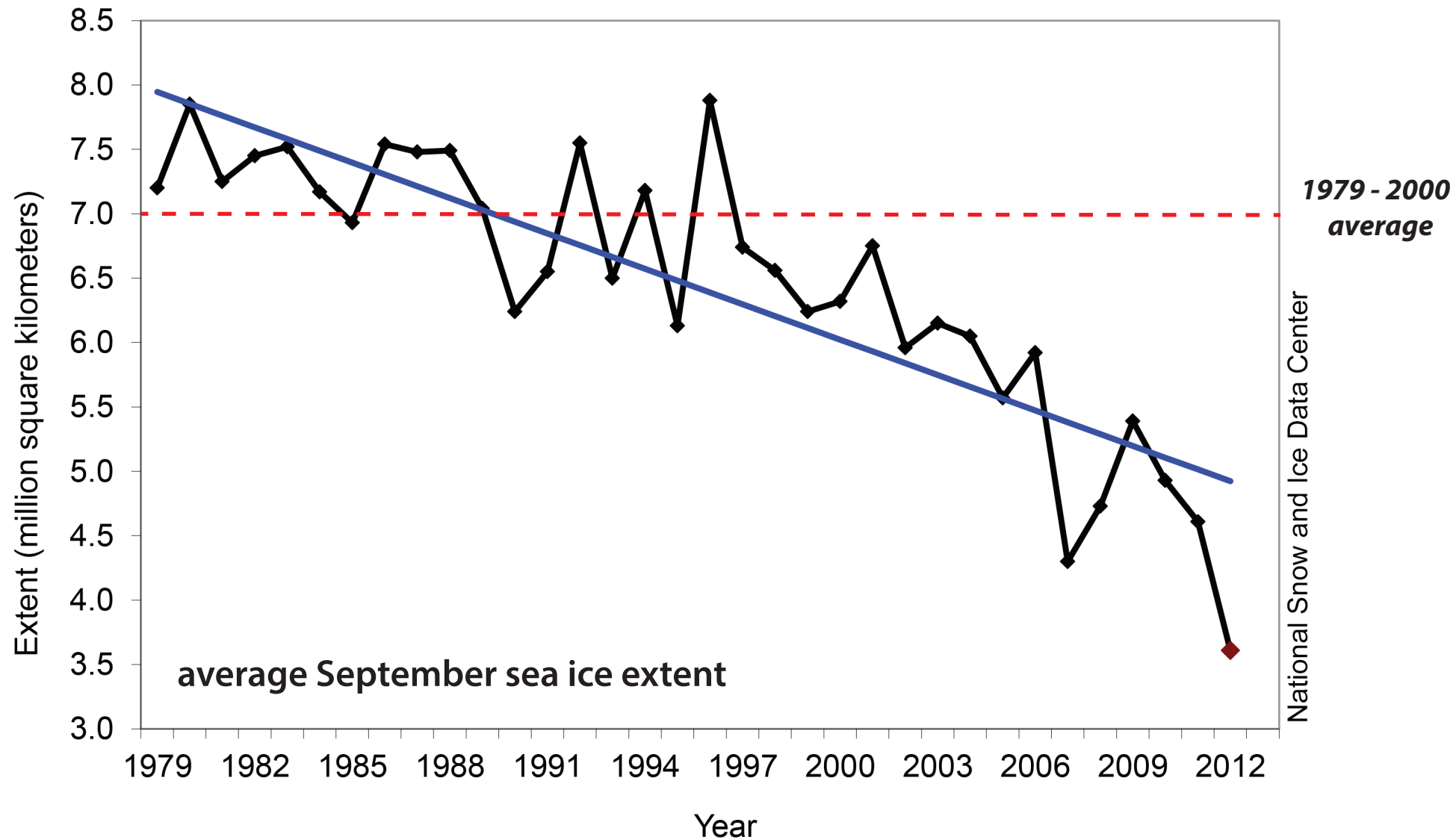
white snow and ice
reflect



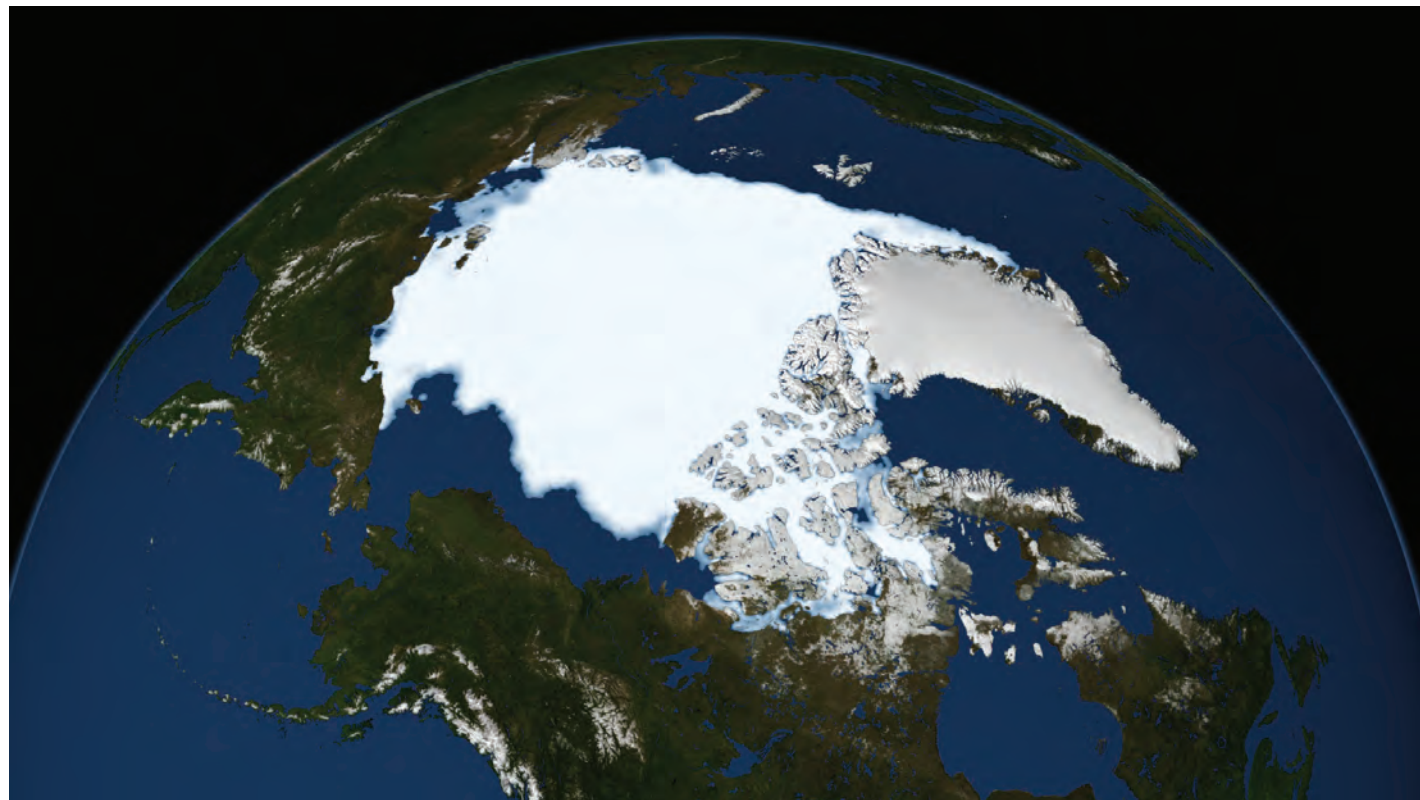
dark water and land
absorb

$$\text{albedo } \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

the summer Arctic sea ice pack is melting



21 September 1979

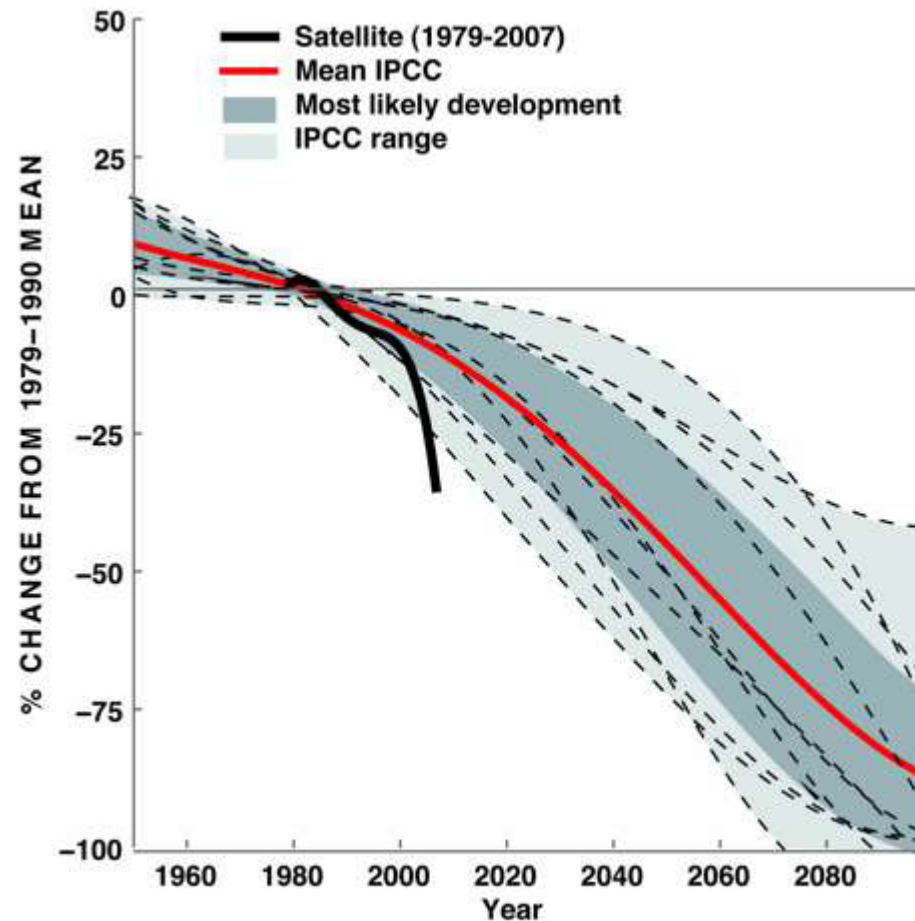


13 September 2012



Intergovernmental Panel on Climate Change (IPCC) 2007 projections

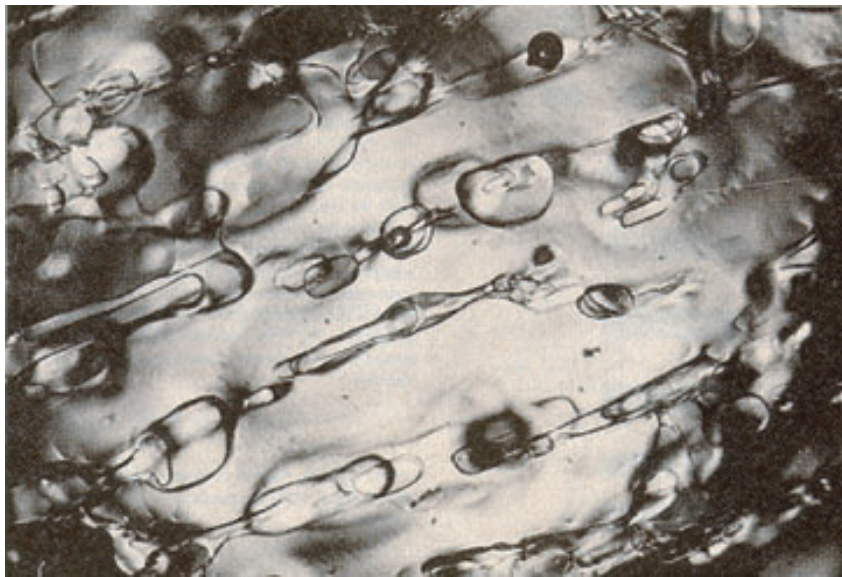
*observed decline in summer Arctic sea ice
outpacing global climate models*



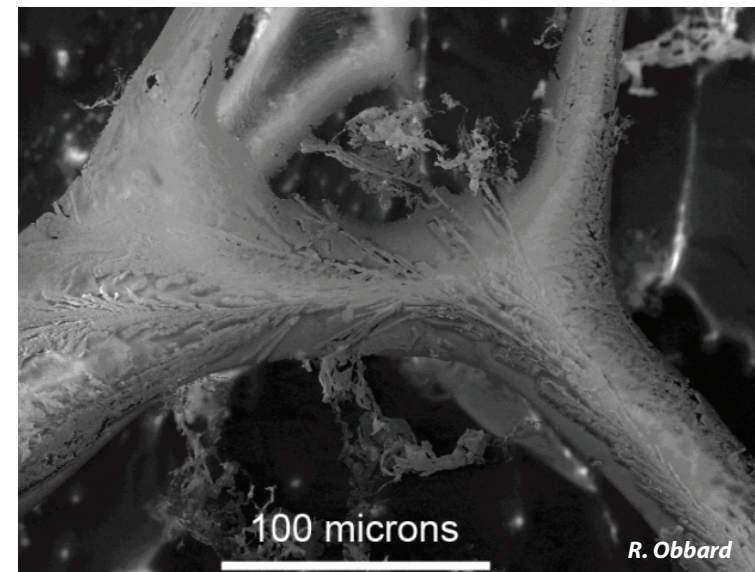
Arctic sea ice loss compared to IPCC models



*sea ice may appear to be a
barren, impermeable cap ...*



brine inclusions in sea ice (mm)



micro - brine channel (SEM)

***sea ice is a
porous composite***

pure ice with brine, air, and salt inclusions

brine channels (cm)



horizontal section



vertical section

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems:

evolution of Arctic melt ponds and sea ice albedo



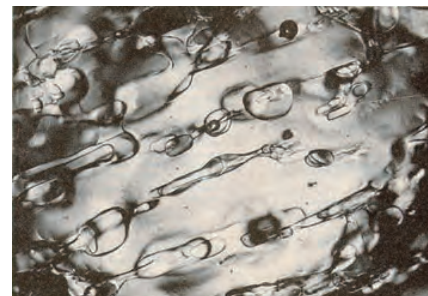
nutrient flux for algal communities



- *drainage of brine and melt water*
- *ocean-ice-air exchanges of heat, CO₂*



linkage of scales



What is this talk about?

Using the mathematics of composite materials and statistical physics to study sea ice structures and processes ... to improve projections of climate change.

- 1. Fluid flow through sea ice***
- 2. Arctic and Antarctic experiments***
- 3. Fractal melt ponds***
- 4. Multiscale homogenization***

small scales



large scales

critical behavior

linkage of scales

cross-pollination

.... develop rigorous representations of sea ice in climate models.

sea ice microphysics

fluid transport

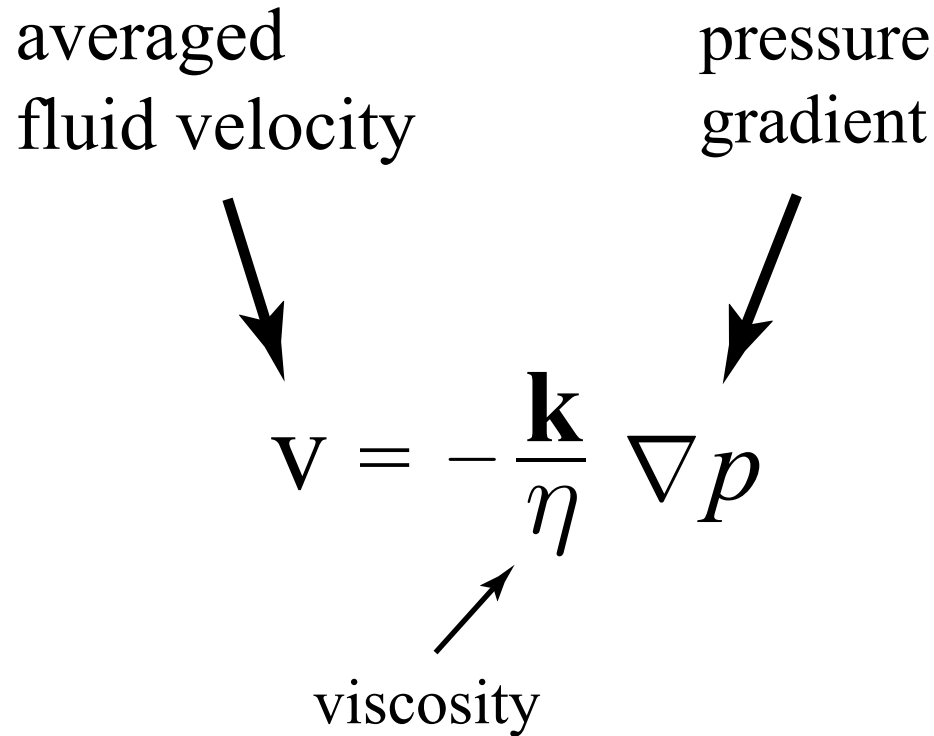
Darcy's Law for slow viscous flow in a porous medium

averaged
fluid velocity

pressure
gradient

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$

viscosity

The diagram shows the equation $\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$ centered on the page. Three labels with arrows point to parts of the equation: 'averaged fluid velocity' points to \mathbf{v} , 'pressure gradient' points to ∇p , and 'viscosity' points to η .

