Emergence and annihilation of localized structures in a phytoplankton–nutrient model

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Abstract. Co-limitation of marine phytoplankton by light and nutrient leads to complex dynamic behavior and a wide array of coherent patterns. The building blocks of this array can be considered to be deep chlorophyll maxima, or DCMs, which are structures localized in the vertical direction. From an ecological point of view, DCMs are evocative of a balance between the inflow of light from the water surface and of nutrients from the sediment. From a (linear) bifurcational point of view, they appear through a transcritical bifurcation in which the trivial, no-plankton steady state is destabilized. This article is devoted to the analytic investigation of the weakly nonlinear dynamics of these DCM patterns, and it has two overarching themes. The first of these concerns the fate of the destabilizing stationary DCM mode beyond the linear regime. Exploiting the natural singularly perturbed nature of the model, we derive an explicit reduced model of asymptotically high dimension which fully captures these dynamics. Our subsequent and fully detailed study of this model—which involves a subtle asymptotic analysis necessarily transgressing the boundaries of a local center manifold reduction—establishes that the bifurcating pattern is stable, albeit it persists in an asymptotically small region of parameter space before it is annihilated in a saddle-node bifurcation. The development of the method underpinning this work—which, we expect, shall prove useful for a larger class of models—forms the second theme of this article.

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1. Introduction

Phytoplanktonic photosynthesis provides the major transport mechanism of atmospheric carbon dioxide into the deep ocean. Concurrently, plankton forms the basis of the aquatic food chain. As a consequence, phytoplankton growth and decay plays a crucial role in understanding climate dynamics [10] and forms an integral part of oceanographic research. Conversely, climate changes—such as global temperature variations—have a direct impact on the aquatic ecosystem and thus also on phytoplankton [3, 20]: there is a subtle and certainly under-explored interplay between the dynamics of phytoplankton concentrations and climate variability. At the same time, phytoplankton concentrations exhibit surprisingly rich spatio-temporal dynamics. The character of those dynamics is determined in an intricate fashion by (changes in) the external conditions—see [15] and the references therein. The building blocks for the observed complex patterns are *'deep chlorophyll maxima'* (DCMs) or *phytoplankton blooms*, in which the phytoplankton concentration exhibits a maximum at a certain, well-defined depth of the basin. These patterns are the manifestation of a fundamental balance between the supply of light from the surface and of nutrients from the depths of the basin. For the simplest models, in which spatiotemporal fluctuations in the nutrient concentration are omitted (*eutrophic* environment), it has been shown that there can only be a stationary global attractor. In particular, if the trivial state (no phytoplankton) is unstable, then there can only be a stationary globally attracting phytoplankton bloom with its maximum either at the surface (a surface layer), at the bottom (a benthic layer), or in between (a DCM) [9, 12, 13, 17]. This is no longer the case in coupled phytoplankton–nutrient systems (*oligotrophic* environment), although DCMs do tend to appear in those systems, also, for certain parameter combinations [6, 7, 11, 13, 16, 18]. The detailed numerical studies reported in [15], however, show that the appearance of a DCM only triggers a complex sequence of bifurcations: as parameters vary, a DCM may be time periodic, undergo a sequence of period doubling bifurcations, and eventually behave chaotically.

In this paper, we focus on the effect that varying environmental conditions have on the dynamics generated by the one-dimensional model for phytoplankton (W)–nutrient (N) interactions originally introduced in [15],

$$\begin{cases} W_t = D\,W_{zz} - V\,W_z + [\mu\,P(L,N) - l]\,W, \\ N_t = D\,N_{zz} - Y^{-1}\,\mu\,P(L,N)\,W. \end{cases} \tag{1.1}$$

In this model, the vertical coordinate z measures the depth in a water column spanned by $[0, z_B]$, while $W(z,t)$ and $N(z,t)$ are the phytoplankton and nutrient concentrations, respectively, at depth z and time t. As in [15, 23], the system is assumed to be in the turbulent mixing regime [9, 13], so that the diffusion coefficient D is identically the same for phytoplankton and nutrient. The phytoplankton is characterized by its sinking speed V, its (species-specific) loss rate l, its maximum specific production rate μ, and its yield Y on light and nutrient. The model is equipped with natural no-flux boundary conditions at the surface for both phytoplankton and nutrients; the bottom is a source of nutrients, while it is (also) impenetrable for phytoplankton,

$$D\,W_z - V\,W|_{z=0,z_B} = 0, \quad N_z|_{z=0} = 0, \quad \text{and} \quad N|_{z=z_B} = N_B. \tag{1.2}$$

The nonlinear expression $P(L,N)$ models phytoplankton growth due to light and nutrient,

$$P(L,N) = \frac{LN}{(L+L_H)(N+N_H)}, \tag{1.3}$$

in which L_H and N_H are the half-saturation constants of light and nutrient, respectively (see [23] for a short discussion on the nature and specificity of $P(L,N)$). The light intensity L at depth z and time t is determined by the total amount of planktonic and non-planktonic components in the column $[0,z]$,

$$L(z,t) = L_I\,\mathrm{e}^{-K_{bg}z - R\int_0^z W(s,t)ds}. \tag{1.4}$$

Hence, the system is non-local—a typical feature of most realistic phytoplankton models. The light intensity term introduces an extra three parameters: L_I, the intensity of the incident light at the water surface; K_{bg}, the light absorption coefficient due to non-planktonic, background components; and R, the light absorption coefficient due to plankton (*self-shading*). The first two of these parameters, together with z_B, D, Y, and N_B quantify the effect that the environment has on the planktonic population. It is by varying these parameters that we examine the effect of changing environmental conditions on plankton.

It is shown in [23] that system (1.1) has a natural singularly perturbed nature. This can be seen by rescaling time and space via $\tau = \mu t$ and $x = z/z_B$, and the phytoplankton concentration W, nutrient concentration N, and light intensity L by

$$\omega^+(x,\tau) = \frac{l z_B^2}{DYN_B} W(z,t), \quad \eta(x,\tau) = 1 - \frac{N(z,t)}{N_B}, \quad \text{and} \quad j(x,\tau) = \frac{L(z,t)}{L_I}.$$

Substitution into (1.1) then yields,

$$\left\{ \begin{array}{l} \omega_\tau^+ = \varepsilon \omega_{xx}^+ - 2\sqrt{\varepsilon A}\, \omega_x^+ + (p(\omega^+,\eta,x) - \ell)\, \omega^+, \\ \eta_\tau = \varepsilon \left(\eta_{xx} + \ell^{-1} p(\omega^+,\eta,x)\, \omega^+\right), \end{array} \right. \tag{1.5}$$

with boundary conditions,

$$\left\{ \begin{array}{l} (\omega_x^+ - 2\sqrt{A/\varepsilon}\, \omega^+)(0) = (\omega_x^+ - 2\sqrt{A/\varepsilon}\, \omega^+)(1) = 0, \\ \eta_x(0) = \eta(1) = 0. \end{array} \right. \tag{1.6}$$

For realistic choices of the original parameters of (1.1),

$$\varepsilon = \frac{D}{\mu z_B^2} \approx 10^{-5},$$

cf. [15, 23]. In this paper, we follow [23] and treat the parameter ε as an asymptotically small parameter, *i.e.*, we assume that $0 < \varepsilon \ll 1$ so that (1.5) has, indeed, a singularly perturbed character. The nonlinearity p in (1.5) is given by

$$p(\omega^+,\eta,x) = \frac{1-\eta}{(\eta_H + 1 - \eta)\,(1 + j_H/j(\omega^+,x))}, \tag{1.7}$$

with rescaled light intensity

$$j(\omega^+,x) = \exp\left(-\kappa x - r\int_0^x \omega^+(s,\tau)ds\right). \tag{1.8}$$

The remaining six rescaled parameters of (1.5),

$$A = \frac{V^2}{4\mu D}, \quad \ell = \frac{l}{\mu}, \quad j_H = \frac{L_H}{L_I}, \quad \eta_H = \frac{N_H}{N_B}, \quad \kappa = K_{bg} z_B, \quad \text{and} \quad r = \frac{RDYN_B}{l z_B}, \tag{1.9}$$

are all considered to be O(1) with respect to ε in the forthcoming analysis (cf. [23]).

The system (1.5) may be written compactly in the form

$$u_\tau^+ = \mathcal{T}^+(u^+) = \begin{pmatrix} \varepsilon\, \omega_{xx}^+ - 2\sqrt{\varepsilon A}\, \omega_x^+ + (p(\omega^+,\eta,x) - \ell)\, \omega^+ \\ \varepsilon\, \eta_{xx} + \varepsilon\, \ell^{-1}\, p(\omega^+,\eta,x)\, \omega^+ \end{pmatrix}, \tag{1.10}$$

where

$$u^+ = \begin{pmatrix} \omega^+ \\ \eta \end{pmatrix}.$$

Here, the nonlinear operator $\mathcal{T}^+$ is densely defined in $\mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$. In terms of the original system (1.1), the origin $u^+ = (0,0)^{\mathrm{T}}$ corresponds to the trivial steady-state solution of (1.10) in which there is no phytoplankton—$W(z,t) \equiv 0$—and the nutrient concentration remains constant throughout the column—$N(z,t) \equiv N_B$, the value at the bottom of the basin (1.2). Our attempt to comprehend the mechanism underpinning the appearance of phytoplankton patterns, as well as the character of such patterns, begins with the determination of the spectral stability of this state, *i.e.*, with the study of the linearization of (1.10) around $u^+ = (0,0)^{\mathrm{T}}$,

$$\mathcal{DT}^+ = \begin{pmatrix} \varepsilon\,\partial_{xx} - 2\sqrt{\varepsilon A}\,\partial_x + f - \ell & 0 \\ \varepsilon\,\ell^{-1} f & \varepsilon\,\partial_{xx} \end{pmatrix}, \tag{1.11}$$

in which

$$f(x) = \frac{\nu}{1 + j_H \mathrm{e}^{\kappa x}} \quad \text{and} \quad \nu = \frac{1}{1+\eta_H} \in (0,1). \tag{1.12}$$

The associated spectral problem has been investigated in full asymptotic detail in [23]. The spectrum $\sigma(\mathcal{DT}^+)$ of the operator $\mathcal{DT}^+$ consists of two distinct, real parts, $\sigma(\mathcal{DT}^+) = \{\nu_n\}_{n\ge 0} \cup \{\lambda_n\}_{n\ge 0}$. Here, the eigenvalues $\{\nu_n\}_{n\ge0}$ are negative and independent of all parameters, $\nu_n = -\varepsilon\,(n+1/2)^2\,\pi^2$, so the spectral stability of the trivial state is governed solely by $\{\lambda_n\}_{n\ge0}$. In [23], we identified two different linear destabilization mechanisms. In the regime $A < f(0) - f(1)$ (cf. (1.9) and (1.12)), the ω^+-component of the eigenfunction w_0^+ associated with the critical eigenvalue λ_0 (*i.e.*, the phytoplankton component of the profile) has the character of a DCM: w_0^+ is localized and attains its maximal value at a certain depth x_* which can be determined explicitly: to leading order, $f(x_*) = f(0) - A$ [23]. In the complementary case $A > f(0) - f(1)$, the ω^+-component of the critical eigenfunction destabilizing the trivial state has the character of a benthic layer (*i.e.*, it increases monotonically with depth).

In this article, we focus exclusively on the regime in which DCMs may appear, *i.e.*, we assume throughout the article that $A < f(0) - f(1)$. In that regime, we investigate the nature of the bifurcation associated with the destabilization mechanism of DCM type. We know from [23] that, in this case,

$$\lambda_n = \lambda_* - \varepsilon^{1/3}\sigma_0^{2/3}\,|A_{n+1}| + \mathrm{O}(\varepsilon^{1/2}), \tag{1.13}$$

with

$$\lambda_* = f(0) - \ell - A = \frac{\nu}{1+j_H} - \ell - A \tag{1.14}$$

and where

$$\sigma_0 = F'(0) = -f'(0) = \frac{\kappa\,\nu\,j_H}{(1+j_H)^2}, \quad \text{with} \quad F(x) = f(0) - f(x). \tag{1.15}$$

Here, $A_n < 0$ is the $n-$th root of Ai, the Airy function of the first kind. The bifurcation occurs as λ_0 crosses zero, yielding the bifurcation diagram in the left panel of Figure 1.

In this paper, then, we focus on the (weakly nonlinear) dynamics generated by (1.10) for parameter choices such that

$$\lambda_0 = \frac{\nu}{1+j_H} - \ell - A - \varepsilon^{1/3}\sigma_0^{2/3}\,|A_{n+1}| + \mathrm{O}(\varepsilon^{1/2}) = \varepsilon^{\alpha}\Lambda_0, \tag{1.16}$$

where $\alpha > 0$ is $\mathcal{O}(1)$ and Λ_0 is allowed to be at most logarithmically large with respect to ε. Note that one can *tune* the appearance of a destabilization of DCM type (*i.e.*, of the simplest phytoplankton pattern) by choosing appropriately the parameters in (1.10);

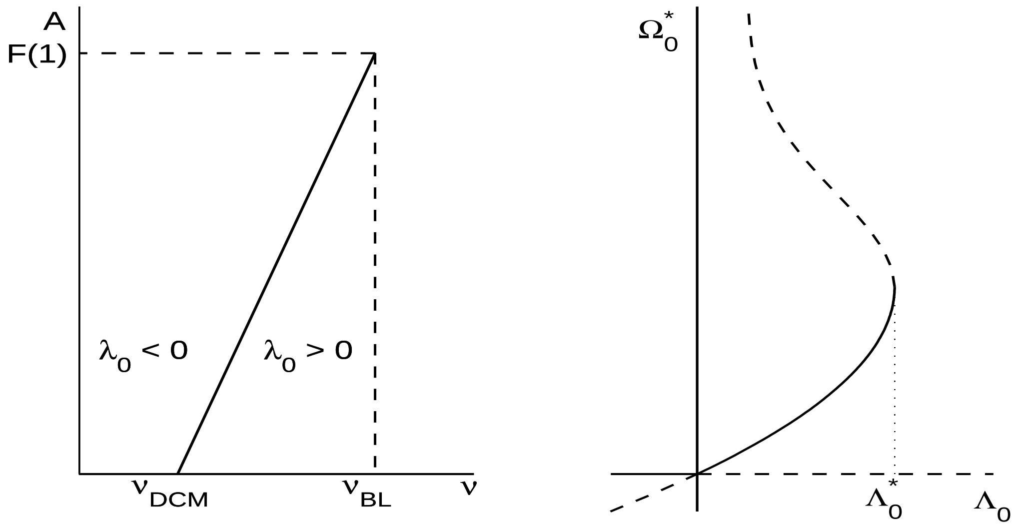

Figure 1. *Left panel*: the bifurcation diagram for the trivial steady state of (1.5) in the regime $A < f(0) - f(1) = F(1)$. The trivial steady state is stable in the region $\lambda_0 < 0$ and unstable in the region $\lambda_0 > 0$. Here, $\nu_{DCM} = \ell(1 + j_H)$ and $\nu_{BL} = \ell(1 + \mathrm{e}^{\kappa} j_H)$. *Right panel*: the bifurcation diagram for the small-amplitude DCM. The origin marks the transcritical bifurcation through which the trivial steady state is destabilized and the small-amplitude DCM pattern emerges. The value Λ_0^* marks the saddle-node bifurcation in which this pattern is annihilated.

also, that λ_0 depends on all parameters with the exception of r, see the definitions of f and σ_0 in (1.12) and (1.15). We remark, finally, that the parameter A depends on the diffusion coefficient D (cf. (1.9)), the main parameter varied in [15] and the one that most strongly depends on varying external conditions such as global temperature [20].

The first step in analyzing the dynamics generated by a linear destabilization mechanism is to perform a center manifold analysis and thus determine the local character of the bifurcation associated with the destabilization (see, for instance, [1, 4]). This is a well-established procedure. In the setting of (1.16), this amounts to assuming that λ_0 is (asymptotically) smaller than all other eigenvalues, and it corresponds to the case $\alpha > 1$ and $\Lambda_0 = \mathrm{O}(1)$. In this regime, the remaining eigenvalues are negative and asymptotically larger than λ_0, so that the local flow near the trivial pattern $(0,0)^{\mathrm{T}}$ is determined by the flow on the center manifold. This manifold—or, rather, its tangent space at the steady state—is spanned by the eigenfunction w_0^+ associated with λ_0. Hence, this flow can be determined by expanding u^+ as $u^+(x,\tau) = \varepsilon^{\alpha}\,\Omega_0(\tau)\,w_0^+(x) +$ *higher order terms*, with Ω_0 being an unknown, time-dependent amplitude and the higher order remainder encapsulating the component of u^+ along directions associated with the stable eigenvalues. A straightforward projection procedure—which can nevertheless be highly technical, especially in a PDE setting—

yields an equation for the unknown amplitude Ω_0. In the case at hand, this equation reads

$$\dot{\Omega}_0 = \Lambda_0 \Omega_0 - a_{000}(0)\Omega_0^2, \tag{1.17}$$

to leading order. Thus, the procedure reveals the existence of a nontrivial fixed point which is stabilized through a standard, co-dimension one transcritical bifurcation. This fixed point corresponds to an asymptotically small DCM pattern, the amplitude of which grows linearly with Λ_0,

$$\omega^+(x) \sim \varepsilon^\alpha \, \Omega_0^* \, \omega_0^+(x), \quad \text{with } \Omega_0^* = \frac{\Lambda_0}{a_{000}(0)}. \tag{1.18}$$

In general, one cannot expect to be able to compute the coefficient $a_{000}(0)$ explicitly. Here, we exploit the singularly perturbed nature of (1.10) and the localized character of the eigenfunction w_0^+ to do exactly that; in particular, it follows from the analysis to be presented in this article that

$$a_{000}(0) = \varepsilon^{1/6}(1-\nu)(1-x_*) \, \frac{\sigma_0^{1/3} \, f(0) \, \exp(|A_1|^{3/2})}{\left(F'(x_*) \int_{A_1}^\infty \mathrm{Ai}^2(s) \, ds\right)^{1/2}} > 0, \tag{1.19}$$

see Section 3 and, especially, Remark 4.2. Additionally to yielding an explicit, leading order formula for the amplitude of the emerging (stable) DCM, this result also implies that this DCM is ecologically relevant ($\omega^+ > 0$). Moreover, one cannot hope to extend the analysis beyond the (one-dimensional) center manifold reduction discussed above and into the regime where λ_0 is *not* asymptotically closer to zero than all other eigenvalues. In other words, it is generically the case that the sole analytical insight into the dynamics of the flow near the destabilization that one can obtain is the confirmation that the DCMs indeed appear through a transcritical bifurcation.

Let us look into this last point in more detail. For $\lambda_0 = \mathcal{O}(\varepsilon)$—equivalently, for $\alpha = 1$ in (1.16)—one can no longer 'project away' the directions corresponding to the eigenvalues ν_n associated with the operator $\mathcal{DT}^+$. Indeed, these are $\mathcal{O}(\varepsilon)$ for $\mathcal{O}(1)$ values of n (cf. Section 2.1), and hence of the same asymptotic magnitude as λ_0. As a result, the center manifold reduction approach yields a leading order system in at least asymptotically many dimensions. In general, such a system cannot be studied analytically, and one has to abandon the idea of performing an asymptotically accurate analysis. A brute alternative would be, for instance, to proceed by studying low-dimensional truncations; nevertheless, the relation of such truncations to the full system is, generally speaking, unclear. The main aim of this paper is to develop an analytic approach through which one can tackle such difficulties and thus go beyond the direct, finite-dimensional center manifold reduction outlined above. The original ideas underlying this approach—namely, the method of *weakly nonlinear stability analysis*—qualify as classical [21]. However, this particular method does not always provide more insight than the rigorously established center manifold reduction method: for instance, it also reduces the flow to a one-dimensional ODE of the form (1.17). The situation is strikingly different here, as we can exploit the singularly perturbed nature of (1.10), in conjunction with the asymptotic information on the eigenfunctions of $\mathcal{DT}^+$ obtained in [23], to explicitly compute to leading order all relevant parameters in the asymptotically high-dimensional system derived by our weakly nonlinear stability analysis, see Sections 6–8. Upon having achieved this, we establish that the system is structured enough to allow us to study in full analytic detail the case $\lambda_0 = \mathcal{O}(\varepsilon)$—see Section 4—and even extend our analysis to the regime $\lambda_0 = \mathcal{O}(\varepsilon \log^2 \varepsilon)$—see Section 5. This way, we can analytically trace the fate of the bifurcating DCM pattern well into the regime where its amplitude depends nonlinearly on Λ_0 and, in fact, all the way to a regime where it undergoes a saddle–node bifurcation and disappears.

The outcome of our (asymptotic) analysis is summarized in the right panel of Figure 1. The localized DCM that bifurcates as λ_0 crosses zero is a stable attractor of the flow generated by (1.1), for all $\alpha > 1$ and $\Lambda_0 = \mathcal{O}(1)$ with respect to ε, cf. (1.16). As we remarked above, the amplitude Ω_0^* of this localized DCM, and thus also the biomass associated with it, grows linearly with Λ_0 in that regime, cf. (1.18)–(1.19). Beyond this center manifold reduction regime, however—and in particular in the regime $\alpha = 1$ and $\Lambda_0 = \mathrm{O}(1)$—$\Omega_0^*$ turns out to depend nonlinearly on Λ_0,

$$\Omega_0^* = \varepsilon^{-1/6}\, C\, \frac{\Lambda_0^{3/2}\, \cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)},$$

for an explicitly known O(1) constant C, see (4.10). The corresponding biomass turns out to be

$$\int_0^1 \omega^+(x)\, dx = \varepsilon \frac{\nu}{(1-\nu)\,(1+j_H)}\, \frac{\Lambda_0^{3/2}\, \cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}. \tag{1.20}$$

First, this establishes that the DCM pattern grows with ν—a parameter measuring nutrient availability in the water column (see (1.9), (1.12))—in the $\lambda_0 = \mathcal{O}(\varepsilon)$ regime, as Λ_0 is proportional to ν (see (1.16)); this fact certainly reinforces our ecological intuition. Further, the pattern grows exponentially with Λ_0, and hence it is natural to attempt an extension of our analysis into a logarithmic region for Λ_0. Our analysis in Section 5 yields that the relevant scaling is $\lambda_0 = \mathcal{O}(\varepsilon \log^2 \varepsilon)$. In that regime, we establish the existence of a *second* localized DCM-type pattern: the associated reduced system has two critical points. Although this second localized structure is unstable, it affects significantly the dynamics of the flow generated by (1.1) for small λ_0, as it *annihilates* the stable DCM in a saddle–node bifurcation. The appearance of this second bifurcation can be determined explicitly by our methods.

Hence, our analysis yields that the stationary, stable, localized DCM pattern emerging at the transcritical bifurcation through which the trivial state becomes unstable only persists in an asymptotically small, $\mathcal{O}(\varepsilon \log^2 \varepsilon)$ region in parameter space. On the one hand, this outcome is quite surprising: one would expect that the appearance of this stationary DCM is the first step in a rather standard cascade of bifurcations leading to the chaotic dynamics reported in [15]—see also our discussion in [23]. On the other hand, it is not the case that the numerical observations reported in [15] could only be explained by means of such a standard cascade: the results in [15] do not suffice to decipher the origin of the DCM-patterns eventually exhibiting chaotic dynamics. In fact, even the amplitudes of the simplest localized DCM-patterns identified numerically in [15] appear to be *larger* than what our findings here would suggest. In light of this, our analytical findings do not contradict these numerical findings but, instead, enlighten a neglected aspect of theirs. In the same vein, our findings here suggest that the chaotic dynamics *cannot* be traced back to the small amplitude patterns emerging from the destabilization of the trivial steady state. (Of course, one must always exercise caution in interpreting numerical observations from an asymptotic point of view, especially when these simulations concern an *unscaled* system as is the case here: the authors of [15] have simulated the original system (1.1) and *not* the scaled system (1.5).)

Naturally, the questions on the nature of the attractor beyond the saddle–node bifurcation (at which the small amplitude DCM patterns are annihilated) and on the origin of the patterns observed in [15] are intriguing. At present, this is the subject of ongoing research. We do not pursue these questions further in this article, apart from a short discussion in its concluding section.

Parallel to understanding the character and fate of the linear destabilization mechanism established in [23], this article has a second—and from a mathematical point of view at least equally important—theme. Here, we have developed a powerful approach by which we can study the weakly nonlinear dynamics generated by (1.5) in full asymptotic detail and far from the region covered by more standard techniques (such as the center manifold reduction method). Our ability to explicitly determine all relevant parameters in the reduced system (of asymptotically high dimension) is crucial to this approach. The sometimes remarkably subtle analysis by which these parameters can be computed provides the foundation for the strength and success of our program. Therefore, this analysis is a central ingredient of our approach and forms the core of the forthcoming presentation, see especially Sections 3 and 6–8. This is also the reason we focus on the destabilization mechanism underpinning the emergence of DCM-like patterns. Our analysis in [23] strongly suggests that, for realistic choices of the parameters, patterns of benthic layer (BL) type are equally relevant to the dynamics generated by (1.1) as the DCM patterns considered here. In fact, preliminary numerical simulations of (1.1) strongly suggest that patterns having a co-dimension 2-type DCM/BL character play an important role in the region where the trivial state is unstable. The approach developed in the present paper will be used—and if necessary extended—in forthcoming work investigating these issues. For a broader outlook, see also the discussion in Section 9.

2. Evolution of the Fourier coefficients

Our aim in this section is to write the PDE system (1.10) as an infinite-dimensional system of nonlinear ODEs and to mod out the fast stable directions. To achieve this, we need explicit formulas for the (point) spectrum $\sigma(\mathcal{DT}^+)$, as well as for the corresponding eigenbasis and its dual. The spectrum and the eigenbasis have been determined in [23];

we summarize the relevant formulas in Section 2.1 below. We then obtain the dual basis in Section 2.2 by solving the eigenproblem for the adjoint operator $(\mathcal{DT}^+)^*$. Finally, in Section 2.3, we derive the desired ODEs for the Fourier coefficients close to the bifurcation point.

2.1. The spectrum and the corresponding eigenbasis of $\mathcal{DT}^+$

For completeness, we let $\mathcal{H}_{\omega^+}$ and $\mathcal{H}_\eta$ be the subspaces of $L^2(0,1)$ associated with the boundary conditions (1.6), $\mathcal{H}_\omega$ be associated with the boundary conditions

$$(\partial_x\omega - \sqrt{A/\varepsilon}\,\omega)(0) = (\partial_x\omega - \sqrt{A/\varepsilon}\,\omega)(1) = 0, \tag{2.1}$$

and we write $\mathcal{H}_+ = \mathcal{H}_{\omega^+} \times \mathcal{H}_\eta$ and $\mathcal{H} = \mathcal{H}_\omega \times \mathcal{H}_\eta$. Both product spaces can be equipped with the inner product

$$\langle u_1^+, u_2^+\rangle = \left\langle \begin{pmatrix} \omega_1^+ \\ \eta_1 \end{pmatrix}, \begin{pmatrix} \omega_2^+ \\ \eta_2 \end{pmatrix} \right\rangle = \int_0^1 \Big(\omega_1^+(x)\,\omega_2^+(x) + \eta_1(x)\,\eta_2(x)\Big)dx.$$

Subsequently, we define the function $E(x) = \exp(\sqrt{A/\varepsilon}\,x)$ and the operator $\mathcal{E} : \mathcal{H} \to \mathcal{H}_+$ corresponding to an application of the Liouville transform,

$$\mathcal{E}u = \begin{pmatrix} E\,\omega \\ \eta \end{pmatrix} = \begin{pmatrix} \omega^+ \\ \eta \end{pmatrix} = u^+ \tag{2.2}$$

with inverse

$$u = \begin{pmatrix} \omega \\ \eta \end{pmatrix} = \begin{pmatrix} \omega^+/E \\ \eta \end{pmatrix} = \mathcal{E}^{-1}u^+. \tag{2.3}$$

(It is straightforward to check that the boundary conditions (1.6) for u yield the boundary conditions (2.1) for u^+.) Both $\mathcal{E}$ and $\mathcal{E}^{-1}$ are self-adjoint and bounded and

$$\mathcal{DT} = \mathcal{E}^{-1}\mathcal{DT}^+\mathcal{E} = \begin{pmatrix} \varepsilon\partial_{xx} + f - \ell - A & 0 \\ \varepsilon\ell^{-1}fE & \varepsilon\partial_{xx} \end{pmatrix}, \tag{2.4}$$

with $\mathcal{DT}$ densely defined and having self-adjoint diagonal blocks.

