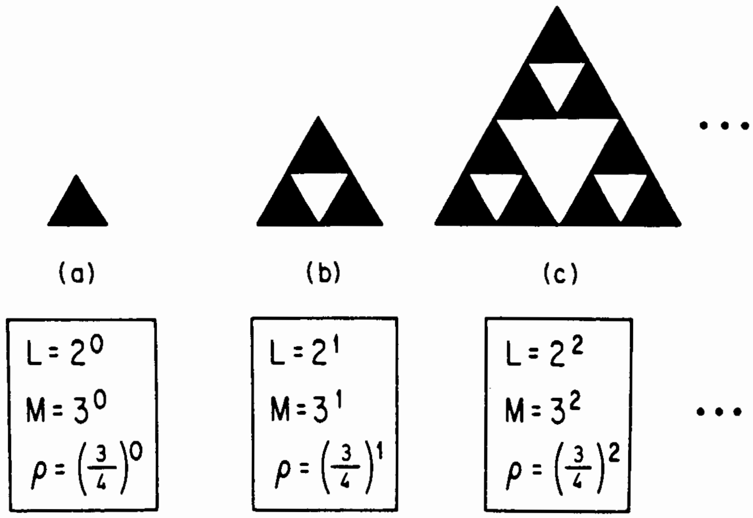

Fig. 1.1. First few stages in the aggregation rule which is iterated to form a Sierpiński gasket fractal. After [1.16]

from physics and chemistry on the one hand to fluid dynamics and meteorology on the other. The purpose of this opening chapter is to provide the nonspecialist with a brief introduction to fractal and multifractal phenomena.

Although there are many different types of fractal and multifractal phenomena, we shall concentrate on a few examples which we hope will prove useful to the reader.

1.2 Nonrandom Fractals

Fractals fall naturally into two categories, *nonrandom* and *random.* Fractals in physics belong to the second category, but it is instructive to first discuss a much-studied example of a nonrandom fractal—the Sierpiński gasket. We simply iterate a *growth rule*, much as a child might assemble a castle from building blocks. Our basic unit is a triangular-shaped tile shown in Fig. 1.1a, which we take to be of unit *mass* ($M = 1$) and of unit edge *length* ($L = 1$).

The Sierpiński gasket is defined operationally as an "aggregation process" obtained by a simple iterative process. In stage one, we join three tiles together to create the structure shown in Fig. 1.1b, an object of mass $M = 3$ and edge $L = 2$. The effect of stage one is to produce a unit with a lower density. If we define the density to be

$$\varrho(L) \equiv M(L)/L^2, \tag{1.1}$$

then the density decreases from unity to 3/4 as a result of stage one.

Now simply iterate—i.e., repeat this rule over and over *ad infinitum*. Thus in stage two, join together—as in Fig. 1.1c—three of the $\varrho = 3/4$ structures constructed in stage one, thereby building an object with $\varrho = (3/4)^2$. In stage

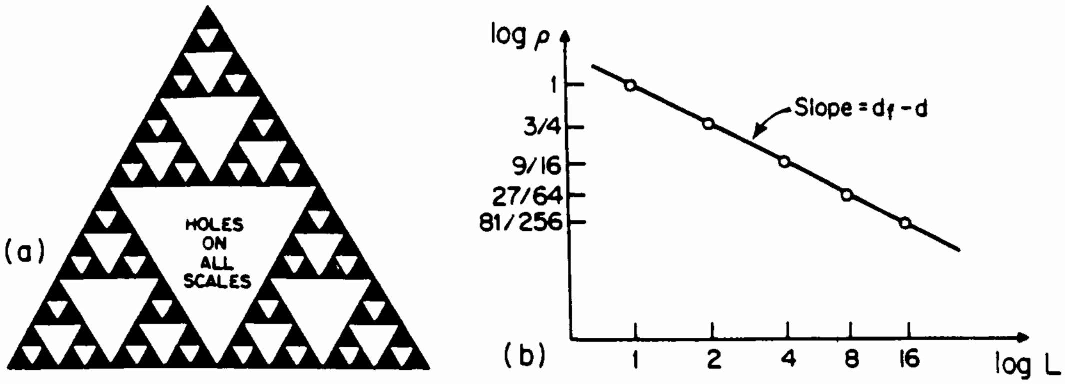

Fig. 1.2. (a) Sierpiński gasket fractal after four stages of iteration. (b) A log-log plot of ϱ, the fraction of space covered by black tiles, as a function of L, the linear size of the object. After [1.18]

three, join three objects identical to those constructed in stage two. Continue until you run out of tiles (if you are a physicist) or until the structure is infinite (if you are a mathematician!). The result after stage four—with 81 black tiles and $27+36+48+64$ white tiles (Fig. 1.2a) may be seen in floor mosaics of the church in Anagni, Italy, which was built in the year 1104. Thus although this fractal is named after the 20th century Polish mathematician W. Sierpiński, it was universally known some eight centuries earlier to every churchgoer of this village!