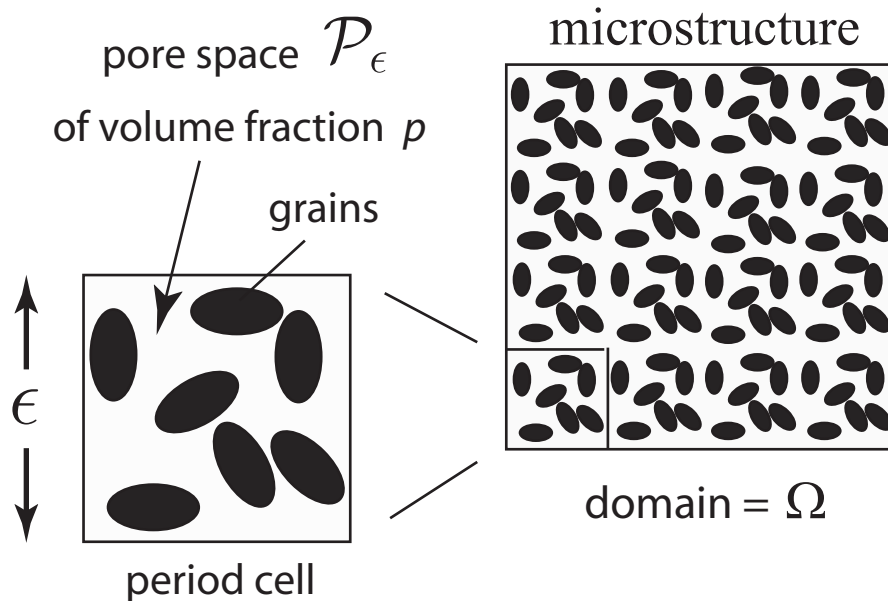
\mathbf{k} = fluid permeability tensor

example of *homogenization*

mathematics for analyzing effective behavior of heterogeneous systems

HOMOGENIZE as $\epsilon \rightarrow 0$

Stokes equations for fluid velocity \mathbf{v}^ϵ , pressure p^ϵ , force \mathbf{f} :



$$\nabla p^\epsilon - \epsilon^2 \eta \Delta \mathbf{v}^\epsilon = \mathbf{f}, \quad x \in \mathcal{P}_\epsilon$$

$$\nabla \cdot \mathbf{v}^\epsilon = 0, \quad x \in \mathcal{P}_\epsilon$$

$$\mathbf{v}^\epsilon = 0, \quad x \in \partial \mathcal{P}_\epsilon$$

η = fluid viscosity

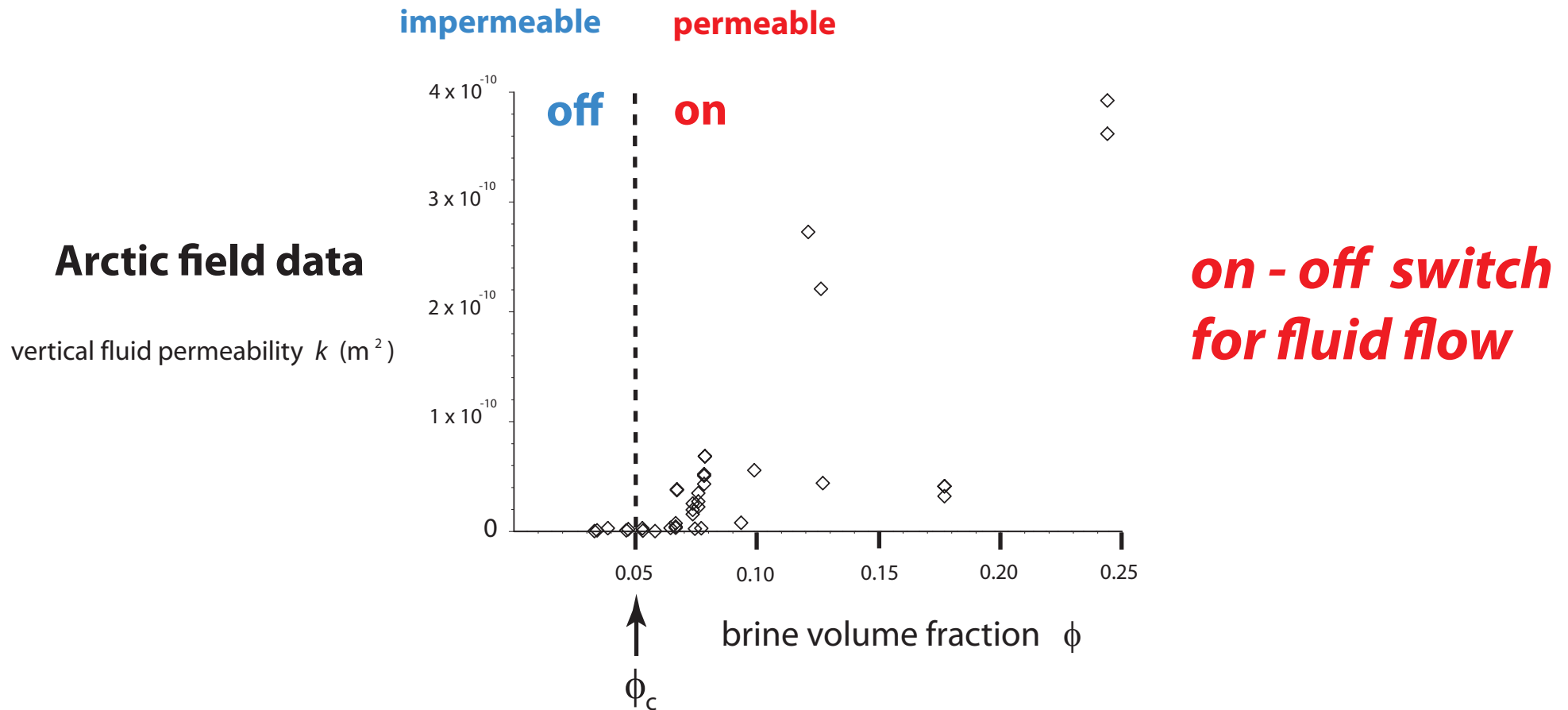
via two-scale expansion

MACROSCOPIC EQUATIONS $\mathbf{v}^\epsilon \rightarrow \mathbf{v}$, $p^\epsilon \rightarrow p$ as $\epsilon \rightarrow 0$

Darcy's law $\mathbf{v} = -\frac{1}{\eta} \mathbf{k} \nabla p, \quad x \in \Omega$ $\mathbf{k}(x)$ = effective fluid permeability tensor

($\mathbf{f} = 0$) $\nabla \cdot \mathbf{v} = 0, \quad x \in \Omega$

Critical behavior of fluid transport in sea ice



critical brine volume fraction $\phi_c \approx 5\%$ \longleftrightarrow $T_c \approx -5^\circ \text{C}$, $S \approx 5 \text{ ppt}$

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

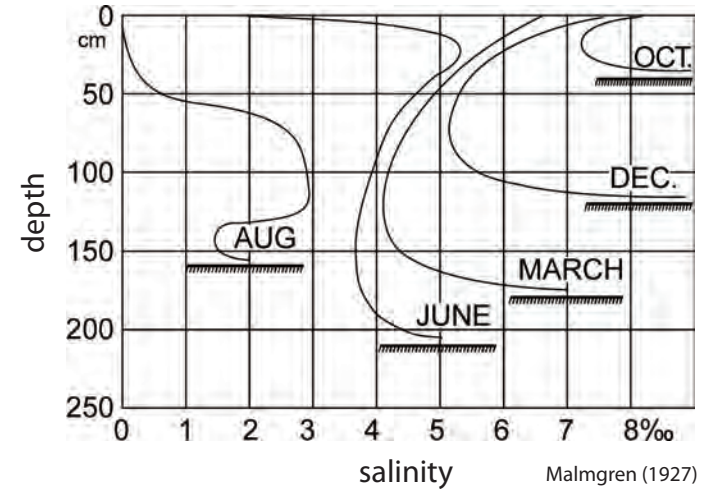
Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

rule of fives constrains:

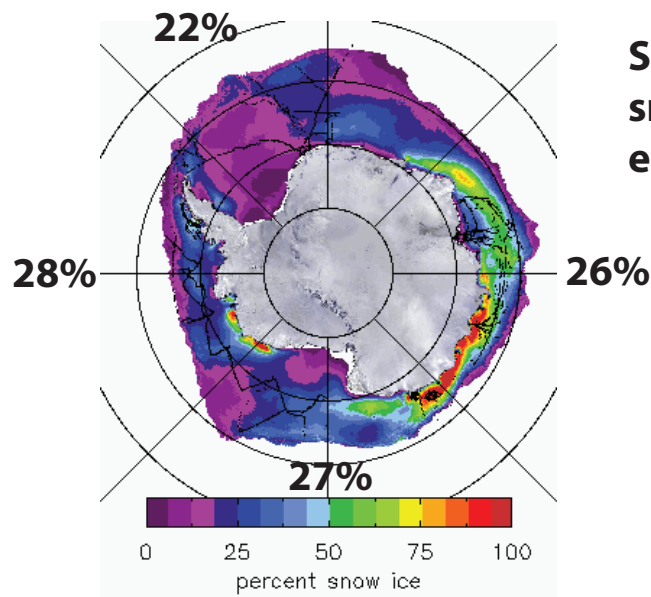
Antarctic surface flooding and snow-ice formation



evolution of salinity profiles



currently assumed constant in climate models



September
snow-ice
estimates

convection - enhanced thermal conductivity

Lytle and Ackley, 1996

Trodahl, et. al., 2000, 2001

Wang, Zhu, Golden, 2012



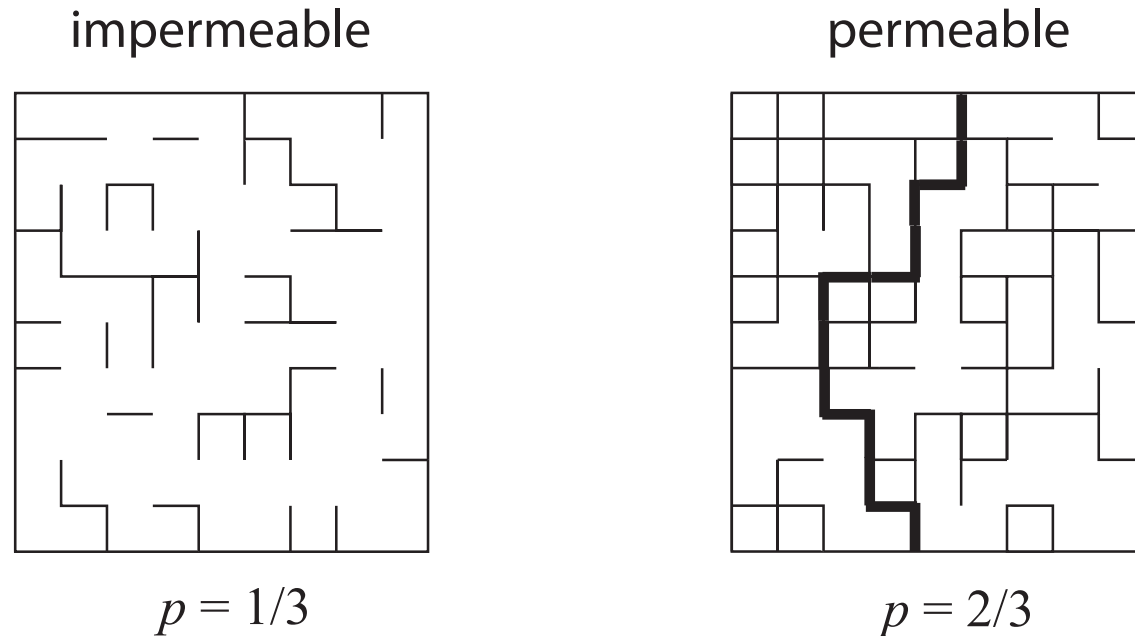
Antarctic snow-to-ice conversion from passive microwave imagery

T. Maksym and T. Markus, 2008

Why is the rule of fives true?

percolation theory

mathematical theory of connectedness



bond \longrightarrow *open* with probability p
closed with probability $1-p$

percolation threshold

$$p_c = 1/2 \quad \text{for } d = 2$$

first appearance of infinite cluster

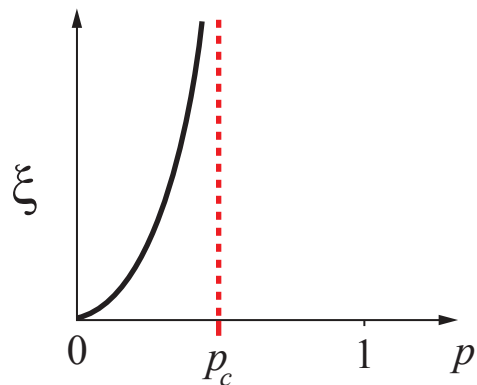
“tipping point” for connectivity

order parameters in percolation theory

geometry

correlation length

characteristic scale
of connectedness

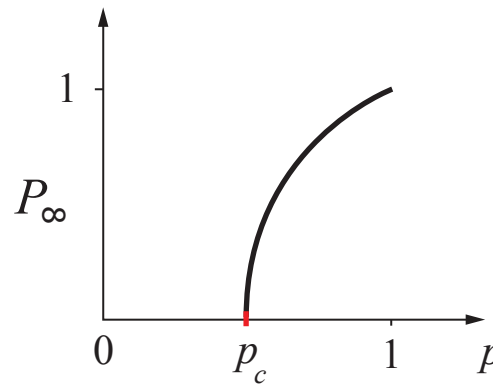


$$\xi(p) \sim |p - p_c|^{-\nu}$$

$$p \rightarrow p_c$$

infinite cluster density

probability the origin
belongs to infinite cluster

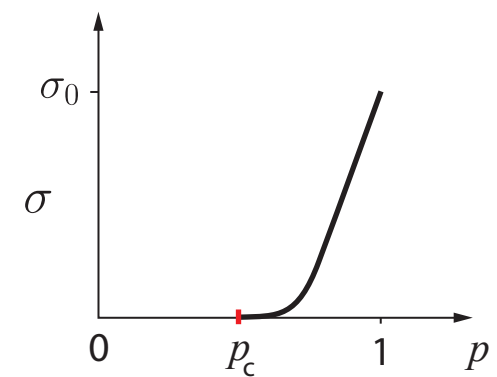


$$P_\infty(p) \sim (p - p_c)^\beta$$

$$p \rightarrow p_c^+$$

transport

effective conductivity
or fluid permeability



$$\sigma(p) \sim \sigma_0 (p - p_c)^t$$

$$p \rightarrow p_c^+$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

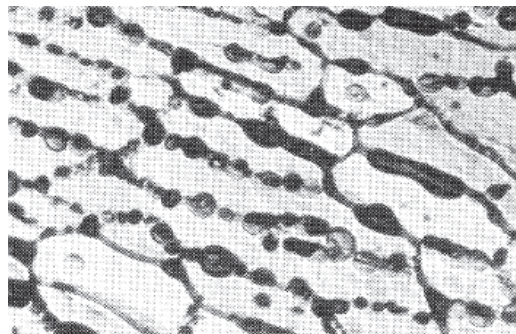
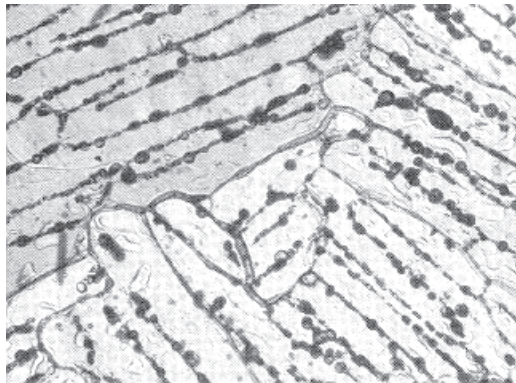
($1 \leq t \leq 2$, Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992)

non-universal behavior in continuum

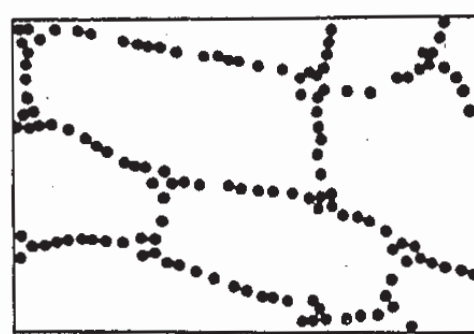
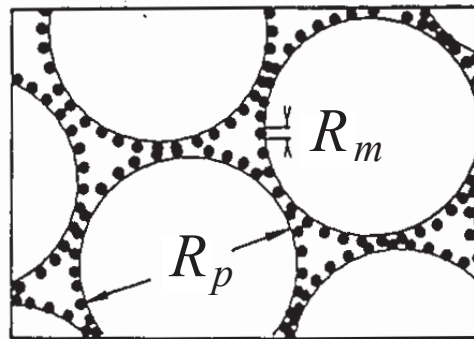
Continuum percolation model for **stealthy** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on **ice production** and **algal growth**