The eigenvalues ν_n associated with $\mathcal{DT}^+$ correspond to the pure diffusion problem for the nutrient in the absence of plankton. In particular, they are solutions to the eigenvalue problem

$$\varepsilon\partial_{xx}\zeta_n = \nu_n\zeta_n, \quad \text{with} \quad \partial_x\zeta_n(0) = \zeta_n(1) = 0$$

and may be calculated explicitly,

$$\nu_n = -\varepsilon N_n = -\frac{\varepsilon}{\epsilon_n^2}, \quad \text{where} \quad N_n = \frac{1}{\epsilon_n^2} = (\pi/2 + n\pi)^2 \quad \text{for } n \geq 0. \tag{2.5}$$

The corresponding eigenfunctions have a zero ω^+-component, and they are

$$v_n = \begin{pmatrix} 0 \\ \zeta_n \end{pmatrix}, \quad \text{where } \zeta_n(x) = \sqrt{2}\cos(\sqrt{N_n}\,x), \tag{2.6}$$

normalized so that $||\zeta_n||_2 = 1$.

The eigenvalues λ_n, on the other hand, correspond to the eigenvectors

$$w_n^+ = \begin{pmatrix} \omega_n^+ \\ \eta_n \end{pmatrix}.$$

Here, the functions ω_n^+ and η_n are solutions to

$$\begin{aligned} \varepsilon\,\partial_{xx}\omega_n^+ - 2\sqrt{\varepsilon A}\,\partial_x\omega_n^+ + (f(x) - \ell - \lambda_n)\,\omega_n^+ &= 0, \\ (\partial_x\omega_n^+ - 2\sqrt{A/\varepsilon}\,\omega_n^+)(0) = (\partial_x\omega_n^+ - 2\sqrt{A/\varepsilon}\,\omega_n^+)(1) &= 0, \end{aligned}$$

cf. (1.11), together with the self-adjoint, inhomogeneous, boundary value problem for the component η_n,

$$\varepsilon\,\partial_{xx}\eta_n - \lambda_n\,\eta_n = -\varepsilon\ell^{-1} f\,\omega_n^+, \quad \text{where } \partial_x\eta_n(0) = \eta_n(1) = 0. \tag{2.7}$$

Equivalently, they are solutions to the self-adjoint, Sturm–Liouville problem

$$\begin{aligned} \varepsilon\,\partial_{xx}\omega_n + (f(x) - \ell - A - \lambda_n)\,\omega_n &= 0, \\ (\partial_x\omega_n - \sqrt{A/\varepsilon}\,\omega_n)(0) &= (\partial_x\omega_n - \sqrt{A/\varepsilon}\,\omega_n)(1) = 0, \end{aligned} \tag{2.8}$$

cf. (2.2)–(2.4). As already stated, in [23] we derived the asymptotic expressions

$$\lambda_n = \lambda_* - \varepsilon^{1/3}\sigma_0^{2/3}\,|A_{n+1}| + \mathrm{O}(\varepsilon^{1/2}), \quad \text{with } n \geq 0,$$

cf. (1.13). Here, $\lambda_* = f(0) - \ell - A$, $\sigma_0 = F'(0) = -f'(0)$, and $A_n < 0$ is the $n-$th root of the Airy function Ai, cf. (1.15). A formula for the $n-$th eigenfunction ω_n can also be derived using the WKB method, cf. [23]. The corresponding eigenfunctions for $\mathcal{DT}^+$ are $w_n^+ = (\omega_n^+, \eta_n)^{\mathrm{T}}$, where $\omega_n^+ = E\,\omega_n$—cf. (2.2). As we will see in the next section, it is natural to impose the normalization condition $||\omega_n||_2 = 1$.

2.2. The dual eigenbasis of $\mathcal{DT}^+$

To carry out the weakly nonlinear stability analysis of the bifurcating DCM profile, we also need to obtain the dual eigenbasis $\{\hat{w}_n^+\}_{n\geq 0} \cup \{\hat{v}_n\}_{n\geq 0}$ uniquely determined by the conditions

$$\langle w_n^+, \hat{w}_m^+\rangle = \langle v_n, \hat{v}_m\rangle = \delta_{nm} \quad \text{and} \quad \langle w_n^+, \hat{v}_m\rangle = \langle v_n, \hat{w}_m^+\rangle = 0,$$

for all $n, m \geq 0$. In this section, we show that

$$\hat{w}_n^+ = \begin{pmatrix} \omega_n^- \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{v}_n = \begin{pmatrix} \psi_n^- \\ \zeta_n \end{pmatrix}. \tag{2.9}$$

Here, $\omega_n^- \equiv \omega_n/E$, where ω_n solves the eigenvalue problem (2.8) and satisfies the normalization condition $||\omega_n||_2 = 1$. Further, expressions for the functions $\{\zeta_n\}_n$ were reported in (2.6), while the functions $\{\psi_n^-\}_n$ may be found by solving the inhomogeneous problem

$$\begin{aligned} \varepsilon\,\partial_{xx}\psi_n^- + 2\sqrt{\varepsilon A}\,\partial_x\psi_n^- + (f(x) - \ell - \nu_n)\,\psi_n^- &= -\varepsilon\ell^{-1} f\,\zeta_n, \\ \partial_x\psi_n^-(0) = \partial_x\psi_n^-(1) &= 0. \end{aligned} \tag{2.10}$$

Alternatively, $\psi_n^- = \psi_n/E$, where ψ_n solves the self-adjoint inhomogeneous problem

$$\begin{aligned} \varepsilon\,\partial_{xx}\psi_n + (f(x) - \ell - A - \nu_n)\,\psi_n &= -\varepsilon\ell^{-1} f E\,\zeta_n, \\ (\partial_x\psi_n - \sqrt{A/\varepsilon}\,\psi_n)(0) &= (\partial_x\psi_n - \sqrt{A/\varepsilon}\,\psi_n)(1) = 0. \end{aligned} \tag{2.11}$$

To verify the above, we start from the observation that the dual basis may be obtained by solving the corresponding eigenvalue problem for $(\mathcal{DT}^+)^*$, the adjoint of the operator $\mathcal{DT}^+$. To calculate $(\mathcal{DT}^+)^*$, we write $v^- = \mathcal{E}^{-1}v$, recall (2.4), and note that

$$\langle \mathcal{DT}^+ u^+, v^-\rangle = \langle \mathcal{DT}^+ \mathcal{E} u, v^-\rangle = \langle \mathcal{E}\mathcal{DT} u, v^-\rangle = \langle \mathcal{DT} u, \mathcal{E} v^-\rangle = \langle \mathcal{DT} u, v\rangle = \langle u, \mathcal{DT}^* v\rangle.$$

This implies, further, that

$$\langle \mathcal{DT}^+ u^+, v^-\rangle = \langle u, \mathcal{DT}^* v\rangle = \langle \mathcal{E}^{-1} u^+, \mathcal{DT}^* \mathcal{E} v^-\rangle = \langle u^+, \mathcal{E}^{-1}\mathcal{DT}^* \mathcal{E} v^-\rangle,$$

whence $(\mathcal{DT}^+)^* = \mathcal{E}^{-1}\mathcal{DT}^*\mathcal{E}$. Here, u^+ satisfies the boundary conditions (1.6), whereas the boundary conditions for v^- are determined from $v^- = \mathcal{E}^{-1}v$ and the boundary conditions (2.1) for v—in particular,

$$\partial_x \psi^-(0) = \partial_x \psi^-(1) = 0 \quad \text{and} \quad \partial_x \zeta(0) = \zeta(1) = 0, \quad \text{where } v = \begin{pmatrix} \psi^- \\ \zeta \end{pmatrix}. \tag{2.12}$$

It is straightforward to show that

$$\mathcal{DT}^* = \begin{pmatrix} \varepsilon\partial_{xx} + f - \ell - A & \varepsilon\ell^{-1} f E \\ 0 & \varepsilon\partial_{xx} \end{pmatrix},$$

and, since also $(\mathcal{DT}^+)^* = \mathcal{E}^{-1}\mathcal{DT}^*\mathcal{E}$,

$$(\mathcal{DT}^+)^* = \begin{pmatrix} \varepsilon\,\partial_{xx} + 2\sqrt{\varepsilon A}\,\partial_x + f - \ell & \varepsilon\ell^{-1} f \\ 0 & \varepsilon\,\partial_{xx} \end{pmatrix}. \tag{2.13}$$

In view of (2.13), the eigenvalue problem $(\mathcal{DT}^+)^*\hat{w}_n^+ = \lambda_n \hat{w}_n^+$ for $\hat{w}_n^+ = (\hat{\omega}_n^+, \hat{\eta}_n)^{\mathrm{T}}$ reads

$$\begin{aligned} &\varepsilon\,\partial_{xx}\hat{\omega}_n^+ + 2\sqrt{\varepsilon A}\,\partial_x\hat{\omega}_n^+ + (f - \ell - \lambda_n)\,\hat{\omega}_n^+ = -\varepsilon\ell^{-1} f\,\hat{\eta}_n, \\ &\varepsilon\partial_{xx}\hat{\eta}_n = \lambda_n\,\hat{\eta}_n, \end{aligned}$$

subject to the boundary conditions (2.12). The latter equation yields immediately $\hat{\eta}_n \equiv 0$, so that the former equation becomes homogeneous. It is now trivial to check that $\hat{\omega}_n^+ = \omega_n^- \equiv \omega_n/E$, where ω_n solves the eigenvalue problem (2.8). This establishes the first part of (2.9).

Similarly, (2.13) shows that the eigenvalue problem $(\mathcal{DT}^+)^*\hat{v}_n = \nu_n \hat{v}_n$ has solutions

$$\hat{v}_n = \begin{pmatrix} \psi_n^- \\ \zeta_n \end{pmatrix},$$

where the functions $\{\psi_n^-\}_n$ satisfy the boundary value problem (2.10). An application of the Liouville transform $\psi_n = E\psi_n^-$ leads directly to the self-adjoint problem (2.11).

2.3. Evolution of the Fourier coefficients

Our aim in this section is to write the PDE system (1.10) as an infinite-dimensional system of nonlinear ODEs. We start by expanding the solution of $\partial_\tau u^+ = \mathcal{T}^+(u^+)$ in terms of the eigenbasis associated with the linear stability problem,

$$u^+(x,\tau) = \varepsilon^c\,\delta \sum_{n\geq 0} \Omega_n(\tau)\, w_n^+(x) + \varepsilon^c \sum_{n\geq 0} \Psi_n(\tau)\, v_n(x), \tag{2.14}$$

where $c > 0$ is yet undetermined. Here, the coefficients Ω_n and Ψ_n are determined by

$$\Omega_n = \varepsilon^{-c}\,\delta^{-1}\langle u^+, \hat{w}_n^+\rangle \quad \text{and} \quad \Psi_n = \varepsilon^{-c}\langle u^+, \hat{v}_n\rangle. \tag{2.15}$$

Moreover, we have introduced the exponentially small parameter

$$\delta = \exp\left(\frac{-J_-(x_*)}{\sqrt{\varepsilon}}\right) \ll 1, \tag{2.16}$$

where

$$J_\pm(x) = \sqrt{A}\,x \pm I(x) \quad \text{and} \quad I(x) = \int_{\bar{x}_0}^{x} \sqrt{F(s) - F(\bar{x}_0)}\,ds. \tag{2.17}$$

Here, the $\mathrm{O}(\varepsilon^{1/3})-$parameter $\bar{x}_0$ corresponds to the turning point of (2.8),

$$\bar{x}_0 = F^{-1}(\lambda_* - \lambda_0) = \varepsilon^{1/3}\sigma_0^{-1/3}\,|A_1| + \mathrm{O}(\varepsilon^{1/2}), \tag{2.18}$$

while x_* is the location of the DCM, the unique point where $J_-(\cdot)$ attains its (positive) maximum ([23]—see also Appendix A), *i.e.*,

$$x_* = F^{-1}(A + F(\bar{x}_0)) = F^{-1}(A) + \mathrm{O}(\varepsilon^{1/3}). \tag{2.19}$$

Thus, δ is a measure for the amplitude of the ω-component of the (linear) mode associated to a bifurcating DCM. The introduction of δ in the decomposition (2.14) allows us to identify *small patterns* ($u^+ \ll 1$) and is motivated by the observation that this decomposition yields

$$\begin{aligned} \omega^+(x,\tau) &= \varepsilon^c\,\delta \textstyle\sum_{n\geq 0} \Omega_n(\tau)\,\omega_n^+(x), \\ \eta(x,\tau) &= \varepsilon^c\,\delta \textstyle\sum_{n\geq 0} \Omega_n(\tau)\,\eta_n(x) + \varepsilon^c \sum_{n\geq 0} \Psi_n(\tau)\,\zeta_n(x). \end{aligned} \tag{2.20}$$

The principal part of ω_0^+ is derived in Appendix A, while asymptotic formulas for ω_n^+, with $n \geq 1$, can be derived in a similar manner. For $\mathrm{O}(1)$ values of n, it follows that ω_n^+ is exponentially small everywhere apart from an asymptotically small neighborhood of x_* where it attains its maximum value of asymptotic magnitude at most $\mathrm{O}(\varepsilon^{-1/12}\delta^{-1})$. Similarly, the principal part of η_0 is given in Appendix B, together with an $\mathrm{L}^\infty-$estimate which shows that η_0 is at most $\mathrm{O}(\varepsilon^{1/6}\delta^{-1})$ in $[0,1]$. As a result, the coefficients of the eigenmodes Ω_n ($n \geq 0$) in (2.14) are bounded uniformly in $L^\infty(0,1)$ by an $\mathrm{O}(\varepsilon^{c-1/12})$ constant, while those of Ψ_n ($n \geq 0$) are $\mathrm{O}(1)$. In what follows, we derive the ODEs governing the evolution of these eigenmodes.

2.3.1. Eigenbasis decomposition of $\mathcal{T}^+(u^+)$ To derive the ODEs for the eigenmodes, we need to express $\mathcal{T}^+(u^+)$ in the eigenbasis $\{w_n^+\}_{n\geq 0} \cup \{v_n\}_{n\geq 0}$. In particular, we show that

$$\begin{aligned}\mathcal{T}^+(u^+) = \varepsilon^c\,\delta \sum_{k\geq 0}\left[\lambda_k\,\Omega_k - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0}\left(a_{mnk}\,\Omega_m\Omega_n + b_{mnk}\,\Psi_m\Omega_n\right)\right] w_k^+ \\ + \varepsilon^c \sum_{k\geq 0}\left[\nu_k\Psi_k - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0}\left(a'_{mnk}\,\Omega_m\Omega_n + b'_{mnk}\,\Psi_m\Omega_n\right)\right] v_k, \qquad (2.21)\end{aligned}$$

where we have omitted an $\mathrm{O}(\varepsilon^{3c})$ remainder. The coefficients appearing in this equation are given by the formulas

$$\begin{aligned} a_{mnk} &= \left\langle \begin{pmatrix} 1 \\ \varepsilon\ell^{-1}\end{pmatrix} a_m\,\omega_n^+, \hat{w}_k^+ \right\rangle = \langle a_m\omega_n, \omega_k\rangle, \\ a'_{mnk} &= \left\langle \begin{pmatrix} 1 \\ \varepsilon\ell^{-1}\end{pmatrix} \delta\,a_m\,\omega_n^+, \hat{v}_k \right\rangle = \delta\,\langle a_m\omega_n, \psi_k\rangle + \varepsilon\delta\,\ell^{-1}\langle a_m\omega_n^+, \zeta_k\rangle, \\ b_{mnk} &= \left\langle \begin{pmatrix} 1 \\ \varepsilon\ell^{-1}\end{pmatrix} b_m\,\omega_n^+, \hat{w}_k^+ \right\rangle = \langle b_m\omega_n, \omega_k\rangle, \\ b'_{mnk} &= \left\langle \begin{pmatrix} 1 \\ \varepsilon\ell^{-1}\end{pmatrix} \delta\,b_m\,\omega_n^+, \hat{v}_k \right\rangle = \delta\,\langle b_m\omega_n, \psi_k\rangle + \varepsilon\delta\,\ell^{-1}\langle b_m\omega_n^+, \zeta_k\rangle. \end{aligned} \qquad (2.22)$$

Here, we have defined the functions

$$\begin{aligned} a_m = \delta\,\left[(1-\nu)\eta_m + r(1-\nu^{-1}f)s_m\right] f \quad \text{and} \quad & b_m = (1-\nu)f\zeta_m, \\ \text{with} \quad & s_n(x) = \textstyle\int_0^x \omega_n^+(s)ds. \end{aligned} \qquad (2.23)$$

Note that we use $\langle\cdot,\cdot\rangle$ to denote all inner products—in $\mathcal{H}$, $\mathcal{H}_{\omega^+}$, and $\mathcal{H}_\eta$—as there is no danger of confusion. We remark further that (2.21) remains valid for $o(\varepsilon^{1/12-c})$ values of Ω_n ($n\geq 0$) and $o(1)$ values of Ψ_n ($n \geq 0$).

We start by decomposing $\mathcal{T}^+(u^+)$ into linear and nonlinear terms by means of

$$\mathcal{T}^+(u^+) = \mathcal{D}\mathcal{T}^+u^+ + \mathcal{N}(u^+), \quad \text{where } \mathcal{N}(u^+) = \begin{pmatrix} 1 \\ \varepsilon\ell^{-1}\end{pmatrix}(p-f)\,\omega^+. \qquad (2.24)$$

Substitution of the decomposition (2.14) into the linear term yields the eigendecomposition of that linear term,

$$\begin{aligned}\mathcal{D}\mathcal{T}^+u^+ &= \varepsilon^c\,\delta\sum_{k\geq 0}\Omega_k\,\mathcal{D}\mathcal{T}^+\,w_k^+ + \varepsilon^c\sum_{k\geq 0}\Psi_k\,\mathcal{D}\mathcal{T}^+\,v_k \\ &= \varepsilon^c\,\delta\sum_{k\geq 0}\lambda_k\,\Omega_k w_k^+ + \varepsilon^c\sum_{k\geq 0}\nu_k\Psi_k v_k, \end{aligned} \qquad (2.25)$$

where we have also used that w_n^+ and v_n are eigenvectors of $\mathcal{D}\mathcal{T}^+$ (see Section 2.1). It remains to express the nonlinearity $\mathcal{N}(u^+)$ with respect to that same eigenbasis. First, since $p-f$ contains the nonlocal term $\int_0^x \omega^+(s)ds$, see (1.7)–(1.8), we write (cf. (2.20))

$$S(x,\tau) := \varepsilon^{-c}\int_0^x \omega^+(s,\tau)ds = \delta\sum_{n\geq 0}\Omega_n(\tau)\,s_n(x), \qquad (2.26)$$

where s_n was introduced in (2.23). We subsequently obtain, by (1.7) and (1.12),

$$\begin{aligned} p &= \frac{1-\eta}{\nu^{-1}-\eta}\,\frac{1}{1+j_H\exp(\kappa x)\exp(\varepsilon^c\,r\,S)} \\ &= f\,\frac{1-\eta}{1-\nu\eta}\,\frac{1}{1+(1-\nu^{-1}f)(\exp(\varepsilon^c\,r\,S)-1)}. \end{aligned}$$

Substituting from (2.20) for ω^+ and η into this formula and expanding asymptotically, we find further

$$p(\omega^+,\eta,x) = f - \varepsilon^c\sum_{m\geq 0} a_m\Omega_m - \varepsilon^c\sum_{m\geq 0} b_m\Psi_m + \mathrm{O}\left(\varepsilon^{2c}\right), \tag{2.27}$$

with a_m and b_m as defined in (2.23). We remark for later use that this asymptotic expansion remains valid for $o(\varepsilon^{1/12-c})$ values of Ω_n and Ψ_n, see our discussion following (2.20). Next, (2.20) and (2.27) yield

$$(p-f)\,\omega^+ = -\varepsilon^{2c}\,\delta\sum_{m\geq 0}\sum_{n\geq 0} a_m\omega_n^+\,\Omega_m\Omega_n - \varepsilon^{2c}\,\delta\sum_{m\geq 0}\sum_{n\geq 0} b_m\omega_n^+\,\Psi_m\Omega_n,$$

where we have again omitted an $\mathrm{O}\left(\varepsilon^{3c}\right)$ remainder. By virtue of (2.24), then,

$$\begin{aligned} \mathcal{N}(u^+) = &-\varepsilon^{2c}\,\delta\sum_{m\geq 0}\sum_{n\geq 0}\begin{pmatrix}1\\ \varepsilon\ell^{-1}\end{pmatrix} a_m\omega_n^+\,\Omega_m\Omega_n \\ &-\varepsilon^{2c}\,\delta\sum_{m\geq 0}\sum_{n\geq 0}\begin{pmatrix}1\\ \varepsilon\ell^{-1}\end{pmatrix} b_m\omega_n^+\,\Psi_m\Omega_n + \mathrm{O}\left(\varepsilon^{3c}\right). \end{aligned}$$

This last result directly leads to the decompositions

$$\begin{aligned} \begin{pmatrix}1\\ \varepsilon\ell^{-1}\end{pmatrix}\delta\,a_m\,\omega_n^+ &= \sum_{k\geq 0}\left(\delta\,a_{mnk}w_k^+ + a'_{mnk}v_k\right), \\ \begin{pmatrix}1\\ \varepsilon\ell^{-1}\end{pmatrix}\delta\,b_m\,\omega_n^+ &= \sum_{k\geq 0}\left(\delta\,b_{mnk}w_k^+ + b'_{mnk}v_k\right), \end{aligned}$$

where the coefficients a_{mnk}, a'_{mnk}, b_{mnk}, and b'_{mnk} are found by means of (2.22). Using this decomposition, we finally write

$$\begin{aligned} \mathcal{N}(u^+) = &-\varepsilon^{2c}\sum_{m\geq 0}\sum_{n\geq 0}\sum_{k\geq 0}\left(\delta\,a_{mnk}\,w_k^+ + a'_{mnk}\,v_k\right)\Omega_m\Omega_n \\ &-\varepsilon^{2c}\sum_{m\geq 0}\sum_{n\geq 0}\sum_{k\geq 0}\left(\delta\,b_{mnk}\,w_k^+ + b'_{mnk}\,v_k\right)\Psi_m\Omega_n + \mathrm{O}\left(\varepsilon^{3c}\right). \end{aligned} \tag{2.28}$$

Combining (2.25) and (2.28), then, we arrive at the desired result (2.21).

2.3.2. ODEs near the bifurcation point We are now in a position to derive the ODEs for the amplitudes $\{\Omega_n\}_{n\geq 0}$ and $\{\Psi_n\}_{n\geq 0}$. Differentiating both members of (2.14) with respect to time, we find

$$\partial_\tau u^+ = \varepsilon^c\,\delta\sum_{k\geq 0}\dot{\Omega}_k\,w_k^+ + \varepsilon^c\sum_{k\geq 0}\dot{\Psi}_k\,v_k, \tag{2.29}$$

where the overdot denotes differentiation with respect to τ. Next, $\partial_\tau u^+ = \mathcal{T}^+(u^+)$ and hence, combining (2.21) with (2.29), we obtain the ODEs for the amplitudes,

$$\dot{\Omega}_k = \lambda_k \Omega_k - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0} \left(a_{mnk}\, \Omega_m\Omega_n + b_{mnk}\, \Psi_m\Omega_n\right) + \mathrm{O}\left(\varepsilon^{2c}\right),$$

$$\dot{\Psi}_k = \nu_k \Psi_k - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0} \left(a'_{mnk}\, \Omega_m\Omega_n + b'_{mnk}\, \Psi_m\Omega_n\right) + \mathrm{O}\left(\varepsilon^{2c}\right).$$

We now tune the bifurcation parameter λ_* so that the largest eigenvalue, λ_0, is the only positive eigenvalue. In particular, we write (cf. (1.16))

$$\begin{aligned}
\lambda_0 &= \varepsilon^\alpha \Lambda_0, \text{ where } \Lambda_0 > 0 \text{ and } \alpha > 1/3,\\
\nu_k &= -\,\varepsilon N_k, \text{ where } N_k > 0 \text{ is } \mathrm{O}(1) \text{ and } k = 0, 1, \ldots,\\
\lambda_k &= -\,\varepsilon^{1/3}\Lambda_k, \text{ where } \Lambda_k > 0 \text{ is } \mathrm{O}(1) \text{ and } k = 1, 2, \ldots.
\end{aligned}$$

(As we will see shortly, the scaling of particular interest will turn out to be the scaling $\alpha = 1$, with Λ_0 either O(1) or logarithmically large.) Then, the evolution equations for the amplitudes become

$$\dot{\Omega}_0 = \varepsilon^\alpha \Lambda_0 \Omega_0 - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0} a_{mn0}\Omega_m\Omega_n - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0} b_{mn0}\Psi_m\Omega_n, \tag{2.30}$$

$$\dot{\Psi}_k = -\varepsilon N_k \Psi_k - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0} a'_{mnk}\Omega_m\Omega_n - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0} b'_{mnk}\Psi_m\Omega_n, \quad k \geq 0, \tag{2.31}$$

$$\dot{\Omega}_k = -\varepsilon^{1/3}\Lambda_k \Omega_k - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0} a_{mnk}\Omega_m\Omega_n - \varepsilon^c \sum_{m\geq 0}\sum_{n\geq 0} b_{mnk}\Psi_m\Omega_n, \quad k \geq 1, \tag{2.32}$$

where we have omitted all higher order terms.

3. Application of Laplace's method on a_{000}

Explicit asymptotic expressions for the coefficients in the ODEs (2.30)–(2.32) obtained in the previous section can be derived by applying Laplace's method and the principle of stationary phase to the integrals in (2.22). In this section, we demonstrate the use of the former in deriving an asymptotic formula for a_{000}. Asymptotic expressions for the remaining coefficients will be derived independently in Sections 6–8, after we have thoroughly analyzed the bifurcations that our system undergoes. Although the analysis in those sections is substantially more involved, our approach there is very similar to that in the present section.