The citizens of Anagni did not have double-logarithmic graph paper in the 12th century. If they had possessed such a marvelous invention, then they might have plotted the dependence of ϱ on L. They would have found Fig. 1.2b, which displays two striking features:

- $\varrho(L)$ decreases monotonically with L, without limit, so that by iterating sufficiently we can achieve an object of *as low a density as we wish*, and
- $\varrho(L)$ decreases with L in a *predictable* fashion—a simple power law.

Power laws have the generic form $y = \mathcal{A}x^{\alpha}$ and, as such, have two parameters, the "amplitude" $\mathcal{A}$ and the exponent α. The amplitude is not of intrinsic interest, since it depends on the choice we make for the definitions of M and L. The exponent, on the other hand, depends on the process itself—i.e., on the "rule" that we follow when we iterate. Different rules give different exponents. In the present example, $\varrho(L) = L^{\alpha}$ so the amplitude is unity. The exponent is given by the slope of Fig. 1.2b,

$$\alpha = \text{slope} = \frac{\log 1 - \log(3/4)}{\log 1 - \log 2} = \frac{\log 3}{\log 2} - 2. \tag{1.2}$$

Finally we are ready to define the fractal dimension d_f, through the equation

$$M(L) \equiv \mathcal{A}\, L^{d_f}. \tag{1.3}$$

If we substitute (1.3) into (1.1), we find

$$\varrho(L) = \mathcal{A}\, L^{d_f - 2}. \tag{1.4}$$

Comparing (1.2) and (1.4), we conclude that the Sierpiński gasket is indeed a fractal object with fractal dimension

$$d_f = \log 3 / \log 2 = 1.58\ldots \quad . \tag{1.5}$$

Classical (Euclidean) geometry deals with regular forms having a dimension the same as that of the embedding space. For example, a line has $d = 1$, and a square $d = 2$. We say that the Sierpiński gasket has a dimension intermediate between that of a line and an area—a kind of "fractional" dimension—and hence the term *fractal*.

1.3 Random Fractals: The Unbiased Random Walk

Real systems in nature do not resemble the floor of the Anagni church—in fact, *nonrandom* fractals are not found in nature. Nature exhibits numerous examples of objects which by themselves are not fractals but which have the remarkable feature that, if we form a *statistical average* of some property such as the density, we find a quantity that decreases linearly with length scale when plotted on double logarithmic paper. Such objects are termed *random fractals*, to distinguish them from the nonrandom *geometric fractals* discussed in the previous section.

Consider the following prototypical problem in statistical mechanics. At time $t = 0$ an ant[1] is parachuted to an arbitrary vertex of an infinite one-dimensional lattice with lattice constant unity: we say $x_{t=0} = 0$. The ant carries an *unbiased* two-sided coin, and a clock. The dynamics of the ant is governed by the following rule. At each "tick" of the clock, it tosses the coin. If the coin is heads, the ant steps to the neighboring vertex on the east $[x_{t=1} = +1]$. If the coin is tails, it steps to the nearest vertex on the west $[x_{t=1} = -1]$.

There are *laws of nature* that govern the position of this drunken ant. For example, as time progresses, the average of the *square* of the displacement of the ant increases monotonically. The explicit form of this increase is contained in the following "law" concerning the *mean square displacement*:

$$\langle x^2 \rangle_t = t. \tag{1.6}$$

Equation (1.6) may be proved by induction, by demonstrating that (1.6) implies $\langle x^2 \rangle_{t+1} = t + 1$.

Additional information is contained in the expectation values of higher powers of x, such as $\langle x^3 \rangle_t$, $\langle x^4 \rangle_t$, and so forth. We can immediately see that

[1] The use of the term *ant* to describe a random walker is used almost universally in the theoretical physics literature—perhaps the earliest reference to this colorful animal is a 1976 paper of de Gennes that succeeded in formulating several general physics problems in terms of the motion of a "drunken" ant with appropriate rules for motion.