$$\phi_c \approx 5 \%$$

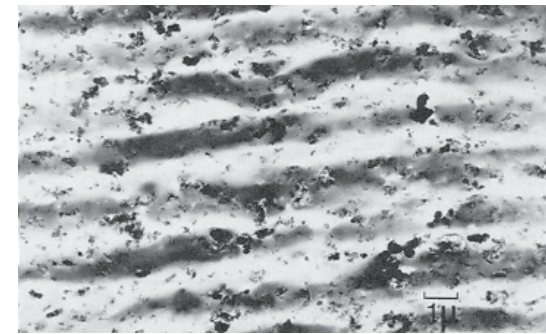
Golden, Ackley, Lytle, *Science*, 1998



sea ice



compressed
powder



radar absorbing
composite

sea ice is radar absorbing



***rigorous bounds
percolation theory
hierarchical model
network model***

field data

X-ray tomography for
brine inclusions

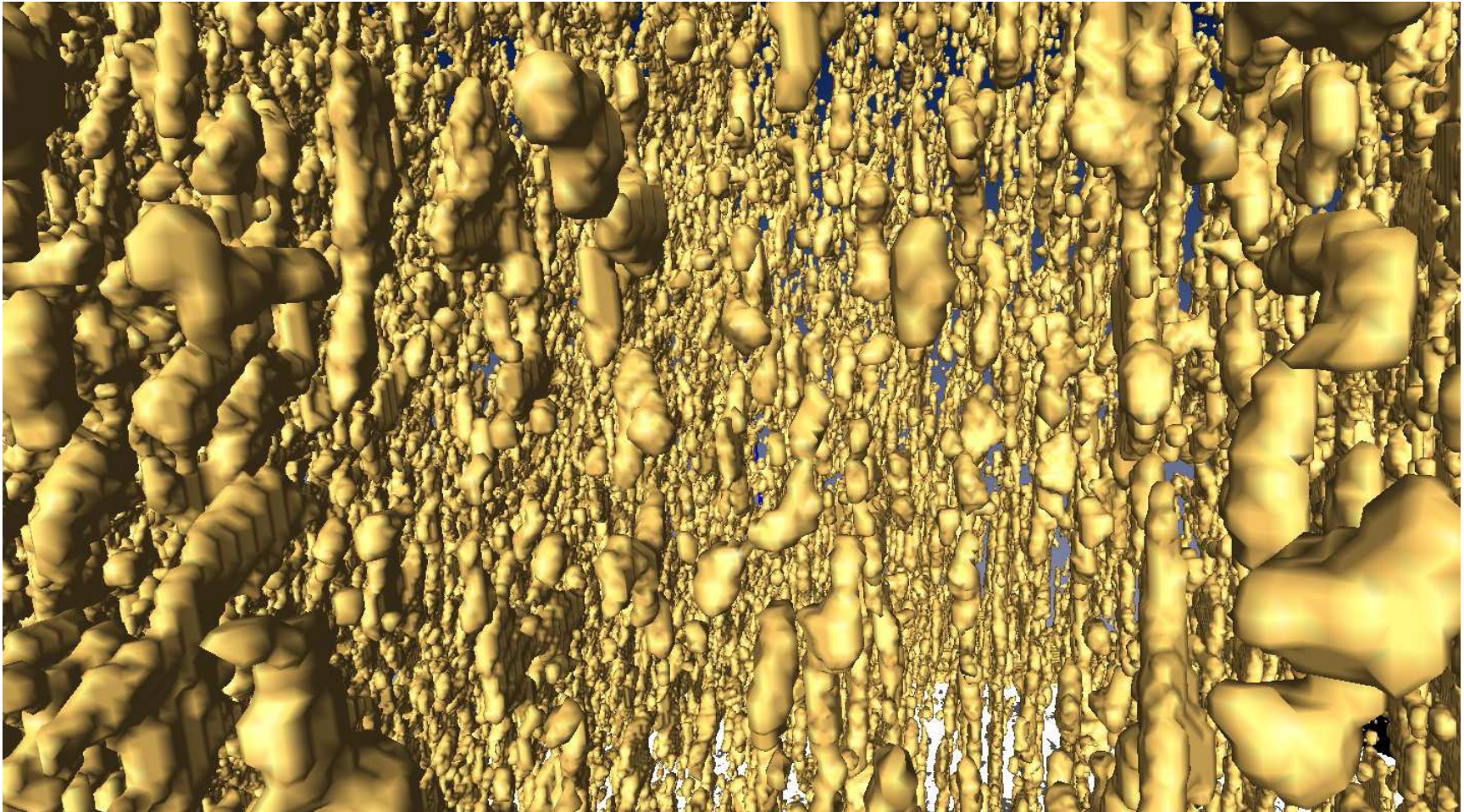
***unprecedented look
at thermal evolution
of brine phase and
its connectivity***

A unified approach to understanding permeability in sea ice • Solving the mystery of
booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

micro-scale
controls
macro-scale
processes

X-ray computed tomography of brine inclusions in sea ice

~ 1 cm across

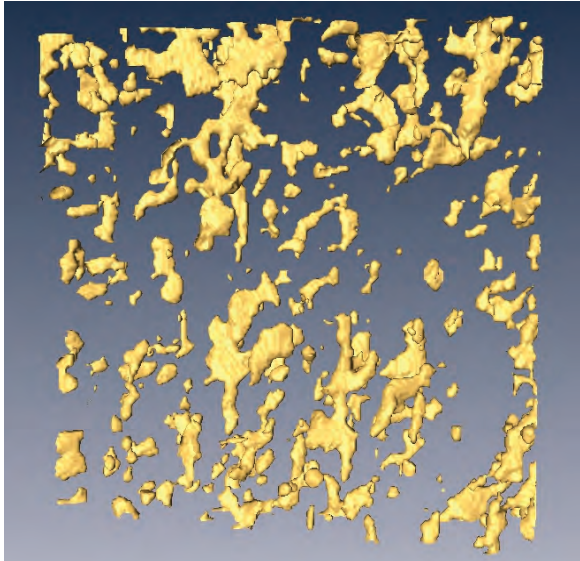


brine volume fraction $\phi = 5.7 \%$ $T = -8^{\circ}\text{C}$

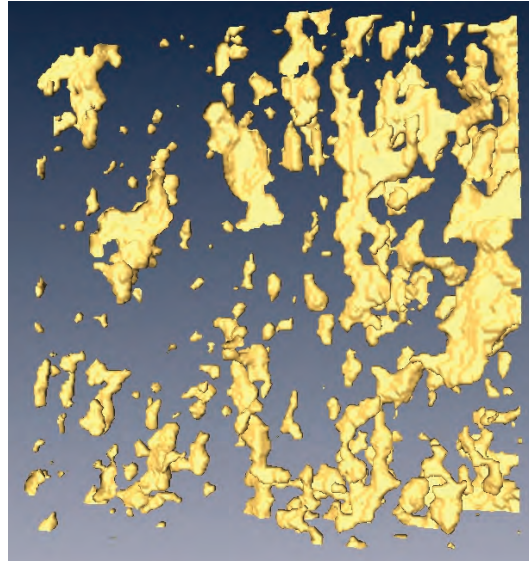
Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

brine connectivity (over cm scale)

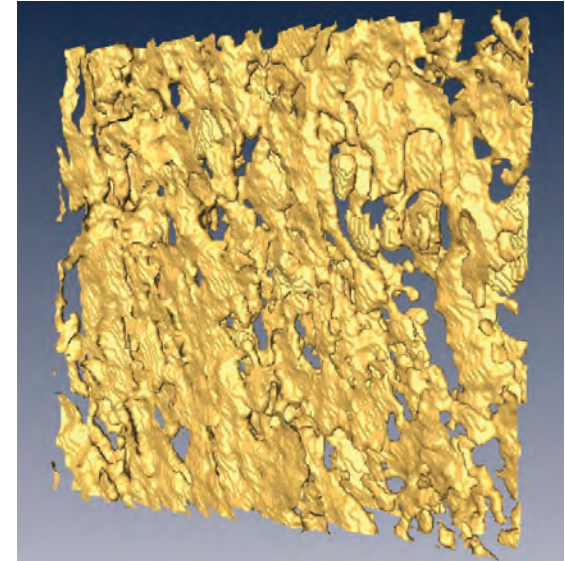
8 x 8 x 2 mm



-15 °C, $\phi = 0.033$



-6 °C, $\phi = 0.075$



-3 °C, $\phi = 0.143$

X-ray tomography confirms percolation threshold

3-D images
pores and throats



3-D graph
nodes and edges

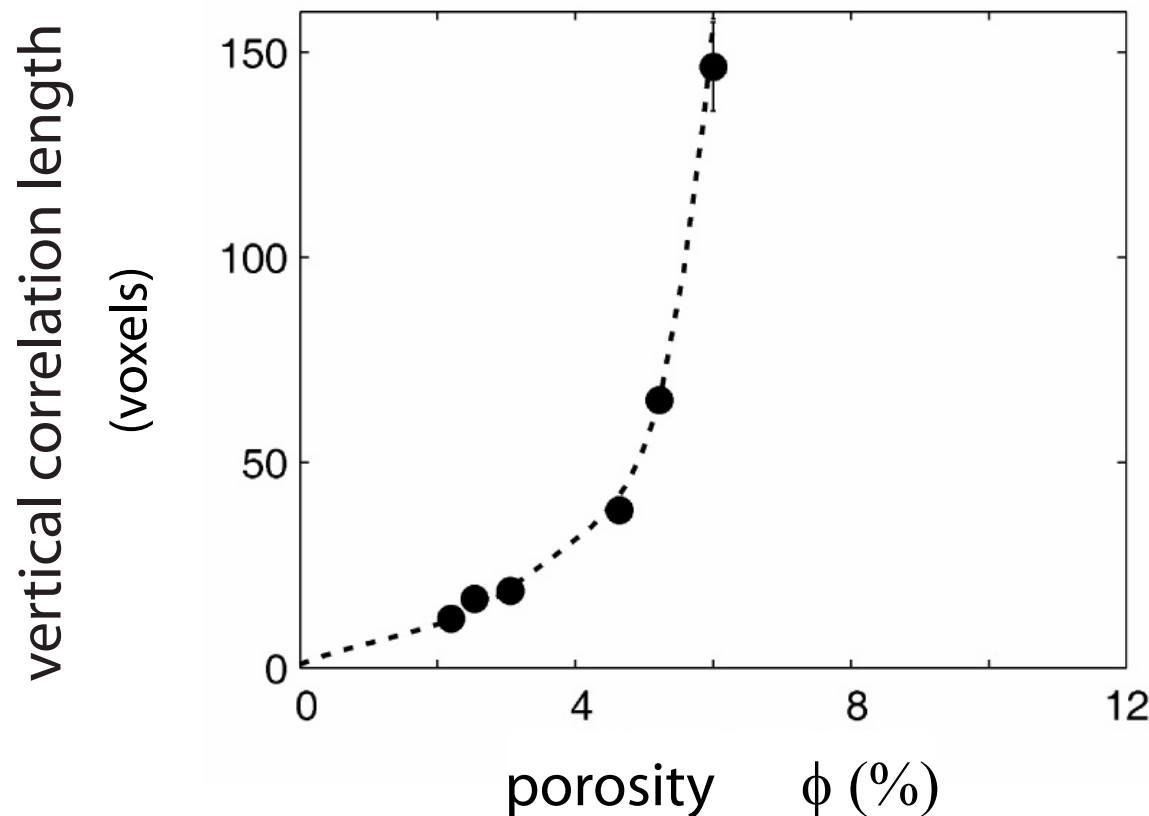
analyze graph connectivity as function of temperature and sample size

- ***use finite size scaling techniques to confirm rule of fives***
- ***order parameter data from a natural material***

The key connectivity functions of percolation theory have been computed **extensively** for many lattice models, but NOT for natural materials.

We have calculated them for sea ice single crystals -- a first for percolation investigations -- and estimated anisotropic percolation thresholds.

Pringle, Miner, Eicken, Golden, JGR (Oceans) 2009



divergence of
correlation length
for single crystal data

use finite size scaling

$$P_{\infty}(p_c, L) \sim L^{-\beta/\nu} \quad (= L^{-0.48})$$

anisotropic percolation threshold estimates

vertical

$$\phi_c = 4.6 \pm 0.7 \%$$

horizontal { parallel to layers

$$\phi_c = 9 \pm 2 \%$$

perpendicular to layers

$$\phi_c = 14 \pm 4 \%$$

lattice and continuum percolation theories yield:

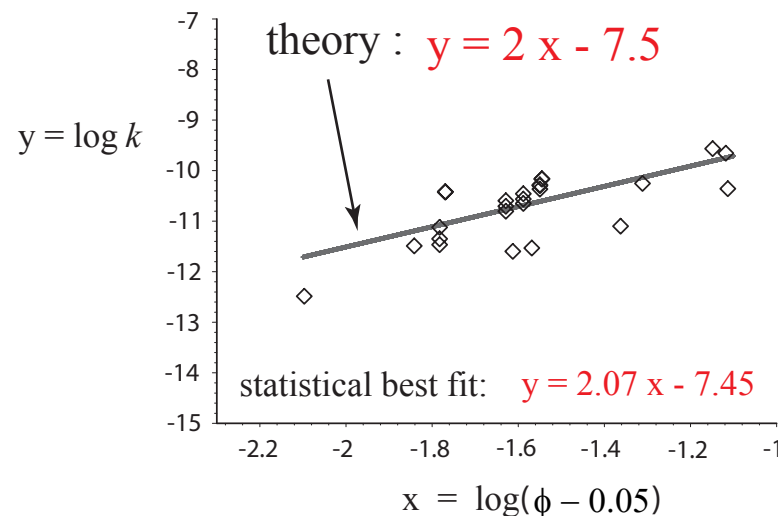
$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical
exponent

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

t

- exponent is **UNIVERSAL** lattice value $t \approx 2.0$
- **sedimentary rocks** like sandstones also exhibit universality
- **critical path analysis** -- developed for electronic hopping conduction -- yields scaling factor k_0



Arctic and Antarctic field experiments

*develop electromagnetic methods
of monitoring fluid transport and
microstructural transitions*

extensive measurements of fluid and
electrical transport properties of sea ice:

2007 Antarctic SIPEX

2010 Antarctic McMurdo Sound

2011 Arctic Barrow AK

2012 Arctic Barrow AK

2012 Antarctic SIPEX II

2013 Arctic Barrow AK



Notices

of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and
the Mathematics of
Transport in Sea Ice

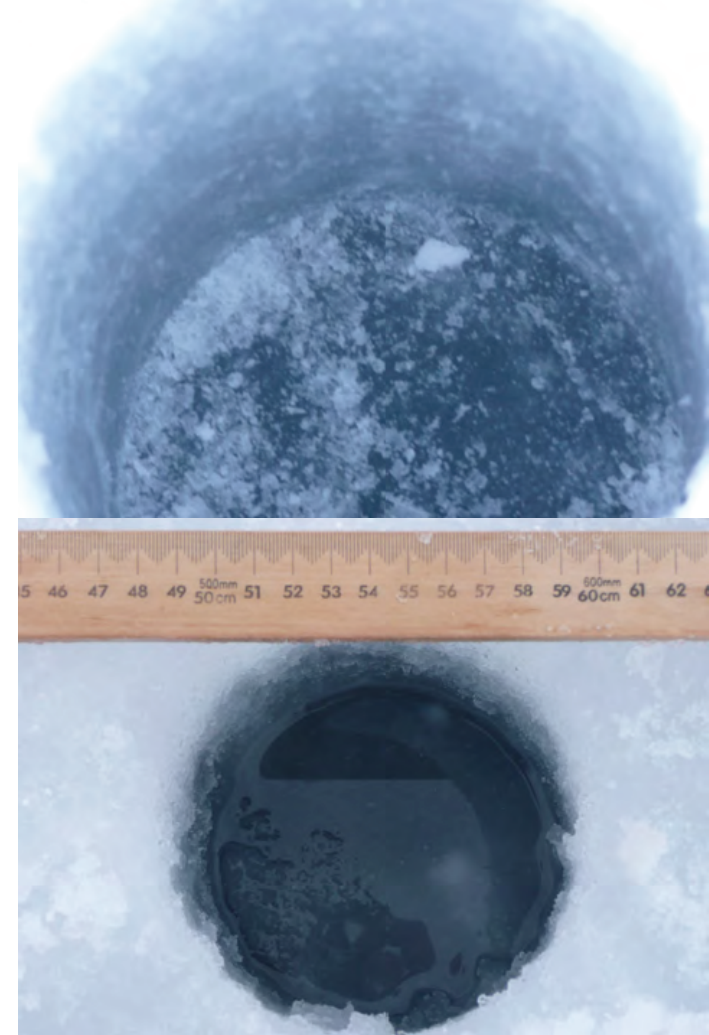
page 562

Mathematics and the
Internet: A Source of
Enormous Confusion
and Great Potential

page 586

photo by Jan Lieser

Real analysis in polar coordinates (see page 613)