The main result of this section is the leading order approximation

$$a_{000} = a_{000}(\Lambda_0) = \varepsilon^{1/6}\, C_a, \quad \text{where } C_a = \bar{C}_a \frac{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}{\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)} \tag{3.1}$$

and we have defined the O(1), Λ_0−independent constant

$$\bar{C}_a = (1-\nu)\, f(0)\, C_1\, C_2\, \sigma_*^{-1/2}\sigma_0^{1/3} > 0. \tag{3.2}$$

Here, σ_0 as defined in (1.15), while

$$C_1 = \left(\int_{A_1}^{\infty} \mathrm{Ai}^2(s)\, ds\right)^{-1/2}, \quad C_2 = \exp(|A_1|^{3/2}), \quad \text{and} \quad \sigma_* = F'(x_*), \tag{3.3}$$

see [23] and Appendix A. We start by recalling that the coefficient a_{000} is given by

$$a_{000} = \int_0^1 a_0(x)\,\omega_0^2(x)\,dx, \tag{3.4}$$

cf. (2.22), where

$$a_0(x) = \delta\left[r(1-\nu^{-1}f(x))s_0(x) + (1-\nu)\eta_0(x)\right]\,f(x).$$

Employing (2.23), (2.26), using the explicit approximation (B.5) for η_0 from Appendix B, and defining the functions

$$h_1(x,y) = f(x)\,\left[r\,\left(1-\frac{f(x)}{\nu}\right) - \frac{1-\nu}{\ell\sqrt{\Lambda_0}}\,\sinh\left(\sqrt{\Lambda_0}(x-y)\right)\,f(y)\right], \tag{3.5}$$

$$h_2(x,y) = \frac{(1-\nu)\,f(x)\,\cosh\left(\sqrt{\Lambda_0}\,x\right)}{\ell\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)}\,f(y)\,\sinh\left(\sqrt{\Lambda_0}(1-y)\right), \tag{3.6}$$

we find further

$$a_0(x) = \delta\int_0^x h_1(x,y)\,\omega_0^+(y)\,dy + \delta\int_0^1 h_2(x,y)\,\omega_0^+(y)\,dy.$$

Thus,

$$\begin{aligned} a_{000} &= \delta\int_0^1\int_0^x h_1(x,y)\,\omega_0^2(x)\,\omega_0^+(y)\,dydx + \delta\int_0^1\int_0^1 h_2(x,y)\,\omega_0^2(x)\,\omega_0^+(y)\,dydx \\ &= \delta(\mathcal{I}_1+\mathcal{I}_2), \end{aligned} \tag{3.7}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are the two double integrals appearing in this expression.

We can obtain the principal parts of $\mathcal{I}_1$ and $\mathcal{I}_2$ using Theorem Appendix D.2 in Appendix D (which is based on [22]). We start with the latter integral which, as we will see, fully determines the leading order behavior of a_{000}. First, the normalization condition $\|\omega_0\|_2 = 1$ yields $\int_0^1 h_2(x,y)\,\omega_0^2(x)\,dx = h_2(0,y)$ to leading order. Since, also, ω_0^+ has a unique maximum at the interior critical point x_*, Theorem Appendix D.2.I (with $\lambda = \varepsilon^{-1/2}$, $\Pi = -J_-$, and $\Xi = h_2(0,\cdot)$) yields

$$\mathcal{I}_2 = \int_0^1 h_2(0,y)\,\omega_0^+(y)\,dy = \frac{1}{(\varepsilon^{-1/2})^{1/2}}\,\frac{\sqrt{2\pi}\,h_2(0,x_*)}{\sqrt{-J_-''(x_*)}}\,\omega_0^+(x_*) = \varepsilon^{1/6}\,\delta^{-1}\,C_3 \tag{3.8}$$

to leading order, where we have used the explicit leading order approximation (A.2) of ω_0^+ from [23] (see also Appendix A), recalled the definition (2.16) of δ, defined

$$C_3 = \frac{\sqrt{2\pi}\,h_2(0,x_*)}{\sqrt{-J_-''(x_*)}}\,\frac{C_1\,C_2\,\sigma_0^{1/3}}{2\sqrt{\pi}\,F^{1/4}(x_*)} = C_1\,C_2\,\sigma_0^{1/3}\,\sigma_*^{-1/2}\,h_2(0,x_*), \tag{3.9}$$

and employed the identity $J_-'' = -2^{-1}F^{-1/2}F'$.

Next, we show $\mathcal{I}_1$ to be exponentially smaller than $\mathcal{I}_2$. First, we rewrite it as

$$\mathcal{I}_1 = \varepsilon^{-1/4}\,\frac{C_1^3\,C_2^3\,\sigma_0}{8\pi^{3/2}}\,\sum_{j=1}^{6}\theta_j\int\int_D \frac{h_1(x,y)}{\sqrt{F(x)}\,F^{1/4}(y)}\,\exp\left(\frac{\Pi_j(x,y)}{\sqrt{\varepsilon}}\right)\,dA_{xy}, \tag{3.10}$$

where we have used (A.2) and (A.1). Here, $D = \{(x,y)|0 \le y \le x,\ 0 \le x \le 1\}$ and

$$\begin{array}{rcllrcl} \Pi_1(x,y) &=& J_-(y) - 2I(x) & \text{and} & \theta_1 &=& 1, \\ \Pi_2(x,y) &=& J_-(y) - 2I(1) & \text{and} & \theta_2 &=& 2\theta, \\ \Pi_3(x,y) &=& J_-(y) + 2I(x) - 4I(1) & \text{and} & \theta_3 &=& \theta^2, \\ \Pi_4(x,y) &=& J_+(y) - 2I(x) - 2I(1) & \text{and} & \theta_4 &=& \theta, \\ \Pi_5(x,y) &=& J_+(y) - 4I(1) & \text{and} & \theta_5 &=& 2\theta^2, \\ \Pi_6(x,y) &=& J_+(y) + 2I(x) - 6I(1) & \text{and} & \theta_6 &=& \theta^3, \end{array} \tag{3.11}$$

where $I(x)$ and $J_{\pm 1}(y)$ have been defined in (2.17), and

$$\theta = \frac{\sqrt{\sigma_1} + \sqrt{A}}{\sqrt{\sigma_1} - \sqrt{A}} \quad \text{with} \quad \sigma_1 = F(1). \tag{3.12}$$

Theorem Appendix D.1 yields, for each integral, a result proportional to $\exp(\max_D \Pi_j/\sqrt{\varepsilon})$ (where the maximum is taken over the domain D). We first identify $\max \Pi_1$ and then show that $\max \Pi_1 > \max \Pi_j$, for $j = 2, \ldots, 6$; it follows that the dominant term in (3.10) corresponds to Π_1 and the rest are exponentially smaller with respect to it. Now, Π_1 has no critical points in D, and thus its global maximum lies on

$$\begin{aligned} \partial D &= (\partial D)_1 \cup (\partial D)_2 \cup (\partial D)_3 \\ &= \{(1,y)|0 \le y \le 1\} \cup \{(x,x)|0 \le x \le 1\} \cup \{(x,0)|0 \le x \le 1\}. \end{aligned}$$

First, the global maximum cannot be on on $(\partial D)_1$; indeed, $\mathring{D}$ lies to the left of $(\partial D)_1$ and $\partial_x \Pi_1(x,y) = -2\sqrt{\bar{F}(x)} \le 0$, where we have introduced $\bar{F}(x) = F(x) - F(\bar{x}_0)$, so that Π_1 assumes higher values in $\mathring{D}$ than on $(\partial D)_1$. Next, $\Pi_1(x,x) = \sqrt{A}x - 3I(x)$ on $(\partial D)_2$, and thus $\max \Pi_1(x,x) = \Pi_1(x^{**},x^{**})$ with $0 < x^{**} = \bar{F}^{-1}(A/9) < x_*$ (recall (2.19) and note that $\bar{F} > 0$ is increasing). Finally, $\Pi_1(x,0) = -2I(x) \le 0$ on $(\partial D)_3$, and thus $\max_{(\partial D)_3} \Pi_1 \le 0 < \Pi_1(x^{**},x^{**})$. In total, then, we find that $\max \Pi_1 = \Pi_1(x^{**},x^{**}) > 0$. Next, $\Pi_2(x,y) \le \Pi_1(x,y) \le \Pi_1(x^{**},x^{**})$. Since the leftmost equality holds only in an $\mathrm{O}(\varepsilon^{1/2})$-neighborhood of $x = 1$, we find that $\max \Pi_2 < \Pi_1(x^{**},x^{**})$, as desired. Additionally, $\Pi_3 \le \Pi_2$ on D, and thus also $\max_D \Pi_3 < \max_D \Pi_1$. Next, Π_4 has no critical points in $\mathring{D}$, and hence we need to examine its behavior on ∂D. First, the maximum cannot be on $(\partial D)_1$ by the same argument we used for Π_1. Next, $\Pi_4(x,x) = J_-(x) - 2I(1)$ on $(\partial D)_2$, and thus $\max_{(\partial D)_2} \Pi_4 = \Pi_4(x_*,x_*) = J_-(x_*) - 2I(1)$. Finally, $\Pi_4 \le -2I(1) < \Pi_4(x_*,x_*)$ on $(\partial D)_3$, and hence $\max \Pi_4 = J_-(x_*) - 2I(1) = \max \Pi_2 < \max \Pi_1$, as desired. Finally, $\Pi_5 \le \Pi_4$ and $\Pi_6 \le \Pi_4$, and the desired result follows.

These estimates show, then, that $\max \Pi_1 = \Pi(x^{**},x^{**}) > \max \Pi_j$, for $j = 2, \ldots, 6$. Since $(x^{**},x^{**}) \in \partial D$ and its Jacobian satisfies $D\Pi_1(x^{**},x^{**}) \ne 0$, Theorem Appendix D.1 yields for (3.10) the asymptotic formula

$$\mathcal{I}_1 = \varepsilon^{3/4}\, C_1' \left(\varepsilon^{-1/4} \frac{C_1^3\, C_2^3}{8\pi^{3/2}} \exp\left(\frac{\Pi_1(x^{**},x^{**})}{\sqrt{\varepsilon}} \right) \right) = \varepsilon^{1/2}\, C_1'' \exp\left(\frac{\Pi_1(x^{**},x^{**})}{\sqrt{\varepsilon}} \right),$$

for some O(1) constants $C_1', C_1'' > 0$. Since $\mathcal{I}_2 = \mathcal{O}(\varepsilon^{1/6}\delta^{-1})$ (3.8) and, by (2.16),

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} = \varepsilon^{1/3} \frac{C_1''}{C_3} \exp\left(\frac{\Pi_1(x^{**},x^{**}) - J_-(x_*)}{\sqrt{\varepsilon}} \right)$$

with

$$\Pi_1(x^{**}, x^{**}) - J_-(x_*) = [J_-(x^{**}) - J_-(x_*)] - 2I(x^{**}) < 0$$

(recall that x_* is defined as the location of the maximum of J_- (2.19)), it indeed follows that $\mathcal{I}_1$ is exponentially small compared to $\mathcal{I}_2$.

We conclude that a_{000} is given by $\delta\,\mathcal{I}_2$ at leading order. Combining the expressions (3.8)–(3.9) with the definition of h_2 in (3.6), we obtain the leading order result (3.1) by using the fact that $f(x_*) = \ell$ to leading order. To derive this last identity, observe that—in the regime $\lambda_0 \ll 1$—it holds that $\lambda_* = 0$ at O(1) (1.14), (1.16), or equivalently that $A = f(0) - \ell$; further, and also to leading order, $F(x_*) = A$ (2.19), so that the desired identity follows from the definition $F(x) = f(0) - f(x)$ (1.16). Finally, we note that higher order terms in formula (3.1) may be obtained solely by considering $\mathcal{I}_2$, as $\mathcal{I}_1$ is exponentially smaller than $\mathcal{I}_2$.

4. Emergence of a stable DCM

The trivial (zero) state is, by construction, a fixed point of the ODEs (2.30)–(2.32) for the Fourier coefficients. In this and the next section, we identify the remaining fixed points of (2.30)–(2.32) and determine their stability. In this entire section, we work exclusively in the regime $\alpha = 1$ and $\Lambda_0 = \mathrm{O}(1)$.

4.1. Asymptotic expressions for b_{m00}, a'_{00k}, and b'_{m0k}

As stated in the previous section, where we derived an asymptotic expression for a_{000}, asymptotic expressions for the coefficients b_{m00}, a'_{00k}, and b'_{m0k} appearing in (2.30)–(2.32) are derived independently in Sections 6–8 below. Here, we summarize the leading order behavior of these coefficients, including also (3.1) for completeness:

$$\begin{aligned} a_{000} &= \varepsilon^{1/6}\, C_a, && \\ b_{m00} &= C_b, && \text{for } m \ll \varepsilon^{-1/3}, \\ a'_{00k} &= \varepsilon^{4/3}\, C_a\, (\alpha_{-1} - \beta_k + \gamma)\, Z_k, && \text{for } 0 \neq k \ll \varepsilon^{-1/3}, \\ b'_{m0k} &= \varepsilon^{7/6}\, C_b\, (\alpha_m - \beta_k)\, Z_k, && \text{for } 0 \neq k, m \ll \varepsilon^{-1/3}. \end{aligned} \tag{4.1}$$

The positive O(1) constant C_a was reported in (3.1), whereas $C_b = \sqrt{2}\,(1-\nu)\,f(0)$. Further, we have introduced the O(1) (with respect to ε) constants

$$Z_m = \sqrt{2}\, C_2\, \sigma_0^{1/3}\, \sigma_*^{-1/2}\, \cos\left(\sqrt{N_m}\, x_*\right), \quad \alpha_{-1} = \frac{C_1 \cosh\left(\sqrt{\Lambda_0}\, x_*\right)}{f(0)} \tag{4.2}$$

$$\alpha_m = \frac{C_1\, \cos\left(\sqrt{N_m}\, x_*\right)}{f(0)}, \quad \beta_k = \frac{1}{(\mathrm{Ai}'(A_1))^2\, C_1\, N_k}, \tag{4.3}$$

$$\gamma = \frac{r\, C_1\, (1 - \nu^{-1}\ell)\sqrt{\Lambda_0}\cosh\left(\sqrt{\Lambda_0}\right)}{2(1-\nu) f(0) \sinh\left(\sqrt{\Lambda_0}(1 - x_*)\right)}, \tag{4.4}$$

with σ_0, σ_*, C_1 and C_2 as defined in (1.15), (3.3). Asymptotic formulas for higher values of m and k can be derived similarly. However, seeing as such formulas only contribute higher order terms in our analysis below, we refrain from presenting the details. In what follows, instead, we treat (4.1) as being valid for all values of k and m.

4.2. The reduced system

The system (2.30)–(2.32) exhibits asymptotically disparate timescales depending on the value of α and associated with the asymptotic magnitudes of the eigenvalues. In this section, we investigate the case $\alpha = 1$, in which regime Ω_0 and Ψ_0, Ψ_1, ... evolve on a slow timescale and the higher-order modes Ω_1, Ω_2, ... become slaved to them. Setting, then, $\alpha = 1$ and rescaling time (with a slight abuse of notation) as $t = \varepsilon\tau$, the evolution equations become

$$\dot{\Omega}_0 = \Lambda_0\Omega_0 - \varepsilon^{c-1}\sum_{m\geq 0}\sum_{n\geq 0} a_{mn0}\Omega_m\Omega_n - \varepsilon^{c-1}\sum_{m\geq 0}\sum_{n\geq 0} b_{mn0}\Psi_m\Omega_n, \tag{4.5}$$

$$\dot{\Psi}_k = -N_k\Psi_k - \varepsilon^{c-1}\sum_{m\geq 0}\sum_{n\geq 0} a'_{mnk}\Omega_m\Omega_n - \varepsilon^{c-1}\sum_{m\geq 0}\sum_{n\geq 0} b'_{mnk}\Psi_m\Omega_n, \quad k \geq 0, \tag{4.6}$$

$$\varepsilon^{2/3}\dot{\Omega}_k = -\Lambda_k\Omega_k - \varepsilon^{c-1/3}\sum_{m\geq 0}\sum_{n\geq 0} a_{mnk}\Omega_m\Omega_n - \varepsilon^{c-1/3}\sum_{m\geq 0}\sum_{n\geq 0} b_{mnk}\Psi_m\Omega_n, \quad k \geq 1. \tag{4.7}$$

(Here also, the overdot denotes differentiation with respect to t.) It is natural to introduce slaving relations for the latter modes in this system,

$$\Omega_k = \varepsilon^{c_k}\, G_k(\Omega_0, \Psi_1, \Psi_2, \ldots), \quad \text{for all} \quad k \geq 1, \tag{4.8}$$

where the positive constants $c_1, c_2, \ldots$ and the O(1) functions (with O(1) partial derivatives) G_1, G_2, ... are to be determined. To do so, we first write the evolution equations for Ω_0 and Ψ_1, Ψ_2, ... under these slaving relations; we find

$$\dot{\Omega}_0 = \Lambda_0\Omega_0 - \varepsilon^{c-1}a_{000}\Omega_0^2 - \varepsilon^{c-1}\Omega_0\sum_{m\geq 0} b_{m00}\Psi_m,$$

$$\dot{\Psi}_k = -N_k\Psi_k - \varepsilon^{c-1}a'_{00k}\Omega_0^2 - \varepsilon^{c-1}\Omega_0\sum_{m\geq 0} b'_{m0k}\Psi_m, \quad k \geq 0,$$

where we have retained only the leading order terms from each sum. Dominant balance yields, then, $c = 1$. Next, the invariance equation for Ω_k yields that the right member of (4.7) must vanish to leading order. Dominant balance yields $c_k = 2/3$ and

$$G_k(\Omega_0, \Psi_1, \Psi_2, \ldots) = -\frac{a_{00k}}{\Lambda_k}\Omega_0^2 - \frac{\Omega_0}{\Lambda_k}\sum_{m\geq 0} b_{m0k}\Psi_m,$$

whereas the evolution equations become

$$\begin{aligned} \dot{\Omega}_0 &= \Lambda_0\Omega_0 - a_{000}\Omega_0^2 - \Omega_0\textstyle\sum_{m\geq 0} b_{m00}\Psi_m, \\ \dot{\Psi}_k &= -N_k\Psi_k - a'_{00k}\Omega_0^2 - \Omega_0\textstyle\sum_{m\geq 0} b'_{m0k}\Psi_m, \quad k \geq 0. \end{aligned} \tag{4.9}$$

Here also, we have retained only the leading order terms from each sum.

4.3. The bifurcating steady state

In this section, we identify the nontrivial fixed point of the reduced system (4.9). In particular, we show that this fixed point is given to leading order by the formulas

$$\Omega_0^* = \varepsilon^{-1/6}\,\bar{C}_a^{-1}\,\frac{\Lambda_0^{3/2}\,\cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}, \tag{4.10}$$

$$\Psi_k^* = -\varepsilon\,\frac{Z_k}{\bar{C}_a\,N_k}\,\frac{\Lambda_0^{5/2}\,\cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}\,\left(\alpha_{-1}(\Lambda_0)+\gamma(\Lambda_0)-\beta_k\right), \quad k\geq 0, \tag{4.11}$$

where we have made explicit the dependence of α_{-1} and γ on Λ_0 (cf. (4.3)). As will become clear in the remainder of this section, this fixed point corresponds to a DCM with an $\mathrm{O}(\varepsilon)$ biomass and an associated $\mathrm{O}(\varepsilon)$ nutrient depletion.

4.3.1. Derivation of (4.10)–(4.11) First, we substitute the formulas collected in (4.1) into (4.9), set the left members to zero, and rescale Ω_0 by

$$\Omega_0 = \varepsilon^{-1/6}\,\bar{\Omega}_0 \tag{4.12}$$

to obtain the system

$$\Lambda_0 - C_a\,\bar{\Omega}_0 - C_b\sum_{m\geq 0}\Psi_m = 0, \tag{4.13}$$

$$N_k\Psi_k + \varepsilon\,Z_k\,\bar{\Omega}_0\,\left[\left(\alpha_{-1}-\beta_k+\gamma\right)\,C_a\,\bar{\Omega}_0 + C_b\sum_{m\geq 0}\left(\alpha_m-\beta_k\right)\,\Psi_m\right] = 0. \tag{4.14}$$

Here, $k\geq 0$ and we have removed a superfluous factor of $\bar{\Omega}_0$ in (4.13). Solving this equation for $\bar{\Omega}_0$ and substituting into (4.14), we obtain the equivalent formulation

$$\frac{\Lambda_0}{C_a} - \frac{C_b}{C_a}\sum_{m\geq 0}\Psi_m = \bar{\Omega}_0, \tag{4.15}$$

$$N_k\Psi_k + \varepsilon\,Z_k\,\bar{\Omega}_0\,\left[\left(\alpha_{-1}-\beta_k+\gamma\right)\,\Lambda_0 + C_b\sum_{m\geq 0}\left(\alpha_m-\alpha_{-1}-\gamma\right)\,\Psi_m\right] = 0. \tag{4.16}$$

There are three possible dominant balance scenarios for this system of equations. The first one we consider—and the only one that leads to a bounded solution, as we shall see—is between the term in the right member of (4.15) and the first term in its left member. This scenario yields, for the leading order behavior of the $\bar{\Omega}_0-$component of the fixed point, the formula

$$\bar{\Omega}_0^* = \frac{\Lambda_0}{C_a} = \bar{C}_a^{-1}\,\frac{\Lambda_0^{3/2}\,\cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}, \tag{4.17}$$

with $\bar{C}_a$ defined earlier. Here, we have recalled the definitions (3.1) and (3.2) of C_a and $\bar{C}_a$. In particular, $\bar{\Omega}_0^*$ is $\mathrm{O}(1)$ and hence it follows that $\Psi_0, \Psi_1, \ldots$ are $\mathrm{O}(\varepsilon)$, $\Psi_m = \varepsilon\,\bar{\Psi}_m$. Thus, (4.16) yields, to leading order and for each $k\geq 0$,

$$N_k\bar{\Psi}_k + \left(\alpha_{-1}-\beta_k+\gamma\right)\,\Lambda_0\,Z_k\,\bar{\Omega}_0^* = 0,$$

whence

$$\bar{\Psi}_k^* = -\frac{Z_k}{\bar{C}_a\,N_k}\,\frac{\Lambda_0^{5/2}\,\cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}\,(\alpha_{-1}+\gamma-\beta_k)\,. \tag{4.18}$$

Using (4.17)–(4.18) and recalling that $\Omega_0^* = \varepsilon^{-1/6}\,\bar{\Omega}_0^*$ and $\Psi_k^* = \varepsilon\,\bar{\Psi}_k^*$, we obtain the desired formulas (4.10)–(4.11).

Remark 4.1. It remains to investigate the two remaining dominant balances for (4.15)–(4.16) or, equivalently, for (4.13)–(4.14). First, the balancing between the first and the third terms in the left member of (4.13) yields no solution. Indeed, let Ψ_m be O(1). It follows that $\bar{\Omega}_0 = \varepsilon^{\beta}\,\bar{\Omega}_0'$, for some $\beta \geq 0$ and and an O(1) quantity $\bar{\Omega}_0'$. Substituting into (4.14), we obtain, to leading order and for each $k \geq 0$,

$$N_k\bar{\Psi}_k + \varepsilon^{1+\beta}\,Z_k\,\bar{\Omega}_0'\left[\varepsilon^{\beta}\,\left(\alpha_{-1}-\beta_k+\gamma\right)\,C_a\,\bar{\Omega}_0' + C_b\sum_{m\geq 0}\left(\alpha_m-\beta_k\right)\,\bar{\Psi}_m\right] = 0.$$

Thus, $\bar{\Psi}_k = 0$ for all k, which violates our explicit assumption that Ψ_k is O(1). As for the remaining balancing—between the second and third terms in (4.13)—we let $\bar{\Omega}_0 = \varepsilon^{\beta}\,\bar{\Omega}_0'$ and $\Psi_m = \varepsilon^{\beta}\,\bar{\Psi}_m$. Substituting this Ansatz into (4.14), we obtain, to leading order and for each $k \geq 0$,

$$\varepsilon^{\beta}\,N_k\,\bar{\Psi}_k + \varepsilon^{1+2\beta}\,Z_k\,\bar{\Omega}_0'\left[\,\left(\alpha_{-1}-\beta_k+\gamma\right)C_a\,\bar{\Omega}_0' + C_b\sum_{m\geq 0}\left(\alpha_m-\beta_k\right)\bar{\Psi}_m\right] = 0,$$

whence $\beta = -1$. This scaling violates our explicit assumption that Ω_n is at most $o(\varepsilon^{-11/12})$ and Ψ_n is at most $o(1)$—cf. Section 2.3.1, in particular—and thus it does not lead to a consistent solution. We will see in the next section, however, that this pattern becomes tractable and comes into play when Λ_0 becomes logarithmically large; in that regime, the pattern interacts with the small DCM pattern we identified above.

4.3.2. Ecological interpretation We now proceed to show that the steady state (*stationary pattern*) we identified above corresponds to an O(ε) biomass with a corresponding O(ε) depletion of the nutrient. Indeed, (2.20) and the rescaling (4.12) yield the leading order expression

$$\int_0^1 \omega^+(x)\,dx = \varepsilon\,\delta\,\Omega_0^*\int_0^1 \omega_0^+(x)\,dx \tag{4.19}$$

for the biomass. (Here, we have also recalled that $c = 1$ and that $\Omega_1^*, \Omega_2^*, \ldots$ are higher order, cf. (4.8).) Recalling the definition of δ in (2.16) and using the explicit leading order formula (A.2) for ω_0^+, we obtain

$$\delta\int_0^1 \omega_0^+(x)\,dx = \varepsilon^{-1/12}\,\frac{C_1C_2\sigma_0^{1/3}}{2\sqrt{\pi}}\int_0^1 F^{-1/4}(x)\exp\left(\frac{J_-(x)-J_-(x_*)}{\sqrt{\varepsilon}}\right)\,dx.$$

As mentioned in Section 3, $J_-(\cdot)$ has a sole, locally quadratic maximum at x_*, and hence the integrand above is exponentially small except in an asymptotically small neighborhood of that point. Hence, the integral is of the type considered in Appendix D; Theorem Appendix D.1 yields, to leading order,

$$\delta\int_0^1 \omega_0^+(x)\,dx = \varepsilon^{-1/12}\,\frac{C_1\,C_2\,\sigma_0^{1/3}}{2\sqrt{\pi}}\left(\varepsilon^{1/4}\,\frac{\sqrt{2\pi}}{F^{1/4}(x_*)\,\sqrt{-J_-''(x_*)}}\right)$$
$$= \varepsilon^{1/6}\,C_1\,C_2\,\sigma_*^{-1/2}\,\sigma_0^{1/3}, \tag{4.20}$$

where we have also recalled that $J_-'' = -2^{-1}F^{-1/2}F'$. Substituting back into (4.19), together with the formula for Ω_0^* given in (4.10), we finally recover the expression (1.20) for the total biomass given in the Introduction. Similarly, (2.20) yields the leading order formula

$$\int_0^1 \eta(x)\,dx = \varepsilon\left[\delta\,\Omega_0^*\int_0^1 \eta_0(x)\,dx + \sum_{k\geq 0}\Psi_k^*\int_0^1 \zeta_k(x)\,dx\right]. \tag{4.21}$$

Now, $\Psi_k^* = \varepsilon\bar{\Psi}_k^*$ with $\bar{\Psi}_k^*$ given to leading order by (4.18), and $\int_0^1 \zeta_k(x)\,dx = (-1)^k/N_k$ by (2.6). The integral $\int_0^1 \eta_0(x)\,dx$ can be calculated using (B.1): integrating both members over $[0,x]$ and using the boundary condition at zero, we find

$$\ell\,\Lambda_0\int_0^1 \eta_0(x)\,dx = \ell\,\partial_x\eta_0(1) + \int_0^1 f(x)\,\omega_0^+(x)\,dx. \tag{4.22}$$

The derivative $\partial_x\eta_0(1)$ can be estimated at leading order by (B.5). Differentiating both members of that formula, we find

$$\ell\,\partial_x\eta_0(1) = \int_0^1 \left[\tanh\left(\sqrt{\Lambda_0}\right)\sinh\left(\sqrt{\Lambda_0}(1-y)\right) - \cosh\left(\sqrt{\Lambda_0}(1-y)\right)\right] f(y)\,\omega_0^+(y)\,dy.$$

It follows from (4.22), then, that

$$\ell\,\Lambda_0\int_0^1 \eta_0(x)\,dx$$
$$= \int_0^1 \left[1 + \tanh\left(\sqrt{\Lambda_0}\right)\sinh\left(\sqrt{\Lambda_0}(1-y)\right) - \cosh\left(\sqrt{\Lambda_0}(1-y)\right)\right]\,f(y)\,\omega_0^+(y)\,dy.$$

Applying Theorem Appendix D.1, we obtain

$$\int_0^1 \eta_0(x)\,dx = \varepsilon^{1/6}\delta^{-1}\,\frac{C_1C_2\,\sigma_*^{-1/2}\,\sigma_0^{1/3}}{\Lambda_0}\left(1 - \frac{\cosh\left(\sqrt{\Lambda_0}\,x_*\right)}{\cosh\left(\sqrt{\Lambda_0}\right)}\right), \tag{4.23}$$

which is the desired formula for $\int_0^1 \eta_0(x)\,dx$. Hence, (4.21) becomes to leading order

$$\int_0^1 \eta(x)\,dx = \varepsilon\left[\bar{\Omega}_0^*\,\frac{C_1C_2\,\sigma_*^{-1/2}\,\sigma_0^{1/3}}{\Lambda_0}\left(1 - \frac{\cosh\left(\sqrt{\Lambda_0}\,x_*\right)}{\cosh\left(\sqrt{\Lambda_0}\right)}\right) + \varepsilon\sum_{k\geq 0}\frac{(-1)^k}{N_k}\bar{\Psi}_k^*\right]$$
$$= \varepsilon\bar{C}_a^{-1}\frac{\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}\left[C_1C_2\,\sigma_*^{-1/2}\,\sigma_0^{1/3}\left(1 - \frac{\cosh\left(\sqrt{\Lambda_0}\,x_*\right)}{\cosh\left(\sqrt{\Lambda_0}\right)}\right)\right.$$
$$\left. -\varepsilon\left[\sum_{k\geq 0}\frac{(-1)^k Z_k}{N_k^2}\right]\Lambda_0^2\,\left(\alpha_{-1}(\Lambda_0) + \gamma(\Lambda_0) - \beta_k\right)\right]. \tag{4.24}$$

Since the second term in the right member of this equation is $\mathrm{O}(\varepsilon)$, we finally find that the total nutrient depletion level is, to leading order,

$$\int_0^1 \eta(x)\,dx = \frac{\varepsilon\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)\left(1-\cosh\left(\sqrt{\Lambda_0}\,x_*\right)/\cosh\left(\sqrt{\Lambda_0}\right)\right)}{(1-\nu)\,f(0)\,\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}. \quad (4.25)$$

4.4. Stability of the small pattern

In this section, we examine the stability of the DCM-like fixed point $(\Omega_0^*, \Psi^*) = (\Omega_0^*, \Psi_0^*, \Psi_1^*, \ldots)$ which we identified in the previous section. In particular, we show that this fixed point is stabilized through a transcritical bifurcation at $\Lambda_0 = 0$ and that it remains stable for all positive and $\mathrm{O}(1)$ values of Λ_0.

Before linearizing around (Ω_0^*, Ψ^*) to determine the stability of this non-trivial fixed point, we rescale Ω_0 and Ψ in (4.9) by means of (4.12) and of $\Psi = \varepsilon\bar{\Psi}$. Then, we substitute for b_{m00} and a'_{00k} from (4.1) to find

$$\begin{aligned}
\dot{\bar{\Omega}}_0 &= \Lambda_0\,\bar{\Omega}_0 - C_a\,\bar{\Omega}_0^2 - \varepsilon\,C_b\,\bar{\Omega}_0\textstyle\sum_{m\geq 0}\bar{\Psi}_m,\\
\dot{\bar{\Psi}}_k &= -N_k\bar{\Psi}_k - C_a Z_k(\alpha_{-1}-\beta_k+\gamma)\bar{\Omega}^2 - \varepsilon C_b Z_k\bar{\Omega}_0\textstyle\sum_{m\geq 0}(\alpha_m-\beta_k)\bar{\Psi}_m,
\end{aligned}$$

with $k \geq 0$. Letting $\bar{\Omega}_0 = \bar{\Omega}_0^* + \mathrm{d}\bar{\Omega}_0$ and $\bar{\Psi}_k = \bar{\Psi}_k^* + \mathrm{d}\bar{\Psi}_k$, and recalling that $C_a\,\bar{\Omega}_0^* = \Lambda_0$ to leading order (cf. (4.17)), we find that the corresponding linearized problem reads

$$\begin{aligned}
\mathrm{d}\dot{\bar{\Omega}}_0 &= -\left[\Lambda_0 + \varepsilon\,C_b\textstyle\sum_{m\geq 0}\bar{\Psi}_m^*\right]\mathrm{d}\bar{\Omega}_0 - \varepsilon\,C_b\,\bar{\Omega}_0^*\textstyle\sum_{m\geq 0}\mathrm{d}\bar{\Psi}_m,\\
\mathrm{d}\dot{\bar{\Psi}}_k &= -\left[N_k + \varepsilon\,C_b\,Z_k\,\bar{\Omega}_0^*\,(\alpha_k-\beta_k)\right]\mathrm{d}\bar{\Psi}_k\\
&\quad -\varepsilon\,C_b\,Z_k\,\bar{\Omega}_0^*\textstyle\sum_{m\neq k}(\alpha_m-\beta_k)\,\mathrm{d}\bar{\Psi}_m\\
&\quad -Z_k\left[2\,\Lambda_0\,(\alpha_{-1}-\beta_k+\gamma) - \varepsilon\,C_b\textstyle\sum_{m\geq 0}(\alpha_m-\beta_k)\bar{\Psi}_m\right]\mathrm{d}\bar{\Omega}_0,
\end{aligned}$$

where we have only retained the leading order component from each term. Here,

$$\sum_{m\geq 0}\bar{\Psi}_m^* = -\bar{C}_a^{-1}\,\frac{\Lambda_0^{5/2}\,\cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}\sum_{m\geq 0}\frac{(\alpha_{-1}+\gamma-\beta_m)\,Z_m}{N_m},$$

$$\sum_{m\geq 0}(\alpha_m-\beta_k)\,\bar{\Psi}_m^* = -\bar{C}_a^{-1}\,\frac{\Lambda_0^{5/2}\,\cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}\sum_{m\geq 0}\frac{(\alpha_{-1}+\gamma-\beta_m)\,(\alpha_m-\beta_k)Z_m}{N_m}$$

by (4.18), and thus both sums are $\mathrm{O}(1)$ and finite by (2.5) and (4.2).

To leading order, then, this system reads $\delta\dot{\hat{\Phi}} = \mathcal{L}_0\,\delta\hat{\Phi}$, where $\delta\hat{\Phi} = (\mathrm{d}\bar{\Omega}_0, \mathrm{d}\bar{\Psi}_0, \mathrm{d}\bar{\Psi}_1, \ldots)^{\mathrm{T}}$ and

$$\mathcal{L}_0 = \begin{pmatrix}
-\Lambda_0 & 0 & 0 & \ldots & 0 & \ldots\\
c_0 & -N_0 & 0 & \ldots & 0 & \ldots\\
c_1 & 0 & -N_1 & \ldots & 0 & \ldots\\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
c_k & 0 & 0 & \ldots & -N_k & \ldots\\
\vdots & \vdots & \vdots & & \vdots & \ddots
\end{pmatrix},$$

with $c_k = -2\,\Lambda_0\, Z_k\,(\alpha_{-1} - \beta_k + \gamma)$. It is plain to show that $\mathcal{L}_0$ can be transformed to $-\mathrm{diag}(\Lambda_0, N_0, N_1, \ldots)$ under a similarity transformation which, together with its inverse, is a compact perturbation of the identity. It follows that $\sigma_p(\mathcal{L}_0) = \{-\Lambda_0, -N_0, -N_1, \ldots\} \subset \mathbf{R}^-$, and hence the bifurcating steady state $(\bar\Omega^*, \bar\Psi^*)$—or, equivalently, (Ω^*, Ψ^*)—is stable.

This section concludes our discussion of the DCM-like steady state for O(1) values of Λ_0. In the next section, we investigate a logarithmic scaling for Λ_0 in which the number of steady states that the system (4.14)–(4.14) yields and the stability properties of these states are drastically different.

Remark 4.2. The ODE (1.17)—describing the flow on the one-dimensional center manifold in the regime where $\lambda_0 = \varepsilon^\alpha \Lambda_0 \ll \varepsilon$—can be obtained from the system (4.9) as its $\Lambda_0 \to 0$ limit. Indeed, the Ψ-modes become slaved to the mode Ω_0 in this limit, and (4.9) reduces to (1.17) with $a_{000} = a_{000}(\Lambda_0)$ (cf. (3.1)) replaced by $\lim_{\Lambda_0\to 0} a_{000}(\Lambda_0)$. It is plain to check that, indeed, $\lim_{\Lambda_0\to 0} a_{000}(\Lambda_0) = a_{000}(0)$ (cf. (1.19)).

5. Annihilation of the DCM

Our work in Section 4 was based on the explicit assumption that $\Lambda_0 = \mathrm{O}(1)$. Here, we analyze a logarithmic scaling for Λ_0 and show that the stable, DCM-like small pattern (Ω_0^*, Ψ^*) identified in that section is annihilated in a saddle–node bifurcation.

To motivate our discussion, we note that our dominant balance scenario for (4.15) breaks down when $\bar\Omega_0^*$ and $\Psi_m^* = \varepsilon\,\bar\Psi_m^*$ attain the same asymptotic magnitude. For $\Lambda_0 \gg 1$, (4.17) yields, to leading order,

$$\bar\Omega_0^* = \bar C_a^{-1}\,\Lambda_0^{3/2}\exp(\sqrt{\Lambda_0}\,x_*).$$

Further, (4.2) and (4.4) become to leading order

$$\alpha_{-1} = \frac{C_1}{2f(0)}\exp\left(\sqrt{\Lambda_0}\,x_*\right), \tag{5.1}$$

$$\gamma = \frac{r\,C_1\,[1-\nu^{-1}\ell]}{2(1-\nu)\,f(0)}\sqrt{\Lambda_0}\exp\left(\sqrt{\Lambda_0}\,x_*\right) = \bar\gamma\,\sqrt{\Lambda_0}\exp\left(\sqrt{\Lambda_0}\,x_*\right), \tag{5.2}$$

while α_m and β_k are independent of Λ_0 and hence remain O(1) in this regime. Thus, $\beta_k \ll \alpha_{-1} \ll \gamma$ and (4.18) becomes

$$\Psi_k^* = -\varepsilon\frac{\bar\gamma\,Z_k}{\bar C_a\,N_k}\Lambda_0^3\exp\left(2\sqrt{\Lambda_0}\,x_*\right).$$

It now becomes plain that $\bar\Omega_0^*$ and Ψ_k^* are of the same asymptotic order when Λ_0 becomes logarithmically large, $\Lambda_0 = \mathrm{O}(\log^2\varepsilon)$. In that regime, the unscaled quantities $\Omega_0 = \mathrm{O}(\varepsilon^{-2/3}\log\varepsilon)$ and $\Psi_k = \mathrm{O}(\varepsilon\log^2\varepsilon)$, and hence both safely remain $o(\varepsilon^{-11/12})$ so that (2.20) is still valid.

5.1. Emergence of a second equilibrium

To substantiate our observations above, we first show that (4.1) yields the correct asymptotic behavior of a_{000}, b_{m00}, a'_{00k}, and b'_{m0k} also for logarithmic scalings of Λ_0. It suffices to demonstrate that our work in Sections 3 and 6–8, which leads to (4.1), remains valid in this case also. First, formulas (A.1) for ω_0 and (A.2) for ω_0^+ in Appendix A are independent of Λ_0 and thus also hold in this regime; in a related note, formula (B.5) for η_0 is exact and hence also holds in this regime. Finally, the definition of μ_0 in (C.2) shows that a change in the asymptotic magnitude of Λ_0 is solely reflected in the scalings for ϵ_n (2.5) that have to be considered. The logarithmic scaling of Λ_0 considered in this section change the scalings for ϵ_n only marginally and in a manner that does not affect the aforementioned formulas. Further, (3.1) yields to leading order

$$C_a = \bar{C}_a \Lambda_0^{-1/2} \exp\left(-\sqrt{\Lambda_0}\, x_*\right), \tag{5.3}$$

so that (4.1) becomes

$$\begin{array}{rcll} a_{000} & = & \varepsilon^{1/6} \bar{C}_a \Lambda_0^{-1/2} \exp\left(-\sqrt{\Lambda_0}\, x_*\right), & \\ b_{m00} & = & C_b, & \text{for } m \ll \varepsilon^{-1/3}, \\ a'_{00k} & = & \varepsilon^{4/3} \bar{\gamma} \bar{C}_a Z_k, & \text{for } m \ll \varepsilon^{-1/3}, \\ b'_{m0k} & = & \varepsilon^{7/6} C_b Z_k \left(\alpha_m - \beta_k\right), & \text{for } m, k \ll \varepsilon^{-1/3}. \end{array} \tag{5.4}$$

Here, we have used again that $\beta_k \ll \alpha_{-1} \ll \gamma$. We remark, finally, that (5.2) also holds.

We commence our analysis in this regime by rescaling Λ_0 so as to reflect our logarithmic scaling. We set

$$\Lambda_0 = \frac{1}{4{x_*}^2} \log^2 \varepsilon + \frac{2}{{x_*}^2} \log\varepsilon \, \log(-\log\varepsilon) - \frac{\log\mu}{{x_*}^2} \log\varepsilon, \tag{5.5}$$

where μ is a free (positive) parameter that controls the value of Λ_0 in the range relevant for the subsequent analysis. It follows that

$$\exp\left(-\sqrt{\Lambda_0}\, x_*\right) = \left(\varepsilon^{1/2} \log^2 \varepsilon\right) \mu^{-1} + \mathrm{O}\left(\varepsilon^{1/2} \log\varepsilon \, \log^2(-\log\varepsilon)\right),$$

$$\sqrt{\Lambda_0} = -\frac{\log\varepsilon}{2x_*} + \mathrm{O}\left(\log(-\log\varepsilon)\right),$$

whence (5.4) becomes, to leading order,

$$\begin{array}{rcl} a_{000} & = & -2\left(\varepsilon^{2/3} \log\varepsilon\right) \bar{C}_a\, x_*\, \mu^{-1}, \\ b_{m00} & = & C_b, \\ a'_{00k} & = & \varepsilon^{4/3} \bar{\gamma} \bar{C}_a Z_k, \\ b'_{m0k} & = & \varepsilon^{7/6} C_b Z_k \left(\alpha_m - \beta_k\right). \end{array} \tag{5.6}$$

Then, with a slight abuse of notation, we rescale Ω_0 and Ψ_k by means of

$$\Omega_0 = -\varepsilon^{-2/3} \log\varepsilon\, \bar{\Omega}_0 \quad \text{and} \quad \Psi_k = \log^2\varepsilon\, \bar{\Psi}_k.$$

Substituting into (4.9), together with the rescaling (5.5) and the expressions collected in (5.6), we obtain

$$\begin{array}{rcl} \frac{1}{\log^2\varepsilon} \dot{\bar{\Omega}}_0 & = & \bar{\Omega}_0 \left[\frac{1}{4{x_*}^2} - 2\left(\bar{C}_a\, x_*\, \mu^{-1}\right) \bar{\Omega}_0 - C_b \sum_{m\geq 0} \bar{\Psi}_m\right], \\ \dot{\bar{\Psi}}_k & = & -N_k \bar{\Psi}_k - \bar{\gamma} \bar{C}_a Z_k \bar{\Omega}_0^2 \\ & & + \left(\varepsilon^{1/2} \log\varepsilon\right) C_b Z_k \bar{\Omega}_0 \sum_{m\geq 0} \left(\alpha_m - \beta_k\right) \bar{\Psi}_m, \ k \geq 0. \end{array} \tag{5.7}$$

The corresponding steady-state problem can be solved as in the previous section; matching the first two terms in the right member of the second equation and substituting into the first one, we obtain to leading order

$$\bar{\Psi}_k = -\bar{\gamma}\,\bar{C}_a\,Z_k\,N_k^{-1}\,\bar{\Omega}_0^2 \quad \text{and} \quad (\bar{\gamma}\,\sigma\,C_b)\,\bar{\Omega}_0^2 - \left(2x_*\mu^{-1}\right)\bar{\Omega}_0 + \left(4\bar{C}_a x_*{}^2\right)^{-1} = 0. \tag{5.8}$$

Here, we have defined the constant $\sigma = \sum_{m\geq 0} Z_m/N_m = (1-x_*)\,C_2\,\sigma_0^{1/3}\,\sigma_*^{-1/2}/\sqrt{2} > 0$ (this formula is derived in Remark 5.1 at the end of this section). The first of these equations yields two positive solutions,

$$\bar{\Omega}_0^{*,\pm} = \frac{x_*}{C_b\,\bar{\gamma}\,\sigma\,\mu}\left[1 \pm \sqrt{1-\frac{\mu^2}{\mu_*^2}}\right], \quad \text{whence also } \bar{\Psi}_k^{*,\pm} = -\frac{\bar{\gamma}\,\bar{C}_a\,Z_k}{N_k}\,\left(\bar{\Omega}_0^{*,\pm}\right)^2, \quad k \geq 0. \tag{5.9}$$

These two steady-states exist for all values of μ up to $\mu_* = 2\,x_*^2\,\sqrt{\bar{C}_a}/\sqrt{\bar{\gamma}\,\sigma\,C_b}$, to leading order, while $\bar{\Omega}_0$ and $\bar{\Psi}$ remain O(1) for all O(1) values of μ. For such values of μ, the biomasses corresponding to the two steady states are $\mathrm{O}(\varepsilon^{1/2}\log\varepsilon)$, with an $\mathrm{O}(\varepsilon^{1/2}\log^{-1}\varepsilon)$ nutrient depletion. The steady-state $(\bar{\Omega}_0^{*,-}, \bar{\Psi}^{*,-})$, which limits to zero as $\mu \downarrow 0$, corresponds to the stable DCM identified in the previous section. The steady-state $(\bar{\Omega}_0^{*,+}, \bar{\Psi}^{*,+})$ on the other hand, which grows unboundedly as $\mu \downarrow 0$, corresponds to the asymptotically large steady-state discussed in Remark 4.1. As we shall shortly see, the value μ_* marks a saddle–node bifurcation: the (stable) steady-state $(\bar{\Omega}_0^{*,-}, \bar{\Psi}^{*,-})$ is annihilated by the (unstable) steady-state $(\bar{\Omega}_0^{*,+}, \bar{\Psi}^{*,+})$.

Remark 5.1. We can obtain a closed formula for the Dirichlet series σ in the following way. First, repeated integration by parts yields

$$\int_0^1 \phi(x)\cos(\sqrt{N_m}\,x)\,dx = \left[\frac{\phi(x)\sin(\sqrt{N_m}\,x)}{\sqrt{N_m}}\right]_0^1 + \left[\frac{\phi'(x)\cos(\sqrt{N_m}\,x)}{N_m}\right]_0^1 - \frac{1}{N_m}\int_0^1 \phi''(x)\cos(\sqrt{N_m}\,x)\,dx.$$

Selecting $\phi(x) = 1-x$, we find $N_m^{-1} = \int_0^1 (1-x)\cos(\sqrt{N_m}x)\,dx$ (recall (2.5)). Since the Fourier decomposition of ϕ is $\phi(x) = \sum_{m\geq 0}\phi_m\cos\left(\sqrt{N_m}\,x\right)$, where $\phi_m = 2\int_0^1 \phi(x)\cos\left(\sqrt{N_m}x\right)\,dx = 2/N_m$, we obtain

$$\sum_{m\geq 0}\frac{\cos(\sqrt{N_m}\,x_*)}{N_m} = \frac{1}{2}\sum_{m\geq 0}\phi_m\cos(\sqrt{N_m}\,x_*) = \frac{\phi(x_*)}{2} = \frac{1-x_*}{2}. \tag{5.10}$$

It now follows (recall (4.2)) that $\sigma = (1-x_*)\,C_2\,\sigma_0^{1/3}\,\sigma_*^{-1/2}/\sqrt{2}$.

5.2. *The saddle–node bifurcation*

In this section, we prove our assertion on the stability types of the two equilibria identified above. In particular, we show that $(\bar{\Omega}_0^{*,-}, \bar{\Psi}^{*,-})$ and $(\bar{\Omega}_0^{*,+}, \bar{\Psi}^{*,+})$ remain stable and unstable, correspondingly, for all values of μ up to μ_*.

First, we rewrite (5.7) in the form

$$\begin{aligned} \tfrac{1}{\log^2\varepsilon}\,\dot{\bar\Omega}_0 &= \bar\Omega_0\left[\tfrac{1}{4\,x_*^2} - \bar\mu^{-1}\,\bar\Omega_0 - C_b\,\textstyle\sum_{m\geq 0}\bar\Psi_m\right], \\ \dot{\bar\Psi}_k &= -N_k\,\bar\Psi_k - \gamma_k\,\bar\Omega_0^2 + \left(\varepsilon^{1/2}\log\varepsilon\right)\bar\Omega_0\textstyle\sum_{m\geq 0}\delta_{mk}\,\bar\Psi_m, \quad k\geq 0, \end{aligned} \tag{5.11}$$

where we have defined the parameters

$$\bar\mu = \frac{\mu}{2\,\bar C_a\,x_*}, \qquad \gamma_k = Z_k\,\bar C_a\,\bar\gamma, \quad \text{and} \quad \delta_{mk} = C_b\,Z_k\,(\alpha_m - \beta_k)\,. \tag{5.12}$$

Writing further $\bar\mu_* = \mu_*/(2\,\bar C_a\,x_*)$ and $\bar\sigma = \bar C_a\,\bar\gamma\,\sigma$, we find that the fixed points identified in (5.8)–(5.9) become

$$\bar\Omega_0^{*,\pm} = \frac{1}{2C_b\,\bar\sigma\,\bar\mu\,p_\pm(\bar\mu)} \quad \text{and} \quad \bar\Psi_k^{*,\pm} = -\frac{\gamma_k}{N_k}\,\left(\bar\Omega_0^{*,\pm}\right)^2, \tag{5.13}$$

with $k\geq 0$ and

$$p_\pm(\bar\mu) = \left[1 \pm \sqrt{1 - \frac{\bar\mu^2}{\bar\mu_*^2}}\right]^{-1}. \tag{5.14}$$

Here, $p_+ : (0,\mu_*) \to (1/2,1)$ and $p_- : (0,\mu_*) \to (1,\infty)$. Now, we observe that $\bar\Omega_0$ evolves on a faster timescale than $\bar\Psi_0,\ldots,\bar\Psi_M$ due to the prefactor of $1/\log^2\varepsilon$; here, $M = \mathrm{O}(\log\varepsilon)$. Working as in Section 4.2, then, we can derive the leading order slaving relation

$$\bar\Omega_0 = \bar\mu\,\left[\frac{1}{4\,x_*^2} - C_b\sum_{m=0}^{M}\bar\Psi_m\right]. \tag{5.15}$$

(Note that both fixed points respect this relation.) Under this constraint, and upon defining $\bar\gamma_k = C_b\,\gamma_k\,\bar\mu$, the linearized problem around either one of the two fixed points becomes

$$\mathrm{d}\dot{\bar\Psi}_k = -N_k\,\mathrm{d}\bar\Psi_k + 2\,\bar\gamma_k\,\bar\Omega_0^{*,\pm}\sum_{m=0}^{M}\mathrm{d}\bar\Psi_m, \quad \text{for} \quad 0\leq k\leq M, \tag{5.16}$$

where we have only retained the leading order components. The matrix corresponding to this linear problem is

$$\mathcal{L}_0 = \begin{pmatrix} -N_0 + 2\,\bar\gamma_0\,\bar\Omega_0^{*,\pm} & 2\,\bar\gamma_0\,\bar\Omega_0^{*,\pm} & \ldots & 2\,\bar\gamma_0\,\bar\Omega_0^{*,\pm} \\ 2\,\bar\gamma_1\,\bar\Omega_0^{*,\pm} & -N_1 + 2\,\bar\gamma_1\,\bar\Omega_0^{*,\pm} & \ldots & 2\,\bar\gamma_1\,\bar\Omega_0^{*,\pm} \\ \vdots & \vdots & \ddots & \vdots \\ 2\,\bar\gamma_M\,\bar\Omega_0^{*,\pm} & 2\,\bar\gamma_M\,\bar\Omega_0^{*,\pm} & \ldots & -N_M + 2\,\bar\gamma_M\,\bar\Omega_0^{*,\pm} \end{pmatrix}. \tag{5.17}$$

Since the higher Ψ_k modes ($k > M$) are stable (cf. (5.11)), the spectrum $\sigma(\mathcal{L}_0)$ determines the stability of the equilibrium in question. To characterize this spectrum, we first derive a formula for the characteristic polynomial $\det(\mathcal{L}_0 - \lambda I)$. First, we note that $\bar\gamma_0 \neq 0$ for all $\bar\mu \in (0,\bar\mu_*)$: indeed, $\bar\gamma_0 = C_b\,\gamma_0\,\bar\mu$; substituting from (4.2), (5.2)–(5.3), and (5.12) into this definition, we obtain

$$\bar\gamma_0 = \mu\,\frac{1-\nu^{-1}\ell}{2\,x_*}\,r\,C_1\,C_2\,\sigma_0^{1/3}\,\sigma_*^{-1/2}\,\cos\left(\frac{\pi\,x_*}{2}\right) \neq 0,$$

for all $x_* \in (0,1)$. Thus, we can use the first row of $\mathcal{L}_0 - \lambda I$ to eliminate the off-diagonal entries of all other rows. In this way, we find that the equation $\det(\mathcal{L}_0 - \lambda I) = 0$ is equivalent to setting to zero the determinant

$$\begin{vmatrix} -(N_0+\lambda) + 2\,\bar{\gamma}_0\,\bar{\Omega}^{*,\pm}_{0,0} & 2\,\bar{\gamma}_0\,\bar{\Omega}^{*,\pm}_{0,0} & 2\,\bar{\gamma}_0\,\bar{\Omega}^{*,\pm}_{0,0} & \dots & 2\,\bar{\gamma}_0\,\bar{\Omega}^{*,\pm}_{0,0} \\ \frac{\bar{\gamma}_1}{\bar{\gamma}_0}(N_0+\lambda) & -(N_1+\lambda) & 0 & \dots & 0 \\ \frac{\bar{\gamma}_2}{\bar{\gamma}_0}(N_0+\lambda) & 0 & -(N_2+\lambda) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\bar{\gamma}_M}{\bar{\gamma}_0}(N_0+\lambda) & 0 & 0 & \dots & -(N_M+\lambda) \end{vmatrix}.$$

Next, we can use the $(m+1)$−st column to eliminate the $(m+1)$−st entry of the first column (for $1 \le m \le M$), as long as $\lambda \neq -N_m$. Since $\lambda = -N_m$ if and only if $\bar{\gamma}_m = 0$ (as can be shown by expanding the determinant along the $(m+1)$−st row), we can eliminate all entries of the first column. (Note that $\bar{\gamma}_m$ may indeed be zero: indeed, working as for $\bar{\gamma}_0$, one obtains that $\bar{\gamma}_m$ is proportional to $\cos((m+1/2)\pi\, x_*)$, which may or may not be zero depending on the values of m and x_*.) Defining $\mathcal{M} = \{m : \bar{\gamma}_m \neq 0\} \subset \{0,\dots,M\}$, $\mathcal{M}_k = \mathcal{M} - \{k\}$, and eliminating the entries of the first column as detailed above, we obtain

$$\begin{vmatrix} -Q(\lambda) & 2\,\bar{\gamma}_0\,\bar{\Omega}^{*,\pm}_{0,0} & 2\,\bar{\gamma}_0\,\bar{\Omega}^{*,\pm}_{0,0} & \dots & 2\,\bar{\gamma}_0\,\bar{\Omega}^{*,\pm}_{0,0} \\ 0 & -(N_1+\lambda) & 0 & \dots & 0 \\ 0 & 0 & -(N_2+\lambda) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -(N_M+\lambda) \end{vmatrix} = 0. \tag{5.18}$$