***measuring
fluid permeability
of Antarctic sea ice***

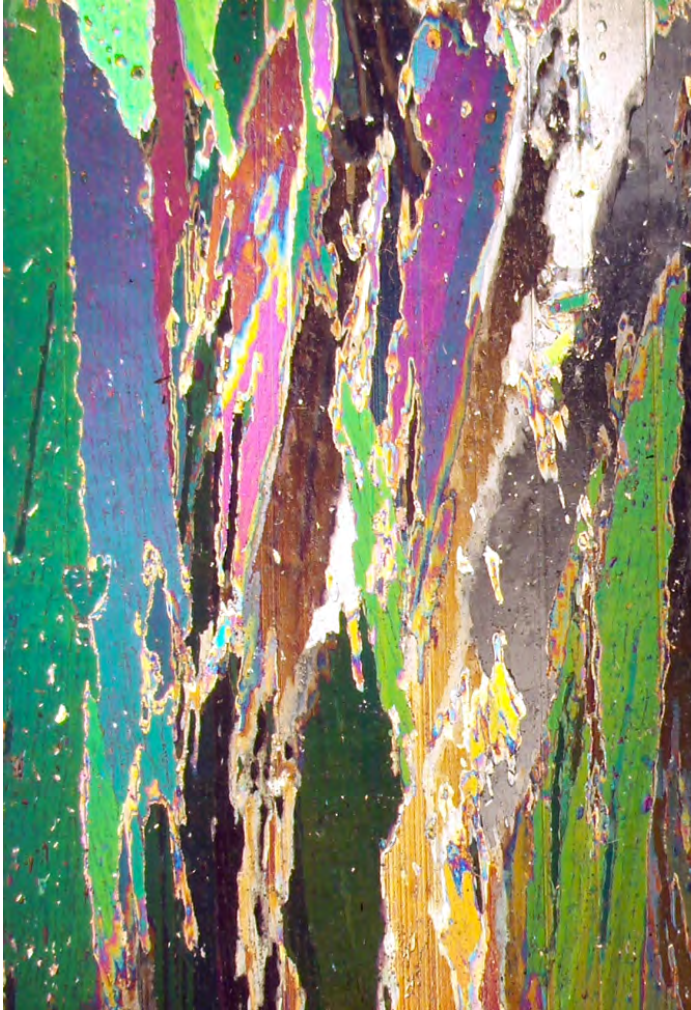
SIPEX 2007

higher threshold for fluid flow in Antarctic granular sea ice

columnar

granular

5%

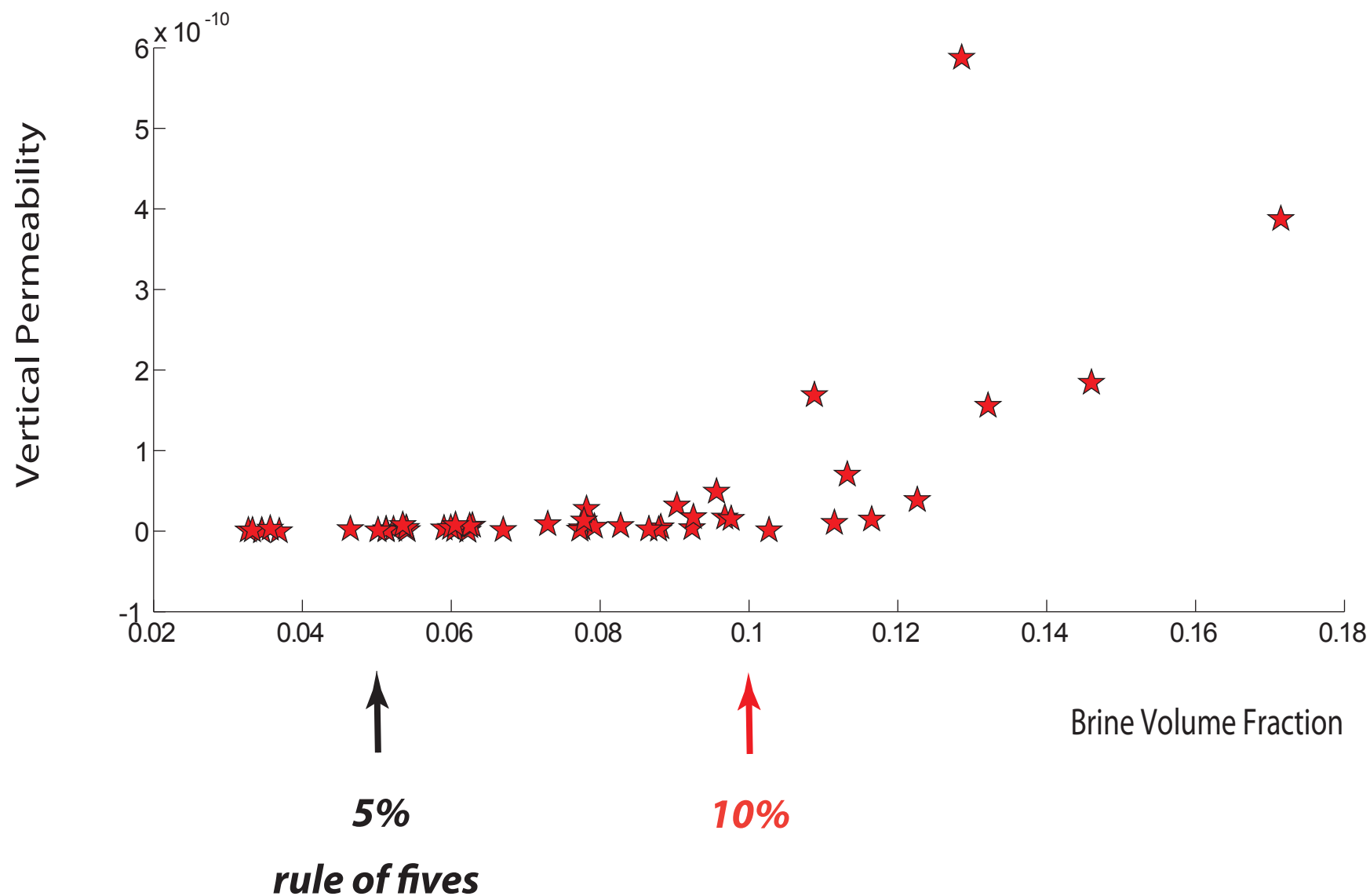


10%



different microstructure -- different threshold

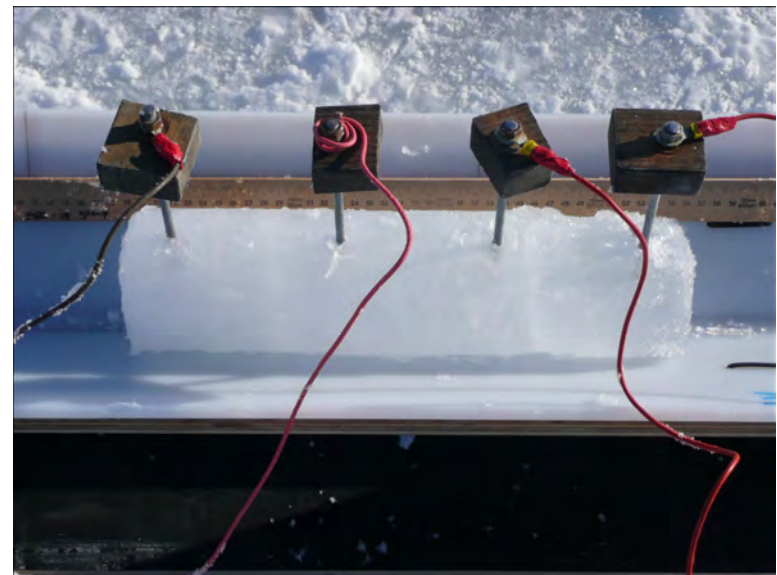
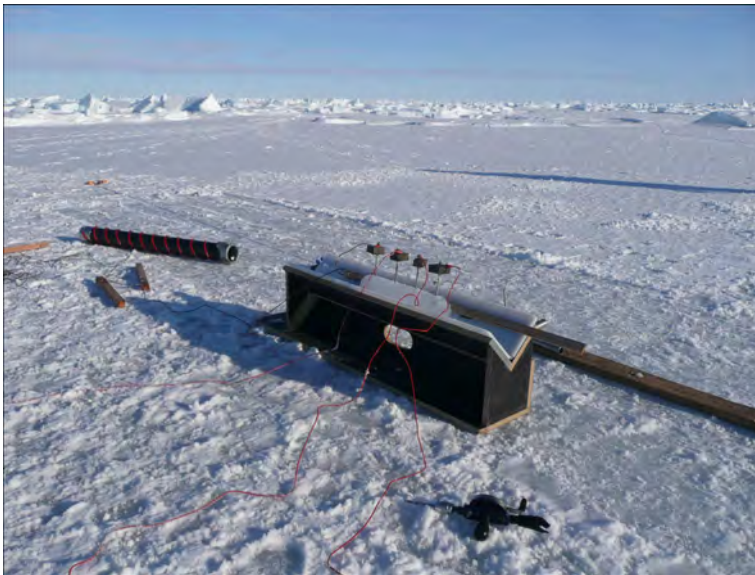
granular ice common in Arctic surface layer



electrical measurements



Wenner array



vertical conductivity

Zhu, Golden, Gully, Sampson *Physica B* 2010

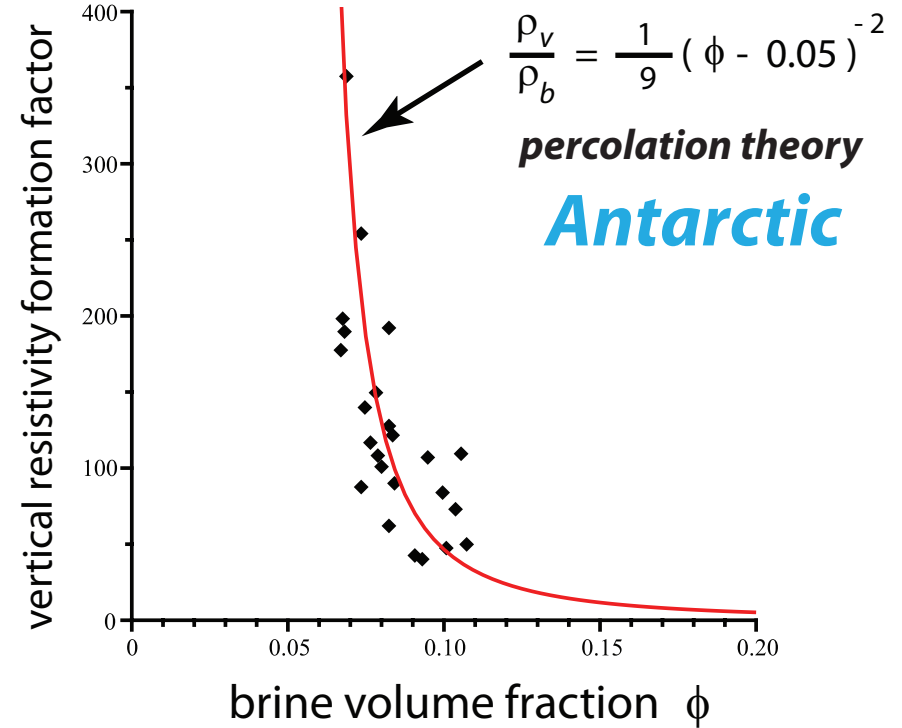
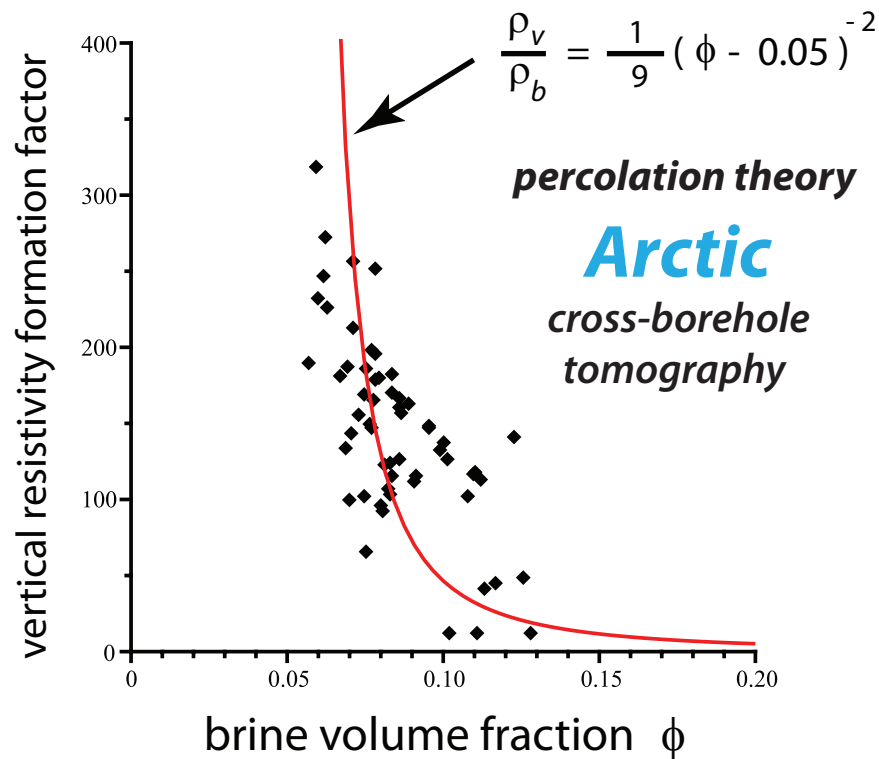
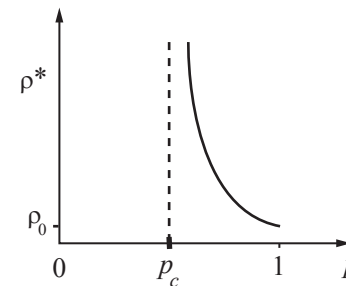
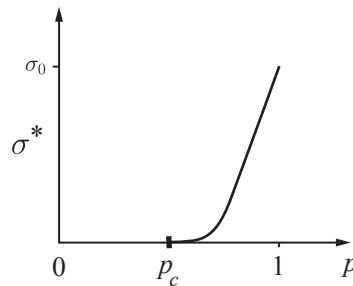
Sampson, Golden, Gully, Worby *Deep Sea Research* 2011

critical behavior of electrical transport in sea ice

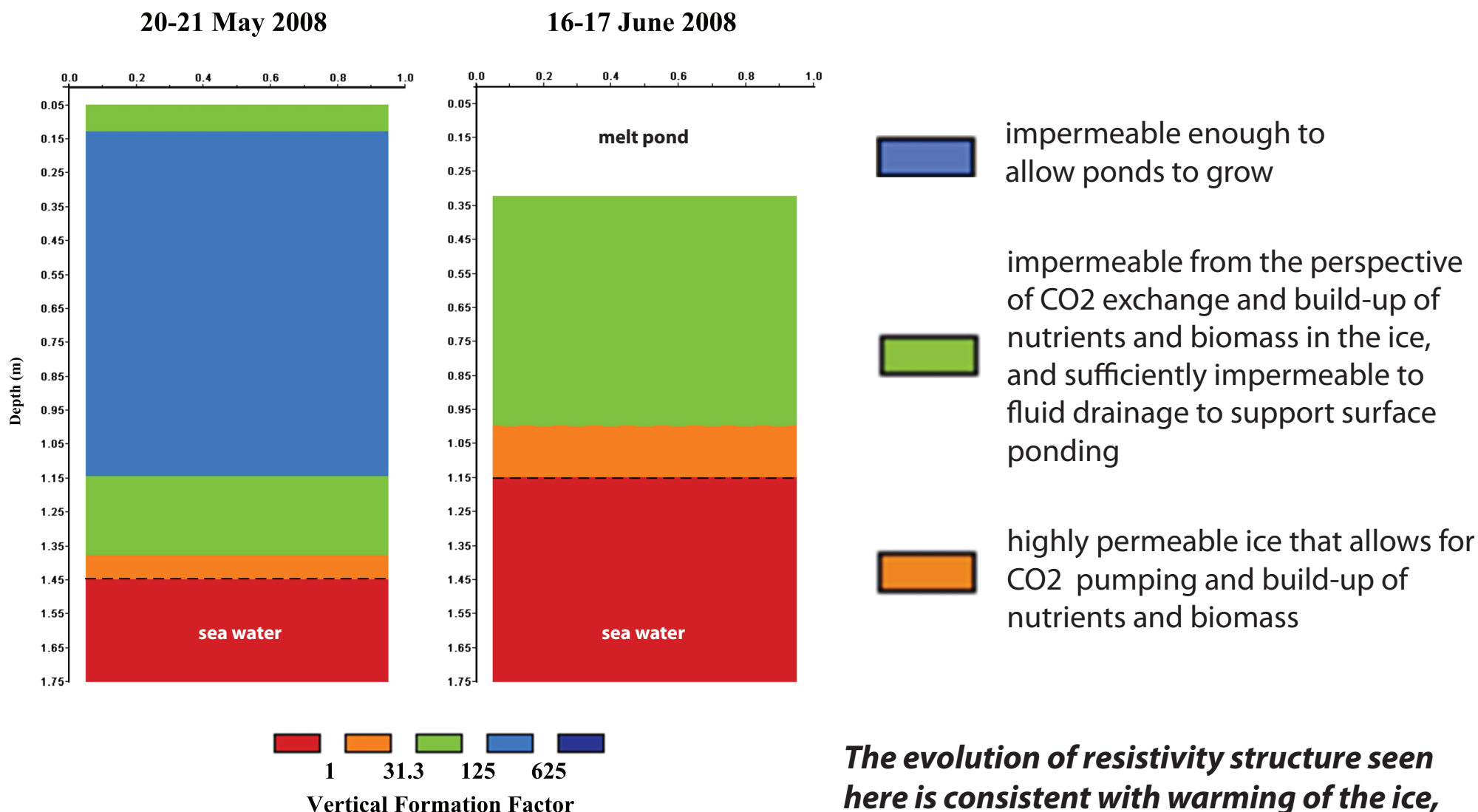
electrical signature of the on-off switch for fluid flow

same universal critical exponent as for fluid permeability

studied for over 50 years but no previous observations or theory of critical behavior