Here,

$$\begin{aligned} Q(\lambda) &= (N_0 + \lambda - 2\,\bar{\gamma}_0\,\bar{\Omega}^{*,\pm}_0) \prod_{m\in\mathcal{M}_0} (N_m+\lambda) - 2\,\bar{\Omega}^{*,\pm}_0 \sum_{k\in\mathcal{M}_0} \bar{\gamma}_k \prod_{m\in\mathcal{M}_k} (N_m+\lambda) \\ &= \prod_{m\in\mathcal{M}} (N_m+\lambda) - 2\,\bar{\Omega}^{*,\pm}_0 \sum_{k\in\mathcal{M}} \bar{\gamma}_k \prod_{m\in\mathcal{M}_k} (N_m+\lambda) \\ &= \left[1 - 2\,\bar{\Omega}^{*,\pm}_0 \sum_{k\in\mathcal{M}} \frac{\bar{\gamma}_k}{N_k+\lambda}\right] \prod_{m\in\mathcal{M}} (N_m+\lambda). \end{aligned}$$

As detailed above, $\lambda = -N_k$ solves (5.18) if and only if $\bar{\gamma}_k = 0$ (equivalently, if and only if $k \notin \mathcal{M}$), and hence we can extend the set over which we sum in the formula above to the entire set $\{0,\dots,M\}$. In particular, we can rewrite $Q(\lambda)$ the equation as

$$Q(\lambda) = \left[1 - 2\,\bar{\Omega}^{*,\pm}_0 \sum_{k=0}^{M} \frac{\bar{\gamma}_k}{N_k+\lambda}\right] \prod_{m=0}^{M} (N_m+\lambda).$$

As we just noted, the elements of the set $\{-N_m\}_{m\notin\mathcal{M}}$ are eigenvalues of $\mathcal{L}_0$, thus the remaining eigenvalues are solutions to the equation

$$2\,\bar{\Omega}_0^{*,\pm}\sum_{k=0}^{M}\frac{\bar{\gamma}_k}{N_k+\lambda}=1. \tag{5.19}$$

Recalling (4.2), Remark 5.1, and (5.12)–(5.13), and writing $c=1/(1-x_*)\in(1,\infty)$, we calculate

$$\bar{\gamma}_k\,\bar{\Omega}_0^{*,\pm}=\frac{Z_k}{2\sigma\,p_\pm(\mu)}=c\,\frac{\cos\left(\sqrt{N_k}\,x_*\right)}{p_\pm(\mu)}.$$

It follows that (5.19) can be rewritten as

$$P_M(\lambda):=\sum_{k=0}^{M}\frac{\cos\left(\sqrt{N_k}\,x_*\right)}{N_k+\lambda}=\frac{p_\pm(\bar{\mu})}{2c}. \tag{5.20}$$

Now, we write

$$P_M(\lambda)=P(\lambda)-\sum_{k\geq M+1}\frac{\cos\left(\sqrt{N_k}\,x_*\right)}{N_k+\lambda},\quad \text{where } P(\lambda)=\sum_{k=0}^{\infty}\frac{\cos\left(\sqrt{N_k}\,x_*\right)}{N_k+\lambda}.$$

The remainder in this formula can be bounded uniformly: since we are primarily interested in locating unstable eigenvalues, we estimate, for all λ with $\mathrm{Re}(\lambda)\geq 0$,

$$\left|\sum_{k\geq M+1}\frac{\cos\left(\sqrt{N_k}x_*\right)}{N_k+\lambda}\right|\leq\sum_{k\geq M+1}\frac{1}{N_k+\mathrm{Re}(\lambda)}\leq\frac{1}{\pi^2}\sum_{k\geq M+1}\frac{1}{k^2}\leq\frac{1}{\pi^2}\int_M^\infty\frac{dx}{x^2}=\frac{1}{M\pi^2}.$$

Recalling that $M=\mathrm{O}(\log\varepsilon)$, then, we find that $P_M=P$ to leading order and for all λ with $\mathrm{Re}(\lambda)\geq 0$, and hence (5.20) becomes

$$P(\lambda)=\frac{p_\pm(\bar{\mu})}{2c}. \tag{5.21}$$

Now, given any value $\bar{\mu}<\bar{\mu}_*$, this equation is satisfied by some λ if and only if it is also satisfied by its conjugate λ^*, as the right member is real and $P^*(\lambda)=P(\lambda^*)$; hence, we can further restrict $\arg(\lambda)$ to lie in $[0,\pi/2]$. Since, also, $P(0)=(1-x_*)/2=1/(2c)$ by virtue of (5.10), (5.21) reads to leading order

$$P(\lambda)=P(0)\,p_\pm(\bar{\mu}),\quad \text{with } \arg(\lambda)\in[0,\pi/2]. \tag{5.22}$$

The series P appearing in this last equation is a Mittag-Lefler expansion; analytic formulas for such expansions can often be obtained by means of the Fourier transform. In particular, [19, Eq. (1.63)] (with $a=\pi$, $b=\mathrm{i}\sqrt{\lambda}$, and $\ell=1$) yields the explicit formula

$$P(\lambda)=\frac{\sin[\mathrm{i}(1-x_*)\sqrt{\lambda}]}{2\mathrm{i}\sqrt{\lambda}\cos(\mathrm{i}\sqrt{\lambda})}=\frac{\sinh[(1-x_*)\sqrt{\lambda}]}{2\sqrt{\lambda}\cosh\sqrt{\lambda}}=\frac{\sinh(c^{-1}\sqrt{\lambda})}{2\sqrt{\lambda}\cosh\sqrt{\lambda}},$$

where any one of the two opposite branches of the square root can be chosen, as the rightmost expression is an even function of $\sqrt{\lambda}$. For concreteness, we select the principal branch, so that $0\leq\arg(\sqrt{\lambda})\leq\pi/4$. Hence, the eigenvalue equation (5.22) becomes

$$P(\lambda)=P(0)\,p_\pm(\bar{\mu}),\quad \text{with } P(\lambda)=\frac{\sinh(c^{-1}\sqrt{\lambda})}{2\sqrt{\lambda}\cosh\sqrt{\lambda}} \text{ and } \arg(\sqrt{\lambda})\in[0,\pi/4].$$

Further rescaling λ by means of $\lambda = c^2\bar{\lambda}^2$, we arrive at the desired form of the characteristic equation,

$$\bar{P}(\bar{\lambda}) = p_{\pm}(\bar{\mu}), \quad \text{for } \arg\left(\sqrt{\bar{\lambda}}\right) \in [0, \pi/4] \text{ and } \bar{P}(\bar{\lambda}) = \frac{\sinh \bar{\lambda}}{\bar{\lambda}\cosh(c\bar{\lambda})}. \quad (5.23)$$

We remark that $\lim_{\bar{\lambda}\to 0} \bar{P}(\bar{\lambda}) = 1$, so that $\bar{P}$ is well-defined in the entire $[0, \infty)$.

It now follows directly from (5.23) that $(\bar{\Omega}^{*,+}, \bar{\Psi}^{*,+})$ is unstable for all $\bar{\mu} < \bar{\mu}_*$ (equivalently, $\mu < \mu_*$). Indeed, $\bar{P}$ is continuous and monotonically decreasing in $[0, \infty)$, with $\lim_{\bar{\lambda}\to\infty} \bar{P}(\bar{\lambda}) = 0$. Further, $1/2 < p_+(\bar{\mu}) < 1$ for all $\bar{\mu} < \bar{\mu}_*$. Hence, for every value of $\bar{\mu} \in (0, \bar{\mu}_*)$, there is a unique real root $\bar{\lambda}_* \in \mathbf{R}^+$—that is, a positive eigenvalue. (In fact, it can be shown that this is the only unstable eigenvalue corresponding to this fixed point.) Consequently, $(\bar{\Omega}^{*,+}, \bar{\Psi}^{*,+})$ is unstable for all such values of $\bar{\mu}$.

The characterization of the stability of $(\bar{\Omega}^{*,-}, \bar{\Psi}^{*,-})$ is more involved. First, the possibility of non-negative real eigenvalues can be immediately excluded: by the same token as above, $p_-(\bar{\mu}) > 1$ for all $\bar{\mu} < \bar{\mu}_*$. On the other hand, $\bar{P}$ is decreasing in $\mathbf{R}^+$, so that $\bar{P}(\bar{\lambda}) < \bar{P}(0) = 1$ for $\bar{\lambda} \in \mathbf{R}^+$. Hence, there are no real and non-negative values of $\bar{\lambda}$ that satisfy (5.23), and as a consequence $\mathcal{L}_0$ does not have real and non-negative spectrum for $\bar{\mu} < \bar{\mu}_*$ (equivalently, for $\mu < \mu_*$).

It remains to eliminate the possibility of complex eigenvalues with non-negative real parts. To achieve this, we first observe that $\mathcal{L}_0 \to \operatorname{diag}(-N_0, \ldots, -N_M)$ as $\bar{\mu} \to 0$, by virtue of $\bar{\Omega}_0^{*,-} \to 0$ (cf. (5.13)–(5.14) and (5.17)). Hence, $\sigma(\mathcal{L}_0) \subset \mathbb{R}_-$ in that limit case. As a result of this observation, $\sigma(\mathcal{L}_0)$ can cross into the right half-plane only through the imaginary axis (excepting the origin, as we just showed). That is, $\sigma(\mathcal{L}_0)$ is not confined to the left half-plane if and only if there exist $\bar{\mu} < \bar{\mu}_*$ and $\bar{\lambda}$ with $\arg(\bar{\lambda}) = \pi/4$ satisfying (5.23). Rescaling by $\bar{\lambda} = \sqrt{\mathrm{i}/2}\,\xi$, we arrive at the equation

$$\frac{2\sinh\left[(1+\mathrm{i})\,\xi/2\right]}{(1+\mathrm{i})\,\xi\cosh\left[(1+\mathrm{i})\,c\,\xi/2\right]} = p_-(\bar{\mu}), \quad \xi \in \mathbf{R}^+.$$

Since $p_-(\bar{\mu}) > 1$ for all $\bar{\mu} < \bar{\mu}_*$, it suffices to show that the modulus of the left member is uniformly bounded above by one. Then, $\bar{\Omega}_0^{*,-}$ remains stable up to the bifurcation at $\bar{\mu}_*$, as there can be no crossings into the right half-plane. A straightforward calculation, now, shows that

$$\left|\frac{2\sinh\left[(1+\mathrm{i})\,\xi/2\right]}{(1+\mathrm{i})\,\xi\cosh\left[(1+\mathrm{i})\,c\,\xi/2\right]}\right|^2 = \frac{2}{\xi^2}\,\frac{\cosh\xi - \cos\xi}{\cosh(c\xi) + \cos(c\xi)} < \frac{2}{\xi^2}\,\frac{\cosh\xi - \cos\xi}{\cosh\xi + \cos\xi}. \quad (5.24)$$

This last inequality is due to the facts that $c \geq 1$ and that $\cosh\xi + \cos\xi$ is increasing in $\mathbf{R}^+$, whence $\cosh(c\xi) + \cos(c\xi) > \cosh\xi + \cos\xi > 0$ for all $(c, \xi) \in (1, \infty) \times \mathbf{R}^+$. Now, integrating repeatedly the identities $\cos s < 1$ and $\cosh s > 1$ over $[0, \xi]$, we reach the estimates

$$\cos\xi > 1 - \frac{\xi^2}{2} + \frac{\xi^4}{24} - \frac{\xi^6}{720} \quad \text{and} \quad \cosh\xi < 1 + \frac{\xi^2}{2} + \frac{\xi^4}{24} + \frac{\xi^6}{720},$$

for $\xi \in \mathbf{R}^+$. Hence,

$$\frac{\cos\xi}{\cosh\xi} > \frac{1 - \frac{\xi^2}{2} + \frac{\xi^4}{24} - \frac{\xi^6}{720}}{1 + \frac{\xi^2}{2} + \frac{\xi^4}{24} + \frac{\xi^6}{720}} \quad \text{or, equivalently,} \quad \frac{\cosh\xi - \cos\xi}{\cosh\xi + \cos\xi} < \frac{\frac{\xi^2}{2} - \frac{\xi^6}{720}}{1 + \frac{\xi^4}{24}}.$$

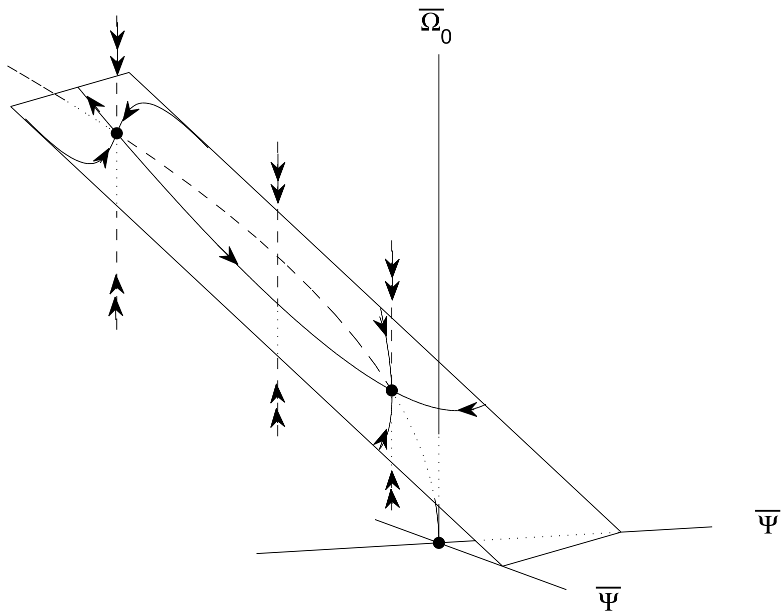

Figure 2. Sketch of the phase space corresponding to (5.11). The horizontal directions represent the (asymptotically many) $\bar{\Psi}-$directions and the vertical one $\bar{\Omega}_0$. The dashed parabola corresponds to the $\bar{\Psi}-$nullcline. The slanted plane and the $\bar{\Psi}-$coordinate-plane correspond to the $\bar{\Omega}_0-$nullcline, with the former also coinciding with the leading order slaving relation (5.15) and being normally attracting. The intersection of the nullclines determines the triad of fixed points (circled) given in (5.13). The stable/unstable manifolds of $(\bar{\Omega}_0^{*,+}, \bar{\Psi}^{*,+})$ restricted to the plane and the flow on the plane near the attractor $(\bar{\Omega}_0^{*,-}, \bar{\Psi}^{*,-})$ are also sketched.

To obtain this last inequality we have cross-multiplied and separated terms. The desired inequality now follows directly,

$$\left|\frac{2\sinh\left[(1+\mathrm{i})\xi/2\right]}{(1+\mathrm{i})\,\xi\cosh\left[(1+\mathrm{i})\,c\,\xi/2\right]}\right|^2 = \frac{2}{\xi^2}\,\frac{\cosh\xi-\cos\xi}{\cosh\xi+\cos\xi} < \frac{1-\frac{\xi^4}{360}}{1+\frac{\xi^4}{24}} < 1, \quad \text{for all } \xi \in \mathbf{R}^+.$$

The analysis in this section is summarized geometrically in Figure 2, which depicts the flow generated by (5.11). To leading order, the co-dimension one hyperplane described by the slaving relation (5.15) is pivoted about the points where it intersects the $\Psi-$axes: the value of $\bar{\mu}$ only controls the zenith angle of its normal vector. As $\bar{\mu} \downarrow 0$—equivalently, as we move into the regime analyzed in the previous section—this angle decreases, so that $(\bar{\Omega}^{*,-}, \bar{\Psi}^{*,-})$ approaches the trivial steady state and $(\bar{\Omega}^{*,+}, \bar{\Psi}^{*,+})$ becomes asymptotically large. As $\bar{\mu} \uparrow \bar{\mu}_*$, on he other hand, this angle increases and the two points approach each other. For $\bar{\mu} > \bar{\mu}_*$, the plane does not intersect the parabola anymore: we have moved past the saddle-node bifurcation analyzed in this section.

6. An asymptotic formula for b_{m00}

In this section, we derive the asymptotic formula for b_{m00} given in (4.1), where $m \in \mathbf{N}$ and

$$b_{m00} = (1-\nu) \int_0^1 f(x)\, \zeta_m(x)\, \omega_0^2(x)\, dx. \tag{6.1}$$

As detailed earlier, the function ω_0, appearing in (6.1), decays exponentially outside an $\mathrm{O}(\varepsilon^{1/3})$–neighborhood of the origin (cf. (A.1)), whereas the period of the sinusoidal term ζ_m is equal to $2\pi/\sqrt{N_m} = 4/(2m+1)$. Below, we analyze the three different regimes—in which the integrand is predominantly localized, concurrently localized and oscillatory, and predominantly oscillatory—and we derive the leading order, uniform asymptotic expansion

$$b_{m00} = \begin{cases} C_b, \quad \text{for } m \ll \varepsilon^{-1/3}, \\ C_b\, C_1^2 \displaystyle\int_0^\infty \cos\left(\frac{\varepsilon^{1/3}\sqrt{N_m}\,\tau}{\sigma_0^{1/3}}\right) \mathrm{Ai}^2\left(\tau + A_1\right)\, d\tau, \quad \text{for } m = \mathrm{O}(\varepsilon^{-1/3}), \\ -C_b\, \dfrac{6\,[\mathrm{Ai}'(A_1)]^2\, C_1^2\, \sigma_0^2}{\varepsilon\, N_m^2\, f(0)}, \quad \text{for } m \gg \varepsilon^{-1/3}. \end{cases} \tag{6.2}$$

Here, $C_b = \sqrt{2}\,(1-\nu)\, f(0)$, cf. Section 4.1.

6.1. *The case* $m \ll \varepsilon^{-1/3}$

Here, $2\pi/\sqrt{N_m} \gg \varepsilon^{1/3}$ and hence the integrand is predominantly localized around $x = 0$. Thus, ζ_m may be approximated to leading order by $\zeta_m(0) = \sqrt{2}$ in that neighborhood. Since $\|\omega_0\|_2 = 1$ (cf. our discussion in Sections 2.1–2.2), we obtain the desired formula

$$b_{m00} \sim C_b. \tag{6.3}$$

6.2. *The case* $m = \mathrm{O}(\varepsilon^{-1/3})$

Here, $2\pi/\sqrt{N_m} = \mathrm{O}(\varepsilon^{1/3})$, and hence the neighborhood of the origin outside which ω_0 decays exponentially and the period of the sinusoidal term are of the same asymptotic magnitude. Defining the new variable $\tau = \tau_1\, x$ in (6.1), with $\tau_1 = |A_1|/\bar{x}_0$ (2.18), we obtain

$$b_{m00} = \frac{\sqrt{2}\,(1-\nu)}{\tau_1} \int_0^{\tau_1} f\left(\frac{\tau}{\tau_1}\right) \cos\left(\sqrt{N_m}\,\frac{\tau}{\tau_1}\right) \omega_0^2\left(\frac{\tau}{\tau_1}\right) d\tau. \tag{6.4}$$

Now, (A.1) yields, to leading order and for any $\tau_0 \ll \varepsilon^{-1/3}$,

$$\omega_0\left(\frac{\tau}{\tau_1}\right) = \begin{cases} \varepsilon^{-1/6}\, C_1\, \sigma_0^{1/6} \mathrm{Ai}\left(\tau + A_1\right), & \text{for } \tau \in [0, -A_1), \\ \frac{\varepsilon^{-1/12}\, C_1\, C_2\, \sigma_0^{1/3}}{2\sqrt{\pi}\, F^{1/4}(\tau/\tau_1)} \exp\left(-\frac{1}{\sqrt{\varepsilon}\tau_1} \int_0^{\tau + A_1} \sqrt{F(\frac{t}{\tau_1} + \bar{x}_0) - F(\bar{x}_0)\, dt}\right), & \\ & \text{for } \tau \in (-A_1, \tau_0], \end{cases}$$

where we have also changed the integration variable by means of $s = t/\tau_1 + \bar{x}_0$. These two formulas agree—as, indeed, they should by construction—in the regime $1 \ll \tau_0 \ll \varepsilon^{-1/3}$.

Indeed, recalling the asymptotic expansion of Ai in a neighborhood of infinity [2], we find that the first branch of the formula above yields

$$\varepsilon^{-1/6}\,\frac{C_1\,\sigma_0^{1/6}}{2\,\sqrt{\pi}\,\tau^{1/4}}\,\exp\left(-\frac{2}{3}(\tau+A_1)^{3/2}\right).$$

Similarly, the formula in the case $\tau \in (|A_1|\,,\tau_0]$ becomes, upon Taylor-expanding F,

$$\varepsilon^{-1/12}\,\frac{C_1\,C_2\,\sigma_0^{1/12}\,\tau_1^{1/4}}{2\sqrt{\pi}\,\tau^{1/4}}\,\exp\left(-\frac{2}{3}\sqrt{\frac{\sigma_0}{\varepsilon}}\left(\frac{\tau+A_1}{\tau_1}\right)^{3/2}\right).$$

That the two formulas agree now follows from the definition $\tau_1 = |A_1|/\bar{x}_0$ and the formulas (2.18) and (3.3) for $\bar{x}_0$ and C_2. Hence, we may write

$$\omega_0\left(\frac{\tau}{\tau_1}\right) \sim \varepsilon^{-1/6}\,C_1\,\sigma_0^{1/6}\mathrm{Ai}\,(\tau+A_1)\,, \quad \text{for} \quad \tau \ll \varepsilon^{-1/3}.$$

Since the contribution to the integral in (6.4) of greater values of τ may be estimated to be exponentially small, we can write

$$\begin{aligned} b_{m00} &= \varepsilon^{-1/3}\,\frac{\sqrt{2}\,(1-\nu)\,C_1^2\,\sigma_0^{1/3}}{\tau_1}\int_0^\infty f\left(\frac{\tau}{\tau_1}\right)\cos\left(\sqrt{N_m}\,\frac{\tau}{\tau_1}\right)\,\mathrm{Ai}^2\,(\tau+A_1)\,d\tau \\ &= C_b\,C_1^2\int_0^\infty \cos\left(\frac{\varepsilon^{1/3}\,\sqrt{N_m}\,\tau}{\sigma_0^{1/3}}\right)\,\mathrm{Ai}^2\,(\tau+A_1)\,d\tau, \end{aligned} \tag{6.5}$$

to leading order, as desired. Note that this formula reduces to (6.3), for $m \ll \varepsilon^{-1/3}$.

6.3. The case $m \gg \varepsilon^{-1/3}$

Here, $2\pi/\sqrt{N_m} \ll \varepsilon^{1/3}$. Similarly to our work in the previous section, we define the new variable $\tau = \varepsilon^{-1/3}x$. We find, then,

$$b_{m00} = \sqrt{2}\,\varepsilon^{1/3}\,(1-\nu)\int_0^{\varepsilon^{-1/3}} g(\tau)\,\cos\left(\varepsilon^{1/3}\,\sqrt{N_m}\,\tau\right)\,d\tau,$$

where $g(\tau) = f\left(\varepsilon^{1/3}\tau\right)\,\omega_0^2\left(\varepsilon^{1/3}\tau\right)$. Using Theorem Appendix D.4 (with $\lambda = \varepsilon^{1/3}\,\sqrt{N_m}$, $\Phi(t) = t = \tau$, and $h(\tau) = g(\tau)$) and the fact that the right-boundary term is exponentially smaller than the left one, as $\omega_0(1)$ is exponentially smaller than $\omega_0(0)$ (cf. Appendix A), we obtain

$$\begin{aligned} b_{m00} &= \sqrt{2}\,\varepsilon^{1/3}\,(1-\nu)\,\mathrm{Re}\left(\sum_{k=0}^{\infty} g^{(k)}(0)\left(\frac{\mathrm{i}}{\varepsilon^{1/3}\,\sqrt{N_m}}\right)^{k+1}\right) \\ &= \sqrt{2}\,\frac{1-\nu}{\varepsilon^{1/3}\,N_m}\sum_{k=0}^{\infty}(-1)^{k+1}g^{(2k+1)}(0)\left(\frac{1}{\varepsilon^{1/3}\,\sqrt{N_m}}\right)^{2k}. \end{aligned} \tag{6.6}$$

Recalling the definition of g, and employing (A.1) and that $\mathrm{Ai}(A_1) = \mathrm{Ai}''(A_1) = 0$, we calculate

$$g'(0) = 0 \quad \text{and} \quad g'''(0) = -6\,[\mathrm{Ai}'(A_1)]^2\,C_1^2\,\sigma_0^2.$$

The desired result now follows, while (6.6) also reduces to (6.5) for $\epsilon_m = \mathrm{O}(\varepsilon^{1/3})$ (2.5).

7. An asymptotic formula for a'_{00k}

In this section, we derive the asymptotic formula for a'_{00k},

$$a'_{00k} = \varepsilon^{4/3}\, C_a\, (\alpha_{-1} - \beta_k + \gamma)\, Z_k, \quad \text{for } 0 \neq k \ll \varepsilon^{-1/3}, \tag{7.1}$$

as was already given in (4.1). We remark, here, that this result is only valid for those values of k for which $\zeta_k(x_*) \neq 0$. For the remaining values of k, Theorem Appendix D.1 yields an (algebraically) higher order result. Also, we note that asymptotic formulas for higher values of k can be derived as in the previous section, albeit at considerable extra computational cost.

We first write out explicitly the expression for a'_{00k} yielded by (2.22),

$$a'_{00k} = \delta \int_0^1 a_0(x)\, \omega_0(x)\, \psi_k(x)\, dx + \varepsilon\delta\ell^{-1} \int_0^1 a_0(x)\, \omega_0^+(x)\, \zeta_k(x)\, dx.$$

Recalling the definition of a_0 from (2.23) and working as in Section 3, we obtain further

$$\begin{aligned} a'_{00k} = {} & \delta^2 \int_0^1 \int_0^x h_1(x,y)\, \omega_0^+(y)\, \omega_0(x)\, \psi_k(x)\, dydx \\ & + \delta^2 \int_0^1 \int_0^1 h_2(x,y)\, \omega_0^+(y)\, \omega_0(x)\, \psi_k(x)\, dydx \\ & + \varepsilon\delta^2\ell^{-1} \int_0^1 \int_0^x h_1(x,y)\, \zeta_k(x)\, \omega_0^+(y)\, \omega_0^+(x)\, dydx \\ & + \varepsilon\delta^2\ell^{-1} \int_0^1 \int_0^1 h_2(x,y)\, \zeta_k(x)\, \omega_0^+(y)\, \omega_0^+(x)\, dydx. \end{aligned}$$

Substituting, finally, from (C.21), we obtain an integral formula for a'_{00k} which is amenable to the sort of asymptotic analysis employed in Sections 3 and 6,

$$\begin{aligned} a'_{00k} = \frac{\varepsilon\delta^2}{\ell} \Bigg[& (W_\psi)^{-1} \int_0^1 \int_0^x \int_0^x h_{1,k}(x,y,z)\, \omega_0(x)\, \psi_{k,-}(x)\, \omega_0^+(y)\, \psi_{k,+}^+(z)\, dzdydx \\ & + (W_\psi)^{-1} \int_0^1 \int_0^1 \int_0^x h_{2,k}(x,y,z)\, \omega_0(x)\, \psi_{k,-}(x)\, \omega_0^+(y)\, \psi_{k,+}^+(z)\, dzdydx \\ & - (W_\psi)^{-1} D_k \int_0^1 \int_0^x \int_0^1 h_{1,k}(x,y,z)\, \omega_0(x)\, \psi_{k,-}(x)\, \omega_0^+(y)\, \psi_{k,-}^+(z)\, dzdydx \\ & - (W_\psi)^{-1} D_k \int_0^1 \int_0^1 \int_0^1 h_{2,k}(x,y,z)\, \omega_0(x)\, \psi_{k,-}(x)\, \omega_0^+(y)\, \psi_{k,-}^+(z)\, dzdydx \\ & + (W_\psi)^{-1} \int_0^1 \int_0^x \int_x^1 h_{1,k}(x,y,z)\, \omega_0(x)\, \psi_{k,+}(x)\, \omega_0^+(y)\, \psi_{k,-}^+(z)\, dzdydx \\ & + (W_\psi)^{-1} \int_0^1 \int_0^1 \int_x^1 h_{2,k}(x,y,z)\, \omega_0(x)\, \psi_{k,+}(x)\, \omega_0^+(y)\, \psi_{k,-}^+(z)\, dzdydx \\ & + \int_0^1 \int_0^x h_1(x,y)\, \zeta_k(x)\, \omega_0^+(x)\, \omega_0^+(y)\, dydx \\ & + \int_0^1 \int_0^1 h_2(x,y)\, \zeta_k(x)\, \omega_0^+(x)\, \omega_0^+(y)\, dydx \Bigg]. \end{aligned} \tag{7.2}$$

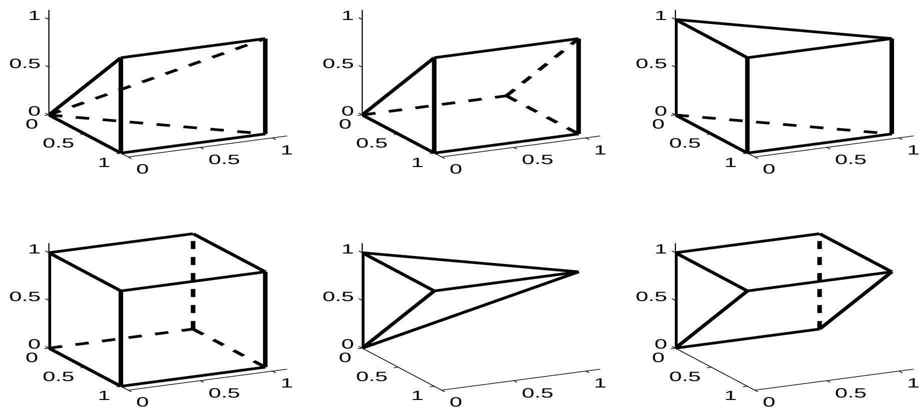

Figure 3. The domains of integration for the integrals $\mathcal{I}_1, \dots, \mathcal{I}_6$ in (7.2).

Here, $h_{i,k}(x,y,z) = h_i(x,y)\, f(z)\, \zeta_k(z)$, for $i = 1, 2$, and the constants D_k are reported in (C.19)–(C.20). Let $\mathcal{I}_1, \dots, \mathcal{I}_8$ denote the integrals in the right member of (7.2) in the order that they appear in it (the three-dimensional domains of integration for $\mathcal{I}_1, \dots, \mathcal{I}_6$ are sketched in Figure 3). In what follows, we omit the term $\theta\, \omega_{0,-}^2(1;\bar{x}_0)\, \omega_{0,+}(x;\bar{x}_0)$ in the expression (A.1) for ω_0, as one can show that its contribution is exponentially small compared to the leading order terms (see also Sections 3 and 6).

We start with rewriting the term $-(W_\psi)^{-1} D_k \mathcal{I}_4 + (W_\psi)^{-1} \mathcal{I}_6 + \mathcal{I}_8$. First,

$$\mathcal{I}_4 = \int\!\!\int_{\pi_x D_4} \left(\int_0^1 h_{2,k}(x,y,z)\, \omega_0(x)\, \psi_{k,-}(x)\, dx \right) \omega_0^+(y)\, \psi_{k,-}^+(z)\, dA_{yz},$$

where π_x is the orthogonal projection on the $yz-$plane—and hence $\pi_x D_4 = [0,1]^2$—and dA_{yz} is the area element on that plane. Since $\psi_{k,-} = \varepsilon^{1/3} C_1^{-1} \sigma_0^{-1/3} \omega_0$ in a neighborhood of the origin (cf. (A.1) and (C.12)), ω_0 is exponentially small outside this neighborhood, and $\|\omega_0\|_2 = 1$, we write

$$\mathcal{I}_4 = \varepsilon^{1/3}\, C_1^{-1}\, \sigma_0^{-1/3} \int\!\!\int_{\pi_x D_4} h_{2,k}(0,y,z)\, \omega_0^+(y)\, \psi_{k,-}^+(z)\, dA_{yz}.$$

Recalling that $\psi_{k,-}^+ = E\, \psi_{k,-}$, according to our convention in Section 2, and substituting into the formula above from (A.2) and (C.12), we obtain

$$\mathcal{I}_4 = \varepsilon^{1/2}\, \frac{C_2^2}{4\pi} \int\!\!\int_{\pi_x D_4} \frac{h_{2,k}(0,y,z)}{F^{1/4}(y)\, F^{1/4}(z)} \exp\left(\frac{J_-(y) + J_-(z)}{\sqrt{\varepsilon}} \right) dA_{yz},$$

whence, employing also (C.19), we find

$$(W_\psi)^{-1} D_k\, \mathcal{I}_4 = \varepsilon^{-1/6} \int\!\!\int_{\pi_x D_4} \Xi_4(y,z) \exp\left(\frac{\Pi_4(y,z)}{\sqrt{\varepsilon}} \right) dA_{yz}. \qquad (7.3)$$

Here,

$$\Xi_4(y,z) = \frac{C_2^2\, d_k}{4\pi\, W_\psi}\, \frac{h_2(0,y)\, f(z)\, \zeta_k(z)}{F^{1/4}(y)\, F^{1/4}(z)} \quad \text{and} \quad \Pi_4(y,z) = J_-(y) + J_-(z). \qquad (7.4)$$

Next, we rewrite $\mathcal{I}_6$,

$$(W_\psi)^{-1}\,\mathcal{I}_6 = (W_\psi)^{-1} \int\!\!\int_{\pi_x D_6} \left(\int_0^z h_{2,k}(x,y,z)\,\omega_0(x)\,\psi_{k,+}(x)\,dx \right) \omega_0^+(y)\,\psi_{k,-}^+(z)\,dA_{yz}.$$

Employing (A.1) and (C.13), now, we obtain

$$(W_\psi)^{-1}\,\mathcal{I}_6 = \varepsilon^{1/6}\,\frac{C_1\,\sigma_0^{1/3}}{2\pi\,W_\psi} \int\!\!\int_{\pi_x D_6} \left(\int_0^z \frac{h_{2,k}(x,y,z)}{\sqrt{F(x)}} dx \right) \omega_0^+(y)\,\psi_{k,-}^+(z)\,dA_{yz}.$$

Further using (A.2) and, once again, (C.13), we find

$$(W_\psi)^{-1}\,\mathcal{I}_6 = \varepsilon^{1/3} \int\!\!\int_{\pi_x D_6} \Xi_6(y,z)\,\exp\left(\frac{\Pi_6(y,z)}{\sqrt{\varepsilon}} \right) dA_{yz}. \qquad (7.5)$$

Here, $\pi_x D_6 = \pi_x D_4$, $\Pi_6(y,z) = \Pi_4(y,z)$, and

$$\Xi_6(y,z) = \frac{C_1^2\,C_2^2\,\sigma_0^{2/3}}{8\pi^2\,W_\psi}\,\frac{f(z)\,\zeta_k(z)}{F^{1/4}(y)\,F^{1/4}(z)} \int_0^z \frac{h_2(x,y)}{\sqrt{F(x)}}\,dx.$$

Similarly, renaming (x,y) as (y,z) in $\mathcal{I}_8$, we derive the formula

$$\mathcal{I}_8 = \varepsilon^{-1/6} \int\!\!\int_{D_8} \Xi_8(y,z)\,\exp\left(\frac{\Pi_8(y,z)}{\sqrt{\varepsilon}} \right) dA_{yz}, \qquad (7.6)$$

where $D_8 = \pi_x D_6 = \pi_x D_4$, $\Pi_8(y,z) = \Pi_6(y,z) = \Pi_4(y,z)$, and

$$\Xi_8(y,z) = \frac{C_1^2\,C_2^2\,\sigma_0^{2/3}}{4\pi}\,\frac{h_2(y,z)\,\zeta_k(y)}{F^{1/4}(y)\,F^{1/4}(z)}. \qquad (7.7)$$

Combining (7.3)–(7.7), we obtain

$$-(W_\psi)^{-1}\,D_k\,\mathcal{I}_4 + (W_\psi)^{-1}\,\mathcal{I}_6 + \mathcal{I}_8 = \varepsilon^{-1/6} \int\!\!\int_{D_8} \tilde{\Xi}_8(y,z)\,\exp\left(\frac{\Pi_4(y,z)}{\sqrt{\varepsilon}} \right) dA_{yz}, \qquad (7.8)$$

where, to leading order and uniformly over D_8,

$$\begin{aligned} \tilde{\Xi}_8(y,z) &= \Xi_8(y,z) - \Xi_4(y,z) \\ &= \frac{C_2^2}{4\pi\,F^{1/4}(y)\,F^{1/4}(z)} \left(C_1^2\,\sigma_0^{2/3}\,h_2(y,z)\,\zeta_k(y) - \frac{d_k}{W_\psi}\,h_2(0,y)\,f(z)\,\zeta_k(z) \right). \qquad (7.9) \end{aligned}$$

Next, we rewrite the term $(W_\psi)^{-1}\mathcal{I}_5 + \mathcal{I}_7$. We write first

$$\mathcal{I}_5 = \int\!\!\int_{\pi_x D_5} \left(\int_y^z h_{1,k}(x,y,z)\,\omega_0(x)\,\psi_{k,+}(x)\,dx \right) \omega_0^+(y)\,\psi_{k,-}^+(z)\,dA_{yz},$$

where $\pi_x D_5 = \{(y,z) | 0 \le y \le z,\ 0 \le z \le 1\}$. Now, (A.1)–(A.2) and (C.12) yield further

$$\begin{aligned} (W_\psi)^{-1}\,\mathcal{I}_5 &= \varepsilon^{1/6}\,\frac{C_1\,C_2\,\sigma_0^{1/3}}{2\pi\,W_\psi} \int\!\!\int_{\pi_x D_5} \left(\int_y^z \frac{h_{1,k}(x,y,z)}{\sqrt{F(x)}}\,dx \right) \omega_0^+(y)\,\psi_{k,-}^+(z)\,dA_{yz} \\ &= \varepsilon^{1/3} \int\!\!\int_{\pi_x D_5} \Xi_5(y,z)\,\exp\left(\frac{\Pi_5(y,z)}{\sqrt{\varepsilon}} \right) dA_{yz}, \qquad (7.10) \end{aligned}$$

where we have defined the functions

$$\Xi_5(y,z) = \frac{C_1^2\,C_2^3\,\sigma_0^{2/3}}{8\pi^2\,W_\psi}\,\frac{f(z)\,\zeta_k(z)}{F^{1/4}(y)\,F^{1/4}(z)} \int_y^z \frac{h_1(x,y)}{\sqrt{F(x)}}\,dx \qquad (7.11)$$

and $\Pi_5(y,z) = \Pi_4(y,z)$. Next, renaming x into z in $\mathcal{I}_7$, we find

$$\mathcal{I}_7 = \varepsilon^{-1/6} \int\int_{D_7} \Xi_7(y,z) \exp\left(\frac{\Pi_7(y,z)}{\sqrt{\varepsilon}}\right) dA_{yz}, \tag{7.12}$$

where

$$D_7 = \pi_x D_5, \quad \Xi_7(y,z) = \frac{C_1^2\, C_2^2\, \sigma_0^{2/3}}{4\pi} \frac{h_1(z,y)\,\zeta_k(z)}{F^{1/4}(y)\, F^{1/4}(z)}, \quad \text{and} \quad \Pi_7(y,z) = \Pi_4(y,z). \tag{7.13}$$

Combining (7.10)–(7.13), we find, to leading order and uniformly over D_8,

$$(W_\psi)^{-1}\,\mathcal{I}_5 + \mathcal{I}_7 = \varepsilon^{-1/6} \int\int_{D_7} \tilde{\Xi}_7(y,z) \exp\left(\frac{\Pi_4(y,z)}{\sqrt{\varepsilon}}\right) dA_{yz}, \tag{7.14}$$

where $\tilde{\Xi}_7(y,z) = \Xi_7(y,z)$.

Next, we rewrite $(W_\psi)^{-1}\mathcal{I}_2$. First,

$$\mathcal{I}_2 = \int\int_{\pi_y D_2} \tilde{H}_2(x)\, f(z)\, \zeta_k(z)\, \omega_0(x)\, \psi_{k,-}(x)\, \psi_{k,+}^{+}(z)\, dA_{xz},$$

where $\tilde{H}_2(x) = \int_0^1 h_2(x,y)\,\omega_0^{+}(y)\,dy$. Substituting for $\omega_0^{+}(y)$ from (A.2), we find further

$$\mathcal{I}_2 = \varepsilon^{-1/12}\, \frac{C_1 C_2 \sigma_0^{1/3}}{2\sqrt{\pi}} \int\int_{\pi_y D_2} H_2(x)\,\omega_0(x)\,\psi_{k,-}(x)\, f(z)\,\zeta_k(z)\,\psi_{k,+}^{+}(z)\,dA_{xz},$$

where $H_2(x) = \int_0^1 h_2(x,y)\, F^{-1/4}(y)\, \exp(J_-(y)/\sqrt{\varepsilon})\,dy$. Using Theorem Appendix D.1, now, we obtain

$$\begin{aligned}(W_\psi)^{-1}\,\mathcal{I}_2 \quad &= \varepsilon^{1/6}\delta^{-1}\, C_2'' \int\int_{\pi_y D_2} h_2(x,x_*)\,\omega_0(x)\,\psi_{k,-}(x)\,f(z)\,\zeta_k(z)\,\psi_{k,+}^{+}(z)\,dA_{xz} \\ &= \varepsilon^{7/12} \int\int_{\pi_y D_2} \Xi_2(x,z) \exp\left(\frac{\Pi_2(x,z)}{\sqrt{\varepsilon}}\right) dA_{xz}, \end{aligned}\tag{7.15}$$

where C_2'' is an O(1) constant, $\pi_y D_2 = \{(x,z) | 0 \le z \le x,\ 0 \le x \le 1\}$, $\Pi_2(x,z) = J_-(x_*) + J_+(z) - 2I(x)$,

$$\Xi_2(x,z) = C_2' \,\frac{h_2(x,x_*)\, f(z)\,\zeta_k(z)}{\sqrt{F(x)}\,F^{1/4}(z)}, \quad \text{with } C_2' \text{ an O(1) constant.} \tag{7.16}$$

We finally rewrite $(W_\psi)^{-1}D_k\,\mathcal{I}_3$. First,

$$\mathcal{I}_3 = \left(\int_0^1 f(z)\,\zeta_k(z)\,\psi_{k,-}^{+}(z)\,dz\right) \int\int_{\pi_z D_3} h_1(x,y)\,\omega_0(x)\,\psi_{k,-}(x)\,\omega_0^{+}(y)\,dA_{xy}.$$

Substituting from (A.1)–(A.2) and (C.12) into this formula and interchanging the roles of y and z in the single and double integrals, we find

$$(W_\psi)^{-1}\, D_k\,\mathcal{I}_3 = \varepsilon^{-1/3}\,\tilde{I}_3 \int\int_{\pi_y D_2} \tilde{\Xi}_3(x,z) \exp\left(\frac{\tilde{\Pi}_3(x,z)}{\sqrt{\varepsilon}}\right) dA_{xz}, \tag{7.17}$$

where $\tilde{I}_3 = \int_0^1 F^{-1/4}(y)\, f(y)\,\zeta_k(y)\, \exp\left(J_-(y)/\sqrt{\varepsilon}\right)\, dy$ and

$$\tilde{\Xi}_3(x,z) = \tilde{C}_3\, \frac{h_1(x,z)}{\sqrt{F(x)}\,F^{1/4}(z)} \quad \text{and} \quad \tilde{\Pi}_3(x,z) = J_-(z) - 2I(x), \tag{7.18}$$

for some O(1) constant $\tilde{C}_3$.

Next, we estimate the various terms derived above, starting from $-(W_\psi)^{-1} D_k \mathcal{I}_4 + (W_\psi)^{-1} \mathcal{I}_6 + \mathcal{I}_8$ (cf. (7.8)–(7.9)). The exponent Π_4 becomes maximum at the interior critical point (x_*, x_*), and thus Theorem Appendix D.1 yields

$$\begin{aligned}-(W_\psi)^{-1} D_k \mathcal{I}_4 + (W_\psi)^{-1} \mathcal{I}_6 + \mathcal{I}_8 &= \varepsilon^{1/2} \frac{2\pi}{|J_-''(x_*)|} \left(\varepsilon^{-1/6} \tilde{\Xi}_8(x_*, x_*) \exp\left(\frac{\Pi_4(x_*, x_*)}{\sqrt{\varepsilon}} \right) \right) \\ &= \varepsilon^{1/3} \delta^{-2} \tilde{C}_8,\end{aligned}$$

where

$$\tilde{C}_8 = C_2^2 \sigma_*^{-1} \zeta_k(x_*) \left(C_1^2 \sigma_0^{2/3} h_2(x_*, x_*) - \frac{d_k}{W_\psi} h_2(0, x_*) f(x_*) \right).$$

Next, we estimate $(W_\psi)^{-1} \mathcal{I}_5 + \mathcal{I}_7$, cf. (7.14). The sole maximum of Π_4 in D_7 lies at the critical point $(x_*, x_*) \in \partial D_7$, and hence Theorem Appendix D.1 yields

$$\begin{aligned}(W_\psi)^{-1} \mathcal{I}_5 + \mathcal{I}_7 &= \varepsilon^{1/2} \frac{\pi}{|J_-''(x_*)|} \left(\varepsilon^{-1/6} \tilde{\Xi}_7(x_*, x_*) \exp\left(\frac{\Pi_4(x_*, x_*)}{\sqrt{\varepsilon}} \right) \right) \\ &= \varepsilon^{1/3} \delta^{-2} \tilde{C}_7,\end{aligned}$$

where

$$\tilde{C}_7 = \frac{C_1^2 C_2^2 \sigma_0^{2/3}}{2\sigma_*} h_1(x_*, x_*) \zeta_k(x_*).$$

We now estimate the remaining three integrals starting with $(W_\psi)^{-1} \mathcal{I}_2$, cf. (7.15)–(7.16). The exponent Π_2 has a sole maximum at the point $(x_*, x_*) \in \partial(\pi_y D_2)$ which is not a critical point (compare to the maximization of Π_4 in Section 3). Hence, Theorem Appendix D.1 yields

$$(W_\psi)^{-1} \mathcal{I}_2 = \varepsilon^{3/4} C_2' \left(\varepsilon^{7/12} \Xi_2(x_*, x_*) \exp\left(\frac{\Pi_2(x_*, x_*)}{\sqrt{\varepsilon}} \right) \right) = \varepsilon^{4/3} \delta^{-2} C_2,$$

for some O(1) constants C_2 and C_2'. Next, since $D_1 \subset D_2$ and the integrands of $\mathcal{I}_1$ and $\mathcal{I}_2$ differ only by an O(1) multiple, the above analysis also yields that $(W_\psi)^{-1}\mathcal{I}_1$ is at most of the same order as $(W_\psi)^{-1}\mathcal{I}_2$. Finally, we estimate $(W_\psi)^{-1} D_k \mathcal{I}_3$, cf. (7.17)–(7.18). First, we estimate

$$\tilde{I}_3 = \int_0^1 \frac{f(y) \zeta_k(y)}{F^{1/4}(y)} \exp\left(\frac{J_-(y)}{\sqrt{\varepsilon}} \right) dy = \varepsilon^{1/4} \delta^{-1} C_3'',$$

for some O(1) constant C_3''. Substituting into (7.17), then, we obtain

$$(W_\psi)^{-1} D_k \mathcal{I}_3 = \varepsilon^{-1/12} \int\int_{\pi_y D_2} \Xi_3(x, z) \exp\left(\frac{\Pi_3(x, z)}{\sqrt{\varepsilon}} \right) dA_{xz},$$

where

$$\Xi_3(x, z) = \tilde{C}_3' \frac{h_1(x, z)}{\sqrt{F(x)} F^{1/4}(z)} \quad \text{and} \quad \Pi_3(x, z) = J_-(x_*) + J_-(z) - 2I(x),$$

for some O(1) constant $\tilde{C}_3'$. The exponent Π_3 has a sole maximum at the point $(x^{**}, x^{**}) \in \partial(\pi_y D_2)$ which is not a critical point (compare to the maximization of Π_1 in Section 3). Hence, Theorem Appendix D.1 yields

$$(W_\psi)^{-1}\,\mathcal{I}_2 = \varepsilon^{3/4} C_3' \left(\varepsilon^{-1/12}\Xi_3(x_*, x_*)\exp\left(\frac{\Pi_3(x_*, x_*)}{\sqrt{\varepsilon}}\right)\right) = \varepsilon^{2/3} C_3 \exp\left(\frac{\Pi_3(x^{**}, x^{**})}{\sqrt{\varepsilon}}\right),$$

for some O(1) constants C_3 and $\tilde{C}_3'$ and where $\Pi_3(x^{**}, x^{**}) < 2J_-(x_*)$.

In total, then, and to leading order, we obtain the leading order formula

$$a'_{00k} = \varepsilon^{4/3}\, C'_{a,00k}, \quad \text{for } k \ll \varepsilon^{-1/3}. \tag{7.19}$$

Here,

$$C'_{a,00k} = \frac{C_2^2\,\zeta_k(x_*)}{\ell\,\sigma_*}\left(C_1^2\,\sigma_0^{2/3}\left(\frac{1}{2}\,h_1(x_*, x_*) + h_2(x_*, x_*)\right) - \frac{d_k}{W_\psi}\,f(x_*)\,h_2(0, x_*)\right), \tag{7.20}$$

while h_1 and h_2 are given in (3.5)–(3.6) and (cf. (C.14) and (C.20))

$$d_k = \frac{\sigma_0^{2/3}\,|\mathrm{Bi}(A_1)|}{N_k \mathrm{Ai}'(A_1)} \quad \text{and} \quad W_\psi = \mathrm{Ai}'(A_1)\,|\mathrm{Bi}(A_1)|\,.$$

To derive the desired formula (7.1) from (7.19), we note that (cf. (3.5)–(3.6) and that $f(x_*) = \ell$ to leading order)

$$\begin{aligned} h_1(x_*, x_*) &= r\,\ell\,\left[1 - \nu^{-1}\ell\right], \\ h_2(x_*, x_*) &= (1-\nu)\,\ell\,\frac{\cosh\left(\sqrt{\Lambda_0}\,x_*\right)\,\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}{\sqrt{\Lambda_0}\cosh\left(\sqrt{\Lambda_0}\right)}, \\ h_2(0, x_*) &= (1-\nu)\,f(0)\,\frac{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}{\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)}. \end{aligned} \tag{7.21}$$

Hence, (7.20) becomes

$$\begin{aligned} C'_{a,00k} = \frac{C_2^2\,\zeta_k(x_*)}{\sigma_*} \Bigg(& C_1^2\,\sigma_0^{2/3}\,(1-\nu)\,\frac{\cosh\left(\sqrt{\Lambda_0}\,x_*\right)\,\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}{\sqrt{\Lambda_0}\cosh\left(\sqrt{\Lambda_0}\right)} \\ & - \frac{(1-\nu)\,f(0)\,d_k}{W_\psi}\,\frac{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}{\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)} + \frac{r\,C_1^2\,\sigma_0^{2/3}}{2}\,\left(1 - \frac{\ell}{\nu}\right)\Bigg). \end{aligned}$$

Using this formula and (3.1), we calculate further,

$$\begin{aligned} \frac{C'_{a,00k}}{C_a} = \quad & \frac{C_1\,C_2\,\sigma_0^{1/3}\,\zeta_k(x_*)}{\sigma_*^{1/2}\,f(0)}\,\cosh\left(\sqrt{\Lambda_0}\,x_*\right) - \frac{C_2\,\sigma_0^{1/3}\,\zeta_k(x_*)}{C_1\,\sigma_*^{1/2}\,N_k\,(\mathrm{Ai}'(A_1))^2} \\ & + \frac{r\,C_1\,C_2\,\sigma_0^{1/3}\,\zeta_k(x_*)\,(1-\nu^{-1}\ell)}{2(1-\nu)\,f(0)\,\sigma_*^{1/2}}\,\frac{\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)}{\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}. \end{aligned}$$

The desired formula (7.1) may now be derived from this equation by recalling (2.6) and the definitions (4.2)–(4.4).

8. An asymptotic formula for b'_{m0k}

Finally, we derive the asymptotic formula for b'_{m0k},

$$b'_{m0k} = \varepsilon^{7/6}\, C_b\, (\alpha_m - \beta_k)\, Z_k, \quad \text{for } 0 \neq k, m \ll \varepsilon^{-1/3}, \tag{8.1}$$

as was already given in (4.1). We also remark that, here also, this result is valid for those values of k for which $\zeta_k(x_*) \neq 0$ and $\zeta_m(x_*) \neq 0$. Theorem Appendix D.1 yields an (algebraically) higher order result for the remaining values of k.

Definition (2.22) and (C.21) yield the expression

$$\begin{aligned}
b'_{m0k} &= \delta\,(1-\nu) \int_0^1 f(x)\,\zeta_m(x)\,\omega_0(x)\,\psi_k(x)\,dx \\
&\quad + \varepsilon\,\delta\,\ell^{-1}(1-\nu) \int_0^1 f(x)\,\zeta_m(x)\,\omega_0^+(x)\,\zeta_k(x)\,dx \\
&= \frac{\varepsilon\,\delta\,(1-\nu)}{\ell} \left[\frac{1}{W_\psi} \int_0^1 \int_0^x f(x)\,f(y)\,\zeta_m(x)\,\zeta_k(y)\,\omega_0(x)\,\psi_{k,-}(x)\psi^+_{k,+}(y)\,dydx \right. \\
&\qquad - \frac{D_k}{W_\psi} \int_0^1 \int_0^1 f(x)\,f(y)\,\zeta_m(x)\,\zeta_k(y)\,\omega_0(x)\,\psi_{k,-}(x)\psi^+_{k,-}(y)\,dydx \\
&\qquad + \frac{1}{W_\psi} \int_0^1 \int_x^1 f(x)\,f(y)\,\zeta_m(x)\,\zeta_k(y)\,\omega_0(x)\,\psi_{k,+}(x)\,\psi^+_{k,-}(y)\,dydx \\
&\qquad \left. + \int_0^1 f(x)\,\zeta_m(x)\,\zeta_k(x)\,\omega_0^+(x)\,dx \right].
\end{aligned} \tag{8.2}$$

Let $\mathcal{I}_1, \ldots, \mathcal{I}_4$ denote the integrals in the right member of this formula in the order that they appear in it. We will derive the leading order terms in the asymptotic expansions of these integrals using Theorem Appendix D.1, as in the previous section and also for $k, m \ll \varepsilon^{-1/3}$. In what follows, we omit the terms $\theta\omega^2_{0,-}(1;\bar{x}_0)\omega_{0,+}(1;\bar{x}_0)$ and $\theta\omega^2_{0,-}(1;\bar{x}_0)\omega^+_{0,+}(1;\bar{x}_0)$ in (A.1) and (A.2), respectively, as one can show that their contribution is exponentially small compared to the leading order terms (see also Section 3 and 6–7).