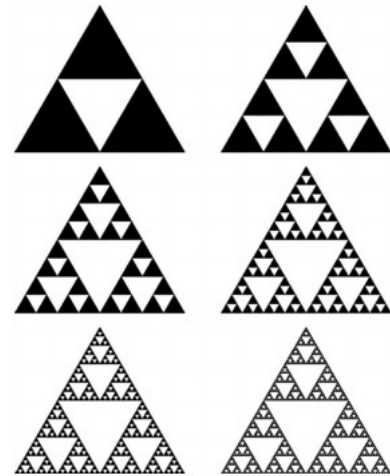


Cross-borehole tomographic reconstructions of the vertical resistivity formation factor for Arctic sea ice before and after melt pond formation



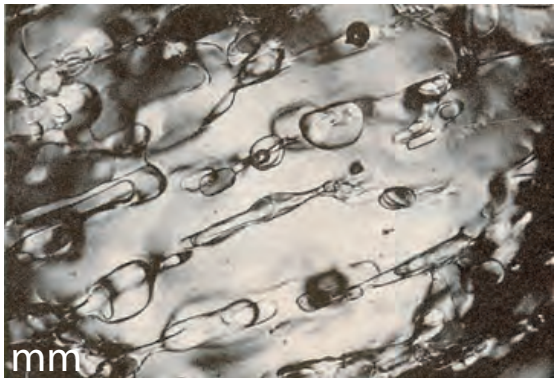
The evolution of resistivity structure seen here is consistent with warming of the ice, thus increasing the fluid permeability.

fractals and multiscale structure



sea ice displays *multiscale* structure over 10 orders of magnitude

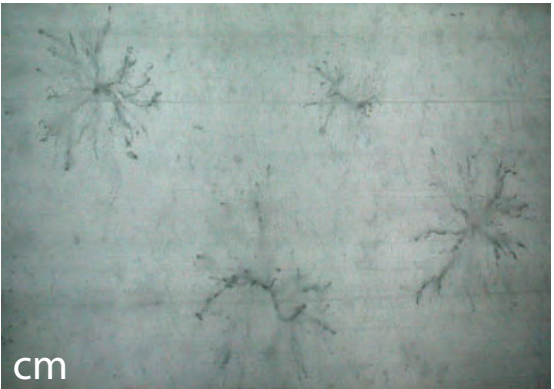
0.1 millimeter



brine inclusions



polycrystals



horizontal

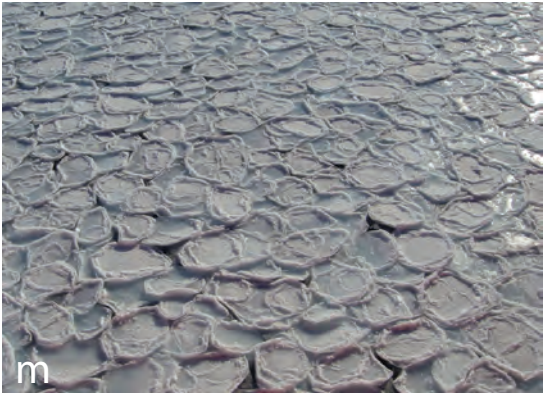


brine channels



vertical

1 meter

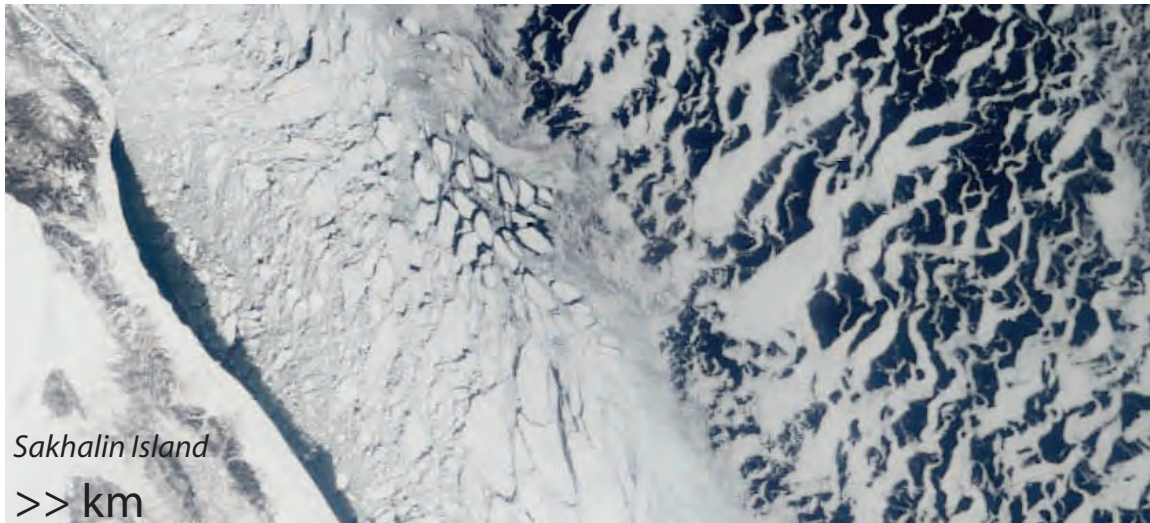


pancake ice

1 meter



100 kilometers



melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

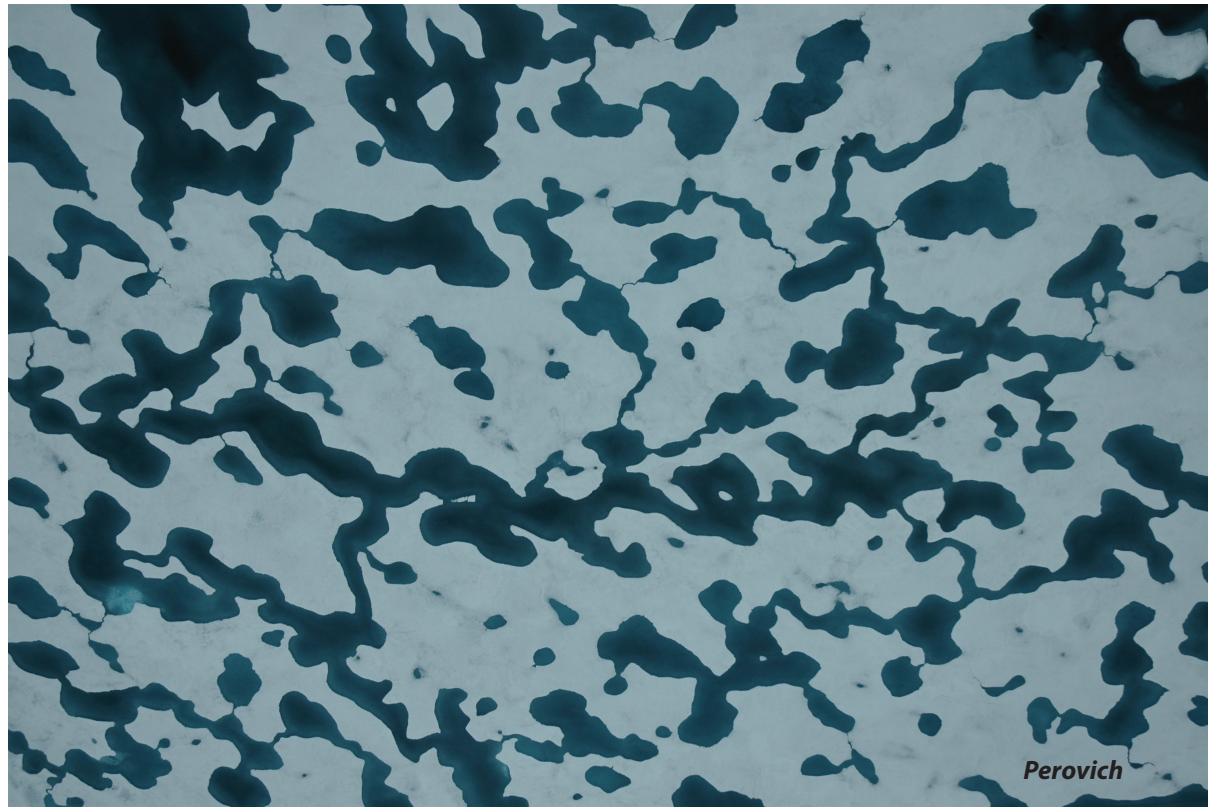
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham,
Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson,
Perovich 2009

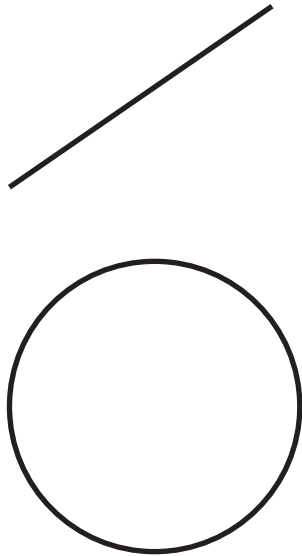
Flocco, Feltham,
Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

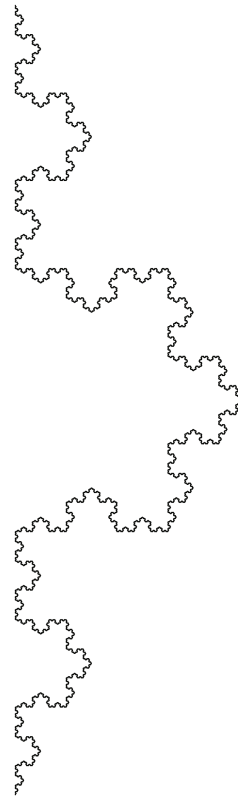
fractal curves in the plane

they wiggle so much that their dimension is >1



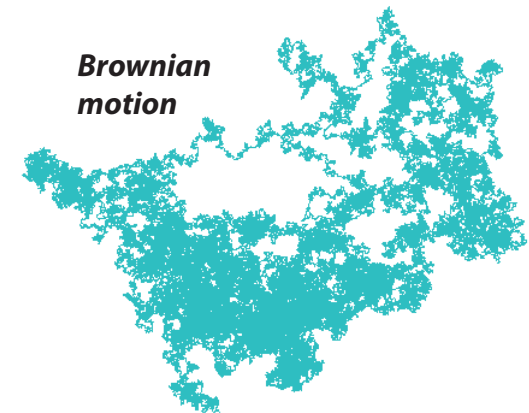
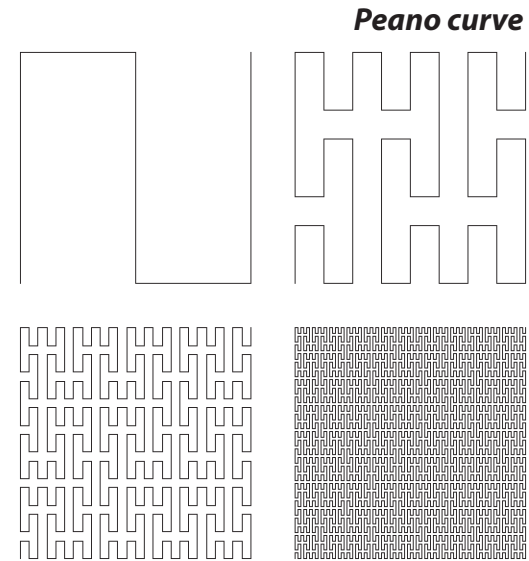
simple curves

$$D = 1$$



Koch snowflake

$$D = 1.26$$



space filling curves

$$D = 2$$

clouds exhibit fractal behavior from 1 to 1000 km

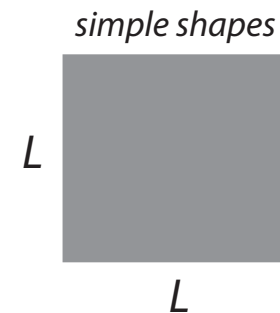
use **perimeter-area** data to find that
cloud and rain boundaries are fractals