First, we derive a formula for $-(W_\psi)^{-1}D_k\,\mathcal{I}_2 + (W_\psi)^{-1}\mathcal{I}_3 + \mathcal{I}_4$. We write

$$\begin{aligned}
\mathcal{I}_2 &= \int_0^1 \left(\int_0^1 f(x)\,\zeta_m(x)\,\omega_0(x)\,\psi_{k,-}(x)\,dx \right) f(y)\,\zeta_k(y)\,\psi^+_{k,-}(y)\,dy \\
&= \varepsilon^{1/3}\,\sqrt{2}\,f(0)\,C_1^{-1}\,\sigma_0^{-1/3} \int_0^1 f(y)\,\zeta_k(y)\,\psi^+_{k,-}(y)\,dy,
\end{aligned}$$

where we have used that $\psi_{k,-} = \varepsilon^{1/3}\,C_1^{-1}\sigma_0^{-1/3}\omega_0$ in a neighborhood of the origin, that ω_0 is exponentially small outside this neighborhood, the identity $\|\omega_0\|_2 = 1$, and (2.6). Employing (C.12), next, we obtain

$$\mathcal{I}_2 = \varepsilon^{7/12}\,\frac{f(0)\,C_2}{\sqrt{2\pi}\,C_1\,\sigma_0^{1/3}} \int_0^1 \frac{f(y)\,\zeta_k(y)}{F^{1/4}(y)}\,\exp\left(\frac{J_-(y)}{\sqrt{\varepsilon}}\right)\,dy.$$

Substituting for D_k from (C.19), we obtain

$$(W_\psi)^{-1}\,D_k\,\mathcal{I}_2 = \varepsilon^{-1/12} \int_0^1 \Xi_2(y)\,\exp\left(\frac{\Pi_2(y)}{\sqrt{\varepsilon}}\right)\,dy, \tag{8.3}$$

where we have defined the functions

$$\Xi_2(y) = \frac{C_2\, f(0)\, d_k}{\sqrt{2\pi}\, C_1\, W_\psi\, \sigma_0^{1/3}}\, \frac{f(y)\, \zeta_k(y)}{F^{1/4}(y)} \quad \text{and} \quad \Pi_2(y) = J_-(y). \tag{8.4}$$

Next, we change the order in which integration is carried out in $\mathcal{I}_3$ and use (A.1) and (C.12)–(C.13) to rewrite this integral as

$$\begin{aligned}(W_\psi)^{-1}\, \mathcal{I}_3 &= (W_\psi)^{-1} \int_0^1 \left(\int_0^y f(x)\, \zeta_m(x)\, \omega_0(x)\, \psi_{k,+}(x)\, dx \right) f(y)\, \zeta_k(y)\, \psi_{k,-}^+(y)\, dy \\ &= \varepsilon^{5/12} \int_0^1 \Xi_3(y)\, \exp\left(\frac{\Pi_3(y)}{\sqrt{\varepsilon}}\right) dy, \end{aligned} \tag{8.5}$$

where $\Pi_3(y) = \Pi_2(y)$ and

$$\Xi_3(y) = \frac{C_1\, C_2\, \sigma_0^{1/3}}{4\pi^{3/2}\, W_\psi} \left(\int_0^y \frac{f(x)\, \zeta_m(x)}{\sqrt{F(x)}}\, dx \right) \frac{f(y)\, \zeta_k(y)}{F^{1/4}(y)}. \tag{8.6}$$

Finally, using (A.2) and renaming the integration variable x into y, we obtain

$$\mathcal{I}_4 = \varepsilon^{-1/12} \int_0^1 \Xi_4(y)\, \exp\left(\frac{\Pi_4(y)}{\sqrt{\varepsilon}}\right) dy, \tag{8.7}$$

where

$$\Xi_4(y) = \frac{C_1\, C_2\, \sigma_0^{1/3}}{2\sqrt{\pi}}\, \frac{f(y)\, \zeta_m(y)\, \zeta_k(y)}{F^{1/4}(y)} \quad \text{and} \quad \Pi_4(y) = \Pi_3(y) = \Pi_2(y). \tag{8.8}$$

Combining (8.3)–(8.8), we obtain

$$-(W_\psi)^{-1} D_k\, \mathcal{I}_2 + (W_\psi)^{-1} \mathcal{I}_3 + \mathcal{I}_4 = \varepsilon^{-1/12} \int_0^1 \tilde{\Xi}_2(y)\, \exp\left(\frac{\Pi_2(y)}{\sqrt{\varepsilon}}\right) dy, \tag{8.9}$$

where, to leading order and uniformly over $[0, 1]$,

$$\tilde{\Xi}_2(y) = \Xi_4(y) - \Xi_2(y) = \frac{C_2\, f(y)\, \zeta_k(y)}{2\sqrt{\pi}\, F^{1/4}(y)} \left(C_1\, \sigma_0^{1/3}\, \zeta_m(y) - \frac{\sqrt{2}\, f(0)\, d_k}{W_\psi\, C_1\, \sigma_0^{1/3}} \right). \tag{8.10}$$

Regarding $\mathcal{I}_1$, we use (A.2) and (C.12)–(C.13), to write it in the form

$$(W_\psi)^{-1}\, \mathcal{I}_1 = \varepsilon^{5/12} \int\!\!\int_D \Xi_1(x, y)\, \exp\left(\frac{\Pi_1(x, y)}{\sqrt{\varepsilon}}\right) dA, \tag{8.11}$$

where $D = \{(x, y) | 0 \le x \le 1 \,\text{and}\, 0 \le y \le x\}$, $\Pi_1(x, y) = J_+(y) - 2I(x)$, and

$$\Xi_1(x, y) = \frac{C_1\, C_2\, \sigma_0^{1/3}}{4\pi^{3/2}\, W_\psi}\, \frac{f(x)\, f(y)\, \zeta_m(x)\, \zeta_k(y)}{\sqrt{F(x)}\, F^{1/4}(y)}. \tag{8.12}$$

First, we estimate $-(W_\psi)^{-1}\, D_k\, \mathcal{I}_2 + (W_\psi)^{-1}\, \mathcal{I}_3 + \mathcal{I}_4$, cf. (8.9)–(8.10). The exponent Π_2 assumes its maximum at the interior critical point $x_* \in (0, 1)$, and hence Theorem Appendix D.1 yields

$$-(W_\psi)^{-1}\, D_k\, \mathcal{I}_2 + (W_\psi)^{-1}\, \mathcal{I}_3 + \mathcal{I}_4 = \varepsilon^{1/4}\, \frac{\sqrt{2\pi}}{\sqrt{-J_-''(x_*)}} \left(\varepsilon^{-1/12}\, \delta^{-1}\, \tilde{\Xi}_2(x_*) \right) = \varepsilon^{1/6}\, \delta^{-1}\, \tilde{C}_2.$$

Here,

$$\tilde{C}_2 = C_2\,\sigma_*^{-1/2}\,f(x_*)\,\zeta_k(x_*)\left(C_1\,\sigma_0^{1/3}\,\zeta_m(x_*) - \frac{\sqrt{2}\,\sigma_0^{1/3}\,f(0)}{C_1\,N_k\,(\mathrm{Ai}'(A_1))^2}\right).$$

Next, we estimate $\mathcal{I}_1$, cf. (8.11)–(8.12). The exponent Π_1 assumes its maximum at the point $(x_*, x_*) \in \partial D$ which is not a critical point of Π_1 (compare to the maximization of Π_4 in Section 3). As a result, Theorem Appendix D.1 yields

$$(W_\psi)^{-1}\,\mathcal{I}_1 = \varepsilon^{3/4}\,C_1'\,\left(\varepsilon^{5/12}\,\delta^{-1}\,\Xi_1(x_*, x_*)\right) = \varepsilon^{7/6}\,\delta^{-1}\,C_1''$$

to leading order, and with C_1' and C_1'' being O(1) constants.

In total, then, and to leading order, we obtain

$$b_{m0k}' = \varepsilon^{7/6}\,C_{b,m0k}', \quad \text{for } m, k \ll \varepsilon^{-1/3},$$

with the constant $C_{b,m0k}'$ given by

$$C_{b,m0k}' = (1-\nu)\,C_2\,\sigma_*^{-1/2}\,\zeta_k(x_*)\left(C_1\,\sigma_0^{1/3}\,\zeta_m(x_*) - \frac{\sqrt{2}\,\sigma_0^{1/3}\,f(0)}{C_1\,N_k\,(\mathrm{Ai}'(A_1))^2}\right).$$

Finally, (8.1) is immediately derived from these formulas by observing that

$$\frac{C_{b,m0k}'}{C_b} = C_2\,\sigma_0^{1/3}\,\sigma_*^{-1/2}\,\zeta_k(x_*)\,\left(\frac{C_1\,\cos(\sqrt{N_m}x_*)}{f(0)} - \frac{1}{C_1\,N_k\,(\mathrm{Ai}'(A_1))^2}\right) = (\alpha_m - \beta_k)\;Z_k.$$

9. Discussion

As argued in the Introduction, there are two contextual themes central to this article. The first one relates to understanding the *nonlinear*, *long-term* dynamics of small patterns of DCM type generated through the linear destabilization mechanism identified in [23]. The second theme concerns the development of a concrete approach to studying the dynamics generated by the (rescaled) PDE model (1.5) *near* a linear destabilization but *beyond* the region of applicability of the center manifold reduction. In this article, we have reported significant results (outlined in the Introduction) touching on both themes. These results, in turn, inspire further investigation within this dual context.

Regarding our first focal point, and in view of our discovery that the bifurcating small-amplitude DCM pattern is soon annihilated in a saddle-node bifurcation, the central question is what happens *beyond* this bifurcation. One would like, in particular, to understand the nature of the attractor in that regime. We intend to address this call in forthcoming work, both analytically and numerically. For the time being, we note that this requires one to examine the regime $\lambda_0 = \mathcal{O}(\varepsilon^{1/3})$, in which only the diffusive, linearly stable $\Psi-$modes are excited (recall Section 2, especially). In fact, this a priori seemed to be the natural scaling for λ_0, as other DCM-like modes associated with the eigenvalues $\lambda_1, \lambda_2, \ldots$ may then be destabilized. Viewed from this perspective, the existence of the relatively rich dynamics reported here for the regime $\lambda_0 = \mathcal{O}(\varepsilon)$—in which the aforementioned DCM-like modes remain linearly stable—acts as a paradigmatic manifestation of the effects of nonlinear interactions. The $\Psi-$modes manage to have a decisive impact on the dynamics solely through nonlinear couplings, while a *strictly linear* point of view dictates that these modes should be utterly irrelevant. It is natural to expect similar nonlinear phenomena in the regime $\lambda_0 = \mathcal{O}(\varepsilon^{1/3})$.

As for the second theme, the approach we developed here can and will be applied to resolve the remaining issues pertaining to our linear bifurcation results in [23]—namely, determining the nonlinear behavior associated with the destabilization of BL-type and analyzing the co-dimension 2 dynamics, see also the Introduction. The same methodology can also be applied to *extended* models. A natural extension of (1.1) is a multi-species model, *i.e.*, a model similar to (1.1) in which *several* phytoplankton species compete for the same nutrient. At the linear level, the species evolution decouples [23]. Nonlinear coupling, however, is present through shadowing (light limitation) and nutrient uptake (nutrient limitation), and hence the presence of extra species affects the life cycle of each species. Reaction–diffusion models of this sort for eutrophic environments—*i.e.*, in the presence of an ample nutrient supply—have been developed and investigated both numerically [14] and theoretically [8]. The *oligotrophic* case, on the other hand—where these multi-species models are coupled to a PDE for the nutrient—has so far only been investigated numerically [15]. Another natural, if not outright necessary, extension is the inclusion of horizontal spatial directions. Plainly, the dynamics generated by (1.1) will be strongly influenced by the flow in directions perpendicular to the one-dimensional z-column considered here: oceanic currents are bound to mix neighboring z-columns and thus also spice up the panel of planktonic patterns. A first attempt to use the methods developed here and in [23] to quantify the impact of this horizontal flow is presented in [5].

Appendix A. An asymptotic formula for ω_0

The formula for the principal part in the asymptotic expansion of ω_0 reads

$$\omega_0(x) \sim \begin{cases} \varepsilon^{-1/6}\,\sigma_0^{1/6}\,C_1 \mathrm{Ai}\left(A_1(1-\bar{x}_0^{-1}x)\right), & \text{for } x\in[0,\bar{x}_0), \\ \frac{\varepsilon^{-1/12}\,C_1\,C_2\,\sigma_0^{1/3}}{2\sqrt{\pi}\,F^{1/4}(x)}\left[\omega_{0,-}(x;\bar{x}_0)+\theta\,\omega_{0,-}^2(1;\bar{x}_0)\,\omega_{0,+}(x;\bar{x}_0)\right], & \text{for } x\in(\bar{x}_0,1], \end{cases} \tag{A.1}$$

cf. [23], where $\bar{x}_0$, C_1, C_2, F, σ_0 and θ have been defined in (1.15), (2.18), (3.3), and (3.12). We remark that C_1 is a normalizing constant ensuring that $||\omega_0||_2 = 1$. (This factor does not appear in the formula for ω_0 we give in [23], since ω_0 was not normalized there.) Also,

$$\omega_{0,\pm}(x;\bar{x}_0) = \exp\left(\pm\frac{I(x)}{\sqrt{\varepsilon}}\right),$$

where I has been defined in (2.17). An asymptotic formula for $\omega_0^+ = E\,\omega_0$ is readily derived using (A.1) above,

$$\omega_0^+(x) \sim \begin{cases} \varepsilon^{-1/6}\,\sigma_0^{1/6}\,C_1 \mathrm{e}^{\sqrt{A/\varepsilon}\,x}\mathrm{Ai}\left(A_1(1-\bar{x}_0^{-1}x)\right), & \text{for } x\in[0,\bar{x}_0), \\ \frac{\varepsilon^{-1/12}\,C_1\,C_2\,\sigma_0^{1/3}}{2\sqrt{\pi}\,F^{1/4}(x)}\left[\omega_{0,-}^+(x;\bar{x}_0)+\theta\,\omega_{0,-}^2(1;\bar{x}_0)\,\omega_{0,+}^+(x;\bar{x}_0)\right], & \text{for } x\in(\bar{x}_0,1], \end{cases} \tag{A.2}$$

where we have defined the functions

$$\omega_{0,\pm}^{+}(x;\bar{x}_0) = E(x)\,\omega_{0,\pm}(x;\bar{x}_0) = \exp\left(\frac{J_\pm(x)}{\sqrt{\varepsilon}}\right),$$

with $J_\pm$ as in (2.17). We remark that J_- becomes maximum at the unique point $x_* \in (0,1)$—the location of the DCM—defined in (2.19), whereas J_+ increases monotonically. Also, the terms involving $\omega_{0,+}$ in (A.1) and $\omega_{0,+}^{+}$ in (A.2) are exponentially smaller than the terms $\omega_{0,-}(x)$ and $\omega_{0,-}^{+}(x)$, respectively, everywhere except for an $\mathrm{O}(\sqrt{\varepsilon})-$region of $x = 1$. Indeed, for all $x < 1$,

$$J_+(x) - 2I(1) = J_-(x) - 2(I(1) - I(x)) < J_-(x). \tag{A.3}$$

In particular, $\|\omega_0^{+}\|_\infty$ can be bounded by an $\mathrm{O}(\varepsilon^{-1/12}\delta^{-1})$ constant, where $\delta = \exp(-J_-(x_*))/\sqrt{\varepsilon}$ is an exponentially small parameter (cf. (2.16)).

Appendix B. An asymptotic formula for η_0

We recall that η_0 is the solution to the boundary value problem (2.7),

$$\varepsilon\,\partial_{xx}\eta_0 - \lambda_0\,\eta_0 = -\varepsilon\ell^{-1} f\,\omega_0^{+}, \quad \text{where } \partial_x\eta_0(0) = \eta_0(1) = 0.$$

Recalling that $\lambda_0 = \varepsilon\Lambda_0$ in our bifurcation analysis, we find that

$$\partial_{xx}\eta_0 - \Lambda_0\eta_0 = -\ell^{-1} f\omega_0^{+}, \quad \text{where } \partial_x\eta_0(0) = \eta_0(1) = 0. \tag{B.1}$$

The functions $\eta_{0,\pm}(x) = \mathrm{e}^{\pm\sqrt{\Lambda_0}x}$ form a pair of fundamental solutions to the homogeneous problem. Using variation of constants, then, we obtain a special solution to the inhomogeneous ODE,

$$\eta_{0,sp}(x) = (2\ell\sqrt{\Lambda_0})^{-1}\left[\Gamma_0\left(\eta_{0,+}f\,;x\right)\eta_{0,-}(x) - \Gamma_0\left(\eta_{0,-}f\,;x\right)\eta_{0,+}(x)\right].$$

Here, we have defined the family of functionals

$$\Gamma_n\left(\cdot\,;x\right) = \int_0^x \cdot(s)\,\omega_n^{+}(s)\,ds, \quad \text{parameterized by } x \in [0,1] \text{ and } n \geq 0. \tag{B.2}$$

The solution to (B.1) is, then,

$$\left\{\begin{array}{rcl} \eta_0(x) & = & \left[C_\eta^{+} - (2\ell\sqrt{\Lambda_0})^{-1}\Gamma_0\left(\eta_{0,-}f\,;x\right)\right]\eta_{0,+}(x) \\ & & +\left[C_\eta^{-} + (2\ell\sqrt{\Lambda_0})^{-1}\Gamma_0\left(\eta_{0,+}f\,;x\right)\right]\eta_{0,-}(x). \end{array}\right. \tag{B.3}$$

Imposing the boundary conditions for η_0 and using the identity $\Gamma_0\left(\cdot\,;0\right) = 0$, we find that the constants C_η^{-} and C_η^{+} satisfy the linear system

$$\begin{aligned} &\sqrt{\Lambda_0}\;C_\eta^{+} - \sqrt{\Lambda_0}\;C_\eta^{-} = 0, \\ &\left[2\ell\sqrt{\Lambda_0}C_\eta^{+} - \Gamma_0\left(\eta_{0,-}f\,;1\right)\right]\mathrm{e}^{\sqrt{\Lambda_0}} + \left[2\ell\sqrt{\Lambda_0}C_\eta^{-} + \Gamma_0\left(\eta_{0,+}f\,;1\right)\right]\mathrm{e}^{-\sqrt{\Lambda_0}} = 0, \end{aligned}$$

the solution to which is $C_\eta^{+} = C_\eta^{-} = C_\eta/(2\ell\sqrt{\Lambda_0})$, with

$$C_\eta = \frac{\Gamma_0\left(\eta_{0,-}f\,;1\right)\eta_{0,+}(1) - \Gamma_0\left(\eta_{0,+}f\,;1\right)\eta_{0,-}(1)}{2\cosh\left(\sqrt{\Lambda_0}\right)}.$$

Thus, (B.5) becomes

$$\eta_0(x) = (2\ell\sqrt{\Lambda_0})^{-1}\left[2C_\eta \cosh\left(\sqrt{\Lambda_0}\,x\right) + \Gamma_0\left(\eta_{0,+}f\,;x\right)\eta_{0,-}(x) - \Gamma_0\left(\eta_{0,-}f\,;x\right)\eta_{0,+}(x)\right]. \text{(B.4)}$$

Further employing the definition (B.2), we calculate

$$\begin{aligned}
&\Gamma_0\left(\eta_{0,+}f\,;x\right)\eta_{0,-}(x) - \Gamma_0\left(\eta_{0,-}f\,;x\right)\eta_{0,+}(x)\\
&\qquad= \int_0^x \left[\eta_{0,-}(x)\eta_{0,+}(y) - \eta_{0,+}(x)\eta_{0,-}(y)\right]\, f(y)\,\omega_0^+(y)\,dy\\
&\qquad= -2\int_0^x \sinh\left(\sqrt{\Lambda_0}(x-y)\right)\, f(y)\,\omega_0^+(y)\,dy\\
&\qquad= -2\,\Gamma_0\left(\sinh\left(\sqrt{\Lambda_0}(x-\cdot)\right)f\,;x\right).
\end{aligned}$$

Additionally,

$$\begin{aligned}
C_\eta &= \frac{\Gamma_0\left(\eta_{0,-}f\,;1\right)\eta_{0,+}(1) - \Gamma_0\left(\eta_{0,+}f\,;1\right)\eta_{0,-}(1)}{2\cosh\left(\sqrt{\Lambda_0}\right)}\\
&= \frac{1}{2\cosh\left(\sqrt{\Lambda_0}\right)}\int_0^1 \left[\eta_{0,-}(y)\,\eta_{0,+}(1) - \eta_{0,+}(y)\,\eta_{0,-}(1)\right]\, f(y)\,\omega_0^+(y)\,dy\\
&= \frac{1}{\cosh\left(\sqrt{\Lambda_0}\right)}\int_0^1 \sinh\left(\sqrt{\Lambda_0}(1-y)\right)\, f(y)\,\omega_0^+(y)\,dy\\
&= \frac{1}{\cosh\left(\sqrt{\Lambda_0}\right)}\,\Gamma_0\left(\sinh\left(\sqrt{\Lambda_0}(1-\cdot)\right)f\,;1\right),
\end{aligned}$$

and hence (B.4) becomes

$$\begin{aligned}
\eta_0(x) &= \frac{1}{\ell\sqrt{\Lambda_0}}\left[\frac{\cosh\left(\sqrt{\Lambda_0}\,x\right)}{\cosh\left(\sqrt{\Lambda_0}\right)}\,\Gamma_0\left(\sinh\left(\sqrt{\Lambda_0}(1-\cdot)\right)f\,;1\right)\right.\\
&\qquad\left. -\Gamma_0\left(\sinh\left(\sqrt{\Lambda_0}(x-\cdot)\right)f\,;x\right)\right]\\
&= \frac{1}{\ell\sqrt{\Lambda_0}}\left[\frac{\cosh\left(\sqrt{\Lambda_0}\,x\right)}{\cosh\left(\sqrt{\Lambda_0}\right)}\int_0^1 \sinh\left(\sqrt{\Lambda_0}(1-y)\right)\, f(y)\,\omega_0^+(y)\,dy\right.\\
&\qquad\left. -\int_0^x \sinh\left(\sqrt{\Lambda_0}(x-y)\right)\, f(y)\,\omega_0^+(y)\,dy\right].
\end{aligned} \tag{B.5}$$

To estimate $\|\eta_0\|_\infty$ over $[0,1]$, we first show that η_0 is positive and that it assumes its maximum in an $\mathrm{O}(\varepsilon^{1/4})$ neighborhood of x_*. First, an estimate based on (B.5) establishes readily that $\eta_0(x) > 0$ for all $x \in (0,1)$,

$$\begin{aligned}
\eta_0(x) &\geq \int_0^1 \left[\frac{\cosh\left(\sqrt{\Lambda_0}\,x\right)}{\cosh\left(\sqrt{\Lambda_0}\right)}\sinh\left(\sqrt{\Lambda_0}(1-y)\right) - \sinh\left(\sqrt{\Lambda_0}(x-y)\right)\right]\frac{f(y)\,\omega_0^+(y)}{\ell\sqrt{\Lambda_0}}\,dy\\
&= \frac{\sinh\left(\sqrt{\Lambda_0}(1-x)\right)}{\ell\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)}\int_0^1 \cosh\left(\sqrt{\Lambda_0}\,y\right)\, f(y)\,\omega_0^+(y)\,dy > 0,
\end{aligned}$$

for $x \in (0,1)$. To locate the maximum, we differentiate both members of (B.5) and obtain

$$\ell\,\partial_x\eta_0(x) = \frac{\sinh\left(\sqrt{\Lambda_0}\,x\right)}{\cosh\left(\sqrt{\Lambda_0}\right)}\int_0^1 \sinh\left(\sqrt{\Lambda_0}(1-y)\right)\, f(y)\,\omega_0^+(y)\,dy$$

$$
\begin{aligned}
&- \int_0^x \cosh\left(\sqrt{\Lambda_0}(x-y)\right) f(y)\,\omega_0^+(y)\,dy \\
&- \frac{\sinh\left(\sqrt{\Lambda_0}(x-y)\right)}{\sqrt{\Lambda_0}} f(x)\,\omega_0^+(x).
\end{aligned} \tag{B.6}
$$

Theorem Appendix D.1 can be used to yield the principal part of the two integrals in this formula, whereas the term proportional to ω_0^+ can be estimated via (A.2). For the values of Λ_0 we are interested in, the localized term in either integrand is ω_0^+, while the Λ_0-dependent terms vary on an asymptotically larger length scale. Thus,

$$
\partial_x \eta_0(x) = \frac{\sinh\left(\sqrt{\Lambda_0}\,x\right)}{\ell \cosh\left(\sqrt{\Lambda_0}\right)} \int_0^1 \sinh\left(\sqrt{\Lambda_0}(1-y)\right) f(y)\,\omega_0^+(y)\,dy > 0,
$$

to leading order and for $x < x_*$ and $|x - x_*| \gg \varepsilon^{1/4}$, since the second and third terms in the right member of (B.6) are exponentially small compared to the first one. Similarly,

$$
\begin{aligned}
\partial_x \eta_0(x) = \frac{1}{\ell \cosh\left(\sqrt{\Lambda_0}\right)} \Bigg[&\sinh\left(\sqrt{\Lambda_0}\,x\right) \int_0^1 \sinh\left(\sqrt{\Lambda_0}(1-y)\right) f(y)\,\omega_0^+(y)\,dy \\
&- \cosh\left(\sqrt{\Lambda_0}\right) \int_0^x \cosh\left(\sqrt{\Lambda_0}(x-y)\right) f(y)\,\omega_0^+(y)\,dy \Bigg],
\end{aligned}
$$

for $x > x_*$ and $|x - x_*| \gg \varepsilon^{1/4}$, since the second and third terms in the same formula are of the same asymptotic order and the third one is exponentially smaller. Changing the upper limit of the second integral to one (and thus only introducing an exponentially small error) and combining the two integrals, we find

$$
\partial_x \eta_0(x) = -\frac{\cosh\left(\sqrt{\Lambda_0}\,(1-x)\right)}{\ell \cosh\left(\sqrt{\Lambda_0}\right)} \int_0^1 \cosh\left(\sqrt{\Lambda_0}\,y\right) f(y)\,\omega_0^+(y)\,dy < 0,
$$

Since $\eta_0 \in \mathrm{C}^2(0,1)$, now, it follows that $\eta_0'(x_1) = 0$ at a point x_1 such that $|x_* - x_1| = \mathrm{O}(\varepsilon^{1/4})$, as desired. Hence, we can now use (B.5) to estimate further

$$
\|\eta_0\|_\infty \le \eta_0(x_1) \le \frac{\cosh\left(\sqrt{\Lambda_0}\,x_1\right)}{\ell \sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)} \int_0^1 \sinh\left(\sqrt{\Lambda_0}(1-y)\right) f(y)\,\omega_0^+(y)\,dy.
$$

Using our asymptotic estimate on x_1 and Theorem Appendix D.1, we find

$$
\|\eta_0\|_\infty \le C\,\varepsilon^{1/6}\,\delta^{-1}\,\frac{\cosh\left(\sqrt{\Lambda_0}\,x_*\right)\,\sinh\left(\sqrt{\Lambda_0}(1-x_*)\right)}{\sqrt{\Lambda_0}\,\cosh\left(\sqrt{\Lambda_0}\right)}.
$$

for some Λ_0-independent, $\mathrm{O}(1)$ constant C. Since the Λ_0-dependent quantity in the bound above remains bounded by an $\mathrm{O}(1)$ constant also for $\Lambda_0 \gg 1$, we finally conclude that $\|\eta_0\|_\infty$ can be bounded by an $\mathrm{O}(\varepsilon^{1/6}\delta^{-1})$ constant.