$$D \approx 1.35$$

S. Lovejoy, Science, 1982



$$P \sim \sqrt{A}$$



$$A = L^2$$
$$P = 4L = 4\sqrt{A}$$

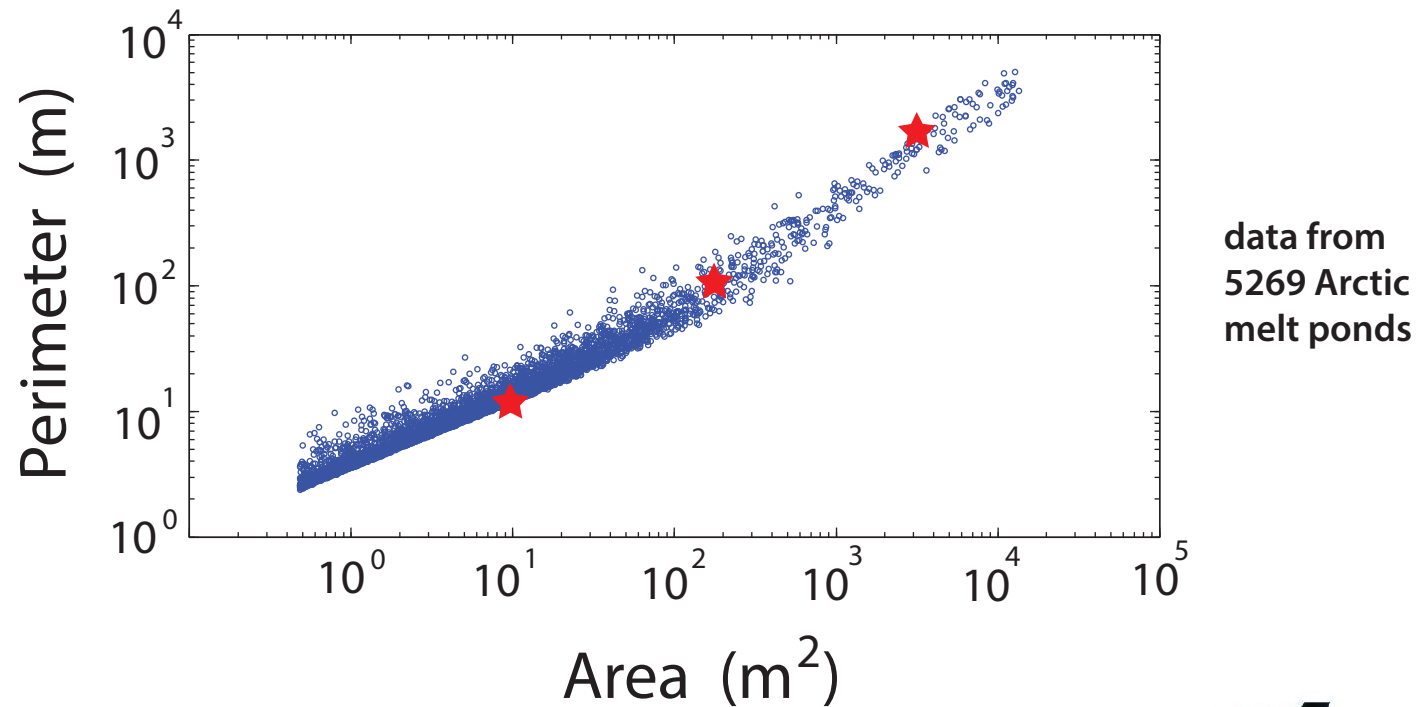
$$P \sim \sqrt{A}^D$$



for fractals with
dimension D

$D = 1.52...$

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden



~ 30 m



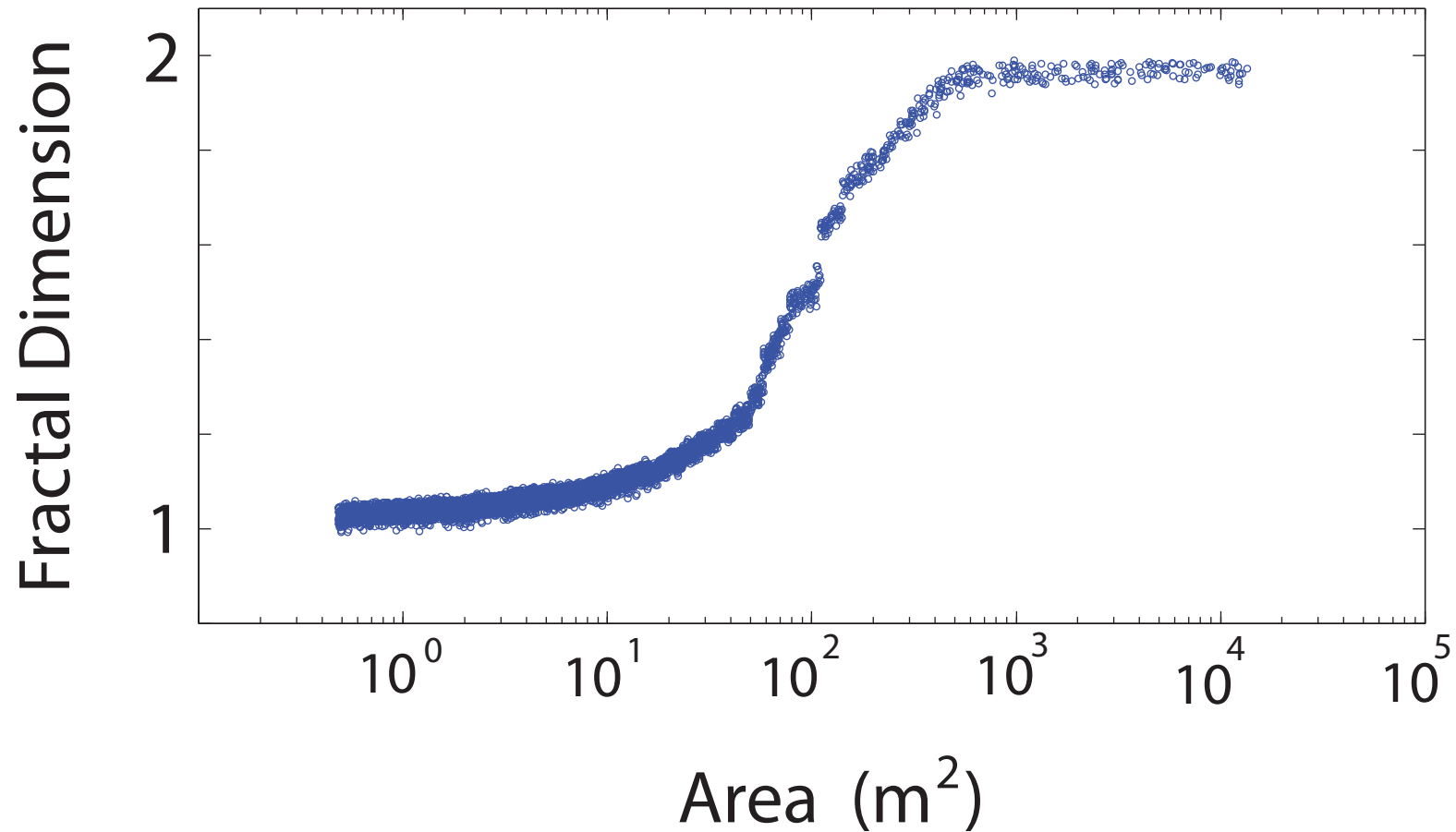
simple pond

transitional pond

complex pond

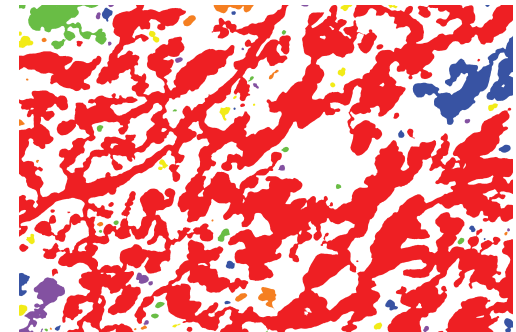
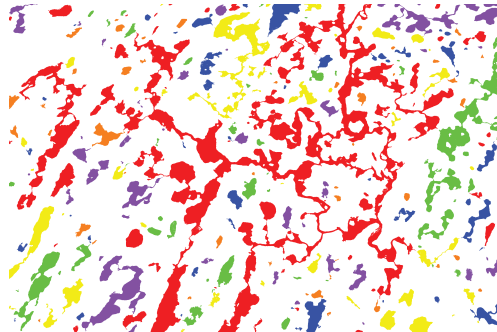
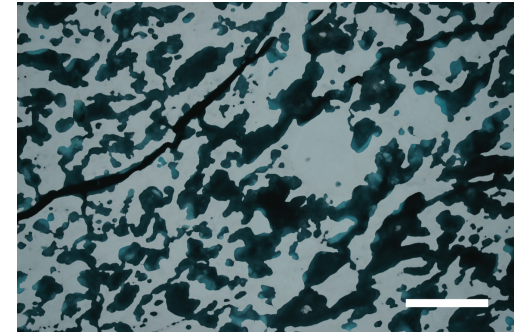
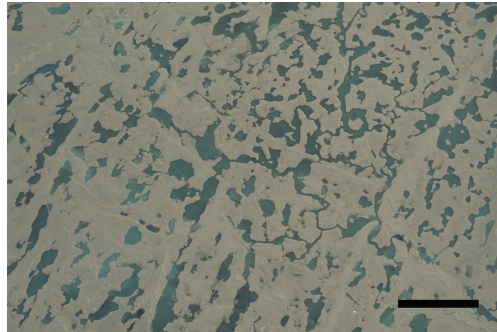
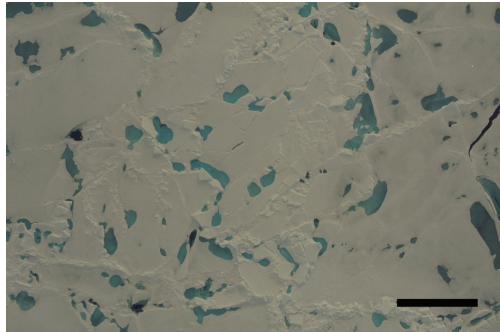
transition in the fractal dimension

complexity grows with length scale



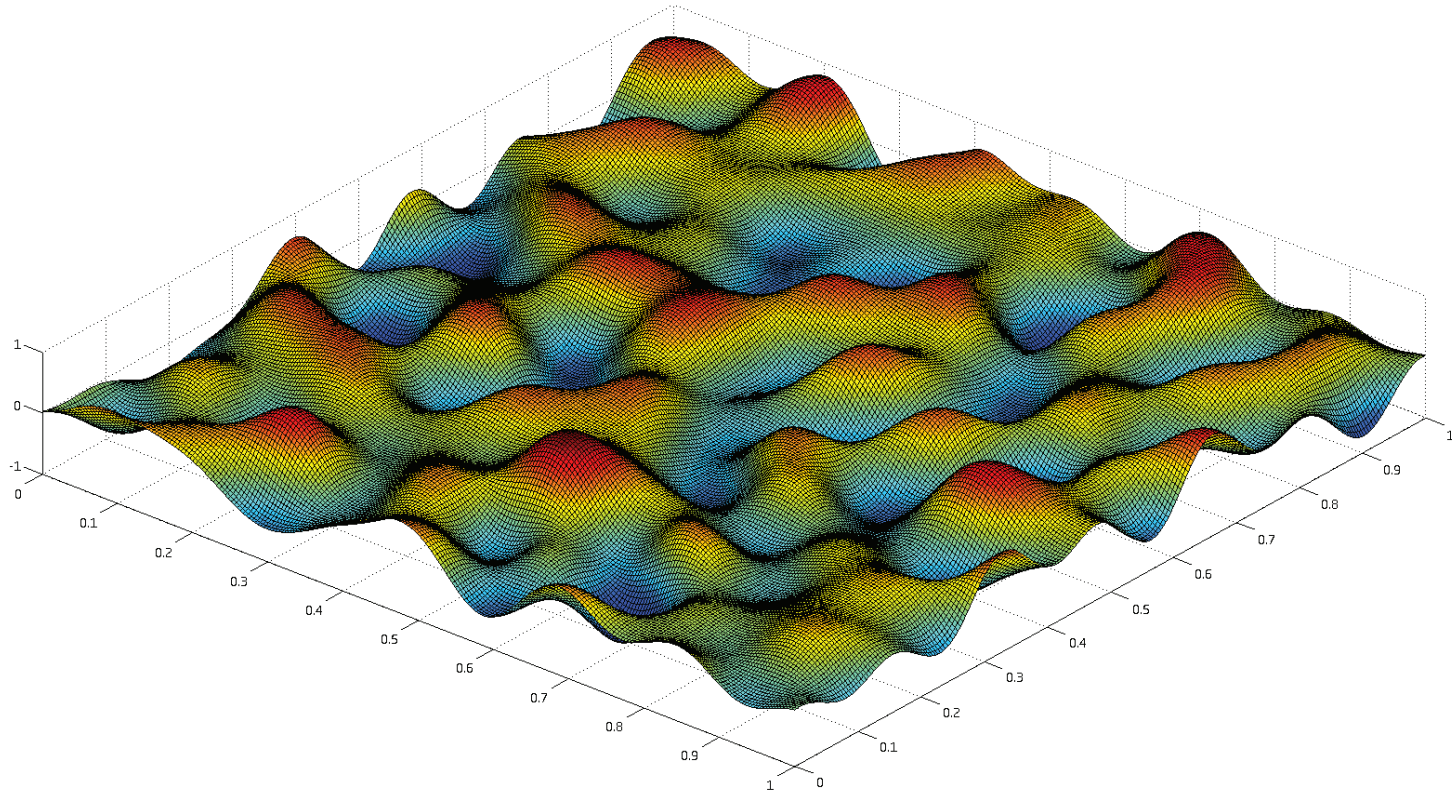
compute “derivative” of area - perimeter data

***small simple ponds coalesce to form
large connected structures with complex boundaries***



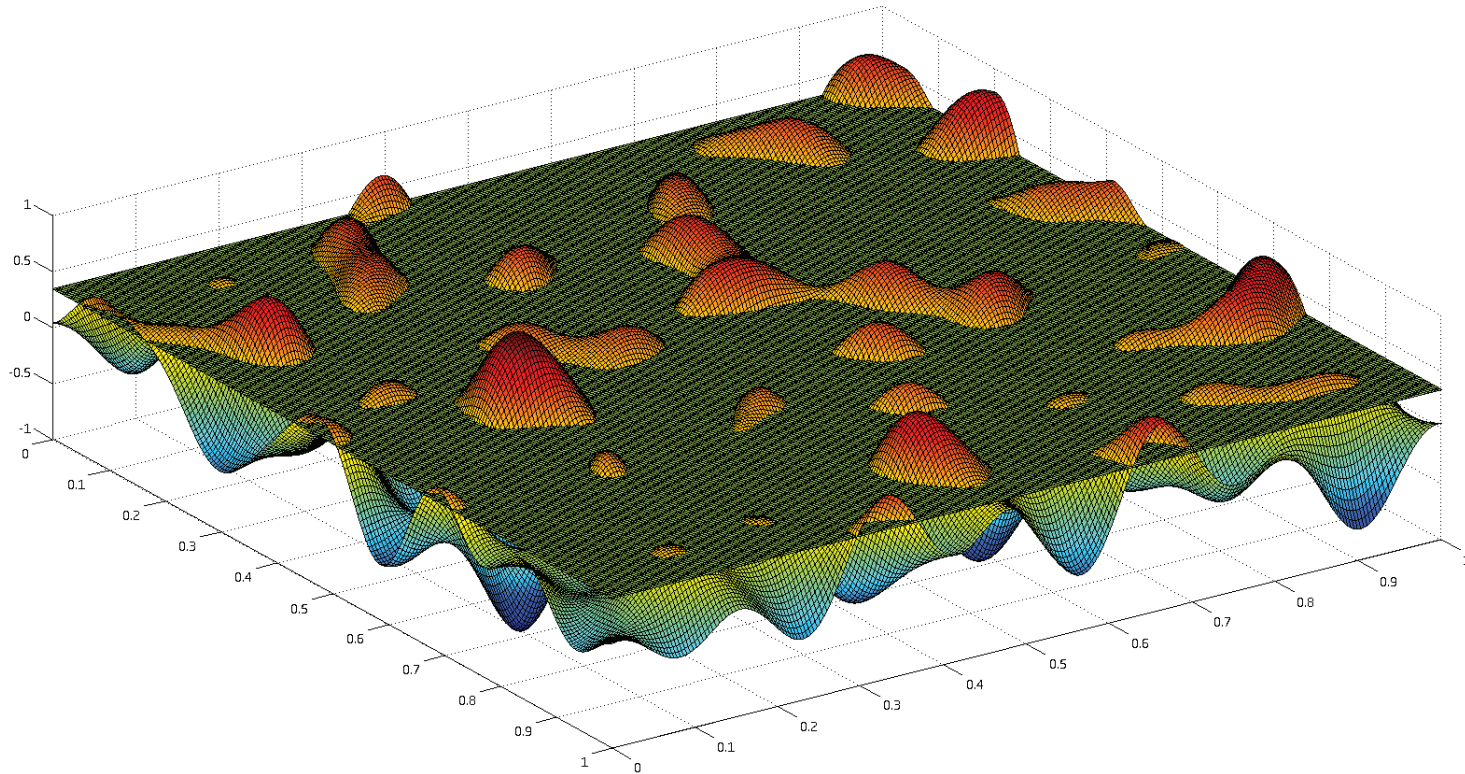
melt pond percolation

Continuum percolation model for melt pond evolution

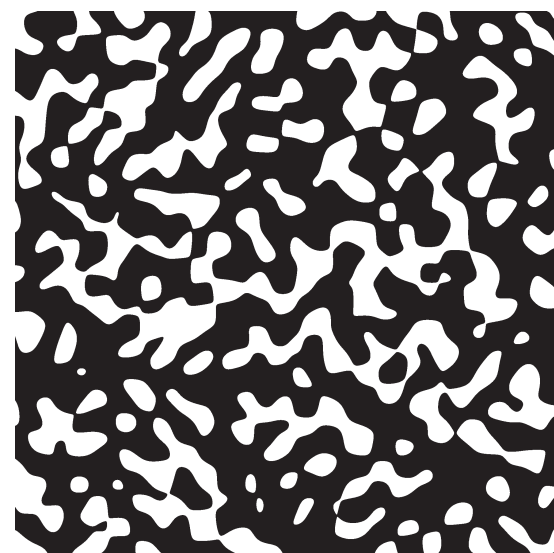
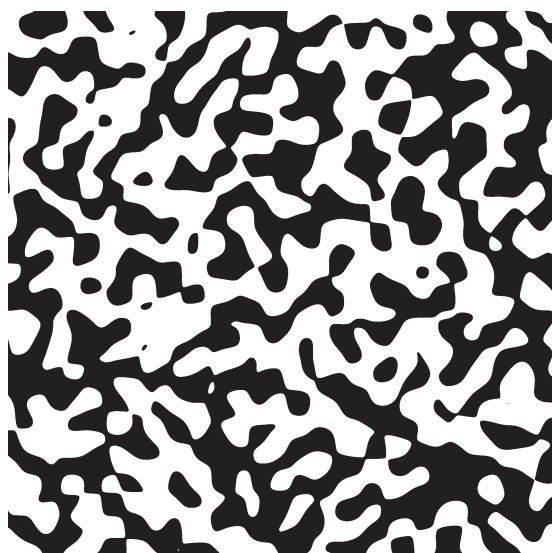
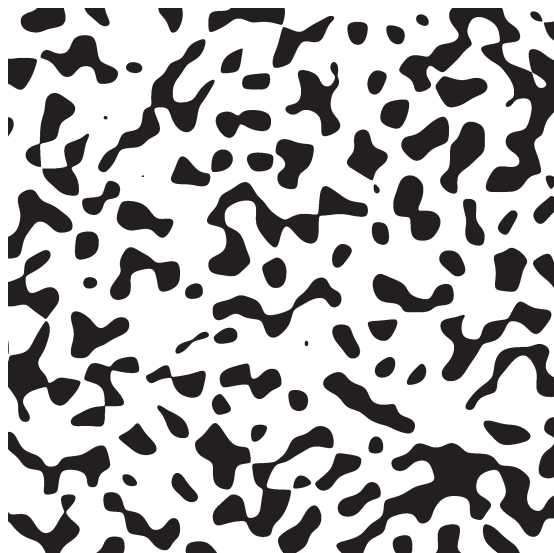
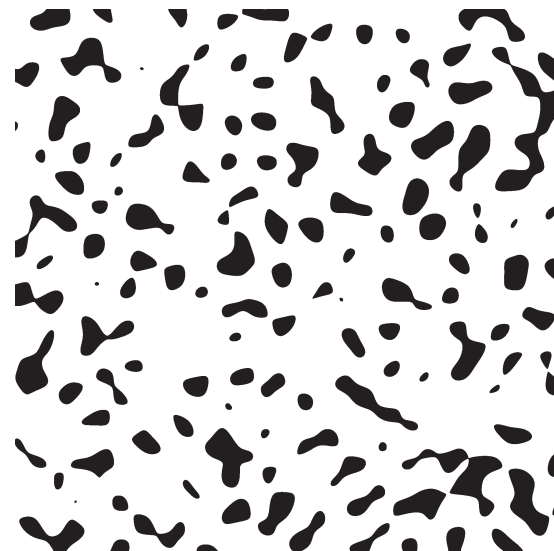
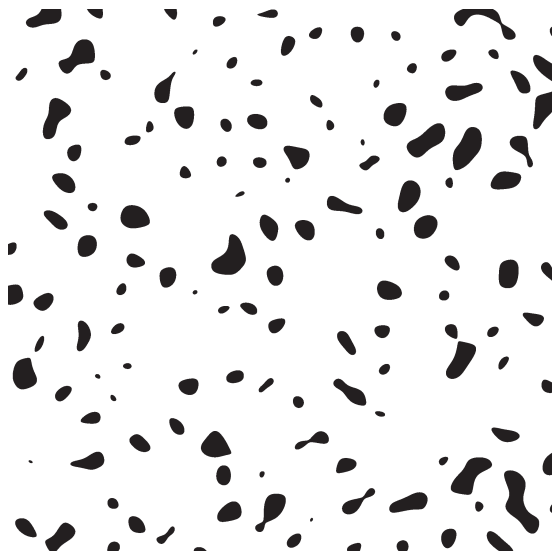
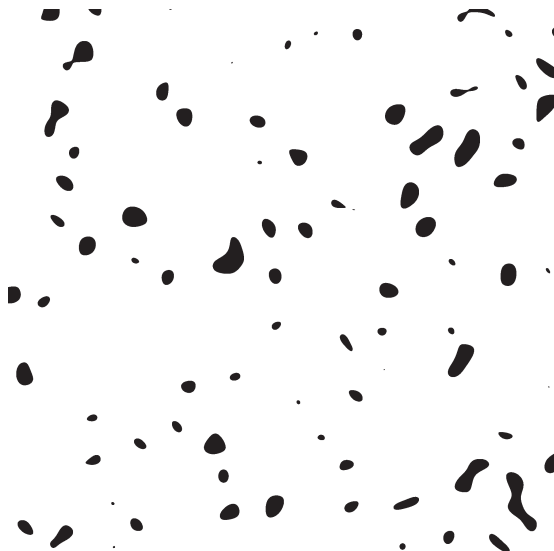