Appendix C. Asymptotic formulas for ψ_n, $n \ge 0$

The function ψ_n is the solution to the boundary value problem

$$
\varepsilon\,\partial_{xx}\psi_n + \left(f(x) - \ell - A - \nu_n\right)\psi_n = -\varepsilon \ell^{-1} f E\,\zeta_n, \quad \text{where } \mathcal{G}\left(\psi_n\,;0\right) = \mathcal{G}\left(\psi_n\,;1\right) = 0,
$$

cf. (2.11). Here, $\mathcal{G}\left(\psi_n\,;x\right) = \psi_n(x) - \sqrt{\varepsilon/A}\,\partial_x\psi_n(x)$ and we recall that

$$
\zeta_n(x) = \sqrt{2}\cos(\sqrt{N_n}\,x) = \sqrt{2}\cos(\epsilon_n^{-1}\,x). \tag{C.1}
$$

(2.5). Recalling also the definitions $F(x) = f(0) - f(x)$ and $\lambda_* = f(0) - \ell - A$, we write $f(x) - \ell - A = \lambda_* - F(x)$. Recalling also (1.13), $\lambda_0 = \lambda_* - \varepsilon^{1/3}\mu_0$ with $\mu_0 = \sigma_0^{2/3}\,|A_1| + \mathrm{O}(\varepsilon^{1/6})$, we obtain

$$f(x) - \ell - A = \lambda_0 - F(x) + \varepsilon^{1/3}\mu_0.$$

Finally, since $\lambda_0 = \varepsilon\Lambda_0$ and $\nu_n = -\varepsilon/\epsilon_n^2$, where $\Lambda_0 > 0$ is $\mathrm{O}(1)$, we may rewrite (2.11) in the final form

$$\varepsilon\,\partial_{xx}\psi_n - \left[F(x) - \varepsilon^{1/3}\mu_0(\Lambda_0) - \frac{\varepsilon}{\epsilon_n^2}\right]\psi_n = -\frac{\varepsilon\,E\,f\,\zeta_n}{\ell}, \quad \text{with } \mu_0(\Lambda_0) = \mu_0 + \varepsilon^{2/3}\Lambda_0, \tag{C.2}$$

together with the boundary conditions $\mathcal{G}(\psi_n\,;0) = \mathcal{G}(\psi_n\,;1) = 0$. In what follows, we derive asymptotic formulas for ψ_n and for values of n satisfying $\epsilon_n \gg \varepsilon^{1/3}$.

In that case, $\varepsilon\epsilon_n^{-2} \ll \varepsilon^{1/3}$ and the term $\varepsilon\epsilon_n^{-2}$ is perturbative to $\varepsilon^{1/3}\mu_0(\Lambda_0)$. Hence, we may write

$$\varepsilon^{1/3}\mu_0(\Lambda_0) + \varepsilon\epsilon_n^{-2} = F(x_n), \quad \text{where} \quad x_n = \bar{x}_0(1 + o(1)) \tag{C.3}$$

is a turning point for (C.2). Then, (C.2) becomes

$$\varepsilon\,\partial_{xx}\psi_n - [F(x) - F(x_n)]\,\psi_n = -\varepsilon\ell^{-1} f E\,\zeta_n, \tag{C.4}$$

equipped with the boundary conditions (2.11). The solution to this boundary-value problem may be found by variation of constants,

$$\psi_n(x) = \left[C_\psi^+ - \varepsilon(\ell W_\psi)^{-1}G_-(x)\right]\psi_{n,+}(x) + \left[C_\psi^- + \varepsilon(\ell W_\psi)^{-1}G_+(x)\right]\psi_{n,-}(x). \tag{C.5}$$

Here, $\psi_{n,\pm}$ is a pair of fundamental solutions to $\varepsilon\,\partial_{xx}\psi_n = [F(x) - F(x_n)]\,\psi_n$ and $W_\psi = \psi_{n,-}\partial_x\psi_{n,+} - \psi_{n,+}\partial_x\psi_{n,-}$ is the associated Wronskian. (To derive the result above, we have used the identity $\partial_x W_\psi(x) = 0$, for all $x \in [0,1]$, which follows from the definition of W_ψ and the ODE that $\psi_\pm$ satisfy.) Further,

$$G_\pm(x) = \int_0^x f(y)\zeta_n(y)\,\psi_{n,\pm}^+(y)dy, \tag{C.6}$$

where $\psi_{n,\pm}^+ = E\,\psi_{n,\pm}$. Using (C.5), we further obtain

$$\partial_x\psi_n(x) = \left[C_\psi^+ - \varepsilon(\ell W_\psi)^{-1}G_-(x)\right]\partial_x\psi_{n,+}(x) + \left[C_\psi^- + \varepsilon(\ell W_\psi)^{-1}G_+(x)\right]\partial_x\psi_{n,-}(x),$$

and thus the boundary conditions yield the system

$$C_\psi^+\mathcal{G}(\psi_{n,+}\,;0) + C_\psi^-\mathcal{G}(\psi_{n,-}\,;0) = 0,$$
$$\left[C_\psi^+ - \frac{\varepsilon}{\ell W_\psi}G_-(1)\right]\mathcal{G}(\psi_{n,+}\,;1) + \left[C_\psi^- + \frac{\varepsilon}{\ell W_\psi}G_+(1)\right]\mathcal{G}(\psi_{n,-}\,;1) = 0.$$

The solution to this system is

$$C_\psi^+ = -\frac{\varepsilon}{\ell W_\psi}\,D_\psi\,\mathcal{G}(\psi_{n,-}\,;0) \quad \text{and} \quad C_\psi^- = \frac{\varepsilon}{\ell W_\psi}\,D_\psi\,\mathcal{G}(\psi_{n,+}\,;0), \tag{C.7}$$

where

$$D_\psi = \frac{G_-(1)\,\mathcal{G}(\psi_{n,+}\,;1) - G_+(1)\,\mathcal{G}(\psi_{n,-}\,;1)}{\mathcal{G}(\psi_{n,+}\,;0)\,\mathcal{G}(\psi_{n,-}\,;1) - \mathcal{G}(\psi_{n,-}\,;0)\,\mathcal{G}(\psi_{n,+}\,;1)}. \tag{C.8}$$

Thus, also, (C.5) becomes

$$\psi_n(x) = \varepsilon(\ell W_\psi)^{-1}\,\left[\Gamma_-(x)\,\psi_{n,-}(x) - \Gamma_+(x)\,\psi_{n,+}(x)\right], \tag{C.9}$$

where

$$\Gamma_-(x) = G_+(x) + D_\psi\,\mathcal{G}\,(\psi_{n,+}\,;0) \tag{C.10}$$

$$\Gamma_+(x) = G_-(x) + D_\psi\,\mathcal{G}\,(\psi_{n,-}\,;0)\,. \tag{C.11}$$

These formulas hold for an arbitrary pair $\psi_{n,\pm}$ of fundamental solutions. Working as in [23], where the problem was considered in detail in the absence of the perturbative term $\varepsilon(\Lambda_0 + N_n)$, we can derive the following leading order formulas for a specific pair of solutions $\psi_{n,\pm}$:

$$\psi_{n,-}(x) = \begin{cases} \varepsilon^{1/6}\,\sigma_0^{-1/6}\,\mathrm{Ai}\left(A_1(1-\bar{x}_0^{-1}x)\right), & \text{for } x\in[0,\bar{x}_0),\\ \varepsilon^{1/4}\,\frac{C_2}{2\sqrt{\pi}\,F^{1/4}(x)}\,\omega_{0,-}(x;\bar{x}_0), & \text{for } x\in(\bar{x}_0,1], \end{cases} \tag{C.12}$$

$$\psi_{n,+}(x) = \begin{cases} \varepsilon^{1/6}\,\sigma_0^{-1/6}\,\mathrm{Bi}\left(A_1(1-\bar{x}_0^{-1}x)\right), & \text{for } x\in[0,\bar{x}_0),\\ \varepsilon^{1/4}\,\frac{1}{\sqrt{\pi}\,C_2\,F^{1/4}(x)}\,\omega_{0,+}(x;\bar{x}_0), & \text{for } x\in(\bar{x}_0,1]. \end{cases} \tag{C.13}$$

Here, we have used that $x_n = \bar{x}_0 + o(\sqrt{\varepsilon})$. The identity $\partial_x W_\psi = 0$, which was reported earlier, leads to

$$W_\psi(x) = W_\psi(0) \sim -\mathrm{Ai}'(A_1)\mathrm{Bi}(A_1) = \mathrm{Ai}'(A_1)\,|\mathrm{Bi}(A_1)| > 0, \tag{C.14}$$

for this particular pair. Next, we simplify the formula (C.8) by investigating the asymptotic magnitude of the terms in its right member. By definition (2.11),

$$\mathcal{G}\,(\psi_{n,\pm}\,;0) = \psi_{n,\pm}(0) - \sqrt{\varepsilon/A}\,(\partial_x\psi_{n,\pm})(0).$$

Equations (C.3) and (C.12)–(C.13) yield

$$\begin{aligned}\mathcal{G}\,(\psi_{n,-}\,;0) \;&=\; -\,\varepsilon^{5/6}\,\sigma_0^{-5/6}\,N_n\,\mathrm{Ai}'(A_1) + \mathrm{O}(\varepsilon^{7/6})\\ \mathcal{G}\,(\psi_{n,+}\,;0) \;&=\varepsilon^{1/6}\,\sigma_0^{-1/6}\,\mathrm{Bi}(A_1) + \mathrm{O}(\varepsilon^{1/3}).\end{aligned}$$

(Here, we have Taylor expanded $\mathrm{Ai}(A_1(1-\bar{x}_0^{-1}x))$ around its zero $x=0$.) Next,

$$\mathcal{G}\,(\psi_{n,\pm}\,;1) \;\sim \varepsilon^{1/4}\frac{(1\mp\sqrt{\sigma_1/A})\,c_\pm}{\sigma_1^{1/4}}\,\exp\left(\pm\frac{I(1)}{\sqrt{\varepsilon}}\right) \tag{C.15}$$

(recall (3.12)). These formulas imply that $\mathcal{G}\,(\psi_{n,+}\,;0)\,\mathcal{G}\,(\psi_{n,-}\,;1)$ is exponentially smaller than $\mathcal{G}\,(\psi_{n,-}\,;0)\,\mathcal{G}\,(\psi_{n,+}\,;1)$, and thus

$$D_\psi = \frac{D_1\,G_+(1) - G_-(1)}{\mathcal{G}\,(\psi_{n,-}\,;0)}, \quad \text{where} \quad D_1 = \frac{\mathcal{G}\,(\psi_{n,-}\,;1)}{\mathcal{G}\,(\psi_{n,+}\,;1)} \tag{C.16}$$

and down to exponentially small terms. Next, the relative asymptotic magnitudes of the terms in $G_-(1) - D_1 G_+(1)$ may be derived using the definitions (2.11) and (C.6) together with Laplace's approximation (cf. Theorem Appendix D.1). One finds that $G_-(1)$ is dominated by $\exp(\varepsilon^{-1/2}J_-(x_*))$, whereas $D_1 G_+(1)$ by $\exp(\varepsilon^{-1/2}J_-(1))$, and hence the latter is exponentially smaller than the former. Hence,

$$D_\psi = -\frac{G_-(1)}{\mathcal{G}\,(\psi_{n,-}\,;0)}. \tag{C.17}$$

It follows, then, that

$$\Gamma_-(x) = G_+(x) - D_n\, G_-(1) \quad \text{and} \quad \Gamma_+(x) = G_-(x) - G_-(1), \tag{C.18}$$

and down to exponentially small terms. Here,

$$D_n = \frac{\mathcal{G}\,(\psi_{n,+}\,;0)}{\mathcal{G}\,(\psi_{n,-}\,;0)} = \varepsilon^{-2/3}\, d_n, \tag{C.19}$$

where

$$d_n = -\frac{\sigma_0^{2/3}\,\mathrm{Bi}(A_1)}{N_n\,\mathrm{Ai}'(A_1)} = \frac{\sigma_0^{2/3}\,\,|\mathrm{Bi}(A_1)|}{N_n\,\mathrm{Ai}'(A_1)} > 0. \tag{C.20}$$

Combining this formula with (C.9), we find

$$\begin{aligned}\psi_n(x) &= \varepsilon(\ell W_\psi)^{-1}\,\left[G_+(x)\psi_{n,-}(x) - G_-(x)\psi_{n,+}(x) + G_-(1)\,(\psi_{n,+}(x) - D_n\psi_{n,-}(x))\right] \\ &= \varepsilon(\ell W_\psi)^{-1}\,\left[(G_+(x) - D_n G_-(1))\,\psi_{n,-}(x) + (G_-(1) - G_-(x))\,\psi_{n,+}(x)\right] \\ &= \varepsilon(\ell W_\psi)^{-1}\,\Bigg[\psi_{n,-}(x)\left(\int_0^x f(y)\zeta_n(y)\,\psi^+_{n,+}(y)dy - D_n\int_0^1 f(y)\zeta_n(y)\,\psi^+_{n,-}(y)dy\right) \\ &\qquad\qquad + \psi_{n,+}(x)\int_x^1 f(y)\zeta_n(y)\,\psi^+_{n,-}(y)dy\Bigg].\end{aligned} \tag{C.21}$$

Appendix D. Asymptotic approximation of integrals

Appendix D.1. Localized integrals

Our main tool in this section will be Laplace's method and, in particular, the following three theorems based on [22, Ch. II, VIII, IX].

Theorem Appendix D.1 *([22, Theorem IX.3]) Let $n \in \mathbf{N}$, $D \subset \mathbf{R}^n$ be a domain with piecewise smooth boundary ∂D, and $u_0 \in \bar{D}$. Let, also, the functions $\Pi \in C^2(\bar{D}, \mathbf{R})$ and $\Xi \in C(\bar{D}, \mathbf{R})$ satisfy the conditions*

(a) $\inf_{\bar{D}-B(u_0;\delta)} \Pi(u) > \Pi(u_0)$, *for all* $\delta > 0$,

(b) $\sigma\left(D^2\Pi(u_0)\right) \subset \mathring{\mathbf{R}}_+$,

(c) the integral $\mathcal{I}_D(\lambda) := \int\cdots\int_D \Xi(u)\,\mathrm{e}^{-\lambda\Pi(u)}du$ *converges absolutely for all sufficiently large* λ.

(Here, $D^2\Pi$ denotes the Hessian matrix of Π.) Then,

$$\mathcal{I}_D(\lambda) \sim \mathrm{e}^{-\lambda\Pi(u_0)} \sum_{k=0}^{\infty} c_k\, \lambda^{-(k+n/2)} \quad (\lambda \to \infty),$$

where one may derive explicit formulas for the constants $\{c_k\}_k$. In particular,

(I) $\mathcal{I}_D(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{\Xi(u_0)\,\mathrm{e}^{-\lambda\Pi(u_0)}}{\sqrt{\det D^2\Pi(u_0)}}$, *if* $u_0 \in \mathring{D}$ *and* $\Xi(u_0) \neq 0$,

(II) $\mathcal{I}_D(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{(n+2)/2} C_0\,\mathrm{e}^{-\lambda\Pi(u_0)}$, *if* $u_0 \in \mathring{D}$ *and* $\Xi(u_0) = 0$,

$$\text{(III)}\quad \mathcal{I}_D(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{\Xi(u_0)\,\mathrm{e}^{-\lambda\Pi(u_0)}}{2\sqrt{\det D^2\Pi(u_0)}}, \quad \textit{if} \quad u_0 \in \partial D,\ \Xi(u_0) \neq 0,\ \textit{and}\ D\Pi(u_0) = 0,$$

$$\text{(IV)}\quad \mathcal{I}_D(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{(n+1)/2} \frac{\Xi(u_0)\,\mathrm{e}^{-\lambda\Pi(u_0)}}{2\pi\sqrt{\det J}}, \quad \textit{if} \quad u_0 \in \partial D,\ \Xi(u_0) \neq 0,\ \textit{and}\ D\Pi(u_0) \neq 0,$$

as $\lambda \to \infty$, *for some constant* C_0 *which is at most* $\mathrm{O}(1)$ *with respect to* λ *and under the assumption that* ∂D *is smooth around* u_0 *in the cases where* $u_0 \in \partial D$. *Here,* J *is a matrix related to* $D^2\Pi(u_0)$ *and to the local characteristics of* ∂D *around* u_0.

Theorem Appendix D.2 *Let* $a < b$ *and* $u_0 \in [a,b]$. *Let, also, the functions* $\Pi \in C^2([a,b],\mathbf{R})$ *and* $\Xi \in C([a,b],\mathbf{R})$ *satisfy the conditions*

(a) $\inf_{[a,b]-B(u_0;\delta)} \Pi(u) > \Pi(u_0)$, *for all* $\delta > 0$,

(b) the integral $\mathcal{I}(\lambda) := \int_a^b \Xi(u)\,\mathrm{e}^{-\lambda\Pi(u)} du$ *converges absolutely for all sufficiently large* λ.

Then,

$$\mathcal{I}(\lambda) \sim \mathrm{e}^{-\lambda\Pi(u_0)} \sum_{k=1}^{\infty} c_k\,\lambda^{-k/2} \quad (\lambda \to \infty),$$

where one may derive explicit formulas for the constants $\{c_k\}_k$. *In particular, as* $\lambda \to \infty$,

$$\text{(I)}\quad \mathcal{I}(\lambda) \sim \frac{\mathrm{e}^{-\lambda\Pi(u_0)}}{\lambda^{1/2}} \frac{\sqrt{2\pi}\,\Xi(u_0)}{\sqrt{\Pi''(u_0)}}, \quad \textit{if} \quad u_0 \in (a,b) \quad \textit{and} \quad \Xi(u_0) \neq 0,$$

$$\text{(II)}\quad \mathcal{I}(\lambda) \sim \frac{\mathrm{e}^{-\lambda\Pi(u_0)}}{\lambda^{3/2}} \frac{\sqrt{\pi}\left(\Xi''(u_0) - \frac{\Xi'(u_0)\,\Pi'''(u_0)}{\Pi''(u_0)}\right)}{\sqrt{2}\,[\Pi''(u_0)]^{3/2}}, \quad \textit{if} \quad u_0 \in (a,b) \quad \textit{and} \quad \Xi(u_0) = 0,$$

$$\text{(III)}\quad \mathcal{I}(\lambda) \sim \frac{\mathrm{e}^{-\lambda\Pi(u_0)}}{\lambda} \frac{\Xi(u_0)}{|\Pi'(u_0)|}, \quad \textit{if} \quad u_0 \in \{a,b\},\ \Xi(u_0) \neq 0, \quad \textit{and} \quad \Pi'(u_0) \neq 0,$$

$$\text{(IV)}\quad \mathcal{I}(\lambda) \sim \frac{\mathrm{e}^{-\lambda\Pi(u_0)}}{\lambda^{1/2}} \frac{\sqrt{\pi}\,\Xi(u_0)}{\sqrt{2\Pi''(u_0)}}, \quad \textit{if} \quad u_0 \in \{a,b\},\ \Xi(u_0) \neq 0, \quad \textit{and} \quad \Pi'(u_0) = 0,$$

$$\text{(V)}\quad \mathcal{I}(\lambda) \sim \frac{\mathrm{e}^{-\lambda\Pi(u_0)}}{\lambda^2} \frac{\pm\Xi'(u_0)}{[\Pi'(u_0)]^2}, \quad \textit{if} \quad u_0 = \begin{cases} a\ (+) \\ b\ (-) \end{cases}, \ \Xi(u_0) = 0, \quad \textit{and} \quad \Pi'(u_0) \neq 0,$$

$$\text{(VI)}\quad \mathcal{I}(\lambda) \sim \frac{\mathrm{e}^{-\lambda\Pi(u_0)}}{\lambda} \frac{\pm\Xi'(u_0)}{\Pi''(u_0)}, \quad \textit{if} \quad u_0 = \begin{cases} a\ (+) \\ b\ (-) \end{cases}, \ \Xi(u_0) = 0, \quad \textit{and} \quad \Pi'(u_0) = 0.$$

Theorem Appendix D.3 *Let* $D \subset \mathbf{R}^2$ *be a two-dimensional domain with piecewise smooth boundary* ∂D *and* $u_0 = (x_0, y_0) \in \partial D$. *Let, also, the functions* $\Pi \in C^2(\bar{D},\mathbf{R})$ *and* $\Xi \in C(\bar{D},\mathbf{R})$ *satisfy the conditions*

(a) $\inf_{\bar{D}-B(u_0;\delta)} \Pi(u) > \Pi(u_0)$, *for all* $\delta > 0$,

(b) the integral $\mathcal{I}_D(\lambda) := \int \cdots \int_D \Xi(u)\,\mathrm{e}^{-\lambda\Pi(u)} du$ *converges absolutely for all sufficiently large* λ.

Assume, further, that ∂D has a corner at u_0 and, in particular, that ∂D is given (locally around u_0) by the curves $k(x,y)=0$ and $h(x,y)=0$ with $Dk(u_0)\times Dh(u_0)\neq 0$. Let the vectors v_k and v_h satisfy

$$v_k \perp Dk(u_0), \quad v_h \perp Dh(u_0), \quad \text{and} \quad \|v_k \times v_h\| = 1.$$

If v_k and v_h can be selected to further satisfy the conditions

$$\Pi_k := \langle v_k, D\Pi(u_0)\rangle > 0 \quad \text{and} \quad \Pi_h := \langle v_h, D\Pi(u_0)\rangle > 0, \tag{D.1}$$

then

$$\mathcal{I}_D(\lambda) \sim \mathrm{e}^{-\lambda\Pi(u_0)} \sum_{k=0}^{\infty} c_k\, \lambda^{-(k+2)} \quad (\lambda \to \infty),$$

where one may derive explicit formulas for the constants $\{c_k\}_k$. In particular,

$$\text{(I)} \quad \mathcal{I}_D(\lambda) \sim \frac{1}{\lambda^2}\, \frac{\Xi(u_0)\,\mathrm{e}^{-\lambda\Pi(u_0)}}{2\Pi_k\Pi_h\sqrt{\Pi_k^2+\Pi_h^2}}, \quad \text{if} \quad \Xi(u_0)\neq 0,$$

$$\text{(II)} \quad \mathcal{I}_D(\lambda) \sim \frac{1}{\lambda^3}\, \frac{(\Pi_k\Xi_h + \Pi_h\Xi_k)\ \mathrm{e}^{-\lambda\Pi(u_0)}}{\Pi_k^2\Pi_h^2\sqrt{\Pi_k^2+\Pi_h^2}}, \quad \text{if} \quad \Xi(u_0) = 0,$$

as $\lambda\to\infty$. Here, $\Xi_k := \langle v_k, D\Xi(u_0)\rangle$ and $\Xi_h := \langle v_h, D\Xi(u_0)\rangle$, compare to (D.1).

Appendix D.2. Oscillatory integrals

Theorem Appendix D.4 *Let $a<b$, $\Xi \in C([a,b],\mathbf{R})$, and $\Phi \in C^2([a,b],\mathbf{R})$. Assume that*

$$\Phi(t) = \Phi(a) + (t-a)\,\Phi_1(t) \quad \text{and} \quad \Phi'(t) > 0, \quad \text{for all } t\in[a,b] \text{ and with } \Phi_1(a)\neq 0.$$

Then, the integral $\mathcal{I}(\lambda) := \int_a^b \Xi(t)\,\mathrm{e}^{\mathrm{i}\lambda\Phi(t)}dt$ has the following asymptotic expansion:

$$\mathcal{I}(\lambda) \sim \sum_{k=0}^{\infty} \left[h^{(k)}(0)\,\mathrm{e}^{\mathrm{i}\lambda\Phi(a)} - h^{(k)}(\Phi(b)-\Phi(a))\,\mathrm{e}^{\mathrm{i}\lambda\Phi(b)}\right] \left(\frac{\mathrm{i}}{\lambda}\right)^{k+1} \quad (\lambda\to\infty),$$

where we have defined the function

$$h(\tau) = \Xi(t(\tau))\,t'(\tau).$$

Here, $\tau(t) = \Phi(t)-\Phi(a)$ or, equivalently, $t(\tau) = \Phi^{-1}(\Phi(a)+\tau)$.

[1] Bates P W and Jones C K R T 1989 Invariant manifolds for semilinear partial differential equations *Dynamics Reported* vol 2 (*Series in Dynamical Systems* vol 2) ed U Kirchgraber and H O Walther (Chichester: Wiley) pp 1–38

[2] Bender C M and Orszag S A 1999 *Advanced Mathematical Methods for Scientists and Engineers* (New York: Springer-Verlag)

[3] Boyce D G, Lewis M R and Worm B 2010 *Nature* **466** 591–6

[4] Carr J 1981 *Applications of Centre Manifold Theory* (*Applied Mathematical Sciences* vol 35) ed S S Antman, J E Marsden and L Sirovich (New York: Springer-Verlag)

[5] Doelman A, de Swart H E, Zagaris A and Zijlstra J 2010 Effects of horizontal diffusion on marine phytoplankton growth *Preprint*

[6] Du Y and Hsu S-B 2008 *SIAM J. Math. Anal.* **40**(4) 1419–40

[7] Du Y and Hsu S-B 2008 *SIAM J. Math. Anal.* **40**(4) 1441–70

[8] Du Y and Hsu S-B 2010 *SIAM J. Math. Anal.* **42**(3) 1305–33
[9] Ebert U, Arrayás M, Temme N, Sommeijer B-P and Huisman J 2001 *Bull. Math. Biology* **63**(6) 1095–124
[10] Falkowski P G, Barber R T and Smetacek V 1998 *Science* **281**(5374) 200–6
[11] Fennel K and Boss E 2003 *Limnol. Oceanogr.* **48**(4) 1521–34
[12] Ghosal S and Mandre S 2003 *J. Math. Biology* **46**(4) 333–46
[13] Huisman J, van Oostveen P and Weissing F J 1999 *Limnol. Oceanogr.* **44**(7) 1781–7
[14] Huisman J, van Oostveen P and Weissing F J 1999 *Am. Nat.* **154**(1) 1 46–68
[15] Huisman J, Pham Thi N N, Karl D M and Sommeijer B P 2006 *Nature* **439** 322–5
[16] Huisman J and Sommeijer B P 2002 *J. Sea Research* **48**(2) 83–96
[17] Ishii H and Takagi I 1982 *J. Math. Biology* **16**(1) 1–24
[18] Klausmeier C A and Litchman E 2001 *Limnol. Oceanogr.* **46**(8) 1998–2007
[19] Oberhettinger F 1973 *Fourier Expansions: a Collection of Formulas* (New York: Academic Press)
[20] Sarmiento J L, Hughes T H C, Stouffer R J and Manabe S 1998, *Nature* **393** 245–9
[21] Wollkind D J, Manoranjan V S and Zhang L 1994, *SIAM Review* **36**(2) 176–214
[22] Wong R 2001 *Asymptotic Approximations of Integrals* (Philadelphia: SIAM)
[23] Zagaris A, Doelman A, Pham Thi N N and Sommeijer B P 2009 *SIAM J. Appl. Math.* **69**(4) 1174–204