random Fourier surface

intersections of a plane with the surface define melt ponds



as the plane varies in height the regions evolve like melt ponds
at a critical height h_c ponds **percolate** and form an infinite ocean



h_c

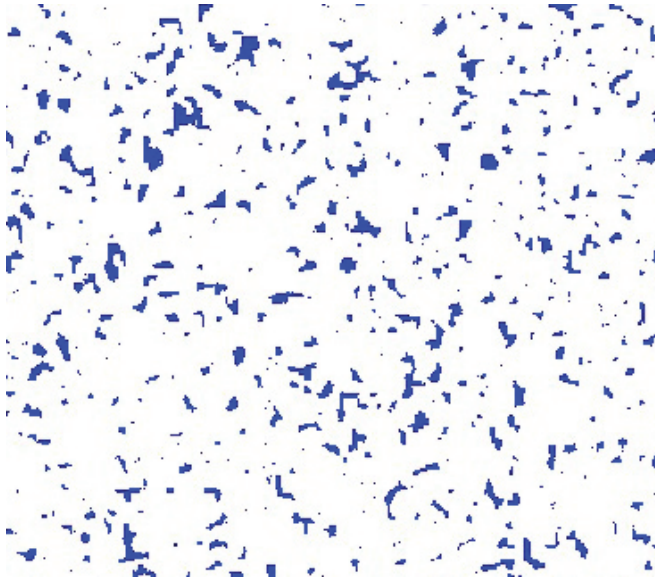
percolation threshold

Ising model for ferromagnets

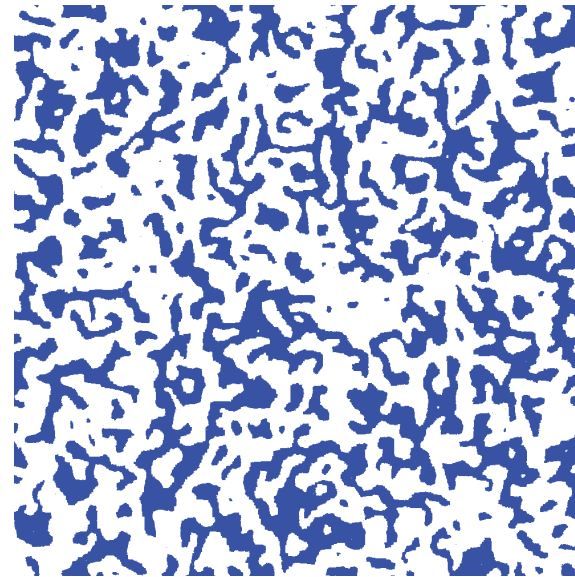


Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{\langle i,j \rangle}^N s_i s_j - H \sum_i^N s_i \quad s_i = \begin{cases} \uparrow & +1 & \text{ice} \\ \downarrow & -1 & \text{water} \end{cases} \quad M = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$

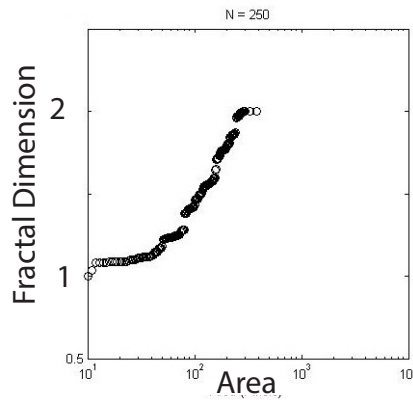


COLD



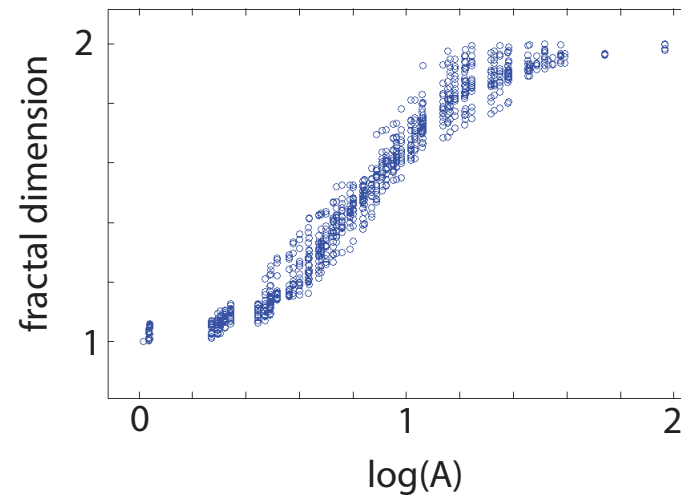
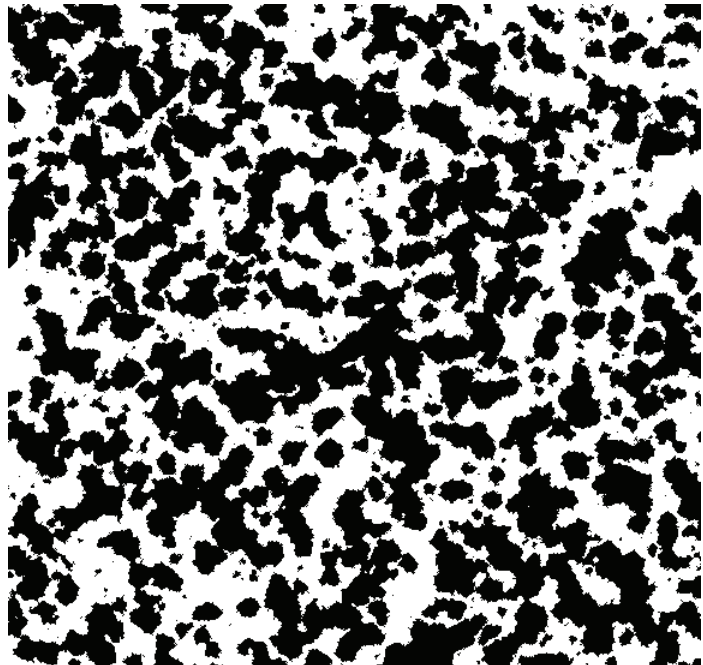
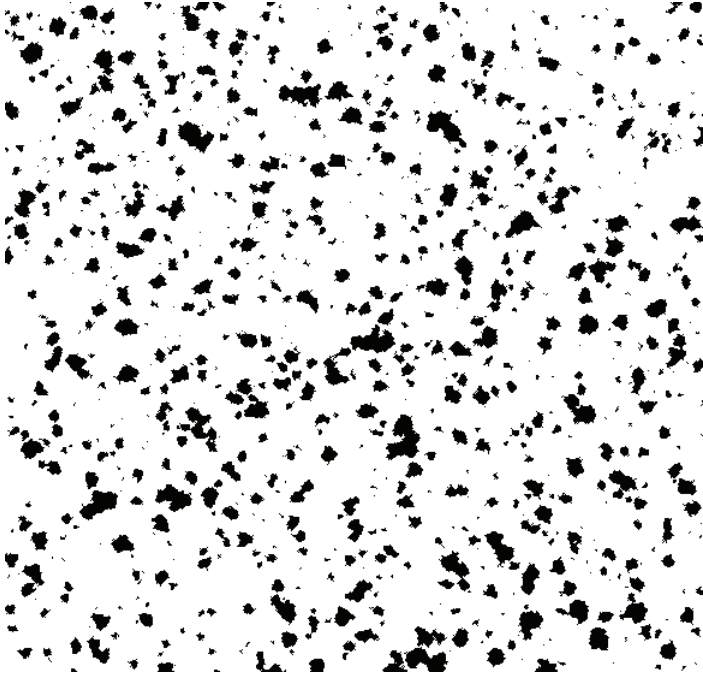
WARM

“melt ponds” are clusters of magnetic spins that align with the applied field

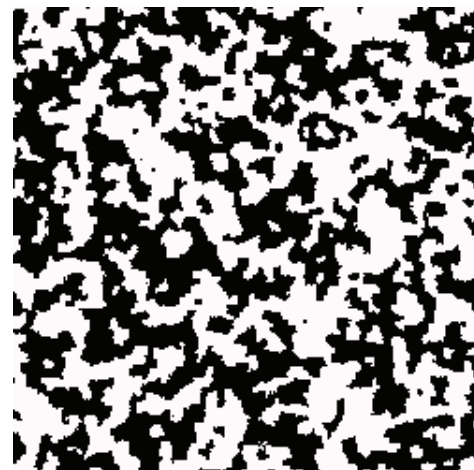


clusters exhibit transition in fractal dimension

simple stochastic growth model of melt pond evolution



a square is more likely to melt
if its neighbors have melted

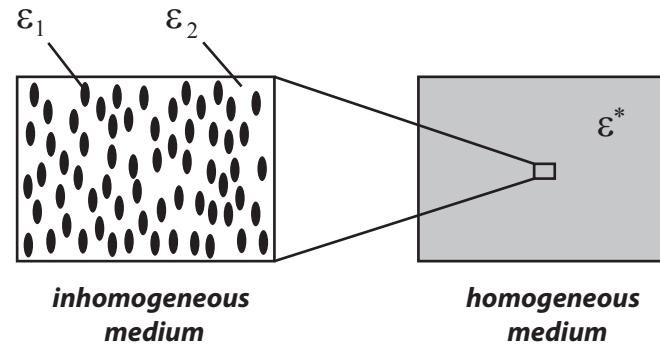


voter
model

multiscale homogenization

Theory of Effective Electromagnetic Behavior of Composites

analytic continuation method



Forward Homogenization Bergman (1978), Milton (1979), Golden and Papanicolaou (1983)

composite geometry
(spectral measure μ) \longrightarrow ϵ^*

integral representations, rigorous bounds, approximations, etc.

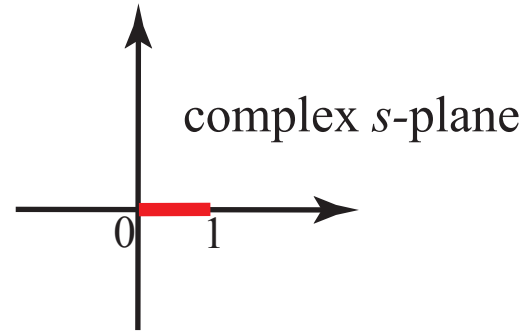
Inverse Homogenization Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001)
(McPhedran, McKenzie, and Milton, 1982)

ϵ^* \longrightarrow **composite geometry**
(spectral measure μ)

recover brine volume fraction, connectivity, etc.

Stieltjes integral representation

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$



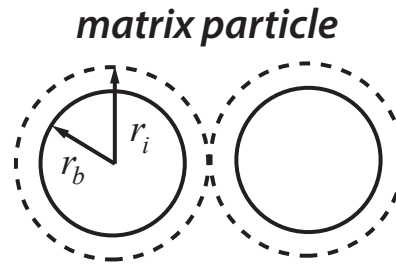
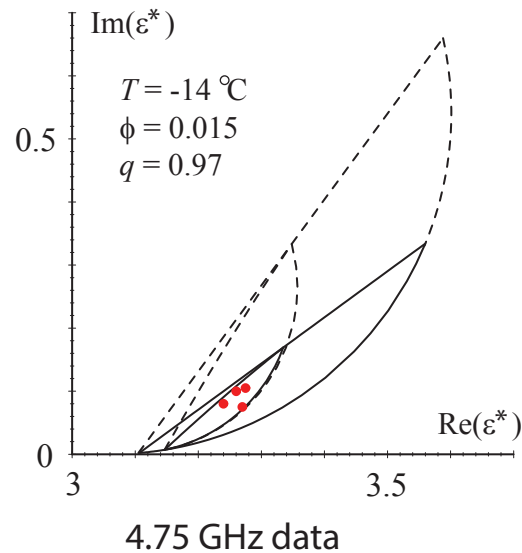
$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

- μ /
- spectral measure of self adjoint operator $\Gamma\chi$
 - mass = p_1
 - higher moments depend on n -point correlations

**separation of geometry
from parameters**

forward and inverse bounds for sea ice

forward bounds

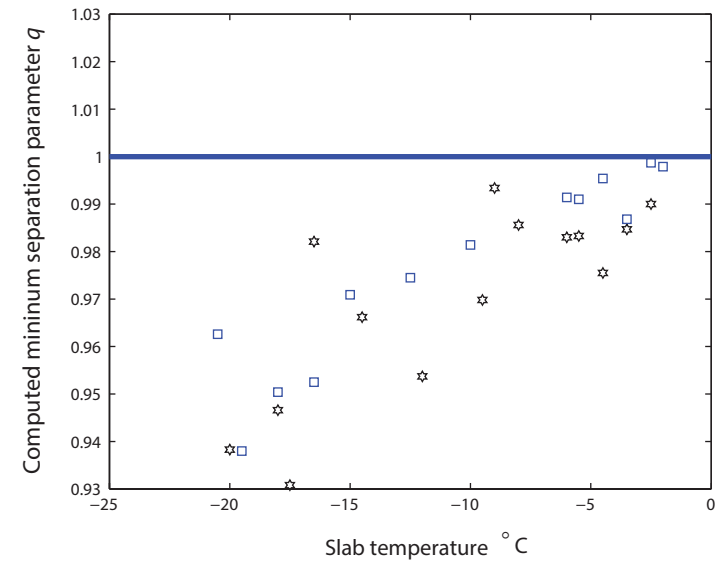


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

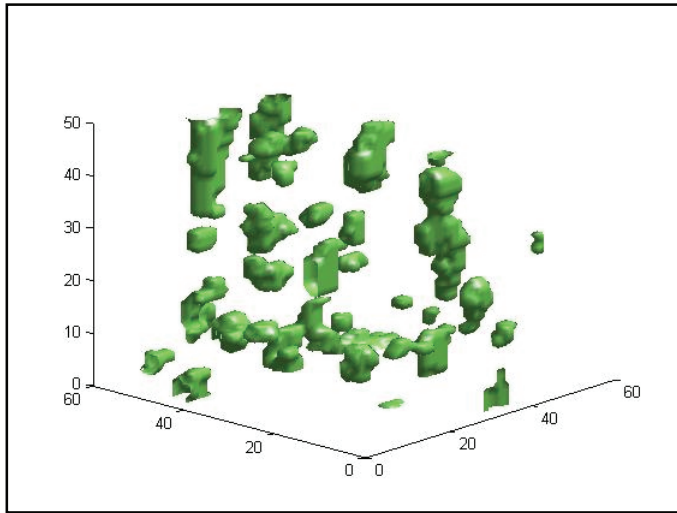
polycrystalline bounds

Gully, Lin, Cherkaev, Golden, 2013

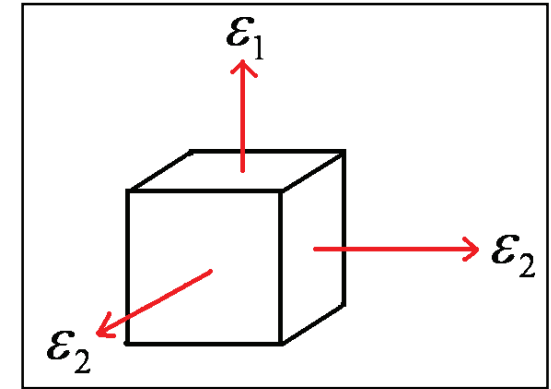
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ϵ^*

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

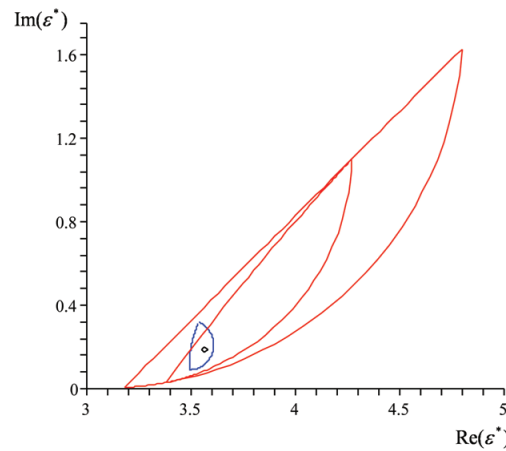
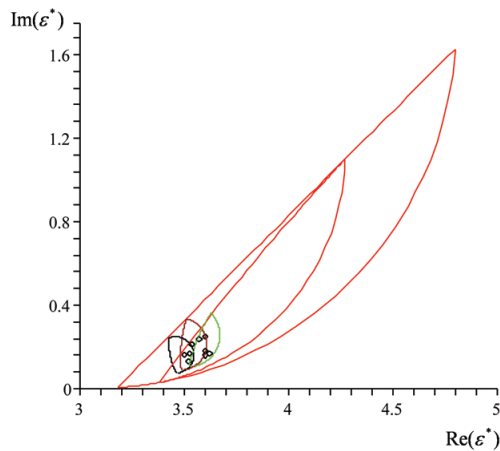
two scale homogenization for polycrystalline sea ice



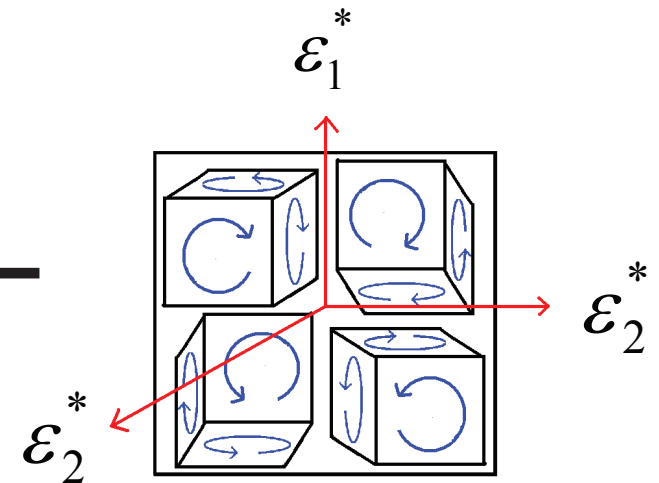
numerical homogenization
for single crystal



analytic continuation
for polycrystals

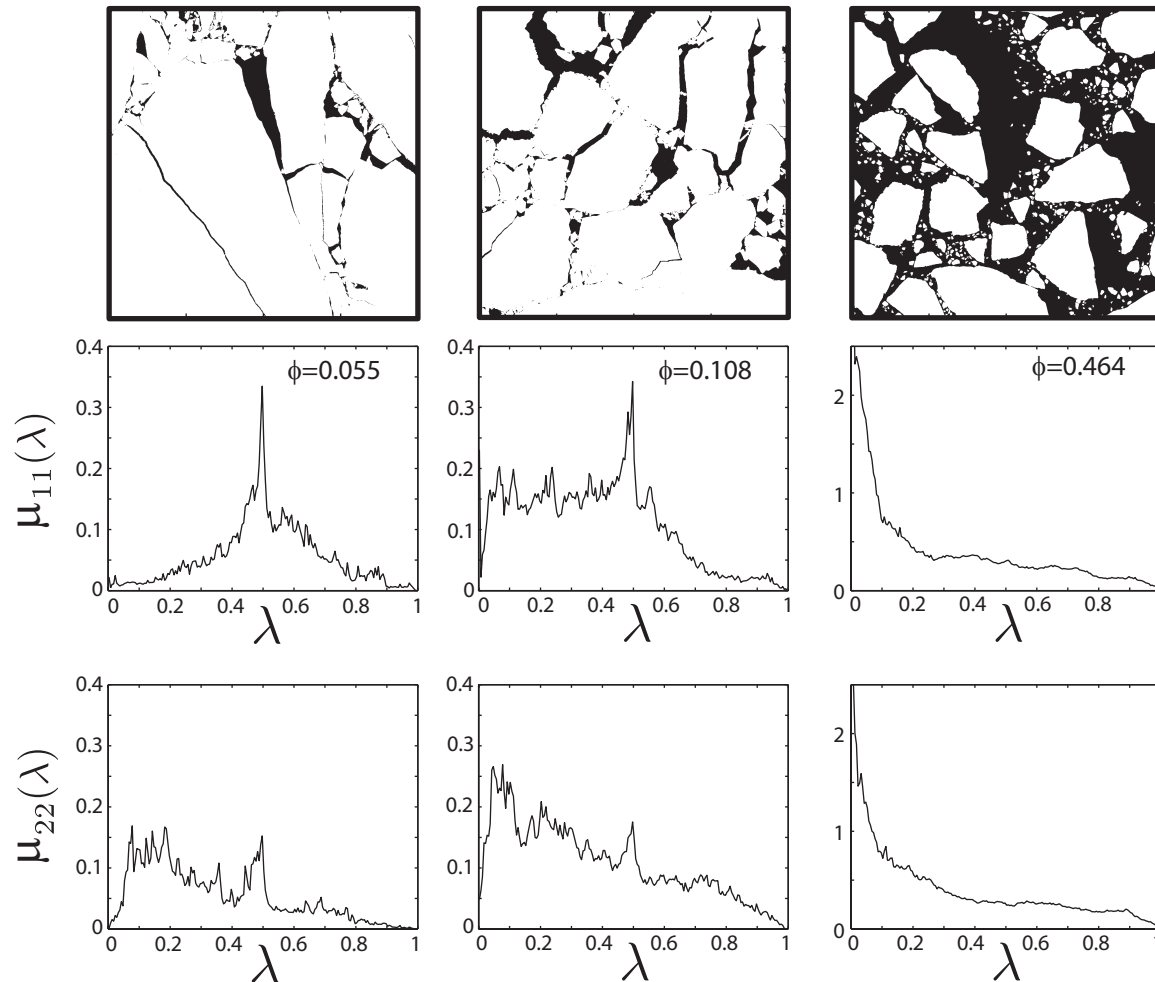


bounds



spectral measures provide a path toward rigorously incorporating
“composite microstructure” into calculations of effective behavior on larger scales

spectral measures for the Arctic sea ice pack



area under curve = ϕ = open water fraction

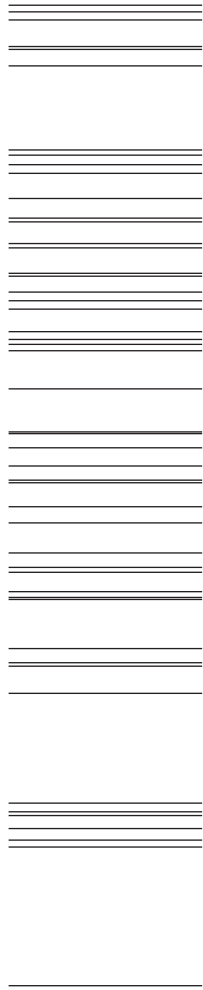
spectral gap closes as ocean phase becomes connected

Transitions in Eigenvalue Correlations

$$P(z) = \exp(-z)$$

Eigenvalue Spacing Distribution

**Poisson
Spectra**

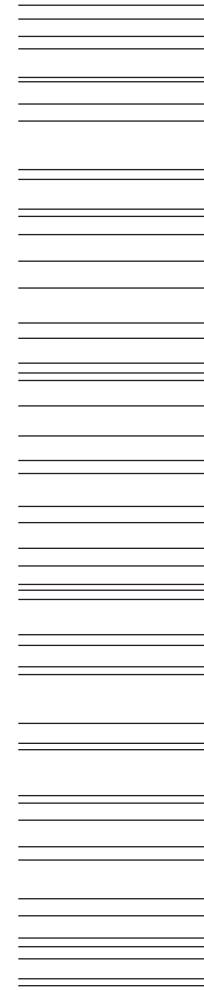


Uncorrelated

$$P(z) \approx \frac{\pi z}{2} \exp(-\pi z^2/4)$$

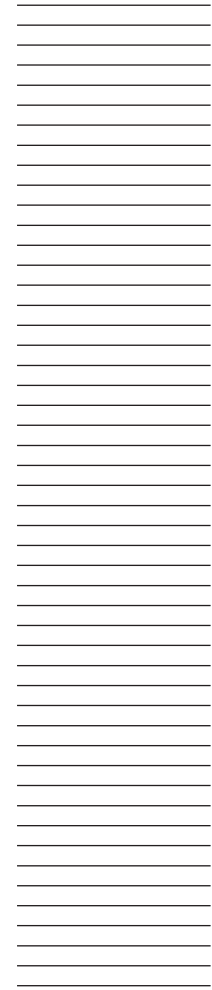
Eigenvalue Spacing Distribution

**WD
Spectra**



**Highly
Correlated**

**Picket
Fence**



**Completely
Correlated**

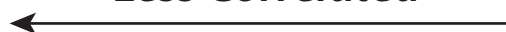
Phase Transition



Less Level Repulsion

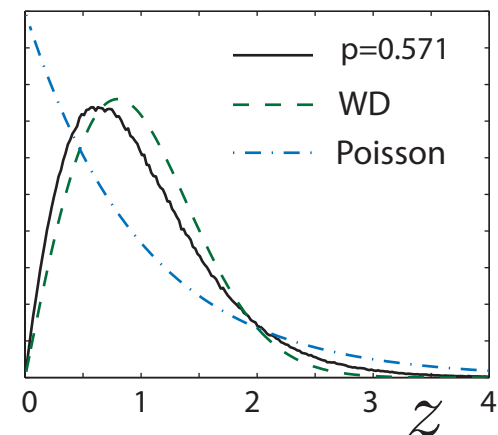
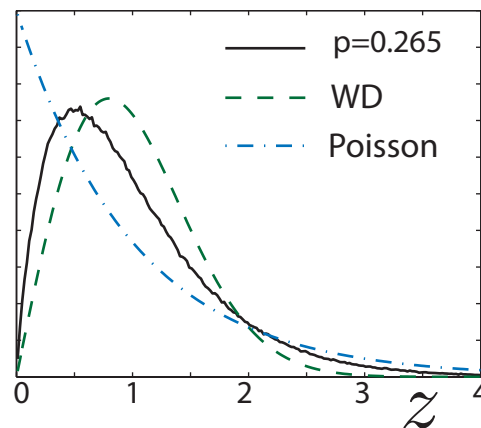
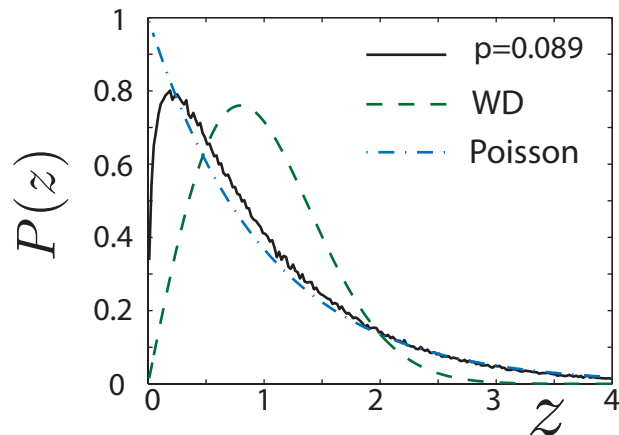
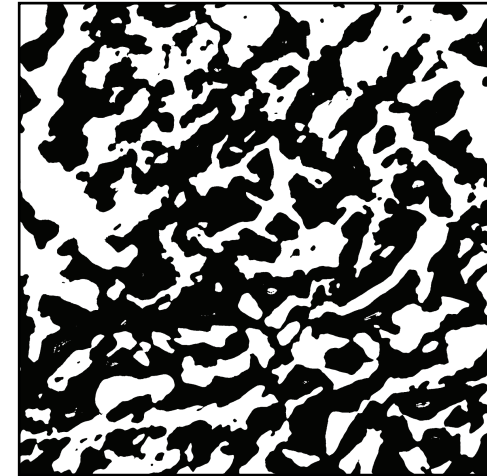
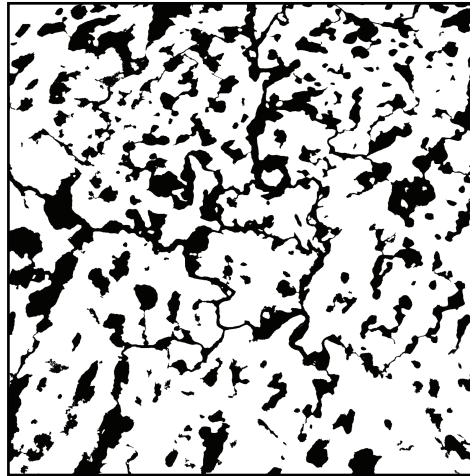
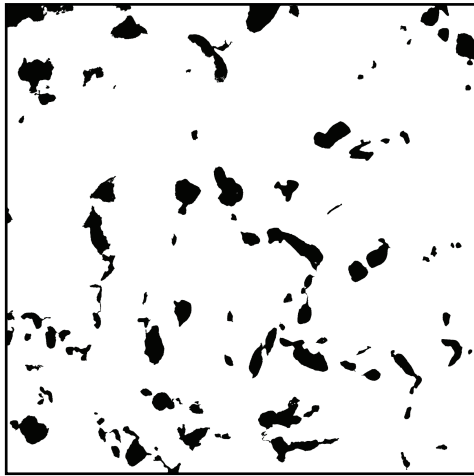


Less Correlated



Unfolded Eigenvalue Spacing Distribution

ARCTIC MELT PONDS



*eigenvalue statistics for transport exhibit **UNIVERSALITY** as the “conducting” phase becomes connected over large scales*

Conclusions

1. Sea ice exhibits composite structure on many length scales.
2. Fluid flow through sea ice governs many processes of importance to understanding climate change and the response of polar ecosystems.
3. Mathematical models of composite materials and statistical physics help unravel the complexities of sea ice structure and processes.
4. Homogenization theory and upscaling methods can provide a rigorous path to representing large scale effective behavior in coarse models.
5. Random matrix theory can help characterize transitions important for climate science and composite materials.

THANK YOU

National Science Foundation

Division of Mathematical Sciences

Arctic Natural Sciences

Office of Polar Programs

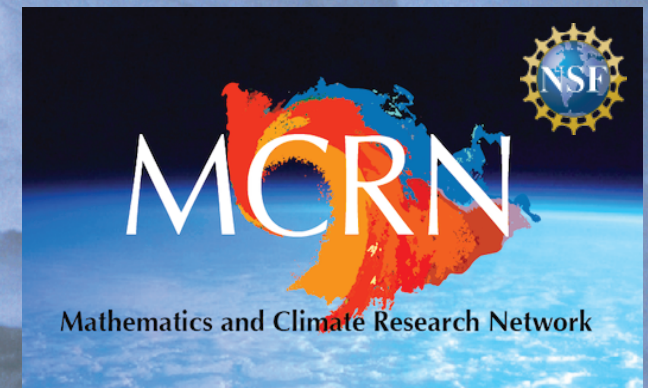
CMG Program

(Collaboration in Mathematical Geosciences)

Office of Naval Research

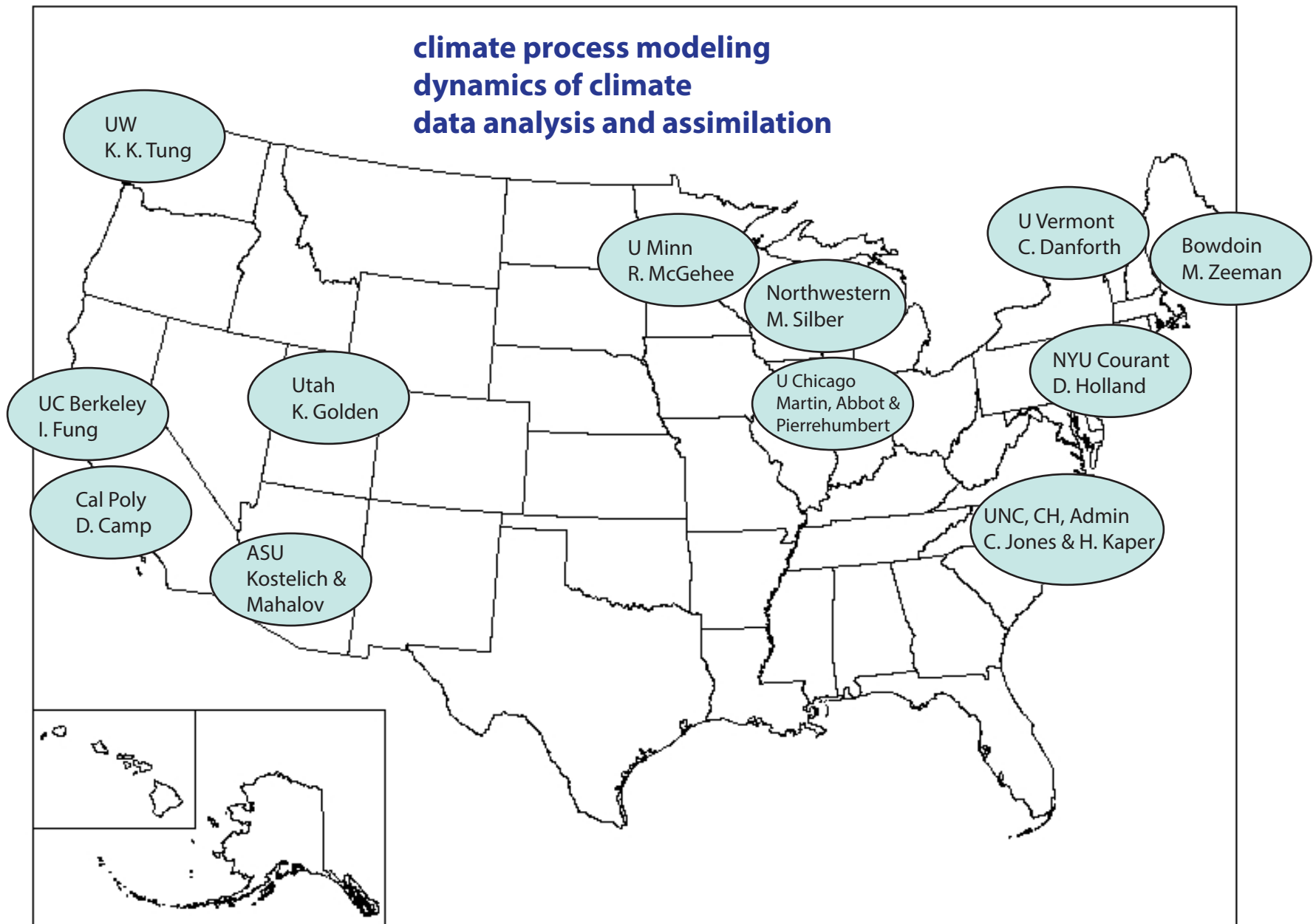
Applied Computational Analysis Program

Arctic and Global Prediction Program



Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999

Mathematics and Climate Research Network (MCRN)



NSF DMS 2010-2015, Lorenz postdocs, grad, undergrad, polar expeditions

Jones, Golden, Kaper, Zeeman